THE PATH TO GELFAND DUALITY AND SPECTRAL THEORY

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Contents

Conventions

$\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{F}$ denotes any arbitrary field.
$\mathbb{N}$ denotes the set of natural numbers (excluding 0) $\mathbb{N}_0$ is the set of natural numbers including 0
$\mathbb{R}_0^+$ denotes the set of all non-negative real numbers. $\mathbb{R}^+$ denotes the set of all positive real numbers.
If A, B are non-empty sets and $f : A \to B$ is a map, then the range of f is denoted by $f(A)$ or $\text{im}(f)$.
Let V, W be $\mathbb{F}$-vector spaces. The set of all linear maps $f : V \to W$ will be denoted by $\text{Hom}(V, W)$. If V and W are $\mathbb{K}$-normed spaces, then the set of bounded linear maps $T : V \to W$ is denoted by $\mathcal{L}(V, W)$
If (X, τ) is a topological space and $A \subseteq X$, then $\overline{A}$ is the closure of A and $\text{Int}(A)$ is the interior of A.
If (X, d) is a metric space, $a \in X$ and $r > 0$ is a constant, the $B(a, r)$ denotes the open ball of radius r centered at a.
If there are multiple metric spaces or any other ambiguities, we denote the open ball by $B_X(a, r)$

Abstract

We provide an introduction to Banach Algebras and $C^\star$-algebras. The discussion of $C^\star$-algebras leads to the two central theorems of this thesis: Gelfand Duality & the Gelfand-Naimark-Segal (GNS) Theorem. The latter tells us that every $C^\star$-algebra may be treated as some (special) subalgebra of bounded endomorphisms of a $\mathbb{C}$-Hilbert space, while the former tells us that every commutative $C^\star$-algebra can be realized as some space of continuous functions. Gelfand Duality is used, alongside some von Neumann algebra theory, as a stepping stone to the proof of the Spectral Theorem for Bounded Normal Operators. In particular, the proof depends on the crucial fact that the Gelfand spectrum of an abelian von Neumann algebra is a Stonian space.

Acknowledgements

I would like to thank my advisors Sandra Pott and Jörg Weber for their useful feedback about the bigger picture, correcting the gazillion (tiny) mistakes I made in my drafts and preventing me from going nuts with the number of topics covered in this thesis. I would also like to thank my friends (and teachers) for consistently reminding me that changing thesis topics every week was not a great idea[a] I also want to thank them for being ~~unwilling~~ **very willing** participants in conversations about my thesis topic. Trying to explain a whole number of (seemingly) disparate ideas to them helped me to piece it together in my head much better.

I would like to give special thanks to my friend, Daniel Falkowski, for providing the beautiful artwork used as the cover page for this thesis. Finally, I would like to thank my friend, Senan Sekhon, for the beautiful LaTeX template that he provided[b]. I would also like to thank him for the (multitude of) insightful discussions that we had on the topic. Part of that definitely came from our innate desire to outdo each other and that helped a lot in pushing the discussion forward to topics that were of interest to both of us. Thank you, truly, for your willingness to always indulge me, whether it was in my desire for big-picture discussions or in my need to inform you of the 126th proof of **BlahBlah Theorem** that I found while perusing the literature randomly at 3am.

[a]Mainly because I would find something new that was also really interesting that I would have wanted to include in my thesis. Of course, it is not possible to do that while keeping the thesis at a reasonable length.

[b]See https://www.overleaf.com/latex/templates/fun-template-2/sgqjnsftvysf.

Popular Scientific Summary

Fate whispers to the warrior, "A storm is coming.". The warrior whispers, "I am the storm.".

And you had best believe that $C^\star$-algebras caused a storm when they were first introduced. The story of these objects is actually rather inspiring and I want to tell you about it. Indeed, it serves as a reminder that we are only human and that we do have our limitations BUT we shouldn't let those limitations hinder us. In fact, it would do us a whole lot of good if we just accepted those limitations as part of our being. Let me paint a picture for you.

Imagine yourself as a (high school) physics student sitting in a lab, having been forced by your teacher to raise a ramp at various angles and measure the speed of a ball when it has rolled down the ramp. Initially, you think that it's a boring exercise but you've gone at it for four hours straight so you feel like you can take an hour more. Now, your teacher comes into the room and throws a curve-ball at you; they hand you a ramp which is the size of the Empire State Building[a] and you're asked to measure the speed of the ball after it has rolled down that ramp at various angles.

Don't get me wrong; this is a hard task for sure but you're not going to be able to measure the speed of that ball accurately by using any kind of measuring instrument that's available in your high school. With a disappointed sigh, you gather your belongings and mentally prepare yourself to break the bad news to your teacher; some speeds are just too large for you to measure. Unbeknownst to you, however, your teacher happens to have had a fair amount of training in mathematics at university and they tell you that you shouldn't view the limitations of your measuring device as a bad thing. Rather, one needs to accept those limitations **and** accommodate for them in their theory. Indeed, this is precisely what $C^\star$-algebras.

By modelling the world as a $C^\star$-algebra of sorts, we admit that there are actual limitations in the way that we can measure physical observables but those limitations are actively incorporated into the theory. That also means that we have a precise way of modelling those limitations. In some sense, those limitations are incorporated directly into the structure of the theory we build to describe the world. Isn't that just so inspiring? Doesn't it make you just want to dive headlong into their magnificent theory?

[a]I know this wouldn't happen in real life but just roll with it.

Chapter 1

Introduction

Welcome! It would likely not be a stretch to just assume that you're reading this because the title sounds cool. But don't you want to know what the thesis is *really* about? Beyond just the topics discussed, don't you want to understand how I actively went about **deciding** the structure of this thesis? These questions are "big picture" questions that I want to address (among other things). To do this in an organized fashion, I'm going to put forth a list of reasonable questions that one might have when wanting to read a dissertation [1] of any kind. I will address these questions as best as I possibly can. In the process, you will get a good sense of the answers to the two questions above (and much more). In this way, I also won't have to worry so much about keeping this introduction organized; it sort of organizes itself through this format. Is that fair?

What is this thesis **really** about/How is this thesis structured?

The topics covered here are a natural extension of basic Functional Analysis. Specifically, I discuss Banach Algebras, $C^\star$-algebras (which are just specific cases of Banach algebras) and Spectral Theory. Let me give a slightly more detailed outline of the coverage of each chapter.

1. **Chapter 2** focuses on the theory of Banach algebras. I expect that some of this material is familiar for readers who have studied some Functional Analysis, so I've mainly just quoted relevant results that the reader should be familiar with and I've mainly focused on the development of Gelfand's Theory of Commutative Banach Algebras.
2. **Chapter 3** covers the theory of $C^\star$-algebras, leading up to a (detailed) discussion of Gelfand Duality in both the commutative and non-commutative cases. In the commutative case, the main result is Theorem 3.5.7 and in the non-commutative case, the main result is the GNS Theorem given by Theorem 3.5.16. These are the two **main** theorems of this thesis and I spend a fair amount of time on each of them.
3. **Chapter 4** covers Spectral Theory, leading up to an (unconventional & not well-known) proof of the spectral theorem for bounded normal operators on a $\mathbb{C}$-Hilbert space. This is done by traversing through some basic operator theory (as is necessary) and some basic von Neumann Algebra theory. The latter culminates in an important topological result about the structure space of an abelian von Neumann algebra; this result is crucial for the proof of the spectral theorem.

Of course, I've tried to list the results above that are truly "central" to the thesis. On the other hand, I like to think that many other results (and sections, even) are quite impressive. Some of my absolute favorites are given below and I think you'll enjoy learning about them too:

1. Gelfand's Theory of Commutative Banach Algebras, Section 2.3.
2. The (Self-Adjoint) Continuous Functional Calculus, Theorem 3.2.4.
3. The Characterization of Positivity in $C^\star$-algebras, Theorem 3.3.8.
4. (Commutative) Gelfand Duality, Theorem 3.5.7.
5. Vigiers Theorem, Proposition 4.2.8.
6. The Spectral Theorem for Bounded Normal Operators, Theorem 4.3.10.

I don't really want to sound like I dislike any of the results in this thesis. The above are just a selection of some of my favorite results, for reasons that have to do with the aesthetic of the given result and the insight it provides about the theory. At this point, I should mention that I arranged the topics of this thesis and the material so that it would make a great precursor to the field of **Non-commutative Geometry**. In particular, our development of the spectral theorem (which proceeds by way of some von Neumann algebra theory) has closer ties to non-commutative Geometry than it does to the classical way that the spectral theorem is presented in the literature. I won't say anything more about this here but you can certainly read [25, Chapters 1 & 2, p.2-20] if you are interested in a more detailed elaboration of that point.

What are the prerequisites for reading this thesis?

[1] I have been informed that "dissertation" is not actually synonymous with "thesis". Nevertheless, I shall commit the first cardinal sin within this thesis by kindly ignoring that piece of information.

This depends on how you would like to read the text. If you want to read the thesis and understand all of the details, then there are typically five different subjects that go hand-in-hand with the topics discussed here and I will list them in decreasing order of importance:

1. General Topology [44];
2. Linear Algebra [2];
3. Functional Analysis [5];
4. Measure Theory [36];
5. Complex Analysis [35];

The first three requirements are quite prevalent; if you are not familiar with those three subjects, then you're essentially not going to be able to understand the details because I've tried to draw upon your knowledge of those topics at various points. I will say, however, that I have written appendices which contain a fairly complete overview of all the tools that I will need. For example, I don't imagine that a student who has studied General Topology would be so familiar with nets; that's why I've written an appendix that deals with nets. I've treated other bits of these subjects in a similar way.

Measure Theory is needed for the portion on the spectral theorem. However, you'll only really need it in Chapter 4.3 and in fact, the specific proof that I've chosen to give is rather devoid of Measure Theory. I don't actually use any results that are not, at the very least, plausible so just knowing the basics is fine. Complex analysis is needed to understand most of the proofs given for the statements in Section 2.1; I've provided references for them. However, I should say that it is possible to avoid the use of complex analysis for the most part; see [22] for proofs of many of these results which do not use much complex analysis. In any event, I do think that most of those results are "intuitively obvious" so you may not even really need the proofs to understand what I'm getting at in most cases.

But is it really necessary for you to understand every single technical detail? After all, the proofs themselves are not going to bring any insight if there is no discussion about the merits of the theory and its place with respect to other topics in mathematics and physics. Particularly with regards to the Gelfand-Naimark Theorems, it is relatively easy to understand what the theorems are saying once you know the definition of a $C^\star$-algebra and some examples that go along with the definition.

If you are only interested in these connections and are looking for ways in which this material connects to whatever you might be learning/working on right now, then a technical reading of this thesis is not necessary. In other words, you can skip the "technical" bits and focus on results which have a similar flavor to things you've already encountered[2]. In this case, I'd say that a background in basic analysis and abstract linear algebra is necessary to profitably read this thesis.

How should one read this thesis?

This will depend on your personal interests and it would be impossible for me to give advice for every possible combination of interests that an individual might have. If you're a mathematician, then **everything** here is good to know because there are connections between this material & almost everything else in mathematics. It is not appropriate, at this stage, for me to give a lengthy defense of that assertion so I leave you with a more sophisticated discussion available in [26, Appendix C].

Let me, however, comment on the role that this thesis plays for physicists. If you're a physicist who wishes understand the mathematical frameworks that Quantum Mechanics(QM) & (Algebraic) Quantum Field Theory(QFT) are built on, then the material in this thesis (and much more) is absolutely essential. Indeed, the theory of operator algebras (which this thesis partially deals with) goes hand-in-hand with the modern development of QM, QFT, Quantum Statistical Mechanics etc. Of course, there are many books that have been written on the topic but a good starting point for this development is [39] and the references therein.

What are the appendices all about?

There are three appendices that I've included in this thesis. All of them have one thing in common; they include material which did not quite fit in the main text. Of course, I do give descriptions at the very beginning of each appendix so it's not worth repeating them here. My hope, however, is that there is something in the appendices that you haven't encountered before & that you're compelled to learn it as you read the thesis.

With all of this being said, I do hope that you enjoy reading this thesis. Mathematics should be serious (indeed, the mathematics in this thesis is pretty serious) but it should also be fun & enjoyable to read. I have made a conscious effort to write in a way that would appeal to your child-like wonder of mathematics while also providing insights about the subject. I hope that I have been successful at this endeavour.

[2] I do make a point of being vocal about this when a theorem does come along and has those qualities. For example, there will be several points in the thesis where I talk about how $C^\star$-algebras behave in a similar way to complex numbers and it absolutely does show in some of the theorems we'll prove about them.

Chapter 2

The Theory of Banach Algebras

As the reader might know, an algebra is nothing but a vector space which has been equipped with an additional map that has some nice properties resembling "multiplication". A normed algebra is nothing but a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$) which has additional structure on it. A Banach algebra .... The focus of this thesis is on the theory of $C^\star$-algebras; as such, we will not spend a lot of time on a detailed exposition of the theory of Banach algebras. This chapter is divided into three main subsections:

1. **The Crucial Basics.** This is, essentially, the "prerequisite" knowledge section for this chapter. We will simply state all the definitions we need and none of the theorems will be given explicit proofs.
2. **Homomorphisms, Representations & Ideals.** In this section, we will discuss the theory of Banach algebras with a special emphasis on the algebraic point of view on the subject.
3. **Gelfand's Theory of Commutative Banach Algebras.** In this section, we will focus on commutative Banach algebras and prove both of the Gelfand Representation Theorems.

In all cases, the reader will not have to worry because we will provide references for any proofs that are omitted. With all this being said, let us begin.

2.1 The Crucial Basics

We begin with the definition of an $\mathbb{F}$-algebra, slowly working our way up to the definition of a $\mathbb{K}$-Banach algebra. We remind the reader of the convention that $\mathbb{F}$ refers to any field and $\mathbb{K}$ refers to $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1

Let V be a $\mathbb{F}$-vector space. Consider a map:

$$\cdot : V \times V \to V \ , \ (x, y) \mapsto x \cdot y$$

This map is said to be a multiplication (or vector multiplication) on V if it is bilinear and associative[a]. An algebra (or $\mathbb{F}$-algebra) is the pair $(V, \cdot)$ of a $\mathbb{F}$-vector space and a multiplication. A $\mathbb{K}$-normed algebra is a $\mathbb{K}$-algebra $(V, \cdot)$ with V being a $\mathbb{K}$-normed vector space such that:

$$\forall x, y \in V : \|x \cdot y\| \leq \|x\| \cdot \|y\|.$$

This is known as the triangle inequality for vector multiplication. Finally, a $\mathbb{K}$-Banach algebra is a $\mathbb{K}$-normed algebra that is also a $\mathbb{K}$-Banach space. A subalgebra of V is a subset of V that is also an algebra under the restricted operations.

[a] Some authors do not require associativity from an algebra. Non-associative algebras are explored further in [17, Chapter V].

Let us give some examples of normed algebras. We wish to focus mainly on three key examples.

Example 1. Let V be a $\mathbb{K}$-Banach space. Let $\mathcal{L}(V)$ be the vector space of bounded $\mathbb{K}$-linear maps $T : V \to V$. Equipped with the operator norm and composition as multiplication, this is a $\mathbb{K}$-Banach algebra. The case where V is a $\mathbb{C}$-Hilbert space will be of the greatest interest to us later.

Example 2. Let (X, τ) be a compact topological space. Then, $\mathcal{C}(X, \mathbb{C})$ is the $\mathbb{C}$-Banach algebra of all continuous functions $f : X \to \mathbb{C}$, equipped with point-wise operations and the uniform norm.

Example 3. Let (X, τ) be a locally compact Hausdorff space and let $\mathcal{C}_0(X)$ be the set of all continuous, complex-valued functions $f : X \to \mathbb{C}$ which vanish at infinity [a]. With point-wise operations and under the uniform norm, this is a commutative $\mathbb{C}$-Banach algebra.

[a] For any $\epsilon > 0$, there is a compact set K such that $|f(x)| < \epsilon$ whenever $x \in X \setminus K$.

Having introduced the notion of multiplication of vectors on a vector space, it seems reasonable to also speak of an identity element with respect to this operation. Actually, what we are really interested in is the notion of "division" or "inverses" in an algebra. In order to introduce that notion, we first need to talk about identity elements.

Definition 2.1.2

Let V be an $\mathbb{F}$-algebra. V is said to be an $\mathbb{F}$-algebra with identity if:

$$\exists e \in V : \forall x \in V : x \cdot e = e \cdot x = x.$$

V is said to be a $\mathbb{K}$-normed algebra with identity if it is a $\mathbb{K}$-normed algebra with an identity element. V is a unital $\mathbb{K}$-normed algebra if it is a $\mathbb{K}$-normed algebra with identity e such that $\|e\| = 1$.

It is clear that the Banach algebras in Example 1 and Example 2 have identity elements. On the other hand, the Banach algebra in Example 3 does not have an identity element. This is important to point out because having a (multiplicative) identity on an $\mathbb{F}$-algebra makes it possible to speak of multiplicative inverses.

Definition 2.1.3

Let V be an $\mathbb{F}$-algebra with identity e.

1. An element $x \in V$ is left invertible if there is a $y \in V$ such that $yx = e$.
2. An element $x \in V$ is right invertible if there is a $y \in V$ such that $xy = e$.

An element $x \in V$ is invertible if it is left-invertible and right-invertible. If an element $x \in V$ is not invertible, then it is said to be singular. The set of invertible elements will be denoted by V^{-1} and the set of singular elements will be denoted by S_V.

The study of invertibility of elements in normed algebras (and, in particular, in $\mathbb{C}$-Banach algebras) will constitute a significant part of the theory discussed in this section. A tool that is indispensable for this purpose is the spectrum of an element. This has a purely algebraic definiton, as seen below.

Definition 2.1.4

Let V be an $\mathbb{F}$-algebra with identity e. Let $x \in V$. The spectrum of x is defined as the following set:

$$\sigma_V(x) := \{\lambda \in \mathbb{F} : \lambda e - x \notin V^{-1}\}.$$

The resolvent of x is defined as $\rho_V(x) := \mathbb{F} \setminus \sigma_V(x)$ [a].

[a] We have chosen to define the resolvent here explicitly because it is standard to do this. However, the reader will find that we will not use this terminology very often because it is not necessary for what we want to do.

It is also possible for us to define the spectrum when V is an $\mathbb{F}$-algebra without identity; we will talk about how to do this later. Let us give two examples that include explicit computation of the spectrum.

Example 4. Let V be a finite-dimensional $\mathbb{F}$-vector space and consider $\mathcal{L}(V)$. Let $T : V \to V$ be a linear map. As before, we note that:

$$\lambda \in \sigma_{\mathcal{L}(V)}(T) \Leftrightarrow \det(T - \lambda \mathrm{Id}) = 0 \Leftrightarrow \lambda \text{ is an eigenvalue of } T$$

The theory of the spectrum of a linear map, in this case, simply reduces to the standard theory of eigenvalues/eigenvectors in linear algebra.

Example 5. Let (X, τ) be a compact topological space. Let $V = \mathcal{C}(X)$ be the $\mathbb{C}$-Banach algebra of complex-valued continuous functions with the sup norm. Let $f \in V$ and let $\lambda \in \mathbb{C}$. Then, $f - \lambda$ is not invertible if $f(x) - \lambda = 0$ for some $x \in X$. This implies that $\sigma_V(f) = f(X)$.

What we want to do is to reduce the computation of the spectrum as much as we possibly can for complicated combinations of elements in an $\mathbb{F}$-algebra. It would be nice if we had a nice set of formulas that helped with calculating, say, $\sigma_V(f^2 + f)$, with $f : X \to \mathbb{C}$ being a continuous function on a compact topological space. To make this more precise, we need a definition.

Definition 2.1.5

Let $p \in \mathbb{F}[z]$ be a polynomial in a field $\mathbb{F}$ with coefficients in $\mathbb{F}$. Let V be a $\mathbb{F}$-algebra with identity e. Then, we can write:

$$\forall z \in \mathbb{F} : p(z) = \sum_{k=0}^{n} a_k z^k.$$

Now, we define the associated polynomial function to be:

$$p : V \to V \ , \ x \mapsto p(x) := \sum_{k=0}^{n} a_k x^k$$

where $x^0 := e$. The multiplication of the coefficients with each x^k is scalar multiplication with a well-defined vector.

It is worth reminding the reader that there is an easy way of computing the spectrum of a polynomial of elements in a $\mathbb{C}$-algebra. This approach is given by the following theorem.

Theorem 2.1.6. The Spectral Mapping Theorem for Polynomials

Let V be a $\mathbb{C}$-algebra with identity and let $x \in V$. Let $p \in \mathbb{C}[z]$ be a complex polynomial with complex coefficients. Suppose that $\sigma_V(x) \neq \varnothing$. Then, $\sigma_V(p(x)) = p(\sigma_V(x))$.

Proof. See [21, p.181, Proposition 3.2.10] for the proof. □

Of course, the point of the discussion above was to illustrate the fact that establishing a functional calculus for the elements of a Banach algebra does require a bit of effort. Nevertheless, there is an easy "power series" of Banach algebra elements that we can deal with and it is going to be quite useful for future proofs.

Theorem 2.1.7. Neumann Series Theorem

Let V be a $\mathbb{C}$-Banach algebra with identity e and let $x \in V$ be such that $\|x\| < 1$. Then, $e - x$ is invertible and:

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

Proof. For the proof, see [21, p.175, Lemma 3.1.5] □

The Neumann Series Theorem is crucial in the assertion that the spectrum of an element in a unital $\mathbb{C}$-Banach algebra is non-empty and compact. The fact that the spectrum is compact does not automatically tell us that it should be empty.

Theorem 2.1.8

Let V be a unital $\mathbb{C}$-Banach algebra. Let $x \in V$. Then, $\sigma_V(x) \neq \varnothing$ and $\sigma_V(x)$ is a compact[a] subset of $\mathbb{C}$.

[a] In particular, it can be shown that $|\lambda| \leq \|x\|$ for every $\lambda \in \sigma_V(x)$; this will follow quickly from the Neumann Series Theorem.

Proof. A nice proof may be found in [21, Theorem 3.2.3, p.179]. □

It is natural for the reader to be curious about what happens when all the elements of a unital $\mathbb{C}$-Banach algebra are invertible. In this case, we actually get the following nice result.

Theorem 2.1.9. The Gelfand-Mazur Theorem

Let V be a unital $\mathbb{C}$-Banach algebra in which every non-zero element is invertible. If e is the identity, then $V = \mathbb{C}e$.

Proof.
Suppose that there is an $x \in V$ such that $x \notin \mathbb{C}e$. So, for every $\lambda \in \mathbb{C}$, $x - \lambda e \neq 0$. This implies that $x - \lambda e$ is invertible for every λ. In particular, $\sigma_V(x) = \varnothing$ and that is a contradiction. □

Let V be a unital $\mathbb{C}$-Banach algebra and let $x \in V$ be fixed. The fact that $\sigma_V(x)$ is compact is all well and good. On the other hand, we would like to know a "measure" of its size, in some sense. To that end, we define the spectral radius of normed algebra element.

Definition 2.1.10

Let V be a unital $\mathbb{K}$-normed algebra. Let $x \in V$ be fixed. The spectral radius of x is given by:

$$\nu(x) = \sup_{\lambda \in \sigma_V(x)} |\lambda|.$$

x is said to be quasi-nilpotent or topologically nilpotent if $\nu(x) = 0$.

It would also be nice if we had an explicit formula for the spectral radius that depended on x. This is motivated as follows. Let V be a unital $\mathbb{K}$-normed algebra and let $x \in V$, with e being the identity. If $\lambda \in \sigma_V(x)$ and $n \in \mathbb{N}$, then $\lambda^n \in \sigma_V(x^n)$ by the Spectral Mapping Theorem for Polynomials. It follows that:

$$\forall n \in \mathbb{N} : |\lambda|^n \leq \|x^n\|,$$

and this implies that:

$$\forall n \in \mathbb{N}; |\lambda| \leq \|x^n\|^{\frac{1}{n}}.$$

Since those holds for any $\lambda \in \sigma_V(x)$, it follows that:

$$\nu(x) \leq \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

This short derivation raises two reasonable questions:

1. Is it possible that $\liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} < \infty$?
2. Is it the case that $\nu(x) = \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}$?

We will attempt to address both questions in just a moment. Before we do that, let us state the following useful result.

Proposition 2.1.11. Jacobson's Lemma

Let V be a unital $\mathbb{C}$-Banach algebra. The following statements hold:

1. $\forall x, y \in V : \sigma_V(xy) \cup \{0\} = \sigma_V(yx) \cup \{0\}$.
2. $\forall x \in V^{-1} : \sigma_V(x^{-1}) = \sigma_V(x)^{-1}$ [a].

[a] This is the set of all reciprocals of elements in $\sigma_V(x)$.

Proof. An easy proof of (1) can be found in [21, Proposition 3.2.8, p. 180-181]. The proof of (2) is a standard computation so we omit it. □

We can safely return to the two questions above. Based on the first line of inquiry, the following definition is warranted.

Definition 2.1.12

Let V be a $\mathbb{K}$-normed algebra. Define the function:

$$\phi : V \to \mathbb{C} \ , \ x \mapsto \phi(x) = \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

For lack of creativity on the part of the author, this will just be known as the ϕ-function.

This is just temporary notation because it can be shown that $\phi(x) = \nu(x)$ for every element x in a unital $\mathbb{C}$-normed algebra. Before we state that result, let us consider the following result concerning the function in Definition 2.1.12.

Theorem 2.1.13

Let V be a $\mathbb{K}$-normed algebra. Then, the following statements hold:

1. $\forall x \in V : \phi(x) = \inf\{\|x^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}$.
2. $\forall x \in V : 0 \le \phi(x) \le \|x\|$.
3. $\forall x \in V : \forall \mu \in \mathbb{K} : \phi(\mu x) = |\mu|\phi(x)$.
4. $\forall x, y \in V : \phi(xy) = \phi(yx)$.
5. $\forall x \in V : \forall k \in \mathbb{N} : \phi(x^k) = \phi(x)^k$.
6. $\forall x, y \in V : xy = yx \Rightarrow \phi(xy) \le \phi(x)\phi(y)$.
7. $\forall x \in V : \left\|x^2\right\| = \|x\|^2 \Leftrightarrow \phi(x) = \|x\|$.

Proof. A fairly straightforward proof of this may be found in [13, Proposition B.1.1, p.298-299]. □

The following result, again, shows that there is a nice "limit" formula for computing the spectral radius of some element in a unital $\mathbb{C}$-normed algebra.

Theorem 2.1.14. The Spectral Radius Formula

Let V be a unital $\mathbb{C}$-Banach algebra. Then, the spectral radius is given by the formula:

$$\forall x \in V : \nu(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

Proof. An accessible proof of this may be found in [21, Theorem 3.3.3, p.204-205]. □

There is one last thing which needs to be discussed in this context before we move on. This is the problem of unitization. This is an important problem because we would like to be able to use the "unital" theory to deal with Banach algebras without identity. To do this, we will need the so-called "Unitization Theorem".

Definition 2.1.15

Let V be a $\mathbb{K}$-normed algebra. The unitization of V is the product vector space $V_e := V \times \mathbb{K}$ where the product is defined by the following prescription:

$$\forall x, y \in V : \forall \lambda, \mu \in \mathbb{K} : (x, \lambda) \cdot (y, \mu) := (xy + \lambda y + \mu x, \lambda\mu).$$

The norm on V_e is defined by:

$$\forall (x, \lambda) \in V_e : \|(x, \lambda)\| := \|x\| + |\lambda|.$$

This turns V_e into a $\mathbb{K}$-normed algebra.

It certainly has to be checked that the "norm" is **actually** a norm; these are routine checks that are not worth the space we might spend on them. There are two points that we should take note of:

1. The unitization V_e has identity $(0, 1)$ and this has norm 1 in V_e. So, V_e is certainly a unital $\mathbb{K}$-normed algebra.
2. V can be naturally "embedded" into V_e. Indeed, we can define a map:

$$\Psi : V \to V_e \ , \ x \mapsto \Psi(x) := (x, 0)$$

This embedding is norm-preserving. The embedding is the image of Ψ, $V \times \{0\}$.

Linear functionals can be extended quite easily in this framework too. Let V be a $\mathbb{K}$-normed algebra and let $f : V \to \mathbb{K}$ be a linear functional. We extend f to the linear functional $f_e : V_e \to \mathbb{K}$:

$$\forall x \in V : \forall \lambda \in \mathbb{K} : f_e((x, \lambda)) := f(x) + \lambda.$$

It is easy to check that this is a linear functional. In fact, an even stronger result can be shown.

Proposition 2.1.16

Let V be a $\mathbb{K}$-normed algebra and let $f : V \to \mathbb{K}$. The following are equivalent:

1. f is continuous.
2. f_e, the extension of f to the unitization, is continuous.

Proof. The implication (1) ⇒ (2) is obvious. Assume that f_e is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V that converges to some $x \in V$. Then, the sequence $((x_n, 0))_{n\in\mathbb{N}}$ in V_e converges to $(x, 0)$. It follows that:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f_e((x_n, 0)) = f_e((x, 0)) = f(x)$$

as was desired. □

We will discuss a similar result concerning multiplicative linear functionals in the last section of this chapter. We should mention that it is possible to "unitize" a unital $\mathbb{K}$-normed algebra; indeed, the construction we did allows for it and that would produce a totally different algebra. With all this being said, we can now state the unitization theorem in full.

Theorem 2.1.17. The Unitization Theorem (Normed Algebras)

Let V be a $\mathbb{K}$-normed algebra, possibly without identity. Then, there exists a unital $\mathbb{K}$-normed algebra which has V embedded as a closed subspace within it. This is denoted by V_e and V_e is complete with respect to this norm if V is complete.

Proof. Let $V_e := V \times \mathbb{K}$ and give V_e the structure as defined in Definition 2.1.15. These prescriptions turn V_e into a unital $\mathbb{K}$-normed algebra, with the identity being $e = (0, 1)$. Showing that V is a $\mathbb{K}$-normed algebra is just a matter of doing some computations, while showing that V_e is unital just involves showing that $\|e\| = 1$, as we do below:

$$\|e\| := \|0\| + |1| = 1.$$

Moreover, it is obvious that V can be embedded into V_e by the map $x \mapsto (x, 0)$. In fact, this embedding is norm-preserving, as we show now:

$$\forall x \in V : \|(x, 0)\| = \|x\| + |0| = \|x\|.$$

This implies that the embedding of V into V_e is closed. Now, assume that V is complete; let us show that V_e is complete. Let $((x_n, \lambda_n))_{n\in\mathbb{N}}$ be a Cauchy sequence in V_e. This implies, by the definition of the norm on V_e, that $(x_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ are Cauchy sequences. It follows that they converge to x and λ respectively. But note that:

$$\lim_{n\to\infty} \|(x_n, \lambda_n) - (x, \lambda)\| = \lim_{n\to\infty} \|(x_n - x, \lambda_n - \lambda)\| = \lim_{n\to\infty} (\|x_n - x\| + \|\lambda_n - \lambda\|) = 0.$$

This implies that $(x_n, \lambda_n) \to (x, \lambda)$ and we are done. □

One should notice that the unitization of an algebra creates an entirely new algebra. In particular, it is possible to "unitize" a normed algebra which already has an identity element. Another consequence of this theorem is that we can (and we will) define the spectrum of an element in a $\mathbb{K}$-algebra without identity. Since we carried out the unitization construction above for $\mathbb{K}$-normed algebras, let us just keep things simple and define the spectrum in that special case.

Definition 2.1.18

Let V be a $\mathbb{K}$-normed algebra without identity. Let $x \in V$. We define the spectrum of x by:

$$\sigma_V(x) := \sigma_{V_e}((x, 0)).$$

where we compute, on the right-hand side, the spectrum of $(x, 0)$ in the unitization of V.

Unitization becomes a far more serious issue for $C^\star$-algebras. Ideally, we would want V_e, as defined above, to be the unitization of a $C^\star$-algebra V but we still need to make sure that the norm given above actually satisfies the so-called "$C^\star$-condition" (we will introduce this later). As it turns out, this is not possible so we will have to come up with a new norm which does fit the bill. The great news here is that this new norm (which we will introduce later) is equivalent to the one given above. In any event, we will cross that bridge when we have to; for now, we should move forward with a discussion of homomorphisms & ideals.

2.2 Homomorphisms & Ideals

In this section, we would like to focus on the algebraic aspects of the theory in preparation for our exposition of Gelfand's Theory of Commutative Banach Algebras. This section will be a mix of algebraic and analytic results, starting off with an introduction of relevant algebraic notions. We start by considering structure-preserving maps between $\mathbb{F}$-algebras.

Definition 2.2.1

Let V, W be $\mathbb{F}$-algebras. Let $\phi : V \to W$ be a map. ϕ is said to be an algebra homomorphism if:

1. ϕ is a linear map and;
2. $\forall v_1, v_2 \in V : \phi(v_1 v_2) = \phi(v_1)\phi(v_2)$. (Multiplicativity)

If both V and W have identities e_V and e_W respectively, then ϕ is a unital algebra homomorphism if ϕ is an algebra homomorphism which satisfies $\phi(e_V) = e_W$. If ϕ is bijective, then we say that ϕ is an algebra isomorphism. When $W = \mathbb{F}$, with its field multiplication, we say that ϕ is a multiplicative linear functional.

As we know from linear algebra and ring theory, the correct way to study homomorphisms is to study their action on the "correct substructures" of the space that one starts with. In the case of algebras, these "correct substructures" are ideals and those will be immensely useful to us in all of our work in this section. Let us give the definition.

Definition 2.2.2

Let V be an $\mathbb{F}$-algebra. A set $I \subseteq V$ is said to be a left ideal of V if:

1. I is a subspace.
2. $\forall x \in I : \forall y \in V : xy \in I$.

Right ideals are defined similarly. A two-sided ideal (or just an ideal) is a subspace which is both a left ideal and a right ideal. If I is a left/right ideal and $I \neq V$, then I is said to be a proper left/right ideal.

A simple computation tells us that if V is an $\mathbb{F}$-algebra and $I \subseteq V$ is an ideal, then V/I is an $\mathbb{F}$-algebra with the usual operations. Let us give two examples that illustrate where the notion of an ideal arises naturally in analysis.

Example 6. Let V be a $\mathbb{K}$-normed space. Then, the subspace $\mathcal{K}_{\mathbb{K}}(V)$ of compact operators on V is an ideal of $\mathcal{L}(V)$.

Example 7. Let $S \subseteq \mathbb{C}$ be a non-empty set with no accumulation point. Let T be the set of all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ which are zero on S. This is an ideal.

Ideals may possess additional properties that might be interesting to study and we wish to gain an understanding of what these properties are. Before we get into that, the following result should be of interest.

Proposition 2.2.3

Let V be a $\mathbb{K}$-normed algebra and let $I \subseteq V$ be a closed proper (two-sided) ideal. Consider the $\mathbb{K}$-normed space V/I equipped with the quotient norm. The following statements hold:

1. V/I is a $\mathbb{K}$-normed algebra.
2. If V is commutative, then V/I is commutative.
3. If V is unital, then V/I is unital.
4. If V is a $\mathbb{K}$-Banach algebra, then V/I is a $\mathbb{K}$-Banach algebra.

Proof. Since I is closed, V/I is a $\mathbb{K}$-normed space in the standard way. The given statements, then, follow directly as a consequence of the discussion we have been having thus far. □

Let V be an $\mathbb{F}$-algebra without identity and suppose that $I \subseteq V$ is an ideal. Then, V/I is guaranteed to be an $\mathbb{F}$-algebra. Now, suppose that V/I has an identity element. Explicitly, this means that:

$$\exists e \in V : \forall x \in V : (e + I)(x + I) = (x + I)(e + I) = x + I$$

This is equivalent to the statement that there exists an $e \in V$ such that $ex + I = xe + I = x + I$ for every $x \in V$. In other words, the ideal I must satisfy $ex - x \in I$ and $xe - x \in I$ for every $x \in V$. This motivates the next definition.

Definition 2.2.4

Let V be an $\mathbb{F}$-algebra. Let $I \subseteq V$ be an ideal.

1. I is a left-modular ideal if:
$$\exists e \in V : \forall x \in V : ex - x \in I.$$
2. I is a right-modular ideal if:
$$\exists e \in V : \forall x \in V : xe - x \in I.$$
3. I is said to be a modular ideal if it is a left-modular ideal and a right-modular ideal.

In each case, one may refer to e as a [-]-identity modulo I, where the blank is supposed to be modified depending on whether I is left modular, right modular or modular.

Obviously, if V has an identity element, then every ideal must be modular. In constructing Definition 2.2.4, we have actually shown that V/I has an identity implies that I must be modular. On the other hand, we should also show that I being modular implies that V/I has an identity.

Proposition 2.2.5

Let V be an $\mathbb{F}$-algebra and let $I \subseteq V$ be an ideal. The following are equivalent:

1. V/I has an identity element.
2. I is modular.

Proof. The statement (1) $\Rightarrow$ (2) follows from the preceding discussion. So, let us prove (2) $\Rightarrow$ (1). Since I is modular, there exists an $e \in V$ such that for all $x \in V$, $ex - x \in I$. We claim that $e + I$ is the identity element for V/I. Indeed, let $x \in V$. Then, $(x + I)(e + I) = x + I$ if $xe - x \in I$. But this follows from I being a modular ideal. Similarly, $(e + I)(x + I) = x + I$ if $ex - x \in I$ and this follows from I being a modular ideal so we are done. □

Let us also speak of the so-called maximal ideals; these are the second type of ideals that we will be interested in. We will obtain a working definition in a different way, based on the following result.

Proposition 2.2.6

Let V be an $\mathbb{F}$-algebra. Let $(I_\alpha)_{\alpha \in A}$ be a directed family of left ideals, totally ordered by inclusion[a]. Then, $\bigcup_{\alpha \in A} I_\alpha$ is a left ideal.

[a]See Definition C.0.1.

Proof. We have to show that the union is a subspace and that for every $x \in V$ and $y \in \bigcup_{\alpha \in A} I_\alpha$, it is the case that $xy \in \bigcup_{\alpha \in A} I_\alpha$. Indeed, let $x, y \in \bigcup_{\alpha \in A} I_\alpha$ and let $\alpha \in \mathbb{F}$. Then, there exist $\beta, \gamma \in A$ such that $x \in I_\beta$ and $y \in I_\gamma$. Since A is directed, there is a $\kappa \in A$ with $\beta \leq \kappa$ and $\gamma \leq \kappa$ so that $I_\beta \subseteq I_\kappa$ and $I_\gamma \subseteq I_\kappa$. This implies that $x, y \in I_\gamma$ so $\alpha x + y \in I_\gamma$. This proves that $\bigcup_{\alpha \in A} I_\alpha$ is a subspace. Next, let $x \in V$ and let $y \in \bigcup_{\alpha \in A} I_\alpha$. Then, there is a $\beta \in A$ such that $y \in I_\beta$. Therefore, $xy \in I_\beta$ and this implies that $xy \in \bigcup_{\alpha \in A} I_\alpha$. This completes the proof. □

This proposition gives us a "picture" of maximal ideals that is encapsulated in the following definition.

Definition 2.2.7

Let V be an $\mathbb{F}$-algebra. An ideal $I \subseteq V$ is maximal if it is proper and for any proper ideal J with $I \subseteq J$, $I = J$.

There is an important existence theorem which can be proven for maximal ideals using the rough discussion we just had and we will prove it in just a moment. Before that, it is useful to have a list of permanence properties which relate ideals and algebra homomorphisms, just so that we can freely reference these properties without deriving them from scratch.

Proposition 2.2.8

Let V, W be $\mathbb{F}$-algebras. Let $f : V \to W$ be a surjective algebra homomorphism. The following statements hold:

1. Let J be an ideal in W. Then, $f^{-1}(J)$ is an ideal in V and $\ker(f) \subseteq f^{-1}(J)$. Moreover, $J = f(f^{-1}(J))$.
2. Let I be an ideal in V. Then, $f(I)$ is an ideal in W. If $\ker(f) \subseteq I$, then $I = f^{-1}(f(I))$.
3. There is a bijection between the set of ideals I in V that contain $\ker(f)$ and the set of ideals in W.
4. If J is a modular ideal in W, then $f^{-1}(J)$ is a modular ideal in V.
5. If I is a modular ideal in V with $\ker(f) \subseteq I$, then $f(I)$ is a modular ideal in W.
6. There is a bijection between the set of modular ideals I in V which satisfy $\ker(f) \subseteq I$ and the set of modular ideals in W.
7. There is a bijection between the set of maximal modular ideals I in V which satisfy $\ker(f) \subseteq I$ and the set of maximal modular ideals in W.

Proof. Statements (1)-(5) are fairly standard computations which we omit. We will prove (6) and the proof of (7) is similar. Let $\mathcal{I}_V$ be the set of modular ideals in V that satisfy $\ker(f) \subseteq I$ for every $I \in \mathcal{I}_V$. Let $\mathcal{I}_W$ be the set of modular ideals in W. Define the map:

$$\phi : \mathcal{I}_V \to \mathcal{I}_W \ , \ I \mapsto \phi(I) := f(I)$$

By the strength of (5), it is the case that $f(I)$ must be a modular ideal in W. Let us show that this is a bijection. Indeed, let $\phi(I_1) = \phi(I_2)$, with $I_1, I_2 \in \mathcal{I}_V$. Then, $f(I_1) = f(I_2)$. By statement (2), it follows that $I_1 = f^{-1}(f(I_1)) = f^{-1}(f(I_2)) = I_2$. This proves injectivity. Next, let $J \in \mathcal{I}_W$. Then, $f^{-1}(J)$ is a modular ideal in V and $\ker(f) \subseteq f^{-1}(J)$ so $f^{-1}(J) \in \mathcal{I}_V$. By statement (1), it follows that $\phi(f^{-1}(J)) = f(f^{-1}(J)) = J$. This proves the result. □

Going back to the infinitely more interesting existence result we mentioned, let us make use of the reasoning we alluded to earlier using partial orders and Zorn's lemma to prove the following (familiar) result.

Theorem 2.2.9. Krull's Theorem

Let V be an $\mathbb{F}$-algebra with identity. Let I be a proper ideal in V. Then, I is contained in some maximal ideal.

Proof. The proof proceeds by Zorn's Lemma. Define:

$$\mathcal{I}_I : \{J \subseteq V : J \text{ is a proper ideal} \wedge I \subseteq J\}.$$

We can partially order this set by inclusion and it is, then, possible to consider a chain $\mathcal{C}$ within this set. Of course, this chain is a directed family of ideals so its union must be an ideal. The only thing left to do is to show that this union is proper. If it was not proper, then it would contain the identity. Therefore, one of the ideals in $\mathcal{C}$ would contain the identity and, as a consequence, be improper. This would contradict the assumption that $\mathcal{C}$ only contains proper ideals. Therefore, every chain has an upper bound in $\mathcal{I}_I$ and this implies that $\mathcal{I}_I$ must have a maximal element. □

For our purposes, it would be good to obtain a characterization of maximal modular ideals. To do this, we introduce the so-called division algebras; they are essential to this characterization.

Definition 2.2.10

Let V be an $\mathbb{F}$-algebra. V is a division algebra if V has an identity e and every non-zero element in V is invertible.

The following lemma is a characterization of division algebras and a key step in allowing us to characterize maximal modular ideals. The approach below is similar to how we characterize maximal ideals in ring theory.

Lemma 2.2.11 Let R be an $\mathbb{F}$-algebra. The following are equivalent:

1. R is a division algebra.
2. The only ideals in R are (0) and R.
3. Every non-zero $\mathbb{F}$-algebra homomorphism $f : R \to S$, with S being an non-zero $\mathbb{F}$-algebra, is injective.

Proof.

$(1) \Rightarrow (2)$: Let R be a divison algebra. If $I \subseteq R$ is an ideal that contains any non-zero element, then the invertibility of this element implies that I also contains the identity. This implies that $I = R$ itself.

$(2) \Rightarrow (3)$: Let $f : R \to S$ be a non-zero algebra homomorphism into an $\mathbb{F}$-algebra. Then, $\ker(f)$ is certainly an ideal in R. It is either the zero ideal or it is R itself. If $\ker(f) = R$, then f would be zero everywhere and that is impossible. It follows that $\ker(f)$ must be trivial.

$(3) \Rightarrow (1)$: We need to show that every non-zero element in R is invertible. Suppose that some $x \in R$ is not invertible. Then, (x) is a proper ideal and we define $S = R/(x)$. Define a map:

$$\phi : R \to S \ , \ y \mapsto y + (x)$$

Since (x) is proper, it follows that ϕ is non-zero. This means that $\ker(\phi) = (x) = (0)$ so $x = 0$. □

Theorem 2.2.12

Let V be an $\mathbb{F}$-algebra and let I be an ideal in V. The following are equivalent:

1. I is a maximal modular ideal.
2. V/I is a division algebra.

Proof.

$(1) \Rightarrow (2)$: The fact that I is modular implies that V/I is an $\mathbb{F}$-algebra with identity. Consider the algebra homomorphism $\pi : V \to V/I$. Let us count the ideals in V which contain $\ker(\pi) = I$. If there is a proper ideal in V that contains π, then it must be I itself. In other words, there are only two ideals in V that contain $\ker(\pi)$. This implies that V/I has only two ideals; that is, it is a divison algebra.

$(2) \Rightarrow (1)$: Assume that V/I is a division algebra. Since V/I has an identity, it follows that I must be modular so it remains to be shown that I is maximal. Let $J \subseteq V$ be a proper ideal which contains I. Again, consider the projection mapping $\pi : V \to V/I$. Since V/I is a division algebra, it has only two ideals so V has only two ideals which contain $\ker(\pi) = I$. As a consequence, it follows that $I = J$ and that I is maximal. □

These results will be relevant later and we will reference them whenever necessary. So far, we have mainly focused on establishing algebraic results about ideals. In order to properly apply this material to $\mathbb{C}$-Banach algebras, we need to also collect some topological properties of ideals when they are subsets of $\mathbb{C}$-Banach algebras.

Proposition 2.2.13

Let V be a $\mathbb{C}$-Banach algebra. The following statements hold:

1. If $I \subseteq V$ is an ideal, then $\overline{I}$ is also an ideal.
2. If $I \subseteq V$ is a proper modular ideal, then $\overline{I}$ is a proper modular ideal.
3. If $I \subseteq V$ is a maximal modular ideal, then I is closed.
4. If $I \subseteq V$ is a proper modular ideal of V, then it is contained in a maximal modular ideal.

Proof. We will prove each statement one-by-one.

1. The fact that $\overline{I}$ is a subspace is standard. Let $x \in V$ and $y \in \overline{I}$. Then, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in I that converges to y. By continuity, it follows that $xy_n \to xy$. But $xy_n \in I$ for every $n \in \mathbb{N}$. Hence, $xy \in \overline{I}$.
2. Observe that $\overline{I}$ is an ideal. It remains to be shown that it is a proper modular ideal. The fact that it is modular just follows from an approximation argument. Now, we just need to show that it is proper. Let $e + I$ be the identity of V/I. Then, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in I that converges to e. Observe that:
$$\|(e + I) - (x_n + I)\| = \inf_{y\in I} \|(e - x_n) + y\| \leq \|e - x_n\|.$$
But this implies that $\|(e + I) - (x_n + I)\| < 1$ for large enough n. By Theorem 2.1.7, it follows that $x_n + I$ is invertible for large enough n. Since $x_n + I = I$ for each $n \in \mathbb{N}$, it follows that $x_n + I$ can never be invertible. It follows that $\overline{I}$ is a proper ideal.
3. Let I be a maximal modular ideal. Then, it is a proper modular ideal. So, $\overline{I}$ is a proper modular ideal. By the maximality of I, it follows that $I = \overline{I}$. This implies that it is closed.
4. An argument that proceeds similarly to Krull's theorem establishes the result. We omit the details. □

Obviously, there are many other statements about ideals and homomorphisms that we could prove. It was never our intent to give an exhaustive treatment. Nevertheless, readers who are more interested in a detailed look at the content relating to the material in this section are encouraged to see [13, Appendix B, p.297-372]. The introduction given there is particularly detailed and interesting. For now, we set our sights on giving a good exposition of Gelfand's theory.

2.3 The Gelfand Theory of Commutative Banach Algebras

While Gelfand's theory mainly deals with commutative $\mathbb{C}$-Banach algebras, we will not always assume that we are working with $\mathbb{C}$-Banach algebras. This is just to make sure that we are not including superfluous assumptions that confuse the reader. This section is (very) loosely based off of [13, Appendix B.6, p.337-342].

Definition 2.3.1

Let V be a $\mathbb{F}$-algebra. A linear functional $\phi : V \to \mathbb{F}$ is multiplicative if it is non-zero and satisfies:
$$\forall x, y \in V : \phi(xy) = \phi(x)\phi(y).$$
We also, sometimes, say that ϕ is a character. The space of all characters is denoted by $\widehat{V}$.

Hopefully, the reader agrees that it is natural to study a $\mathbb{C}$-normed algebra by looking at non-trivial linear functionals that preserve multiplication. Before we get to understanding that, let us deal with unitization in relation to multiplicative linear functionals. If V is a $\mathbb{K}$-normed algebra, then we would like to know the relationship between $\widehat{V}$ and $\widehat{V_e}$.

Proposition 2.3.2

Let V be a $\mathbb{K}$-normed algebra and let V_e be its unitization. Let $\phi_e : V_e \to \mathbb{K}$ be a multiplicative linear functional. Define the map:
$$\phi : V \to \mathbb{K}\ ,\ x \mapsto \phi(x) := \phi_e((x, 0))$$
Then, ϕ is a multiplicative linear functional and ψ_e (the extension of ϕ to V_e) is ϕ_e.

Proof. The multiplicativity of ϕ can be easily checked using the definition. Next, we have that:
$$\forall x \in V : \forall \lambda \in \mathbb{K} : \psi_e((x, \lambda)) := \phi(x) + \lambda = \phi_e((x, 0)) + \lambda.$$
Since ϕ_e must map the identity to the identity, it follows that $\phi_e((0, 1)) = 1$. But this means that $\lambda = \phi_e((0, \lambda))$. As a consequence, $\psi_e((x, \lambda)) = \phi_e((x, 0)) + \phi_e((0, \lambda)) = \phi_e((x, \lambda))$ for every $x \in V$ and $\lambda \in \mathbb{K}$ so we are done. □

The proposition tells us that it is possible to view $\widehat{V}$ as a subset of $\widehat{V_e}$. The next result tells us that we can associate a maximal modular ideal to every multiplicative linear functional.

Proposition 2.3.3

Let V be an $\mathbb{F}$-algebra and let $\phi : V \to \mathbb{F}$ be multiplicative. Then, $\ker(\phi)$ is a maximal modular ideal.

Proof. Let ϕ be multiplicative. It follows that ϕ is a surjective algebra homomorphism. It follows that $V/\ker(\phi) \equiv \phi(V) = \mathbb{F}$. Since $\mathbb{F}$ is a division algebra, it follows by Theorem 2.2.12 that $\ker(\phi)$ is a maximal modular ideal. □

If we, then, started with an ideal I of an $\mathbb{F}$-algebra V, is it possible for I to be the kernel of some multiplicative linear functional? The answer appears to be in the affirmative for commutative $\mathbb{C}$-Banach algebras.

Theorem 2.3.4. Correspondence Theorem for Multiplicative Linear Functionals

Let V be a commutative $\mathbb{C}$-Banach algebra. If I is a maximal modular ideal in V, then there exists a unique multiplicative linear functional $\phi : V \to \mathbb{C}$ such that $\ker(\phi) = I$.

Proof. Let us show uniqueness first. Let ϕ, ϕ' be two multiplicative linear functionals with $\ker(\phi) = \ker(\phi') = I$. Since I is modular, there exists an identity $e + I$ for V/I. Let $x \in V$. Now, observe that:

$$\phi(x - \phi(x)e) = \phi(x) - \phi(x)\phi(e) = \phi(x) - \phi(xe) = \phi(x - xe) = 0,$$

where the last equality follows because $x - xe \in I$ and $\ker(\phi) = I$. It follows that $x - \phi(x)e \in \ker(\phi)$. Observe that $\phi(x) = \phi(x)\phi(e)$ for all $x \in V$. Since ϕ is multiplicative, it follows that $\phi(a) \neq 0$ for some $a \in V$. Hence, $\phi(e) = 1$ and by similar considerations, we know that $\phi'(e) = 1$. Since $x - \phi(x)e \in I$, it follows that $x - \phi(x)e \in \ker(\phi')$. Hence:

$$0 = \phi'(x - \phi(x)e) = \phi'(x) - \phi(x)\phi'(e) = \phi'(x) - \phi(x),$$

which implies that $\phi(x) = \phi'(x)$. Now, we will prove existence. There are two steps in the proof:

1. We need to show that V/I is a field. Since I is maximal modular, V/I is a $\mathbb{C}$-division algebra. Since I is maximal, it is closed so V/I is also a commutative $\mathbb{C}$-Banach algebra. By replacing the norm on V/I by an appropriate equivalent norm, we can assume that V/I is a unital commutative $\mathbb{C}$-Banach algebra which is also a division algebra. In fact, this just means that it is a field.
2. Let $f : V/I \to \mathbb{C}$ be an isomorphism. Observe that if $\pi : V \to V/I$ is the projection map, then $f \circ \pi$ is a multiplicative linear functional with $\ker(f \circ \pi) = I$.

This proves the desired result. □

We also have the result that multiplicative linear functionals on $\mathbb{C}$-normed algebras are automatically continuous.

Proposition 2.3.5

Let V be a $\mathbb{C}$-normed algebra and let $\phi \in \widehat{V}$. If V is unital, then ϕ is continuous and $\|\phi\| = 1$. If V does not have an identity, then ϕ is continuous and $\|\phi\| \leq 1$.

Proof. Suppose that V is unital and has an identity e. Then, $\phi(e) = 1$. Moreover, $\phi(x^{-1}) \neq 0$ for every $x \in V^{-1}$ due to the multiplicativity of ϕ. Suppose that there was an $x \in V$ such that $\phi(x) \notin \sigma_V(x)$. Then, $x - \phi(x)e$ must be invertible. But we have that $\phi(x - \phi(x)e) = \phi(x) - \phi(x)\phi(e) = \phi(x) - \phi(x) = 0$ and this is a contradiction. As a consequence, it must be true that $\phi(x) \in \sigma_V(x)$ for every $x \in V$. This means that:

$$\forall x \in V : |\phi(x)| \leq \|x\|,$$

which implies that $\|\phi\| \leq 1$. Since $\phi(e) = 1$ and $\|e\| = 1$, it follows that $\|\phi\| = 1$. If V does not have an identity, then take its unitization V_e and the extended functional $\phi_e : V_e \to \mathbb{R}$. Then:

$$\forall x \in V : |\phi(x)| = |\phi_e((x, 0))| \leq \|\phi_e\| \cdot \|(x, 0)\| = \|(x, 0)\| = \|x\|,$$

which proves that $\|\phi\| \leq 1$[a]. □

[a] Proposition 2.3.3 gives us a "soft" proof of this proposition. Let V be a $\mathbb{C}$-normed algebra and let $\phi : V \to \mathbb{C}$ be a multiplicative linear functional. We know that maximal modular ideals are closed; since $\ker(\phi)$ is closed, it follows that ϕ is continuous.

The next proposition deals with the relationship between multiplicative linear functionals and points in the spectrum.

Proposition 2.3.6

Let V be a commutative $\mathbb{C}$-Banach algebra. Let $x \in V$.

1. If V has an identity e, then $\mu \in \sigma_V(x)$ iff there is a $\phi \in \widehat{V}$ such that $\phi(x) = \mu$.
2. If V has no identity and $\mu \neq 0$, then $\mu \in \sigma_V(x)$ iff there is a $\phi \in \widehat{V}$ such that $\phi(x) = \mu$.

Proof. We will prove each statement one-by-one.

1. Let V have an identity e. Suppose that there is a $\phi \in \widehat{V}$ such that $\phi(x) = \mu$. If $x - \mu e$ is invertible, then $\phi(x - \mu e) \neq 0$. But we have that $\phi(x - \mu e) = \phi(x) - \mu\phi(e) = 0$. It follows that $x - \mu e$ is not invertible and $\mu \in \sigma_V(x)$. On the other hand, suppose that $\mu \in \sigma_V(x)$. Then, $x - \mu e$ is singular. Define:

   $$I = \{(x - \mu e)y : y \in V\}.$$

   This is a proper ideal in V. Therefore, it is contained in a maximal ideal J [a]. In light of Theorem 2.3.4, there exists a $\phi \in \widehat{V}$ such that $\ker(\phi) = J$. But that means that $0 = \phi(x - \mu e) = \phi(x) - \mu\phi(e) = \phi(x) - \mu$, which gives the result.
2. Suppose that V has no identity. Let $\mu \neq 0$. Consider the unitization V_e. Then, $\mu \in \sigma_V(x)$ iff $\mu \in \sigma_{V_e}(x, 0)$ iff there exists a multiplicative linear functional $\widehat{\phi} \in \widehat{V_e}$ such that $\widehat{\phi}(x, 0) = \mu$ if there exists a multiplicative linear functional $\phi \in \widehat{V}$ such that $\phi(x) = \mu$. All of these equivalences hold due to the validity of the unitization procedure and statement (1) above. □

[a] J is a modular ideal as well, owing to the fact that V is commutative and has an identity element.

At this point, we are ready to dive deep into the crux of Gelfand's theory. This theory really just begins with a study of the Gelfand transform, defined as follows. We will also simultaneously define the Gelfand representation, a natural map that should be considered alongside the Gelfand transform.

Definition 2.3.7

Let V be a $\mathbb{C}$-normed algebra and let $x \in V$. Then, the Gelfand transform of x is the map:

$$\widehat{x} : \widehat{V} \to \mathbb{C}\ ,\ \phi \mapsto \widehat{x}(\phi) := \phi(x)$$

The Gelfand topology on $\widehat{V}$ is the weakest topology on $\widehat{V}$ such that the family of maps $(\widehat{x})_{x\in V}$ is continuous. $\widehat{V}$ equipped with the Gelfand topology is known as the structure space/character space/Gelfand spectrum of V. We also can define the Gelfand representation of V to be the map:

$$\mathcal{G} : V \to \mathcal{F}(\widehat{V})\ ,\ x \mapsto \hat{x}$$

where $\mathcal{F}(\widehat{V})$ is the space of all functions $f : \widehat{V} \to \mathbb{C}$ and is regarded as a $\mathbb{C}$-algebra with the usual point-wise operations.

As the reader can expect, the rest of this section is devoted to clarifying the properties of the following three objects:

1. The Gelfand Transform.
2. The Gelfand Topology.
3. The Gelfand Representation

In fact, the section will conclude with a proof of the Gelfand Representation Theorem. This theorem one of the keys to Gelfand Duality. We begin by stating some important computational facts about the Gelfand transform.

Proposition 2.3.8

Let V be a $\mathbb{C}$-normed algebra. Then, the following hold:

1. The Gelfand representation $\mathcal{G} : V \to \mathcal{F}(\widehat{V})$ is a $\mathbb{C}$-algebra homomorphism.
2. $\forall x \in V : \forall \phi \in \widehat{V} : |\widehat{x}(\phi)| \leq \|x\|$.
3. If V has no identity, then for every $x \in V$ we have that $\sigma_V(x) = \widehat{x}(\widehat{V}) \cup \{0\}$. If V is unital with identity e, then $\sigma_V(x) = \widehat{x}(\widehat{V})$ for every $x \in V$.
4. Suppose that V has an identity. Then, $x \in V^{-1}$ if $\widehat{x}(\phi) \neq 0$ for every $\phi \in \widehat{V}$.

Proof. The proofs of all of these statements are either computations that can be carried out in a single line or they are consequences of previous results that we have already proved. Therefore, we will not include the proofs here. □

Let us turn to the Gelfand topology. As with the study of any topology, it is natural to gain an understanding of convergence in that topology. With the Gelfand topology, we have the following characterization result.

Proposition 2.3.9

Let V be a $\mathbb{C}$-normed algebra and let $(\phi_\delta)_{\delta\in D}$ be a net in $\widehat{V}$, with $\phi \in \widehat{V}$ as well. Then, the following are equivalent[a]:

1. $\phi_\delta \to \phi$ in the Gelfand topology.
2. $\forall x \in V : \phi_\delta(x) \to \phi(x)$.
3. $\forall x \in V : \widehat{x}(\phi_\delta) \to \widehat{x}(\phi)$

[a] SeeAppendix C for an introduction to nets in General Topology. We will use this material throughout our work.

Proof.

$(1) \Rightarrow (2)$: This follows by $(1) \Rightarrow (2)$ in Proposition C.0.14.

$(2) \Rightarrow (3)$: This follows by definition.

$(3) \Rightarrow (1)$: Explicitly, $\phi_\delta(x) \to \phi(x)$ for every $x \in V$. The result follows by $(2) \Rightarrow (1)$ in Proposition C.0.14. □

This result allows us to relate the Gelfand topology to another topology that should be familiar to the reader.

Proposition 2.3.10

Let V be a $\mathbb{C}$-normed algebra. The Gelfand topology on $\widehat{V}$ is the relative topology on $\widehat{V}$ as a subset of $V^\star$, equipped with the weak*-topology.

Proof. We appeal to Theorem C.0.11 for this. We will show that both the Gelfand topology and the subset topology described in the statement of the proposition have the same convergent nets with the same limits. Let $(\phi_\delta)_{\delta\in D}$ be a net in $\widehat{V}$ and let $\phi \in \widehat{V}$ be fixed. Then, $\phi_\delta \to \phi$ in the Gelfand topology if $\phi_\delta(x) \to \phi(x)$ for every $x \in V$ and this is equivalent to $\phi_\delta \to \phi$ in the relative topology, since convergence in the weak*-topology is described by point-wise convergence of evaluation functionals. □

The next step is to establish other properties of $\widehat{V}$ with respect to the Gelfand topology. To do that, we need to have a preliminary discussion about one-point compactifications. If (X, τ) is a non-compact, locally compact Hausdorff space and $f : X \to \mathbb{C}$ is a continuous function, then we want to extend this function to the one-point compactification of X and determine precisely when this extension is continuous. The following result accomplishes that for us.

Proposition 2.3.11

Let (X, τ) be a non-compact, locally compact Hausdorff space and let $X^+ = X \cup \{\infty\}$ denote its one-point compactification. Let $f : X \to \mathbb{C}$ be a continuous function. The following are equivalent:

1. For every $\epsilon > 0$, there exists a compact set $K \subseteq X$ such that $|f(x)| < \epsilon$ for every $x \in X \setminus K$.
2. The function $f^+ : X^+ \to \mathbb{C}$ defined by:
$$f^+(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}$$
is continuous.

Proof.

$(1) \Rightarrow (2)$: We just have to prove continuity at $x = \infty$. Let $\epsilon > 0$ be given. Then, there exists a compact set $K \subseteq X$ such that $|f(x)| < \epsilon$ for every $x \in X \setminus K$. Note that $X^+ \setminus K$ is an open neighborhood of ∞. But now, the result follows by the definition of continuity.

$(2) \Rightarrow (1)$: Assume that the given function is continuous. Let $\epsilon > 0$ be given. Since f^+ is continuous at ∞, there exists an open neighborhood G of ∞ such that for all $x \in G$, we have that $|f(x)| < \epsilon$. But note that $X^+ \setminus G$ is a closed, compact subset of X so we are done. □

It is now possible to show that the unitization procedure has a rather natural topological interpretation if we consider spaces of continuous functions. This fact is presented in the form of the following proposition that we will not prove.

Proposition 2.3.12

Let (X, τ) be a locally compact, Hausdorff space. Then, the unitization of $\mathcal{C}_0(X)$ is isomorphic to $\mathcal{C}(X^+)$, where $X^+ = X \cup \{\infty\}$ is the one-point compactification of (X, τ).

Proof. See the proof of [26, Lemma C.38, p.664]. □

We are almost ready to deal with the Gelfand Representation Theorem. In fact, we will be presenting two theorems that deal with the unital and non-unital cases separately. This is to prevent our exposition from being cluttered with annoying details and references. Before we get to this, let us separately prove the following important theorem about the structure space of a commutative $\mathbb{C}$-Banach algebra.

Theorem 2.3.13

Let V be a commutative $\mathbb{C}$-Banach algebra. Then, $\widehat{V}$ is a locally compact, Hausdorff space. If V has an identity element, $\widehat{V}$ is compact. if V does not possess an identity element, each of the functions $\widehat{x}$ vanishes at infinity.

Proof. There are a number of assertions being made, so we will have to organize the proof to make it easy to read. Let us, first, show that $\widehat{V}$ is Hausdorff. For this, we will use Theorem C.0.15. Let $(\phi_\delta)_{\delta \in D}$ be a net in $\widehat{V}$ that converges to $\phi, \psi \in \widehat{V}$. Then:
$$\forall x \in V : \phi(x) = \lim_\delta(\phi_\delta(x)) = \psi(x).$$
Hence, $\phi = \psi$. It follows that $\widehat{V}$ is a Hausdorff space, since every convergent net has a unique limit. By the Banach-Alaoglu Theorem(Theorem B.2.4), the closed unit ball B in $V^\star$ is weak dual compact. By the strength of Proposition 2.3.5, it follows that $\widehat{V} \subseteq B$. This is true whether or not V has an identity element. Consider the weak dual closure of $\widehat{V}$, denoted by W for convenience. Let $\phi \in W$. Then, there is a net $(\phi_\delta)_{\delta \in D}$ in $\widehat{V}$ that converges in the weak dual topology to ϕ. But now, notice that:
$$\forall x, y \in V : \phi(xy) = \lim_\delta(\phi_\delta(xy)) = \lim_\delta(\phi_\delta(x)\phi_\delta(y)) = \phi(x)\phi(y).$$
In a similar way, it is easy to also prove linearity. This implies that ϕ is an algebra homomorphism. This implies that either $\phi = 0$ or $\phi \in \widehat{V}$. There are, then, two possibilities.

1. If the zero functional is not in the weak dual closure W, then $\widehat{V}$ is a weak dual closed subset of B and that implies that it is a weak dual compact Hausdorff space.
2. If the zero functional is in the weak dual closure W, then $W = \widehat{V} \cup \{0\}$. Since W is a compact Hausdorff space and $\widehat{V}$ is obtained by removing one element from W, it follows that $\widehat{V}$ is locally compact and Hausdorff.

If V has an identity, then $\phi(e) = 1$ due to the net-theoretic argument presented above so the second case cannot occur. If $\widehat{V}$ is not compact, then W is the one-point compactification of $\widehat{V}$ and $\hat{x}$ must vanish at ∞. □

We will first state (and prove) the result for unital $\mathbb{C}$-Banach algebras. Then, we will extend that result to $\mathbb{C}$-Banach algebras that are non-unital. The reader will actually find that most of the work has already been done so there is little left for us to do other than to cite the results that we have already shown.

Theorem 2.3.14. The Gelfand Representation Theorem I

Let V be a commutative unital $\mathbb{C}$-Banach algebra. Then, the following statements hold:

1. The Gelfand representation $\mathcal{G}$ is an algebra homomorphism into $\mathcal{C}(\widehat{V})$.
2. $\forall x \in V : \widehat{x}(\widehat{V}) = \sigma_V(x)$.
3. $\forall x \in V : \|\widehat{x}\|_\infty = \nu(x)$[a] $\leq \|x\|$. In particular, $\mathcal{G}$ is continuous.
4. $\forall x \in V : x \notin V^{-1} \Leftrightarrow (\exists \phi \in \widehat{V} : \widehat{x}(\phi) = 0)$.

[a] Recall that this is the spectral radius of x.

Proof. We will deal with each statement one-by-one.

1. Let us first show that $\mathcal{G}(V) \subseteq \mathcal{C}(\widehat{V})$. Let $x \in V$ and consider $\widehat{x}$. We want to show that this is a continuous map. So, let $(\phi_\delta)_{\delta \in D}$ be a net in $\widehat{V}$ that converges to some $\phi \in \widehat{V}$. By Proposition 2.3.9, this means that $\phi_\delta(y) \to \phi(y)$ for every $y \in V$. But this means that:
$$\lim_\delta (\widehat{x}(\phi_\delta) = \lim_\delta \phi_\delta(x) = \phi(x) = \widehat{x}(\phi),$$
which proves continuity. By statement (1) of Proposition 2.3.8, $\mathcal{G}$ is an algebra homomorphism.
2. This is statement (3) in Proposition 2.3.8.
3. Let $x \in V$ be fixed. Since V is unital, every $\phi \in \widehat{V}$ has norm 1. Observe that $\widehat{x}(\widehat{V}) = \sigma_V(x)$. It follows that:
$$\nu(x) = \sup_{c \in \sigma_V(x)} |c| = \sup_{c \in \widehat{x}(\widehat{V})} |c| = \sup_{\phi \in \widehat{V}} |\phi(x)| = \|\widehat{x}\|_\infty.$$
Since $\nu(x) \leq \|x\|$, the given result about $\mathcal{G}$ follows immediately.
4. This is statement (4) in Proposition 2.3.8. □

In fact, the proof of the non-unital case is quite easy because the argument proceeds in almost entirely the same way.

Theorem 2.3.15. The Gelfand Representation Theorem II/The Abstract Riemann-Lebesgue Lemma

Let V be a commutative $\mathbb{C}$-Banach algebra without identity. Then, the following statements hold:

1. The map $\mathcal{G}$ is an algebra homomorphism into $\mathcal{C}_0(\widehat{V})$ [a].
2. $\forall x \in V : \widehat{x}(\widehat{V}) \setminus \{0\} = \sigma_V(x) \setminus \{0\}$.
3. $\forall x \in V : \|\widehat{x}\| = \nu(x)$.

[a] This is the space of all continuous functions $f : \widehat{V} \to \mathbb{C}$ which vanish at infinity; see Example 3.

Proof. We will prove each statement one-by-one.

1. Let $x \in V$. We know that $\widehat{x}$ must be continuous and we also know that it must vanish at ∞, as given by Theorem 2.3.13. By statement (1) of Proposition 2.3.8, $\mathcal{G}$ is an algebra homomorphism.
2. This follows immediately from statement (3) in Proposition 2.3.8.
3. If we work in the unitization, the equality holds. Nothing changes when we remove the unit so we are done. □

We should actually explain why we have chosen to give the name "The Abstract Riemann-Lebesgue Lemma" for the non-unital version of the Gelfand Representation Theorem. For the reader who does not know any Fourier analysis, it is quite alright to skip the discussion that follows and move on to the next chapter. Let $L^1(\mathbb{R})$ denote the set of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ which are integrable and let $\mathcal{L}^1(\mathbb{R})$ denote the quotient of $L^1(\mathbb{R})$ modulo almost everywhere equality. Then, the following facts hold:

1. If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is continuous.
2. If $f \in L^1(\mathbb{R})$, then $\lim_{|t| \to \infty} \widehat{f}(t) = 0$. (The Riemann-Lebesgue Lemma)

These two facts indicate that we can define the following map:
$$\mathcal{F} : \mathcal{L}^1(\mathbb{R}) \to \mathcal{C}_0(\mathbb{R}) \ , \ [f] \mapsto \mathcal{F}([f]) := \widehat{f}$$
This map is known as the Fourier representation and it is generalized by the Gelfand representation. Let us, in quick succession, point out certain facts of interest that can be shown if one is willing to work hard enough. We would be doing a disservice to the reader if we did not talk about the connections of the theory in this chapter to Fourier analysis.

1. $\mathcal{L}^1(\mathbb{R})$ can be equipped with the structure of a $\mathbb{C}$-Banach algebra, through the convolution product. In particular, this becomes a $\mathbb{C}$-Banach algebra without identity (it can be shown that it does not have an identity using the Convolution Theorem & the Riemann-Lebesgue Lemma). The Convolution Theorem, then, proves that the Fourier representation is an algebra homomorphism.
2. The Fourier representation suggests that $\widehat{\mathcal{L}^1(\mathbb{R})}$ is homeomorphic to $\mathbb{R}$. Indeed, this is true but it also requires some effort to prove; see [47, Theorem 7.7, p.46-47]. In the next chapter, it will be shown that $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(\widehat{\mathcal{L}^1(\mathbb{R})})$ are "isomorphic" in a suitable sense (they are, at the very least, algebra isomorphic). But this means that there is a direct way to link the Fourier representation to the Gelfand representation, indicating that the Fourier representation can be viewed as a special case of the Gelfand representation.

3. It is natural to wonder if the Fourier representation is injective/surjective. It can be shown that the Fourier representation is injective; the proof relies on the Fourier Inversion Theorem. On the other hand, one can show that the Fourier representation is not surjective. The argument roughly proceeds as follows; if it were surjective, then it is a bijective continuous map between Banach spaces. By Theorem B.2.6, it has a bounded inverse but it can be shown explicitly that such an inverse can never be bounded.

This marks the end of our development of Gelfand's Theory of Commutative Banach Algebras. In the next chapter, our main goal will be to study the Gelfand representation when our normed algebras are given the extra structure of a so-called involution. For further information about Banach Algebras, the reader may refer to [11], [33] and [30]. For now, we should move on to the greener patch that is the theory of $C^\star$-algebras.

Chapter 3

The Theory of $C^\star$-Algebras

This chapter will be focused on developing $C^\star$-algebra theory with the purpose of proving two representation theorems for $C^\star$-algebras. Indeed, we will have to deal with the commutative and non-commutative cases separately, just to keep the chapter well-organized. The reader will notice that both of these representation theorems are discussed in the last section of this chapter; this has been done on purpose even if it is different from the way that other texts may present this material. The way we have chosen to present the theory attempts to maximize the number of insights that can be obtained when natural questions are posed, rather than worrying about the efficiency with which crucial results are obtained.

3.1 The Basic Set-up

In the spirit of maintaining an algebraic grip over our theory, we do not begin by defining $C^\star$-algebras right away. Instead, we define the notion of a $\star$-algebra first. From now on, we will mainly be working with $\mathbb{C}$-algebras. We hope to justify these choices throughout this chapter, both in explicit justifications and also in the results we prove.

Definition 3.1.1

Let V be a $\mathbb{C}$-algebra. A map $\star : V \to V$ is said to be an involution if the following properties hold:

1. $\forall x, y \in V : (x+y)^\star = x^\star + y^\star$.
2. $\forall \alpha \in \mathbb{C} : \forall x \in V : (\alpha x)^\star = \overline{\alpha} x^\star$.
3. $\forall x, y \in V : (xy)^\star = y^\star x^\star$.
4. $\forall x \in V : (x^\star)^\star = x$.

The pair $(V, \star)$ is said to be a $\star$-algebra. Let $x \in V$ be fixed. Then, we define the following:

1. $x^\star$ is said to be the adjoint of x. x is said to be self-adjoint if $x = x^\star$.
2. x is said to be normal if $x^\star x = x x^\star$.
3. If V also has an identity e, then x is said to be unitary if $xx^\star = x^\star x = e$.

Of course, V is a normed $\star$-algebra if it is a $\mathbb{C}$-normed algebra with an involution $\star : V \to V$. It is a Banach $\star$-algebra if it is a complete normed $\star$-algebra.

So far, we have not introduced any "compatibility condition" between the involution and the norm. This is a rather important point for the discussion that we will have right now. Normally, we would want to develop some theory about $\star$-algebras first, and then work our way up the ladder of "$\star$"-structures to obtain the definition of a $C^\star$-algebra. However, we will choose to not walk that path for a number of reasons:

1. The thesis would become far too long if we worked through every rung of that ladder.
2. Most of the results we will prove that are of significance in this chapter pertain **only** to $C^\star$-algebras[1].

As a consequence, we will find ourselves working exclusively with $C^\star$-algebras. By the end of this section, we will place some restrictions even on that but that will take some further explaining and it is best to deal with that in its natural context. Nevertheless, we should say that we will try to provide general definitions for many of the notions introduced here. For instance, if a definition does make sense even when considering a $\star$-algebra, then we will give it for an arbitrary $\star$-algebra. Keeping all of this in mind, let us give the definition of a $C^\star$-algebra.

Definition 3.1.2

A $\mathbb{C}$-Banach $\star$-algebra V is a $C^\star$-algebra iff it satisfies the following condition:

$$\forall x \in V : \|x^\star x\| = \|x\|^2.$$

This is known as the $C^\star$-condition. A $C^\star$-algebra with identity e is called a unital $C^\star$-algebra[a].

[a]We will assume that every $C^\star$-algebra (and all $\star$-algebras) are not trivial; that is, none of them just contain the zero vector.

[1]Some of those results might be true for structures that are lower on the ladder but the proofs might be entirely different and may require techniques that need to be discussed in further detail.

The following proposition collects some of the basic properties of the involution that we will constantly be using throughout this chapter. Most of these properties will be used without further reference

Proposition 3.1.3

Let V be a $C^\star$-algebra with identity e. Then:

1. The involution is a bijection.
2. $e = e^\star$ and $\|e\| = 1$. That is, V is a unital $\mathbb{C}$-Banach algebra.
3. $\forall x \in V : x \in V^{-1} \Rightarrow (x^\star)^{-1} = (x^{-1})^\star$.
4. $\forall x \in V : \sigma_V(x^\star) = \overline{\sigma_V(x)}$ [a].
5. Let $U(V)$ be the set of unitary elements of V. This is a subgroup of V^{-1}.
6. $\forall x \in V : \|x^\star\| = \|x\|$. In particular, the involution is continuous.

[a] If $S \subseteq \mathbb{C}$, then $\overline{S} := \{z \in \mathbb{C} : \overline{z} \in S\}$. Once we introduce topological structure, it is easy to confuse this with the closure of S in whatever topology we provide. In those instances, the author will strive to be as clear as possible so that the confusion is minimal.

Proof. We prove each statement one-by-one.

1. This follows from the fact that the involution is its own inverse.
2. We have that:
$$e = (e^\star)^\star = (ee^\star)^\star = (e^\star)^\star e^\star = ee^\star = e^\star.$$
Next, observe that:
$$\|e\|^2 = \|e^\star e\| = \|e\|.$$
If $\|e\| = 0$, then $\|x\| = \|x \cdot e\| \leq \|x\| \cdot \|e\| = 0$ for every $x \in V$. This implies that $x = 0$ for every $x \in V$ and that contradicts the fact that V is not trivial. It follows that $\|e\| = 1$ and that V is a unital $\mathbb{C}$-Banach algebra.
3. Let $x \in V^{-1}$. Then:
$$x^\star (x^{-1})^\star = (x^{-1}x)^\star = e^\star = e.$$
On the other hand:
$$(x^{-1})^\star x^\star = (xx^{-1})^\star = e^\star = e.$$
In other words, $(x^\star)^{-1} = (x^{-1})^\star$.
4. Let $x \in V$ be fixed and let $\lambda \in \sigma_V(x^\star)$. Then, $x^\star - \lambda e$ is singular. It follows that its adjoint, given by $x - \overline{\lambda} e$ is singular too. This implies that $\overline{\lambda} \in \sigma_V(x)$. Hence, $\lambda \in \overline{\sigma_V(x)}$. On the other hand, let $\lambda \in \overline{\sigma_V(x)}$. Then, $\overline{\lambda} \in \sigma_V(x)$. So, $x - \overline{\lambda} e$ is singular. Therefore, its adjoint, given by $x^\star - \lambda e$, is singular too. Hence, $\lambda \in \sigma_V(x^\star)$.
5. This is a fairly routine computation so we omit it.
6. Let $x \in V$. If $x = 0$, then we are done. So, assume that $x \neq 0$. Then:
$$\|x\|^2 = \|x^\star x\| \leq \|x^\star\| \|x\|$$
This implies that $\|x\| \leq \|x^\star\|$. This identity holds for all $x \in V$; in particular, it holds for $x^\star$. Therefore, $\|x^\star\| \leq \|x\|$. This implies that $\|x\| = \|x^\star\|$. To prove continuity, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V that converges to some $x \in V$. Then, we know that:
$$\forall n \in \mathbb{N} : \|{x_n}^\star - x^\star\| = \left\|(x_n - x)^\star\right\| = \|x_n - x\|.$$
As $n \to \infty$, it follows that that ${x_n}^\star \to x^\star$ and we are done. □

The last property listed in Proposition 3.1.3 is deserving of some comment. Since we do have a fair amount to say about it, we will leave the matter till the end of this section. For now, we need to provide examples of $C^\star$-algebras. To make the examples a bit easier to deal with, let us prove a rather useful lemma.

Lemma 3.1.4 Let V be a Banach $\star$-algebra which satisfies the following condition:
$$\forall x \in V : \|x^\star x\| \geq \|x\|^2.$$
Then, V is a $C^\star$-algebra.

Proof. The given inequality can be used to show that $\|x^\star\| = \|x\|$ for every $x \in V$. But this means that:
$$\forall x \in V : \|x^\star x\| \leq \|x^\star\| \cdot \|x\| = \|x\|^2$$
which proves that $\|x^\star x\| = \|x\|^2$ for every $x \in V$. □

The usefulness of the lemma lies in the fact that there is less for us to prove if we need to show that a given Banach $\star$-algebra is a $C^\star$-algebra. This is seen in the following example.

Example 8. Let (X, τ) be a locally compact, Hausdorff space and let $\mathcal{C}_0(X)$ denote the $\mathbb{C}$-Banach algebra of all complex-valued continuous maps $f : X \to \mathbb{C}$ which vanish at ∞, equipped with the supremum norm. Define the map:
$$\star : \mathcal{C}_0(X) \to \mathcal{C}_0(X) \ , \ f \mapsto f^\star := \overline{f}$$
It is easy to see that this is an involution. Let us show that this is a $C^\star$-algebra. Indeed, let $f \in \mathcal{C}_0(X)$. Then:
$$\forall x \in X : |f(x)|^2 = |\overline{f(x)} \cdot f(x)| = |f^\star(x) \cdot f(x)|.$$

In particular, it follows that:

$$\forall x \in X : |f(x)|^2 \leq \sup_{x\in X} |f^\star(x) \cdot f(x)| = \|f^\star f\|_\infty.$$

This implies that $\|f\|_\infty^2 \leq \|f^\star f\|_\infty$. But this implies, by Lemma 3.1.4, that V is a $C^\star$-algebra. In fact, this is a commutative $C^\star$-algebra without identity.

Example 9. Let (X, τ) be a compact Hausdorff space and let $\mathcal{C}(X)$ be the set of all complex-valued continuous functions $f : X \to \mathbb{C}$, equipped with the supremum norm. Define the map:

$$\star : \mathcal{C}(X) \to \mathcal{C}(X)\ ,\ f \mapsto f^\star := \overline{f}$$

It is easy to see that this is an involution. An argument that is similar to the one given in the example above shows that $\mathcal{C}(X)$ is a commutative, unital $C^\star$-algebra.

Let us provide an important example of a non-commutative $C^\star$-algebra.

Example 10. Let H be a Hilbert space. Let $\mathcal{B}(H)$ denote the set of all bounded linear maps $T : H \to H$. This is a $\mathbb{C}$-Banach algebra under point-wise addition, scalar multiplication and composition, as well as the operator norm. The involution here is the adjoint operation[a]. This involution turns $\mathcal{B}(H)$ into a non-commutative, unital $C^\star$-algebra[b].

[a] This is defined from scratch in Section 3.5.2. In particular, the reader is advised to look at Definition 3.5.8.
[b] This will be proved in Section 3.5.2.

Let us "fill out" the basic theory a little bit by introducing the usual algebraic structures that are of interest. We will begin by introducing $\star$-subalgebras.

Definition 3.1.5

Let V be a $\star$-algebra and let $U \subseteq V$ be a set. The adjoint of U is the set:

$$U^\star := \{u^\star : u \in U\}$$

U is said to be self-adjoint if $U = U^\star$. U is said to be a $\star$-subalgebra if it is a self-adjoint subalgebra. Finally, if V is a $C^\star$-algebra and $U \subseteq V$, we say that U is a $C^\star$-subalgebra iff it is a closed $\star$-subalgebra of V.

The following proposition tells us that the closure of a $\star$-subalgebra in a $C^\star$-algebra is a $C^\star$-subalgebra.

Proposition 3.1.6

Let V be a $C^\star$-algebra and let $U \subseteq V$ be a $\star$-subalgebra. Then, $\overline{U}$ is a $C^\star$-subalgebra.

Proof. We need to show that $\overline{U}$ is a $\star$-subalgebra and that will prove the result. Let $x \in \overline{U}$. Then, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in U that converges to x. Since $x_n^\star \in U$ for every $n \in \mathbb{N}$, it follows that:

$$\lim_{n\to\infty} x_n^\star = \big(\lim_{n\to\infty} x_n\big)^\star = x^\star,$$

which implies that $x^\star \in \overline{U}$ so we are done. □

A great number of proofs that we will give are actually going to involve the self-adjoint elements in a $C^\star$-algebra. The usual strategy will be to prove a result for self-adjoint elements (where the going is easy) and then use that to prove a result for all other elements (where the going gets tough). The following lemma is a gift in making that strategy work. We should also point out that it is the first sign that $C^\star$-algebras actually exhibit behavior that is very similar to the complex numbers.

Lemma 3.1.7 Let V be a $C^\star$-algebra.

1. Every $a \in V$ can be written uniquely as a sum $a = x + iy$, with x, y being self-adjoint.
2. Let $a \in V$ and suppose that $a = x + iy$, with x, y being self-adjoint. a is normal iff $xy = yx$.
3. Let $a \in V$ and let $a = x + iy$, with x, y being self-adjoint. Then, a is self-adjoint iff $y = 0$.

If $a \in V$ and $a = x + iy$ for self-adjoint elements x & y, then x is called the real part of a and y is called the imaginary part of a[a].

[a] Sometimes, we denote x by $\mathrm{Re}(a)$ and y by $\mathrm{Im}(a)$. We will find it convenient to adopt this notation for specific problems.

Proof. We will prove each statement one-by-one.

1. Let $a \in V$. To motivate the expressions for x and y, we will first assume that a can be written in that form and then try to deduce expressions for x and y from this assumption. So, observe that:

$$a^\star = x^\star - iy^\star = x - iy$$

This implies that the following formulas hold:

$$2x = a + a^\star$$
$$2iy = a - a^\star$$

It follows that $x = \frac{1}{2}(a + a^\star)$ and that $y = -\frac{i}{2}(a - a^\star)$. It is, now, obvious that these are self-adjoint elements so that $x + iy = a$. Coincidentally, the same argument above also proves uniqueness[a].

2. Suppose that a is normal. Then:
$$a^\star a = aa^\star \Rightarrow (x - iy)(x + iy) = (x + iy)(x - iy)$$
Expanding both products gives us:
$$x^2 - iyx + ixy + y^2 = x^2 + iyx - ixy + y^2$$
which tells us that $xy = yx$. On the other hand, if $xy = yx$, then every implication above can be reversed and we obtain normality.
3. Let $a \in V$ with $a = x + iy$, with x, y being self-adjoint. If a is self-adjoint, then $a = a + i0$. Since such a decomposition is unique, it follows that $y = 0$. Next, assume that $y = 0$. Then, $a = x$ so a is self-adjoint.

This completes the proof. □

[a] After all, we have just shown that if such a decomposition does exist, then x and y must be of the form given above.

The lemma suggests (rather strongly) that elements of $C^\star$-algebras can be "treated" like the complex numbers. This analogy will be prevalent in our work with $C^\star$-algebras, particularly when we discuss positivity later. Of course, when these analogies do arise, we will comment on them as best as we can.

The next natural step in the development of our theory is to discuss maps between $\star$-algebras. $\star$-algebras are algebras which have an involution, so it seems like the correct maps to consider would be algebra homomorphisms which preserve the involution. This is precisely the content of the following definition.

Definition 3.1.8

Let V and W be $\star$-algebras[a]. A map $\phi : V \to W$ is a $\star$-homomorphism if it is an algebra homomorphism such that:
$$\forall x \in V : \phi(x^\star) = \phi(x)^\star$$
If V and W have identities, then the $\star$-homomorphism will also be required to preserve the identity elements. In this case, we will refer to it as a unital $\star$-homomorphism.

[a] We will not make any effort to distinguish between the involutions on two different $\star$-algebras; the reader must keep track of this by themselves.

Of course, all of the usual terminology transfers over to this setting. For instance, two $\star$-algebras are $\star$-isomorphic iff there is a bijective $\star$-isomorphism between them. Of course, there are some permanence properties of $\star$-homomorphisms (with easy proofs) that are worth noting right away. Let us collect them (they do not form an exhaustive list of all properties we want) in the following proposition.

Proposition 3.1.9

Let V and W be $\star$-algebras. Let $f : V \to W$ be a $\star$-homomorphism.
1. If $M \subseteq V$ is a $\star$-subalgebra of V, then $f(M)$ is a $\star$-subalgebra of W.
2. If $N \subseteq W$ is a $\star$-subalgebra of W, then $f^{-1}(N)$ is a $\star$-subalgebra of V.
3. $\ker(f)$ is a $\star$-ideal of V; that is, it is a self-adjoint ideal of V[a].
4. $V/\ker(f)$ is $\star$-isomorphic to $f(V)$.

[a] $\star$-ideals are not going to be so relevant to our development of the theory so we will not prove many results about them.

Proof. Statements (1) – (3) consist of routine verifications. Statement (4) is a consequence of the First Isomorphism Theorem, when $V/\ker(f)$ has been equipped with the obvious structure of a $\star$-algebra. □

We refer the reader to the exercises in [13, p.88] for a number of extensions of the algebraic ideas above. In particular, there is a fair amount of further discussion on the role that $\star$-ideals play within the theory. We end this section off by looking at the problem of unitization. To make this exposition more clear, we need the following (familiar) definition.

Definition 3.1.10

Let (X, d) and (Y, h) be metric spaces. Let $f : X \to Y$ be a map. f is an isometry iff:
$$\forall x_1, x_2 \in X : h(f(x_1), f(x_2)) = d(x_1, x_2).$$
f is also said to be an isometric mapping.

Before we state the unitization theorem, let us prove that the involution in a $C^\star$-algebra is an isometry. We have already noted that the involution in a $C^\star$-algebra is automatically continuous but this does not have to be true for Banach $\star$-algebras (see [13, p.61] for a non-trivial example). Many of the results that we will prove rely heavily on the $C^\star$-condition, though we should say that there have been (successful) attempts at proving the results discussed in this chapter for Banach $\star$-algebras with discontinuous involutions. For more information about these developments, see [13].

Proposition 3.1.11

Let V be a $C^\star$-algebra. Then, $\star$ is an isometry.

Proof. Let $x_1, x_2 \in V$. Then:

$$\left\|(x_1 - x_2)^\star\right\| = \|x_1^\star - x_2^\star\| = \|x_1 - x_2\|,$$

where we have used the fact that $\star$ satisfies condition (2). The result now follows. □

Let us finally state (and prove) the unitization result that is relevant to us. This will provide justification for us working (mainly) with unital $C^\star$-algebras.

Theorem 3.1.12. The Unitization Theorem ($C^\star$-Algebras)

If V is a $C^\star$-algebra without identity, then it can be isometrically embedded into a unital $C^\star$-algebra V_e as an ideal satisfying $\dim(V_e/V) = 1$.

Proof. The proof is rather long and involves checking quite a number of details. Therefore, we will not give it in the main text of this thesis. The reader should see Theorem A.0.1 for the proof. □

One thing that we do have to comment on is the fact that the norm introduced in the proof of the Unitization Theorem for $C^\star$-algebras is **not** the same as the one introduced in Theorem 2.1.17. So, one might worry that the (norm) topologies induced are different. As it stands, there is nothing to worry about.

Proposition 3.1.13

Let V be a $C^\star$-algebra without identity. Let V_e^1 be the unitization as described in Theorem 2.1.17 and let V_e^2 be the unitization as described in Theorem 3.1.12[a]. Then, both V_e^1 and V_e^2 have the same topologies.

[a]Obviously, we do not include the $C^\star$-structure.

Proof. Let $\|\cdot\|_1$ denote the norm on V_e^1 and let $\|\cdot\|_2$ denote the norm on V_e^2. Let $x \in V$ and let $\lambda \in \mathbb{C}$. Then, we have that:

$$\|(x,\lambda)\|_2 = \sup_{y \in V, \|y\| \leq 1} \|xy + \lambda y\| \leq \|x\| + |\lambda| = \|(x,\lambda)\|_1$$

Consider the identity map:

$$I : V_e^2 \to V_e^1 \ , \ x \mapsto I(x) := x$$

By the estimate above, this is a continuous bijection between Banach spaces. By Theorem B.2.6, it follows that I^{-1} is continuous. In particular:

$$\exists K > 0 : \forall x \in V : \forall \lambda \in \mathbb{C} : \|(x,\lambda)\|_1 \leq K\|(x,\lambda)\|_2$$

Since both norms are equivalent, they induce the same topology and we are done. □

We needed to let V be a $C^\star$-algebra in the proposition above. This is mainly because the norm introduced in the proof of Definition 2.1.15 was shown to be a complete norm by using the properties of the involution. With all of this established, we can safely proceed by proving most of our preparatory results for unital $C^\star$-algebras. Let us proceed to the next section where we establish a key tool that will be of much use to us.

3.2 Functional Calculus for $C^\star$-algebras

The title for this section could be construed as rather dismissive but, really, it should be the slogan for what we want to do. The idea behind this section is to develop a tool, known broadly as "functional calculus", which will aid us in the proofs of many results later. We assume that the reader is "familiar" with functional calculus in the setting of Banach algebras[2]. Indeed, one of the results we are aiming to obtain is a "spectral mapping theorem" within this setting[3]. Most of the results in this section have been taken from [21, Chapter 4.1].

Proposition 3.2.1

Let V be a unital $C^\star$-algebra and let $x \in V$. The following statements hold:

1. If x is normal, then $\nu(x) = \|x\|$, where $\nu(x)$ is the spectral radius of x.
2. If x is self-adjoint, then $\sigma_V(x)$ is a compact subset of $\mathbb{R}$ and contains at least one of the two real numbers $\pm\|x\|$.
3. Suppose that V is unital with identity e and x is unitary. Then, $\|x\| = 1$ and $\sigma_V(x)$ is a compact subset of the unit circle.

Proof.
We will prove each result one-by-one.

1. Let x be normal. If x was self-adjoint, then $\|x\|^2 = \|x^\star x\| = \left\|x^2\right\|$. By statement (7) of Theorem 2.1.13 and

[2]By "familiar", we mean that the reader should know about the results in that direction and the motivation for having such a calculus. We will not use those results in any of the proofs here, though they are certainly helpful by way of useful connections that the reader can make.

[3]See Theorem 2.1.6 for a reminder of what such a result should look like.

Theorem 2.1.14, it follows that $\nu(x) = \|x\|$. Next, assume that x is normal. Then, $x^\star x$ is self-adjoint. Using statement (6) of Theorem 2.1.13 and the normality of x, we have that:

$$\|x\|^2 = \|x^\star x\| = \nu(x^\star x) \leq \nu(x^\star)\nu(x) = \nu(x)^2 \leq \|x\|^2$$

which proves that $\nu(x) = \|x\|$.

2. Assume that x is self-adjoint. If we can show that $\sigma_V(x)$ is a subset of $\mathbb{R}$, then the normality of x implies that:

$$\nu(x) = \|x\| = \sup_{t \in \sigma_V(x)} |t|$$

and the supremum is achieved because $\sigma_V(x)$ is compact. That is, there is a $c \in \sigma_V(x)$ such that $|c| = \|x\|$, so it immediately follows that $\|x\| \in \sigma_V(x)$ or $-\|x\| \in \sigma_V(x)$. Next, let $c \in \sigma_V(x)$. Then, we can write $c = a + ib$ for $a, b \in \mathbb{R}$. For every $n \in \mathbb{N}$, define $x_n = x - ae + inbe$. Observe that:

$$\forall n \in \mathbb{N} : x_n - i(n+1)be = x - ce.$$

Since $x - ce$ is singular, it follows that $i(n+1)b \in \sigma_V(x_n)$ for each $n \in \mathbb{N}$. As a consequence, it follows that:

$$(n^2 + 2n + 1)b^2 = |i(n+1)b|^2 \leq \nu(x_n)^2 \leq \|x_n\|^2.$$

Using the $C^\star$-condition, we have that:

$$\forall n \in \mathbb{N} : (n^2 + 2n + 1)b^2 \leq \|x_n\|^2 = \|x_n{}^\star x_n\| = \|(x - ae - inbe)(x - ae + inbe)\| = \left\|(x - ae)^2 + n^2 b^2 e\right\|.$$

Finally, we can use the triangle inequality to deduce the following inequality:

$$\forall n \in \mathbb{N} : (n^2 + 2n + 1)b^2 \leq \left\|(x - ae)^2\right\| + n^2 b^2 \leq \|x - ae\|^2 + n^2 b^2,$$

which implies that $(2n+1)b^2 \leq \|x - ae\|^2$ for every $n \in \mathbb{N}$. Dividing throughout by $2n+1$ and taking the limit as $n \to \infty$ implies that $b = 0$. This means that $c \in \mathbb{R}$ and it implies that $\sigma_V(x) \subseteq \mathbb{R}$.

3. Let x be unitary. Then:

$$\|x\|^2 = \|x^\star x\| = \|e\| = 1,$$

which implies that $\|x\| = 1$. Next, since x is unitary, it follows that it is invertible. In particular:

$$\sigma_V(x^{-1}) = \sigma_V(x)^{-1}.$$

But since $x^{-1} = x^\star$, it follows that $\sigma_V(x^\star) = \sigma_V\overline{(x)}^{-1}$. If $\lambda \in \sigma_V(x)$, then $\lambda^{-1} \in \sigma_V\overline{(x)}^{-1}$ and that means that $x^\star - \lambda^{-1}e$ is singular. But this means that $x - \overline{\lambda^{-1}}e$ is singular. Since $\lambda \in \sigma_V(x)$ and $\overline{\lambda^{-1}} \in \sigma_V(x)$, it must be the case that:

$$|\lambda| \leq \|x\| = 1 \wedge |\lambda^{-1}| = |\lambda|^{-1} \leq \|x\| = 1.$$

This implies that $|\lambda| = 1$. Hence, $\sigma_V(x)$ is a compact subset of the unit circle. □

The proposition we have just shown will prove to be rather useful in a number of proofs later. The next step is to establish the continuous functional calculus for self-adjoint elements in a $C^\star$-algebra; we will need the following (plausible) lemmas lemmas to do this. The first is a generalization of the classical Weierstrass Polynomial Approximation Theorem.

Lemma 3.2.2 Let $K \subseteq \mathbb{R}$ be a non-empty compact set and let $\mathcal{P}(K)$ be the set of all restrictions of polynomial functions $p : \mathbb{C} \to \mathbb{C}$ to K. Then, $\mathcal{P}(K)$ is dense in $\mathcal{C}(K)$.

Proof. Sums, scalar multiples and products of polynomials are polynomials so $\mathcal{P}(K)$ is clearly a $\mathbb{C}$-algebra. The strategy is to show that the algebra $\mathcal{P}(K)$ satisfies all of the required conditions of the Stone-Weierstrass Theorem (see Theorem B.1.8). We will just verify them one-by-one.

1. Let $x \in K$ be fixed. Then, any non-zero constant polynomial will be non-zero when evaluated at x so $\mathcal{P}(K)$ satisfies the condition (1) in the theorem.
2. Let $x, y \in K$ with $x \neq y$. Then, define $f(t) = t^3$ for $t \in K$. Clearly, $f(x) \neq f(y)$ so $\mathcal{P}(K)$ satisfies condition (2) of the Stone-Weierstrass Theorem.
3. Clearly, the conjugate of any polynomial $f \in \mathcal{P}(K)$ is still in $\mathcal{P}(K)$ because only the coefficients of f are changed (since f is defined on a subset of $\mathbb{R}$).

Since $\mathcal{P}(K)$ satisfies all three conditions of the Stone-Weierstrass Theorem, it follows that $\overline{\mathcal{P}(K)} = \mathcal{C}(K)$. □

The next lemma is an extension result which will be used later in the thesis as well.

Lemma 3.2.3. The Uniform Extension Lemma Let (X, d) and (Y, h) be metric spaces. Suppose that (Y, h) is complete and that $A \subseteq X$ is dense. If $f : A \to Y$ is a uniformly continuous map, then f can be extended uniquely to a uniformly continuous map $g : X \to Y$.

Proof. See [38, Theorem D, p.78] for the proof. Even though the proof itself is relatively long, that is only the case because there is a lot to check. The idea (which we will need in the next result) is fairly simple. Indeed, let $x \in X$. If $x \in A$, then we can define $g(x) := f(x)$. On the other hand, if $x \notin A$, then there is a sequence $(x_n)_{n \in \mathbb{N}}$ in A which converges to x. The idea is to define:

$$g(x) := \lim_{n \to \infty} f(x_n)$$

Of course, one needs to show that the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges. One also needs to check that using a different sequence which approximates x will not change the limit above. □

The proof for the functional calculus is intuitive. If V is a unital $C^\star$-algebra and $a \in V$ is self-adjoint, then it is easy to define $f(a)$ when f is a complex polynomial. Since complex polynomials can be used to "construct" continuous functions, it follows that $f(a)$ can be defined for any continuous function $f : \sigma_V(a) \to \mathbb{C}$ through an approximation.

Theorem 3.2.4. The (Self-Adjoint) Continuous Functional Calculus

Let V be a unital $C^\star$-algebra and let $a \in V$ be a self-adjoint element. Then, there is a unique continuous map:

$$\phi : \mathcal{C}(\sigma_V(a)) \to V \ , \ f \mapsto f(a)$$

such that $f(a)$ has its elementary meaning when f is a polynomial. Moreover:

1. $\forall f \in \mathcal{C}(\sigma_V(a)) : \|f(a)\| = \|f\|$.
2. $\forall f, g \in \mathcal{C}(\sigma_V(a)) : \forall \lambda, \mu \in \mathbb{C} : (\mu f + \lambda g)(a) = \mu f(a) + \lambda g(a)$.
3. $\forall f, g \in \mathcal{C}(\sigma_V(a)) : (fg)(a) = f(a)g(a)$.
4. $\forall f \in \mathcal{C}(\sigma_V(a)) : \overline{f}(a) = f(a)^\star$. In this case, $\overline{f}$ is the conjugate function associated with f. In particular, $f(a)$ is self-adjoint if f is real-valued throughout $\sigma_V(a)$.
5. $\forall f \in \mathcal{C}(\sigma_V(a)) : f(a)$ is normal.
6. $\forall b \in V : \forall f \in \mathcal{C}(\sigma_V(a)) : ab = ba \Rightarrow f(a)b = bf(a)$.

Proof. Let $\mathcal{P}(\sigma_V(a))$ denote the set of all polynomial functions $p : \sigma_V(a) \to \mathbb{C}$. Define a map:

$$\phi : \mathcal{P}(\sigma_V(a)) \to V \ , \ p \mapsto \phi(p) := p(a)$$

Since $\sigma_V(a)$ is a compact subset of $\mathbb{R}$, it follows that $\mathcal{P}(\sigma_V(a))$ is dense in $\mathcal{C}(\sigma_V(a))$ by the strength of Lemma 3.2.2. In fact, the proof will be completed when we show two things:

1. ϕ is uniformly continuous; this will be established when we prove properties (1) and (2) for polynomials. Indeed, that property just says that:
$$\forall p \in \mathcal{P}(\sigma_V(a)) : \|\phi(p)\| = \|p\|,$$
which immediately implies uniform continuity of ϕ due to the fact that ϕ is linear. Since $\mathcal{P}(\sigma_V(a))$ is dense in $\mathcal{C}(\sigma_V(a))$, it follows that ϕ can be extended to the latter space by the Uniform Extension Lemma (Lemma 3.2.3).
2. We prove that properties $(3) - (6)$ hold for polynomials. Once this is shown to be true, we can show that the properties will hold in general as well.

With this in mind, let us first show uniform continuity. Observe that for $p \in \mathcal{P}(\sigma_V(a))$, $p(a)$ is normal. By statement (1) of Proposition 3.2.1, it follows that:

$$\|\phi(p)\| = \|p(a)\| = \nu(p(a)) = \max\{|t| : t \in \sigma_V(p(a))\}.$$

By the spectral mapping theorem for polynomials (Theorem 2.1.6), it is the case that $\sigma_V(p(a)) = p(\sigma_V(a))$. Therefore:

$$\|\phi(p)\| = \|p(a)\| = \max\{|p(s)| : s \in \sigma_V(a)\} = \|p\|.$$

It is easily seen that ϕ is linear, so it remains to be shown that $(3) - (6)$ hold for polynomials and it also remains to be shown how the extension process works. The only ones for which this is not so clear is (4) and (6) so let $p(t) = \sum_{k=0}^{n} a_k t^k$. Since a is self-adjoint it follows that $t \in \mathbb{R}$. As a consequence:

$$(\overline{p})(a) = \sum_{k=0}^{n} \overline{a_k} a^k = (p(a))^\star.$$

As for (6), it becomes clear if we observe that $a^n b = ba^n$ for every $n \in \mathbb{N}$. This can be proved by induction and by using the associativity of the multiplication. Therefore, $(3) - (6)$ hold for polynomials. Now, the extension procedure given to us in Lemma 3.2.3 tells us an explicit way of extending ϕ. Namely, for any $f \in \mathcal{C}(\sigma_V(a))$, there is a sequence $(p_n)_{n \in \mathbb{N}}$ of polynomials that converges uniformly to f. Therefore, we can define:

$$\Phi(f) := \lim_{n \to \infty} \phi(p_n) = \lim_{n \to \infty} p_n(a)$$

and this extension is unique. Now, by the continuity of multiplication, it follows that properties $(3) - (6)$ are satisfied whenever we are dealing with continuous functions as well. This completes the proof. □

It is possible to use Theorem 3.5.1 to derive a continuous functional calculus for normal elements of a $C^\star$-algebra. Moreover, we can obtain extensions of all other results we will derive in the remainder of this section to normal elements in a $C^\star$-algebra. Since we will not need this, we refrain from including a proof (or even a formulation) here.

In the remainder of this section, we will strive to establish some neat consequences of the self-adjoint functional calculus. The highlight of this discussion will be the Spectral Mapping Theorem for $C^\star$-algebras. Let us begin with the following result whose name is self-explanatory.

Corollary 3.2.5 (Weierstrass Approximation Theorem)**.** Let V be a unital $C^\star$-algebra and let $a \in V$ be self-adjoint. Define the set:

$$V_a = \{f(a) : f \in \mathcal{C}(\sigma_V(a))\}.$$

V_a is a commutative unital $C^\star$-subalgebra of V and is the smallest closed subalgebra of V that contains the identity and a. Each element of V_a is the limit of a sequence of polynomials in a.

Proof. The fact that V_a is commutative unital $\star$-subalgebra of V is a consequence of statements (2), (3), (4) of Theorem 3.2.4. To show that V_a is a $C^\star$-algebra, let us show that it is closed. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(\sigma_V(a))$ such that $(f_n(a))_{n\in\mathbb{N}}$ converges to some $x \in V$. Then, $(f_n(a))_{n\in\mathbb{N}}$ is a Cauchy sequence in V_a and by statement (1) of Theorem 3.2.4, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}(\sigma_V(a))$. It follows that this sequence converges to a continuous function $g \in \mathcal{C}(\sigma_V(a))$. We claim that $g(a) = x$. Indeed, note that:

$$\forall n \in \mathbb{N} : \|g(a) - x\| \leq \|g(a) - f_n(a)\| + \|f_n(a) - x\| = \|g - f_n\| + \|f_n(a) - x\|.$$

As $n \to \infty$, the right-hand side goes to 0 and we are left with the fact that $g(a) = x$. This proves that $x \in V_a$. Finally, a subalgebra of V which contains the identity and a will consist of the polynomials in a. The closure will, then, contain all limits of polynomials. By the self-adjoint functional calculus, V_a is precisely that closure. □

The corollary above forms the first step for us in trying to establish a version of the Spectral Mapping Theorem for unital $C^\star$-algebra. The next step is to show that the spectrum of an element in a $C^\star$-algebra does not change when one considers a $C^\star$-subalgebra instead. To do that, we need the following lemma.

Lemma 3.2.6 Let V be a unital $C^\star$-algebra with identity e and let $U \subseteq V$ be a unital $C^\star$-subalgebra. Suppose that $a \in U$ has an inverse a^{-1} in V. Then, $a^{-1} \in U$.

Proof. Assume, first, that a is self-adjoint. Since a^{-1} exists, it follows that $0 \notin \sigma_V(a)$. Therefore, the continuous map $f : t \mapsto t^{-1}$ is well-defined on $\sigma_V(a)$. But this means that $f(t)t = tf(t) = 1$ for every $t \in \sigma_V(a)$. By the continuous functional calculus, it follows that $af(a) = f(a)a = e$. That is, $a^{-1} = f(a)$. By Corollary 3.2.5, we know that V_a is the smallest, closed subalgebra of V which contains both a and e. Therefore, $f(a) \in V_a \subseteq U$ and that proves the result when a is self-adjoint. Now, assume that a is not self-adjoint. Then, $a^\star a$ is self-adjoint and is invertible. Therefore, its inverse (which is $a^{-1}(a^\star)^{-1}$) lies in U. But since U is a $C^\star$-subalgebra, it follows that $a^\star \in U$. This implies that $a^{-1} \in U$. □

Now, Let us get to a proof of the second ingredient. The proposition we will prove now is also of independent interest.

Proposition 3.2.7. Spectral Permanence Theorem

Let V be a unital $C^\star$-algebra with identity e and let $U \subseteq V$ be a unital $C^\star$-subalgebra. For each $a \in U$, $\sigma_V(a) = \sigma_U(a)$.

Proof. Let $\lambda \in \sigma_V(a)$. Then, $a - \lambda e$ is singular in V. Therefore, it is singular in U. As a consequence, $\lambda \in \sigma_U(a)$. Now, we have to show that $\sigma_U(a) \subseteq \sigma_V(a)$. Suppose that $\lambda \notin \sigma_V(a)$. Then, $a - \lambda e \in U$ and it is invertible in V. By the lemma, it follows that $(a - \lambda e)^{-1} \in U$. But this implies that $\lambda \notin \sigma_U(a)$. That proves the inclusion. □

We are ready to give the proof of the spectral mapping theorem. Note that we will only prove this for self-adjoint elements but it can be extended to normal elements, as we will show later.

Theorem 3.2.8. Spectral Mapping Theorem for $C^\star$-algebras

Let V be a unital $C^\star$-algebra. If $a \in V$ is self-adjoint and $f \in \mathcal{C}(\sigma_V(a))$, then:

$$\sigma_V(f(a)) = f(\sigma_V(a)).$$

Proof. Let V_a be the $C^\star$-subalgebra as in Corollary 3.2.5. Then, $\sigma_V(f(a)) = \sigma_{V_a}(f(a))$. Now, the map ϕ as defined in Theorem 3.2.4 is a $\star$-isomorphism from V onto V_a[a]. Therefore, $\sigma_{V_a}(f(a)) = \sigma_{\mathcal{C}(\sigma_V(a))}(f) = f(\sigma_V(a))$. But this means that $\sigma_V(f(a)) = f(\sigma_V(a))$ and we are done. □

[a] Indeed, ϕ is injective since it is norm-preserving (in fact, its linearity implies that it is isometric).

Let us finish our work in this section by proving some additional properties of $\star$-homomorphisms that are rather interesting in their own right.

Proposition 3.2.9

Let V, W be unital $C^\star$-algebras and let $f : V \to W$ be a unital $\star$-homomorphism. Then, the following hold:

1. $\forall a \in V : \sigma_W(f(a)) \subseteq \sigma_V(a) \wedge \|f(a)\| \leq \|a\|$.
2. Let $a \in V$ be self-adjoint and let $\phi \in \mathcal{C}(\sigma_V(a))$. Then, $\phi(f(a)) = f(\phi(a))$.
3. If f is injective, $\sigma_W(f(a)) = \sigma_V(a)$ for all $a \in V$. Moreover, $\|f(a)\| = \|a\|$ for all $a \in V$. Finally, $f(V)$ is a unital $C^\star$-subalgebra.

Proof. We will prove each statement one-by-one.

1. Let $a \in V$ & let e_V, e_W denote the identity elements of V and W respectively. Let us show that $\sigma_W(f(a)) \subseteq \sigma_V(a)$. Let $\lambda \in \sigma_W(f(a))$. Then, $f(a) - \lambda e_W$ is singular. But we can write $f(a) - \lambda e_W = f(a - \lambda e_V)$ since f is a unital $\star$-homomorphism. Since unital $\star$-homomorphisms map invertible elements to invertible elements, it follows that $a - \lambda e_V$ is singular and that $\lambda \in \sigma_V(a)$. Next, we will prove that $\|f(a)\| \leq \|a\|$. Observe that:

$$\|f(a)\|^2 = \left\|f(a)^\star f(a)\right\| = \|f(a^\star)f(a)\| = \|f(a^\star a)\|.$$

If a is normal, then $f(a)$ is normal. Since $\sigma_W(f(a)) \subseteq \sigma_V(a)$, it follows that $\nu(f(a)) \leq \nu(a)$. Assume that a is normal. Then:

$$\|f(a)\| = \nu(f(a)) \leq \nu(a) = \|a\|.$$

Now, suppose that a is arbitrary. Then, $a^\star a$ is self-adjoint (which implies that it is normal). Therefore:

$$\|f(a)\|^2 = \|f(a^\star a)\| \leq \|a^\star a\| = \|a\|^2.$$

as was desired.

2. If ϕ is a polynomial, then this is obviously true because f is a unital $\star$-homomorphism. By approximation by polynomials, it must be the case that this holds for all continuous functions $\phi : \sigma_V(a) \to \mathbb{C}$.
3. Let $a \in V$ be self-adjoint first. We know that $\sigma_W(f(a)) \subseteq \sigma_V(a)$, from (1). Suppose that this inclusion is strict. Then, there is a $\lambda \in \sigma_V(a)$ such that $\lambda \notin \sigma_W(f(a))$. By statement (4) of Theorem B.1.7, there exists a continuous function $g \in \mathcal{C}(\sigma_V(a))$ such that $g|_{\sigma_W(f(a))} = 0$ and such that $g(\lambda) = 1$. By statement (2) of this proposition, it is the case that $g(f(a)) = f(g(a))$. Since g is zero on $\sigma_W(f(a))$, it follows that $g(f(a)) = 0$. This means that $g(a) \in \ker(f)$. Since f is injective, it follows that $g(a) = 0$. But this means that $\|g\| = 0$ from statement (1) of Theorem 3.2.4 so g is identically zero and that is impossible. It follows that:

$$\sigma_W(f(a)) = \sigma_V(a).$$

We will show this result for all other elements of V towards the end of this proof. Let us establish the second statement in (3). If a is self-adjoint, then $f(a)$ is normal and we have:

$$\|a\| = \nu(a) = \nu(f(a)) = \|f(a)\|.$$

Their spectral radii are equal because $\sigma_W(f(a)) = \sigma_V(a)$ for all self-adjoint elements. If $a \in V$ is arbitrary, then $a^\star a$ is self-adjoint so:

$$\|a\|^2 = \|a^\star a\| = \|f(a^\star a)\| = \|f(a^\star)f(a)\| = \left\|f(a)^\star f(a)\right\| = \|f(a)\|^2,$$

which proves that $\|f(a)\| = \|a\|$. Moreover, we know that $f(V)$ is a unital $\star$-subalgebra of W so all we need to do is to show that $f(V)$ is closed. Indeed, let $(f(a_n))_{n\in\mathbb{N}}$ be a sequence in $f(V)$ that converges to $y \in W$. Then:

$$\forall m, n \in \mathbb{N} : \|f(a_n) - f(a_m)\| = \|f(a_n - a_m)\| = \|a_n - a_m\|.$$

It follows that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V and converges to some $a \in V$. We claim that $y = f(a)$. Indeed, the continuity of f implies this so that proves that $f(V)$ is closed and, therefore, a $C^\star$-subalgebra. Finally, we know that $f : V \to f(V)$ is a $\star$-isomorphism of $C^\star$-algebras. Therefore, we have that $\sigma_W(f(a)) = \sigma_{f(V)}(f(a)) = \sigma_V(a)$, as was desired. Here, we have used the Spectral Permanence Theorem. □

A consequence of the proposition and the Unitization Theorem is the following result.

Corollary 3.2.10. Let V be a unital $C^\star$-algebra, with the identity denoted by e. Let $W \subseteq V$ be a $C^\star$-subalgebra not containing e. Then, W_e is $\star$-isomorphic to the $C^\star$-algebra generated by $W \cup \{e\}$. Moreover, it also is true that:

$$\forall x \in W : \forall \lambda \in \mathbb{C} : \|(x, \lambda)\|_{W_e} = \|x + \lambda e\|_V$$

Proof. Let $C^\star(W, e)$ denote the $C^\star$-subalgebra generated by $W \cup \{e\}$. I claim that:

$$C^\star(W, e) = \{a + \lambda e : a \in W \wedge \lambda \in \mathbb{C}\}$$

The set on the right-hand side is clearly a subset of $C^\star(W, e)$. On the other hand, the set on the right-hand side above is also a $C^\star$-subalgebra which contains e and W. Therefore, $C^\star(W, e)$ is a subset of this set and that proves equality. Define the following map:

$$\Phi : W_e \to C^\star(W, e)\ ,\ (x, \lambda) \mapsto \Phi(x, \lambda) := x + \lambda e$$

Showing that this is a $\star$-homomorphism is relatively easy. Surjectivity is obvious, so it remains for us to show injectivity. Let $\Phi(x, \lambda) = 0$. This means that $x + \lambda e = 0$ so if we assume that $\lambda \neq 0$, then $-\frac{x}{\lambda} = e$. Since W is a $C^\star$-algebra, it follows that $-\frac{x}{\lambda} \in W$. But this means that $e \in W$ and that is impossible. Therefore, $\lambda = 0$. But this means that $x = 0$ so $(x, \lambda) = (0, 0)$. Since Φ is a $\star$-isomorphism, statement (3) of Proposition 3.2.9 implies that:

$$\forall x \in W : \forall \lambda \in \mathbb{C} : \|x + \lambda e\|_V = \|\Phi(x, \lambda)\|_V = \|(x, \lambda)\|_{W_e},$$

as was desired. □

The reader might be confused about where exactly we used the Unitization Theorem in the proof of the corollary above. In fact, we did not use it explicitly. It is simply necessary to make sense of W_e as a $C^\star$-algebra. The reader should definitely strive to familiarize themselves with the self-adjoint functional calculus and some of its consequences as proved above. We will be repeatedly referring back to the results in this section within future proofs.

Now, recall that Lemma 3.1.7 provides some indication that $C^\star$-algebras should be viewed as "generalized complex numbers". In the next section, we attempt to take this analogy a little further through the notion of "positivity" on $C^\star$-algebras. We will define precisely what it means for an element in a $C^\star$-algebra to be "positive" and this will lead to a characterization result which cements the analogy above a lot more than we currently have.

3.3 Order & Involution

The chosen title describes exactly what we would like to discuss in this section. To facilitate a good introduction to the ideas we will present, let us set the scene with the following example.

Example 11. Let (X, τ) be a compact Hausdorff space[a]. We define a partial order $\prec$ on $\mathcal{C}(X)$ as follows:

$$\forall f, g \in \mathcal{C}(X) : f \prec g \Leftrightarrow ((g-f)(X) \subseteq [0, \infty)),$$

where $(g-f)(X)$ denotes the image of the continuous function $x \mapsto g(x) - f(x)$. This defines a partial order on $\mathcal{C}(X, \mathbb{C})$. In particular, we note three properties that $\prec$ possesses:

1. $\forall f, g, h \in \mathcal{C}(X) : f \prec g \Rightarrow f + h \prec g + h$.
2. $\forall \alpha \in \mathbb{C} : \forall f \in \mathcal{C}(X) : (0 \leq \alpha \wedge 0 \prec f) \Rightarrow 0 \prec \alpha f$.
3. $\forall f \in \mathcal{C}(X) : 0 \prec f \Leftrightarrow (f = \overline{f} \wedge f(X) \subseteq [0, \infty))$[b].

All of these are very easy to establish using the definition of $\prec$. Notice that properties (1) and (2) are of a distinctively different character than property (3). This is mainly because properties (1) and (2) mainly have to do with the compatibility between the partial order on $\mathcal{C}(X)$ and addition/scalar multiplication, while property (3) has to do with a characterization of "positivity", where we say that a function $f \in \mathcal{C}(X)$ is positive if $0 \prec f$.

[a] For readers who are not familiar with topology, feel free to think of X as some compact subset of $\mathbb{R}^n$. The gist of this example should still be fairly appreciable in that case.

[b] The reader may have noticed that if we require $f(X) \subseteq [0, \infty)$, then it is a bit redundant to also require that $f = \overline{f}$. We chose to distinguish both conditions because one is (usually) much easier to check than the other. We also want to use this exact formulation as motivation for the abstract definition of positivity in a $C^\star$-algebra.

Let us start by focusing on properties (1) and (2), as listed in Example 11. As we mentioned, these are compatibility conditions which can be made abstract through the following definition.

Definition 3.3.1

An ordered vector space is a pair $(V, \prec)$ of a real/complex vector space V and a partial order [a] $\prec$ on V which satisfies the following properties:

1. $\forall v \in V : \forall \lambda \in \mathbb{R} : (0 \prec v \wedge 0 \leq \lambda) \Rightarrow 0 \prec \lambda v$.
2. $\forall u, v, w \in V : u \prec v \Rightarrow u + w \prec v + w$.

The set $P := \{v \in V : 0 \prec v\}$ is called the positive cone of V.

[a] We remind the reader that a partial order on a set is a relation which is reflexive, anti-symmetric and transitive. For the reader who may have an interest in studying partial orders from a set-theoretic perspective, they may see [20, Chapter 2.5, Chapter 6].

The two conditions given in Definition 3.3.1 are compatibility conditions with the operations defined on a vector space. With the terminology introduced by this definition, it should be noted that $\mathcal{C}(X)$ is an ordered vector space when (X, τ) is a compact topological space. Our goal is not to delve deep into the study of ordered vector spaces; we will simply be using the terminology for convenience.

With all of this being said, let us now direct our focus to property (3) as listed in Example 11. We will freely use the notation in that example. Recall that in Example 5, we computed the spectrum of an $f \in \mathcal{C}(X)$. In particular, $\sigma_{\mathcal{C}(X)}(f) = f(X)$. But now, we can re-phrase property (3) as follows:

$$\forall f \in \mathcal{C}(X) : 0 \prec f \Leftrightarrow (\sigma_{\mathcal{C}(X)}(f) \subseteq [0, \infty) \wedge f \text{ is self-adjoint})$$

The reader should, now, be able to see a way to generalize the notion of "positivity" within an arbitrary $\star$-algebra. This is accomplished by the following definition.

Definition 3.3.2

Let V be a $\star$-algebra. An element $x \in V$ is said to be positive if x is self-adjoint and $\sigma_V(x) \subseteq \mathbb{R}_0^+$[a]. V^+ will be used to denote the set of all positive elements of V and for any $x \in V$, we write $0 \prec x$ to denote that x is positive in the sense described above.

[a] The spectrum of an element of a $\star$-algebra (possibly without identity) is defined as its spectrum in the unitization.

The use of the word "positive" indicates that some kind of partial order is lurking around the corner. We need to show that V does have a partial ordering such that V^+ is a positive cone within that partial ordering. To that end, we need the following lemma about self-adjoint elements in a $C^\star$-algebra. A fair number of proofs, from now on, will make use of the continuous functional calculus that we established in the last section.

Lemma 3.3.3 Let V be a unital $C^\star$-algebra. Let $x \in V$ be self-adjoint. The following are equivalent:

1. $x \in V^+$.
2. $\forall a \in \mathbb{R} : a \geq \|x\| \Rightarrow \|x - ae\| \leq a$.
3. $\exists a \in \mathbb{R} : a \geq \|x\| \Rightarrow \|x - ae\| \leq a$.

Proof.

(1) $\Rightarrow$ (2) : Let $x \in V^+$. Observe that $\sigma_V(x) \subseteq [0, \|x\|] \subseteq [0, a]$. Define the function:
$$f : \sigma_V(x) \to \mathbb{R}\ ,\ t \mapsto f(t) := t - a$$
By (1) in Theorem 3.2.4, it is the case that:
$$\|x - ae\| = \|f(x)\| = \|f\| \leq a,$$
as was desired.

(2) $\Rightarrow$ (3) : This is obvious.

(3) $\Rightarrow$ (1) : Consider the function f defined in the proof of (1) $\Rightarrow$ (2) above. By Theorem 2.1.6, we have that $\sigma_V(f(x)) = f(\sigma_V(x))$. This means that $a + \sigma_V(f(x)) = \sigma_V(x)$. We know that $\sigma_V(f(x)) \subseteq [-a, a]$. It follows that $\sigma_V(x) \subseteq [0, 2a]$. This establishes that $x \in V^+$. □

This lemma gives us the following permanence theorem concerning V^+, when V is a $C^\star$-algebra.

Theorem 3.3.4

Let V be a unital $C^\star$-algebra. Then, the following hold:
1. V^+ is closed.
2. $\forall x \in V^+ : \forall a \in [0, \infty) : ax \in V^+$.
3. $\forall x, y \in V^+ : x + y \in V^+$.
4. $\forall x, y \in V^+ : xy = yx \Rightarrow xy \in V^+$.
5. Let $x \in V$ be fixed. If $x \in V^+$ and $-x \in V^+$, then $x = 0$.

Proof. We will prove each statement one-by-one.
1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V^+ that converges to $x \in V$. We will show that x is a positive element. Indeed:
$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} {x_n}^\star = (\lim_{n\to\infty} x_n)^\star = x^\star,$$
where we have used the continuity of the involution. Next, observe that:
$$\forall n \in \mathbb{N} : \|x_n - \|x_n\|e\| \leq \|x_n\|.$$
Taking the limit as $n \to \infty$ and from the continuity of the norm (and the operations on V), we have that:
$$\|x - \|x\|e\| \leq \|x\|.$$
which proves, by Lemma 3.3.3, that $x \in V^+$.
2. Let $x \in V^+$ and let $a \in [0, \infty)$. If $a = 0$, then we are done. So, assume that $a > 0$. Then:
$$(ax)^\star = ax^\star = ax.$$
Let $\lambda \in \sigma_V(x)$. Then, $ax - \lambda e$ is singular. But that means that $x - \frac{\lambda}{a}e$ is singular. Since $x \in V^+$ and $a > 0$, it follows that $\lambda > 0$. It follows that $\sigma_V(ax) \subseteq [0, \infty)$.
3. Let $x, y \in V^+$. Then:
$$(x + y)^\star = x^\star + y^\star = x + y.$$
Next, observe that $\|x + y\| \leq \|x\| + \|y\|$. By Lemma 3.3.3, it is the case that:
$$\|x - \|x\|e\| \leq \|x\| \wedge \|y - \|y\|e\| \leq \|y\|.$$
As a consequence, we have that:
$$\|(x + y) - (\|x\| + \|y\|)e\| = \|(x - \|x\|e) + (y - \|y\|e)\| \leq \|x - \|x\|e\| + \|y - \|y\|e\| \leq \|x\| + \|y\|.$$
It follows that $x + y \in V^+$.
4. Let $x, y \in V^+$ and $xy = yx$. Then:
$$(xy)^\star = y^\star x^\star = yx = xy.$$
Let W be the $C^\star$-algebra generated by x, y, e. This will be commutative and so:
$$\sigma_W(xy) \subseteq \sigma_W(x)\sigma_W(y) \subseteq [0, \infty).$$
Since the spectrum in the $C^\star$-subalgebra is the same as the spectrum in V, it follows that $xy \in V^+$.
5. Since $\sigma_V(-x) = -\sigma_V(x)$ and since $x, -x \in V^+$, it follows that $\sigma_V(x) = \{0\}$[a]. Since $\|x\| = \nu(x) = 0$, it follows that $x = 0$. □

[a] The spectrum could not be empty since we are working in a unital $\mathbb{C}$-Banach algebra (Theorem 2.1.8).

Even if it is glaringly obvious right now, we still cannot refer to V^+ as a "positive cone" when V is a $C^\star$-algebra. Indeed, one needs a partial order to define a positive cone in the first place; let us deal with this next.

Proposition 3.3.5

Let V be a unital $C^\star$-algebra. Define a relation $\prec$ by:
$$\forall x, y \in V : x \prec y \Leftrightarrow y - x \in V^+$$
This is a partial order on V and turns V into an ordered vector space. In this case, V^+ is a positive cone.

Proof. We will first show that $\prec$ defines a partial order on V. Then, we will show that the compatibility conditions in Definition 3.3.1 hold. The fact that V^+ is a positive cone is obvious from how the relation $\prec$ has been defined.

1. It is obvious that $x \prec x$ for any $x \in V$. This is because the zero vector is a positive element.
2. Let $x \prec y$ and $y \prec x$. Then, $y - x \in V^+$ and $x - y \in V^+$. Since $\sigma_V(x-y) = -\sigma_V(y-x)$, it follows that $\sigma_V(x-y) = \{0\}$. But $x - y$ is self-adjoint so $\|x-y\| = \nu(x-y) = 0$. As a consequence, $x = y$.
3. Let $x \prec y$ and $y \prec z$. Then, $y - x \in V^+$ and $z - y \in V^+$. Then, $z - x \in V^+$ from (3) in Theorem 3.3.4.

Now, we will prove that the compatibility conditions hold.

1. Let $x \prec y$ and let $z \in V$. Then, $y - x = (y+z) - (x+z) \in V^+$. This implies that $x + z \prec y + z$.
2. Let $0 \prec x$ and let $a \geq 0$ be a real number. By (2) in Theorem 3.3.4, it follows that $ax \in V^+$ and that $0 \prec ax$.

This completes the proof. □

At this stage, we should also attempt to get a better control over the partial order introduced above. This comes in the form of the following decomposition result.

Proposition 3.3.6

Let V be a unital $C^\star$-algebra and let $a \in V$ be self-adjoint. Let $f \in \mathcal{C}(\sigma_V(a))$.

1. $f(a) \in V^+$ iff $f(t) \geq 0$ for all $t \in \sigma_V(a)$.
2. $\|a\|e \pm a \in V^+$.
3. There exist $a^+, a^- \in V^+$ such that $a^+a^- = a^-a^+ = 0$ and $a = a^+ - a^-$. These conditions determine a^+ and a^- uniquely. Moreover, $\|a\| = \max(\|a^+\|, \|a^-\|)$.

Proof. We will prove each statement one-by-one.

1. Let $f(t) \geq 0$ for every $t \in \sigma_V(a)$. By Theorem 3.2.8, we have that $\sigma_V(f(a)) = f(\sigma_V(a))$. As a consequence, $\sigma_V(f(a)) \subseteq [0, \infty)$. Since f is real-valued throughout $\sigma_V(a)$, $f(a)$ is self-adjoint. This implies that $f(a) \in V^+$. Next, assume that $f(a) \in V^+$. This implies that $\sigma_V(f(a)) \subseteq [0, \infty)$ and that $f(\sigma_V(a)) \subseteq [0, \infty)$.
2. Clearly, $\|a\|e \pm a$ are self-adjoint elements. Define two polynomials:
$$\forall t \in \mathbb{C} : f^{\pm}(t) := \|a\| \pm t$$
Since a is self-adjoint, it follows that $\sigma_V(a) \subseteq \mathbb{R}$. If $t \in \sigma_V(a)$, then $|t| \leq \|a\|$ so $-\|a\| \leq t \leq \|a\|$. But this just translates to the fact that the functions $f^{\pm}$ are non-negative on $\sigma_V(a)$ so $f^{\pm}(a) = \|a\|e \pm a \in V^+$ by statement (1), as was desired.
3. Define two functions:
$$f : \mathbb{R} \to \mathbb{R}\ ,\ t \mapsto f(t) := \frac{1}{2}(t + |t|)$$
$$g : \mathbb{R} \to \mathbb{R}\ ,\ t \mapsto g(t) := \frac{1}{2}(-t + |t|)$$
Both of these functions are continuous, even when restricted to $\sigma_V(a)$ [a]. By the self-adjoint functional calculus, we define $a^+ := f(a)$ and $a^- := g(a)$. We will prove that these satisfy all of the desired properties.
   (a) Since $f \geq 0$ and $g \geq 0$, it follows by statement (1) that $f(a) \in V^+$ and $g(a) \in V^+$.
   (b) Notice that $f(t) - g(t) = t$ for every $t \in \mathbb{R}$. As a consequence, $a = f(a) - g(a) = a^+ - a^-$.
   (c) It should also be clear that $f(t)g(t) = g(t)f(t) = 0$ for all $t \in \mathbb{R}$. It follows, from the self-adjoint functional calculus, that $f(a)g(a) = a^+a^- = g(a)f(a) = a^-a^+ = 0$.
   (d) Next, we have that:
$$\|a\| = \|f(a) - g(a)\| = \|f - g\| = \max(\|f\|, \|g\|) = \max(\|a^+\|, \|a^-\|)$$
   as was desired.

We still need to show that those given conditions determine a^+ and a^- uniquely. Suppose that $a = b - c$, where $b, c \in V^+$ and $bc = cb = 0$. Then, we have that:
$$\forall n \in \mathbb{N} : a^n = (b-c)^n = b^n + (-c)^n.$$
The above implies that for every complex polynomial $p \in \mathbb{C}[z]$, we have that $p(a) = p(b) + p(-c)$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of polynomials that converge to the function f (defined above) uniformly on $\sigma_V(a) \cup \sigma_V(b) \cup \sigma_V(-c)$. Then:
$$f(a) = \lim_{n\to\infty} p_n(a) = \lim_{n\to\infty} (p_n(a) + p_n(-c)) = f(b) + f(-c).$$
Since c is a positive element, $-c$ has non-positive numbers in its spectrum so $f(s) = 0$ whenever $s \in \sigma_V(-c)$. On the other hand, $f(s) = s$ when $s \in \sigma_V(b)$. Therefore, $f(b) = b$ and $f(-c) = 0$. But this means that $a^+ = f(a) = f(b) = b$. This implies that $c = a^-$ and we are done. □

[a] Note that this is a subset of $\mathbb{R}$ since a is self-adjoint.

The next step is to use this decomposition theorem to derive a useful characterization theorem for positivity. To do this, we will need the following lemma.

Lemma 3.3.7 Let V be a unital $C^\star$-algebra and let $a \in V$. Suppose that $-a^\star a \in V^+$. Then, $a = 0$.

Proof. Let $a = x + iy$, where x and y are self-adjoint. Then, $a^\star = x - iy$. We have the following formulas:

$$a^\star a = (x - iy)(x + iy) = x^2 + i(xy - yx) + y^2.$$
$$aa^\star = (x + iy)(x - iy) = x^2 + i(yx - xy) + y^2.$$

By the the spectral mapping theorem (Theorem 2.1.6), it follows that x^2 and y^2 are positive elements. Therefore:

$$aa^\star + a^\star a = 2(x^2 + y^2).$$

Bringing $a^\star a$ to the right-hand side and using the hypothesis, it follows that $aa^\star$ is a positive element. By Jacobson's Lemma (Proposition 2.1.11), it follows that $a^\star a$ is a positive element as well. As a consequence of statement (5) of Theorem 3.3.4, it follows that $a^\star a = 0$. By the $C^\star$-condition, we have that:

$$0 = \|a^\star a\| = \|a\|^2,$$

which implies that $a = 0$. □

With this lemma, we are ready to prove the desired characterization of positivity of elements in a $C^\star$-algebra.

Theorem 3.3.8. Characterization of Positivity

Let V be a $C^\star$-algebra and let $a \in V$. The following are equivalent:

1. a is positive.
2. $a = b^2$ for some self-adjoint $b \in V$.
3. $a = bb^\star$ for some $b \in V$
4. $a = b^\star b$ for some $b \in V$.

If b in (2) is positive, then it is also the unique element whose square is a. We refer to it as the square root of a and denote it by $\sqrt{a}$ or $a^{\frac{1}{2}}$.

Proof.

(1) ⇒ (2) : Assume that a is positive. Then, a is self-adjoint and its spectrum is a subset of the non-negative real numbers. Define the function:

$$f : [0, \infty) \to \mathbb{R} \ , \ x \mapsto f(x) := \sqrt{x}$$

Notice that f is continuous and that $(f(x))^2 = x$ for every $x \in [0, \infty)$. By the self-adjoint functional calculus, $f(a)$ is well-defined since the spectrum of a lies in $[0, \infty)$. Letting $b = f(a)$, we know that b must be self-adjoint because f is real-valued[a].

(2) ⇒ (3) : Since b is self-adjoint, there is no issue in just writing $a = b^2 = bb^\star$.

(3) ⇒ (4) : In this case, there is no problem in writing $a = bb^\star = (b^\star)^\star b^\star$. This is in the same form as the one given in the actual statement of the result.

(4) ⇒ (1) : Let $a = a^+ - a^-$, with a^+, a^- being the unique positive elements which satisfy $a^+a^- = 0 = a^-a^+$. Next, let $c = ba^-$. Then:

$$c^\star c = a^- b^\star b a^- = a^- a a^- = a^-(a^+ - a^-)a^- = -(a^-)^3.$$

Since a^- is positive, it follows that $-c^\star c$ is positive because $(a^-)^3$ is positive. As a consequence of the preceding lemma, $c = 0$. But this means that $(a^-)^3 = 0$. Since a^- is self-adjoint, it is normal so $a^- = 0$ by the strength of statement (1) of Proposition 3.2.1. This implies that $a = a^+$ and this certainly means that a is positive.

Finally, suppose that $a = b^2$ with $b := f(a)$. Suppose that $a = c^2$ for some $c \in V^+$ and consider the function f as given above, restricted to $\sigma_V(a)$. We can find a sequence $(p_n)_{n \in \mathbb{N}}$ of polynomials that converge uniformly on $\sigma_V(a)$ to f. Define, for each $n \in \mathbb{N}$, the functions:

$$q_n : \sigma_V(a) \to \mathbb{R} \ , \ t \mapsto q_n(t) := p_n(t^2)$$

Since $\sigma_V(a) = \sigma_V(c^2) = (\sigma_V(c))^2$, it follows that $q_n(t) \to t$ uniformly for $t \in \sigma_V(c)$. As a consequence:

$$c = \lim_{n\to\infty} q_n(c) = \lim_{n\to\infty} p_n(c^2) = \lim_{n\to\infty} p_n(a) = f(a) = b$$

which proves the desired result. □

[a]This is a part of statement (4) of Theorem 3.2.4.

We should also point out that Lemma 3.3.3, combined with Theorem 3.3.8, gives a useful characterization of positivity for self-adjoint elements. Symbolically, Theorem 3.3.8 should remind one of a similar result that holds in $\mathbb{C}$. The involution on a $C^\star$-algebra should not be conflated with complex conjugation. Nevertheless, we have derived a number of results in this section (and the last) which suggest that $C^\star$-algebras are "generalized" complex numbers. This is a bold claim but there does seem to be an element of truth to it given everything we have learned so far.

3.4 Positive Linear Functionals

The time has arrived for us to acknowledge that a large part of the study of vector spaces with additional structure has to do with the study of the corresponding dual space. This is certainly true even in linear algebra, where the dual space is often used to provide information about the original vector space. Of course, if V is a $\mathbb{K}$-normed space, then one must account for the extra structure of a norm by looking at the *bounded/continuous* linear functionals on V instead.

In this section, we want to investigate the consequences of linear functionals being equipped with a structure that is compatible with the involution. Since the involution induces a a natural (partial) ordering on a $C^\star$-algebra, it is also necessary to equip linear functionals with a structure that is compatible with this order. To this end, we start with the following general definition. This section is adapted from [21, Section 4.3].

Definition 3.4.1

Let V be a $\star$-algebra with identity e. Let $M \subseteq V$ be a self-adjoint subspace of V containing e and let M^+ denote the set of all elements of M that are positive in V. A linear functional $f : M \to \mathbb{C}$ is positive if:

$$\forall x \in M^+ : f(x) \geq 0.$$

f is a state if it is a positive linear functional such that $f(e) = 1$. We also define a function:

$$f^\star : M \to \mathbb{C} \ , \ x \mapsto f^\star(x) := \overline{f(x^\star)}$$

This function is known as the conjugate of f. f is Hermitian/self-adjoint if $f = f^\star$.

Let us provide three examples of positive functionals, all three being fairly straightforward.

Example 12. Let (X, τ) be a compact Hausdorff space and let $a \in X$ be fixed. Define a map:

$$\phi_a : \mathcal{C}(X) \to \mathbb{C} \ , \ f \mapsto \phi_a(f) := f(a)$$

This is obviously a positive linear functional.

Example 13. Let H be a Hilbert space and let $x \in H$ be fixed. Define:

$$\omega_x : \mathcal{B}(H) \to \mathbb{C} \ , \ T \mapsto \langle Tx, x\rangle$$

This is a positive linear functional with respect to the $C^\star$-orders. Indeed, let $T \in \mathcal{B}(H)$ be positive. By Theorem 3.3.8, there is a $S \in \mathcal{B}(H)$ such that $T = S^\star S$. Therefore:

$$\omega_x(T) = \langle S^\star S x, x\rangle = \langle S(x), S(x)\rangle = \|S(x)\|^2 \geq 0$$

which proves that this is a positive linear functional.

Example 14. Let $\mathcal{C}[0,1]$ be the set of all complex-valued continuous functions $f : [0,1] \to \mathbb{C}$. Define the function:

$$\phi : \mathcal{C}[0,1] \to \mathbb{C} \ , \ f \mapsto \phi(f) := \int_0^1 f(x)\ dx$$

This is a positive linear functional.

Let us begin with a simple lemma that deals with the set M^+ when M is a particular subspace of a $C^\star$-algebra.

Lemma 3.4.2 Let V be a $C^\star$-algebra and let M be a self-adjoint subspace of V containing the identity. Let M^+ be the set of all positive elements of M.

1. M is the span of M^+.
2. If $M \subseteq W \subseteq V$ with W being a $C^\star$-subalgebra, then M^+ is also the set of positive elements with respect to W.

Proof. We will prove each statement one-by-one.

1. Let $a \in M$. Then, $a = x + iy$ with x and y being self-adjoint. Since x and y can be written as linear combinations of a and $a^\star$, it follows that $x, y \in M$ because $a^\star \in M$. But notice that both x and y can be decomposed into positive elements by Proposition 3.3.6 and these positive elements will lie in M^+. As a consequence, a can be written as a linear combination of elements in M^+ so we are done.
2. Observe that:
$$M^+ = M \cap V^+ = M \cap W \cap V^+ = M \cap (W \cap V^+) = M \cap W^+,$$
so we are done. □

Before doing anything else, let us establish a decomposition result for linear functionals and some other useful properties of functionals that behave "nicely" with respect to an involution.

Proposition 3.4.3

Let V be a $\star$-algebra with identity e. Let $M \subseteq V$ be a self-adjoint subspace containing e.

1. Define a map:
$$\star : \mathrm{Hom}(M, \mathbb{C}) \to \mathrm{Hom}(M, \mathbb{C}) \ , \ f \mapsto f^\star$$
This is a conjugate linear map.
2. Self-adjoint linear functionals are linear maps that preserve involution.
3. Let $f : M \to \mathbb{C}$ be a linear functional. Then, there exist unique self-adjoint linear functionals $f_1, f_2 : M \to \mathbb{C}$ such that $f = f_1 + i f_2$.
4. Let $f : M \to \mathbb{C}$ be a linear functional. f is self-adjoint iff $f(x) \in \mathbb{R}$ for every self-adjoint $x \in M$.

Proof. (3) and (4) are the only parts which are not very clear. For (3), we can mimic the argument used in Lemma 3.1.7 to find explicit expressions for f_1 and f_2. This will prove uniqueness and existence simultaneously. For (4), we start by assuming that f is self-adjoint. Then, $f(x) = f(x^\star) = \overline{f(x)}$ for all $x \in M$ where x is self-adjoint. As a consequence, $f(x) \in \mathbb{R}$ when x is self-adjoint. Next, assume that f is real on the self-adjoint elements of M. Let $f = f_1 + if_2$, where f_1, f_2 are self-adjoint linear functionals. Let $x \in M$ be fixed. Then, $x + x^\star \in M$ so:

$$f(x) + f(x^\star) = f(x + x^\star) = f_1(x + x^\star) + if_2(x + x^\star).$$

The point is that $f_1(x + x^\star), f_2(x + x^\star) \in \mathbb{R}$ since they are f_1 & f_2 are both self-adjoint and $x + x^\star$ is self-adjoint. In particular, $f_2(x + x^\star) = 0$ since $f(x + x^\star)$ is real. Now, we compute $f_2(x + x^\star)$ as follows:

$$0 = f_2(x + x^\star) = f_2(x) + f_2(x^\star) = f_2(x) + \overline{f_2(x)}.$$

This implies that $f_2(x) = -\overline{f_2(x)}$. It follows that $\mathrm{Re}(f_2(x)) = 0$. On the other hand, note that $i(x - x^\star)$ is self-adjoint. Therefore, we have that:

$$f(i(x - x^\star)) = f_1(i(x - x^\star)) + if_2(i(x - x^\star)).$$

Since f_1 and f_2 are self-adjoint, it follows that $f_1(i(x - x^\star)), f_2(i(x - x^\star)) \in \mathbb{R}$ so $f_2(i(x - x^\star)) = 0$. This implies $f_2(x) = \overline{f_2(x)}$ and this means that $\mathrm{Im}(f_2(x)) = 0$. That is, $f_2(x) = 0$ It follows that $f(x) = f_1(x)$ for every $x \in M$ and this proves that f is self-adjoint. □

Due to statement (4) of this proposition, we actually obtain the following nice corollary about positive functionals.

Corollary 3.4.4. Let V be a unital $C^\star$-algebra. If $f : V \to \mathbb{C}$ is a positive linear functional, then f is self-adjoint.

Proof. Let $x \in M$ be self-adjoint. By Proposition 3.3.6, we have $x = x^+ - x^-$ with $x^+, x^- \in V^+$. This means that:

$$f(x) = f(x^+) - f(x^-).$$

Since f is a positive linear functional, it follows that $f(x^+), f(x^-) \in \mathbb{R}$. This implies that $f(x) \in \mathbb{R}$. By statement (4) of the preceding proposition, it follows that f is self-adjoint. □

It was shown previously that multiplicative linear functionals defined on unital Banach algebras are continuous. The hope is that positivity of a linear functional is enough to guarantee continuity; this hope is well-founded if we work with $C^\star$-algebras. Before we show it, we prove the following estimate which is of independent interest.

Proposition 3.4.5. The Cauchy-Schwarz Inequality

Let V be a $\star$-algebra and let $f : V \to \mathbb{C}$ be a linear functional which satisfies:

$$\forall x \in V : f(x^\star x) \geq 0.$$

It is, then, the case that:

1. $\forall x, y \in V : f(x^\star y) = \overline{f(y^\star x)}$.
2. $\forall x, y \in V : |f(y^\star x)|^2 \leq f(x^\star x) f(y^\star y)$.

Proof. Let $x, y \in V$ be fixed. Then:

$$\forall t \in \mathbb{C} : 0 \leq f((tx + y)^\star (tx + y)) = f((\bar{t}x^\star + y^\star)(tx + y)) = f(|t|^2 x^\star x + ty^\star x + \bar{t}x^\star y + y^\star y).$$

Using linearity, it follows that:

$$\forall t \in \mathbb{C} : 0 \leq |t|^2 f(x^\star x) + tf(y^\star x) + \bar{t}f(x^\star y) + f(y^\star y).$$

This will be the main inequality we will use to prove both statements. To make sure that the argument is not so messy, let $A = f(y^\star x)$ and $B = f(x^\star y)$.

1. If $t = 1$, then $A + B \in \mathbb{R}$ so:

   $$\mathrm{Im}(A) + \mathrm{Im}(B) = 0.$$

   If $t = i$, then $iA - iB \in \mathbb{R}$. Therefore, $\mathrm{Re}(A) - \mathrm{Re}(B) = 0$. But this means that:

   $$A = \mathrm{Re}(A) + i\,\mathrm{Im}(A) = \mathrm{Re}(B) - i\,\mathrm{Im}(B) = \overline{B}.$$

   This implies that $f(y^\star x) = \overline{f(x^\star y)}$ and this gives the desired result.
2. Using statement (1), let us simplify the inequality we have been given to:

   $$\forall t \in \mathbb{R} : 0 \leq |t|^2 f(x^\star x) + 2\,\mathrm{Re}(tf(y^\star x)) + f(y^\star y) = t^2 f(x^\star x) + 2t\,\mathrm{Re}(f(y^\star x)) + f(y^\star y).$$

   Notice that we have a quadratic in t which always lies above the t-axis. In particular, this means that its discriminant is non-positive. It follows that:

   $$4|\,\mathrm{Re}(f(y^\star x))|^2 - 4f(x^\star x)f(y^\star y) \leq 0,$$

   which gives us the inequality $|\,\mathrm{Re}(f(y^\star x))|^2 \leq f(x^\star x)f(y^\star y)$. Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. If we replace x with αx in the given inequality, then the right-hand side remains unchanged and we get that:

   $$|\,\mathrm{Re}(\alpha f(y^\star x))|^2 \leq f(x^\star x)f(y^\star y).$$

   Now, observe that there exists a $\theta \in \mathbb{R}$ such that $f(y^\star x) = |f(y^\star x)| \exp(i\theta)$. Select $\alpha = \exp(-i\theta)$. It follows, then, that:

   $$|f(y^\star x)|^2 \leq f(x^\star x)f(y^\star y),$$

   as was desired[a]. □

[a]The proof given here is adapted from the homogenization trick that Terence Tao uses to prove the usual Cauchy-Schwarz Inequality for inner product spaces.

The readers will notice that the two properties shown in the result above are rather reminiscent of the properties of an inner product. In particular, let V be a $\star$-algebra and let $f : V \to \mathbb{C}$ be a linear functional which satisfies:

$$\forall x \in V : f(x^\star x) \geq 0$$

Now, we can define a conjugate-linear map as follows:

$$s : V \times V \to \mathbb{C} \ , \ (x, y) \mapsto s(x, y) := f(y^\star x)$$

The point, now, is this *nearly* induces an inner product on V; $s(x, x) = 0$ does not necessarily imply that $x = 0$. Since this will be extremely important in the GNS construction, we will come back to it in the next section. Instead, we will direct our focus to the following crucial theorem involving positive linear functionals.

Theorem 3.4.6

Let V be a unital $C^\star$-algebra and let $M \subseteq V$ be a self-adjoint subspace containing the identity e. Let $f : M \to \mathbb{C}$ be a linear functional. The following are equivalent:

1. f is positive.
2. f is bounded and $\|f\| = f(e)$.

Proof.

$(1) \Rightarrow (2)$: Let f be positive. Let $x \in M$ and suppose that it is self-adjoint. Notice that $\|x\|e \pm x \in M^+$, by statement (2) of Proposition 3.3.6. As a consequence:

$$\|x\|f(e) + f(x) \geq 0 \wedge \|x\|f(e) - f(x) \geq 0.$$

This implies that $|f(x)| \leq f(e)\|x\|$. Next, we use the characterization of positivity (Theorem 3.3.8) of elements in a $C^\star$ algebra to conclude that f being positive means that:

$$\forall x \in V : x^\star x \in M \Rightarrow f(x^\star x) \geq 0.$$

Let $x \in M$ be arbitrary. Then, $x^\star x$ is self-adjoint so we can use the Cauchy-Schwarz Inequality and the estimate calculated above to derive the following chain of inequalities:

$$|f(x)|^2 = |f(e^\star x)|^2 \leq f(e^\star e) f(x^\star x) \leq f(e) f(x^\star x) \leq f(e)^2 \|x x^\star\| = f(e)^2 \|x\|^2,$$

which tells us that f is bounded with $\|f\| \leq f(e)$. Since $\|e\| = 1$, it follows that $\|f\| = f(e)$.

$(2) \Rightarrow (1)$: It suffices to prove this for the case where $f(e) = 1$, i.e. when f is a state. Let $x \in M^+$. We need to show that $f(x) \geq 0$. Since $f(x) = a + ib$ for real numbers a and b, showing that f is positive is tantamount to proving that $a \geq 0$ and that $b = 0$. Let $p_s(t) := 1 - st$ be a family of polynomials parametrized by $s > 0$. By the Spectral Mapping Theorem for polynomials (Theorem 2.1.6), it is the case that:

$$\sigma_V(e - sx) = \sigma_V(p_s(x)) = p_s(\sigma_V(x)).$$

Since $\sigma_V(x) \subseteq [0, \infty)$ (because $x \in M^+$), it follows that having s be small enough results in $\sigma_V(e - sx) \subseteq [0, 1]$. Since $e - sx$ is self-adjoint, it is normal so:

$$\|e - sx\| = \nu(e - sx) \leq 1.$$

This means that:

$$1 - sa \leq |1 - s(a + ib)| \leq \nu(e - sx) \leq 1,$$

which implies that $0 \leq a$ since $s > 0$. Let $y_n = x - ae + inbe$ for every $n \in \mathbb{N}$. Then:

$$\|y_n\|^2 = \|{y_n}^\star y_n\| = \left\|(x - ae)^2 + n^2 b^2 e\right\| \leq \|x - ae\|^2 + n^2 b^2.$$

Notice that $f(y_n) = f(x) - af(e) + inbf(e) = a + ib - a + inb = i(n+1)b$ for every $n \in \mathbb{N}$. Therefore:

$$\forall n \in \mathbb{N} : |f(y_n)|^2 = (n^2 + 2n + 1)b^2 \leq \|f\|^2 \cdot \|y_n\|^2 \leq \|x - ae\|^2 + n^2 b^2.$$

Subtracting $n^2 b^2$ from both sides, we obtain that:

$$\forall n \in \mathbb{N} : (2n + 1)b^2 \leq \|x - ae\|^2.$$

It follows that $b^2 = 0$ and that $b = 0$. This completes the proof. □

An important corollary of this theorem is that multiplicative linear functionals on unital $C^\star$-algebras are states.

Corollary 3.4.7. Let V be a unital $C^\star$-algebra. Then, every multiplicative linear functional $f : V \to \mathbb{C}$ is a state.

Proof. Since V is unital with identity e, $\|f\| = 1$ and $f(e) = 1$. By Theorem 3.4.6, it follows that f is positive. □

We will end our exposition of positive linear functionals with the following analogue of the Hahn-Banach Theorem as well as a useful corollary.

Theorem 3.4.8. The Hahn-Banach Theorem for States

Let V be a $C^\star$-algebra with identity e. Let $M \subseteq V$ be a self-adjoint subspace containing the identity. Let $x \in M$ and let $t \in \sigma_V(x)$. Then, there is a state $f : M \to \mathbb{C}$ such that $f(x) = t$.

Proof. Let $S \subseteq M$ be the span of $\{x, e\}$. Define a linear functional on S by setting $g(x) = t$ and $g(e) = 1$, extending g to the rest of S by linearity. Note that g is bounded because we have that:

$$\forall a, b \in \mathbb{C} : |g(ax + be)| = |ag(x) + bg(e)| = |at + b|.$$

We claim that $at + b \in \sigma_V(ax + be)$. Indeed, $ax + be - (at + b)e = a(x - te)$. Since $x - te$ is singular, it follows that $ax + be - (at + b)e$ is singular. Therefore:

$$\forall a, b \in \mathbb{C} : |g(ax + be)| = |at + b| \leq \|ax + be\|,$$

which proves that $\|g\| \leq 1$. But since $g(e) = 1$ and $\|e\| = 1$, it follows that $\|g\| = 1$. By the Hahn-Banach Theorem[a], we can extend g to a bounded linear functional $f : V \to \mathbb{C}$ with $f(e) = 1$ and $\|f\| = 1 = f(e)$. By Theorem 3.4.6, f is positive and we are done. □

[a] See Theorem B.2.1.

The value of the theorem above is that it guarantees the existence of plenty of states. The theorem above also has the value of allowing us to determine the properties of elements in a $C^\star$-algebra by looking at the effect that states have on them. This is the "useful corollary" we were referring to above.

Corollary 3.4.9. Let V be a unital $C^\star$-algebra and let $M \subseteq V$ be a self-adjoint subspace containing the identity. Let $a \in M$. Then:

1. If $f(a) = 0$ for every state $f : M \to \mathbb{C}$, then $a = 0$.
2. If $f(a) \in \mathbb{R}$ for every state $f : M \to \mathbb{C}$, then a is self-adjoint.
3. If $f(a) \geq 0$ for every state $f : M \to \mathbb{C}$, then $a \in M^+$.
4. If a is normal, then there is a state $f : M \to \mathbb{C}$ such that $|f(a)| = \|a\|$.

Proof. We will prove each statement one-by-one.

1. Let $f(a) = 0$ for every state $f : M \to \mathbb{C}$. Suppose that a is self-adjoint. By statement (2) of Proposition 3.2.1, we know that at least one of the two real numbers $\pm\|a\|$ is contained in $\sigma_V(a)$. By Theorem 3.4.8, there exists a state $f : M \to \mathbb{C}$ such that $f(a) = \pm\|a\|$[a]. But now, we know that $f(a) = 0$ so $a = 0$. Next, assume that $a \in M$ is arbitrary. Then, $a = x + iy$ where $x, y \in M$ are self-adjoint. Then, we have that:
$$f(x) + if(y) = 0$$
for every state $f : M \to \mathbb{C}$. Let $f : M \to \mathbb{C}$ be a state. Then, $f(x) \in \mathbb{R}$ and $f(y) \in \mathbb{R}$ because x, y and f are self-adjoint. It follows that $f(x) = f(y) = 0$. But since this is true for every state on M, it follows that $x = y = 0$. This gives the result.
2. Let $f(a) \in \mathbb{R}$ for every state $f : M \to \mathbb{C}$. Let $a = x + iy$ with x, y being self-adjoint in M. Let $f : M \to \mathbb{C}$ be a state. Then:
$$f(a) = f(x) + if(y).$$
Since $f(x), f(y) \in \mathbb{R}$, it follows that $f(y) = 0$. Since that holds for every state, we conclude by (1) that $y = 0$. This implies that $a = x$ and the self-adjointness of x gives the result.
3. Let $f(a) \geq 0$ for every state $f : M \to \mathbb{C}$. Then, $f(a) \in \mathbb{R}$ for every state so a is self-adjoint, by (2). Next, let $\lambda \in \sigma_V(a)$. By Theorem 3.4.8, it follows that there is a state $f : M \to \mathbb{C}$ such that $f(a) = \lambda$. But this means that $\lambda \geq 0$ and this proves that a is positive.
4. Let a be normal. Then, $\nu(a) = \|a\|$, by statement (1) of Proposition 3.2.1. In particular, there is a $\lambda \in \sigma_V(a)$ such that $|\lambda| = \|a\|$. By Theorem 3.4.8, we know that there is a state $f : M \to \mathbb{C}$ such that $f(a) = \lambda$. As a consequence, this state satisfies $|f(a)| = \|a\|$ and we are done. □

[a] Over here, we obviously pick the number which lies in $\sigma_V(a)$.

With all of this preparatory work being complete, we are now ready to focus on an (extremely) detailed discussion of Gelfand Duality. Please note that the material we have covered above is the **bare** minimum that we needed to discuss the upcoming content in a satisfactory way. For instance, we did not talk about pure states while [21] actually does spend a fair amount of time on them and uses them to derive stronger results than we have.

3.5 Gelfand Duality

Gelfand Duality is in the title of this thesis and we have finally reached the point where we are going to discuss it in some depth. As the reader can see, this section itself is divided into three further subsections which deal with different aspects of Gelfand Duality. This will, hopefully, allow us to be a lot more organized in our presentation of this material. We will start by talking about the commutative theory, where Gelfand Duality expresses itself most vibrantly.

3.5.1 The Commutative Theory

Most of this section is loosely adapted from [26, Appendix C.2, Appendix C.3] and [21, Section 4.4]. We repeat, however, that we have taken liberties with the way that theorems are stated and the way that their proofs are presented. The reader must definitely keep that in mind if they choose to consult the resources presented above[4].

We promised that we would prove the commutative Gelfand-Naimark Theorem here and so, we shall deliver on that first because it is the standard bit of the material that we need to discuss. To make this easy for us and to reduce the clutter within the thesis, we will split the theorem up into two theorems which deal with the unital and non-unital cases separately. Let us begin with the unital case.

Theorem 3.5.1. The Commutative Gelfand-Naimark Theorem I

Let V be a commutative unital $C^\star$-algebra. Then, there exists a compact Hausdorff space (X, τ) and an isometric $\star$-isomorphism $f : V \to \mathcal{C}(X)$.

Proof. Recall that Theorem 2.3.14 states that the Gelfand representation is given by the map:

$$\mathcal{G} : V \mapsto \mathcal{C}(\widehat{V}) \ , \ x \mapsto \mathcal{G}(x) := \widehat{x}$$

with $\widehat{V}$ being a compact Hausdorff space. Let $X = \widehat{V}$ and equip this with the Gelfand topology. If we can show that $\mathcal{G}$ is an isometric $\star$-isomorphism, then we will be done. Let us verify this in a number of simple steps.

1. Let us show that $\mathcal{G}$ is injective. By statement (2) of Theorem 2.3.14, we have that:
$$\forall x \in V : \|\widehat{x}\|_\infty = \nu(x).$$
Since V is commutative, every element is normal. Therefore, statement (1) of Proposition 3.2.1 tells us that $\nu(x) = \|x\|$ for every $x \in V$. This implies that for every $x \in V$, we have that $\|\widehat{x}\|_\infty = \|x\|$.
2. We will show that it is a $\star$-homomorphism. The fact that it is a homomorphism has already been established. On the other hand, we know that every multiplicative linear functional on V is a state and, therefore, a self-adjoint linear functional[a]. Therefore:
$$\forall x \in V : \forall \phi \in \widehat{V} : \widehat{x^\star}(\phi) = \phi(x^\star) = \overline{\phi(x)} = \overline{\widehat{x}(\phi)}.$$
as was desired.
3. It remains to be shown that $\mathcal{G}$ is surjective. By statement (3) of Proposition 3.2.9, it is the case that $\mathcal{G}(V)$ is a $C^\star$-subalgebra. The strategy is to appeal to Theorem B.1.8; if we can show that $\mathcal{G}(V)$ satisfies all of the listed conditions, we can see that $\overline{\mathcal{G}(V)} = \mathcal{C}(\widehat{V}) = \mathcal{G}(V)$ because $\mathcal{G}(V)$ is closed. Let us do this one-by-one as follows:
   (a) Let $\phi \in \widehat{V}$. Since ϕ is multiplicative, it follows that there is an $x \in V$ such that $\phi(x) \neq 0$. This implies that $\widehat{x}(\phi) \neq 0$; that is, there is a function $f \in \mathcal{G}(V)$ such that $f(\phi) \neq 0$.
   (b) Let $\phi, \psi \in \widehat{V}$ and suppose that $\phi \neq \psi$. Then, there is an $x \in V$ such that $\phi(x) \neq \psi(x)$. But this just means that $\widehat{x}(\phi) \neq \widehat{x}(\psi)$ and that is what we wanted.
   (c) For each $x \in V$, it is the case that $\overline{\widehat{x}} = \widehat{x^\star}$. It follows that $\overline{\widehat{x}} \in \mathcal{G}(V)$.

   By the Stone-Weierstrass Theorem (see Theorem B.1.8), we have that $\overline{\mathcal{G}(V)} = \mathcal{G}(V) = \mathcal{C}(X)$.

This completes the proof. □

[a]See Corollary 3.4.4.

Most of the work has already been done in the proof of the theorem above. One might be inclined to say that this is good enough to establish the non-unital case. To make sure we are being as precise as possible in our arguments, let us endeavor to establish two more results that are of independent interest. We begin with the following "uniqueness" result.

Proposition 3.5.2

Let V, W be commutative, unital $C^\star$-algebras. Let $\mathcal{A}$ be the set of all unital $\star$-homomorphisms from V into W and let $\mathcal{B}$ be the set of all continuous maps from $\widehat{W}$ into $\widehat{V}$. Then, there is a bijective correspondence between $\mathcal{A}$ and $\mathcal{B}$. In particular, $\widehat{V}, \widehat{W}$ are homeomorphic iff V and W are $\star$-isomorphic.

Proof. Let $\mathcal{G}_V$ and $\mathcal{G}_W$ denote the Gelfand representations on V and W respectively. If $\phi : \widehat{W} \to \widehat{V}$ is a continuous map, then we define the so-called pullback of ϕ as follows:

$$\phi^\star : \mathcal{C}(\widehat{V}) \to \mathcal{C}(\widehat{W}) \ , \ f \mapsto \phi^\star(f) := f \circ \phi$$

Given this, one can form a map from V to W. The best way to see how to do this is to draw an arrow diagram:

$$V \to \mathcal{C}(\widehat{V}) \to \mathcal{C}(\widehat{W}) \to W$$

In particular, $\mathcal{G}_W^{-1} \circ \phi^\star \circ \mathcal{G}_V$ is a $\star$-homomorphism from V into W. Now, we define a map between $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\alpha : \mathcal{B} \to \mathcal{A} \ , \ \phi \mapsto \alpha_\phi := \mathcal{G}_W^{-1} \circ \phi^\star \circ \mathcal{G}_V$$

The claim is that α is a bijection. We assume this and prove the second part of the assertion first.

$\Rightarrow$: Let $\phi : \widehat{W} \to \widehat{V}$ be a homeomorphism. Observe that $\phi^\star$ is a $\star$-isomorphism. Define $f : V \to W$ by the composition $f = \mathcal{G}_W^{-1} \circ \phi^\star \circ \mathcal{G}_V$. Since f is a composition of $\star$-isomorphisms, it follows that V and W are $\star$-isomorphic.

$\Leftarrow$: Assume that V and W are $\star$-isomorphic and let $f : V \to W$ be a $\star$-isomorphism. We claim that α_f is a

[4]For instance, [21] spends a fair amount of time discussing pure states and the relationship between pure states & multiplicative linear functionals. We have not done that at all.

homeomorphism. Clearly, α_f is a continuous bijection between compact Hausdorff spaces. It follows that it is a homeomorphism.

It remains to be shown that α is a bijection. Let $\phi_1, \phi_2 \in \mathcal{B}$ such that $\alpha_{\phi_1} = \alpha_{\phi_2}$. Then, it is clear that ${\phi_1}^\star = {\phi_2}^\star$. That is, for every $f \in \mathcal{C}(\widehat{V})$, it is the case that $f \circ \phi_1 = f \circ \phi_2$. Since $\mathcal{C}(\widehat{V})$ separates distinct points of $\widehat{V}$, it follows that $\phi_1 = \phi_2$. This shows injectivity. Finally, let $f : V \to W$ be a $\star$-homomorphism. Define:

$$f^\star : \widehat{W} \to \widehat{V} \ , \ \omega \mapsto f^\star(\omega) = \omega \circ f$$

It is, now, a simple enough computation to show that $\alpha_{f^\star} = f$ and we are done. □

The proposition above tells us that we can obtain information about the topological properties of the structure spaces just by looking at the "algebraic" properties of the $C^\star$-algebras themselves and vice versa. In hindsight, this development is not too surprising considering that we have mainly been deriving many algebraic results via analysis.

We mentioned that this was a "uniqueness" result. If V is a commutative, unital $C^\star$-algebra, then the Gelfand representation $\mathcal{G} : V \to \mathcal{C}(\widehat{V})$ is an isometric $\star$-isomorphism. Suppose that V were also $\star$-isomorphic to $\mathcal{C}(X)$ when X is some compact Hausdorff space. Then, $\mathcal{C}(X)$ is $\star$-isomorphic to $\mathcal{C}(\widehat{V})$ so one can say that $\widehat{\mathcal{C}(X)}$ is homeomorphic to $\widehat{V}$. Ideally, we would like to be able to make the reduction that $\widehat{\mathcal{C}(X)}$ is homeomorphic to X.

Proposition 3.5.3

Let (X, τ) be a compact Hausdorff space. Define the map:

$$\Phi : X \to \widehat{\mathcal{C}(X)} \ , \ x \mapsto \Phi(x)$$

where $\Phi(x)(f) := f(x)$ for every $f \in \mathcal{C}(X)$. It is the case that Φ is a homeomorphism.

Proof. Let us show bijectivity first. Let $x_1, x_2 \in X$ such that $\Phi(x_1) = \Phi(x_2)$. Then:

$$\forall f \in \mathcal{C}(X) : f(x_1) = f(x_2)$$

On the other hand, we know by Theorem B.1.7 that continuous functions separate distinct points[a]. It follows that $x_1 = x_2$. Next, let us prove surjectivity by contradiction; assume that there is a $\phi \in \widehat{\mathcal{C}(X)}$ such that $\Phi(x) \neq \phi$ for all $x \in X$. To ensure that the argument isn't cluttered and is easy to follow, we will break it up into a few steps.

1. We claim that $\forall x \in X : \ker(\phi) \neq \ker(\Phi(x))$. We know, by the strength of Theorem 2.3.4, that there is a bijective correspondence between maximal modular ideals and multiplicative linear functionals on commutative $\mathbb{C}$-Banach algebras[b]. If $\ker(\phi) = \ker(\Phi(x))$ for some $x \in X$, then $\phi = \Phi(x)$ and that is a contradiction.
2. We claim that $\ker(\phi)$ is not a subset of $\ker(\Phi(x))$ for any $x \in X$. Indeed, both of these are maximal ideals. If $\ker(\phi) \subseteq \ker(\Phi(x))$ for some $x \in X$ then it would be the case that $\ker(\phi) = \ker(\Phi(x))$ and that is impossible. As a consequence, for each $x \in X$, there is a $g_x \in \ker(\phi)$ such that $\Phi(x)(g_x) = g_x(x) \neq 0$. Without loss of generality, we can assume that $g_x(x)$ is strictly positive for every $x \in X$ and that each g_x is real-valued[c].
3. For each g_x, define $U_x := g_x^{-1}(0, \infty)$. This gives us a family of open sets $(U_x)_{x \in X}$ which forms an open cover for X. By the compactness of X we can extract a finite subcover $(U_{x_n})_{n=1}^m$. Define the function $g = \sum_{n=1}^m g_{x_n}$. This function is strictly positive so it is invertible. But note that $g_x \in \ker(\phi)$ for every $x \in X$ so $g \in \ker(\phi)$. Since $\ker(\phi)$ is an ideal that contains an invertible element, it follows that it must be the whole of $\mathcal{C}(X)$. This contradicts the fact ϕ is multiplicative.

Let us, now, prove that Φ is continuous. Let $(x_\delta)_{\delta \in D}$ be a net in X that converges to x. Then, $(\Phi(x_\delta))_{\delta \in D}$ is a net of multiplicative linear functionals and we want to know if this net converges to $\Phi(x)$ in the Gelfand topology. Indeed, that just means that for every $f \in \mathcal{C}(X)$, we must have that $\Phi(x_\delta)(f) \to \Phi(x)(f)$. But this is tantamount to showing that $f(x_\delta) \to f(x)$ and that is certainly true because f is a continuous function. This proves the continuity of Φ. The continuity of Φ^{-1} follows from the fact that a continuous bijection between compact Hausdorff spaces is a homeomorphism. This completes the proof. □

[a] That is for any two distinct points x and y, there exists a continuous function $f : X \to \mathbb{C}$ such that $f(x) \neq f(y)$.
[b] Recall, of course, that $\mathcal{C}(X)$ is a commutative $\mathbb{C}$-Banach algebra.
[c] This can be done by multiplying by the appropriate complex number and taking the real part, for instance.

The proposition above gives us a variation of a well-known result about rings of continuous functions.

Corollary 3.5.4 (The Gelfand-Kolmogorov Theorem)**.** Let (X, τ_X) and (Y, τ_Y) be compact Hausdorff spaces. Then, there is a bijective correspondence between unital $\star$-homomorphisms from $\mathcal{C}(Y)$ into $\mathcal{C}(X)$ and continuous maps from X into Y. In particular, $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are $\star$-isomorphic iff X and Y are homeomorphic.

Proof. For the first statement, we may apply Proposition 3.5.2 to the commutative unital $C^\star$-algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$. For the second part, note that $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are $\star$-isomorphic iff $\widehat{\mathcal{C}(X)}$ and $\widehat{\mathcal{C}(Y)}$ are homeomorphic[a] iff X and Y are homeomorphic[b]. □

[a] This equivalence follows from Proposition 3.5.2.
[b] This equivalence follows directly from Proposition 3.5.3. In particular, $\widehat{\mathcal{C}(X)}$ and $\widehat{\mathcal{C}(Y)}$ are homeomorphic to X and Y respectively so it follows that X and Y are homeomorphic.

In particular, a commutative unital $C^\star$-algebra V is $\star$-isomorphic to $\mathcal{C}(X)$ for some compact Hausdorff space (X,τ) and this space is unique up to homeomorphism. The value of this result is in the fact that we can "recover" a given topological space by studying its space of continuous functions. With this, we are finally ready to deal with the non-unital version of the commutative Gelfand-Naimark Theorem in a satisfactory way as compared to most texts. We should note that [12, Theorem 1.3.1, p.7-8] provides an overall more convincing argument than most other texts while [26, p.664] provides a proof which is brushed off to the side but is essentially the same as ours.

Theorem 3.5.5. The Commutative Gelfand-Naimark Theorem II

Let V be a commutative $C^\star$-algebra without identity. Then, V is isometrically $\star$-isomorphic to $\mathcal{C}_0(X)$ for some locally compact Hausdorff space (X,τ).

Proof. Let us split the argument into a few steps. For this argument, if V is a commutative $C^\star$-algebra, then we will denote its structure space by $\mathcal{M}(V)$.

1. Observe that V_e, the unitization of V, is $\star$-isomorphic to $\mathcal{C}(\mathcal{M}(V_e))$. Since V_e has an identity element, $\mathcal{M}(V_e)$ is a compact Hausdorff space. Removing the identity element from V_e is tantamount to turning $\mathcal{M}(V_e)$ into a locally compact Hausdorff space.
2. By Theorem B.1.13, it follows that $\mathcal{M}(V_e)$ is homeomorphic to the one-point compactification of $\mathcal{M}(V)$.
3. Let us denote the one-point compactification of $\mathcal{M}(V)$ by $\mathcal{M}(V)^+$. The corollary above implies that $\mathcal{C}(\mathcal{M}(V_e))$ is $\star$-isomorphic to $\mathcal{C}(\mathcal{M}(V)^+)$.
4. Note that $\mathcal{C}(\mathcal{M}(V_e))$ is $\star$-isomorphic to V_e while $\mathcal{C}(\mathcal{M}(V)^+)$ is $\star$-isomorphic to $(\mathcal{C}_0(\mathcal{M}(V)))_e$, by Proposition 2.3.12. It follows that V_e is $\star$-isomorphic to $(\mathcal{C}_0(\mathcal{M}(V)))_e$.
5. Since the $\star$-isomorphism above is unital, we can remove the identity elements and still have an isometric $\star$-isomorphism from V onto $\mathcal{C}_0(\widehat{V})$. This proves the desired result.

Note that it is the Gelfand representation which is the isometric $\star$-isomorphism in this situation. □

While taking a bit of a breather, let us note the following interesting result about the Gelfand representation in a commutative unital $C^\star$-algebra. We will use this result in the proof of Theorem 4.2.10 and we will also use it to understand "orthogonal projections"; these are particular maps that are crucial because of their role in the statement of the Spectral Theorem for Bounded Normal Operators (we will cover some operator theory in the next subsection).

Proposition 3.5.6

Let V be a commutative unital $C^\star$-algebra. Then, the Gelfand representation $\mathcal{G} : V \to \mathcal{C}(\widehat{V})$ satisfies the following properties:

1. $\forall x \in V : 0 \prec x \Leftrightarrow 0 \prec \mathcal{G}(x)$.
2. If $f \in \mathcal{C}(\widehat{V})$ satisfies $f(\phi) \in [0,1]$ for all $\phi \in \widehat{V}$, then $0 \prec \mathcal{G}^{-1}(f) \prec e$, where e is the identity on V.

Here, $\prec$ refers to the $C^\star$-orders on the respective $C^\star$-algebras above[a].

[a] Again, we make no effort to distinguish between these orders as it should be fairly clear from context.

Proof. Let us prove each statement one-by-one.

1. Let $x \in V$. Suppose that $0 \prec x$. Now, $\sigma_{\mathcal{C}(\widehat{V})}(\widehat{x}) = \widehat{x}(\widehat{V})$. Statement (5) of Proposition 2.3.8 tells us that $\widehat{x}(\widehat{V}) = \sigma_V(x) \subseteq [0,\infty)$. It follows that $\mathcal{G}(x)$ is a positive element of $\mathcal{C}(\widehat{V})$. Now, assume that $0 \prec \mathcal{G}(x)$. Then, $\widehat{x}(\widehat{V}) = \sigma_V(x) \subseteq [0,\infty)$. It remains to be shown that x is self-adjoint. Indeed:
$$\forall \phi \in \widehat{V} : \widehat{x}(\phi) = \phi(x) = \overline{\phi(x)} = \widehat{x^\star}(\phi)$$
The injectivity & linearity of $\mathcal{G}$ implies that $x^\star = x$ as was desired. That is, $0 \prec x$.
2. Let $f \in \mathcal{C}(\widehat{V})$. Then, there is an $x \in V$ such that $f = \widehat{x}$. Therefore, we have that:
$$\forall \phi \in \widehat{V} : 0 \leq f(\phi) = \widehat{x}(\phi) = \phi(x) \leq 1 = \mathcal{G}(e)(\phi)$$
But this corresponds to the inequality $\mathcal{G}(0) \prec f \prec \mathcal{G}(e)$. Using part (1) on this inequality gives the result. □

At this point, we should tell the reader that if their main interest was in spectral theory, then they have almost everything they need to be able to fruitfully read the next chapter. Readers who fall into this category may feel free to skip the material that follows [5] and go straight to the next chapter. Of course, we do not recommend doing this because we were just about to present the key result which is typically named "Gelfand Duality".

Theorem 3.5.7. (Commutative) Gelfand Duality

Let (X,τ_x) be a compact Hausdorff space. Then, $\mathcal{C}(X)$ is a commutative unital $C^\star$-algebra. If (Y,τ_Y) is a compact Hausdorff space and $\phi : X \to Y$ is a continuous map, then we define the (topological) pullback of ϕ as the map:
$$C(\phi) := \phi^\star : \mathcal{C}(Y) \to \mathcal{C}(X) \ , \ f \mapsto \phi^\star(f) := f \circ \phi$$

[5] With the exception of Definition 3.5.8 and Proposition 3.5.9.

This is a unital $\star$-homomorphism which satisfies the following properties:

1. If ϕ is the identity, then $\phi^\star$ is also the identity.
2. If (Z, τ_Z) is another compact Hausdorff space and $\psi : Y \to Z$ is a another continuous map, then $C(\psi \circ \phi) = C(\phi) \circ C(\psi)$.

If V is a commutative unital $C^\star$-algebra, then $\widehat{V}$ is a compact Hasudorff space. If V and W are commutative unital $C^\star$-algebras and $\alpha : V \to W$ is a unital $\star$-homomorphism, then define the (algebraic) pullback of α as the map:

$$\alpha^\star : \widehat{W} \to \widehat{V} \ , \ f \mapsto \alpha^\star(f) := f \circ \alpha$$

This is a continuous function which satisfies the following properties:

1. If α is the identity, then $\alpha^\star$ is also the identity.
2. If U is a commutative unital $C^\star$-algebra and $\beta : W \to U$ is a unital $\star$-homomorphism, then $(\beta \circ \alpha)^\star = \alpha^\star \circ \beta^\star$.

Let (X, τ) be a compact Hausdorff space and let V be a commutative unital $C^\star$-algebra. Define the map:

$$\text{ev}_X : X \to \widehat{\mathcal{C}(X)} \ , \ x \mapsto \text{ev}_X(x)$$

where $\text{ev}_X(x) : \mathcal{C}(X) \to \mathbb{C}$ is an evaluation functional for each $x \in X$. The map ev_X is a homeomorphism. This map, along with the Gelfand representation $\mathcal{G}_V : V \to \mathcal{C}(\widehat{V})$, satisfy the following properties:

1. If (Y, τ) is a compact Hausdorff space with $\phi : X \to Y$ being a continuous map and if $C(\phi)^\star : \widehat{\mathcal{C}(X)} \to \widehat{\mathcal{C}(Y)}$ is the (topological) pullback of the (algebraic) pullback $C(\phi) : \mathcal{C}(Y) \to \mathcal{C}(X)$, then we have that:
$$(C(\phi))^\star \circ \text{ev}_X = \text{ev}_Y \circ \phi$$
2. If W is a commutative unital $C^\star$-algebra and $\alpha : V \to W$ is a unital $\star$-homomorphism, then the (topological) pullback $C(\alpha^\star) : \mathcal{C}(\widehat{V}) \to \mathcal{C}(\widehat{W})$, with $\alpha^\star$ being the (algebraic) pullback of α, satisfies the following property:
$$C(\alpha^\star) \circ \mathcal{G}_V = \mathcal{G}_W \circ \alpha$$

If V is a commutative unital $C^\star$-algebra and we define $\mathcal{M}(V) := \widehat{V}$, then we have that $\mathcal{C}(\mathcal{M}(V)) \equiv V$. That is, they are $\star$-isomorphic. If (X, τ) is a compact Hausdorff space, then $\mathcal{M}(\mathcal{C}(X)) \equiv X$. That is, they are homeomorphic.

Proof. Let us go through each claim one-by-one.

1. The fact that $\mathcal{C}(X)$ is a commutative unital $C^\star$-algebra was established in Example 9.
2. Let us establish the two given properties of the pullback:
   (a) Assume that $\phi : X \to X$ is the identity. Then, $\phi^\star(f) = f \circ \phi = f$. This means that $\phi^\star : \mathcal{C}(X) \to \mathcal{C}(X)$ is the identity.
   (b) Let $f \in \mathcal{C}(Z)$. Then, $(\psi \circ \phi)^\star(f) = f \circ (\psi \circ \phi) = (f \circ \psi) \circ \phi) = \psi^\star(f) \circ \phi = \phi^\star(\psi^\star(f)) = (\phi^\star \circ \psi^\star)(f)$. The result now follows.
3. Let V be a commutative unital $C^\star$-algebra. Then, $\widehat{V}$ is a compact Hausdorff space by Theorem 2.3.13.
4. Let us establish the two given properties of the pullback for $\star$-homomorphisms:
   (a) Let α be the identity map and let $f \in \widehat{V}$. Then, $\alpha^\star(f) = f \circ \alpha = f$. That is, $\alpha^\star$ is the identity.
   (b) Let $f \in \widehat{U}$. Then: $(\beta \circ \alpha)^\star(f) = f \circ (\alpha \circ \beta) = (f \circ \alpha) \circ \beta = \alpha^\star(f) \circ \beta = \beta^\star(\alpha^\star(f)) = (\beta^\star \circ \alpha^\star)(f)$. The result now follows.
5. The fact that ev_X is a homeomorphism was established in Proposition 3.5.3.
6. Let us prove the two additional properties of ev_X.
   (a) Consider the map $C(\phi)^\star \circ \text{ev}_X : X \to \widehat{\mathcal{C}(Y)}$. Let $x \in X$ be fixed. Then:
   $$C(\phi)^\star(\text{ev}_X(x)) = \text{ev}_X(x) \circ C(\phi)$$
   Let $f \in \mathcal{C}(Y)$. Then:
   $$C(\phi)^\star(\text{ev}_X(x))(f) = (\text{ev}_X(x) \circ C(\phi))(f) = \text{ev}_X(x)(f \circ \phi) = (f \circ \phi)(x) = f(\phi(x)) = \text{ev}_Y(\phi(x))(f)$$
   Varying over every $x \in X$ and every $f \in \mathcal{C}(Y)$, the validity of the given equality holds.
   (b) Consider the map $C(\alpha^\star) \circ \mathcal{G}_V : V \to \mathcal{C}(\widehat{W})$. Let $x \in V$ be fixed. Then:
   $$(C(\alpha^\star) \circ \mathcal{G}_V)(x) = C(\alpha^\star)(\widehat{x})$$
   Let $f \in \widehat{W}$. Then:
   $$C(\alpha^\star)(\widehat{x})(f) = (\widehat{x} \circ \alpha^\star)(f) = \widehat{x}(f \circ \alpha) = (f \circ \alpha)(x) = f(\alpha(x)) = \widehat{\alpha(x)}(f) = [(\mathcal{G}_W \circ \alpha)(x)](f)$$
   The desired equality now follows immediately.

Finally, the last two statements follow immediately from Theorem 3.5.1 and Proposition 3.5.3[a]. □

[a] The result was actually found in [26, Theorem C.23, p. 656]. We have provided the details within the proof (the author claimed that the proof was just a combination of previously proved statements). We have merely filled in the details.

There is a similar duality result for non-unital $C^\star$-algebras but trying to establish that here would have been a more technical endeavor. Certainly, it would have taken away from the main point so we chose to not deal with that. Nevertheless, information about that is actually provided in [26].

One can actually significantly simplify the claim of Theorem 3.5.7 through the language of Category Theory. We will not go into the details here because that would constitute an entire separate thesis topic unto itself. However, we should say that one can define two categories CommCAlg (the category of unital commutative $C^\star$-algebras) and CompHaus (the category of compact Hausdorff spaces). Then, Gelfand Duality reduces to the assertion that CommCAlg and CompHaus are contravariantly equivalent categories. For further information about this, see [31].

We are, at this point, ready to leave the commutative theory behind and try to deal with Gelfand Duality for arbitrary $C^\star$-algebras. This, of course, means that we have to drop the assumption of commutativity to get the general theory. Before we do that, some preparation in the form of an exposition on representation theory is needed.

3.5.2 The Representation Theory

Let us begin by saying something about the theory of bounded operators on a Hilbert space. The starting point is the definition of the adjoint of a bounded linear map.

Definition 3.5.8

Let H be a $\mathbb{C}$-Hilbert space[a] and let $T \in \mathcal{B}(H)$. Define the Banach adjoint of T as the following map:

$$T' : H^\star \to H^\star \ , \ f \mapsto T'(f) := f \circ T$$

Let $J : H \to H^\star$ denote the conjugate-linear map between H and $H^\star$ as given by the Riesz Representation Theorem (Theorem B.2.2). Consider the following diagram:

$$\begin{array}{ccc} H & \xrightarrow{T^\star} & H \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ H^\star & \xrightarrow{T'} & H^\star \end{array}$$

We define $T^\star : H \to H$, the adjoint of T, as the unique[b] map such that the diagram above commutes.

[a] We assume that the inner product is linear in the first argument.

[b] This should be fairly obvious; $T^\star$ will be defined by an explicit formula that only depends on T' and J.

The first thing to do is to verify that the adjoint is, truly, the unique map such that the diagram in Definition 3.5.8 commutes. In fact, we will take it upon ourselves to verify some other properties of interest simultaneously.

Proposition 3.5.9

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$. The following hold:

1. $T^\star$ is a bounded linear map.
2. $\forall x, y \in H : \langle T(x), y \rangle = \langle x, T^\star(y) \rangle$.
3. Let $S : H \to H$ be a map such that $\langle T(x), y \rangle = \langle x, S(y) \rangle$ for all $x, y \in H$, then $S = T^\star$.
4. $\|T\| = \|T^\star\|$.
5. The map $T \mapsto T^\star$ is an involution on $\mathcal{B}(H)$ and turns $\mathcal{B}(H)$ into a unital $C^\star$-algebra.
6. $\ker(T^\star) = T(H)^\perp$ and $\ker(T) = T^\star(H)^\perp$.

Proof. We prove each of these assertions one-by-one.

1. T' is bounded because:
$$\forall f \in H^\star : \|T'(f)\| = \|f \circ T\| \leq \|f\| \cdot \|T\|$$
where we have used the continuity of f and T. Using this, one can show that $T^\star$ is bounded and linear; we omit this computation.
2. Let $x, y \in H$. Then:
$$\langle x, T^\star(y) \rangle = [J(T^\star(y))](x) = [(J \circ T^\star)(y)](x) = [(T' \circ J)(y)](x) = [T'(J(y))](x) = \langle T(x), y \rangle$$
as was desired.
3. We know, as a consequence of (2), that:
$$\forall x, y \in H : \langle x, S(y) \rangle = \langle x, T^\star(y) \rangle$$
This implies that:
$$\forall x, y \in H : \langle x, (S - T^\star)(y) \rangle = 0$$
Let $y \in H$ be fixed and select $x := (S - T^\star)(y)$. Then, the above tells us that $\|(S - T^\star)(y)\|^2 = 0$ and this implies that $S(y) = T^\star(y)$. Since y was arbitrary, it follows that $S = T^\star$.
4. To prove that $\|T\| = \|T^\star\|$, let us make use of (2) and (3). Observe that:
$$\forall x \in H : \|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle x, T^\star(T(x)) \rangle \leq \|x\| \cdot \|T^\star(T(x))\| \leq \|x\|^2 \cdot \|T\| \cdot \|T^\star\|$$
This implies that:
$$\|T\|^2 \leq \|T\| \cdot \|T^\star\|$$

which implies that $\|T\| \leq \|T^\star\|$. The same argument being used (but this time, we work with $T^\star(y)$ instead) gives us that $\|T^\star\| \leq \|T\|$. This proves the desired equality.

5. Proving most of this requires the reader to simply verify that properties of the involution are satisfied and we omit the details of this verification because they are just tedious to work through. The hard part is just showing that the $C^\star$-condition holds. We proceed by using (2) and (4). Indeed, we have that:
$$\forall x \in H : \|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^\star(T(x))\rangle \leq \|x\|^2 \|T^\star T\|$$
where we have used the Cauchy-Schwarz Inequality and the boundedness of $T^\star T$. It follows that $\|T\|^2 \leq \|T^\star T\|$. By Lemma 3.1.4, we have that $\mathcal{B}(H)$ is a $C^\star$-algebra.

6. Let $T^\star(x) = 0$. Let $y \in T(H)$. Then, $y = T(u)$ for some $u \in H$. So:
$$\langle y, x\rangle = \langle T(u), x\rangle = \langle u, T^\star(x)\rangle = 0$$
It follows that $x \in T(H)^\perp$. Next, let $x \in T(H)^\perp$. Then:
$$\forall y \in H : \langle T(y), x\rangle = \langle y, T^\star(x)\rangle = 0$$
Selecting $y = T^\star(x)$ gives us that $x \in \ker(T^\star)$. The second set equality is proved in precisely the same way. □

There is just one more construction we need involving Hilbert spaces. This is the so-called "direct sum" construction and it is fairly tedious. Since we do not assume explicit familiarity with this construction but do not want to make space for it over here, we will relegate it to the appendix. The reader may have a look at the discussion right after Theorem B.2.6 for further information about this; we will freely make use of notation introduced within that discussion. Let us, now, provide the definition of a representation of a $\star$-algebra.

Definition 3.5.10

Let V be a $\star$-algebra. A representation of V is a pair (H, π) consisting of a $\mathbb{C}$-Hilbert space H and a $\star$-homomorphism $\pi : V \to \mathcal{B}(H)$. If V has an identity element e and (H, π) is a representation which satisfies $\pi(e) = \mathrm{Id}_H$, then (H, π) is said to be a unital representation of V. A representation is faithful iff it is injective.

The motivation for the definition above actually comes from the following corollary of the Gelfand-Mazur Theorem.

Corollary 3.5.11. Let V be a unital $\mathbb{C}$-Banach algebra such that every non-zero element in V is invertible. Then, V is algebra isomorphic to $\mathcal{L}(V)$.

Proof. By the Gelfand-Mazur Theorem(Theorem 2.1.9, we know that V is algebra isomorphic to $\mathbb{C}$ and we also know that $\mathbb{C}$ is algebra-isomorphic to $\mathcal{L}(\mathbb{C})$. This is certainly algebra isomorphic to $\mathcal{L}(V)$. Since a composition of algebra isomorphisms is an algebra isomorphism, it follows that V is algebra isomorphic to $\mathcal{L}(V)$. □

The hope is that information about $C^\star$-algebras can be better obtained by studying it within some "concrete" algebra(like an algebra of operators). Let us provide examples of representations.

Example 15. Let H be $\mathbb{C}$-Hilbert space and let V be a $\star$-subalgebra of $\mathcal{B}(H)$. Define:
$$\pi : V \to \mathcal{B}(H) \ , \ T \mapsto \pi(T) := T$$
This is clearly a (faithful) representation of V.

Example 16. Let (X, Σ, μ) be a measure space[a]. Let $\mathcal{L}^\infty(X)$ be the set of equivalence classes of essentially bounded measurable functions $f : X \to \mathbb{C}$ modulo almost everywhere equality. It is known that this is a Banach space[b] and, when equipped with point-wise multiplication and conjugation as its involution, it becomes a (commutative) $C^\star$-algebra. For each $[f] \in \mathcal{L}^\infty(X)$, we define:
$$M_f : \mathcal{L}^2(X) \to \mathcal{L}^2(X) \ , \ [\phi] \mapsto M_f([\phi]) := [f\phi]$$
This is a bounded linear operator. Indeed, linearity should be fairly clear. To show boundedness, observe that:
$$\forall [\phi] \in \mathcal{L}^2(X) : \|M_f(\phi)\|_2^2 := \int_X |f\phi|^2 \, d\mu \leq \|f\|_\infty^2 \int_X |\phi|^2 \, d\mu = \|f\|_\infty^2 \|\phi\|_2^2$$
M_f is sometimes called the multiplication operator associated with f. In particular, we can now define:
$$\pi_\mu : \mathcal{L}^\infty(X) \to \mathcal{B}(\mathcal{L}^2(X)) \ , \ f \mapsto \pi_\mu(f) := M_f$$
This is a representation. Let us check that this is the case:

1. The linearity of the integral implies that π_μ is linear.
2. Let $f, g \in \mathcal{L}^\infty(X)$. Then:
$$\forall \phi \in \mathcal{L}^2(X) : M_{fg}(\phi) = (fg)\phi = f(g\phi) = M_f(g\phi) = M_f(M_g(\phi)) = (M_f \circ M_g)(\phi)$$
which implies that $\pi_\mu(fg) = M_{fg} = \pi_\mu(f) \circ \pi_\mu(g)$. That is, π_μ is an algebra homomorphism.
3. Let us calculate the adjoint of M_f when $f \in \mathcal{L}^\infty$. Indeed:
$$\forall \phi, \psi \in \mathcal{L}^2(X) : \langle M_f(\phi), \psi\rangle = \int_X \overline{f\phi}\psi \, d\mu = \int_X \overline{\phi} \cdot (\overline{f}\psi) \, d\mu = \langle \phi, M_{\overline{f}}(\psi)\rangle$$

which implies that $M_f^\star = M_{\overline{f}}$. We will use this to show that π_μ is a $\star$-homomorphism. Indeed:

$$\forall f \in \mathcal{L}^\infty(X) : \pi_\mu(\overline{f}) = M_{\overline{f}} = M_f^\star = \pi_\mu^\star(f)$$

as was desired.

This proves that π_μ is a representation.

[a]This example requires some knowledge of measure theory but it is not necessary for much of the material covered later.
[b]See [3, Theorem 9.17, p.212-215].

The next example will be rather crucial in the discussion of the GNS construction in the next section; certainly, the reader should return to it when we signal for them to do so.

Example 17. Let V be a $\star$-algebra. Let $((\pi, H_i))_{i\in I}$ be a family of representations of V. Then, we can define the direct sum of these representations as the following representation:

$$\pi : V \to \mathcal{B}\left(\bigoplus_{i\in I} H_i\right) \quad , \; x \mapsto \pi(x) := \bigoplus_{i\in I} \pi_i(x)$$

where the right-hand side is the direct sum of the family of bounded linear maps $(\pi_i(x))_{i\in I}$ for each $x \in V$[a]. We claim that $\left(\pi, \bigoplus_{i\in I} H_i\right)$ is a representation of V. We need to show that this is a $\star$-homomorphism:

1. The linearity of π follows from the linearity of each π_i as well as statement (3) of Proposition B.2.9.
2. Let us show that π is product-preserving. Indeed, using statement (4) of Proposition B.2.9, we have that:

$$\forall x, y \in V : \pi(xy) = \bigoplus_{i\in I} \pi_i(xy) = \bigoplus_{i\in I} \pi_i(x)\pi_i(y) = \left(\bigoplus_{i\in I} \pi_i(x)\right)\left(\bigoplus_{i\in I} \pi_i(y)\right) = \pi(x)\pi(y)$$

as was desired.
3. Let us show that π is a $\star$-homomorphism. Indeed:

$$\forall x \in V : \pi(x^\star) = \bigoplus_{i\in I} \pi_i(x^\star) = \bigoplus_{i\in I} \pi_i^\star(x) = \left(\bigoplus_{i\in I} \pi_i(x)\right)^\star$$

where we have used statement (2) of Proposition B.2.9 for that last equality.

This proves that π is a $\star$-homomorphism and, therefore, a representation of V. We refer to this as the direct sum of the representations $((\pi_i, H_i))_{i\in I}$ and it is denoted by $\bigoplus_{i\in I} \pi_i$.

[a]The reader should see Proposition B.2.8 for the definition of $\bigoplus_{i\in I} H_i$ and Proposition B.2.9 for the definition of $\bigoplus_{i\in I} \pi_i(x)$.

In fact, the Gelfand-Naimark Theorem (in the non-commutative setting) **is** about faithful representations. Having the following characterization of faithful representations is useful.

Proposition 3.5.12

Let V be a unital $C^\star$-algebra and let $\pi : V \to \mathcal{B}(H)$ be a representation, for some $\mathbb{C}$-Hilbert space H. The following are equivalent:

1. π is isometric.
2. π is faithful.

If either one of the conditions above holds, then $\pi : V \to \pi(V)$ is a unital $\star$-isomorphism of $C^\star$-algebras.

Proof. The proof that (1) $\Rightarrow$ (2) is standard. The fact that (2) $\Rightarrow$ (1) follows from statement (3) of Proposition 3.2.9. Finally, we also know from statement (3) of Proposition 3.2.9 that $\pi(V)$ is a unital $C^\star$-subalgebra of $\mathcal{B}(H)$ and, therefore, $\pi : V \to \pi(V)$ is a unital $\star$-isomorphism in this case. □

There is just one more notion that we have to introduce. Suppose that V is a $\star$-algebra and (H, π) is a representation of V. Then, for every $x \in H$, we can compute the following set:

$$\pi(V)x := \{\pi(v)x : v \in V\}$$

In some sense, $\pi(V)x$ represents the "action" of the representation on x. One might view the above as being somewhat motivated by the notion of the orbit of an element in a set with respect to some group action on that set. This serves as the basis for the following definition.

Definition 3.5.13

Let V be a $\star$-algebra and let (H, π) be a representation of V. For each $x \in H$, define the orbit of x as the following set:

$$\pi(V)x := \{\pi(v)x : v \in V\}$$

A vector $x \in H$ is cyclic iff $\pi(V)x$ is dense in H. A representation which has a cyclic vector is said to be a cyclic representation.

Cyclic vectors should be of interest to us because, intuitively, their orbit gives us "all" of the information about the underlying Hilbert space through the representation. For readers who are unsatisfied with this bit of intuition, see [32, Chapter VII]. For us, cyclic vectors will just be a useful tool that gives us the non-commutative version of Gelfand Duality. With all of this preliminary material on representations out of the way, we are finally ready to use all of it to obtain an understanding of (non-commutative) Gelfand Duality.

3.5.3 The Non-commutative Theory

Our presentation of the non-commutative theory (in its limited extent) was formulated by taking the best bits from several different sources; it is best if we reference the sources individually when we prove the relevant results. With this being said, we urge the reader to recall the point we made about there being a natural way to induce "inner products" or structures similar to that on $C^\star$-algebras using positive linear functionals; see the discussion right after the proof of Proposition 3.4.5. Let us continue from that point with the following proposition.

Proposition 3.5.14

Let V be a unital $C^\star$-algebra and let $f : V \to \mathbb{C}$ be a positive linear functional. Define:

$$\ker_p(f) := \{x \in V : f(x^\star x) = 0\}$$

This is a closed left ideal and $f(y^\star x) = 0$ whenever $x \in \ker_p(f)$ and $y \in V$. The following relation:

$$s : (V/\ker_p(f)) \times (V/\ker_p(f)) \to \mathbb{C}\ ,\ \ (x + \ker_p(f), y + \ker_p(f)) \mapsto s(x + \ker_p(f), y + \ker_p(f)) := f(y^\star x)$$

is a function and defines an inner product on the quotient space $V/\ker_p(f)$[a].

[a] This was mostly taken from [21, Proposition 4.5.1, p.277], though we found that the proof also worked for positive linear functionals.

Proof. Let us show that $\ker_p(f)$ is a closed left ideal which satisfies $f(y^\star x) = 0$ whenever $x \in \ker_p(f)$ and $y \in V$.

1. Let $x \in \ker_p(f)$ and $y \in V$. By the Cauchy-Schwarz Inequality, it follows that:
$$|f(y^\star x)|^2 \leq f(x^\star x) f(y^\star y) = 0$$
which implies that $f(y^\star x) = 0$.
2. Let $x, y \in \ker_p(f)$ and let $\mu \in \mathbb{C}$. Then:
$$f((\mu x + y)^\star(\mu x + y)) = |\mu|^2 f(x^\star x) + \mu f(y^\star x) + \overline{\mu} f(x^\star y) + f(y^\star y) = 0$$
which proves that $\mu x + y \in \ker_p(f)$ and that $\ker_p(f)$ is a subspace.
3. Let us show that $\ker_p(f)$ is a left ideal. Let $x \in \ker_p(f)$ and let $y \in V$. Then, notice that:
$$|f((yx)^\star yx)|^2 = |f(x^\star y^\star yx)|^2 \leq f(x^\star x) f((y^\star yx)^\star(y^\star yx)) = 0$$
by the Cauchy-Schwarz Inequality and the fact that $f(x^\star x) = 0$. It follows that $yx \in \ker_p(f)$. This proves that $\ker_p(f)$ is a left ideal.
4. To show that $\ker_p(f)$ is closed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\ker_p(f)$ that converges to some $x \in V$. Then, the continuity of the involution, the vector multiplication and f itself implies that:
$$f(x^\star x) = \lim_{n\to\infty} f(x_n^\star x_n) = 0$$
which means that $x \in \ker_p(f)$. This proves that $\ker_p(f)$ is closed.

Let us show that the relation s is actually a map. Let $x_1 \in x + \ker_p(f)$ and $y_1 \in y + \ker_p(f)$. Then, $x_1 - x \in \ker_p(f)$ and $y_1 - y \in \ker_p(f)$. Observe that:

$$\begin{aligned} f({y_1}^\star x_1) - f(y^\star x) &= f({y_1}^\star x_1 - y^\star x) \\ &= f({y_1}^\star x_1 - {y_1}^\star x + {y_1}^\star x - y^\star x) \\ &= f({y_1}^\star(x_1 - x)) + f(({y_1}^\star - y^\star)x) \end{aligned}$$

Observe that $f({y_1}^\star(x_1 - x)) = 0$ because $\ker_p(f)$ is a left-ideal. The second term $f(({y_1}^\star - y^\star)x) = 0$ as well due to the Cauchy-Schwarz Inequality. It follows that $f({y_1}^\star x_1) = f(y^\star x)$ and this shows that s is well-defined as a map. The fact that this is an inner product is now obvious from the preceding discussion. □

This proposition gives us everything we need to carry out the GNS Construction. This will be a key ingredient in the proof of the non-commutative Gelfand-Naimark Theorem. We will only prove the unital version of the GNS construction, commenting on the non-unital version right after.

Proposition 3.5.15. The Gelfand-Naimark-Segal (GNS) Construction

Let V be a unital $C^\star$-algebra and let $f : V \to \mathbb{C}$ be a positive linear functional. Then, there exists a representation (H_f, π_f) and a cyclic vector $x_f \in H_f$ associated with the representation which satisfy $\|x_f\|^2 = \|f\|$ and:

$$\forall a \in V : f(a) = \langle \pi_f(a) x_f, x_f \rangle.$$

The triple (H_f, π_f, x_f) is known as the GNS triplet associated with f[a].

[a] This was mostly taken from [15, Proposition 1.10.3, p.35], though we modified it suitably for positive linear functionals.

Proof. Define:

$$I := \ker_p(f) := \{x \in V : f(x^\star x) = 0\}$$

As we showed in Proposition 3.5.14, V/I is an inner product space. For each $a \in V$, we can define a linear operator $\pi_f(a)$ on V/I by the following prescription:

$$\forall x \in V : \pi_f(a)(x + I) := ax + I$$

Now, we want to show that each $\pi_f(a)$ can be extended to a bounded operator in $\mathcal{B}(H_f)$, where H_f is the completion of the inner product space V/I. For each $x \in V$, let $[x + I]$ denote an element of H_f. Let $a \in V$ be fixed and let $x \in V$ be fixed. Then:

$$\|\pi_f(a)(x + I)\|^2 = \langle ax + I, ax + I\rangle = f((ax)^\star ax) = f(x^\star a^\star ax)$$

For each $\alpha > \|a^\star a\| = \|a\|^2$, it is the case that $\alpha e - a^\star a$ is a positive element (where e is the identity of V)[a]. Since $\alpha e - a^\star a$ is a positive element, there exists a self-adjoint $b \in V$ such that $b^2 = \alpha e - a^\star a$[b]. Hence, it is the case that:

$$\alpha f(x^\star x) - f(x^\star a^\star ax) = f(\alpha x^\star x - x^\star a^\star ax) = f(x^\star(\alpha e - a^\star a)x) = f(x^\star b^2 x) = f((bx)^\star bx) \geq 0$$

It follows, then, that:

$$\|\pi_f(a)(x + I)\|^2 \leq \alpha f(x^\star x) = \alpha\|x + I\|^2$$

This inequality proves that $\pi_f(a)$ is uniformly continuous. By Lemma 3.2.3, it is possible to extend $\pi_f(a)$ to a bounded operator on H_f; this extension will still be denoted by $\pi_f(a)$. In other words, we have successfully defined a representation (H_f, π_f) associated with the state, with the proof that π_f is a $\star$-homomorphism being a computation that we carry out in three steps below. The notation for the elements of H_f should be fairly clear.

1. Let us show that π_f is linear. Let $a, b \in V$ and let $\lambda \in \mathbb{C}$ be fixed and let $x \in V$ be given. Then:

   $$\pi_f(a+\lambda b)([x+I]) = (a+\lambda b)x+I = [(ax+\lambda bx)+I] = [(ax+I)+((\lambda b)x+I)] = \pi_f(a)([x+I])+\lambda\pi_f(b)([x+I])$$

   which implies that $\pi_f(a + \lambda b) = \pi_f(a) + \lambda\pi_f(b)$.
2. Let us show that π_f preserves products. Let $a, b \in V$ be fixed and let $x \in V$ be given. Then:

   $$\pi_f(ab)([x + I]) = [(ab)x + I] = [a(bx) + I] = \pi_f(a)(\pi_f(b)([x + I])) = (\pi_f(a) \circ \pi_f(b))([x + I])$$

   It follows that $\pi_f(ab) = \pi_f(a) \circ \pi_f(b)$, as was desired.
3. Let us show that π_f is a $\star$-homomorphism. Let $a \in V$ be fixed and consider $\pi_f(a^\star)$. Let $x, y \in V$ be given so we have that:

   $$\langle\pi_f(a^\star)([x + I]), [y + I]\rangle = f(y^\star a^\star x) = f((ay)^\star x) = \langle[x + I], \pi_f(a)([y + I])\rangle = \langle\pi_f(a)^\star([x + I]), [y + I]\rangle$$

   It follows that $\pi_f(a^\star) = \pi_f(a)^\star$.

This completes the proof that there exists a representation associated with f. Now, define $x_f := [e + I]$, where e is the identity of V. Indeed:

$$\forall a \in V : \pi_f(a)([e + I]) = [ae + I] = [a + I].$$

But this means that $\pi_f(V)x_f$ is dense in H, so x_f is certainly a cyclic vector. Finally, we have the following:

$$\forall a \in V : \langle\pi_f(a)x_f, x_f\rangle = \langle[a + I], [e + I]\rangle = f(e^\star a) = f(a).$$

When $a = e$, it follows that $\|x_f\|^2 = f(e) = \|f\|$, where the last equality follows from the fact that f is a positive linear functional. □

[a] Indeed, $\|a\|^2 e = \|a^\star a\| e \prec \alpha e$. Using the fact that V is an ordered vector space (by Proposition 3.3.5), we have that $\|a\|^2 e - a^\star a \prec \alpha e - a^\star a$. By statement (2) of Proposition 3.3.6, it follows that $\|a^\star a\| e - a^\star a \in V^+$ and that $\alpha e - a^\star a \in V^+$.

[b] This follows from Theorem 3.3.8.

Recall the functional defined in Example 13. In the proposition above, we should note that the formula given can be written as a composition of two maps:

$$a \mapsto \pi_f(a) \mapsto \langle\pi_f(a)x_f, x_f\rangle$$

That is, $f = \omega_{x_f} \circ \pi_f$. By an appropriate extension of our definition of the adjoint, it can actually be shown that this formula "uniquely" determines the GNS triplet associated with the state. For a precise formulation of this, see [21, Proposition 4.5.3, p.279-280][6]. Let us move on to the proof of the GNS Theorem. The representation-theoretic formulation of this statement was taken from [21, Theorem 4.5.6, p.281] but the proof was adapted from [15, Theorem 1.10.1, p.34-35]. Now, it is possible to prove a non-unital version of the GNS construction; the problem is that it would require u

Theorem 3.5.16. The Gelfand-Naimark-Segal Theorem/Non-Commutative Gelfand Duality

Every $C^\star$-algebra has a faithful representation. In particular, every $C^\star$-algebra is $\star$-isomorphic to a $C^\star$-subalgebra of operators in $\mathcal{B}(H)$, where H is some $\mathbb{C}$-Hilbert space.

Proof. Let us, first, assume that V is a $C^\star$-algebra with identity e. Let $a \in V$ be given. Note that $\sigma_V(a^\star a) \subseteq [0, \infty)$ because $a^\star a$ is positive, by the strength of Theorem 3.3.8. It follows that $\|a^\star a\| = \|a\|^2 \in \sigma_V(a^\star a)$, by statement (2) of Proposition 3.2.1. By Theorem 3.4.8, there exists a state $f_a : V \to \mathbb{C}$ such that $f_a(a^\star a) = \|a^\star a\| = \|a\|^2$. Now, let $(H_{f_a}, \pi_{f_a}, x_{f_a})$ be a GNS triplet associated with f_a. We claim that $\|\pi_{f_a}(a)\| = \|a\|$ for every $a \in V$. Indeed:

$$\|\pi_{f_a}(a)\|^2 \geq \|\pi_{f_a}(a)x_{f_a}\|^2 = \langle\pi_{f_a}(a)x_{f_a}, \pi_{f_a}(a)x_{f_a}\rangle = \langle\pi_{f_a}(a)^\star(\pi_{f_a}(a)x_{f_a}), x_{f_a}\rangle = \langle\pi_{f_a}(a^\star a)x_{f_a}, x_{f_a}\rangle = f_a(a^\star a),$$

[6] We choose not to discuss this here because it is not necessary for our purposes.

where we have used the GNS construction and the fact that π_{f_a} is a $\star$-homomorphism. The above implies that $\|\pi_{f_a}(a)\|^2 \geq \|a\|^2$, while the GNS construction implies that $\|\pi_{f_a}(a)\| \leq \|a\|$. Therefore, $\|\pi_{f_a}(a)\| = \|a\|$. We define the following representation:

$$\pi : V \to B\left(\bigoplus_{a\in V} H_{f_a}\right) \ , \ x \mapsto \pi(x) := \bigoplus_{a\in V} \pi_{f_a}(x)$$

where the above is just the direct sum representation of the family of GNS triplets formed above[a]. Now, note that:

$$\forall x \in V : \|\pi(x)\| = \sup_{a\in V} \|\pi_{f_a}(x)\|.$$

Since π_{f_a} is a representation, we have that $\|\pi_{f_a}(x)\| \leq \|x\|$ for every $x \in V$. Since $\|\pi_{f_x}(x)\| = \|x\|$, it follows that:

$$\forall x \in V : \|\pi(x)\| = \|x\|.$$

By Proposition 3.5.12, it follows that π is injective. That is, π is a faithful representation. By statement (3) of Proposition 3.2.9, it follows that $\pi(V)$ is a $C^\star$-subalgebra and that proves the result for unital $C^\star$-algebras. Now, let V be a $C^\star$-algebra without identity. Let V_e be the unitization of V. Then, there exists a faithful representation (H, π) of V_e. Let $\tau : V \to V_e$ be the isometric embedding prescribed by $x \mapsto (x, 0)$. This is clearly a $\star$-homomorphism. We note that $\pi \circ \tau$ is faithful and it is also isometric. As a consequence, the image $(\pi \circ \tau)(V)$ is a $C^\star$-subalgebra of $\mathcal{B}(H)$ and we are done. □

[a]We discussed this in Example 17. That example also provides references to the appendix for the definition of the direct sum.

Let us make a number of important remarks about what we have shown and its relation to what is available in the literature. In Theorem 3.5.16, we constructed the representation π by using cyclic representations. Even though we took the direct sum over all elements of a, we only included the states which give rise to cyclic representations.

[37, Chapter 3.3] makes use of a similar proof but does not make use of cyclic vectors. In [21, Chapter 4.5], the so-called "pure states" are used in the actual proof of the theorem above. We did not require the notion of a pure state so we simply did not define these at any point in this thesis. One does not have to make use of all the states to obtain a faithful representation via the process above. The following extension of the GNS Theorem makes this more precise.

Theorem 3.5.17

Let V be a $C^\star$-algebra of density character κ[a]. Then, there exists a $\mathbb{C}$-Hilbert space H of density character at most κ and a faithful representation $\pi : V \to \mathcal{B}(H)$.

[a]If (X, τ) is a topological space and κ is a cardinal number, then the density character of X is said to be κ if the minimal cardinality of dense subsets of X is κ. As the reader can tell, showing that this definition even makes sense is certainly an adventure of its own and would make for an entirely separate discussion.

Proof. See [15, Corollary 1.10.4, p.35] for the proof. This text also explores the deeper connections between $C^\star$-algebras and axiomatic set theory; it might be of interest to the reader on its own. □

The theorem just tells us that we do have some measure of control over the Hilbert space which is used to represent a given $C^\star$-algebra. In particular, a separable $C^\star$-algebra can be represented by a separable Hilbert space H. If H is finite-dimensional and $\pi : V \to \mathcal{B}(H)$ is a faithful representation, then $\mathcal{B}(H)$ is certainly finite-dimensional and the first isomorphism theorem tells us that $V/\ker(\pi) = V/\{0\}$ is linearly isomorphic to $\pi(V)$. But $\pi(V)$ is a subspace of $\mathcal{B}(H)$ and, therefore, is finite-dimensional. Since $V/\{0\}$ is finite-dimensional and is linearly isomorphic to V, it follows that V is finite-dimensional. In other words, finite-dimensional Hilbert spaces can only represent finite-dimensional $C^\star$-algebras[7].

The reader might, quite rightfully, be wondering about how all of this relates to the commutative case. Indeed, the commutative theory has a distinct flavor from the non-commutative theory so one might possibly expect to find new insights about commutative $C^\star$-algebras by the methods in this subsection. We leave a more thorough discussion of this to Proposition A.0.2 and related results there for the interested reader, since we will not be needing it for any of the work we will do. With that said, we are finally ready to leave the theory of $C^\star$-algebras behind and focus on (a portion of) spectral theory as an application of the results we have derived in this chapter.

[7]In this paragraph, we have just used standard facts from linear algebra that the reader may find in [2].

Chapter 4

Spectral Theory

This chapter is dedicated to a detailed discussion of spectral theory from the point of view of the material covered in the last two chapters. Since it can be a bit difficult to navigate this material, let us speak of what will be covered in the three main sections comprising this chapter:

1. **The $C^\star$-algebra $\mathcal{B}(H)$ & Operator Topologies**. We expand on the basic material on bounded endomorphisms of a Hilbert space as introduced in Section 3.5, giving a self-contained treatment which has all of the power of $C^\star$-algebra theory behind it.
2. **von Neumann Algebras**. A short (but sweet) section on von Neumann algebras just gives us an excuse to introduce important (topological) tools that will all contribute to the proof of the spectral theorem. For our purposes, it will be sufficient to work with abelian von Neumann Algebras.
3. **The (Not-So-Normal Proof of the) Spectral Theorem** We will use the results derived in Section 4.1 and Section 4.2 to give a rather careful proof of the spectral theorem for bounded normal operators.

We **will** assume that readers are familiar with standard theorems from the theory of Hilbert spaces. We **will not** assume extensive knowledge of the properties of bounded operators. We will, however, refer the reader to the basic introduction to bounded operators on a Hilbert space given in Section 3.5 starting with Definition 3.5.8 and we will also assume that the reader is familiar with a few standard facts from Appendix B.2.

4.1 The $C^\star$-Algebra $\mathcal{B}(H)$

Throughout this section, we will assume that $\mathcal{B}(H)$ (H is a $\mathbb{C}$-Hilbert space) has its $C^\star$-algebraic structure unless otherwise stated. We begin by defining special operators of interest in analogy with the special elements of interest in a $C^\star$-algebra (see Definition 3.1.1). This section is primarily based off of [34].

Definition 4.1.1

Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$.

1. T is a self-adjoint operator if it is a self-adjoint element of $\mathcal{B}(H)$.
2. T is a normal operator if it is a normal element of $\mathcal{B}(H)$.
3. T is a unitary operator if it is a normal element of $\mathcal{B}(H)$ such that $T^\star T = \mathrm{Id}_H$.

Over here, Id_H always refers to the identity operator on H.

It is useful to have operator-theoretic characterizations of each of the notions introduced in Definition 4.1.1. To prove such a characterization result, the following lemma is needed.

Lemma 4.1.2 Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$. Suppose that $\langle T(x), x\rangle = 0$ for every $x \in H$. Then, $T = 0$.

Proof. Let $x, y \in H$. Then, we have the following equations:

$$0 = \langle T(x+y), x+y\rangle = \langle T(x), x\rangle + \langle T(y), x\rangle + \langle T(x), y\rangle + \langle T(y), y\rangle = \langle T(y), x\rangle + \langle T(x), y\rangle,$$
$$0 = \langle T(x+iy), x+iy\rangle = \langle T(x), x\rangle + i\langle T(y), x\rangle - i\langle T(x), y\rangle + \langle T(y), y\rangle = i\langle T(y), x\rangle - i\langle T(x), y\rangle.$$

This implies, altogether, that $\langle T(y), x\rangle = 0$ for every $x, y \in H$. But this implies that $T(y) = 0$ for every $y \in H$ and that proves the result. □

We can define a family of (continuous) linear functionals as follows. Let $x \in H$ be fixed and define:

$$\omega_x : \mathcal{B}(H) \to \mathbb{C}\ ,\ T \mapsto \omega_x(T) := \langle T(x), x\rangle$$

The reader may recall that we actually used this family of linear functionals as examples of positive linear functionals on $\mathcal{B}(H)$ (see Example 13). So, the lemma can be re-phrased as the following statement: if an operator $T \neq 0$, then there is an $x \in H$ such that $\omega_x(T) \neq 0$. To rephrase it in another way, if S, T are two operators such that $T \neq S$, then there is an $x \in H$ such that $\omega_x(T) \neq \omega_x(S)$. In other words, the family $(\omega_x)_{x \in H}$ of linear functionals separates the points of $\mathcal{B}(H)$.

Proposition 4.1.3

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$. The following hold:

1. T is self-adjoint iff $\langle T(x), x\rangle \in \mathbb{R}$ for every $x \in H$.
2. T is normal iff for every $x \in H$, $\|T(x)\| = \|T^\star(x)\|$.
3. T is unitary iff for every $x \in H$, $\|T(x)\| = \|x\|$ and $T(H) = H$.

Proof.

1. Assume that T is self-adjoint. Then:
$$\forall x \in H : \langle T(x), x\rangle = \langle x, T^\star(x)\rangle = \langle x, T(x)\rangle = \overline{\langle T(x), x\rangle},$$
which implies that $\langle T(x), x\rangle \in \mathbb{R}$ for every $x \in H$. Next, assume that $\langle T(x), x\rangle \in \mathbb{R}$ for every $x \in H$. Then:
$$\forall x \in H : \langle T(x), x\rangle = \langle x, T^\star(x)\rangle = \overline{\langle T^\star(x), x\rangle} = \langle T^\star(x), x\rangle.$$
The lemma, then, implies that $T = T^\star$, as was desired.
2. Assume that T is normal. Let $x \in H$. Then:
$$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^\star(T(x))\rangle = \langle x, (T \circ T^\star)(x)\rangle = \langle T^\star(x), T^\star(x)\rangle = \|T^\star(x)\|^2,$$
which implies the desired equality. Next, assume that $\|T(x)\| = \|T^\star(x)\|$ for every $x \in H$. Then, we have the following predicates:
$$\forall x \in H : \|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^\star(T(x))\rangle = \langle x, (T^\star \circ T)(x)\rangle,$$
$$\forall x \in H : \|T^\star(x)\|^2 = \langle T^\star(x), T^\star(x)\rangle = \langle x, T(T^\star(x))\rangle = \langle x, (T \circ T^\star)(x)\rangle.$$
By hypothesis, it follows that $\langle x, (T^\star \circ T)(x)\rangle = \langle x, (T \circ T^\star)(x)\rangle$ for every $x \in H$. This implies that $T^\star \circ T = T \circ T^\star$.
3. Assume that T is unitary. Then:
$$\forall x \in H : \|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^\star(T(x))\rangle = \langle x, (T^\star \circ T)(x)\rangle = \langle x, \mathrm{Id}_H(x)\rangle = \|x\|^2,$$
which gives the desired equality. Obviously, T is surjective because it being unitary implies that it is bijective. Next, assume that $\|T(x)\| = \|x\|$ for every $x \in H$. Then:
$$\forall x \in H : \|x\|^2 = \langle x, x\rangle = \langle x, \mathrm{Id}_H(x)\rangle = \|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^\star(T(x))\rangle = \langle x, (T^\star \circ T)(x)\rangle.$$
By the lemma, it follows that $\mathrm{Id}_H = T^\star \circ T$. Since T has a left inverse, it is injective. Since $T(H) = H$, it is surjective. Therefore, T is a continuous bijection. By the Inverse Operator Theorem (Theorem B.2.6), it follows that T is invertible. in particular, $T^\star \circ T = T \circ T^\star = \mathrm{Id}_H$ and that proves that T is unitary. □

Since we are working towards a proof of the spectral theorem for bounded normal operators, it makes sense for us to understand the behavior of these operators in greater depth. We begin with the following simple result.

Proposition 4.1.4

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be a normal operator. Then, the following hold:

1. $\ker(T) = \ker(T^\star)$.
2. $T(H)$ is dense in H iff T is injective.
3. T is bijective iff there exists $\delta > 0$ such that $\|T(x)\| \geq \delta\|x\|$ for every $x \in H$.

Proof. We will prove each statement one-by-one.

1. Let T be normal. Then:
$$\forall x \in H : x \in \ker(T) \Leftrightarrow T(x) = 0 \Leftrightarrow \|T(x)\| = 0 \Leftrightarrow \|T^\star(x)\| = 0 \Leftrightarrow T^\star(x) = 0 \Leftrightarrow x \in \ker(T^\star)$$
which proves the result.
2. For this statement, we should keep the following statement in mind: if $A \subseteq H$, then $\overline{\mathrm{span}(A)} = A^{\perp\perp}$[a]. Assume that T is injective. By (1), it follows that $\ker(T^\star) = \{0\}$. By property (6) in Proposition 3.5.9[b], we have that $T(H)^\perp = \{0\}$. This implies that $\overline{T(H)} = H$. Next, assume that $T(H)$ is dense in H. Then, $\overline{T(H)} = H = T(H)^{\perp\perp}$. It follows that $T(H)^\perp = \{0\}$. But this means that $\ker(T^\star) = \{0\}$ so $\ker(T) = \{0\}$ and T is injective.
3. Assume that T is bijective. By Theorem B.2.6, it follows that T^{-1} is bounded. Let $\delta = \left\|T^{-1}\right\|^{-1}$. Then:
$$\forall y \in H : \left\|T^{-1}(y)\right\| \leq \left\|T^{-1}\right\|\|y\|$$
which translates to:
$$\forall x \in H : \delta\|x\| \leq \|T(x)\|.$$
Next, assume that there exists a $\delta > 0$ such that $\|T(x)\| \geq \delta\|x\|$ for every $x \in H$. If $T(x) = 0$ for some $x \in H$, then $\|x\| = 0$ so $x = 0$, which implies that T is injective. By (2), $T(H)$ is dense in H and Theorem B.2.5 implies that T must be surjective. This means that T is a continuous bijection. By statement (3) of Theorem B.2.6, it follows that T is surjective and we are done. □

[a] The proof of this is just a standard computation.
[b] Specifically, we are using the fact that $\ker(T^\star) = T(H)^\perp$.

The next proposition gives us a characterization of the spectrum of a normal operator.

Proposition 4.1.5

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be a normal operator. The following are equivalent:

1. $\lambda \in \sigma_{\mathcal{B}(H)}(T)$.
2. There is a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors in H such that $\lim_{n\to\infty} \|(T - \lambda\,\mathrm{Id}_H)x_n\| = 0$.

Proof.

(1) $\Rightarrow$ (2) : Assume that $\lambda \in \sigma_{\mathcal{B}(H)}(T)$. Then, $T - \lambda\,\mathrm{Id}_H$ is not invertible and it is a normal operator. Let B^1 denote the closed unit sphere in H. By statement (3) of Proposition 4.1.4, we have that:

$$\forall n \in \mathbb{N} : \exists x_n \in B^1 : \|(T - \lambda\,\mathrm{Id}_H)x_n\| < \frac{1}{n}$$

The result in (2) now follows from this inequality.

(2) $\Rightarrow$ (1) : Suppose that $T - \lambda\,\mathrm{Id}_H$ is invertible. Since it is a normal operator, it follows that:

$$\exists \delta > 0 : \forall x \in S^1 : \|(T - \lambda\,\mathrm{Id}_H)x\| \geq \delta$$

It follows that:

$$\forall n \in \mathbb{N} : \|(T - \lambda\,\mathrm{Id}_H)x_n\| \geq \delta$$

Taking limits on both sides, we find that $\delta \leq 0$ and this is a contradiction. □

The next result provides a characterization of positivity for elements in $\mathcal{B}(H)$, where H is a $\mathbb{C}$-Hilbert space. We have already derived a characterization of positivity for elements in a $C^\star$-algebra but this one is specific to elements in $\mathcal{B}(H)$.

Proposition 4.1.6

Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$. Then, the following are equivalent:

1. $\forall x \in H : \langle T(x), x\rangle \geq 0$.
2. T is a positive element of $\mathcal{B}(H)$.

Proof.

(1) $\Rightarrow$ (2) : Since $\langle T(x), x\rangle \geq 0$ for every $x \in H$, it follows that T is self-adjoint. Let $\lambda \in \sigma_{\mathcal{B}(H)}(T)$ and suppose that $\lambda < 0$. Then:

$$\forall x \in H : \langle T(x) - \lambda x, x\rangle = \langle T(x), x\rangle - \lambda\|x\|^2 \geq -\lambda\|x\|^2$$

This implies that $T - \lambda\,\mathrm{Id}_H$ is bounded below. By statement (3) of Proposition 4.1.4, it follows that $T - \lambda\,\mathrm{Id}_H$ is bijective. By the Inverse Operator Theorem, it follows that it is invertible in $\mathcal{B}(H)$ and this is impossible because $\lambda \in \sigma_{\mathcal{B}(H)}(T)$. Therefore, $\lambda \geq 0$.

(2) $\Rightarrow$ (1) : Assume that (2) holds. By Theorem 3.3.8, we have that $T = S^\star S$ for some $S \in \mathcal{B}(H)$. Therefore:

$$\forall x \in H : \langle T(x), x\rangle = \langle S^\star(S(x)), x\rangle = \|S(x)\|^2 \geq 0$$

which gives the desired result[a] □

[a] It is extremely interesting to the author that [34, p.330-331, Theorem 12.32] contains a proof of this result where the Spectral Theorem (which we will deal with later) is used to prove (2) $\Rightarrow$ (1). In some sense, this is actually quite understandable; establishing a spectral decomposition is usually the first priority for an analyst and once one has it, they can use it even if it is a genuine sledgehammer.

The next step to take is to define the notion of a *projection* and derive as much about that as we possibly can. The reason for this is that the statement of the spectral theorem will employ the use of so-called "projection-valued measures"; we will talk about these in-depth later. For now, let us provide a definition for them in $\star$-algebras.

Definition 4.1.7

Let V be a $\star$-algebra and let $x \in V$. x is said to be a projection iff $x^2 = x$. x is an orthogonal projection iff it is a self-adjoint projection[a].

[a] We could have defined the notion of a projection much earlier. However, we did not use them at all in the last chapter so it would have been rather pointless.

Let us examine two simple instances where the orthogonal projections are easy to compute. Both of these are crucial in our proof of the spectral theorem (in the sense that they are facts which are very easy to overlook). We did not find either of these propositions stated explicitly in any texts. We will establish further facts about orthogonal projections right after this (short) discussion.

Proposition 4.1.8

Let (X, τ) be a compact Hausdorff space. Then, the only orthogonal projections of $\mathcal{C}(X)$ are the characteristic functions of clopen subsets of X.

Proof. It's fairly easy to check that the characteristic functions of clopen subsets of X are, indeed, orthogonal projections in $\mathcal{C}(X)$. Let $f \in \mathcal{C}(X)$ be an orthogonal projection. Then, f is real-valued and satisfies the equation $f(f-1) = 0$ so $f(x) = 1$ or $f(x) = 0$ for any $x \in X$. Let $G = f^{-1}(\{1\})$. Then, $f = 1_G$ so it remains to be shown that G is clopen. G is obviously closed and $X \setminus G$ is closed so G must be open and we are done. □

This result provides a nice characterization of orthogonal projections via the Gelfand transform. We point out, again, that this result is going to be useful in the proof of the spectral theorem.

Proposition 4.1.9

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be a linear operator. The following are equivalent:

1. T is an orthogonal projection.
2. $\widehat{T}$ is the characteristic function of a clopen set.

Proof.

$(1) \Rightarrow (2):$ Let T be an orthogonal projection. Note that $\widehat{T}$ is an orthogonal projection in $\mathcal{C}(\widehat{\mathcal{B}(H)})$. That is, it is the characteristic function of a clopen set.

$(2) \Rightarrow (1):$ If $\widehat{T}$ is the characteristic function of a clopen set, then it is an orthogonal projection in $\mathcal{C}(\widehat{\mathcal{B}(H)})$. In particular, we have that $\widehat{T^2} = \widehat{T}\widehat{T} = \widehat{T}$ and we also have that $\widehat{T^\star} = \overline{\widehat{T}} = \widehat{T}$. Both of these statements imply that $T^2 = T$ and that $T = T^\star$. This proves that T is an orthogonal projection. □

For the most part, we will only be interested in the use of orthogonal projections. To motivate our interest in them specifically, we needed to distinguish between projections and orthogonal projections. Making this distinction gives us the added benefit of the following characterization theorem for orthogonal projections. Having access to results of this type also gives us several ways of proving that certain concrete operators of interests are orthogonal projections[1].

Theorem 4.1.10

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be a projection. The following are equivalent:

1. T is an orthogonal projection.
2. T is normal.
3. $T(H) = \ker(T)^\perp$.
4. $\forall x \in H : \langle T(x), x\rangle = \|T(x)\|^2$.

Also, orthogonal projections are positive. If $T, S \in \mathcal{B}(H)$ are orthogonal projections, then $T(H) \perp S(H)$ iff $TS = 0$.

Proof.

$(1) \Rightarrow (2):$ Since T is an orthogonal projection, it is self-adjoint so it must be normal.

$(2) \Rightarrow (3):$ Let T be normal. Then, $\ker(T) = \ker(T^\star)$. It follows that:

$$\ker(T)^\perp = \ker(T^\star)^\perp = T(H)^{\perp\perp} = \overline{T(H)}$$

It remains to be shown that $T(H)$ is closed. Since $T^2 = T$, it follows that $(\mathrm{Id}_H - T)T = 0$. Therefore, $T(H) = \ker(\mathrm{Id}_H - T)$ and the latter set is certainly closed due to the continuity of T so we are done.

$(3) \Rightarrow (4):$ Let $x \in H$. We know that $H = \ker(T) \oplus \ker(T)^\perp = \ker(T) \oplus T(H)$. We can write $x = y + z$ with $y \in \ker(T)$ and $z \in \ker(T)^\perp$. Since $T(x) = T(z)$ and since $z \in T(H)$, we have $z = T(u)$ for some $u \in H$. Therefore:

$$T(x) = T(z) = T(T(u)) = T(u)$$

which implies that $x - u \in \ker(T)$. But by the uniqueness of the decomposition of x, it follows that $x - u = y$ and this means that $u = z$. Therefore:

$$\langle T(x), x\rangle = \langle T(z), T(z)\rangle = \|T(z)\|^2 = \|T(x)\|^2$$

as was desired.

$(4) \Rightarrow (1):$ This involves showing that T is self-adjoint; it follows immediately from (4) because $\langle T(x), x\rangle$ is real whenever $x \in H$ and we are done.

The fact that orthogonal projections are positive follows from statement (4) and Proposition 4.1.6. Finally, let $T, S \in \mathcal{B}(H)$ be orthogonal projections. Assume that $TS = 0$. Let $x, y \in H$. Then:

$$\langle T(x), S(y)\rangle = \langle x, T^\star(S(y))\rangle = \langle x, T(S(y))\rangle = 0$$

which implies that $T(H) \perp S(H)$. Assume, now, that $T(H) \perp S(H)$. Then, we have the following:

$$\forall x \in H : \langle (TS)(x), x\rangle = \langle S(x), T^\star(x)\rangle = \langle S(x), T(x)\rangle = 0$$

which implies, by Lemma 4.1.2, that $TS = 0$. □

As the reader can see, statement (3) of Theorem 4.1.10 associates a closed subspace to every orthogonal projection; namely, the range of the orthogonal projection. In fact, the next result shows that there is a correspondence between closed subspaces of a Hilbert space H and orthogonal projections.

[1] Or whatever other notion we are interested in characterizing in that moment.

Proposition 4.1.11

Let H be a $\mathbb{C}$-Hilbert space. Then, there is a one-to-one correspondence between closed subspaces of H and orthogonal projections. An orthogonal projection $T \in \mathcal{B}(H)$ is a positive operator satisfying $\|T\| = 1$ unless $T = 0$.

Proof. Let $Y \subseteq H$ be a closed subspace. Then, we can write $H = Y \oplus Y^\perp$. Define:

$$T : H \to H \ , \ x \mapsto T(x) := y$$

where $x = y + z$ is the (unique) decomposition of x in the direct sum above. This is clearly a projection. To show that it is an orthogonal projection, observe that:

$$\langle T(x), x\rangle = \langle y, x\rangle = \langle y, y\rangle + \langle y, z\rangle = \|y\|^2 = \|T(x)\|^2$$

By statement (4) of Theorem 4.1.10, it follows that T is an orthogonal projection. By statement (3) of Theorem 4.1.10, it is clear that every orthogonal projection gives us a closed subspace of H. The positivity of an orthogonal projection has already been shown so it remains to show that $\|T\| = 1$ if it is a non-zero orthogonal projection. Assume, now, that $T \in \mathcal{B}(H)$ is a non-zero orthogonal projection. Then:

$$\forall x \in H : \|T(x)\|^2 = \langle T(x), x\rangle \leq \|T(x)\| \cdot \|x\|$$

which implies that $\|T\| \leq 1$. Next, let $Y = T(H)$. This is a closed subspace of H because $T(H) = \ker(T)^\perp$ by statement (3) of Theorem 4.1.10. Then, we can write $H = Y \oplus Y^\perp$. We know that:

$$\forall y \in Y : \langle T(y), y\rangle = \|y\|^2 = \|T(y)\|^2$$

It follows that $\|T\| = 1$ and we are done. □

As it turns out, projections happen to have some rather nice properties relative to the $C^\star$-order on $\mathcal{B}(H)$ when H is a $\mathbb{C}$-Hilbert space. We will list some of these out in the following proposition.

Proposition 4.1.12

Let H be a $\mathbb{C}$-Hilbert space and let $Y, Z \subseteq H$ be closed subspaces. Let $T : H \to Y$ and $S : H \to Z$ be surjective orthogonal projections. The following are equivalent:

1. $Y \subseteq Z$.
2. $ST = T$.
3. $TS = T$.
4. $\forall x \in H : \|T(x)\| \leq \|S(x)\|$.
5. $T \prec S$.

Proof.

(1) ⇒ (2) : Let $Y \subseteq Z$. Let $x \in H$ be fixed. Then, $x = y + y'$ with $y \in Y$ and $y' \in Y^\perp$. Therefore, $T(x) = y$. Since $y \in Y$ and $Y \subseteq Z$, it follows that $y \in Z$. In other words, $y = y + 0$ is the decomposition of y in $Z \oplus Z^\perp$. This implies that $S(y) = y$. Hence:

$$(ST)(x) = S(T(x)) = S(y) = y = T(x)$$

which proves that $ST = T$.

(2) ⇒ (3) : Let $ST = T$. Then, we have that:

$$T = T^\star = T^\star S^\star = TS$$

because orthogonal projections are self-adjoint.

(3) ⇒ (4) : We have that:

$$\forall x \in H : \|T(x)\| = \|T(S(x))\| \leq \|T\| \cdot \|S(x)\| \leq \|S(x)\|$$

as was desired.

(4) ⇒ (5) : Let $x \in H$ be fixed. Then:

$$\langle (S - T)(x), x\rangle = \langle S(x) - T(x), x\rangle = \langle S(x), x\rangle - \langle T(x), x\rangle = \|S(x)\|^2 - \|T(x)\|^2 \geq 0$$

By Proposition 4.1.6, it follows that $S - T$ is positive. That is, $T \prec S$.

(5) ⇒ (1) : Assume that $T \prec S$. Let $y \in Y$. Then:

$$\|y\|^2 = \langle T(y), y\rangle \leq \langle S(y), y\rangle \leq \|y\|^2$$

It follows that $\|S(y)\|^2 = \|y\|^2$. Let $y = z + z'$ with $z \in Z$ and $z' \in Z^\perp$. Then, $S(y) = z$. Since $0 = \langle y - S(y), y\rangle = \langle z', z + z'\rangle = \|z'\|^2$, it follows that $z' = 0$ and we are done. □

At this point, we have developed enough to be able to move on to a brief (but careful) study of von Neumann Algebras. We point out that many further results about bounded operators on Hilbert spaces can be found in [8][2].

[2] The interesting thing about this reference is that it takes a similar approach to what is done here; the theory of $C^\star$-algebras is properly developed in [7] BEFORE Hilbert spaces are dealt with. As the reader can imagine, this makes for an more insightful exposition overall because it is (in principle) easier to work with the elements of an arbitrary $C^\star$-algebra first.

4.2 von Neumann Algebras

After a rather thorough study of the basic results associated with $\mathcal{B}(H)$, it seems natural to focus on "special" subsets of $\mathcal{B}(H)$ when H is a $\mathbb{C}$-Hilbert space. The preceding section focused on the study of the individual points in $\mathcal{B}(H)$ and their properties relative to each other, while this section will focus on the study of special subalgebras of $\mathcal{B}(H)$. We prepare for the definition of a von Neumann algebra with the following definition.

Definition 4.2.1

Let V, W be $\mathbb{K}$-normed spaces. Define two family of maps $(T_v)_{v\in V}$ and $(S_{v,f})_{v\in V, f\in W^\star}$ by the following prescriptions:

$$T_v : \mathcal{L}(V,W) \to W \ , \ M \mapsto T_v(M) := M(v)$$

$$S_{v,f} : \mathcal{L}(V,W) \to \mathbb{K} \ , \ M \mapsto S_{v,f}(M) := f(M(v))$$

The strong operator topology (SOT) on $\mathcal{L}(V,W)$ is the weakest topology such that the family of maps $(T_v)_{v\in V}$ is continuous. The weak operator topology (WOT) on $\mathcal{L}(V,W)$ is the weakest topology such that the family of maps $(S_{v,f})_{v\in V, f\in W^\star}$ is continuous.

Let us immediately establish the following result about the topologies on $\mathcal{L}(V,W)$.

Proposition 4.2.2

Let V, W be $\mathbb{K}$-normed spaces. Let τ_{norm}, τ_{WOT} and τ_{SOT} refer to the respective topologies of interest on $\mathcal{L}(V,W)$. Then, we have that $\tau_{\text{WOT}} \subseteq \tau_{\text{SOT}} \subseteq \tau_{\text{norm}}$. Moreover, let $(T_\delta)_{\delta\in D}$ be a net in $\mathcal{L}(V,W)$ and let $T \in \mathcal{L}(V,W)$. Then, the following hold:

1. $T_\delta \to T$ in SOT iff for every $v \in V$, $T_\delta(v) \to T(v)$ in W.
2. $T_\delta \to T$ in WOT iff for every $v \in V$ and every $f \in W^\star$, it follows that $f(T_\delta(v)) \to f(T(v))$.

Proof. The SOT and WOT are both weak topologies. Therefore, we appeal to Proposition C.0.14 and the definitions of the WOT/SOT. Clearly, convergence in SOT implies convergence in WOT. But now, it is obvious that convergence in the norm topology implies convergence in SOT and that implies convergence in WOT. □

For the most part, we will be interested in the behavior of these topologies on $\mathcal{B}(H)$ when H is a $\mathbb{C}$-Hilbert space. In fact, there is a better representation of the weak operator topology on $\mathcal{B}(H)$. Let us provide this in the next result.

Proposition 4.2.3

Let H be a $\mathbb{C}$-Hilbert space. Let $v, w \in H$ be fixed. The weak-operator topology on $\mathcal{B}(H)$ is the weakest topology on $\mathcal{B}(H)$ such that the family of linear functionals $(S_{v,w})_{v,w\in H}$ defined by:

$$S_{v,w} : \mathcal{B}(H) \to \mathbb{C} \ , \ M \mapsto S_{v,w}(M) := \langle M(v), w\rangle$$

is continuous.

Proof. This follows immediately from the Riesz Representation Theorem (Theorem B.2.2) by representing a linear functional on H as an inner product. □

We will be content in proving just a few results about the operator topologies that will be crucial for the proof of the spectral theorem. We will begin with a fairly straightforward continuity result concerning the weak-operator topology, followed by a more difficult compactness result for the same topology.

Proposition 4.2.4

Let H be a $\mathbb{C}$-Hilbert space and let $S \in \mathcal{B}(H)$ be fixed. Then, the involution on $\mathcal{B}(H)$ is weak-operator continuous. Define two maps:

$$R_S : \mathcal{B}(H) \to \mathcal{B}(H) \ , \ T \mapsto R_S(T) := TS$$
$$L_S : \mathcal{B}(H) \to \mathcal{B}(H) \ , \ T \mapsto L_S(T) := ST$$

Then, R_S and L_S are weak-operator continuous.

Proof. Let $(T_\delta)_{\delta\in D}$ be a net in $\mathcal{B}(H)$ which converges to $T \in \mathcal{B}(H)$ in the weak-operator topology. Then:

$$\forall v, w \in H : \lim_\delta \langle T_\delta(v), w\rangle = \langle T(v), w\rangle$$

Now, we have that:

$$\forall v, w \in H : \lim_\delta \langle T_\delta^\star(v), w\rangle = \lim_\delta \langle v, T_\delta(w)\rangle = \lim_\delta \overline{\langle T_\delta(w), v\rangle} = \overline{\langle T(w), v\rangle} = \langle T^\star(v), w\rangle$$

which proves that $\star$ is WOT-continuous. R_S and L_S can be shown to be continuous in the same way. □

The next result is certainly not as simple as the one we have just given. Nevertheless, the reader will notice that the proof is very similar to that of the Banach-Alaoglu Theorem.

Theorem 4.2.5

Let H be a $\mathbb{C}$-Hilbert space. Then, the closed unit ball in $\mathcal{B}(H)$ is weak-operator compact.

Proof. The argument is very similar to that of the Banach-Alaoglu Theorem, so we will not supply every little detail. Observe that for every $v, w \in H$, we have that:

$$\forall T \in \mathcal{B}(H) : |S_{v,w}(T)| \leq \|T\| \cdot \|v\| \cdot \|w\|$$

Let $(\mathcal{B}(H))_1$ denote the closed unit ball in $\mathcal{B}(H)$. Then:

$$\forall T \in (\mathcal{B}(H))_1 : |S_{v,w}(T)| \leq \|v\| \cdot \|w\|$$

In particular, for each $v, w \in H$, it is the case that $S|_{(\mathcal{B}(H))_1} \subseteq \overline{B(0, \|v\| \cdot \|w\|)}$[a]. Now, we observe that the product space $\prod_{v,w\in H} \overline{B(0, \|v\| \cdot \|w\|)}$[b] is compact in the product topology, by Theorem B.1.6. Consider the following map:

$$\Phi : (\mathcal{B}(H))_1 \to \prod_{v,w\in H} \overline{B(0, \|v\| \cdot \|w\|)} \ , \ T_{v,w}|_{(\mathcal{B}(H))_1} \mapsto \Phi(T_{v,w}|_{(\mathcal{B}(H))_1}) := (\langle T(v), w\rangle)_{v,w\in H}$$

Let $X := \Phi((\mathcal{B}(H))_1)$. If we can show that $\Phi : (\mathcal{B}(H))_1 \to X$ is a homeomorphism and that X is compact, then that will prove that $(\mathcal{B}(H))_1$ is weak-operator compact.

1. Let us show that $\Phi : (\mathcal{B}(H))_1 \to X$ is bijective; it is enough to show injectivity. Let $\Phi(T_1) = \Phi(T_2)$. Then, $\langle T_1(v), w\rangle = \langle T_2(v), w\rangle$ for every $v, w \in H$. This implies that $T_1 = T_2$, which proves that Φ is injective.
2. Let us show that Φ is continuous; we appeal to Theorem C.0.12. Let $(T_\delta)_{\delta\in D}$ be a net in $(\mathcal{B}(H))_1$ that converges to some $T \in (\mathcal{B}(H))_1$ in the WOT. This means that $S_{v,w}(T_\delta) \to S_{v,w}(T)$ for every $v, w \in H$. But this is exactly what it means for Φ to be continuous so we are done.
3. Let us show that $\Phi^{-1} : X \to (\mathcal{B}(H))_1$ is continuous; we appeal to Theorem C.0.12 again. Let $(f_\delta)_{\delta\in D}$ be a net of functions in X that converges to $f \in X$ in the product topology. Since the product topology induces point-wise convergence[c], it follows that $f_\delta(v, w) \to f(v, w)$ for $v, w \in H$. For each $\delta \in D$, there is a unique $T_\delta \in (\mathcal{B}(H))_1$ such that $f_\delta(v, w) = \langle T_\delta(v), w\rangle$ for every $v, w \in H$. Similarly, there exists a unique $T \in (\mathcal{B}(H))_1$ such that $f(v, w) = \langle T(v), w\rangle$ for every $v, w \in H$. It follows that $\langle T_\delta(v), w\rangle \to \langle T(v), w\rangle$ for every $v, w \in H$; this implies that $T_\delta \to T$ in the weak-operator topology. This proves that Φ^{-1} to be continuous.
4. Let us show that X is compact. Since X is a subset of a compact Hausdorff space, it is sufficient to show that X is closed. Let $f \in \overline{X}$ (the closure of X). Since $f \in \prod_{v,w\in H} \overline{B(0, \|v\| \cdot \|w\|)}$, it is the case that $|f(v, w)| \leq \|v\| \cdot \|w\|$ for every $v, w \in H$. Then, there is a net $(f_\delta)_{\delta\in D}$ in X that converges to f in the product topology[d]. This means that $f_\delta(v, w) \to f(v, w)$ for $v, w \in H$. In particular, we can prove that f is a conjugate-bilinear functional. To do this, we will make use of the fact that the net above consists of conjugate-bilinear functionals as well:
   (a) Let $v_1, v_2, w \in H$. Then:
   $$f(v_1 + v_2, w) = \lim_\delta f_\delta(v_1 + v_2, w) = \lim_\delta (f_\delta(v_1, w) + f_\delta(v_2, w)) = f(v_1, w) + f(v_2, w)$$
   where we have used linearity in the first argument for each function in the net.
   (b) Let $\alpha \in \mathbb{C}$ and $v, w \in H$. Then:
   $$f(\alpha v, w) = \lim_\delta f_\delta(\alpha v, w) = \lim_\delta \alpha f_\delta(v, w) = \alpha f(v, w)$$
   where we have used compatibility with scalar multiplication for each f_δ.
   (c) Let $v, w \in H$. Then:
   $$f(v, w) = \lim_\delta f_\delta(v, w) = \lim_\delta \overline{f_\delta(w, v)} = \overline{\lim_\delta f_\delta(w, v)} = \overline{f(w, v)}$$
   where we have used the conjugate symmetry of each f_δ.

   This proves that f is conjugate-bilinear. By Corollary B.2.3, there is a $T_0 \in \mathcal{B}(H)$ such that:
   $$\forall v, w \in H : f(v, w) = \langle T_0(v), w\rangle$$
   which proves that $f \in X$.

This completes the proof[e]. □

[a] Over here, we mean that $S|_{(\mathcal{B}(H))_1}$ takes values in the closed ball of radius $\|v\| \cdot \|w\|$, centered at the origin, in $\mathbb{C}$.
[b] This refers to the generalized Cartesian product. Evergreen readers can just rely on the definition of the Cartesian product that they are familiar with to understand what this is.
[c] This follows from the fact that the product topology is a weak topology; see Proposition C.0.14.
[d] This follows from Lemma C.0.8.
[e] The proof here has been inspired by the one given for [21, Theorem 5.1.3, p.306].

With all of this preliminary material out of the way, we can finally give the definition of a von Neumann algebra. Again, we remind the reader that we are not going to treat von Neumann algebras in a lot of detail; we will only deal with the theory that is necessary for our proof of the spectral theorem.

Definition 4.2.6

Let H be a $\mathbb{C}$-Hilbert space and let $V \subseteq \mathcal{B}(H)$ be a set. V is a von Neumann algebra if it is a unital $\star$-subalgebra which is WOT-closed.

Before we provide two examples of von Neumann algebras, let us first prove the following characterization result.

Proposition 4.2.7

Let H be a $\mathbb{C}$-Hilbert space and let $V \subseteq \mathcal{B}(H)$ be a set. The following are equivalent:

1. V is a von Neumann algebra.
2. V is a unital $C^\star$-subalgebra of $\mathcal{B}(H)$ which is WOT-closed.

Proof. (2) $\Rightarrow$ (1) is clear, so we shall prove (1) $\Rightarrow$ (2). It remains to be shown that V is closed in the norm topology. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in V that converges to some $T \in \mathcal{B}(H)$, in the norm topology. Then, $T_n \to T$ in the weak-operator topology. Since V is WOT-closed, it follows that $T \in V$ and we are done. □

We will make use of this result when we prove, in particular, some of the important structure theorems about abelian von Neumann algebras. Before getting into that discussion, we should provide examples of von Neumann Algebras. This constitutes an interesting discussion in itself. Indeed, one can try to also classify von Neumann algebras via a representation theorem similar to the ones we derived for $C^\star$-algebras. Since our focus is not on von Neumann algebras, we have chosen not to include any discussion of the examples that would (naturally) lead to such representation theorems. The reader who is interested in von Neumann algebra theory should see [47].

Example 18. Let H be a $\mathbb{C}$-Hilbert space. Then, $\mathcal{B}(H)$ is certainly a von Neumann algebra.

Example 19. Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be a normal operator. Let V_T be the von Neumann algebra generated by T[a]. Then, V_T is the WOT-closure of the set of all operators of the form $p(T, T^\star)$, where p is some complex polynomial in the variables z and $\overline{z}$. Moreover, V_T is abelian.

[a] The smallest von Neumann Algebra containing T.

As we said earlier, it will be enough for us to restrict our focus to abelian von Neumann Algebras. In particular, the result we want has to do with the structure space of an abelian von Neumann algebra and to prove it, we need the following analogue of the Monotone Convergence Theorem.

Proposition 4.2.8. Vigiers Theorem

Let H be a $\mathbb{C}$-Hilbert space. Let $(T_\delta)_{\delta\in D}$ be a net of self-adjoint operators in $\mathcal{B}(H)$ satisfying the following properties:

1. $\exists k \geq 0 : \forall \delta \in D : T_\delta \prec k\,\mathrm{Id}_H$.
2. $\forall \delta, \delta' \in D : \delta \leq \delta' \Rightarrow T_\delta \prec T_{\delta'}$.

Then, the net converges to a positive operator $T \in \mathcal{B}(H)$ in the strong-operator topology. Moreover, the operator T is the least upper bound for $\{T_\delta : \delta \in D\}$ in $\mathcal{B}(H)$[a].

[a] The theorem statement & proof was taken from [21, Lemma 5.1.4, p.307]. The name of the theorem was given in [42, Lemma 1, p.7], though the statement isn't exactly the same.

Proof. Select a $\delta_0 \in D$. Then, the convergence of the net is tantamount to considering the convergence of $(T_\delta)_{\delta\geq\delta_0,\delta\in D}$. By statement (2) of Proposition 3.3.6 and the given conditions, we have that:

$$\forall \delta \geq \delta_0 : -\|T_{\delta_0}\|\,\mathrm{Id}_H \prec T_\delta \prec k\,\mathrm{Id}_H$$

Given this, we can assume that $(T_\delta)_{\delta\in D}$ is bounded below by some constant multiple of the identity operator. Since closed balls in $\mathcal{B}(H)$ are WO-compact, it follows that there is a subnet $(T_{u(\beta)})_{\beta\in B}$ of the given net of operators which is weak-operator convergent to an operator T in $\mathcal{B}(H)$[a]. For $\delta, \delta' \in D$ with $\delta \geq \delta'$, we have that:

$$\forall x \in H : \langle T_\delta(x), x\rangle \geq \langle T_{\delta'}(x), x\rangle$$

It follows, then, that $T_{u(\beta)} \prec T$ for every $\beta \in B$. Now, suppose that $\delta \geq u(\beta)$ with $\delta \in D$ and $\beta \in B$. Then, it is the case that $T_{u(\beta)} \prec T_\delta$ so $0 \prec T - T_\delta \prec T - T_{u(\beta)}$. But now, we have the following chain:

$$\forall x \in H : 0 \leq \langle (T - T_\delta)(x), x\rangle = \left\|(T - T_\delta)^{\frac{1}{2}}(x)\right\|^2 \leq \langle (T - T_{u(\beta)})(x), x\rangle$$

whenever $\delta \geq u(\beta)$. Taking the limit with respect to β, it follows that the net of operators $((T - T_\delta)^{\frac{1}{2}})_{\delta\in D}$ is strong-operator convergent to 0. By the strong-operator continuity of multiplication, it follows that $(T - T_\delta)_{\delta\in D}$ is strong-operator convergent to 0.

Finally, let T' be an upper bound for $\{T_\delta : \delta \in D\}$. Then, $T_\delta \prec T'$ for every $\delta \in D$. Then, $\langle T'(x), x\rangle \geq \langle T_\delta(x), x\rangle$ for every $x \in H$ and every $\delta \in D$. Taking limits, it follows that $\langle T'(x), x\rangle \geq \langle T(x), x\rangle$, where T is the strong-operator limit of the given net of operators. But this implies that $T \prec T'$ so T is the least upper bound. □

[a] Here, we are making use of statement (3) of Theorem C.0.17 and the fact that closed balls in $\mathcal{B}(H)$ are weak-operator compact(Theorem 4.2.5). In this case, we also point out that the function $u : B \to D$ is a finalizing map, as defined in Definition C.0.16.

Before we prove the main result of this section, let us pause for a moment and derive a corollary of Vigiers Theorem that pertains to increasing nets of orthogonal projections. This corollary will be somewhat important for the proof of the spectral theorem and, of course, is of independent interest on its own.

Corollary 4.2.9. Let H be a $\mathbb{C}$-Hilbert space and let $V \subseteq \mathcal{B}(H)$ be a von Neumann algebra. Let $(T_\delta)_{\delta\in D}$ be an increasing net of orthogonal projections. Then, this net is SOT-convergent to an orthogonal projection $T \in V$.

Proof. By Proposition 4.1.9, we know that $\widehat{T_\delta}$ is the characteristic function of a clopen subset of $\widehat{\mathcal{B}(H)}$. By statement (2) of Proposition 3.5.6, it follows that $0 \prec T_\delta \prec \mathrm{Id}_H$ for every $\delta \in D$. By Vigiers Theorem, it follows that $T_\delta \to T$ in SOT. Since multiplication is SOT continuous on bounded sets, it follows that T is a projection. □

We can, now, finally move on the or proof of the main result of this section and one of the key tools we will use to prove the spectral theorem in the next section. The theorem comes with the major benefit of giving us a (relatively) easy and non-trivial example of an extremally disconnected space. We will speak of a further application of extremally disconnected spaces to analysis towards the end of the next section.

Theorem 4.2.10

Let H be a $\mathbb{C}$-Hilbert space and let $V \subseteq \mathcal{B}(H)$ be an abelian von Neumann algebra. Then, $\widehat{V}$ is a Stonian space[a].

[a]See Definition B.1.9.

Proof. Since V is a unital $C^\star$-algebra, it follows that $\widehat{V}$ is a compact Hausdorff space. It remains to be shown that $\widehat{V}$ is extremally disconnected. Let $U \subseteq \widehat{V}$ be open and define the following set:

$$D_U := \left\{ f \in \mathcal{C}(\widehat{V}) : f(\phi) \in [0,1] \text{ if } \phi \in U \wedge f(\phi) = 0 \text{ if } \phi \notin U \right\}$$

where $\mathcal{C}(\widehat{V})$ is well-defined because V is a unital commutative Banach algebra and, therefore, $\widehat{V}$ is a compact Hausdorff space[a]. Define an order on D_U as follows:

$$\forall f, g \in D_U : f \prec g \Leftrightarrow (\forall \phi \in \widehat{V} : f(\phi) \leq g(\phi))$$

This is clearly a partial order on D_U. Moreover, D_U is actually a directed set[b] with this order; if $f, g \in D_U$, then $\max\{f, g\} \in D_U$ and $f, g \prec \max\{f, g\}$. Since V is a commutative unital $C^\star$-algebra, it follows that the Gelfand representation $\mathcal{G} : V \to \mathcal{C}(\widehat{V})$ is an isometric $\star$-isomorphism by Theorem 3.5.1. In particular, for every $f \in D_U$, there exists a unique $T_f \in V$ such that $\mathcal{G}(T_f) = f$. We, then, have the following claims:

1. $\forall f_1, f_2 \in D_U : T_{f_1} \prec T_{f_2} \Leftrightarrow \mathcal{G}(T_{f_1}) \prec \mathcal{G}(T_{f_2})$. This follows from statement (1) of Proposition 3.5.6.
2. $\forall f \in D_U : 0 \prec T_f \prec \mathrm{Id}_H$. This follows from statement (2) of Proposition 3.5.6.

It follows that the net $(T_f)_{f\in D_U}$ converges to a positive operator $T \in \mathcal{B}(H)$ in the strong operator topology, by Proposition 4.2.8. Of course, this means that the net converges to T in the weak operator topology too and since V is weak operator closed, it follows that $T \in V$. We claim that $\widehat{T} = 1_{\overline{U}}$[c]; if this is the case, then $\overline{U}$ must be an open set by the continuity of $\widehat{T}$. This would prove that $\widehat{V}$ is extremally disconnected and we would be done. Since $T \geq T_f$ for every $f \in D_U$, $\widehat{T}(\phi) \geq \widehat{T_f}(\phi) = f(\phi)$ for every $\phi \in \widehat{V}$. Select an arbitrary $\phi_0 \in \widehat{V}$ and evaluate $\widehat{T}(\phi_0)$ for two cases:

1. Assume that $\phi_0 \in \overline{U}$. In fact, it is enough to show that $\widehat{T}(\phi_0) = 1$ when $\phi_0 \in U$. After all, if $\phi_0 \in \overline{U}$, then there is a net $(\phi_\delta)_{\delta\in D}$ in U which converges to ϕ_0[d]; in that case, we know that $\widehat{T}(\phi_\delta) = 1$ for every $\delta \in D$ so the continuity of $\widehat{T}$ gives us that $\widehat{T}(\phi_0) = 1$. So, we assume that $\phi_0 \in U$. We now apply statement (4) of Theorem B.1.7 to the disjoint closed sets $\{\phi_0\}$ and $X \setminus U$ to form a continuous function $f : \widehat{V} \to \mathbb{C}$ such that $f(\phi_0) = 1$ and $f(\phi) = 0$ for $\phi \in X \setminus U$. This will imply that $f \in D_U$. This implies that $\widehat{T}(\phi_0) \geq f(\phi_0) = 1$. On the other hand, it is known that $T \prec \mathrm{Id}_H$ so $\widehat{T}(\phi_0) \leq \widehat{\mathrm{Id}_H}(\phi_0) = 1$. It follows that $\widehat{T}(\phi_0) = 1$.
2. Suppose that $\phi_0 \notin \overline{U}$. Apply statement (4) of Theorem B.1.7 to construct a continuous function:

$$g : \widehat{V} \to [0,1] \ , \ \phi \mapsto g(\phi) := \begin{cases} 1 & \text{if } \phi \in \overline{U} \\ 0 & \text{if } \phi = \phi_0 \end{cases}$$

   Let $T_g \in V$ be the operator such that $\mathcal{G}(T_g) = g$. Since $g(\phi) \geq f(\phi)$ for every $\phi \in \widehat{V}$, it follows (by a prior statement we made) that $T_g \geq T_f$ for every $f \in D_U$. Since the operator T was an upper bound of the net $(T_f)_{f\in D_U}$, it follows that $T_g \geq T$. This implies that $0 = g(\phi_0) = \widehat{T_g}(\phi_0) \geq \widehat{T}(\phi_0) \geq 0$. But this means that $\widehat{T}(\phi_0) = 0$. In particular, $\widehat{T}(\phi) = 0$ whenever $\phi \notin \overline{U}$.

Both points above show that $\widehat{T} = 1_{\overline{U}}$. This completes the proof[e]. □

[a]See Theorem 2.3.13.
[b]See Definition C.0.1.
[c]This is the characteristic function of $\overline{U}$, where the range of this function is assumed to have the discrete topology.
[d]We are using Lemma C.0.8.
[e]The proof is adapted from [13, Theorem 53.2, p.234].

This marks the end of our short venture into the theory of von Neumann algebras. Obviously, there is a lot more to discuss than what we have provided here and the reader could surely tell that the study of von Neumann algebras is actually motivated particularly by the Theorem 3.5.16. After all, that theorem particularly indicates that subalgebras of bounded linear operators are particularly amenable to further study when extra structure is placed on them[3].

[3]However, we should point out that this was **not** the historical motivation for the study of these algebras. For a detailed look at this, see

4.3 The (Not-So-Normal Proof of the) Spectral Theorem

We are ready to deal explicitly with spectral theory. As we mentioned before, we are approaching the spectral theorem (for bounded normal operators) from a relatively unknown perspective. Given this, we think that a few words about other proofs of this theorem in the literature are necessary so we will list these right now and say something about their general character.

1. [29, Chapter IV.4, p. 100-104] has a proof which is not based in $C^\star$-algebra theory. Instead, it proceeds by a "lemma of Murray, concerning the norm of the polynomial $p_N(A)$"[4]. This source claims that this is "a straightforward and natural proof of the spectral theorem".
2. On the other hand, [13, Section 52, p.222-233] gives us a proof of the theorem using the commutative Gelfand-Naimark Theorem; this is one of the standard proofs. After we have proved the spectral theorem, we will spend some time comparing this proof (and its variants) to our proof.
3. [18, p.69-73] provides a rather "concrete" proof that does not make use of any "abstract spectral theorems" (which the proof in (2) does use through its use of the normal continuous functional calculus). It starts by proving the result for bounded self-adjoint operators and then extends it to bounded normal operators.
4. [46, Chapters 6-7, p.247-291] provides a proof of the spectral theorem for bounded self-adjoint operators (from which the theorem about bounded normal operators follows) by first proving (Hans) Freudenthal's Spectral Theorem for Riesz Spaces[5]. This proof appears to have a similar spirit to the proof in (2) because it relies on an abstract spectral theorem from which the spectral theorem for bounded normal operators can be extracted. The same text also mentions that the "Radon-Nikodym Theorem from Measure Theory and the Poisson formula for bounded harmonic functions in an open circle are special cases of the spectral theorem".

The proof we will give is based off of the paper [14], though this section is an agglomeration of material from that paper, [21, Section 5.2, p. 304-325] and [45]. Let us, now, proceed with the development of the spectral theory of bounded operators. The spectral theorem asserts that two given quantities are equal to each other under some conditions[6]. We need to spend some time making sure that those quantities are well-defined, even before we can prove that they are equal. We will begin doing this with the following bit of terminology (and notation).

Definition 4.3.1

Let (X, τ) be a topological space. The set of all Borel subsets of X is denoted by $\mathrm{Bor}(X)$.

There is a rather natural $C^\star$-algebra involving the Borel subsets of a topological space which arises in this setting. The following proposition aims to set up the structure around this $C^\star$-algebra.

Proposition 4.3.2

Let (X, τ) be a topological space and let $B(X)$ denote the set of all bounded Borel[a] functions $f : X \to \mathbb{C}$. Then, the following maps are well-defined:

$$\|\cdot\| : B(X) \to \mathbb{R}\ ,\ f \mapsto \|f\| := \sup_{x\in X} |f(x)|$$

$$\star : B(X) \to B(X)\ ,\ f \mapsto f^\star := \overline{f}$$

Then, $B(X)$ is a commutative $C^\star$-algebra. Moreover, the simple functions are dense in $B(X)$.

[a] These functions are measurable with respect to the Borel σ-algebras on both the domain and co-domain.

Proof. Most of the proof is just a routine verification. The fact that $\|\cdot\|$ is well-defined is just a consequence of us considering bounded functions. The fact that $\star$ is well-defined is just a consequence of conjugation being continuous (and, therefore, measurable). The completeness of $B(X)$ is a consequence of the fact that the uniform limit of bounded Borel functions is a bounded Borel function. The density of the simple functions in $B(X)$ can be shown by recourse to the proof of a similar (standard) result from Measure Theory (see [36, Theorem 1.26, p.16]) so we are done. □

Let H be a $\mathbb{C}$-Hilbert space and let (X, τ) be a topological space. If $T \in \mathcal{B}(H)$ is a normal operator and $f \in B(X)$, then we certainly would like to have a definition of $f(T)$. The strategy for this is fairly simple:

1. Approximate a bounded Borel function f uniquely by a continuous function g.
2. T generates an abelian von Neumann algebra V_T and this is isometrically $\star$-isomorphic to $\mathcal{C}(\widehat{V_T})$. Since $f \circ \widehat{T}$ is a bounded Borel function on $\widehat{V_T}$, one can "approximate" it by a continuous function g. But now, $f(T)$ can be defined as the unique operator such that $\widehat{f(T)} = g$.

Of course, this is all very vague; extra conditions need to be placed on X in order for this to make sense but the idea should be fairly clear. What follows next is a detailed investigation which attempts to make both steps above precise. We begin with the following theorem.

the first article in [28].

[4] See [29, p.100].

[5] This is Theorem 40.2, located on page 257.

[6] See Theorem 4.3.10 for what we mean.

Theorem 4.3.3

Let (X, τ) be a Stonian space[a]. The following statements hold:

1. If $B \in \mathrm{Bor}(X)$, then there exists a unique compact & open set $K \subseteq X$ so that $B \Delta K$ is meagre in X.
2. For each bounded Borel function $g : X \to \mathbb{C}$, there exists a unique continuous function $f : X \to \mathbb{C}$ such that $f = g$ except on a meagre set.
3. Let $U \subseteq X$ be a dense and open set. Let $f : U \to \mathbb{C}$ be a bounded continuous function. Then, there is unique continuous function $h : X \to \mathbb{C}$ such that $h|_U = f$.
4. Define a map:
$$\Phi : B(X) \to \mathcal{C}(X) \ , \ g \mapsto \Phi(g) := f$$
where f is the unique continuous function from (2). Φ is a surjective $\star$-homomorphism.

[a] See Definition B.1.9. Also, see Definition B.1.1 before reading the statements that comprise this theorem.

Proof. The proof will be rather long but the ideas are straightforward[a]. We will prove each statement one-by-one.

1. Let us show uniqueness first. Let $K_1, K_2 \subseteq X$ be compact and open sets such that $B\Delta K_1$ and $B \Delta K_2$ are meagre. By Proposition B.1.3, $K_1 \setminus K_2$ is meagre; here, we use the fact that $K_1 \setminus K_2 = (K_1 \setminus K_2)\Delta(B \setminus B)$. Similarly, $K_2 \setminus K_1$ is meagre so their union is meagre. But that means that $K_1 \Delta K_2$ is meagre. Since K_1, K_2 are compact subsets of X, they are closed subsets too. This implies that $K_1 \Delta K_2$ is open. If it is non-empty, then it cannot be meagre since (X, τ) is a Baire space. It follows that $K_1 \Delta K_2 = \varnothing$ and this means that $K_1 = K_2$.

   It remains to show existence. Let P denote the property given in the statement; that is, a set $B \in \mathrm{Bor}(X)$ has the property P iff there exists a compact & open set $K \subseteq X$ such that $B \Delta K$ is of the first category in X. Let us define the following set:
$$\mathcal{S} := \{B \in \mathrm{Bor}(X) : B \text{ satisfies } P\}$$
   If we can show that $\mathcal{S}$ is a σ-algebra containing all open sets, then we will be done because this will imply that $\mathrm{Bor}(X) \subseteq \mathcal{S}$. Let us prove this in the following four statements:
   (a) If $U \subseteq X$ is an open set, then $\overline{U}$ is open because (X, τ) is a Stonian space. On the other hand, it is a closed subset of a compact Hausdorff space so it is compact. Finally, $U \Delta \overline{U} = \varnothing \cup (\overline{U} \setminus U) = \overline{U} \setminus U$. We know that $\overline{U} = \mathrm{Int}(U) \cup \partial U$[b] It follows that $\overline{U} \setminus U \subseteq \partial U$. Observe that ∂U is nowhere dense so we are done.
   (b) Clearly, X satisfies P because we can just choose $K := X$. Similarly, $\varnothing$ satisfies P because we can just choose $K := \varnothing$. Therefore, $X, \varnothing \in \mathcal{S}$.
   (c) Let $B \in \mathcal{S}$. Then, there exists a compact and open set K such that $B\Delta K$ is of the first category. Since $X \Delta X$ is of the first category, it follows by Proposition B.1.3 that $(X \setminus B)\Delta(X \setminus K)$ is of the first category. But $X \setminus K$ is compact and open so it follows that $X \setminus B \in \mathcal{S}$.
   (d) Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{S}$ and let B denote the countable union of this sequence. Then, there exists a corresponding sequence of compact and open sets $(K_n)_{n\in\mathbb{N}}$ such that $B_n \Delta K_n$ is of the first category for each $n \in \mathbb{N}$. It follows, by Proposition B.1.3, that $B\Delta\left(\bigcup_{n\in\mathbb{N}} K_n\right)$ is of the first category. Let $K := \bigcup_{n\in\mathbb{N}} K_n$. Since each K_n is open, it follows that K is open. Let $C := \overline{K}$. Observe that C is closed and open because (X, τ) is Stonian. Therefore, C is a compact set. We claim that $B\Delta C$ is meagre. Indeed, we know that $B \Delta K$ is meagre and that $\varnothing\Delta(C \setminus K)$ is meagre. Therefore, $B\Delta C$ is meagre and we are done.

   It follows that every Borel subset of X satisfies the property P, as was desired.
2. Let us deal with uniqueness first; let $f_1, f_2 \in \mathcal{C}(X)$ such that $f_1 = g$ except on a meagre set U_1 and $f_2 = g$ except on a meagre set U_2. Then, $f_1 = f_2$ except on, at most, $U_1 \cup U_2$ and this is certainly meagre. Since (X, τ) is a compact Hausdorff space, it is a Baire space[c]. Let $A = \{x \in X : f_1(x) \neq f_2(x)\}$; this is a meagre set and it is open. If $A \neq \varnothing$, then Proposition B.1.2 tells us that A is non-meagre and that is impossible. Therefore, $A = \varnothing$ and $f_1 = f_2$ everywhere.

   Let us now prove existence; the notation we use here is independent of the notation used for the uniqueness proof. Let $I \subseteq B(X)$ be the set of all bounded Borel functions which vanish except on a set of the first category in X. A standard argument by sequences will show that I is a closed ideal. It follows that $B(X)/I$ is a $\mathbb{C}$-Banach algebra by the usual quotient norm. Define the map:
$$\Phi : \mathcal{C}(X) \to B(X)/I \ , \ f \mapsto \Phi(f) := f + I$$
   A computation shows that Φ is an algebra-homomorphism. To show that Φ is bounded, we have that:
$$\forall f \in \mathcal{C}(X) : \|\Phi(f)\| = \|f + I\| := \inf_{h\in I} \|f + h\| \leq \|f + 0\| = \|f\|.$$
   We shall show that Φ is injective; let $f_1, f_2 \in \mathcal{C}(X)$ such that $f_1 + I = f_2 + I$. Then, $f_1 - f_2 \in I$; $f_1 - f_2$ is a bounded Borel function which vanishes everywhere except on a set of first category. It follows, by statement (1), that $f_1 = f_2$ everywhere and that proves injectivity. It follows that $\Phi(\mathcal{C}(X)) = D$ is a closed subalgebra of $B(X)/I$. We should point out that showing that $D = B(X)/I$ is enough to prove the given result.

   After all, if $g \in B(X)$, then $g + I \in D$ so there is a function $f \in \mathcal{C}(X)$ such that $\Phi(f) = g + I$. But this means that $f = g$ except on a meagre set and that is what we wanted to show. Let $B \subseteq X$ be a Borel set. By statement (1) of this theorem, we know that there is a unique compact and open set $K \subseteq X$ such that $B \Delta K$ is meagre. Note that the function 1_B and 1_K differ only on a meagre set. Also, note that 1_K is continuous. It follows that $1_K + I \in D$ but this coset certainly contains 1_B. It follows that every coset containing a simple function belongs to D. Since the simple functions are dense in $B(X)$ and D is closed, it follows that $D = B(X)/I$.

3. Define a function:

$$g : X \to \mathbb{C} \ , \ x \mapsto g(x) := \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{if } x \in X \setminus U \end{cases}$$

Clearly, g is a bounded Borel function. There is a continuous function $h : X \to \mathbb{C}$ such that $h = g$ everywhere except on a meagre set so $h = f$ everywhere except on a meagre set[d]. If $h \neq f$ for some $x \in U$, then $h \neq f$ on some open set around x and this contradicts the fact that h & f differ on, at most, a meagre set.

4. Every $f \in \mathcal{C}(X)$ is bounded and measurable, so it is clear that Φ is surjective. Let $f, g \in B(X)$ and let $\alpha \in \mathbb{C}$. Then, there exist continuous functions f_1, g_1 such that $f_1 = f$ and $g_1 = g$ except on meagre sets. It follows that $\alpha f + g = \alpha f_1 + g_1$ except on a meagre set. Since $\alpha f_1 + g_1$ is a continuous function, it follows that:

$$\Phi(\alpha f + g) = \alpha f_1 + g_1 = \alpha \Phi(f) + \Phi(g)$$

which proves linearity. Let $f, g \in B(X)$; there exist continuous functions $f_1, g_1 \in \mathcal{C}(X)$ such that $f_1 = f$ and $g_1 = g$ except on meagre sets. It follows that $fg = f_1 g_1$ except on a meagre set. By uniqueness, it follows that:

$$\Phi(fg) = f_1 g_1 = \Phi(f)\Phi(g)$$

Finally, let $f \in B(X)$. Then, there is a continuous function $g \in \mathcal{C}(X)$ such that $f = g$ except on a meagre set. Therefore, $\overline{f} = \overline{g}$ except on a meagre set. Note that $\overline{g}$ is continuous and $\overline{f} \in B(X)$ so we have that:

$$\Phi(\overline{f}) = \overline{g} = \Phi(f)^\star$$

which tells us that Φ is a $\star$-homomorphism. □

[a]Statements (1) and (2) were taken from Lemma 4.1 and Theorem 4.2 in [14]. Statement (3) was taken from Corollary 5.2.11 in [21].
[b]∂U is the boundary of U. Showing this equality is a matter of definition-pushing so we omit an explicit proof.
[c]See Theorem B.1.5.
[d]If K is the meagre set where h and g are (possibly) not the same, then $K \cup (X \setminus U)$ is meagre because $X \setminus U$ is nowhere dense.

The theorem above paves the way for us to define a "bounded Borel functional calculus". Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be a normal operator. Let us note two quick facts in preparation for the next definition.

1. Let V_T be the von Neumann algebra generated by T. If p is a complex polynomial in two variables z and $\overline{z}$, then the corresponding operator polynomial $p(T, T^\star)$ is in V_T. In particular, V_T is the weak operator closure of the set of all such polynomials and it is abelian.
2. Since V_T is an abelian von Neumann algebra, we know that it is isometrically isomorphic to $\mathcal{C}(\widehat{V}_T)$, with $\widehat{V}_T$ being a Stonian space[7]. Let us define:

$$\Lambda := \widehat{T}(\widehat{V}_T),$$

where $\widehat{T}$ is the Gelfand transform of T. By statement (2) of Theorem 2.3.14, it follows that $\widehat{T}(\widehat{V}_T) = \sigma_{V_T}(T)$. But now, we also note that, by spectral permanence (Proposition 3.2.7), it is the case that $\sigma_{\mathcal{B}(H)}(T) = \sigma_{V_T}(T)$ so Λ is the spectrum of T in $\mathcal{B}(H)$.

With the content of these two facts in mind, we can proceed with the following definition. To make things easy and less, let us just make use of the notation above.

Definition 4.3.4

Let $f \in B(\Lambda)$ and note that $f \circ \widehat{T}$ is a bounded Borel function[a] on $\widehat{V}_T$. Let $g \in \mathcal{C}(\widehat{V}_T)$ be the unique continuous function such that $f \circ \widehat{T} = g$ except on a meagre set[b]. We define $f(T) \in \mathcal{B}(H)$ to be the unique operator such that $\widehat{f(T)} = g$.

[a]This is just a consequence of the fact that the composition of bounded Borel functions is a bounded Borel function. Also, see the two paragraphs just before this definition for the definition of Λ.
[b]Such a function does exist, by virtue of statement (2) of Theorem 4.3.3 and the fact that $\widehat{V}_T$ is a Stonian space.

The next proposition marks the end of our little adventure in trying to define a "bounded Borel functional calculus".

Proposition 4.3.5. Bounded Borel Functional Calculus

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be a normal operator. Define the map:

$$\tau : B(\Lambda) \mapsto \mathcal{B}(H) \ , \ f \mapsto \tau(f) := f(T)$$

This is a $\star$-homomorphism[a]. If V_T is the abelian von Neumann algebra generated by T, then $\tau(B(\Lambda)) \subseteq V_T$.

[a]There is a more extensive version of this result which tells us that the bounded Borel functional calculus satisfies many, many more properties. We will not be needing that at all but the reader can find it presented in [5, Theorem 5.6.5, p.267].

[7]This follows from Theorem 3.5.1 and Theorem 4.2.10.

Proof. Showing that this is a $\star$-homomorphism is quite simple but it is a little bit annoying because of the details we need to fill in. So, we will only show linearity and note that the argument for showing the other properties is the same. Let $f, g \in B(\Lambda)$ and let $\alpha \in \mathbb{C}$. Let f_1, g_1 be the unique continuous functions such that $f \circ \widehat{T} = f_1$ and $g \circ \widehat{T} = g_1$ except on meagre sets. Now, we find that $\alpha f_1 + g_1$ is a continuous function that differs from $\alpha f + g$ except on a meagre set. In particular:

$$(\alpha f + g)(T) = \mathcal{G}^{-1}(\alpha f_1 + g_1) = \alpha \cdot \mathcal{G}^{-1}(f) + \mathcal{G}^{-1}(g) = \alpha f(T) + g(T)$$

where $\mathcal{G}$ is the Gelfand representation. Let us show the second part of the claim. In this case, the Gelfand representation is defined on V_T so it follows immediately that $f(T) \in V_T$. □

In other words, the spectral theorem serves as a different way of computing the operator $f(T)$ as defined in Definition 4.3.4. This is an important conceptual point because the it is usually used as a means to **define** a (bounded) Borel functional calculus so our approach clearly deviates from the norm. The next step is to focus on the right-hand side of the equality given in the spectral theorem. We will need to to define the notion of a spectral integral. The treatment here mainly follows [14] and [45, Section 5, p.15-19]; we will begin by giving the definition of a spectral measure.

Definition 4.3.6

Let H be a $\mathbb{C}$-Hilbert space and let (X, Σ) be a measurable space. Let $\mathcal{P}$ be the set of all orthogonal projections in $\mathcal{B}(H)$. A projection-valued measure is a map $\mu : \Sigma \to \mathcal{P}$ such that:

1. $\mu(X) = \mathrm{Id}_H$ and $\mu(\varnothing) = 0$.
2. If $(E_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in Σ, then:

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$$

   where the sum on the right converges in the strong operator topology.

We also call these spectral measures.

The reader has certainly encountered "measures" which do not take on only non-negative values; they have encountered signed measures and even complex measures. Therefore, the structure of the definition above should not be so surprising. On the other hand, we will show that spectral measures are quite similar to the usual "positive" measures.

Proposition 4.3.7

Let H be a $\mathbb{C}$-Hilbert space and let (X, Σ) be a measurable space. Let $\mu : \Sigma \to \mathcal{B}(H)$ be a spectral measure. Then, the following statements hold:

1. $\forall E_1, E_2 \in \Sigma : E_1 \cap E_2 = \varnothing \Rightarrow \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$
2. $\forall E_1, E_2 \in \Sigma : E_1 \subseteq E_2 \Rightarrow \mu(E_1) \prec \mu(E_2)$. (Monotonicity)
3. $\forall E_1, E_2 \in \Sigma : E_1 \subseteq E_2 \Rightarrow \mu(E_1)\mu(E_2) = \mu(E_1)$
4. $\forall E_1, E_2 \in \Sigma : \mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)$
5. The set $\{E \in \Sigma : \mu(E) = 0\}$ is a σ-algebra.
6. For every $x, y \in H$, the map:

$$\mu_{x,y} : \Sigma \to \mathbb{C}\ ,\ E \mapsto \mu_{x,y}(E) := \langle \mu(E)x, y\rangle$$

   is a complex measure.
7. If X is a compact subset of $\mathbb{C}$ and $\Sigma = \mathrm{Bor}(X)$, then the spectral measure μ is regular in the sense that:

$$\forall M_0 \in \Sigma : \mu(M_0) = \sup\{\mu(M) : M \subseteq M_0\ \wedge\ M \text{ is compact.}\}$$

Proof. Let us prove each statement one-by-one.

1. Let $E_1, E_2 \in \Sigma$ be disjoint measurable sets. Let $E_n := \varnothing$ for $n \geq 3$. Then:

$$\forall x \in H : \mu(E_1 \cup E_2)x = \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right)x = \sum_{n\in\mathbb{N}} \mu(E_n)x = \mu(E_1)x + \mu(E_2)x$$

   which gives the desired result.
2. Let $E_1, E_2 \in \Sigma$ be measurable sets such that $E_1 \subseteq E_2$. Then, $E_2 \setminus E_1$ and E_1 are disjoint sets whose union is E_2. Therefore, we can write $\mu(E_2) = \mu(E_2 \setminus E_1) + \mu(E_1)$. This implies that $\mu(E_1) \prec \mu(E_2)$.
3. Let $E_1, E_2 \in \Sigma$ with $E_1 \subseteq E_2$. Then, $\mu(E_1) \prec \mu(E_2)$. By statements (2) and (3) of Proposition 4.1.12, we have that $\mu(E_1)\mu(E_2) = \mu(E_2)\mu(E_1) = \mu(E_1)$.
4. Let $E_1, E_2 \in \Sigma$. Then, $E_1 \cap E_2$, $E_1 \setminus E_2$ and $E_2 \setminus E_1$ are disjoint sets whose union is $E_1 \cup E_2$. Note that $E_1 \setminus E_2 = (E_1 \cup E_2) \setminus E_1$ and $E_2 \setminus E_1 = (E_1 \cup E_2) \setminus E_2$. We know that:

$$\mu(E_1 \cup E_2) = \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1) + \mu(E_1 \cap E_2) = \mu(E_1 \cup E_2) - \mu(E_1) + \mu(E_1 \cup E_2) - \mu(E_2) + \mu(E_1 \cap E_2)$$

   It follows that $\mu(E_1 \cap E_2) + \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$. We compose both sides with $\mu(E_1)$:

$$\mu(E_1 \cap E_2)\mu(E_1) + \mu(E_1 \cup E_2)\mu(E_1) = \mu(E_1)\mu(E_1) + \mu(E_2)\mu(E_1) = \mu(E_1) + \mu(E_2)\mu(E_1)$$

where we have used the fact that $\mu(E_1)$ is an orthogonal projection. On the other hand, we can also compose both sides of the original equation above with $\mu(E_2)$ so that:

$$\mu(E_2)\mu(E_1 \cap E_2) + \mu(E_2)\mu(E_1 \cup E_2) = \mu(E_2)\mu(E_1) + \mu(E_2)\mu(E_2) = \mu(E_2)\mu(E_1) + \mu(E_2)$$

Since $E_1 \cap E_2 \subseteq E_2$, it follows that $\mu(E_2)\mu(E_1 \cap E_2) = \mu(E_1 \cap E_2)$. On the other hand, we know that $E_2 \subseteq E_1 \cup E_2$ so $\mu(E_2)\mu(E_1 \cup E_2) = \mu(E_2)$. Therefore, we obtain:

$$\mu(E_1 \cap E_2) + \mu(E_2) = \mu(E_2)\mu(E_1) + \mu(E_2)$$

which gives the desired result.

5. This is just a routine computation using the definition of a σ-algebra so we omit it.
6. This is just a routine computation using the definition of σ-additivity for spectral measures.
7. This can be reduced to verifying whether the given statement is true for complex measures. Since the latter statement is standard, we will simply show how the reduction can be made. Let $M_0 \in \Sigma$ and let c be the supremum on the right-hand side as given in the statement of the result. Then, for any compact set M which is a subset of M_0, we have that $\mu(M) \prec \mu(M_0)$. It follows that $c \prec \mu(M_0)$. Now, we want to show that $\mu(M_0) \prec c$. By the regularity of complex-valued measures, we have that:

$$\forall x \in H : \mu_{x,x}(M_0) = \langle \mu(M_0)x, x\rangle = \sup_{M \subseteq M_0 \wedge M \text{ is compact}} \langle \mu(M)x, x\rangle$$

Since $\mu(M) \prec c$ for every compact $M \subseteq M_0$, it follows that $\langle \mu(M)x, x\rangle \leq \langle cx, x\rangle$ for every $x \in H$. This implies that $\mu(M_0) \prec c$ and we are done. □

The spectral integral of a bounded Borel function is defined using the following proposition.

Proposition 4.3.8

Let H be a $\mathbb{C}$-Hilbert space and let $X \subseteq \mathbb{C}$ be a non-empty compact set. Let $\mu : \mathrm{Bor}(X) \to \mathcal{B}(H)$ be a spectral measure. For each $x, y \in H$, consider the complex measure:

$$\mu_{x,y} : \mathrm{Bor}(X) \to \mathbb{C} \ , \ E \mapsto \mu_{x,y}(E) := \langle \mu(E)x, y\rangle$$

Then, for each $f \in B(X)$, there exists a unique operator $T_f \in \mathcal{B}(H)$ such that $\langle T_f(x), y\rangle = \int_X f \, d\mu_{x,y}$. The bounded linear map T_f is the spectral integral of f with respect to μ and is denoted by $\int_X f \, d\mu$. Moreover, the following map:

$$\pi : B(X) \to \mathcal{B}(H) \ , \ f \mapsto \pi(f) := T_f =: \int_X f \, d\mu$$

is an isometric unital $\star$-homomorphism which satisfies:

$$\forall f \in B(X) : \forall x \in H : \|\pi(f)x\|^2 = \int_X |f|^2 \, d\mu_{x,x}$$

Finally, an operator $S \in \mathcal{B}(H)$ commutes with every $\mu(M)$ ($M \in \mathrm{Bor}(X)$) iff S commutes with every $\pi(f)$.

Proof. Let $f \in B(X)$ be fixed and let x be fixed. Then, the following map:

$$\phi : H \times H \to \mathbb{C} \ , \ (x, y) \mapsto \phi(x, y) := \int_X f \, d\mu_{x,y}$$

is well-defined and is a bounded conjugate-linear functional. By Corollary B.2.3, we have that:

$$\exists! T_f \in \mathcal{B}(H) : \forall x, y \in H : \langle T_f(x), y\rangle = \phi(x, y) = \int_X f \, d\mu_{x,y},$$

which gives the desired existence result. Let us prove that the map π is an isometric $\star$-homomorphism. Let $S(X)$ be the set of all simple functions $f : X \to \mathbb{C}$; this is a (dense) $\star$-subalgebra of $B(X)$. It can be shown, by some tedious computations, that $\pi|_{S(X)}$ is an isometric unital $\star$-homomorphism. By Lemma 3.2.3, this has an extension to all of $B(X)$; the uniqueness of the extension tells us that it must be π. Now, we can verify that π is a $\star$-homomorphism by using its continuity and the fact that bounded Borel functions can be approximated by simple functions.

Finally, let $S \in \mathcal{B}(H)$. Suppose that S commutes with every $\pi(f)$. Then, S commutes with every $\pi(1_M)$, where $M \in \mathrm{Bor}(X)$. But this just means that S commutes with $\mu(M)$ for every $M \in \mathrm{Bor}(X)$. Now, assume that S commutes with every $\mu(M)$ for every $M \in \mathrm{Bor}(X)$. Then, S commutes with every $\pi(f)$ when $f \in B(X)$ is a simple function. It follows, by continuity, that S commutes with $\pi(f)$ for every $f \in B(X)$. □

There is one more result that we should certainly prove before we give the proof of the spectral theorem. Technically, this result is not needed for the proof of the spectral theorem but it is certainly of independent interest.

Proposition 4.3.9

Let H be a $\mathbb{C}$-Hilbert space and let $\mathcal{P}$ be the set of all orthogonal projections on H. Let $X \subseteq \mathbb{C}$ be compact and let $\mu : \mathrm{Bor}(X) \to \mathcal{P}$ be a spectral measure such that $\mu(\mathrm{Bor}(X)) \subseteq V$, where $V \subseteq \mathcal{B}(H)$ is a von Neumann algebra. Then, for every $f \in B(X)$, the spectral integral $\int_X f \, d\mu$ lies in V.

Proof. Let $M \in \text{Bor}(X)$. Then, $1_M \in B(X)$ and:

$$\forall x, y \in H : \left\langle \left(\int_X 1_M \, d\mu \right) x, y \right\rangle = \int_X 1_M \, d\mu_{x,y} = \mu_{x,y}(M) = \langle \mu(M)x, y \rangle$$

It follows that $\int_X 1_M \, d\mu = \mu(M)$ and this is certainly in V. By linearity, it follows that $\int_X f \, d\mu \in V$ when $f \in B(X)$ is a simple function. Now, let $f \in B(X)$ be arbitrary. There is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions in $B(X)$ which converge uniformly to f. If we can show that $T_{f_n} \to T_f$ in the WOT, then we are done because V is closed in the WOT since it is a von Neumann algebra. But showing this comes down to proving that:

$$\forall x, y \in H : \lim_{n\to\infty} \langle T_{f_n}(x), y \rangle = \lim_{n\to\infty} \int_X f_n \, d\mu_{x,y} = \langle T_f(x), y \rangle = \int_X f \, d\mu_{x,y}$$

Since $f_n \to f$ in $B(X)$, it follows that $f_n \to f$ point-wise. Since $f_n \to f$ uniformly, it follows that $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded. So, we can apply the Lebesgue Dominated Convergence Theorem to show that the equality written above is actually valid. This proves that $T_f \in V$ and we are done. □

We are ready to give the proof of the spectral theorem for bounded normal operators. It might be useful for the reader to keep two things in mind while reading the proof:

1. The bulk of the proof just involves checking if our proposed constructions work. We should mention that the argument we have used to prove countable additivity is non-trivial.
2. Once the spectral measure has been given, the main idea is to show that the two maps π (defined in Proposition 4.3.8) and τ (defined in Proposition 4.3.5) are equal. This is done by showing that they are equal on a dense set, from which we can conclude that they are equal everywhere due to the continuity of both maps.

With all this said, let us proceed with the proof.

Theorem 4.3.10. The Spectral Theorem for Bounded Normal Operators

Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$ be normal. Then, there exists a unique spectral measure $\mu : \text{Bor}(\sigma_{\mathcal{B}(H)}(T)) \to \mathcal{B}(H)$ such that for each f in $B(\sigma_{\mathcal{B}(H)}(T))$, we have that:

$$f(T) = \int_{\sigma_{\mathcal{B}(H)}(T)} f \, d\mu$$

Proof. Let us show uniqueness first. Let μ, ν be spectral measures for which the above holds. Then:

$$\forall f \in B(\sigma_{\mathcal{B}(H)}(T)) : \forall x, y \in H : \int_{\sigma_{\mathcal{B}(H)}(T)} f \, d\mu_{x,y} = \int_{\sigma_{\mathcal{B}(H)}(T)} f \, d\nu_{x,y}$$

Let $M \subseteq \sigma_{\mathcal{B}(H)}(T)$ be a Borel set. Then, we have that:

$$\forall x, y \in H : \mu_{x,y}(M) = \nu_{x,y}(M)$$

which further implies that:

$$\forall x, y \in H : \langle \mu(M)x, y \rangle = \langle \nu(M)x, y \rangle$$

But this implies that $\mu(M) = \nu(M)$. Since M was arbitrary, it follows that $\mu = \nu$. Before we show existence, let us just set up some notation that will indicate where the proof is going. Let V_T be the abelian von Neumann algebra generated by T & let $\mathcal{G} : V_T \to \mathcal{C}(\widehat{V_T})$ be the Gelfand representation. We know that $\widehat{V_T}$ is a Stonian space. If $\mu : \text{Bor}(\sigma_{\mathcal{B}(H)}(T)) \to \mathcal{B}(H)$ is a spectral measure, then let us define two maps:

$$\pi_1 : B(\sigma_{\mathcal{B}(H)}(T)) \to \mathcal{B}(H) \ , \ f \mapsto \pi_1(f) := \int_{\sigma_{\mathcal{B}(H)}(T)} f \, d\mu$$

$$\pi_2 : B(\sigma_{\mathcal{B}(H)}(T)) \to \mathcal{B}(H) \ , \ f \mapsto \pi_2(f) := f(T)$$

We will study these two maps in just a moment; indeed, the spectral measure that we construct will be used particularly for π_1. Before we discuss all of that, let us just construct the desired spectral measure first. Let $M \subseteq \sigma_{\mathcal{B}(H)}(T)$ be a (Borel) measurable set. Then, $\widehat{T}^{-1}(M)$ is Borel measurable. By statement (1) of Theorem 4.3.3, there is a unique compact & open set $K \subseteq \widehat{V_T}$ such that $\widehat{T}^{-1}(M) \Delta K$ is meagre in $\widehat{V_T}$. Define $\mu(M)$ to be the operator in V_T such that $\widehat{\mu(M)} = 1_K$[a]. Let us show that this is a spectral measure in 3 steps; we will freely make use of the facts in Appendix B.1 without referencing all of them directly.

1. Let us show, first of all, that $\mu(M)$ is an orthogonal projection. This follows immediately from the fact that its Gelfand transform, by definition, is the characteristic function of a clopen set (see Proposition 4.1.9).
2. Let $M = \sigma_{\mathcal{B}(H)}(T)$ and let us compute $\mu(M)$. Let $K_1 := \widehat{V_T}$; this is compact and open. Also, note that $\varnothing = \widehat{V_T} \Delta \widehat{V_T} = \widehat{T}^{-1}(M) \Delta K_1$. By uniqueness, it follows that $\widehat{\mu(M)} = 1_{\widehat{V_T}}$. Since the Gelfand representation is unital, it follows that $\mu(M) = \text{Id}_H$. Now, let $K_2 = \varnothing$; this is compact and open. Note that $\varnothing = \varnothing \Delta \varnothing = \widehat{T}^{-1}(\varnothing) \Delta K_2$. By uniqueness, it follows that $\widehat{\mu(\varnothing)} = 1_\varnothing$. That is, this function is zero everywhere so it follows that $\mu(\varnothing) = 0$.
3. Let us show countable additivity[b]. Let $(M_n)_{n\in\mathbb{N}}$ be a disjoint sequence of Borel sets and let M denote their union. For each $n \in \mathbb{N}$, there exists a unique compact & open set $K_n \subseteq \widehat{V_T}$ such that $\widehat{T}^{-1}(M_n) \Delta K_n$ is meagre[c]. Let us prove that this sequence $(K_n)_{n\in\mathbb{N}}$ is pairwise disjoint. The argument proceeds in a few steps:
   (a) Let $n, l \in \mathbb{N}$ be fixed (but arbitrary) and let $X = \sigma_{\mathcal{B}(H)}(T)$ (just for this part of the proof). We know that

$\widehat{V}_T \Delta \widehat{V}_T$ is meagre so $(\widehat{V}_T \setminus \widehat{T}^{-1}(M_l))\Delta(\widehat{V}_T \setminus K_l)$ is meagre. But this is just $\widehat{T}^{-1}(X \setminus M_l)\Delta(\widehat{V}_T \setminus K_l)$ so this set is also meagre. Similarly, $\widehat{T}^{-1}(X \setminus M_n)\Delta(\widehat{V}_T \setminus K_n)$ is also meagre.

(b) It follows that:
$\left(\widehat{T}^{-1}(X \setminus M_l) \cup \widehat{T}^{-1}(X \setminus M_n)\right) \Delta((\widehat{V}_T\setminus K_l)\cup(\widehat{V}_T\setminus K_n)) = \widehat{T}^{-1}((X\setminus M_l)\cup(X\setminus M_n))\Delta((\widehat{V}_T\setminus K_l)\cup(\widehat{V}_T\setminus K_n))$
is meagre. Using the fact that $\widehat{V}_T \Delta \widehat{V}_T$ is meagre, we have that $\widehat{T}^{-1}(M_n \cap M_l)\Delta(K_n \cap K_l)$ is meagre.

(c) Note, now, that $\widehat{T}^{-1}(M_n \cap M_l)$ is empty because $M_n \cap M_l = \varnothing$. This implies that $K_n \cap K_l$ is meagre. But notice that this is an open set and since non-empty open sets are nonmeagre in the Stonian space $\widehat{V}_T$, it follows that $K_n \cap K_l$ is empty and that proves what we wanted.

Observe that for each $n \in \mathbb{N}$, the set $\left(\bigcup_{k=1}^n \widehat{T}^{-1}(M_k)\right) \Delta \left(\bigcup_{k=1}^n K_k\right) = \widehat{T}^{-1}\left(\bigcup_{k=1}^n M_k\right) \Delta \left(\bigcup_{k=1}^n K_k\right)$ is meagre[d]. Define the sequence $S_n := \sum_{k=1}^n \mu(M_k)$ for each $n \in \mathbb{N}$. Applying the Gelfand representation to this, we get:

$$\forall n \in \mathbb{N} : \widehat{S_n} = \sum_{k=1}^n \widehat{\mu(M_k)} = \sum_{k=1}^n 1_{K_k} = 1_{\bigcup_{k=1}^n K_k}$$

where the last equality holds because $(K_n)_{n\in\mathbb{N}}$ is a disjoint sequence. Since $\bigcup_{k=1}^n K_k$ is compact and open for every $n \in \mathbb{N}$, it follows by the uniqueness part of statement (1) of Theorem 4.3.3 that $1_{\bigcup_{k=1}^n K_k} = \widehat{\mu\left(\bigcup_{k=1}^n M_k\right)}$ for each $n \in \mathbb{N}$. It follows, then, that $\mu\left(\bigcup_{k=1}^n M_k\right) = S_n$ for each $n \in \mathbb{N}$; that is, finite additivity holds. Let us observe that two facts about the sequence $(S_n)_{n\in\mathbb{N}}$ hold as follows:

(a) The sequence is increasing in the order on $\mathcal{B}(H)$; this is a consequence of the fact that μ is a spectral measure and that orthogonal projections are positive operators[e].

(b) The sequence is bounded because $S_n = \mu\left(\bigcup_{k=1}^n M_k\right) \prec \mu(\sigma_{\mathcal{B}(H)}(T)) = \mathrm{Id}_H$ for every $n \in \mathbb{N}$.

By Corollary 4.2.9, it follows that S_n converges to an orthogonal projection S in the SOT, with S being the least upper bound of the set $\{S_n : n \in \mathbb{N}\}$ so $S \prec \mu(M)$. Since S is an orthogonal projection, it follows by Proposition 4.1.9 that $\widehat{S} = 1_K$ with K being a clopen subset of $\widehat{V}_T$. Let K' be the compact & open set such that $\widehat{\mu(M)} = 1_{K'}$ and $\widehat{T}^{-1}(M)\Delta K'$ is meagre. Since the Gelfand representation is order-preserving, it follows that $1_K \leq 1_{K'}$. In particular, it means that $K \subseteq K'$. If we can show that $K = K'$, then $\widehat{S} = \widehat{\mu(M)}$ and this will imply that $S = \mu(M)$. Let us define the following set:

$$K'' := \overline{\bigcup_{n\in\mathbb{N}} K_n}$$

where $(K_n)_{n\in\mathbb{N}}$ is the sequence of compact and open sets found at the beginning of this argument. Let us show that $K = K' = K''$. Since $\partial K''$ is meagre, it follows that $\varnothing\Delta\partial K''$ is meagre. Now, $\widehat{T}^{-1}(M)\Delta\left(\bigcup_{n\in\mathbb{N}} K_n\right)$ is meagre so it follows that $\widehat{T}^{-1}(M)\Delta K''$ is meagre. This implies that $K'' \setminus K'$ is meagre but note that this set is certainly open (because K'' is the closure of a union of open sets so it must be open). It follows that $K'' \setminus K'$ is empty and similarly, we can show that $K' \setminus K''$ is empty. As a consequence, $K' = K''$.

If we show that each $K_n \subseteq K$, then $\bigcup_{n\in\mathbb{N}} K_n \subseteq K$ so $K'' \subseteq K$ by the definition of the closure and the fact that K is closed. This will imply that $K'' \subseteq K \subseteq K' = K''$ so $K = K' = K''$. To show that each $K_n \subseteq K$, it suffices to show that $1_{K_n} \leq 1_K$ for each $n \in \mathbb{N}$. Note that $S_n \prec S$ for each $n \in \mathbb{N}$. Therefore:

$$\forall n \in \mathbb{N} : 1_{K_n} \leq \sum_{k=1}^n 1_{K_k} = \widehat{S_n} \leq \widehat{S} = 1_K$$

and this completes the proof of countable additivity.

All that remains is for us to show the given equality; this is clearly equivalent to showing that $\pi_1 = \pi_2$. If we can show that they are equal on a dense subset of $B(\sigma_{\mathcal{B}(H)}(T))$, then we can use continuity to conclude that they are equal everywhere and that will be enough to prove the desired result. By linearity, it is sufficient to prove that $\pi_1 = \pi_2$ on the characteristic functions of Borel sets. Let $M \in \mathrm{Bor}(\sigma_{\mathcal{B}(H)}(T))$. Then, it is the case that $\int_{\sigma_{\mathcal{B}(H)}(T)} 1_M \, d\mu = \mu(M)$.

Let us prove that $\widehat{\mu(M)}$ differs from $1_M \circ \widehat{T}$ only on a meagre set. By definition of the given spectral measure, there is a unique compact & open $K \subseteq \widehat{V}_T$ such that $\widehat{T}^{-1}(M)\Delta K$ is meagre in $\widehat{V}_T$. Note that $\widehat{\mu(M)} = 1_K$ and 1_K differs from $1_{\widehat{T}^{-1}(M)} = 1_M \circ \widehat{T}$[f] on, at most, a meagre set. But this is exactly what we wanted to show and this is enough because, by Definition 4.3.4, we have that $\widehat{\mu(M)}$ differs from $1_M \circ \widehat{T}$ on a meagre set. □

[a]Indeed, such characteristic functions are continuous, as can be verified easily from the definition.

[b]For this part of the proof, the paper by Carl Pearcy and R.G. Douglas [14] says that it is an easy manipulation of sets of the first category. We were not able to formulate an argument based on that hint, so we decided to try and adapt a proof from a paper [16] by John Kelley and J.M.G. Fell. The argument we have given was **not** as simple as it was made out to be in both papers above.

[c]This follows, again, by statement (1) of Theorem 4.3.3.

[d]This follows from a slight modification of the first part of Proposition B.1.3.

[e]See Proposition 4.1.11.

[f]Proving this equality is not so difficult so we omit the details.

Now, we shall give comments about the specific result we have derived and talk a little bit about what it is capable of doing for us. Let us freely make use of the notation in Theorem 4.3.10 in the following remarks:

1. The statement of Theorem 4.3.10 is **not** the usual statement of the spectral theorem. In the usual statement as given in a standard source like Rudin [34, Theorem 12.23, p.324], the equality given is actually:
$$T = \int_{\sigma_{\mathcal{B}(H)}(T)} \lambda \, dE(\lambda)$$
where E is the notation used for the spectral measure. Once this is proved, Rudin actually **defines** $f(T) := \int_{\sigma_{\mathcal{B}(H)}(T)} f(\lambda)\, dE(\lambda)$ for some bounded Borel function f defined on $\sigma_{\mathcal{B}(H)}(T)$. Indeed, this appears to be standard.

   In our approach, this definition is a theorem because we provide a separate meaning of $f(T)$. We should also point out that Lax's text [27, Theorem 9', p.363] proves a result that is similar to what we have proved. The difference, though, is that he proves it for bounded, self-adjoint linear operators and he also assumes continuity of the function f. Moreover, he establishes the spectral theorem for bounded normal operators later in the same form given by Rudin, except that he makes use of the normal continuous functional calculus (we did not use this).
2. In both [27] and [34], the Riesz-Markov-Kakutani Representation Theorem is essential in constructing the proof. It produces the regular complex measure which is, then, used to construct the required spectral measure. We should point out that our approach did not require that result. In fact, it would be of interest for the reader to know that the aforementioned representation theorem can be proved using additional facts about Stonian spaces. For more information about this approach and related material, see [19].
3. The approach we have used here also generalizes to unbounded operators on a $\mathbb{C}$-Hilbert space. In fact, the earliest known paper which uses the ideas in this chapter to look at unbounded operators is [16]. A more modern treatment of this is available in [21, Chapter 5], though we point out that this text works under the assumption of the underlying $\mathbb{C}$-Hilbert space being separable to simplify many technical arguments.
4. The proof of the spectral theorem is based on the (non-trivial) fact that the structure space of an abelian von Neumann algebra is extremally disconnected. Admittedly, the way we introduced Stonian spaces here has been a little unmotivated. We will not attempt to motivate Stonian spaces from a historical point of view; there are many resources which attempt to do this.

   We should mention the so-called AW*-algebras[8]; these are $C^\star$-algebras which are "generalizations" of von Neumann algebras and they were first introduced by Irving Kaplansky in [23]. It can be shown that commutative $C^\star$-algebras are precisely those $C^\star$-algebras whose structure space is a Stonian space [1, Theorem 1, p.40-44].
5. Let us speak of applications. [34, Chapter 12] contains **many** applications of the spectral theorem. In particular, we should point out that [34, p.327-330] contains information about how one can characterize the eigenvalues of normal operators and compact operators by means of the spectral theorem above.

With all of this being said, we hope that the reader has enjoyed our presentation of all of this material. We will spend some time reflecting on the thesis & the value it brings to the table in the next (the last!) section.

4.4 Concluding Remarks

The time has come to drop the pretentiousness and speak to you like you're an actual human being. I want to explain some of the choices I made in writing things the way that I did without fear of being judged for it. There are two main principles that I tried to heavily abide by in my presentation of this material:

1. **Be as insightful as possible**. Admittedly, this is pretty hard to do and I found that it required me to look at different resources to see how different authors presented the material. Beyond that, I always tried to present proofs that could be motivated in a simple manner or led to some analogy that would be valuable later. This also meant that I did not use any "sledgehammers" at all; I did not prove any of the "big" results and use them to bulldoze my way through a huge chunk of the theory.

   This was (partially) my motivation for why I presented Spectral Theory in the way that I did. I wanted to present a set of ideas that would take you back to notions that you might be familiar with but haven't used for a long time in Functional Analysis.
2. **Try to make every result seem like the most natural thing in the world**. Indeed, many of the proofs in Chapters 2 and 3 were constructed precisely to make those ideas seem extremely natural. Of course, space constraints made it a bit difficult to motivate the development at **every** step (which is what I would've wanted to do) but I tried to make the theory seem extremely natural subject to those constraints wherever I could.

I hope that I have succeeded, at least somewhat, in abiding by these two principles and presenting a piece of work that can be appreciated for the connections it provides to other bits of mathematics. Certainly, the references I have used throughout suggest that there are **many** connections that this material has to other bits of mathematics; I certainly encourage you to look through that which interests you. With that, I bid you adieu :D

[8] $W^\star$-algebras are just von Neumann algebras. So, AW* means "Abstract $W^\star$-algebra"; this is based on Kaplansky's original paper where he introduces these objects.

Appendix A

Extras: The Good & The Bad

This chapter consists of results that I would've wanted to include in the main body of the thesis but that I wasn't able to, for a variety of reasons. Sometimes, it would've been because of space reasons. Other times, it might've been because those results would provide little to no insight but would still be relevant to the material discussed. Nevertheless, I still thought that some of these were important enough to deserve an appendix of their own.

Going by the reasoning above, it might be reasonable to argue that I should've included a lot more here than I actually did. This is somewhat fair but I must say that some level of arbitrariness is to be expected in decisions related to these matters. With all of that said, let us begin with the unitization theorem for $C^\star$-algebras.

Theorem A.0.1. The Unitization Theorem ($C^\star$-Algebras)

If V is a $C^\star$-algebra without identity, then it can be isometrically embedded into a unital $C^\star$-algebra V_e as an ideal satisfying $\dim(V_e/V) = 1$.

Proof. We adapt the proofs given in [10, Proposition 1.18, p.6] and [13, Proposition 6.1, p.20-22]. Let $V_e := V \times \mathbb{C}$ be equipped with its product vector space structure and define:

$$\forall x, y \in V : \forall \lambda, \mu \in \mathbb{C} : (x, \lambda) \cdot (y, \mu) := (xy + \mu x + \lambda y, \mu\lambda)$$
$$\forall x \in V : \forall \lambda \in \mathbb{C} : (x, \lambda)^\star := (x^\star, \overline{\lambda})$$
$$\forall x \in V : \forall \lambda \in \mathbb{C} : \|(x, \lambda)\|_1 := \sup_{y \in V, \|y\| \leq 1} \|xy + \lambda y\|.$$

The fact that V_e is a $\star$-algebra is fairly easy to show. Now, we want to show that V_e is a normed algebra which satisfies the $C^\star$-condition. We will do this in a few steps and the last thing we will do is to prove that V_e is complete.

1. Let $\|(x, \lambda)\|_1 = 0$. Then, for any $y \in V$ with $\|y\| \leq 1$, it is the case that $xy + \lambda y = 0$. Assume that $\lambda \neq 0$. Then, $\left(-\frac{x}{\lambda}\right) y = y$ for every $y \in V$. In other words, $-\frac{x}{\lambda}$ is a left identity for V. Now, the adjoint of this is a right identity so we can write:
$$-\frac{x}{\lambda} = \left(-\frac{x}{\lambda}\right)\left(-\frac{x}{\lambda}\right)^\star = -\frac{x^\star}{\overline{\lambda}},$$
where the first equality holds because the adjoint of $-\frac{x}{\lambda}$ is a right identity while the second equality holds because $-\frac{x}{\lambda}$ is a left identity. Since they are equal, it follows that $-\frac{x}{\lambda}$ is an identity element and it lies in V, which is a contradiction. This implies that $\lambda = 0$. But the fact that $xy + \lambda y = 0$ means that $xy = 0$ for any $y \in V$ with $\|y\| \leq 1$. This implies that $xy = 0$ for any $y \in V$. But this means that $xx^\star = 0$ so the $C^\star$-condition implies that $x = 0$. In other words, $(x, \lambda) = (0, 0)$.
2. Let us prove homogeneity. Let $x \in V$ and let $\lambda, \mu \in \mathbb{C}$. Then, we have that:
$$\|\mu(x, \lambda)\|_1 = \|(\mu x, \mu\lambda)\|_1 = \sup_{y \in V, \|y\| \leq 1} \|(\mu x)y + (\mu\lambda)y\| = \sup_{y \in V, \|y\| \leq 1} |\mu| \cdot \|xy + \lambda y\| = |\mu| \|(x, \lambda)\|_1$$
as was desired.
3. Let us now prove the (additive) triangle inequality. Let $x, y \in V$ and let $\lambda, \mu \in \mathbb{C}$ be given. Then, we have that:
$$\|(x, \lambda) + (y, \mu)\|_1 = \|(x + y, \lambda + \mu)\|_1 = \sup_{z \in V, \|z\| \leq 1} \|(x + y)z + (\lambda + \mu)z\|$$
Now, let $z \in V$ with $\|z\| \leq 1$. Observe, then, that:
$$\|(x + y)z + (\lambda + \mu)z\| \leq \|xz + \lambda z\| + \|yz + \mu z\| \leq \|(x, \lambda)\|_1 + \|(y, \mu)\|_1$$
where we have used the triangle inequality for $\|\cdot\|$. It follows that $\|(x, \lambda) + (y, \mu)\|_1 \leq \|(x, \lambda)\|_1 + \|(y, \mu)\|_1$ and we are done.
4. Let us show that $\|\cdot\|_1$ satisfies the multiplicative triangle inequality. Let $x, y \in V$ and let $\lambda, \mu \in \mathbb{C}$. Therefore:
$$\|(x, \lambda)(y, \mu)\|_1 = \|(xy + \lambda y + \mu x, \mu\lambda)\|_1$$
Let $z \in V$ with $\|z\| \leq 1$. Then, we have that:
$$\|(xy + \lambda y + \mu x)z + (\mu\lambda)z\| = \|x(yz + \mu z) + \lambda(yz + \mu z)\|.$$

Now, we assume that $\|(y,\mu)\|_1 \neq 0$. Indeed, the inequality holds trivially in the case where $\|(y,\mu)\|_1 = 0$. It follows, then, that:

$$\|x(yz+\mu z)+\lambda(yz+\mu z)\| = \|(y,\mu)\|_1 \cdot \left\| x \cdot \frac{yz+\mu z}{\|(y,\mu)\|_1} + \lambda \cdot \frac{yz+\mu z}{\|(y,\mu)\|_1} \right\|$$

Since it is true that $\|yz+\mu z\| \leq \|(y,\mu)\|_1$, it follows that:

$$\|(x,\lambda)(y,\mu)\|_1 \leq \|(x,\lambda)\|_1 \|(y,\mu)\|_1$$

as was desired.

5. Let us show that the $C^\star$-condition is satisfied. Let $x \in V$ and let $\lambda \in \mathbb{C}$. Let us, first, show that $\|(x,0)\|_1 = \|x\|$. Indeed, it is clear that $\|(x,0)\|_1 \leq \|x\|$. On the other hand, the $C^\star$-condition on V itself yields that:

$$\|x\| = \left\| x \cdot \frac{x^\star}{\|x\|} \right\| \leq \|(x,0)\|_1 \leq \|x\|$$

because $\frac{\|x^\star\|}{\|x\|} = 1$. It follows that $\|(x,0)\|_1 = \|x\|$. Now, let $c \in (0,1)$ be fixed but arbitrary. Then, there exists a $y \in V$ with $\|y\| = 1$ such that:

$$c\|(x,\lambda)\|_1 \leq \|xy+\lambda y\| = \|(xy+\lambda y, 0)\|_1$$

Therefore, we have, upon squaring both sides, the following:

$$c^2\|(x,\lambda)\|_1^2 \leq \|xy+\lambda y\|^2 = \left\|(xy+\lambda y)^\star(xy+\lambda y)\right\|$$

But now, we know that Observe $(x,\lambda)(y,0) = (xy+\lambda y, 0)$. Therefore, we have that:

$$\left\|(xy+\lambda y)^\star(xy+\lambda y)\right\| = \left\|((xy+\lambda y)^\star(xy+\lambda y),0)\right\|_1 = \left\|(xy+\lambda y,0)^\star(xy+\lambda y,0)\right\|_1$$

We can simplify the last term as follows:

$$c^2\|(x,\lambda)\|_1^2 \leq= \left\|(y,0)^\star(x,\lambda)^\star \cdot (x,\lambda)(y,0)\right\|_1 \leq \left\|(y,0)^\star\right\|_1 \cdot \left\|(x,\lambda)^\star(x,\lambda)\right\|_1 \cdot \|(y,0)\|_1$$

Since $\left\|(y,0)^\star\right\|_1 = \|y^\star\| = \|y\| = \|(y,0)\|_1$ and since $\|y\| = 1$ in this instance, it follows that:

$$c^2\|(x,\lambda)\|_1^2 \leq \left\|(x,\lambda)^\star(x,\lambda)\right\|_1$$

This holds for every $c \in (0,1)$ so taking the limit as $c \to 1$ gives us that $\|(x,\lambda)\|_1^2 \leq \left\|(x,\lambda)^\star(x,\lambda)\right\|_1$. By Lemma 3.1.4, we have that V_e satisfies the $C^\star$-condition. We are aware that the lemma we quoted would require V_e to be a Banach space; this will be proved below, without recourse to the $C^\star$-condition itself.

Let us show that V_e is complete in $\|\cdot\|_1$. Let $((x_n,\lambda_n))_{n\in\mathbb{N}}$ be a Cauchy sequence in V_e. We have two possibilities:

1. $(\lambda_n)_{n\in\mathbb{N}}$ is not bounded. So, we can extract a subsequence of non-zero terms that diverges to ∞. Since $((x_n,\lambda_n))_{n\in\mathbb{N}}$ is bounded, it follows that:

$$\lim_{n\to\infty}\left(\frac{1}{\lambda_n}x_n, 1\right) = \lim_{n\to\infty}\frac{1}{\lambda_n}(x_n,\lambda_n) = 0$$

where we are taking the limit of an appropriate subsequence so that $\lambda_n \to \infty$. This implies that $-\left(\frac{1}{\lambda_n}x_n, 0\right) \to e$. Since V is closed in V_e, we have that $e \in V$ and this is impossible.

2. $(\lambda_n)_{n\in\mathbb{N}}$ is bounded. By the Bolzano-Weierstrass Theorem, we can assume that we are working with a convergent sequence. But now, notice that:

$$\forall n \in \mathbb{N} : (x_n, 0) = (x_n,\lambda_n) - (0,\lambda_n)$$

It follows that $((x_n,0))_{n\in\mathbb{N}}$ is a Cauchy sequence in V and, therefore, it converges to some $(x,0)$. This means that $((x_n,\lambda_n))_{n\in\mathbb{N}}$ converges in V_e and we are done.

This completes the proof. □

Let us, now, switch gears and talk about the GNS construction for commutative $C^\star$-algebras. That is, itself, a fairly interesting discussion that takes up quite a fair amount of space so that is why I did not include it in the main text, at the end of Chapter 3. I will also work mainly with unital $C^\star$-algebras because I did only discuss the GNS construction in the unital case. We will begin by characterizing the states of commutative $C^\star$-algebras, expanding further on the rather short discussion slightly before [9, Theorem 7.7, p.31-32].

Proposition A.0.2

Let (X,τ) be a compact Hausdorff space. Define the map:

$$\pi : \mathcal{C}(X) \to \mathcal{B}(l^2(X)) \ , \ f \mapsto \pi(f)$$

where we define $\pi(f)(v_x)_{x\in X} := (f(x)v_x)_{x\in X}$ for every $(v_x)_{x\in X} \in l^2(X)$. This is the faithful representation of $\mathcal{C}(X)$ that is obtained from the GNS construction.

Proof. The trick is to go through the GNS construction but only take the direct sum over multiplicative linear functionals on $\mathcal{C}(X)$. Let $\phi : \mathcal{C}(X) \to \mathbb{C}$ be a multiplicative linear functional. Then:

$$\exists p \in X : \forall f \in \mathcal{C}(X) : \phi(f) = f(p).$$

Indeed, this follows as a consequence of Proposition 3.5.3. Next, we need to consider the elements of the following set:

$$N_\phi := \{f \in \mathcal{C}(X) : \phi(ff^\star) = 0\}$$

I claim that $N_\phi = \ker(\phi)$. Indeed, if $f \in N_\phi$, then $\phi(ff^\star) = |f(p)|^2 = 0$ so $f(p) = 0$. On the other hand, $f \in \ker(\phi)$ means that $\phi(f) = f(p) = 0$ so $\phi(ff^\star) = |f(p)|^2 = 0$. Now, we note that $\ker(\phi)$ is a maximal ideal so it is the case that $\mathcal{C}(X)/N_\phi$ is isomorphic $\mathbb{C}$ via the following map:

$$g : \mathcal{C}(X)/N_\phi \to \mathbb{C} \ , \ f + N_\phi \mapsto g(f + N_\phi) := f(p)$$

Let us show that this is an isomorphism:

1. Linearity is fairly obvious.
2. Let $g(f + N_\phi) = 0$. Then, $f(p) = 0$ so $f \in N_\phi$ which implies that $f + N_\phi = N_\phi$.
3. Let $c \in \mathbb{C}$ be a complex number. Since ϕ is multiplicative, it is surjective so there exists an $f \in \mathcal{C}(X)$ such that $c = \phi(f) = f(p)$. That is, $g(f + N_\phi) = f(p) = c$ and that proves surjectivity.

In particular, the discussion above implies that $\mathcal{C}(X)/N_\phi$ is complete so we do not need to further take its completion. Now, ϕ is uniquely associated to $p \in X$ so we could just write N_p instead of N_ϕ. For $x \in X$, we define the following map:

$$\pi_x : \mathcal{C}(X) \to \mathcal{B}(\mathbb{C}) \ , \ f \mapsto \pi_x(f) := f(x)$$

where $f(x)$ is a real number but it can be realized as a map in $\mathcal{B}(\mathbb{C})$ through left multiplication. Proving that π_x is a representation amounts to a simple computation. Now, we define $H = \bigoplus_{x\in X} \mathbb{C} = l^2(X)$ and we also define:

$$\forall f \in \mathcal{C}(X) : \forall (v_x)_{x\in X} \in l^2(X) : \pi(f)((v_x)_{x\in X}) := (\pi_x(f)v_x)_{x\in X} = (f(x)v_x)_{x\in X}$$

This defines a representation $\pi : \mathcal{C}(X) \to \mathcal{B}(H)$ and the fact that this is isometric follows from the simple equality:

$$\forall f \in \mathcal{C}(X) : \|\pi(f)\| = \sup_{x\in X} \|\pi_x(f)\| = \sup_{x\in X} |f(x)| = \|f\|_\infty$$

and we are done. □

Of course, the fact that we took the direct sums over every element of X just means that we took them over every multiplicative linear functional. This is an instance in which it was not necessary to use all of the states to construct an isometric representation. In particular, we have the following corollary.

Corollary A.0.3. Let V be a commutative unital $C^\star$-algebra. Then, $l^2(\widehat{V})$ is a Hilbert space that can represent V.

Proof. We have the following chain of maps:

$$V \to \mathcal{C}(\widehat{V}) \to \mathcal{B}(l^2(\widehat{V}))$$

where the second map is the representation from the proposition above and the first map is the Gelfand representation. Since both of these are isometric, it follows that the composition will be isometric and we are done. □

The proposition above also tells us that even the $C^\star$-algebraic structure of $\mathcal{C}(X)$ is accessible to us with multiplicative linear functionals. This point is made to stand out even more when one discusses pure states, as is done in [21, Chapter 4]. We will not pursue this point further because a few short paragraphs would not be able to do it justice; we have provided a sufficiently good reference for this point.

This is all well and good but we still have not actually addressed the content of the short discussion referenced above. In particular, if (X, τ) is a compact Hausdorff space, then we know, by Theorem B.2.7, that every positive linear functional $\Phi : \mathcal{C}(X) \to \mathbb{C}$ has a representation given by:

$$\forall f \in \mathcal{C}(X) : \Phi(f) = \int_X f \ d\mu$$

where $\mu : \mathrm{Bor}(X) \to \mathbb{R}$ is a positive measure satisfying extra conditions as given by the theorem cited above. Since Φ is a positive linear functional, then it is the case that:

$$\|\Phi\| = \Phi(1) = \int_X 1 \ d\mu = \mu(X).$$

If Φ was a state, then $\|\Phi\| = 1$ and we would have that $\mu(X) = 1$. That is, we would have a correspondence between the states of $\mathcal{C}(X)$ and the probability measures defined on $\mathrm{Bor}(X)$. This is quite nice but how does this relate to the GNS construction for the specific case of $\mathcal{C}(X)$? The answer to that lies in the following result.

Proposition A.0.4

Let (X, τ) be a compact Hausdorff space. Then, the multiplicative linear functionals on $\mathcal{C}(X)$ correspond to Dirac measures.

Proof. Let $\Phi : \mathcal{C}(X) \to \mathbb{C}$ be a multiplicative linear functional. Then, Φ is a state so it follows, by Theorem B.2.7, that there exists a positive measure $\mu : \mathrm{Bor}(X) \to \mathbb{R}$ such that:

$$\forall f \in \mathcal{C}(X) : \Phi(f) = \int_X f \ d\mu$$

By the discussion provided above, it follows that μ is a probability measure. Now, we also know that multiplicative linear functionals correspond to points in X. That is, there exists a $p \in X$ such that:

$$\forall f \in \mathcal{C}(X) : \Phi(f) = f(p)$$

It follows, then, that:

$$\forall f \in \mathcal{C}(X) : f(p) = \int_X f \; d\mu$$

The claim, then, is that $\mu = \delta_p$, where the latter is the Dirac measure at p. Indeed, let $A \in \mathrm{Bor}(X)$ such that $p \notin A$. We claim that $\mu(A) = 0$. Note that A can be approximated from below by compact sets (because we are in a compact Hausdorff space). Therefore, let $K \subseteq A$ be a compact set. Then, $p \notin K$ so $\{p\}$ and K are disjoint compact sets (which are, therefore, closed). By Urysohn's Lemma, we can find a continuous map $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(x) = 1$ on all of K. Therefore:

$$0 = f(p) = \int_X f \; d\mu$$

which implies that f is zero almost everywhere. In particular, $\mu(K) = 0$. But this means that $\mu(A) = 0$. Now, let $A \in \mathrm{Bor}(X)$ such that $p \in A$. Then, $X \setminus A$ is a Borel set that does not contain p so $\mu(X \setminus A) = 0$. But this implies that $\mu(A) = \mu(X) = 1$. That is, we have that $\mu = \delta_p$. □

Now, we have a measure available to us so the most natural Hilbert space to consider is $\mathcal{L}^2(X, \delta_x)$, for each $x \in X$. In fact, there is a very natural claim that we can prove here. Let us define the following map:

$$\Phi : \mathcal{L}^2(X, \delta_x) \to \mathbb{C} \; , \; [f] \mapsto \Phi([f]) := f(x)$$

This is well-defined and we claim that this is an isomorphism. Indeed, linearity is obvious. Let $\Phi([f]) = 0$. This means that $f(x) = 0$. But now, note that:

$$\int_X |f|^2 \; d\delta_x = |f(x)|^2 = 0$$

which implies that $f = 0$ almost everywhere. Finally, let $c \in \mathbb{C}$. Then, we define a function:

$$f : X \to \mathbb{C} \; , \; p \mapsto f(p) = \begin{cases} c & \text{if } p = x \\ 0 & \text{if } p \neq x \end{cases}$$

Note that this is measurable and, certainly, a function in $\mathcal{L}^2(X, \delta_x)$. Then, we have that $\Phi([f]) = f(x) = c$ and that proves surjectivity. This is quite nice, because it tells us that even if we did start with Theorem B.2.7 and tried to construct a Hilbert space representation that way, we would still end up with the same construction as before provided that we worked with multiplicative linear functionals rather than all of the states. This concludes our discussion of the GNS construction relative to $\mathcal{C}(X)$.

Appendix B

Background Information

This appendix collects the background information that is needed by the reader so that they can understand all of the details within the proofs that have been given. We should emphasize again that one does **not** need to understand all the details to appreciate the main points of the thesis. For the most part, we have refrained from providing explicit proofs, though there are some exceptions. Proofs have been provided in those cases where there is some benefit to seeing the argument here first.

B.1 General Topology

The thesis will make use of a number of topological notions that are of independent interest. This section is meant to collect those bits of General Topology that are most relevant to the work we will do here[1].

Definition B.1.1

Let (X, τ) be a topological space. (X, τ) is said to be a Baire space iff for any sequence $(U_n)_{n\in\mathbb{N}}$ of dense open sets, $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X. Let $U \subseteq X$ be a set.

1. U is a meagre set /Baire first category set iff it is a countable union of nowhere dense sets[a].
2. U is a Baire second category set iff it is not a meagre set. U is also said to be nonmeagre.

[a] A set $A \subseteq X$ is nowhere dense iff $\operatorname{Int}(\overline{A}) = \varnothing$.

The definition above was inspired by [5, Definition 1.6.1, p.40]. Obviously, the notion of a "dense" set or a "nowhere dense set" carry over without modification to the general setting of topological spaces. The following characterization result for Baire spaces is useful.

Proposition B.1.2

Let (X, τ) be a topological space.

1. X is a Baire space.
2. Every non-empty open subset of X is nonmeagre.

Proof.

$(1) \Rightarrow (2)$: Let X be a Baire space. Let $U \subseteq X$ be a non-empty open set and suppose that it was meagre. Then, there is a sequence $(G_n)_{n\in\mathbb{N}}$ of nowhere dense sets such that $U = \bigcup_{n\in\mathbb{N}} G_n$. Now, $X \setminus G_n$ must contain a dense open set[a]. It follows that $\bigcap_{n\in\mathbb{N}}(X \setminus G_n)$ is dense, due to the fact that X is a Baire space. But this is a closed set because it is precisely the complement of U, so it follows that U is empty and that is a contradiction.

$(2) \Rightarrow (1)$: Assume that every non-empty open subset of X is nonmeagre. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of dense open sets. We know that $X \setminus U_n$ is nowhere dense for each $n \in \mathbb{N}$[b]. Consider $\bigcup_{n\in\mathbb{N}}(X \setminus U_n)$; we claim that this is nowhere dense. If its interior was non-empty, then it would intersect with every U_n because each U_n is dense; this is not possible. But this means that $\bigcap_{n\in\mathbb{N}} U_n$ contains a dense open set and, therefore, the intersection is also dense. □

[a] If $A \subseteq X$ is nowhere dense, then we claim that $X \setminus A$ contains a dense open set. Indeed, let $U = X \setminus \overline{A}$; we claim that U is dense in X. Let $x \in X$; if $x \in U$, then we are gucci. Otherwise, let $x \in \overline{A}$. Since $\overline{A}$ has empty interior, every open set around x intersects with U

[b] Let $U \subseteq X$ be a dense open set. Then, $X \setminus U$ is a closed set. Let x be an element of the interior of $X \setminus U$. Since U is dense, it follows that every open set around x intersects with U because U is dense and that is impossible. Therefore, the interior of $X \setminus U$ is empty; that is, it is nowhere dense.

We should point out that one can prove more detailed characterization results for Baire spaces; for our purposes, this isn't so necessary. The following useful property of first category sets in a topological space is worth noting down.

[1] Of course, we also have Appendix C but that is mainly meant for the development of a theory of convergence in topological spaces which the reader is unlikely to be familiar with.

Proposition B.1.3

Let (X, τ) be a topological space and let $(V_n)_{n\in\mathbb{N}}, (W_n)_{n\in\mathbb{N}}$ be sequences of subsets of the topological space. Suppose that for every $n \in \mathbb{N}$, the set $V_n \Delta W_n$[a] is of the first category. Then, $\left(\bigcup_{n\in\mathbb{N}} V_n\right) \Delta \left(\bigcup_{n\in\mathbb{N}} W_n\right)$ is of the first category. Moreover, for every $m, n \in \mathbb{N}$, it is the case that $(V_n \setminus V_m)\Delta(W_n \setminus W_m)$ is of the first category.

[a] If x and y are two sets, then $x\Delta y = (x \setminus y) \cup (y \setminus x)$ is the symmetric difference of x and y.

Proof. The first claim follows from the purely set-theoretic fact that:

$$\left(\bigcup_{n\in\mathbb{N}} V_n\right) \Delta \left(\bigcup_{n\in\mathbb{N}} W_n\right) \subseteq \bigcup_{n\in\mathbb{N}} (V_n \Delta W_n)$$

Of course, we will not take the time to prove this explicitly; it can be done with the usual arguments. The second assertion is fairly obvious, given the hypothesis so we are done[a]. □

[a] This was Proposition C (p. 395) in [14]. There was no proof provided, so we provided the short argument above which only requires some details to be fully complete.

In our development of spectral theory, we will need to develop a functional calculus involving a particular class of measurable functions. This class of measurable functions can be described rather well using the notion of "category" (in the sense of Definition B.1.1) and we will need a version of the Baire Category Theorem (not the usual version for complete metric spaces) to establish the validity of this description. To prove this, the following "nested interval principle" is useful.

Lemma B.1.4 Let (X, τ) be a compact Hausdorff space. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of non-empty compact sets such that $U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$. Then, $\bigcap_{n\in\mathbb{N}} U_n$ is non-empty.

Proof. Suppose that the given intersection is empty. Then, $\bigcup_{n\in\mathbb{N}}(X \setminus U_n) = X$. Note that each U_n is closed[a]. Therefore, the family $\{X \setminus U_n\}_{n\in\mathbb{N}}$ of sets forms of an open cover for X. Therefore, we can find $n_1, \ldots, n_m \in \mathbb{N}$ such that $\bigcup_{i=1}^m (X \setminus U_{n_i}) = X$. But this means that $\bigcap_{i=1}^m U_{n_i} = \varnothing$ and this is impossible so we are done. □

[a] Let $K \subseteq X$ be compact, where (X, τ) is a Hausdorff space. Let $(x_\delta)_{\delta\in D}$ be a net in K that converges to some $x \in X$. Since K is compact, there is a subnet of the given net which converges (Theorem C.0.17) to some $y \in K$. Since X is Hausdorff, it follows that $x = y$.

Theorem B.1.5. The Baire Category Theorem

Every compact, Hausdorff space is a Baire space.

Proof. Let (X, τ) be a compact Hausdorff space. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of dense open subsets of X and let U denote the intersection $\bigcap_{n\in\mathbb{N}} U_n$. We want to show that U is dense; let K be a non-empty open set and let us show that $K \cap U \neq \varnothing$. We will construct a sequence $(V_n)_{n\in\mathbb{N}}$ of open sets as follows.

1. Define $V_1 := K$.
2. Suppose that a finite sequence $(V_n)_{n=1}^m$ of non-empty open sets have been formed. Since U_{m+1} is dense, there is a non-empty open set V such that $\overline{V}_{m+1} \subseteq U_{m+1} \cap V_m$. Set this V to be V_{m+1}.

This gives us a nested sequence of open sets and we note that $\bigcap_{n\in\mathbb{N}} \overline{V}_n$ is non-empty[a]. The given construction shows that this intersection is a subset of V_1 and a subset of each U_n. In particular, $K \cap U \neq \varnothing$. □

[a] After all, each $\overline{K}_n$ is non-empty, closed and, therefore, compact.

Theorem B.1.6. Tychonoff's Theorem

Let $((X_i, \tau_i))_{i\in I}$ be a family of topological spaces. The following are equivalent:

1. For every $i \in I$, X_i is compact.
2. $\prod_{i\in I} X_i$ is compact in the product topology.

Moreover, if each X_i is a Hausdorff space, then $\prod_{i\in I} X_i$ is a Hausdorff space.

Proof. The interesting direction is (1) ⇒ (2), since (2) ⇒ (1) follows from the fact that the projection maps are all continuous and that the continuous image of a compact set is compact. For an elementary proof of (1) ⇒ (2) which **only** makes use of the material in Appendix C, see [6]. Finally, assume that each X_i is Hausdorff. We will prove that the product space is Hausdorff by appealing to Theorem C.0.15. Let $((x_i)_{i\in I}^\delta)_{\delta\in D}$ be a net in $\prod_{i\in I} X_i$ which converges to $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$. Then, for each $j \in I$, the projection map $\pi_j : \prod_{i\in I} X_i \to X_j$ is continuous so $\pi_j(((x_i)_{i\in I}^\delta)_{\delta\in D})$ converges to both x_j and y_j. Since X_j is Hausdorff, it follows that $x_j = y_j$. Since j was arbitrary, it follows that $(x_i)_{i\in I} = (y_i)_{i\in I}$. □

Theorem B.1.7. Urysohn's Lemma

Let (X, τ) be a topological space. Then, the following are equivalent:

1. Every pair of disjoint closed sets K, L in X can be separated by disjoint open neighborhoods $K \subseteq U$ and $L \subseteq V$.
2. For every closed set K in X and every open neighborhood U of K, there exists an open set V and a closed set L such that $K \subseteq V \subseteq L \subseteq U$.
3. For every closed set K in X and every open neighborhood U of K, there exists a continuous function $f : X \to [0, 1]$ such that $1_K(x) \leq f(x) \leq 1_U(x)$ for every $x \in X$.
4. For every pair of disjoint closed sets K, L in X, there exists a continuous function $f : X \to [0, 1]$ which equals 1 on K and 0 on L.

The function in (4) is said to be an Urysohn function.

Proof. A nice proof of this may be found in [41, Lemma 1.10.1, p.134]. We should point out that condition (4) is the usual "Urysohn's Lemma" which is seen in General Topology texts, while condition (1) is the usual definition of "normal" topological spaces. We also point out that compact Hausdorff spaces are normal, a standard fact which is a guided exercise in [41, Exercise 1.10.2, p.135]. □

Theorem B.1.8. The Stone-Weierstrass Theorem

Let (X, τ) be a locally compact Hausdorff space and let $\mathcal{A}$ be a subalgebra of $\mathcal{C}_0(X)$. Suppose that $\mathcal{A}$ satisfies the following conditions:

1. $\forall x \in X : \exists f \in \mathcal{A} : f(x) \neq 0$.
2. $\forall x, y \in X : \exists f \in \mathcal{A} : x \neq y \Rightarrow f(x) \neq f(y)$.
3. $\forall f \in \mathcal{A} : \overline{f} \in \mathcal{A}$.

Then, $\overline{\mathcal{A}} = \mathcal{C}_0(X)$.

Proof. There is a classical proof that can be found in [40, p.152]. A generalization of this theorem, known as Bishop's Theorem, can be found in [34, p.121]. It is a consequence of the Krein-Milman Theorem, the Banach-Alaoglu Theorem and the Hahn-Banach Theorem, all of which are proven also in the same text. A lengthy discussion of Bishop's Theorem as well as the detailed study of $\mathcal{C}(X)$ by looking at its subalgebras can also be found in [4]. □

Definition B.1.9

Let (X, τ) be a topological space. X is extremally disconnected iff the closure of every open set is open. It is a Stonian topological space/Stonian space iff it is compact, Hausdorff and extremally disconnected.

The last thing we will need is some facts about the one-point compactification of a topological space. Over here, we will just state the definition of a one-point compactification and state the results we need, foregoing any and all proofs. Our treatment of this is based off of [44, Chapter 8.1] and the proofs of all results we state here can be found there, along with many stimulating examples.

Definition B.1.10

Let (X, τ) be a topological space and let ∞ be a set such that $\infty \notin X$. Define $X^+ := X \cup \{\infty\}$. A set $G \subseteq X^+$ is open iff G is an open subset of X or $X^+ \setminus G$ is a closed and compact subset of X. The topological space (X^+, τ_+), where τ_+ is the set of all open subsets of X^+, is called the one-point compactification of X.

Of course, it must be shown that τ_+ is a topology on X^+. Moreover, the use of the word "compactification" suggests that (X^+, τ_+) is compact; this also needs to be shown. These facts are summarized in the following proposition.

Proposition B.1.11

Let (X, τ) be a topological space and let (X^+, τ_+) be its one-point compactification. Then, $X^+ := X \cup \{\infty\}$ is a compact topological space[a] and X is a topological subspace of X^+. If X is compact, then ∞ is an isolated point of X^+. If X is not compact, then X is dense in X^+.

[a] Keep in mind that we will **always** assume that $\infty \notin X$.

Proof. See [44, Theorem 8.1.1, p.138] for the proof. □

In the thesis, we will mainly be interested in Hausdorff spaces. We have a condition which tells us precisely when the one-point compactification is Hausdorff.

Theorem B.1.12

Let (X, τ) be a non-compact Hausdorff space and let $X^+ := X \cup \{\infty\}$ be its one-point compactification. The following are equivalent:

1. X^+ is Hausdorff.
2. X is locally compact.

Proof. See [44, Theorem 8.1.2, p.139] for the proof. The statement proved there is slightly more general and the author uses "T_2-space" in place of "Hausdorff space". □

Finally, we have the following result which tells us that there is essentially only one way to add a point to a locally compact Hausdorff space so that it becomes compact and Hausdorff.

Theorem B.1.13

Let (Y, τ_Y) be a compact Hausdorff space and let $h \in Y$ be a nonisolated point. Let $X = Y \setminus \{h\}$. Then, $Y = X^+$ with $\infty := h$.

Proof. See [44, Theorem 8.1.3, p.140] for the proof. □

B.2 Functional Analysis

Functional Analysis forms the backdrop for this thesis, so it is no surprise that some discussion of it is included in an appendix here. I have only included results that are of real significance to the work we do here. Proofs have been included for fairly non-standard formulations of familiar statements (for instance, the Open Mapping Theorem).

Theorem B.2.1. The Hahn-Banach Theorem

Let V be a $\mathbb{C}$-normed space and let $W \subseteq V$ be a subspace. Let $\psi : W \to \mathbb{C}$ be a complex linear functional and let $c \geq 0$ so that $|\phi(x)| \leq c\|x\|$ for every $x \in W$. Then, there exists a complex linear functional $\Psi : V \to \mathbb{R}$ such that $\Psi|_W = \psi$ and:

$$\forall x \in V : |\Psi(x)| \leq c\|x\|$$

Proof. See [5, Section 2.3]. We should note that this is not the most general form of the Hahn-Banach Theorem which is presented in Functional Analysis texts; nevertheless, it is the form that will be relevant for us. □

Theorem B.2.2. The Fréchet-Riesz Representation Theorem

Let H be a $\mathbb{K}$-Hilbert space. For every $f \in H^\star$, there exists a unique $y \in H$ such that:

$$\forall x \in H : f(x) = \langle x, y \rangle$$

The map $y \mapsto \langle \cdot, y \rangle$ defines a bijective, conjugate-linear map between H and $H^\star$, denoted by J_H. This map is also norm-preserving[a].

[a] That is, $\|J_H(y)\| = \|y\|$.

Proof. See [43, Corollary 3.19, p.81-82]. □

There is a rather nice consequence of the theorem above that is easy to prove and will be used exactly once in the thesis (we promise that it is used for a non-trivial argument).

Corollary B.2.3. Let H be a $\mathbb{C}$-Hilbert space and let $T \in \mathcal{B}(H)$. Define a function $b_T : H \times H \to \mathbb{C}$ by the prescription:

$$\forall x, y \in H : b_T(x, y) := \langle T(x), y \rangle$$

The map b_T is a conjugate-bilinear functional[a] which satisfies $\|b_T\| = \|T\|$[b]. Moreover, every bounded conjugate-bilinear functional arises in this way from a unique element of $\mathcal{B}(H)$.

[a] A function $b : H \times H \to \mathbb{C}$ is a bounded conjugate-bilinear functional iff it is linear in the first variable, conjugate-linear in the second variable and there is a constant $k > 0$ such that $|b(x, y)| \leq k\|x\| \cdot \|y\|$ for every $x, y \in H$.

[b] The norm of such functions is defined in a similar way to linear maps.

Proof. Most of this just comes down to computing things correctly, with the exception of the last statement so we will prove that. Let $b : H \times H \to \mathbb{C}$ be a bounded conjugate-bilinear functional. Let $x \in H$. Define a functional:

$$\phi : H \to \mathbb{C} \ , \ y \mapsto \phi(y) = \overline{b(x, y)}$$

Notice that this is a bounded linear map, so there exists a unique or they $cz \in H$ such that:

$$\forall y \in H : \overline{b(x, y)} = \langle y, z \rangle$$

It follows that $b(x, y) = \langle z, y \rangle$ for every $y \in H$. Now, define a map $T : H \to H$ by defining $T(x) := z$. The continuity of T follows from the continuity of b so we are done. □

It is certainly important for us to mention some of the main principles of Functional Analysis that we will make use of from time to time.

Theorem B.2.4. Banach-Alaoglu Theorem

Let V be a $\mathbb{K}$-normed space. Then, the closed unit ball in $V^\star$ is compact in the weak*-topology.

Proof. See [5, p.126] for a proof of the general case when V is a real vector space. The proof can be adapted to the case of complex normed spaces rather easily. □

Theorem B.2.5. The Open Mapping Theorem

Let V, W be $\mathbb{K}$-Banach spaces and let $T \in \mathcal{L}(V, W)$. Then, the following are equivalent:

1. T is an open map.
2. $\exists r > 0 : B_W(0, r) \subseteq T(B_V(0, 1))$.
3. There exists a dense subspace $W' \subseteq W$ and a constant $c > 0$ such that:
$$\forall w \in W' : \exists v \in V : T(v) = w$$
and $\|v\|_V \leq c\|w\|_W$
4. T is surjective.

Proof. This result is rather interesting because we actually found it as an exercise in [43, Exercise 4.23, p.130]. Since this exercise is quite non-trivial, we will actually prove the equivalence. It should, however, be pointed out that the implication (4) ⇒ (1) is the typical statement of the Open Mapping Theorem in the literature and, therefore, we will not prove it explicitly here.

(1) ⇒ (2) : Since $B_V(0, 1)$ is an open set in V and since T is an open map, it follows that $T(B_V(0, 1))$ is an open set. In particular, $0 \in T(B_V(0, 1))$ and since $T(B_V(0, 1))$ is an open set, it follows that it must contain an open ball centered on 0. This is precisely the desired result.

(2) ⇒ (3) : Let $r > 0$ in (2) be as given. It should be fairly obvious that (2) implies that T is an open map. With this in mind, we note that $T(V)$ is an open set and a subspace of W. Since non-zero, proper subspaces cannot contain open balls, it follows that $T(V) = W$. Now, let $W' = T(V)$; this is clearly a dense subspace in W. Let $w \in W'$ be non-zero. Observe that $\frac{rw}{2\|w\|} \in B_W(0, r)$. It follows that there is a $u \in B_V(0, 1)$ such that:
$$T(u) = \frac{rw}{2\|w\|}$$
In particular, it follows that $T\left(\frac{2\|w\|u}{r}\right) = w$. Let $v = \frac{2\|w\|u}{r}$. Then:
$$\|v\|_V \leq \frac{2}{r}\|w\|$$
with $c = \frac{2}{r}$ being the desired constant.

(3) ⇒ (4) : The proof of this is actually very similar to that of the Open Mapping Theorem, so we will not give it. Instead, we refer the reader to the resource given for the implication (4) ⇒ (1); the one which is the classical version of the Open Mapping Theorem.

(4) ⇒ (1) : As we mentioned before, this is the implication which is typically referred to as the Open Mapping Theorem. We refer the reader to [43, Section 4.2.2, p.128-130] for a detailed proof of this result. □

Theorem B.2.6. The Inverse Operator Theorem

Let V, W be $\mathbb{K}$-Banach spaces and let $T : V \to W$ be a bijective continuous map. Then, T^{-1} is continuous.

Proof. This is a simple consequence of the Open Mapping Theorem. Since T is surjective, it is an open map. This implies that T^{-1} is continuous and we are done. □

Theorem B.2.7. The Riesz-Markov-Kakutani Representation Theorem

Let (X, τ) be a locally compact Hausdorff space and let $\Lambda : \mathcal{C}_c(X) \to \mathbb{C}$ be a positive linear functional. Then, there exists a σ-algebra $\mathcal{M}$ on X which contains all the Borel subsets of X and there exists a unique positive measure $\mu : \mathcal{M} \to [0, \infty]$ which satisfies the following properties:

1. $\forall f \in \mathcal{C}(X) : \Lambda(f) := \int_X f \, d\mu$.
2. $\mu(K) < \infty$ for every compact set $K \subseteq X$.
3. $\forall E \in \mathcal{M} : \mu(E) = \inf\{\mu(V) : E \subseteq V \wedge V \text{ is open}\}$.
4. If E is an open set, then $\mu(E) = \sup\{\mu(K) : K \subseteq E \wedge K \text{ is compact}\}$. This relation also holds when $E \in \mathcal{M}$ and $\mu(E) < \infty$.
5. $\forall E \in \mathcal{M} : \forall A \subseteq E : \mu(E) = 0 \Rightarrow A \in \mathcal{M}$.

Proof. See [35, Theorem 2.14, p.40-47] for an accessible proof. We also talk about this theorem in some detail in the last part of the thesis, where we prove the spectral theorem. This result is **not** used in a major way at any point in the thesis; it is mainly used in an example. □

The content of the final results of this section have to do with the idea that if we have a family of $\mathbb{C}$-Hilbert spaces, then there is a nice way to induce a Hilbert space structure on a "natural" subset of the Cartesian product of this family of Hilbert spaces. The exact features of this structure are handled by the following result.

Proposition B.2.8

Let $(H_i)_{i\in I}$ be a family of $\mathbb{C}$-Hilbert spaces. Define the following set:

$$H := \bigoplus_{i\in I} H_i := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} H_i | \sum_{i\in I} \|x_i\|_i^2 < \infty \right\}$$

Define the following relations:

$$+ : H \times H \to H \ , \ ((x_i)_{i\in I}, (y_i)_{i\in I}) \mapsto (x_i)_{i\in I} + (y_i)_{i\in I} := (x_i + y_i)_{i\in I}$$
$$\cdot : \mathbb{C} \times H \to H \ , \ (\alpha, (x_i)_{i\in I}) \mapsto \alpha \cdot (x_i)_{i\in I} := (\alpha x_i)_{i\in I}$$
$$s : H \times H \to \mathbb{C} \ , \ ((x_i)_{i\in I}, (y_i)_{i\in I}) \mapsto \langle (x_i)_{i\in I}, (y_i)_{i\in I} \rangle_H := \sum_{i\in I} \langle x_i, y_i \rangle_{H_i}$$

The following claims hold:

1. All three relations are, in fact, well-defined maps.
2. $(H, +, \cdot)$ is a $\mathbb{C}$-vector space.
3. The map s defines an inner product on H and H is a $\mathbb{C}$-Hilbert space with respect to the norm induced by s.

H is, then, referred to as the direct sum of the family $(H_i)_{i\in I}$ of Hilbert spaces.

Proof. This has not really been stated as an explicit theorem in any of the references that the author has used; nevertheless, an extended discussion of this construction along with the proofs is available in [21, p.121-124]. □

If we have a family of Hilbert spaces and a a family of associated linear maps, then it becomes possible to speak of the direct sums of linear maps. This is the content of the next result.

Proposition B.2.9

Let $(H_i)_{i\in I}$ be a family of Hilbert spaces and let $(T_i)_{i\in I}$ be a family of linear maps with $T_i \in \mathcal{B}(H_i)$ for every $i \in I$. Suppose that this family of linear maps is uniformly bounded; that is:

$$\exists M > 0 : \forall i \in I : \|T_i\| \leq M$$

Define a map:

$$T : \bigoplus_{i\in I} H_i \to \bigoplus_{i\in I} H \ , \ (x_i)_{i\in I} \mapsto T((x_i)_{i\in I}) := (T_i(x_i))_{i\in I}$$

Then, T is a bounded linear map and it is denoted by $\bigoplus_{i\in I} T_i$. It satisfies the following properties:

1. $\|T\| = \sup_{i\in I} \|T_i\|$.
2. $T^\star = \bigoplus_{i\in I} T_i^\star$.
3. Let $\alpha, \beta \in \mathbb{C}$ and let $(S_i)_{i\in I}$ be a family of uniformly bounded linear maps with $S_i \in \mathcal{B}(H_i)$ for every $i \in I$. Then:
$$\bigoplus_{i\in I} (\alpha T_i + \beta S_i) = \alpha \bigoplus_{i\in I} T_i + \beta \bigoplus_{i\in I} S_i$$
4. Let $(S_i)_{i\in I}$ be a family of uniformly bounded linear maps with $S_i \in \mathcal{B}(H_i)$ for every $i \in I$. Then:
$$\left(\bigoplus_{i\in I} T_i \right) \left(\bigoplus_{i\in I} S_i \right) = \bigoplus_{i\in I} T_i S_i$$

The map $\bigoplus_{i\in I} T_i$ is called the direct sum of the family of linear maps $(T_i)_{i\in I}$.

Proof. The situation here is similar to that of the preceding result; the reader is referred to [21, p.122-124] for a detailed discussion of its proof. The result isn't actually proved in the text we have just cited but the text gives enough information for the reader to deduce the content of the proposition by themselves. □

Appendix C

Moore-Smith Convergence

The purpose of this appendix is to discuss the underpinnings of Moore-Smith Convergence, also known as "net convergence". This is typically not covered in modern textbooks on General Topology (and if it is covered, then it is usually done so in exercises). Nets form a useful generalization of sequences in General Topology and will be used in many places within this thesis. To ensure that this appendix is accessible to the reader, we will switch from the formal & serious tone used throughout the thesis to a more personal tone.

As you should know, sequences often fail to characterize many topological properties of interest (for instance, Hausdorffness and compactness). You will soon find out that topological properties can be characterized using net convergence and that many of the nice arguments we use with sequences can be replicated with nets, albeit with some more technical difficulties[1]. Most of this section is based off of the (standard) material in [44] & [24].

Definition C.0.1

Let D be a non-empty set and let $\leq$ be a relation on D [a]. $(D, \leq)$ is said to be a directed set if it satisfies the following conditions:

1. $\forall x \in D : x \leq x$. (Reflexive)
2. $\forall x, y, z \in D : [(x \leq y) \wedge (y \leq z)] \Rightarrow (x \leq z)$. (Transitive)
3. $\forall x, y \in X : \exists z \in X : x \leq z \wedge y \leq z$. (Directed)

If $\leq$ is also a partial order, then it is said to be a directed partial order.

[a] That is, $\leq \subseteq D \times D$.

As is usual, I will often just say "Let D be a directed set...." rather than denoting the set as an ordered pair, like I did in the preceding definition. Definition C.0.1 is all I need to define the notion of a net.

Definition C.0.2

Let X be any non-empty set. A net in X is a map $f : D \to X$, where D is some directed set.

So, nets are really just maps. This should not be so surprising; a sequence is just a map defined on $\mathbb{N}$ and nets are an attempt at generalizing sequences. Now, let me provide a few key examples.

Example 20. Let $\mathbb{N}$ be equipped with its usual partial order $\leq$ and let X be a non-empty set. A sequence $f : \mathbb{N} \to X$ is a net in X, with $(\mathbb{N}, \leq)$ being a directed set.

Example 21. Let X be any non-empty set. Define a relation $\leq$ in $\mathcal{P}(X)$ by the following prescription:

$$\forall A, B \in \mathcal{P}(X) : A \leq B \Leftrightarrow B \subseteq A$$

It is easily checked that $(\mathcal{P}(X), \leq)$ is a directed set. If Y is any non-empty set, then a map $f : \mathcal{P}(X) \to Y$ is a net.

Before I talk about the convergence of nets, let me introduce some notation that will be somewhat useful.

Definition C.0.3

Let X be a non-empty set and let $f : D \to X$ be a net [a]. The net f will be denoted by $(x_\delta)_{\delta \in D}$. This is the notation that I will use when talking about nets. Also, if $(D, \leq)$ is a directed set, then I will define:

$$\forall x, y \in D : x \geq y :\Leftrightarrow y \leq x$$

Again, this is just to make sure that some common notation that everyone is very familiar with can just be used without further comment.

[a] From now on, whenever I say that some map is a net, it will be implicitly implied that its domain is just some directed set.

[1] These technical difficulties are not to be harped about constantly, for they can be overcome rather quickly.

Notice that the notation above is rather similar to the one used for sequences? This should, to an extent, make the arguments "look" like the same ones that are typically used with sequences.

Definition C.0.4

Let (X, τ) be a topological space and let $(x_\delta)_{\delta \in D}$ be a net in X. The net converges to a point $x \in D$ if for every open set $U \in \tau$ with $x \in U$, we have that:

$$\exists \delta_0 \in D : \forall \delta \geq \delta_0 : x_\delta \in U$$

x is said to be a limit of the net and we write $x = \lim_\delta(x_\delta)$. We also write $x_\delta \to x$ to indicate that the net converges to x.

In general, convergent nets do **not** have unique limits. Here's a simple example where there is a convergent sequence that has infinitely many limits.

Example 22. Let $X = \mathbb{R}^2$ and define a map $d : X \times X \to [0, \infty)$ by the following prescription:

$$\forall x, x', y, y' \in \mathbb{R} : d((x, y), (x', y')) = |x' - x|$$

So, this is the horizontal distance between two points. It is easy to show that (X, d) satisfies all the axioms for a metric space except for the requirement that $d(a, b) = 0 \Rightarrow a = b$ for any $a, b \in X$. Such a space is said to be a semi-metric space. Semi-metric spaces can be turned into topological spaces, through precisely the same approach used with metric spaces. For our specific example above, there exists a sequence in X which converges to infinitely many points. Indeed, the sequence $\left(\left(\frac{1}{n}, 0\right)\right)_{n \in \mathbb{N}}$ converges to $(0, y)$ for any $y \in \mathbb{R}$.

Before I prove any of the main results, there is some terminology that is pretty useful to have, just so that the proofs become more intuitive. It has to do with two behaviors of nets that will be rather important to us (at least, to an extent).

Definition C.0.5

Let X be a non-empty set. Let $(x_\delta)_{\delta \in D}$ be a net in X and let $U \subseteq X$ be a set.

1. We say that the net lies eventually in U if:
$$\exists \delta_0 \in D : \forall \delta \geq \delta_0 : x_\delta \in U$$
2. We say that the net lies frequently in U if:
$$\forall \delta \in D : \exists \delta_0 \geq \delta : x_{\delta_0} \in U$$

Whenever one uses these bits of terminology, the elements δ_0 are not referenced because they are usually not needed.

The following proposition is also pretty useful and just reduces some of the writing in proofs.

Proposition C.0.6

Let X be a non-empty set and let $(x_\delta)_{\delta \in D}$ be a net. Suppose that $(U_i)_{i \in I}$ is a finite family of sets such that the net lies in each U_i eventually. Then, the net lies in $\bigcap_{i \in I} U_i$ eventually.

Proof. This can just be proved by induction on the cardinality of I. □

I should also mention that using this terminology also allows one to re-phrase the definition of convergence in a topological space. This re-phrase is slight but it is conceptually helpful & often makes proofs shorter.

Proposition C.0.7

Let (X, τ) be a topological space. Let $(x_\delta)_{\delta \in D}$ be a net in X and let $x \in X$. The following are equivalent:

1. The net converges to x.
2. For every open set $U \in \tau$ which contains x, the net lies eventually in X.

Proof. This is obvious, as it is just a slight re-phrase of the definition of convergence. □

I am finally ready to start proving results of significance. The first one I will prove is a characterization of closed sets in a topological space. To do that, it will be useful to have the following lemma which characterizes closures using nets.

Lemma C.0.8 Let (X, τ) be a topological space. Let $S \subseteq X$ and let $x \in X$ be fixed. The following are equivalent:

1. $x \in \overline{S}$.
2. There is a net in S that converges to x.

Proof.

$(1) \Rightarrow (2)$: Let $x \in \overline{S}$. Then, every open set around x intersects with S. Define a set:

$$D = \{N \cap S : N \in \mathcal{N}_x\}$$

where $\mathcal{N}_x$ is the set of all neighborhoods of x. Now, $\mathcal{N}_x$ is a directed set by the order:

$$\forall N_1, N_2 \in \mathcal{N}_x : N_1 \leq N_2 :\Leftrightarrow N_2 \subseteq N_1$$

By the Axiom of Choice, we can define a net on $D = \mathcal{N}_x$ by the following prescription: for each $N \in D$, assign $f(N)$ to be an element of $N \cap S$. This is a net that converges to x. Indeed, let N_0 be a neighborhood of x. For every $N_0 \leq N$, we have that $f(N)$ lies in N and, therefore, lies in N_0 so we are done.

(2) ⇒ (1) : Let $(x_\delta)_{\delta \in D}$ be a net in S that converges to x. If $x \notin \overline{S}$, then there is a closed set $K \supseteq S$ such that $x \notin K$. It follows that $x \in X \setminus K$. But this is open so the net must eventually lie in this set. This is impossible, so it follows that $x \in \overline{S}$. □

Proposition C.0.9

Let (X, τ) be a topological space. Let $S \subseteq X$ be a set. The following are equivalent:

1. S is a closed set.
2. Every net in S which converges will converge inside of S.

Proof.

(1) ⇒ (2) : Let S be a closed set. Let $(x_\delta)_{\delta \in D}$ be a net in S that converges to some $x \in X$. Suppose that $x \notin S$. Then, $x \in X \setminus S$ and this set is open. Therefore, the net must eventually be within it and that is impossible.

(2) ⇒ (1) : Let us show that $S = \overline{S}$; it only needs to be shown that $\overline{S} \subseteq S$. Let $x \in \overline{S}$. Then, there is a net in S that converges to x. By hypothesis, it follows that $x \in S$ so we are done. □

A nice consequence of the characterization result for closed sets proved above is that the convergence of nets can even be used to characterize a topology. Let us prepare for the proof of this with the following result.

Lemma C.0.10 Let X be a set and let τ, τ' be topologies on X. The following are equivalent:

1. $\tau \subseteq \tau'$.
2. Let $(x_\delta)_{\delta \in D}$ be a net in X and let $x \in X$. If $x_\delta \to x$ in τ', then $x_\delta \to x$ in τ.

Proof.

(1) ⇒ (2) : Let U be an open set around x in τ. Then, U is an open set around x in τ'. It follows that the net eventually lies in U because it converges to x in τ', which proves that the net converges to x in τ.

(2) ⇒ (1) : To use (2) properly, it might be useful to work with closed sets instead. Since open sets and closed sets are complements of each other, it's enough to show that every closed set in τ is a closed set in τ'. Indeed, let $U \in \tau$. Then, U^c is closed in τ. So, U^c is closed in τ'. But that means that U is open in τ.

So, let C be a closed set in τ. Let $(x_\delta)_{\delta \in D}$ be a net in C which converges to some $x \in X$ in τ'. Then, $x_\delta \to x$ in τ. Since C is closed in τ, it follows that $x \in C$. But this is exactly what it means for C to be closed in τ' so we are done. □

Theorem C.0.11. Unique Topology Lemma

Let X be a set and let τ, τ' be topologies on X. Suppose that both topologies have the same convergent nets with the same limits. Then, $\tau = \tau'$.

Proof. The fact that both topologies have the same convergent nets with the same limits implies that $\tau \subseteq \tau'$ and $\tau' \subseteq \tau$, by Lemma C.0.10. It follows that $\tau = \tau'$. □

This lemma is actually more useful than you might think. I'll explain what I mean in just a moment; first, let me give you a characterization of continuity of maps using nets. Such a characterization is useful to have once you realize that it is analogous to the sequential characterization of continuity that is used all the time in the setting of metric spaces.

Theorem C.0.12

Let (X, τ_X) and (Y, τ_Y) be topological spaces. Let $f : X \to Y$ be a map. The following are equivalent:

1. f is continuous.
2. If $(x_\delta)_{\delta \in D}$ is a net in X which converges to $x \in X$, then $(f(x_\delta))_{\delta \in D}$ is a net in Y that converges to $f(x)$.

Proof.

(1) ⇒ (2) : Let U be an open set in Y so that $f(x) \in U$. Then, $x \in f^{-1}(U)$. Since f is continuous, $f^{-1}(U)$ is an open set in X. This implies that $(x_\delta)_{\delta \in D}$ is eventually in $f^{-1}(U)$. As a consequence, $(f(x_\delta))_{\delta \in D}$ is eventually in U. This proves convergence.

(2) ⇒ (1) : Let $U \subseteq Y$ be a closed set. Let us show that $f^{-1}(U)$ is closed. Let $(x_\delta)_{\delta \in D}$ be a net in $f^{-1}(U)$ that converges to x. Then, $f(x_\delta) \to f(x)$. But $(f(x_\delta))_{\delta \in D}$ is a net in U and U is closed, which implies that $f(x) \in U$. Hence, $x \in f^{-1}(U)$ and we are done. □

These characterization theorems aren't just being proven for fun; I do use them a fair amount throughout the thesis. But now, I want to show you that there is an easy way to understand what a "weak topology" is really doing by making use of nets. I'll give the definition of a weak topology on a set first.

Definition C.0.13

Let X be a set and let $((Y_i, \tau_i))_{i \in I}$ be an indexed family of topological spaces. Let $(f_i : X \to Y_i)_{i \in I}$ be a family of maps. The weak topology on X with respect to this family of maps is the smallest topology on X such that every f_i is continuous.

In particular, Let us note that sets of the form $f_i^{-1}(U_i)$, where $i \in I$ and $U_i \subseteq Y_i$ is open, actually form a subbasis for the weak topology on X. This will be very useful in the proof of the following result. If you've done some functional analysis, then you already have some idea of how weak topologies are supposed to work and the definition above is just a generalization of that. Let me prove the result which will improve upon your understanding of the "weak" topologies you have encountered.

Proposition C.0.14

Let X be a set and let $((Y_i, \tau_i))_{i \in I}$ be an indexed family of topological spaces. Let $(f_i : X \to Y_i)_{i \in I}$ be a family of maps and suppose that X has the weak topology with respect to the given family of maps. Let $(x_\delta)_{\delta \in D}$ be a net in X and let $x \in X$. The following are equivalent:

1. $x_\delta \to x$ in the weak topology.
2. $\forall i \in I : f_i(x_\delta) \to f(x)$.

Proof.

(1) $\Rightarrow$ (2) : This follows immediately from the definition of the weak topology and the characterization of continuity as proved above.

(2) $\Rightarrow$ (1) : Let U be an open neighborhood of x. Then, there exist $i_1, \dots, i_n \in I$ and open sets $U_{i_1}, \dots, U_{i_n}$ such that $x \in \bigcap_{k=1}^n f_{i_k}^{-1}(U_{i_k}) \subseteq U$. Since $f_{i_k}(x_\delta) \to f_{i_k}(x)$ for every k, it follows that $f_{i_k}(x_\delta) \in U_{i_k}$ eventually for each k. It follows that $(x_\delta)_{\delta \in D}$ in U eventually and that proves the result. □

In other words, convergence in a weak topology amounts to point-wise convergence of the maps which generate that topology. This simple characterization will be useful throughout the thesis. As it also happens, we will find it useful to have a characterization of Hausdorff spaces too.

Theorem C.0.15. Characterization of Hausdorff Spaces

Let (X, τ) be a topological space. The following are equivalent:

1. (X, τ) is Hausdorff.
2. Every convergent net in X has a unique limit.

Proof.

(1) $\Rightarrow$ (2) : Let $(x_\delta)_{\delta \in D}$ be a convergent net in X with limits x, y. Since X is Hausdorff, x and y are separated by disjoint open sets U and V. But since the net converges to both x and y, it must eventually lie in both U and V. That is a contradiction.

(2) $\Rightarrow$ (3) : Let $x, y \in X$ be distinct points. Let $\mathcal{N}_x$ and $\mathcal{N}_y$ denote the set of neighborhoods of x and y respectively. Suppose that for every $U \in \mathcal{N}_x$ and every $V \in \mathcal{N}_y$, $U \cap V \neq \varnothing$. In other words, suppose that all of their individual neighborhoods intersect. Define:

$$D = \{U \cap V : U \in \mathcal{N}_x \wedge V \in \mathcal{N}_y\}$$

Turn this into a directed set as follows:

$$\forall U, V \in D : U \leq V \Leftrightarrow V \subseteq U$$

Since every set in D is non-empty, we can select one element from each set [a] and form a net $f : D \to X$. I claim that f converges to x and that it converges to y. Indeed, let N be any neighborhood of x. If $N' \in D$ is such that $N \leq N'$, then $N' \subseteq N$. Since $f(N') \in N'$, it follows that $f(N') \in N$ and that proves that $f \to x$. A similar argument holds for y. By the hypothesis, $x = y$ and that is a contradiction. Hence, D must contain the empty set. That is equivalent to x and y being separated by neighborhoods. □

[a] This requires the Axiom of Choice.

Now, Let us deal with compactness. To do this, I will need to introduce two notions; subnets and universal nets. Subnets are a bit easier to motivate because they can be seen as generalizations of subsequences. Universal nets are (quite a bit) harder to motivate without the use of other set-theoretic constructions like ultrafilters.

Definition C.0.16

Let A, B be two directed sets and let $u : A \to B$ be a map. u is said to be a finalizing map if:

$$\forall b \in B : \exists a_0 \in A : \forall a \geq a_0 : u(a) \geq b$$

Let (X, τ) be a topological space and let $(x_\delta)_{\delta \in D}$ be a net in X. A net $(y_\beta)_{\beta \in B}$ is a subnet of the given net if there exists a finalizing map $u : B \to D$ such that $y_\beta = x_{u(\beta)}$ for every $\beta \in B$.

Now, we have a characterization theorem for compactness which is certainly very valuable in General Topology. There is a particular instance in which we will find use for this result but we have mostly included it for completeness.

Theorem C.0.17. Characterization Theorem for Compactness

Let (X, τ) be a topological space. The following are equivalent:

1. (X, τ) is compact.
2. Each family of closed subsets of X with the finite intersection property[a] has a non-empty intersection
3. Every net in (X, τ) has a convergent subnet.
4. Every net in (X, τ) has a cluster point[b].

[a] The intersection of every set in each finite subfamily of this family is non-empty.
[b] A point $x \in X$ is a cluster point of a net iff the net lies frequently in each neighborhood of x.

Proof.

(1) ⇒ (2) : Let $\mathcal{C}$ be a family of closed sets with the finite intersection property. Suppose that $\bigcap_{C \in \mathcal{C}} C = \varnothing$. Then, $\bigcup_{C \in \mathcal{C}} (X \setminus C) = X$. That is, the indexed family $(X \setminus C)_{C \in \mathcal{C}}$ of open sets is an open cover for X. By compactness, there exist closed sets $C_1, \ldots, C_m$ such that $X = \bigcup_{n=1}^{m} (X \setminus C_n)$. But this means that $\bigcap_{i=1}^{m} C_i = \varnothing$ and this is a contradiction.

(2) ⇒ (3) : Let $(x_\delta)_{\delta \in D}$ be a net in X. Define:

$$\forall \delta_0 \in D : S_{\delta_0} := \{x_\delta : \delta \geq \delta_0\}$$

Then, we find that the family of closures $(\overline{S_{\delta_0}})_{\delta_0 \in D}$ is a family of closed sets which satisfy the finite intersection property. In particular, their intersection is non-empty. Therefore, there is a point $x \in X$ such that $x \in \overline{S_{\delta_0}}$ for every $\delta_0 \in D$. This implies the existence of a subnet that converges to x.

(3) ⇒ (4) : Every net has a convergent subnet; in particular, a limit of that convergent subnet is a cluster point so we are done.

(4) ⇒ (1) : Let us prove that (4) ⇒ (2); the fact that (2) ⇒ (1) is fairly standard. Let $\mathcal{C}$ be a set of closed subsets of X with the finite intersection property and let G be the set of all finite intersections of elements of $\mathcal{C}$. Define a relation on G:

$$\forall x, y \in G : x \prec y \Leftrightarrow y \subseteq x$$

This relation turns G into a directed set. By the Axiom of Choice, we can construct a net $(f_x)_{x \in G}$ such that $f_x \in x$ for every $x \in G$[a]. This net has a cluster point; let us denote that by f. Assume that $\bigcap_{C \in \mathcal{C}} C = \varnothing$. Then, $\bigcup_{C \in \mathcal{C}} (X \setminus C) = X$. Therefore, $f \in X \setminus C$ for some $C \in \mathcal{C}$. Now, there is a $D \in \mathcal{C}$ such that $C \prec D$ $(D \subseteq C)$ and $f_D \in X \setminus C$. But $f_D \in D$ so $f_D \in C$ and that is a contradiction. It follows that $\bigcap_{C \in \mathcal{C}} C = \varnothing$[b]. □

[a] This is possible because each $x \in G$ is non-empty, by the strength of the finite intersection property.
[b] The proof given here is (mostly) inspired by the treatments of Theorems 1 and 2 in [24, p.136].

Let us give a rather nice application of the theorem above.

Proposition C.0.18

Let (X, τ_X) and (Y, τ_Y) be topological spaces, with X being compact and Y being Hausdorff. If $f : X \to Y$ is a continuous bijection, then it is a homeomorphism.

Proof. Let $(y_\delta)_{\delta \in D}$ be a net in Y that converges to some $y \in Y$. For each $\delta \in D$, there is an $x_\delta \in X$ such that $f(x_\delta) = y_\delta$. Moreover, there is an $x \in X$ such that $f(x) = y$; all of these are unique because f is a bijection. Suppose that $(x_\delta)_{\delta \in D}$ does not converge to x; it must have a subnet that does not converge to x. Since X is compact, this subnet must have a convergent subnet with limit x'. By the continuity of f and the fact that Y is Hausdorff, it follows that $x' = x$. But this contradicts the fact that $(x_\delta)_{\delta \in D}$ does not converge to x so we are done. □

This completes our exposition of Moore-Smith Convergence. Of course, there is much more that can be said here and, in general, it is a bit weird to discuss nets without discussing filters but I had to do that for space reasons.

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