MACQUARIE UNIVERSITY

Macquarie University,
New South Wales 2109,
Australia

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Relating Descent Notions

Dear Urs,

It would seem to me that and understanding of the result you seek, regarding the relationship between the descent notions in the related contexts of strict ω-categories and simplicially enriched categories, does indeed arise quite naturally from the kinds of considerations you outline in your recent email to me on this subject.

However, we need to be careful here because a full answer to the question you asked does not arise from pure enriched category theory alone. It has a homotopical component which must also be taken into account.

Having spent the last couple of days mulling over this issue, while recovering from a nasty spider bite, I believe that I have a complete proof of the kind of result you seek. I will outline my full argument supporting this analysis later in this letter, but I makes sense here to telegraph those results in the following capsule statement:

Notation 1. First let us fix some notation by taking $\omega\!-\!\underline{\mathrm{Cat}}_{\mathrm{st}}$ to denote the category of all (small) strict ω-categories and strict ω-functors between them. Furthermore, let $\omega\!-\!\underline{\mathrm{Gpd}}_{\mathrm{st}}$, the category of all strict ω-groupoids, be the full subcategory of $\omega\!-\!\underline{\mathrm{Cat}}_{\mathrm{st}}$ whose objects are those ω-categories all of whose cells are equivalences in some suitable sense.

Furthermore let N denote Street's ω-categorical nerve functor

$$\omega\!-\!\underline{\mathrm{Cat}}_{\mathrm{st}} \xrightarrow{\quad N \quad} \underline{\mathrm{Simp}} \tag{1}$$

which (for the moment) carries each strict ω-category to a simplicial set. Also, from here on we will assume that the category $\underline{\mathrm{Simp}}$ implicitly carries the Kan model structure, under which inclusions are the cofibrations, map possessing the right lifting property with respect to all horns are the fibrations and the weak equivalences are those maps which induce isomorphisms of homotopy groups.

Finally let $\mathrm{Desc}_{\omega-\underline{\mathrm{Cat}}_{\mathrm{st}}}$ denote Street's (lax) descent construction on strict ω categories. Correspondingly, let $\mathrm{Desc}_{\underline{\mathrm{Simp}}}$ denote the standard descent construction on simplicial sets.

Theorem 2. *Suppose that $X\colon \Delta \longrightarrow \omega\!-\!\underline{Cat}_{st}$ is a functor into the category of strict ω-categories, then there exists a canonical comparison map*

$$N(\mathrm{Desc}_{\omega-\underline{Cat}_{st}}(X)) \hookrightarrow \mathrm{Desc}_{\underline{Simp}}(N\circ X) \tag{2}$$

which is natural in $X \in [\Delta, \omega\!-\!\underline{Cat}_{st}]$ and an inclusion of simplicial sets (that is to say a cofibration).

Assume further that on composing this functor with Street's ω-categorical nerve functor $N\colon \omega\!-\!\underline{Cat}_{st} \longrightarrow \underline{Simp}$ we obtain a fibrant object $N\circ X\colon \Delta \longrightarrow \underline{Simp}$ in the Reedy model structure on the functor category $[\Delta, \underline{Simp}]$. Then, under this extra condition, the map in display (2) *is also a weak equivalence of Kan complexes.*

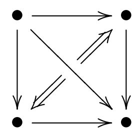
Notice here that I make no specific assumption here that the functor $X\colon \Delta \longrightarrow \omega\!-\!\underline{\mathrm{Cat}}_{\mathrm{st}}$ whose descent is being studied actually takes values in the full subcategory of strict ω-groupoids. However, my fibrancy assumption on $N\circ X$ implies, in particular, that the simplicial set $N(X(n))$ is a Kan complex for each object $n \in \mathrm{obj}(\Delta)$. Then, in turn, it is a routine consequence of my work on strict complicial sets [8] that a strict ω-category $\mathbb{C} \in \omega\!-\!\underline{\mathrm{Cat}}_{\mathrm{st}}$ is a strict ω-groupoid, in the sense alluded to above, if and only if its nerve $N(\mathbb{C})$ is a Kan complex. So it follows that if $N\circ X\colon \Delta \longrightarrow \underline{\mathrm{Simp}}$ is Reedy fibrant in $[\Delta, \underline{\mathrm{Simp}}]$ then the functor $X\colon \Delta \longrightarrow \omega\!-\!\underline{\mathrm{Cat}}_{\mathrm{st}}$ factors through the full subcategory of strict ω-groupoids $\omega\!-\!\underline{\mathrm{Gpd}}_{\mathrm{st}}$ in $\omega\!-\!\underline{\mathrm{Cat}}_{\mathrm{st}}$.

Now, before proving this theorem, I should point out that while your overall argument regarding the limits possessed by the enriched categories whose bases are being changed is correct, the detail of your argument falls a little short in two respects:

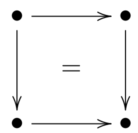
1. To make this kind of argument work it is important to identify exactly how Street's descent construction may be described as a weighted limit is some enriched category theory. It appears that your email

implicitly assumes that we should simply regard $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ as a category enriched in itself relative to the cartesian product on that category. On the other hand, Ross' descent construction captures information about *lax* structures and may thus only be regarded as an enriched limit in $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ if we regard that category as a base of enrichment with respect to its lax Gray tensor product $\otimes$.

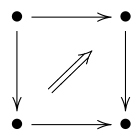
2. As we will see later on, your intuition with regard to strong monoidality of $F\colon \omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}} \longrightarrow \underline{\mathrm{Simp}}$ is quite correct. To ensure that change of enrichment base change along a right adjoint functor like the nerve functor $N\colon \omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}} \longrightarrow \underline{\mathrm{Simp}}$ behaves well with respect to the colimits in $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$-enriched categories it is sufficient to know that its left adjoint is strong monoidal with respect to the chosen monoidal structures on $\underline{\mathrm{Simp}}$ (in this case cartesian product) and $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$.
However, it is unfortunate in this case that this functor $F\colon \omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}} \longrightarrow \underline{\mathrm{Simp}}$ is *not* strong monoidal with respect to the cartesian product on $\underline{\mathrm{Simp}}$ and either the cartesian product or the (lax) Gray tensor product on $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$. To see what fails here it is easiest simply to consider a simple example. Consider the standard 1-simplex Δ^1 in $\underline{\mathrm{Simp}}$ and observe that maps under F to the strict ω-category consisting of two 0-cells with a single 1-cell between then (customarily drawn $\bullet \to \bullet$). Now, it is a matter of straightforward calculation to draw presentations of each of the three strict ω-categories $F(\Delta^1 \times \Delta^1)$, $F(\Delta^1) \times F(\Delta^1)$ and $F(\Delta^1) \otimes F(\Delta^1)$.
Firstly, the simplicial set $\Delta^1 \times \Delta^1$ maybe described as a pair of 2-simplices glued along their 1-dimensional faces corresponding to the face operator δ_0. So, since F preserves colimits we see that $F(\Delta^1 \times \Delta^1)$ may be constructed by gluing together two copies of the second oriental thus:

$F(\Delta^1 \times \Delta^1) =$  (3)

On the other hand, the latter two structures may be presented as

$F(\Delta^1) \times F(\Delta^1) = \{\bullet \to \bullet\} \times \{\bullet \to \bullet\} =$  (4)

and:

$F(\Delta^1) \otimes F(\Delta^1) = \{\bullet \to \bullet\} \otimes \{\bullet \to \bullet\} =$  (5)

It follows, therefore, that no pair of these strict ω-categories are isomorphic to each other.
Furthermore, things are not improved one iota by reflecting these strict ω-categories into strict ω-groupoids, where they still remain stubbornly distinct. Although, in that context we may at least show that the groupoidal reflections of the strict ω-categories in displays (3) and (5) are at least ω-categorically equivalent to each other. It is this latter observation which will, eventually, lead us to proving the result postulated in the statement of theorem 2.

So how might we fix things up here? The first of these points tells us that in order to capture Street's descent notion we should be careful to do our $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$-enriched category theory with respect to the lax Gray tensor product $\otimes$. The second tells us that this won't give us a relationship between the simplicial and Street descent constructions as a piece of pure enriched category theory, because the left adjoint to the nerve construction is not strong monoidal with regard to cartesian product on $\underline{\mathrm{Simp}}$ and lax Gray tensor on $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$.

We can resolve this tension in one of two ways:

- If we wish to remain in the realm of enriched category theory alone we can pick a tensor product on $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ (or more properly $\omega\text{−}\underline{\mathrm{Gpd}}_{\mathrm{st}}$) which makes the functor $F\colon \underline{\mathrm{Simp}} \longrightarrow \omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ strong monoidal. Indeed, there does exist just such a monoidal structure on $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$, which we might think of as a kind of *Gray tensor with diagonals* and whose definition is modelled upon higher dimensional versions of the picture presented in display (3). By enriching over that tensor we obtain a descent notion, as a weighted limit, which is indeed related to simplicial descent in precisely the sense that you gave in your email using exactly the argument you outline there.
This sounds promising, and it is a trivial matter to establish this result using either direct calculation or some abstract enriched category theory, but it does not actually answer the question you asked. As I understand it you are specifically interested in Street's descent notion, and this is quite distinct from the notion you obtain using the Gray tensor with diagonals.

- Concentrate upon understanding Street's descent construction using the lax Gray tensor product on $\omega\text{--}\underline{\mathrm{Cat}}_{\mathrm{st}}$ and accept that we will need to do some homotopy theoretic heavy lifting here. I'll take this approach in the remainder of this letter, mainly just to give us some meat to chew on, and hope to convince you of the voracity of the result I stated as theorem 2 above.

So how do we go about proving this result? Well, as a first step my general preference would be to move away from working with strict ω-categories as quickly as possible and to do most of our work in the context of simplicial sets. By doing so, we avoid having to fiddle about building an explicit homotopy theory for strict ω-groupoids themselves. Furthermore, I generally find it much easier to work with the Gray tensor product in the simplicial (or more precisely complicial) context where it admits a far more explicit presentation.

To make this first step, we may simply apply the kind of "change of enrichment base" argument that you outlined in your email. In principle, the result to which you allude is well studied, although I suspect that it is far less well known than it should be, mainly because the only full presentation of a general such result occurs in my, as yet, unpublished thesis [6]. For reference purposes, I have attached a copy of my thesis as a DVI file and you may be interested to hear that it will be published as a TAC reprint as soon as I have a chance to "tart up" its typesetting.

I certainly wouldn't suggest that you waste any time plowing through this work, since it tends to take an exceedingly abstract approach to change of base problems. As a testament to the ridiculous level of abstract attained, I believe that this is probably the very first place that Gray categories (or at least a bicategorical generalisation of such) were applied in anger. Indeed this use predates the Gordon, Power and Street work on tricategories [2] by 2 or 3 years.

Cutting through the abstraction of my thesis, for current purposes we might paraphrase its primary result on the transformation of enriched limits under change of base (theorem 1.7.1 of [6]) as:

Theorem 3. *Suppose $(\mathcal{V}, \otimes, I)$ and $(\mathcal{W}, \otimes', I')$ are monoidal categories (not necessarily symmetric, braided or closed but possessing small colimits which are preserved by tensoring by objects on either side) and that they are related by an adjunction*

$$\mathcal{V} \underset{U}{\overset{F}{\underset{\perp}{\leftrightarrows}}} \mathcal{W} \tag{6}$$

in which the left adjoint F is strong monoidal.

The strong monoidal structure of F (or more specifically its structure as a comonoidal functor) induces a monoidal structure on its right adjoint U. So it follows that we may apply both of the functors U and F "pointwise" to $\mathcal{V}$-enriched (respectively $\mathcal{W}$-enriched) categories, functors, profunctors, weights and so forth and thereby construct corresponding $\mathcal{W}$-enriched (respectively $\mathcal{V}$-enriched) structures.

For example, if $\mathbf{C}$ is a $\mathcal{V}$-category then $U(\mathbf{C})$ is a $\mathcal{W}$-category which has the same set of objects as $\mathbf{C}$ but which has homsets given by $U(\mathbf{C})(c, c') \overset{def}{=} U(\mathbf{C}(c, c'))$. As another example, if W is a $\mathcal{W}$-weight on a $\mathcal{W}$ category $\mathbf{D}$ then $F(W)$ is a $\mathcal{V}$-weight on the $\mathcal{V}$-category $F(\mathbf{D})$ whose value at an object $d \in \mathrm{obj}(F(\mathbf{D})) = \mathrm{obj}(\mathbf{D})$ is $F(W(d)) \in \mathcal{V}$

Indeed these pointwise actions gives rise to a biadjunction (actually a 2-adjunction) between the 2-categories of $\mathcal{V}$-enriched and $\mathcal{W}$-enriched categories and functors, and to a well behaved and closely interrelated local adjunction *between the bicategories of $\mathcal{V}$-enriched and $\mathcal{W}$-enriched categories and profunctors.*

Now suppose that $\mathbf{C}$ is a $\mathcal{V}$-enriched category, that $W \colon \mathbf{D} \longrightarrow \mathcal{W}$ is a weight for a $\mathcal{W}$-enriched limit and that $D \colon F(\mathbf{D}) \longrightarrow \mathbf{C}$ is a $\mathcal{V}$-diagram in $\mathbf{C}$. Suppose further that the $\mathcal{W}$-diagram $\hat{D} \colon \mathbf{D} \longrightarrow U(\mathbf{C})$ is the adjoint transpose of D under the biadjunction induced by $F \dashv U$ on 2-categories of enriched categories and functors. Then:

- *if the $\mathcal{V}$-enriched weighted limit $\lim(F(W), D)$ exists in $\mathbf{C}$ then the $\mathcal{W}$-enriched weighted limit $\lim(W, \hat{D})$ exists in $U(\mathbf{C})$ and is canonically isomorphic to $\lim(F(W), D)$ when this latter limit is regarded as an object of $U(\mathbf{C})$.*
- *under the further assumption that U is a full and faithful functor, we may strengthen this last result to show that the $\mathcal{V}$-enriched weighted limit $\lim(F(W), D)$ exists in $\mathbf{C}$ if and only if the $\mathcal{W}$-enriched weighted limit $\lim(W, \hat{D})$ exists in $U(\mathbf{C})$.*

Proof. In its most abstract and general form this is just theorem 1.7.1 and lemma 1.7.7 of [6], but in the form stated here this result can easily be verified by hand. □

But this isn't quite the result we need, since the result you appealed to in your email actually looks to relate enriched limits in the base categories $\mathcal{V}$ and $\mathcal{W}$ when regarded as enriched over themselves. For our purposes here, we need to be a little careful about what it means to enrich these categories over themselves because we have not placed any symmetry assumptions upon their tensors. This is a necessary generalisation since, as discussed above, we will obtain Street's descent construction as an enriched limit relative to the non-symmetric Gray tensor on $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$.

From here we will assume that $\mathcal{V}$ (and $\mathcal{W}$ for that matter) is closed on both sides of its tensor, in the sense that for each object $V \in \mathrm{obj}(\mathcal{V})$ both of the functors $V \otimes -, - \otimes V : \mathcal{V} \longrightarrow \mathcal{V}$ have right adjoints, denoted $\mathrm{cls}_l(V, *)$ and $\mathrm{cls}_r(V, *)$ respectively. For the specific example of the lax Gray tensor on $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ its left closure $\mathrm{lax}_l(\mathbf{C}, \mathbf{D})$ is the strict ω-category whose cells are strict ω-functors (from $\mathbf{C}$ to $\mathbf{D}$), lax natural transformations between them, lax modifications between those and so forth. On the other hand, its right closure $\mathrm{lax}_r(\mathbf{C}, \mathbf{D})$ is the corresponding strict ω-category of strict ω-functors, oplax natural transformations, oplax modifications and so forth.

Now, given that we have two closures on $\mathcal{V}$, we find that there are two distinct ways of enriching it to an $\mathcal{V}$-enriched category. The first uses the left closure cls_l to give a $\mathcal{V}$ enriched category $\mathcal{V}_l$ whose homsets are given by $\mathcal{V}_l(V, V') \stackrel{\mathrm{def}}{=} \mathrm{cls}_l(V, V')$ and whose composition is defined analogously to the symmetric case. The second uses the right closure cls_r giving another $\mathcal{V}$ enriched category $\mathcal{V}_r$ whose homsets are given by $\mathcal{V}_r(V, V') \stackrel{\mathrm{def}}{=} \mathrm{cls}_r(V, V')$.

As in the classical case, each of the $\mathcal{V}$-enriched categories $\mathcal{V}_l$ and $\mathcal{V}_r$ will have all (small) $\mathcal{V}$-enriched limits and colimits so long as the underlying category $\mathcal{V}$ has all (small) limits and colimits in the usual sense. However it is important to observe that the cotensors of these two $\mathcal{V}$-categories are in general quite distinct, with that of $\mathcal{V}_l$ being the left closure cls_l whereas that of the dual $\mathcal{V}$-enrichment $\mathcal{V}_r$ is the right closure cls_r. It follows therefore that if $\mathbf{D}$ is a (small) un-enriched category regarded as a $\mathcal{V}$-enriched category in the usual way, W is a $\mathcal{V}$-weight on $\mathbf{D}$, and $D : \mathbf{D} \longrightarrow \mathcal{V}$ is a diagram in $\mathcal{V}$ then we may construct two quite distinct enriched limits in $\mathcal{V}$. The first is the $\mathcal{V}$-enriched limit taken in $\mathcal{V}_l$ and given by the formula

$$\lim_{\mathcal{V}_l}(W, D) = \int_{d \in \mathbf{D}} \mathrm{cls}_l(W(d), D(d)) \tag{7}$$

and the second is the $\mathcal{V}$-enriched limit taken in $\mathcal{V}_r$ and given by the formula:

$$\lim_{\mathcal{V}_r}(W, D) = \int_{d \in \mathbf{D}} \mathrm{cls}_r(W(d), D(d)) \tag{8}$$

In the particular case of Street's lax descent construction, the weight for this limit is (as you suggested) the orientals functor $\mathcal{O} : \Delta \longrightarrow \omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ and the descent category for a cosimplicial strict ω-category $X : \Delta \longrightarrow \omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ is simply the $\mathcal{O}$-weighted limit of X relative to the lax Gray tensor. However, the lax Gray tensor is not symmetric or braided, so we actually obtain two distinct lax descent notions corresponding to the two, left handed and right handed, $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$-enrichments of $\omega\text{−}\underline{\mathrm{Cat}}_{\mathrm{st}}$ relative to that tensor. The first of these is properly called Street's *lax descent construction*

$$\mathrm{Desc}_{\mathrm{lax}}(X) = \int_{n \in \Delta} \mathrm{lax}_l(\mathcal{O}^n, X^n) \tag{9}$$

whereas the second is the dual *oplax descent construction*:

$$\mathrm{Desc}_{\mathrm{oplax}}(X) = \int_{n \in \Delta} \mathrm{lax}_r(\mathcal{O}^n, X^n) \tag{10}$$

Notice here that when talking about cosimiplicial objects such as $\mathcal{O}$ or X I will tend to write superscripts such as $\mathcal{O}^n$ and X^n, rather than using the possibly more standard evaluation notation $\mathcal{O}([n])$ and $U([n])$. This allows me to notationally distinguish cosimplicial and simplicial notions, by adopting the standard tensor convention and writing subscripts for simplicial actions.

It is worth pointing out that these two descent notions will not coincide (be naturally isomorphic) even if we restrict attention only to the full subcategory of strict ω-groupoids $\omega\text{−}\underline{\mathrm{Gpd}}_{\mathrm{st}}$. However, it is the case that they will at the least be equivalent in the obvious ω-categorical sense.

With these notational observations out of the way, we are now able to state and justify the following corollary to the last theorem:

Corollary 4. *Suppose that we are in the situation of theorem 2, without assuming the extra postulate that U should be full and faithful. Then the functor U has an $\mathcal{W}$-enrichment $U : U(\mathcal{V}_l) \longrightarrow \mathcal{W}_l$ which possess a $\mathcal{W}$-enriched right adjoint derived from F.*

Suppose further that $\mathbf{D}$ *is a* $\mathcal{W}$*-category,* W *is a* $\mathcal{W}$*-weight on* $\mathbf{D}$ *and* $D\colon F(\mathbf{D}) \longrightarrow \mathcal{V}_l$ *is a diagram in the left* $\mathcal{V}$*-enrichment. Furthermore let* $\hat{D}$ *denote the dual diagram* $\hat{D}\colon \mathbf{D} \longrightarrow U(\mathcal{V})_l$*. Then, under these conditions, there exists a canonical isomorphism*

$$U(\lim(F(W), D)) \cong \lim(W, \hat{D}) \tag{11}$$

in $\mathcal{W}$*.*

Proof. Since F is strong monoidal, a routine calculation allows us to construct a canonical comparison map $U(\mathrm{cls}_l(A,B)) \longrightarrow \mathrm{cls}'_l(U(A), U(B))$. So we may define the $\mathcal{W}$-enrichment $U\colon U(\mathcal{V}_l) \longrightarrow \mathcal{W}_l$ that we seek by mapping each object $A \in \mathrm{obj}(U(\mathcal{V}_l)) = \mathrm{obj}(\mathcal{V})$ to $U(A) \in \mathrm{obj}(\mathcal{W}_l)$ and acting on homsets using the canonical comparison maps of the last sentence. Furthermore, the strong monoidality of F may also be used to establish a canonical isomorphism $U(\mathrm{cls}_l(F(W), V)) \cong \mathrm{cls}'_l(W, U(V))$, which in turn tells us that F gives rise to a $\mathcal{W}$-enriched left adjoint to $U\colon U(\mathcal{V}_l) \longrightarrow \mathcal{W}_l$ which maps each object $W \in \mathcal{W}_l$ to the object $F(W) \in U(\mathcal{V}_l)$.

Now, in particular, the fact that the $\mathcal{W}$-functor $U\colon U(\mathcal{V}_l) \longrightarrow \mathcal{W}_l$ has an $\mathcal{W}$-enriched left adjoint implies that it preserves all $\mathcal{W}$-enriched limits. Consequently the result we seek is simply a matter of applying theorem 2 to obtain a relationship between enriched limits in the $\mathcal{V}$-category $\mathcal{V}_l$ and the $\mathcal{W}$-category $U(\mathcal{V}_l)$ and then applying the $\mathcal{W}$-limit preserving functor $U\colon U(\mathcal{V}_l) \longrightarrow \mathcal{W}_l$.

Alternatively, if this level of abstraction is still a little disquieting it is again possible to prove this result by a direct calculation. □

Now we are in a position, in one fell swoop, to translate our whole problem into a purely simplicial context. More precisely we are going turn it into a problem in the theory of *simplicial sets with thin elements*, otherwise called *stratified simplicial sets* in my various works on (weak) complicial sets.

My AMS Memoir [8] (or its archive variant [5]) on the topic of (strict) complicial sets provides a fairly thorough introduction to the theory of stratified simplicial sets and a detailed analysis of their relationship to strict ω-categories. For our purposes here, we can probably get away with knowing only the following factoids:

- A *stratified simplicial set* (X, tX) consists of a simplicial set X and a specified subset tX of its simplices. The set tX is called a stratification and its elements are said to be *thin*. We ask that it contain no 0-simplices and that it should contain (at least) all of the degenerate simplices of X. A *stratified simplicial map* $f\colon (X, tX) \longrightarrow (Y, tY)$ is simply a simplicial map $f\colon X \longrightarrow Y$ which *preserves thinness* in the sense that $f(tX) \subseteq tY$.
- The forgetful functor $\overline{(\ )}\colon \underline{\mathrm{Strat}} \longrightarrow \underline{\mathrm{Simp}}$ which forgets stratifications has both a left adjoint and a right adjoint. Its left adjoint $\min\colon \underline{\mathrm{Simp}} \longrightarrow \underline{\mathrm{Strat}}$ equips each simplicial set with a minimal stratification, under which only degenerate simplices are taken as being thin, and its right adjoint $\max\colon \underline{\mathrm{Simp}} \longrightarrow \underline{\mathrm{Strat}}$ equips each simplicial set with a maximal stratification, under which all simplices are taken as being thin. Both of the adjoint min and max are full and faithful.
- We will identify the category $\underline{\mathrm{Simp}}$ with its image in $\underline{\mathrm{Strat}}$ under the minimal stratification functor min. On doing so we will observe customary convention and regard the underlying simplicial set functor $\overline{(\ )}$ as being the endo-functor of $\underline{\mathrm{Strat}}$ which replaces the stratification on each stratified simplicial set by the minimal stratification on its underlying set.
- Under this convention we will need to be a little careful to distinguish the simplicial and stratified simplicial variants of many common constructions. For example, if we regard simplicial sets X and Y as minimally stratified objects in $\underline{\mathrm{Strat}}$ their product $X \times Y$ in there is not, generally, minimally stratified. Consequently the cartesian products of these sets in $\underline{\mathrm{Simp}}$ and $\underline{\mathrm{Strat}}$ are distinct and should be distinguished in our notation. Where necessary, we will use "barred" versions of operator symbols to denote simplicial, rather than stratified simplicial, versions of such operator. So for example the cartesian product in $\underline{\mathrm{Simp}}$ would be denoted $X \mathbin{\bar{\times}} Y$ and the corresponding closure by $\overline{\hom}(X, Z)$.
- Ross' nerve construction [4] may be generalised slightly to give an adjoint pair of functors:

$$\omega\text{--}\underline{\mathrm{Cat}}_{\mathrm{st}} \underset{N}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \underline{\mathrm{Strat}} \tag{12}$$

  If $\mathbf{C}$ is a strict ω-category then the stratified simplicial set $N(\mathbf{C})$ has as its n-simplices the functors $x\colon \mathcal{O}^n \longrightarrow \mathbf{C}$ from the n^{th} oriental to $\mathbf{C}$ and stratification under which the simplex x is thin if and only if it maps the unique non-trivial n-cell of $\mathcal{O}^n$ to a cell of dimension lower than n (an identity) in $\mathbf{C}$.

- The nerve functor $N\colon \omega{-}\underline{\mathrm{Cat}}_{\mathrm{st}} \longrightarrow \underline{\mathrm{Strat}}$ is full and faithful and a stratified simplicial set is in its replete image if and only if it is *strictly complicial*. Here a stratified simplicial set A is strictly complicial if and only if its thin 1-simplices are all degenerate, it possesses *unique thin fillers* for a certain class of *complicial horns* and it satisfies a related thinness stability property called *pre-compliciality*. This theorem, often referred to as the Street-Roberts conjecture, is the ultimate result theorem 266 of [8].
- There is a tensor product $\otimes$ on $\underline{\mathrm{Strat}}$ which is the simplicial analogue of the lax Gray tensor on $\omega{-}\underline{\mathrm{Cat}}_{\mathrm{st}}$. The tensor $X \otimes Y$ of stratified simplicial sets $X, Y \in \underline{\mathrm{Strat}}$ has as its underlying simplicial set the cartesian product of underlying simplicial sets and has a stratification which is defined in precise terms in definition 128 of loc. cit. It is a routine consequence of theorem 255 of loc. cit. that the left adjoint $F\colon \underline{\mathrm{Strat}} \longrightarrow \omega{-}\underline{\mathrm{Cat}}_{\mathrm{st}}$ is strong monoidal with respect to this new tensor on $\underline{\mathrm{Strat}}$ and the lax Gray tensor on $\omega{-}\underline{\mathrm{Cat}}_{\mathrm{st}}$ (as defined in [1]).
- The tensor $\otimes$ is not actually left or right closed as a tensor on $\underline{\mathrm{Strat}}$. To rectify this deficiency, we introduce a full subcategory $\underline{\mathrm{Pcs}}$ of *pre-complicial sets* in $\underline{\mathrm{Strat}}$ (see definition 121 of [8]) is reflective in there with reflector $(\ )^{\diamond}\colon \underline{\mathrm{Strat}} \longrightarrow \underline{\mathrm{Pcs}}$ which expands the tX stratification of each stratified simplicial set X to a smallest pre-complicial stratification $tX^{\diamond}$ containing tX. and we may define a tensor $\otimes^{\diamond}$ on $\underline{\mathrm{Pcs}}$ by $X \otimes^{\diamond} Y \stackrel{\mathrm{def}}{=} (X \otimes Y)^{\diamond}$ which is both left and right closed on $\underline{\mathrm{Pcs}}$, with closures denoted by lax_l and lax_r respectively. This category is large enough for all of our needs since it contains all strict complicial sets along with all of the stratified simplicial sets in the image of the minimal stratification functor $\min\colon \underline{\mathrm{Simp}} \longrightarrow \underline{\mathrm{Strat}}$. It even contains the *weak* complicial sets studied in [9] and [7].

These last few results allow us to apply corollary 4 to give the following result relating Street's lax descent construction to a corresponding lax descent construction for $\underline{\mathrm{Pcs}}$-enriched categories.

Definition 5. Suppose that $\mathbf{C}$ is a category which is $\underline{\mathrm{Pcs}}$-enriched relative to the lax Gray tensor $\otimes^{\diamond}$ and let $X\colon \Delta \longrightarrow \mathbf{C}$ be a cosimplicial object in there. Furthermore, let $\Delta\colon \Delta \longrightarrow \underline{\mathrm{Pcs}}$ denote the Yoneda functor which maps each object $n \in \Delta$ to the standard n-simplex Δ^n with its minimal stratification.

Then the *lax pre-complicial descent object* $\mathrm{Desc}_{\underline{\mathrm{Pcs}}}(X)$ of X is defined to be $\underline{\mathrm{Pcs}}$-limit of X weighted by Δ (if it exists).

If $\mathbf{C}$ is actually the left handed (respectively right handed) enrichment $\underline{\mathrm{Pcs}}_l$ (respectively $\underline{\mathrm{Pcs}}_r$) of $\underline{\mathrm{Pcs}}$ then we obtain the lax (respectively oplax) pre-complicial descent construction for cosimplicial objects in $\underline{\mathrm{Pcs}}$ given by the formulae

$$\mathrm{Desc}_{\mathrm{lax}}(X) = \int_{n\in\Delta} \mathrm{lax}_l(\Delta^n, X^n) \tag{13}$$

and

$$\mathrm{Desc}_{\mathrm{oplax}}(X) = \int_{n\in\Delta} \mathrm{lax}_r(\Delta^n, X^n) \tag{14}$$

Lemma 6. *Suppose that $X\colon \Delta \longrightarrow \omega{-}\underline{Cat}_{st}$ is a cosimplicial strict ω-category then there exists a canonical isomorphism*

$$N(\mathrm{Desc}_{lax} X) \cong \mathrm{Desc}_{lax}(N \circ X) \tag{15}$$

in the category $\underline{Pcs}$ of pre-complicial sets. Dually there also exists a canonical isomorphism:

$$N(\mathrm{Desc}_{oplax} X) \cong \mathrm{Desc}_{oplax}(N \circ X) \tag{16}$$

Proof. First, it is worth mentioning that the uses of the notation $\mathrm{Desc}_{\mathrm{lax}}$ on either side of the isomorphism in display (15) actually refer to distinct constructions. On the left it is applied to a cosimplicial strict ω-category and thus refers to Streets lax descent construction for strict ω-categories. On the right, however, it is applied to a cosimplicial object in $\underline{\mathrm{Pcs}}$ and thus refers to the lax descent construction introduced in definition 5.

The proof here is straightforward. Nerves of strict ω-categories are pre-complicial, so we may restrict the nerve adjunction in display (12) to the subcategory $\underline{\mathrm{Pcs}}$ and thus obtain an adjunction between closed monoidal categories whose left adjoint is strong monoidal (with regard to the lax Gray tensors on either side). Consequently we can apply corollary 4 to the specific example of Street's lax descent construction. However, by definition, the left adjoint $F\colon \underline{\mathrm{Pcs}} \longrightarrow \omega{-}\underline{\mathrm{Cat}}_{\mathrm{st}}$ carries each standard simplex $\Delta^n \in \underline{\mathrm{Pcs}}$ to the corresponding oriental $\mathcal{O}^n \in \omega{-}\underline{\mathrm{Cat}}_{\mathrm{st}}$. So it follows that if we apply F to the weight for the pre-complicial lax descent construction introduced in definition 5 then we obtain the weight for Street's lax descent construction and the stated result follows from corollary 4. □

This dispatches the part of this proof which is purely a matter of enriched category theory, so next we turn to the homotopy theoretic content of theorem 2. Using the result of the last lemma we are at liberty to translate our problem into a purely simplicial (or at least stratified simplicial) context.

Actually we may restate this theorem in a slightly more general context than the strictly complicial one. For technical reasons, it is a little easier to state it in terms of the corresponding weak ω-category theory of *weak complicial sets*. This theory generalises that of the strict complicial sets by retaining pre-compliciality but insisting only on existence, rather than unique existence, of thin fillers for our restricted class of complicial horns. As described in [9], the category $\underline{\text{Strat}}$ of stratified simplicial sets admits a model category structure for which the cofibrations are the inclusions (monomorphisms) of stratified simplicial sets (so all objects are cofibrant there) and the fibrant objects are precisely these weak complicial sets. From here on we will assume that the category $\underline{\text{Strat}}$ implicitly comes equipped with this *complicial model structure*.

Making this restatement, we obtain:

Theorem 7. *Suppose that $X\colon \Delta \longrightarrow \underline{Pcs}$ is a cosimplicial object in $\underline{Pcs}$ and let $\overline{X}\colon \Delta \longrightarrow \underline{Simp}$ denote the cosimplicial object in $\underline{\text{Simp}}$ obtained by composing X with the underlying simplicial set functor $\overline{(\ )}\colon \underline{Strat} \longrightarrow \underline{Simp}$. Then there exists a canonical comparison map*

$$\overline{\text{Desc}_{\text{lax}}(X)} \hookrightarrow \text{Desc}(\overline{X}) \tag{17}$$

in $\underline{Simp}$ which is natural in X and an inclusion (monomorphism) of simplicial sets. Here the use of Desc_{lax} *on the left refers to the lax pre-complicial descent construction introduced in definition 5 and the use of* Desc *on the right refers to the usual simplicial descent construction.*

Suppose further that X is fibrant in the Reedy model structure on $[\Delta, \underline{Strat}]$ and that $\overline{X}$ is fibrant in the Reedy model structure on $[\Delta, \underline{Simp}]$. Then under these extra assumptions, the simplicial map in display (17) *is a weak equivalence of Kan complexes.*

On the whole, a simple application of lemma 6 to the statement of theorem 2 leads us directly to the statement of this new theorem. The only wrinkle here is that we appear to have picked up an extra condition for which there is no direct analogue in our original statement of theorem 2, that being the assumption that X is Reedy fibrant in $[\Delta, \underline{\text{Strat}}]$.

However, this apparently extra condition is easily dispatched by the observation that in order to obtain theorem 2 we apply theorem 7 to the cosimplicial object obtained by applying the nerve functor $N\colon \omega\text{--}\underline{\text{Cat}}_{\text{st}} \longrightarrow \underline{\text{Pcs}}$ to the cosimplicial object of that former theorem. So for this particular X each X^n, for $n \in \Delta$, is actually a strict complicial set. Furthermore the category of strict complicial sets is a reflective full subcategory of $\underline{\text{Strat}}$, so any limit of a diagram whose nodes are all X^ns will again be a strict complicial set and in particular it follows that all matching objects and all pullbacks of matching objects along maps of X are strict complicial sets. So all of the stratified sets involved in the fibrancy conditions which must hold in order for X to be Reedy fibrant are actually strict complicial sets. So finally the Reedy fibrancy of X follows from the easy observation that any stratified simplicial map between *strict* complicial sets is actually a fibration in the complicial model structure on $\underline{\text{Strat}}$.

This latter observation follows trivially from the fact that a map between weak complicial sets is a fibration of the complicial model structure iff it has the RLP property with respect to the horns used to define weak (and strict) complicial sets. So if $f\colon X \longrightarrow Y$ is any stratified simplicial map between strict complicial sets then the unique filler condition that they satisfy immediately allows us to lift any such horn simply by taking a (unique) thin filler in X and appealing to uniqueness in Y to show that our filler maps to the chosen simplex in there.

To prove this theorem, we can start by giving explicit descriptions of the objects on either side of display (17). Firstly, the simplicial descent of $\overline{X}$ may be given by the formula

$$\text{Desc}(\overline{X}) \cong \int_{n\in\Delta} \overline{\text{hom}}(\Delta^n, \overline{X}^n) \tag{18}$$

in $\underline{\text{Simp}}$. Here, as discussed above, $\overline{\text{hom}}(X, *)$ denotes the closure operation right adjoint to the cartesian product endo-functor $X \mathbin{\bar{\times}} -$ on $\underline{\text{Simp}}$.

Well actually this formula glosses over and important point which, in all honesty, I should really bring to light at this point. In truth, the simplicial descent of $\overline{X}$ is usually defined to be its homotopy limit $\text{holim}(\overline{X})$. This in turn is traditionally taken to be the $\underline{\text{Simp}}$-limit of $\overline{X}$ weighted by the functor which takes $n \in \Delta$ to the simplicial nerve $N(\Delta \downarrow n) \in \underline{\text{Simp}}$ or, in other words, it may be written as the end:

$$\text{holim}(\overline{X}) \cong \int_{n\in\Delta} \overline{\text{hom}}(N(\Delta \downarrow n), \overline{X}^n) \tag{19}$$

However, a well know result due to Bousfield-Kan (cf. Hirschhorn [3] theorem 18.7.4) tells us that the coends in displays (19) and (18) are canonically weakly equivalent (in $\underline{\text{Simp}}$) so long as $\overline{X}$ is Reedy fibrant (in $[\Delta, \underline{\text{Simp}}]$).

So given that theorem 7 explicitly assumes that $\overline{X}$ is Reedy fibrant, we take the liberty here of adopting the formula of display (18) as our descent notion in Simp.

On the other hand the lax pre-complicial descent of X is given by the formula in display (13). Now we know that the underlying simplicial set functor $\overline{(\ )}\colon \underline{\text{Strat}} \longrightarrow \underline{\text{Simp}}$ has a left adjoint and thus preserves all limits. But Pcs is a reflective subcategory of Strat and is thus closed in there under all limits, so it follows that $\overline{(\ )}$ preserves the end in display (13) and thus we have

$$\overline{\text{Desc}_{\text{lax}}(X)} \cong \int_{n\in\Delta} \overline{\text{lax}_l(\Delta^n, X^n)} \tag{20}$$

So to construct the map in display (17) it is enough to construct a family of simplicial inclusions $\overline{\text{lax}_l(\Delta^n, Y)} \hookrightarrow \text{hom}(\Delta^n, \overline{Y})$ which are natural in $n \in \Delta$ and $Y \in \underline{\text{Pcs}}$. Given such a family, we could then construct a natural transformation of those inclusions from the diagram whose limit is taken in display (20) to the diagram whose limit is taken in display (18), which in turn would induce the required inclusion of their limits $\overline{\text{Desc}_{\text{lax}}(X)}$ and $\text{Desc}(\overline{X})$.

Now observe that an m-simplex of $\overline{\text{hom}}(\Delta^n, \overline{Y})$ corresponds, by Yoneda's lemma, to a simplicial map $\Delta^m \longrightarrow \overline{\text{hom}}(\Delta^n, \overline{Y})$ which in turn corresponds to a simplicial map $\Delta^n \mathbin{\bar{\times}} \Delta^m \longrightarrow \overline{Y}$ and thus to a stratified simplicial map $\Delta^n \mathbin{\bar{\times}} \Delta^m \longrightarrow Y$. On the other hand, an m-simplex of $\overline{\text{lax}_l(\Delta^n, Y)}$ corresponds, again by Yoneda, to a simplicial map $\Delta^m \longrightarrow \overline{\text{lax}_l(\Delta^n, Y)}$ and thus to a stratified simplicial map $\Delta^m \longrightarrow \text{lax}_l(\Delta^n, Y)$ which in turn corresponds to a stratified simplicial map $\Delta^n \otimes \Delta^m \longrightarrow Y$.

Furthermore the stratified sets $\Delta^n \mathbin{\bar{\times}} \Delta^m$ and $\Delta^n \otimes \Delta^m$ both have the same underlying simplicial set, this being the simplicial cartesian product of these standard simplices, but they have different stratifications. However, by definition, a simplex is thin in $\Delta^n \mathbin{\bar{\times}} \Delta^m$ iff it is degenerate which implies that it must also be thin in $\Delta^n \otimes \Delta^m$, so it follows that $\Delta^n \mathbin{\bar{\times}} \Delta^m$ may be regarded as being a *entire stratified simplicial subset* of $\Delta^n \otimes \Delta^M$. In other words, there is a inclusion map

$$\Delta^n \mathbin{\bar{\times}} \Delta^m \xhookrightarrow{\ \subseteq_e\ } \Delta^n \otimes \Delta^m \tag{21}$$

in Strat which acts as the identity on underlying simplicial sets. So given our description the simplices of $\overline{\text{hom}}(\Delta^n, \overline{Y})$ and $\overline{\text{lax}_l(\Delta^n, Y)}$ as stratified simplicial maps into Y from the stratified simplicial sets $\Delta^n \mathbin{\bar{\times}} \Delta^m$ and $\Delta^n \otimes \Delta^m$ respectively, it follows that pre-composition with members of the natural family of maps in display (21) gives rise to a canonical simplicial map

$$\overline{\text{lax}_l(\Delta^n, Y)} \longrightarrow \overline{\text{hom}}(\Delta^n, \overline{Y}) \tag{22}$$

which is natural in $n \in \Delta$ and $Y \in \underline{\text{Strat}}$.

Finally to complete a proof of the first stanza of theorem 7 we must also prove that the map in display (22) is an inclusion of simplicial sets. To do this, it is enough to remark that a stratified simplicial map is determined by its action on underlying simplicial sets and thus that its is epimorphic whenever its underlying simplicial map is epimorphic. Consequently the maps of display (21) are all epimorphisms, since their underlying simplicial maps are identities, and it follows that pre-composition by those maps acts monomorphically and thus that the map in display (22) is an inclusion as required.

To prove the second part of our theorem let us first examine one way of proving that the simplicial descent construction is well behaved with respect to fibrations and pointwise weak equivalences. In particular, this will allow us to prove that the descent object of a Reedy fibrant cosimplicial space $X\colon \Delta \longrightarrow \underline{\text{Simp}}$ is a Kan complex.

So let $\overline{X}$ be an arbitrary cosimplicial space in $[\Delta, \underline{\text{Simp}}]$, then we know that its descent object may be constructed as the end given in display (18) and this may, in turn, may be presented as a (Set-)limit weighted by the hom-functor:

$$\begin{array}{ccc} \Delta^{\text{op}} \times \Delta & \xrightarrow{\quad\Delta\quad} & \underline{\text{Set}} \\ (n,m) & \longmapsto & \Delta(n,m) \end{array} \tag{23}$$

In turn, we may apply the Grothendieck construction to this weight to give a category $\mathbb{G}(\Delta)$ and a projection functor $\pi\colon \mathbb{G}(\Delta) \longrightarrow \Delta^{\text{op}} \times \Delta$ which allows us to reduce this weighted limit to a conical one:

$$\int_{n\in\Delta} \overline{\text{hom}}(\Delta^n, \overline{X}^n) \cong \lim(\Delta, \overline{\text{hom}}(\Delta^*, \overline{X})) \cong \lim_{\mathbb{G}(\Delta)} (\overline{\text{hom}}(\Delta^*, \overline{X}) \circ \pi) \tag{24}$$

(Notice that this reduction of weighted limits to conical ones is a very specific property of the enrichment base Set).

More explicitly, we may describe $\mathbb{G}(\Delta)$ as having:

- Objects which are arrows $\alpha\colon n \longrightarrow m$ of Δ,
- Arrows $(\beta,\gamma)\colon (\alpha\colon n \longrightarrow m) \longrightarrow (\alpha'\colon n' \longrightarrow m')$ pairs of arrows $\beta\colon n' \longrightarrow n$ and $\gamma\colon m \longrightarrow m'$ (notice the reversed orientation of β) such that the diagram

$$\begin{array}{ccc} n & \xleftarrow{\beta} & n' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ m & \xrightarrow[\alpha']{} & m' \end{array} \tag{25}$$

  commutes.
- Composition and identities given component-wise.

Now, given this description the following lemma is easily established:

Lemma 8. *The category $\mathbb{G}(\Delta)$ is a Reedy category with*

- *Degree function* $\deg(\alpha\colon n \longrightarrow m) = n + m$,
- *Subcategory of "face maps"* $\overrightarrow{\mathbb{G}(\Delta)}$ *consisting of those maps (β,γ) for which β is a degeneracy operator and γ is a face operator in Δ.*
- *Subcategory of "degeneracy maps"* $\overleftarrow{\mathbb{G}(\Delta)}$ *consisting of those maps (β,γ) for which β is a face operator and γ is a degeneracy operator in Δ.*

In other words, the Reedy category structure on $\mathbb{G}(\Delta)$ may be regarded as being constructed by "pulling back" the product Reedy category structure of $\Delta^{\mathrm{op}} \times \Delta$ along the projection $\pi\colon \mathbb{G}(\Delta) \longrightarrow \Delta^{\mathrm{op}} \times \Delta$, in the sense that a map (β,γ) is in $\overrightarrow{\mathbb{G}(\Delta)}$ (respectively $\overleftarrow{\mathbb{G}(\Delta)}$) iff when regarded as a map of $\Delta^{\mathrm{op}} \times \Delta$ it is an arrow of $\overrightarrow{\Delta^{\mathrm{op}} \times \Delta} = \overleftarrow{\Delta}^{\mathrm{op}} \times \overrightarrow{\Delta}$ (respectively $\overleftarrow{\Delta^{\mathrm{op}} \times \Delta} = \overrightarrow{\Delta}^{\mathrm{op}} \times \overleftarrow{\Delta}$)

The latching and matching categories of $\mathbb{G}(\Delta)$ (cf. Hirschhorn [3] definition 15.2.3) have the following properties:

- *The latching category $\partial(\overrightarrow{\mathbb{G}(\Delta)} \downarrow \alpha)$ associated with an object $\alpha\colon n \longrightarrow m$ of $\mathbb{G}(\Delta)$ is either empty (if α is an identity) or has an initial object constructed by taking the face-degeneracy factorisation of α in Δ.*
- *The matching category $\partial(\alpha \downarrow \overleftarrow{\mathbb{G}(\Delta)})$ associated with an object $\alpha\colon n \longrightarrow m$ of $\mathbb{G}(\Delta)$ is isomorphic to the matching category $\partial((n,m) \downarrow \overleftarrow{\Delta^{\mathrm{op}} \times \Delta})$ of the projection (n,m) of α to $\Delta^{\mathrm{op}} \times \Delta$. This in turn is isomorphic to the union of the subcategories $\partial(\overrightarrow{\Delta} \downarrow n)^{\mathrm{op}} \times (m \downarrow \overleftarrow{\Delta})$ and $(\overrightarrow{\Delta} \downarrow n)^{\mathrm{op}} \times \partial(m \downarrow \overleftarrow{\Delta})$ of $(n,m) \downarrow \overleftarrow{\Delta^{\mathrm{op}} \times \Delta} \cong (\overrightarrow{\Delta} \downarrow n)^{\mathrm{op}} \times (m \downarrow \overleftarrow{\Delta})$.*

Notice that the properties of the latching categories of $\mathbb{G}(\Delta)$ given in the last lemma implies that $\mathbb{G}(\Delta)$ has *cofibrant constants* (Hirschhorn [3] definition 15.10.1 and proposition 15.10.2). It follows that if $\mathcal{M}$ is a Quillen model category then the adjunction for limits of $\mathbb{G}(\Delta)$-diagrams in $\mathcal{M}$

$$[\mathbb{G}(\Delta), \mathcal{M}] \underset{\lim}{\overset{\mathrm{const}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{M} \tag{26}$$

is a Quillen pair with respect to the Reedy model structure on $[\mathbb{G}(\Delta), \mathcal{M}]$ (Hirschhorn [3] theorem 15.10.9).

In particular it follows that if $\mathcal{M}$ is the category of simplicial sets Simp then this limit functor carries Reedy fibrations, Reedy trivial fibrations, Reedy fibrant objects and weak equivalences of Reedy fibrant objects to Kan fibrations, Kan trivial fibrations, Kan complexes and weak equivalences of Kan complexes respectively.

On the other hand, the property of the matching categories of $\mathbb{G}(\Delta)$ given in the last lemma immediately tells us that if $\mathcal{M}$ is a Quillen model category and a map $\eta\colon D \longrightarrow D'$ of $[\Delta^{\mathrm{op}} \times \Delta, \mathcal{M}]$ is a Reedy fibration or trivial fibration in there iff the map $\eta \circ \pi\colon D \circ \pi \longrightarrow D' \circ \pi$, obtained by pre-composing with the projection $\pi\colon \mathbb{G}(\Delta) \longrightarrow \Delta^{\mathrm{op}} \times \Delta$, is a Reedy fibration or trivial fibration (respectively) of $[\mathbb{G}(\Delta), \mathcal{M}]$.

Putting these facts together we immediately get the following result:

Corollary 9. *If $\mathcal{M}$ is a Quillen model category then the end functor*

$$\int_{n\in\Delta} : [\Delta^{\mathrm{op}} \times \Delta, \mathcal{M}] \longrightarrow \mathcal{M} \tag{27}$$

is a right Quillen functor with respect to the Reedy model structure on the left hand side. In particular, it carries Reedy fibrations, Reedy trivial fibrations and weak equivalences of Reedy fibrant objects to fibrations, trivial fibrations and weak equivalences of fibrant objects (respectively).

(Actually it might be worth observing that actually the arguments of the last couple of lemmas continue to hold if we replace Δ by any Reedy category $\mathbf{C}$.)

To apply this theorem to the end for $\mathrm{Desc}(\overline{X})$ in display (18) we must show that the simplicial presheaf $\overline{\hom}(\Delta^*,\overline{X})\colon \Delta^{\mathrm{op}}\times\Delta \longrightarrow \underline{\mathrm{Simp}}$ is Reedy fibrant in $[\Delta^{\mathrm{op}}\times\Delta, \underline{\mathrm{Simp}}]$. This result, however, follows directly from the following lemma which allows us to calculate the Reedy matching objects for a bifunctor like $\overline{\hom}(\Delta^*,\overline{X})$:

Lemma 10. *Suppose that $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{L}$ are appropriately complete / cocomplete categories and that the bifunctor* $\mathrm{cls}\colon \mathcal{N}^{\mathrm{op}}\times\mathcal{M}\longrightarrow\mathcal{L}$ *possesses the following preservation properties:*

- *for each $M\in\mathcal{M}$ the contravariant functor* $\mathrm{cls}(*,M)$ *carries colimits of $\mathcal{N}$ to limits in $\mathcal{L}$,*
- *for each $N\in\mathcal{N}$ the covariant functor* $\mathrm{cls}(N,*)$ *carries limits of $\mathcal{M}$ to limits in $\mathcal{L}$,*

Then if $U\colon\Delta\longrightarrow\mathcal{N}$ is object in $[\Delta,\mathcal{N}]$ and $X\colon\Delta\longrightarrow\mathcal{M}$ is a object in $[\Delta,\mathcal{M}]$ then the matching objects of the functor $\mathrm{cls}(U,X)\colon\Delta^{\mathrm{op}}\times\Delta\longrightarrow\mathcal{L}$ *are related to those of U and X by a canonical isomorphism:*

$$M^{(n,m)}(\mathrm{cls}(U,X))\cong\mathrm{cls}(L^n(U),X^m)\times_{\mathrm{cls}(L^n(U),M^m(X))}\mathrm{cls}(U^m,M^m(X)) \tag{28}$$

It follows that the matching map $\mu^{n,m}\,\mathrm{cls}(U^n,X^m)\longrightarrow M^{(n,m)}(\mathrm{cls}(U,X))$ *is isomorphic to the* corner closure *map*

$$\mathrm{cls}(U^n,X^m)\xrightarrow{\mathrm{cls}^c(\iota^n,\mu^m)}\mathrm{cls}(L^n(U),X^m)\times_{\mathrm{cls}(L^n(U),M^m(X))}\mathrm{cls}(U^n,M^m(X)) \tag{29}$$

where $\iota^n\colon L^n(U)\longrightarrow U^n$ is the n^{th} latching map of U and $\mu^m\colon X^m\longrightarrow M^m(X)$ is the m^{th} matching map of X.

Proof. This is essentially a standard result, and is a direct consequence of a simple calculation of limits / colimits to identify the matching objects of the functor $\mathrm{cls}(U,X)\colon\Delta^{\mathrm{op}}\times\Delta\longrightarrow\mathcal{L}$. We have the following expression for the matching object of this functor at $(n,m)\in\Delta^{\mathrm{op}}\times\Delta$:

$$M^{(n,m)}(\mathrm{cls}(U,X))\cong\lim_{(\alpha,\beta)\in\partial((n,m)\downarrow\overleftarrow{\Delta^{\mathrm{op}}\times\Delta})}\mathrm{cls}(U^{\mathrm{dom}(\alpha)},X^{\mathrm{cod}(\beta)}) \tag{30}$$

But we know that $(n,m)\downarrow\overleftarrow{\Delta^{\mathrm{op}}\times\Delta}$ is isomorphic to $(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times(m\downarrow\overleftarrow{\Delta})$ and that $\partial((n,m)\downarrow\overleftarrow{\Delta^{\mathrm{op}}\times\Delta})$ is isomorphic to the union of the subcategories $\partial(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times(m\downarrow\overleftarrow{\Delta})$ and $(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times\partial(m\downarrow\overleftarrow{\Delta})$ of $(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times(m\downarrow\overleftarrow{\Delta})$.This union of categories is actually a pushout in $\underline{\mathrm{Cat}}$ under the common subcategory $\partial(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times\partial(m\downarrow\overleftarrow{\Delta})$, so it follows that we may express the limit in display (30) may be expressed as a pullback of limits taken over each of these categories.

Now applying the limit / colimit preservation properties of cls we may calculate the limit of $\mathrm{cls}(U,X)$ taken over each of these subcategories:

$$\begin{aligned}\lim_{(\alpha,\beta)\in\partial(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times(m\downarrow\overleftarrow{\Delta})}\mathrm{cls}(U^{\mathrm{dom}(\alpha)},X^{\mathrm{cod}(\beta)}) &\cong\lim_{\alpha\in\partial(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}}\left(\lim_{\beta\in(m\downarrow\overleftarrow{\Delta})}\mathrm{cls}(U^{\mathrm{dom}(\alpha)},X^{\mathrm{cod}(\beta)})\right)\\ &\cong\lim_{\alpha\in\partial(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}}\mathrm{cls}\left(U^{\mathrm{dom}(\alpha)},\lim_{\beta\in(m\downarrow\overleftarrow{\Delta})}X^{\mathrm{cod}(\beta)}\right)\\ &\cong\mathrm{cls}\left(\mathrm{colim}_{\alpha\in\partial(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}}U^{\mathrm{dom}(\alpha)},\lim_{\beta\in(m\downarrow\overleftarrow{\Delta})}X^{\mathrm{cod}(\beta)}\right)\\ &\cong\mathrm{cls}(L^n(U),X^m)\end{aligned} \tag{31}$$

and by similar arguments

$$\begin{aligned}\lim_{(\alpha,\beta)\in(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times\partial(m\downarrow\overleftarrow{\Delta})}\mathrm{cls}(U^{\mathrm{dom}(\alpha)},X^{\mathrm{cod}(\beta)})&\cong\mathrm{cls}(U^n,M^m(X))\\ \lim_{(\alpha,\beta)\in\partial(\overrightarrow{\Delta}\downarrow n)^{\mathrm{op}}\times\partial(m\downarrow\overleftarrow{\Delta})}\mathrm{cls}(U^{\mathrm{dom}(\alpha)},X^{\mathrm{cod}(\beta)})&\cong\mathrm{cls}(L^n(U),M^m(X))\end{aligned} \tag{32}$$

whose pullback is the object given on the RHS of display (28) as required. □

Applying this to $\overline{\hom}(\Delta^*, \overline{X})\colon \Delta^{\rm op} \times \Delta \longrightarrow \underline{\rm Simp}$ we get the following result:

Lemma 11. *The functor $\overline{\hom}(\Delta^*, \overline{X})\colon \Delta^{\rm op} \times \Delta \longrightarrow \underline{Simp}$ is a Reedy fibrant object of $[\Delta^{\rm op} \times \Delta, \underline{Simp}]$ so it follows that its end* $\mathrm{Desc}(\overline{X})$ *is a Kan complex.*

Proof. The latching objects $L^n(\Delta^n)$ of the functor $\Delta^*\colon \Delta \longrightarrow \underline{\rm Simp}$ are the simplex boundaries $\partial\Delta^n$ and their latching maps ι^n are the inclusions of those boundaries into the corresponding standard simplices Δ^n, and the cofibrations of $\underline{\rm Simp}$ are precisely the inclusions. By assumption, $\overline{X}$ is Reedy fibrant so its matching maps $\mu^m\colon \overline{X}^m \longrightarrow M^m(\overline{X})$ are all fibrations in $\underline{\rm Simp}$.

Now the last lemma tells us that the matching maps of $\overline{\hom}(\Delta^*, \overline{X})$ are precisely the corner maps

$$\overline{\hom}(\Delta^n, \overline{X}^m) \xrightarrow{\overline{\hom}^c(\iota^n, \mu^m)} \overline{\hom}(\partial\Delta^n, \overline{X}^m) \times_{\overline{\hom}(\partial\Delta^n, M^m(\overline{X}))} \overline{\hom}(\Delta^n, M^m(\overline{X})) \tag{33}$$

associated with $\overline{\hom}$. Furthermore the Kan model structure is monoidal with respect to cartesian products, so it follows that $\overline{\hom}$ possesses the usual *homotopy orthogonality* property by which this corner map is a fibration in $\underline{\rm Simp}$, thus making $\overline{\hom}(\Delta^*, \overline{X})$ Reedy fibrant as postulated. The rest follows by application of corollary 9 to the formula for $\mathrm{Desc}(\overline{X})$ in display (18). ☐

A similar proof also gives us:

Lemma 12. *Suppose that the cosimplicial object $X\colon \Delta \longrightarrow \underline{Strat}$ by the latter clause of the statement of theorem 7. Then the functor $\overline{\mathrm{lax}_l(\Delta^*, X)}\colon \Delta^{\rm op} \times \Delta \longrightarrow \underline{Simp}$ is a Reedy fibrant object of $[\Delta^{\rm op} \times \Delta, \underline{Simp}]$ and it follows that its end $\overline{\mathrm{Desc}_{lax}(X)}$ is a Kan complex.*

Proof. This result may be established by much the same argument as the last one, once we have established appropriate limit / colimit preservation properties and homotopy orthogonality properties for the bifunctor $\overline{\mathrm{lax}_l(*, -)}$.

The that end, it is established in [8] that lax_l satisfies an appropriate limit / colimit preservation property as a bifunctor on $\underline{\rm Strat}^{\rm op} \times \underline{\rm Pcs}$. Furthermore, the underlying simplicial set functor $\overline{(\ )}$ has a left adjoint and so preserves all limits, so the limit / colimit preservation properties of lemma 10 hold for their composite $\overline{\mathrm{lax}_l(*, -)}$.

Notice that the matching maps of X possess two properties, they are fibrations in the complicial model structure and their underlying simplicial maps are Kan fibrations. So to complete this theorem it is enough to show that if $i\colon U \hookrightarrow V$ is an inclusion of simplicial sets and $p\colon X \longrightarrow Y$ is a stratified simplicial map which is both a fibration in the complicial model category and Kan fibration of underlying simplicial sets then the corner map

$$\overline{\mathrm{lax}_l(V, X)} \xrightarrow{\overline{\mathrm{lax}_l^c(i,p)}} \overline{\mathrm{lax}_l(U, X)} \times_{\overline{\mathrm{lax}_l(U,Y)}} \overline{\mathrm{lax}_l(V, Y)} \tag{34}$$

is a Kan fibration.

It is enough to establish this result for only those inclusions which are simplex boundary inclusions $i\colon \partial\Delta^n \hookrightarrow \Delta^n$, since all other simplicial inclusions are transfinite composites of pushouts of such boundary maps. Furthermore, taking adjoint transposes this result is equivalent to demonstrating that any corner product

$$(\partial\Delta^n \otimes \times\Delta^m) \cup (\Delta^n \otimes \overline{\Lambda^{m,k}}) \overset{\subseteq}{\hookrightarrow} \Delta^n \otimes \Delta^m \tag{35}$$

has the left lifting property with respect to those p satisfying the fibration conditions of the last paragraph.

This latter result is a matter of a routine combinatorial construction, along the lines of lemmas 70-72 of [9]. However it should be pointed out that this result is not a direct consequence of lemma 72 of loc. cit. to which it bears a considerable resemblance. The difference is that the horn $\overline{\Lambda^{m,k}}$ we are considering here is a plain simplicial horn with minimal stratification (thus the over line to emphasise this fact) not the complicial horn $\Lambda^{m,k}$ of loc. cit. whose stratification is somewhat more elaborate.

However, to prove the result required here we may apply the proof of lemma 72 of loc. cit. upto and including its third observation. Then in its fourth observation we must substitute a simplicial horns $\overline{\Lambda^{m+n,k}}$ for the complicial horn used there. This final simplicial horn is still guaranteed to have the required lifting property with respect to p by the assumption that its underlying simplicial set is a Kan fibration. ☐

Finally, given that we have established the Reedy fibrancy of $\overline{\hom}(\Delta^*, \overline{X})$ and $\overline{\mathrm{lax}_l(\Delta^*, X)}$ in the past two lemmas we are now in an excellent position to establish the weak equivalence of their ends. Our approach is to show that the maps in display (22) are weak equivalences and thus that the natural transformation of $\overline{\mathrm{hom}}(\Delta^*, \overline{X})$ and $\overline{\mathrm{lax}_l(\Delta^*, X)}$ that they give is a Reedy weak equivalence. Then we can apply corollary 9 to show that the induced map in display (17) is also a weak equivalence.

So to conclude the proof of theorem 7, all we need is the following:

Lemma 13. *If the stratified simplicial set $Y \in \underline{Strat}$ is a weak complicial set and has underlying simplicial set $\overline{Y}$ which is a Kan complex, then the simplicial map of display* (17) *is a weak equivalence.*

Proof. Again this is a matter of a little combinatorial fiddling. We start by factoring the map in display (21) by forming its mapping cylinder

$$\begin{array}{ccccc} \Delta^n \mathbin{\bar{\times}} \Delta^m & \xhookrightarrow{i_1} & (\Delta^n \mathbin{\bar{\times}} \Delta^m) \otimes \Delta^1 & \xhookleftarrow{i_0} & \Delta^n \mathbin{\bar{\times}} \Delta^m \\ {\scriptstyle \subseteq_e} \big\downarrow & & \big\downarrow {\scriptstyle \subseteq_e} & \swarrow {\scriptstyle i_0} & \\ \Delta^n \otimes \Delta^m & \xhookrightarrow[i_1]{} & C^{n,m} & & \end{array} \tag{36}$$

and writing it as a composite of the inclusion $i_0 \colon \Delta^n \mathbin{\bar{\times}} \Delta^m \hookrightarrow C^{n,m}$ and the collapsing map $s \colon C^{n,m} \longrightarrow \Delta^n \otimes \Delta^m$ which is right inverse to i_1.

Now the construction $C^{n,m}$ is functorial in $n, m \in \Delta$, so we can define a simplicial set $T(Y)_n$ whose m-simplices are stratified simplicial maps $t \colon C^{n,m} \longrightarrow Y$. Then pre-composition with the various maps i_0, i_1, s and so forth provides us with a diagram of simplicial maps

$$\begin{array}{ccc} & T(Y)_n & \\ {\scriptstyle p_1} \swarrow \nearrow {\scriptstyle f} & & \searrow {\scriptstyle p_0} \\ \overline{\mathrm{lax}_l(\Delta^n, Y)} & \xhookrightarrow[i]{} & \overline{\mathrm{hom}}(\Delta^n, \bar{Y}) \end{array} \tag{37}$$

in which the bottom horizontal i is the inclusion of display (22), the composite of p_0 and f is equal to i and f is right inverse to p_1. It follows therefore that we can demonstrate that i is a weak equivalence by showing that p_0 and p_1 are trivial fibrations.

However, that latter result follows if we can show that each of the inclusions

$$\begin{array}{ccc} \partial_l C^{n,m} \cup (\Delta^n \mathbin{\bar{\times}} \Delta^m) & \hookrightarrow & C^{n,m} \\ \partial_l C^{n,m} \cup (\Delta^n \otimes \Delta^m) & \hookrightarrow & C^{n,m} \end{array} \tag{38}$$

(where $\partial_l C^{n,m}$ is the mapping cylinder of the map $\partial(\Delta^n \mathbin{\bar{\times}} \Delta^m) \hookrightarrow \partial(\Delta^n \otimes \Delta^m)$) has the left lifting property with respect to all stratified simplicial sets Y which are weakly complicial and have underlying simplicial sets which are Kan complexes. But this result is again a matter of straightforward combinatorial computation using lemmas 70-72 of [9] and taking a little care with thin simplices where necessary, much as in the last lemma. □

In closing, I have thought a little about what would happen to theorem 7 if we were to replace its Reedy fibrancy assumptions and replace them by weaker pointwise fibrancy assumptions. There are model categories structures on our categories of cosimplicial objects which have such pointwise fibrant objects as their fibrant objects, and one might hope that we could use those to replace our reliance upon the Reedy model structure in our arguments. However, after a few hours of thought I am not sure that I can push through a more general result of this kind. Under these weakened conditions we must return to using the homotopy limit description of descent for simplicial object, which only takes us further from a clear relationship with Street's descent construction.

In the end, it could well be that Ross' purely categorical description of the descent construction does not tell the whole strict n-categorical descent story. As Ezra Getzler pointed out to me last year, usually something a little more elaborate, like a fibrant replacement step, is required. In this case, maybe he should have insisted on taking Reedy fibrant replacements relative to the folkloric Quillen model structure on strict ω-groupoids (categories) before applying his construction.

I hope these ramblings have been a help.

Kind Regards

Dom Verity
Centre of Australian Category Theory
Faculty of Science

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