# Spin MTC and Fermionic QH States

a joint-work with N. Read

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# Fermionic QH States

- Laughlin $\nu$=1/3, MR State, Read-Rezayi States (RR M=1, Milovanovic-R. SCFT)

  Bosonic versions (RR M=0) are modeled by unitary modular tensor categories

- Use spin modular tensor categories to model their topological properties

# Category

A category is a directed graph such that (Vertices=objects, edges from x to y are called the morphism set Hom(x,y))

- Each vertex x has a loop $id_x$
- Every two compatible edges $f:x\to y, g: y\to z$ completed to a triangle with $fg:x\to z$, called composition
- $Id_x$ is a two-sided unit for composition

# (Unitary) Fusion Category

- A finite label set L, $i \in L$, dual label (anti-particle type): $i \to i^*$, and $i^{**}=i$, trivial $0=|gs>$, $0^*=0$

- Objects isomorphic to finite sum of labels
- Hom(x,y) is a f.d. Hilbert space

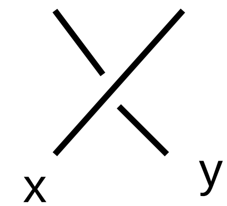
- x simple if dimHom(x,x)=1, labels are iso classes of simple objects or a set of representatives.

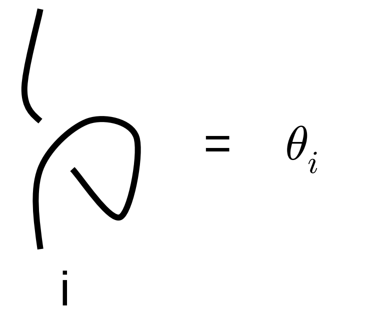
Fusion rules: $i \otimes j = \oplus N_{ij}^k\, k$ (tensor product)

# Ribbon Fusion Category

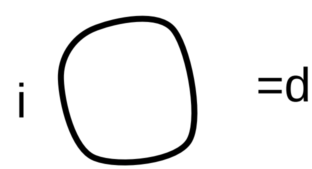
- Braiding: $c_{x,y}$: $x \otimes y \rightarrow y \otimes x$
- A twist $\theta_x$: $x \rightarrow x$
- For a label, $\theta_i = e^{2\pi i h_i}$,

  $h_i$=conformal weight mod 1




Topological invariant Z(L) for labeled oriented links L, e.g.

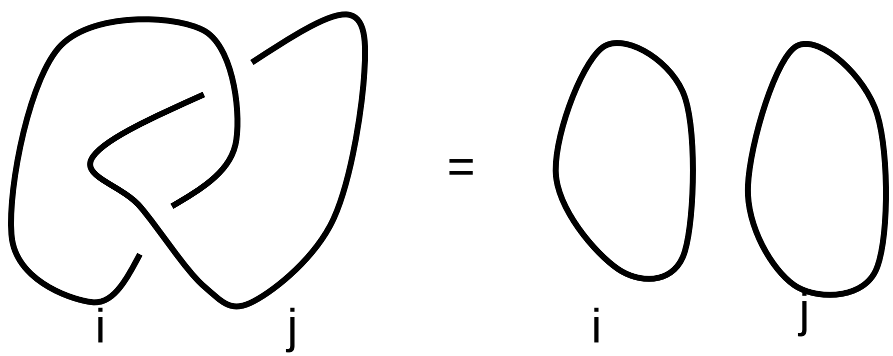


# Modular Tensor Category (MTC)

A ribbon fusion category with no degenerate simple objects:

a nontrivial simple object i (=q.p. or q.h. mod local) is degenerate or transparent if $Ds_{ij}=d_i d_j$ for any other simple j

Hopf Link




$D^2 = \sum d_i^2$

D total quantum dim

$D^2$ total order or index?

# Ribbon Fusion Category

If not modular, braid statistics and invariants of closed 3-manifolds are well-defined,

but the full TQFT structure such as the rep of the modular group "SL(2,Z)" cannot be defined.

If modular, then the modular s-matrix $s=(s_{ij})$ is not singular, and the full (2+1)-TQFT is defined.

E.g. Fusion rules are given by Verlinde formulas.

# Spin MTC

- An MTC with a fermion f
- A fermion is a simple object $f^2=1$ and $\theta_f=-1$
- Note if $f^2=1$, then $\theta_f=1$ (boson), $\theta_f=-1$ (fermion), or $\theta_f=\pm$ i (semion)
- In Ising MTC with simples $1,\sigma,\psi$,

  $\psi$ is a fermion.

# Examples

- Laughlin states $\nu$=1/Q,

  $Z_{4Q}$, L={0,1,…,4Q-1}, Q=odd, f=2Q,

  charge $q_s$=s/2Q, $\theta_s = e^{2\pi i s^2/8Q}$
- MR state

  Ising $\times$ $Z_8$, $Z_8$=Laughlin at Q=2

  L={x$\otimes$ s}, x=1,$\sigma$,$\psi$,s=0,1,…,7

  f= $\psi \otimes$ 4, charge from $Z_8$
- $SU(2)_k$, k=2 mod 4, e.g. k=6, L={0,1,…,6}, f=6

# c-Spin MC

- A c-spin MC is a spin MTC (F, f) which is unitary and covers a fermionic QH state (tentative).

- c-Spin MC

  ↓ Local w.r.t fermion---NS vs R

  Unitary ribbon fusion category

  ↓ Identify fermion with |gs>

  Tensor category

  (without braidings and twists,

  but with pure braidings and double twists)

# f is a charged fermion

Covering theory (F,f) has two components:

- Statistics/Spin sector: B (UMTC)

  Charge sector: C (cyclic UMTC)

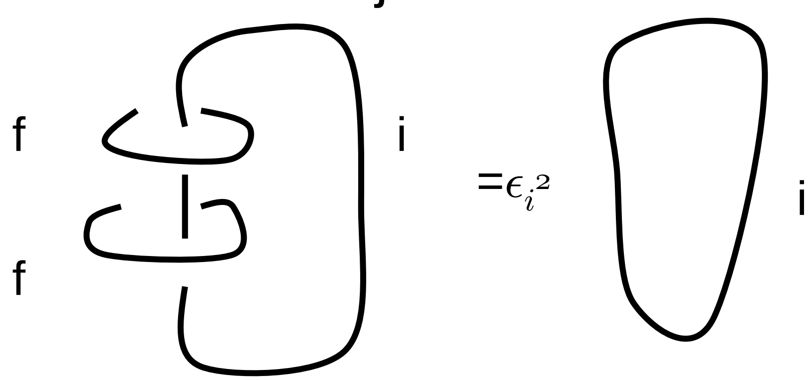
  Topological content is in the quotient Q of $B \otimes C$, where every anyon has a well-defined electric charge mod 1.
- More general, F is not a product.

# $Z_2$ Grading

- Unitary spin MTC, for any $i \in L$,

  $Ds_{fi}=\varepsilon_i d_i$, $\varepsilon_i=\pm 1$

  i is even if $\varepsilon_i=1$, odd otherwise
- $L=L_0 \cup L_1$, $f \in L_0$
- If $N_{ij}^k \neq 0$, then $\varepsilon_i\varepsilon_j\varepsilon_k=1$



# Elementary Properties

- $D_0^2 = D_1^2$, where $D_k^2 = \sum_{i \in L_k} d_i^2$
- $s_{fi,j} = \epsilon_j \, s_{i,j}$
- $\theta_{fi} = -\epsilon_i \, \theta_i$
- f has no fixed points $L_0$
- Proofs:

1st and fth row are orthogonal, so $\sum s_{f,i} d_i = 0$, i.e., $\sum_i \varepsilon_i \, d_i^2 = 0$

# Verlinde s-matrix

Define naive fusion rules

$N_{[a][b]}^{[c]}=N_{ab}^{c}+N_{ab}^{fc}$,

where [a] are quotient labels,

and naive s-matrix,

$s^{Q}_{[a],[b]}=2s_{a,b}$, then

$s^{Q}$ is a unitary matrix and Verlinde formulas hold for the naive fusion rules.

# MR Fusion Rules

Labels $L_Q=\{1, \psi, \sigma, \sigma', \alpha, \alpha'\}$

charges={0, 0,1/4,3/4, 1/2,1/2}

$\alpha\alpha'=1$, $\sigma\sigma'=1+\psi$

$\psi^2=1$, $\alpha^2=\alpha'^2=\psi$, $\sigma^2=\sigma'^2=\alpha+\alpha'$,

$\psi\sigma=\sigma$, $\psi\sigma'=\sigma'$, $\psi\alpha=\alpha'$, $\alpha\sigma=\sigma'$

No braided fusion category realizations!

(P. Bonderson's thesis)

# MR $s^Q$-matrix

$D=2\sqrt{2}$

$s^Q=1/D$

$$\begin{pmatrix} 1 & \sqrt{2} & 1 & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & i\sqrt{2} & 0 & -i\sqrt{2} \\ 1 & -\sqrt{2} & 1 & 1 & -\sqrt{2} & 1 \\ 1 & i\sqrt{2} & 1 & -1 & -i\sqrt{2} & -1 \\ \sqrt{2} & 0 & -\sqrt{2} & -i\sqrt{2} & 0 & i\sqrt{2} \\ 1 & -i\sqrt{2} & 1 & -1 & i\sqrt{2} & -1 \end{pmatrix}$$

# A Puzzle

$SU(2)_6$, labels $L=\{0,1,2,3,4,5,6\}$, $f=6$,
$L_0=\{0,2,4,6\}$, $L_1=\{1,3,5\}$

$$s^Q = 1/D \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix}$$

$[0]=1$, $[2]=x$, then $x^2=1+2x$

Verlinde formulas hold for the fusion rules,
but $\{1,x\}$ with $x^2=1+2x$ does not exist as a fusion category (V. Ostrik)

# (2+1)-TQFTs from MTCs

Two compatible functors (rules)

- A modular functor V:

  surfaces Y to Hilbert spaces V(Y),

  mapping classes b: $Y \to Y$ to unitary maps

  $V(b): V(Y) \to V(Y)$

- A partition functor Z:

  bordisms $M^3$ from $Y_1$ to $Y_2$ to

  linear maps $Z(M^3): V(Y_1) \to V(Y_2)$

$Y_i = \emptyset$, $Z(M^3)$=partition functions in CSW theories

# Spin TQFTs from spin MTCs

- Surfaces and 3-mfds are endowed with compatible spin structures
- Spin structure:

  given an oriented surface Y, a spin structure $\sigma$ on Y is a quadratic enhancement $q_\sigma: H_1(Y,Z_2) \to Z_2$ such that $q(x+y)=q(x)+q(y)+\langle x,y\rangle$ mod 2, where $\langle x,y\rangle$ is the $Z_2$-intersection form of Y.

# Theorem (C. Blanchet)

Given a TQFT, a spin structure $\sigma$ on closed oriented surface Y,

let $V^s(Y,\sigma)=\{v\in V(Y) \mid O_\gamma v=(-1)^{q_\sigma(\gamma)}v, \text{ all } \gamma\}$, where $\gamma$ is a simple closed curve $\gamma$ on Y and $O_\gamma$ is an operator,

Then $V(Y)=\sum_{\text{spin structures } \sigma} V^s(Y,\sigma)$

# Quotient Categories

- Quotient $F \to Q$

  Let $\Gamma = 1 \oplus f$,

  Objects of Q=objects of F,

  Given objects x, y of Q,
  $Hom_Q(x,y) = Hom_F(\Gamma \otimes x, y)$
- Note that in Q, $f \cong 1$
- $\otimes$ is well-defined
- ? direct sum, semi-simplicity, rigidity

# Other Structures

- Braiding, No
- Twist, No
- Pure braidings, Yes
- Double twists, Yes
- Representation of the subgroup of SL(2,Z) generated by s and $t^2$

# Possible Applications

- Entanglement entropy: $-\log D_Q$

- Topological stability