

Parametricity and Strong Dinaturality

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Motivation

- Parametric polymorphism is one of the key features of modern (functional) languages:
 - most commonly in Hindley-Milner style, where type variables are quantified in top level,
 - but more recently also in less restricted form (eg. rank-2 or rank-n polymorphism).
- "Common knowledge" about polymorphism:
 - you get theorems for free! [Wadler, 1989] (but there are some pessimists);
 - polymorphic functions are (di)natural transformations (or maybe vice versa?!);
 - if you have PhD, get a real job! [Eppendhal, 2004]

Logical Relations and Parametricity

- The usual reading of a type is that it's a set of values (maybe with some extra structure):
 - eg. the type `Int` is the set of integers;
 - the type `A × B` is the set of pairs, where components are from the sets corresponding to types `A` and `B` respectively;
 - etc.
- An alternative is to take that a type is a relation:
 - base types are interpreted as identity relations;
 - eg. $(x, y) \in \text{Int} \Leftrightarrow x = y$;
 - every type constructor is interpreted as a corresponding action on relations.
- Relational reading is the key for parametricity results and free theorems.

Logical Relations and Parametricity

Definition

- For any relations $\mathcal{A} : A \leftrightarrow A'$, $\mathcal{B} : B \leftrightarrow B'$, the relation $\mathcal{A} \times \mathcal{B} : (A \times B) \leftrightarrow (A' \times B')$ is defined by:

$$((x, y), (x', y')) \in \mathcal{A} \times \mathcal{B} \quad \text{iff} \quad (x, x') \in \mathcal{A} \ \& \ (y, y') \in \mathcal{B}$$

- For any relations $\mathcal{A} : A \leftrightarrow A'$, $\mathcal{B} : B \leftrightarrow B'$, the relation $\mathcal{A} \rightarrow \mathcal{B} : (A \rightarrow B) \leftrightarrow (A' \rightarrow B')$ is defined by:

$$(f, f') \in \mathcal{A} \rightarrow \mathcal{B} \quad \text{iff} \quad (x, x') \in \mathcal{A} \Rightarrow (fx, f'x') \in \mathcal{B}$$

- For any relation transformer $\mathcal{F} : F \leftrightarrow F'$, the relation $\forall \mathcal{X}. \mathcal{F}(\mathcal{X}) : (\forall X. F(X)) \leftrightarrow (\forall X. F'(X))$ is defined by:

$$(g, g') \in \forall \mathcal{X}. \mathcal{F}(\mathcal{X}) \quad \text{iff} \quad \mathcal{A} : A \leftrightarrow A' \Rightarrow (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A})$$

Logical Relations and Parametricity

Parametricity

If T is a closed type and $t : T$ is a closed term, then $(t, t) \in \mathcal{R}$, where $\mathcal{R} : T \leftrightarrow T$ is the relation corresponding to the type T .

Theorems for free

Given a closed type T

- construct the corresponding relation $\mathcal{R} : T \leftrightarrow T$;
- instantiate relation transformers with graph relations;
- and simplify.

Definition

Given a function $g : A \rightarrow B$, the graph relation $\langle g \rangle : A \leftrightarrow B$ is defined by $\langle g \rangle = \{(u, g\ u) \mid \forall u : A\}$

Logical Relations and Parametricity

Example

$$\begin{aligned}(t, t) \in \forall \mathcal{X}. \mathcal{X} \rightarrow \mathcal{X} &\Leftrightarrow \forall \mathcal{R} : A \leftrightarrow B. (t_A, t_B) \in \mathcal{R} \rightarrow \mathcal{R} \\&\Leftrightarrow \forall \mathcal{R} : A \leftrightarrow B. \forall x : A, y : B. \\&\quad (x, y) \in \mathcal{R} \Rightarrow (t_A x, t_B y) \in \mathcal{R} \\&\Rightarrow \forall g : A \rightarrow B. \forall x : A, y : B. \\&\quad (x, y) \in \langle g \rangle \Rightarrow (t_A x, t_B y) \in \langle g \rangle \\&\Leftrightarrow \forall g : A \rightarrow B. \forall x : A, y : B. \\&\quad y = g x \Rightarrow t_B y = g(t_A x) \\&\Leftrightarrow \forall g : A \rightarrow B. \forall x : A. t_B(g x) = g(t_A x) \\&\Leftrightarrow \forall g : A \rightarrow B. t_B \circ g = g \circ t_A\end{aligned}$$

Note

The equation says that t is a natural transformation $\text{Id} \rightarrow \text{Id}$.

Natural transformations

Definition

Let $G, H : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A **natural transformation** $\tau : G \rightarrow H$ is a family of maps $\tau_X : G(X) \rightarrow H(X)$ in $\mathcal{D}$ such that, for every map $f : X \rightarrow Y$ in $\mathcal{C}$, the following square commutes:

$$\begin{array}{ccc} G(X) & \xrightarrow{\tau_X} & H(X) \\ G(f) \downarrow & & \downarrow H(f) \\ G(Y) & \xrightarrow{\tau_Y} & H(Y) \end{array}$$

Note

- Type may have mixed variant type variables.
- Separate the co- and contravariant instances and use diagonalization to recover the original type.
- Eg. $\forall X. X \rightarrow X = \forall X. H(X, X)$, where $H(A, B) = A \rightarrow B$.

Dinaturality

Definition

Let $G, H : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$ be functors.

A **dinatural transformation** $\theta : G \rightarrow H$ is a family of maps $\theta_X : G(X, X) \rightarrow H(X, X)$ in $\mathcal{D}$ such that, for every map $f : X \rightarrow Y$ in $\mathcal{C}$, the following hexagon commutes:

$$\begin{array}{ccccc} & & G(X, X) & \xrightarrow{\theta_X} & H(X, X) \\ & \nearrow^{G(f, X)} & & & \searrow^{H(X, f)} \\ G(Y, X) & & & & H(X, Y) \\ & \searrow_{G(X, f)} & & & \nearrow_{H(f, Y)} \\ & & G(Y, Y) & \xrightarrow{\theta_Y} & H(Y, Y) \end{array}$$

Strong dinaturality

Definition

A **strong dinatural transformation** $\theta : G \rightarrow H$ is a family of maps $\theta_X : G(X, X) \rightarrow H(X, X)$ in $\mathcal{D}$ such that, for every

- map $f : X \rightarrow Y$ in $\mathcal{C}$,
- object W in $\mathcal{D}$ and
- maps $p_0 : W \rightarrow G(X, X)$, $p_1 : W \rightarrow G(Y, Y)$ in $\mathcal{D}$,

if the square commutes, then so does the hexagon:

$$\begin{array}{ccccc} & G(X, X) & \xrightarrow{\theta_X} & H(X, X) & \\ & \nearrow p_0 & \searrow G(X, f) & \searrow H(X, f) & \\ W & & G(X, Y) & \Rightarrow & H(X, Y) \\ & \searrow p_1 & \nearrow G(f, Y) & \nearrow H(f, Y) & \\ & G(Y, Y) & \xrightarrow{\theta_Y} & H(Y, Y) & \end{array}$$

From parametricity to strong dinaturality

Covariant types

$$\begin{array}{lcl} F(A) & ::= & A \qquad \qquad \qquad | \quad C \\ & | & F(A) \times F(A) \quad | \quad F(A) + F(A) \\ & | & G'(A) \rightarrow F(A) \quad | \quad \forall X. F([X, A]) \\ G'(A) & ::= & C \\ & | & G'(A) \times G'(A) \quad | \quad G'(A) + G'(A) \\ & | & F(A) \rightarrow G'(A) \end{array}$$

From parametricity to strong dinaturality

Contravariant types

$$\begin{array}{lll} G(A) & ::= & C \\ & | & G(A) \times G(A) \quad | \quad G(A) + G(A) \\ & | & F'(A) \rightarrow G(A) \quad | \quad \forall X. G([X, A]) \\ F'(A) & ::= & A \quad | \quad C \\ & | & F'(A) \times F'(A) \quad | \quad F'(A) + F'(A) \\ & | & G(A) \rightarrow F'(A) \end{array}$$

From parametricity to strong dinaturality

Definition

A functor $H : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$ is **weakly cartesian** if for all $f : A \rightarrow B$ the bifactoriality diagram is a weak pullback:

$$\begin{array}{ccc} H(B, A) & \xrightarrow{H(f, A)} & H(A, A) \\ H(B, f) \downarrow & & \downarrow H(A, f) \\ H(B, B) & \xrightarrow{H(f, B)} & H(A, B) \end{array}$$

Weakly cartesian types

$$\begin{array}{lcl} H(A, B) & ::= & G'(A) \quad \quad \quad | \quad F'(B) \\ & | & H(A, B) \times H(A, B) \quad | \quad H(A, B) + H(A, B) \\ & | & C \rightarrow H(A, B) \end{array}$$

From parametricity to strong dinaturality

Definition

A type K is **Eq2R** if for all closed terms $a : K(A, A)$, $b : K(B, B)$ and functions $g : A \rightarrow B$

$$K(A, g) a = K(g, B) b \implies (a, b) \in \mathcal{K}\langle g \rangle$$

Eq2R types

$K(A, B) ::= B$		C
$K(A, B) \times K(A, B)$		$K(A, B) + K(A, B)$
$H(B, A) \rightarrow K(A, B)$		$\forall X. K([X, A], [X, B])$

From parametricity to strong dinaturality

Definition

A type K is **R2Eq** if for all closed terms $a : K(A, A)$, $b : K(B, B)$ and functions $g : A \rightarrow B$

$$(a, b) \in \mathcal{K}\langle g \rangle \implies K(A, g) a = K(g, B) b$$

R2Eq types

$L(A, B) ::= B$		C
$L(A, B) \times L(A, B)$		$L(A, B) + L(A, B)$
$K(B, A) \rightarrow L(A, B)$		$\forall X. L([X, A], [X, B])$

From parametricity to strong dinaturality

Theorem

Let K and L be System F types containing one free type variable and let $t : \forall X. K(X, X) \rightarrow L(X, X)$ be a closed term of closed type.

If K is Eq2R and L is R2Eq, then t is a strongly dinatural transformation.

From parametricity to strong dinaturality

Theorem

Let $F : \mathcal{C} \rightarrow \mathcal{C}$ be a functor and $C \in \mathcal{C}$ an object. If F has an initial algebra μF then:

$$\text{SDinat}(\text{Hom}(F-, -), \text{Hom}(C, -)) \cong \text{Hom}(C, \mu F)$$

Corollary

Let F be a type expression with one covariant type variable, derivable from nonterminal F' . Then

$$\forall X. (F(X) \rightarrow X) \rightarrow X \cong \mu F$$

From parametricity to strong dinaturality

Example

- Fixpoints are not definable in System F :

$$\begin{aligned}\forall X. (X \rightarrow X) \rightarrow X &\cong \forall X. (\text{Id}(X) \rightarrow X) \rightarrow X \\ &\cong \mu \text{Id} \cong 0\end{aligned}$$

- Polymorphic identity:

$$\begin{aligned}\forall X. X \rightarrow X &\cong \forall X. (\mathbf{1}(X) \rightarrow X) \rightarrow X \\ &\cong \mu \mathbf{1} \cong 1\end{aligned}$$

- Empty type:

$$\begin{aligned}\forall X. X &\cong \forall X. \mathbf{1} \rightarrow X \\ &\cong \forall X. (\mathbf{0}(X) \rightarrow X) \rightarrow X \\ &\cong \mu \mathbf{0} \cong 0\end{aligned}$$

Conclusions and Further Work

- We have identified a class of types whose terms are strongly dinatural in every parametric model.
- The class is large enough to cover several important applications;
 - eg. Church encoding of initial algebras.
- Possible directions for the future work include:
 - to investigate the relationship with other formalisms (eg. structural polymorphism [Freyd, 1993], polynomial polymorphism [Jay, 1995], cospan diparametricity [Eppendahl, 2005]);
 - to try to find less syntactic characterization of the suitable classes of types.