FIBRATIONS AND SHEAVES

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ABSTRACT

The purpose of this book is to give a systematic treatment of fibration theory and sheaf theory, the emphasis being on the foundational essentials.

CONTENTS

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STACKS

§0. *CATEGORICAL CONVENTIONS*

In this book, the foundation for category theory is the "one universe" approach taken by Herrlich-Strecker† . The key words are "set", "class", and "conglomerate". Thus the issue is not only one of size but also of membership (every set is a class and every class is a conglomerate).

0.1 DEFINITION A <u>category</u> $\underline{C}$ is a class of objects Ob $\underline{C}$, a class of morphisms Mor $\underline{C}$, a function dom:Mor $\underline{C} \rightarrow$ Ob $\underline{C}$, a function cod:Mor $\underline{C} \rightarrow$ Ob $\underline{C}$, and a function

$$\circ:\{(f,g):f,g \in \text{Mor } \underline{C} \ \& \ \text{cod } f = \text{dom } g\} \rightarrow \text{Mor } \underline{C} \quad (\circ(f,g) = g \circ f)$$

such that... .

0.2 TERMINOLOGY

- <u>Small Category</u>: A category whose morphism class is a set.
- <u>Large Category</u>: A category whose morphism class is a proper class.

[Note: If $\underline{C}$ is a category and if Ob $\underline{C}$ is a proper class, then $\underline{C}$ is large.]

Given a category $\underline{C}$ and objects $X,Y \in$ Ob $\underline{C}$, it is not assumed that the class

$$\text{Mor}(X,Y) = \{f:f \in \text{Mor } \underline{C}, \text{ dom } f = X, \text{ cod } f = Y\}$$

is a set.

0.3 DEFINITION A category $\underline{C}$ is said to be <u>locally small</u> if $\forall$ $X,Y \in$ Ob $\underline{C}$, Mor(X,Y) is a set.

0.4 EXAMPLE <u>SET</u> is a locally small large category.

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† *Category Theory*, Heldermann Verlag, 1979; see also Osborne, *Basic Homological Algebra*, Springer Verlag, 2000.

0.5 EXAMPLE $\underline{\text{TOP}}$ is a locally small large category.

0.6 EXAMPLE $\underline{\text{SCH}}$ is a locally small large category (cf. 23.20).

0.7 REMARK There are abelian categories $\underline{A}$ whose positive derived category $D_+\underline{A}$ is not locally small.

0.8 NOTATION $\underline{\text{CAT}}$ is the locally small category whose objects are the small categories and whose morphisms are the functors.

[Note: $\underline{\text{CAT}}$ is a locally small large category.]

0.9 DEFINITION A <u>metacategory</u> $\underline{C}$ is a conglomerate of objects Ob $\underline{C}$, a conglomerate of morphisms Mor $\underline{C}$, a function dom:Mor $\underline{C} \to$ Ob $\underline{C}$, a function cod:Mor $\underline{C} \to$ Ob $\underline{C}$, and a function

$$\circ:\{(f,g):f,g \in \text{Mor } \underline{C} \ \& \ \text{cod } f = \text{dom } g\} \to \text{Mor } \underline{C} \ \ (\circ(f,g) = g \circ f)$$

such that... .

<u>N.B.</u> Every category is a metacategory.

0.10 NOTATION Given categories $\begin{bmatrix} \underline{C} \\ \underline{D} \end{bmatrix}$, the functor category $[\underline{C},\underline{D}]$ is the metacategory whose objects are the functors $F:\underline{C} \to \underline{D}$ and whose morphisms are the natural transformations Nat(F,G) from F to G.

0.11 REMARK Suppose that $\underline{C}$ and $\underline{D}$ are nonempty.

- If $F:\underline{C} \to \underline{D}$ is a functor, then F:Mor $\underline{C} \to$ Mor $\underline{D}$ is a function, i.e., F is a subclass

$$F \subset \text{Mor } \underline{C} \times \text{Mor } \underline{D}.$$

And F is a proper class iff Mor $\underline{C}$ is a proper class.

• If $F,G:\underline{C} \rightarrow \underline{D}$ are functors and if $\Xi:F \rightarrow G$ is a natural transformation, then $\Xi:\text{Ob }\underline{C} \rightarrow \text{Mor }\underline{D}$ is a function, i.e., Ξ is a subclass

$$\Xi \subset \text{Ob }\underline{C} \times \text{Mor }\underline{D}.$$

And Ξ is a proper class iff Ob $\underline{C}$ is a proper class.

Accordingly, if Ob $\underline{C}$ is a proper class, then $[\underline{C},\underline{D}]$ is a metacategory, not a category.

[Note: If, however, $\underline{C}$ is small, then $[\underline{C},\underline{D}]$ is a category and if $\underline{D}$ is locally small, then $[\underline{C},\underline{D}]$ is locally small.]

0.12 EXAMPLE Let $\underline{ON}$ be the ordered class of ordinals — then $[\underline{ON}^{OP},\underline{SET}]$ is a metacategory, not a category.

0.13 NOTATION **CAT** is the metacategory whose objects are the categories and whose morphisms are the functors.

§1. *2-CATEGORIES*

It is a question here of establishing notation and reviewing the basics.

1.1 DEFINITION A <u>2-category</u> $\mathbf{C}$ consists of a class O and a function that assigns to each ordered pair X,Y ∈ O a category $\mathbf{C}(X,Y)$ plus functors

$$C_{X,Y,Z}:\mathbf{C}(X,Y) \times \mathbf{C}(Y,Z) \longrightarrow \mathbf{C}(X,Z)$$

and

$$I_X:\underline{1} \longrightarrow \mathbf{C}(X,X)$$

satisfying the following conditions.

(2-cat$_1$) The diagram

$$\begin{tikzcd}
\mathbf{C}(X,Y) \times (\mathbf{C}(Y,Z) \times \mathbf{C}(Z,W)) \arrow[r, "\mathrm{id} \times C"] \arrow[d, "A"'] & \mathbf{C}(X,Y) \times \mathbf{C}(Y,W) \arrow[dd, "C"] \\
(\mathbf{C}(X,Y) \times \mathbf{C}(Y,Z)) \times \mathbf{C}(Z,W) \arrow[d, "C \times \mathrm{id}"'] & \\
\mathbf{C}(X,Z) \times \mathbf{C}(Z,W) \arrow[r, "C"'] & \mathbf{C}(X,W)
\end{tikzcd}$$

commutes.

(2-cat$_2$) The diagram

$$\begin{tikzcd}
\underline{1} \times \mathbf{C}(X,Y) \arrow[r, "L"] \arrow[d, "I \times \mathrm{id}"'] & \mathbf{C}(X,Y) \arrow[d, equal] & \mathbf{C}(X,Y) \times \underline{1} \arrow[l, "R"'] \arrow[d, "\mathrm{id} \times I"] \\
\mathbf{C}(X,X) \times \mathbf{C}(X,Y) \arrow[r, "C"'] & \mathbf{C}(X,Y) & \mathbf{C}(X,Y) \times \mathbf{C}(Y,Y) \arrow[l, "C"]
\end{tikzcd}$$

commutes.

1.2 REMARK It is not assumed that the $\mathbf{C}(X,Y)$ are small or even locally small.

1.3 TERMINOLOGY Let $\mathbf{C}$ be a 2-category.

- The elements of the class O are called <u>0-cells</u> (denoted X,Y,Z,...).
- The objects of the category $\mathbf{C}(X,Y)$ are called <u>1-cells</u> (denoted f,g,h,...) (and we write $f:X \longrightarrow Y$ or $X \xrightarrow{f} Y$).
- The morphisms of the category $\mathbf{C}(X,Y)$ are called <u>2-cells</u> (denoted $\alpha,\beta,\gamma,\ldots$) (and we write $\alpha:f \Longrightarrow g$ or $f \overset{\alpha}{\Longrightarrow} g$).

<u>N.B.</u> It is common practice to define a 2-category by simply delineating the 0-cells, the 1-cells, and the 2-cells, leaving implicit the precise definition of the $\mathbf{C}(X,Y)$ (as well as the $C_{X,Y,Z}$ and the I_X).

1.4 EXAMPLE There is a 2-category 2-<u>REL</u> whose 0-cells are the sets, whose 1-cells $f:X \rightarrow Y$ are the subsets f of $X \times Y$, and whose 2-cells $\alpha:f \Longrightarrow g$ ($f,g \subset X \times Y$) are defined by stipulating that there is a unique 2-cell from f to g if $f \subset g$ but no 2-cell from f to g otherwise.

1.5 EXAMPLE There is a 2-category 2-<u>TOP</u> whose 0-cells are the topological spaces, whose 1-cells are the continuous functions, and whose 2-cells are the homotopy classes of homotopies.

1.6 EXAMPLE Let $\underline{C}$ be a locally small finitely complete category -- then there is a 2-category $\mathbf{CAT}(\underline{C})$ whose 0-cells are the internal categories in $\underline{C}$, whose 1-cells are the internal functors, and whose 2-cells are the internal natural transformations.

[Note:

- Take $\underline{C}$ = <u>SET</u> -- then the 0-cells in $\mathbf{CAT}$(<u>SET</u>) are the small categories.

• Take $\underline{C} = \underline{CAT}$ -- then the 0-cells in $\mathbf{CAT}(\underline{CAT})$ are the small double categories.]

1.7 NOTATION

• The composition of

$$f \overset{\alpha}{\Longrightarrow} g \overset{\beta}{\Longrightarrow} h$$

in $\mathbf{C}(X,Y)$ is denoted by $\beta \bullet \alpha$.

[Note: Given a 1-cell f, there is a 2-cell $\mathrm{id}_f : f \Longrightarrow f$ such that $\alpha \bullet \mathrm{id}_f = \alpha$ for all $\alpha : f \Longrightarrow g$ and $\mathrm{id}_f \bullet \beta = \beta$ for all $\beta : h \Longrightarrow f$.]

• The image of 1-cells $f:X \to Y$, $k:Y \to Z$ under $C_{X,Y,Z}$ is denoted by $k \circ f$.

[Note: Let 1_X be the image of the unique object of $\underline{1}$ under I_X (hence $1_X : X \to X$) -- then for any 1-cell $f:X \to Y$,

$$C(1_X, f) = f \circ 1_X = f = 1_Y \circ f = C(f, 1_Y).]$$

• The image of 2-cells $f \overset{\alpha}{\Longrightarrow} g$, $k \overset{\mu}{\Longrightarrow} \ell$ under $C_{X,Y,Z}$ is denoted by $\mu * \alpha$.

[Note: If $\alpha : f \Longrightarrow g$, then

$$\alpha * \mathrm{id}_{1_X} = \alpha = \mathrm{id}_{1_Y} * \alpha.$$

On the other hand, if $f:X \to Y$, $k:Y \to Z$, then

$$\mathrm{id}_k * \mathrm{id}_f = \mathrm{id}_{k \circ f}.]$$

To illustrate, suppose given

$$\begin{cases} f \overset{\alpha}{\Longrightarrow} g \overset{\beta}{\Longrightarrow} h \\ k \overset{\mu}{\Longrightarrow} \ell \overset{\nu}{\Longrightarrow} m. \end{cases}$$

Then

$$\begin{cases} \mu * \alpha : k \circ f \Longrightarrow \ell \circ g \\ \nu * \beta : \ell \circ g \Longrightarrow m \circ h. \end{cases}$$

Therefore

$$\begin{aligned}(\nu * \beta) \bullet (\mu * \alpha) &= C_{X,Y,Z}(\beta,\nu) \bullet C_{X,Y,Z}(\alpha,\mu) \\ &= C_{X,Y,Z}((\beta,\nu) \bullet (\alpha,\mu)) \\ &= C_{X,Y,Z}(\beta \bullet \alpha,\ \nu \bullet \mu) \\ &= (\nu \bullet \mu) * (\beta \bullet \alpha).\end{aligned}$$

1.8 REMARK The equation

$$(\nu * \beta) \bullet (\mu * \alpha) = (\nu \bullet \mu) * (\beta \bullet \alpha)$$

is called the exchange principle.

1.9 EXAMPLE Suppose that

$$\begin{cases} \alpha : f \Longrightarrow g \\ \mu : k \Longrightarrow \ell. \end{cases}$$

Then

$$\mu * \alpha = \begin{cases} (\mu * \mathrm{id}_g) \bullet (\mathrm{id}_k * \alpha) \\ \\ (\mathrm{id}_\ell * \alpha) \bullet (\mu * \mathrm{id}_f). \end{cases}$$

1.10 EXAMPLE Suppose that $\alpha,\beta : 1_X \Longrightarrow 1_X$ -- then

$$\alpha \bullet \beta = \beta \bullet \alpha.$$

In fact,

$$\begin{aligned}
\alpha \bullet \beta &= (\mathrm{id}_{1_X} * \alpha) \bullet (\beta * \mathrm{id}_{1_X}) \\
&= (\mathrm{id}_{1_X} \bullet \beta) * (\alpha \bullet \mathrm{id}_{1_X}) \\
&= \beta * \alpha \\
&= (\beta \bullet \mathrm{id}_{1_X}) * (\mathrm{id}_{1_X} \bullet \alpha) \\
&= (\beta * \mathrm{id}_{1_X}) \bullet (\mathrm{id}_{1_X} * \alpha) \\
&= \beta \bullet \alpha.
\end{aligned}$$

1.11 DEFINITION The <u>underlying category</u> $U\mathbf{C}$ of a 2-category $\mathbf{C}$ has for its class of objects the 0-cells and for its class of morphisms the 1-cells.

[Note: In this context, 1_X serves as the identity in $\mathrm{Mor}(X,X)$.]

1.12 NOTATION Let

$$2\text{-}\underline{CAT} = \mathbf{CAT}(\underline{SET}) \quad (\text{cf. } 1.6).$$

1.13 EXAMPLE We have

$$U2\text{-}\underline{CAT} \approx \underline{CAT}.$$

1.14 EXAMPLE Every category $\underline{C}$ determines a 2-category $\mathbf{C}$ for which $U\mathbf{C} \approx \underline{C}$.

[Let $O = \mathrm{Ob}\ \underline{C}$ and let $\mathbf{C}(X,Y) = \mathrm{Mor}(X,Y)$ (viewed as a discrete category).]

1.15 DEFINITION Let $\mathbf{C}$ be a 2-category — then a 1-cell $f:X \to Y$ is said to be a <u>2-isomorphism</u> if there exists a 1-cell $g:Y \to X$ and invertible 2-cells

$$\left[\begin{array}{l} \phi:1_X \Longrightarrow g \circ f \\ \\ \psi:1_Y \Longrightarrow f \circ g. \end{array}\right.$$

1.16 DEFINITION Let $\mathfrak{C}$ be a 2-category -- then 0-cells X and Y are said to be <u>2-isomorphic</u> if there exists a 2-isomorphism $f:X \to Y$.

1.17 EXAMPLE In 2-<u>TOP</u>, topological spaces X and Y are 2-isomorphic iff they have the same homotopy type.

1.18 EXAMPLE In 2-<u>CAT</u>, small categories $\underline{I}$ and $\underline{J}$ are 2-isomorphic iff they are equivalent.

It is clear that 1.1 admits a "2-meta" formulation (cf. 0.1 and 0.9), thus O may be a conglomerate and $\mathfrak{C}(X,Y)$ may be a metacategory.

1.19 EXAMPLE There is a 2-metacategory $\mathfrak{TOP}$ whose 0-cells are the Grothendieck toposes, whose 1-cells are the geometric morphisms, and whose 2-cells are the geometric transformations.

[Note: The 0-cells in $\mathfrak{TOP}$ constitute a conglomerate. However, if $\underline{E}$, $\underline{F}$ are Grothendieck toposes and if $f,g:\underline{E} \longrightarrow \underline{F}$ are geometric morphisms, then there is just a set of natural transformations $f^* \longrightarrow g^*$ or still, just a set of geometric transformations $(f^*,f_*) \longrightarrow (g^*,g_*)$.]

1.20 NOTATION 2-$\mathfrak{CAT}$ is the 2-metacategory whose 0-cells are the categories, whose 1-cells are the functors, and whose 2-cells are the natural transformations.

[Note: On the other hand, as agreed to above (cf. 1.12), 2-<u>CAT</u> is the 2-category whose 0-cells are the small categories, whose 1-cells are the functors, and whose 2-cells are the natural transformations.]

1.21 DEFINITION Let $\mathbf{C}$ be a 2-category -- then a diagram

$$\begin{array}{ccc} W & \xrightarrow{v} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

of 0-cells <u>2-commutes</u> (or is <u>2-commutative</u>) if the 1-cells

$$\begin{cases} f \circ u : W \longrightarrow Z \\ g \circ v : W \longrightarrow Z \end{cases}$$

are isomorphic, i.e., if there exists an invertible 2-cell ϕ in $\mathbf{C}(W,Z)$ such that

$$\phi : f \circ u \Longrightarrow g \circ v.$$

1.22 EXAMPLE Given categories $\underline{A},\underline{B},\underline{C}$ and functors $F:\underline{A} \to \underline{C}$, $G:\underline{B} \to \underline{C}$, let $\underline{A} \times_{\underline{C}} \underline{B}$ be the category whose objects are the triples (A,B,f), where $\begin{cases} A \in \text{Ob } \underline{A} \\ B \in \text{Ob } \underline{B} \end{cases}$ and $f:FA \to GB$ is an isomorphism in $\underline{C}$, and whose morphisms

$$(A,B,f) \longrightarrow (A',B',f')$$

are the pairs (a,b), where $a:A \to A'$ is a morphism in $\underline{A}$ and $b:B \to B'$ is a morphism in $\underline{B}$, such that the diagram

$$\begin{array}{ccc} FA & \xrightarrow{f} & GB \\ {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Gb} \\ FA' & \xrightarrow[f']{} & GB' \end{array}$$

commutes. Define functors

$$\begin{cases} P:\underline{A} \times_{\underline{C}} \underline{B} \longrightarrow \underline{A} \\ Q:\underline{A} \times_{\underline{C}} \underline{B} \longrightarrow \underline{B} \end{cases}$$

by

$$\begin{cases} P(A,B,f) = A \quad (P(a,b) = a) \\ Q(A,B,f) = B \quad (P(a,b) = b) \end{cases}$$

and define a natural isomorphism

$$\Xi : F \circ P \longrightarrow G \circ Q$$

by

$$\Xi_{(A,B,f)} : FP(A,B,f) = FA \xrightarrow{f} GB = GQ(A,B,f).$$

Then the diagram

$$\begin{tikzcd} \underline{A} \times_{\underline{C}} \underline{B} \arrow[r, "Q"] \arrow[d, "P"'] & \underline{B} \arrow[d, "G"] \\ \underline{A} \arrow[r, "F"'] & \underline{C} \end{tikzcd}$$

of 0-cells in 2-$\mathbf{CAT}$ is 2-commutative.

[Note: $\underline{A} \times_{\underline{C}} \underline{B}$ is called the pseudo pullback of the 2-sink $\underline{A} \xrightarrow{F} \underline{C} \xleftarrow{G} \underline{B}$. In this connection, recall that the pullback $\underline{A} \times_{\underline{C}} \underline{B}$ of (F,G) is the category whose objects are the pairs (A,B) $(A \in \text{Ob } \underline{A},\ B \in \text{Ob } \underline{B})$ such that $FA = GB$ and whose morphisms

$$(A,B) \longrightarrow (A',B')$$

are the pairs (a,b), where $a:A \to A'$ is a morphism in $\underline{A}$ and $b:\underline{B} \to \underline{B}'$ is a morphism in $\underline{B}$, such that $Fa = Gb$, there being, then, a commutative diagram

$$\begin{tikzcd} \underline{A} \times_{C} \underline{B} \arrow[r, "q"] \arrow[d, "p"'] & \underline{B} \arrow[d, "G"] \\ \underline{A} \arrow[r, "F"'] & \underline{C}\ .] \end{tikzcd}$$

1.23 REMARK The comparison functor

$$\Gamma:\underline{A} \times_{\underline{C}} \underline{B} \longrightarrow \underline{A} \underset{\sim}{\times}_{\underline{C}} \underline{B}$$

is the rule that sends (A,B) to (A,B,id) (id the identity per $FA = GB$) and (a,b) to (a,b). While clearly fully faithful, Γ need not have a representative image, hence is not an equivalence in general.

Definition: G has the <u>isomorphism lifting property</u> if $\forall$ isomorphism $\psi:GB \to C$ in $\underline{C}$, $\exists$ an isomorphism $\phi:B \to B'$ in $\underline{B}$ such that $G\phi = \psi$ (so $GB' = C$).

Exercise: Given $G:\underline{B} \to \underline{C}$, the comparison functor Γ is an equivalence for all $F:\underline{A} \to \underline{C}$ if G has the isomorphism lifting property.

Solution: Take an object (A,B,f) in $\underline{A} \underset{\sim}{\times}_{\underline{C}} \underline{B}$, let $\psi:GB \to FA$ be f^{-1}, and get an isomorphism $\phi:B \to B'$ such that $G\phi = f^{-1}$ and $GB' = FA$ -- then

$$(\mathrm{id}_A,\phi):(A,B,f) \longrightarrow \Gamma(A,B')$$

is an isomorphism

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & GB \arrow[d, "G\phi"] \\ FA \arrow[r, "\mathrm{id}"'] & GB'\ , \end{tikzcd}$$

thus Γ has a representative image.

§2. 2-FUNCTORS

Suppose that $\mathbf{C}$ and $\mathbf{C}'$ are 2-categories with 0-cells O and O' -- then a <u>2-functor</u> $F:\mathbf{C} \to \mathbf{C}'$ is the specification of a rule that assigns to each 0-cell $X \in O$ a 0-cell $FX \in O'$ and the specification of a rule that assigns to each ordered pair $X,Y \in O$ a functor

$$F_{X,Y}:\mathbf{C}(X,Y) \longrightarrow \mathbf{C}'(FX,FY)$$

such that the diagram

$$\begin{tikzcd}
\mathbf{C}(X,Y) \times \mathbf{C}(Y,Z) \arrow[r, "C"] \arrow[d, "F_{X,Y} \times F_{Y,Z}"'] & \mathbf{C}(X,Z) \arrow[d, "F_{X,Z}"] \\
\mathbf{C}'(FX,FY) \times \mathbf{C}'(FY,FZ) \arrow[r, "C"'] & \mathbf{C}'(FX,FZ)
\end{tikzcd}$$

commutes and the equality

$$I_{FX} = F_{X,X} \circ I_X$$

obtains.

[Note: The <u>underlying functor</u>

$$UF:U\mathbf{C} \longrightarrow U\mathbf{C}'$$

sends X to FX and $f:X \to Y$ to $Uf:FX \to FY$.]

<u>N.B.</u>

(1) $F_{X,X}1_X = 1_{FX}$;

(2) $F_{X,Y}id_f = id_{F_{X,Y}f}$;

(3) $F_{X,Z}k \circ f = F_{Y,Z}k \circ F_{X,Y}f$;

(4) $F_{X,Y}\beta \bullet \alpha = F_{X,Y}\beta \bullet F_{X,Y}\alpha$;

(5) $F_{X,Z}\beta * \alpha = F_{Y,Z}\beta * F_{X,Y}\alpha$.

2.1 EXAMPLE There is a 2-functor

$$\Pi:2\text{-}\underline{\mathrm{TOP}} \longrightarrow 2\text{-}\underline{\mathrm{CAT}}$$

that sends a topological space X to its fundamental groupoid ΠX.

2.2 EXAMPLE Let $\underline{C}$ and $\underline{C}'$ be locally small finitely complete categories and let $\phi:\underline{C} \to \underline{C}'$ be a functor that preserves finite limits -- then there is an induced 2-functor

$$\mathbf{CAT}(\phi):\mathbf{CAT}(\underline{C}) \longrightarrow \mathbf{CAT}(\underline{C}') \quad \text{(cf. 1.6).}$$

2.3 NOTATION Let $\mathfrak{C}$ be a 2-category.

- $\mathfrak{C}^{1\text{-OP}}$ is the 2-category obtained by reversing the 1-cells but not the 2-cells, thus

$$\mathfrak{C}^{1\text{-OP}}(X,Y) = \mathfrak{C}(Y,X).$$

- $\mathfrak{C}^{2\text{-OP}}$ is the 2-category obtained by reversing the 2-cells but not the 1-cells, thus

$$\mathfrak{C}^{2\text{-OP}}(X,Y) = \mathfrak{C}(X,Y)^{\mathrm{OP}}.$$

- $\mathfrak{C}^{1,2\text{-OP}}$ is the 2-category obtained by reversing both the 1-cells and the 2-cells, thus

$$\mathfrak{C}^{1,2\text{-OP}}(X,Y) = \mathfrak{C}(Y,X)^{\mathrm{OP}}.$$

<u>N.B.</u> Taking opposites defines an isomorphism

$$\mathrm{OP}:\mathbf{CAT} \to \mathbf{CAT}$$

of metacategories. On the other hand, this operation does not define a 2-functor

$$2\text{-}\mathbf{CAT} \longrightarrow 2\text{-}\mathbf{CAT}$$

but it does define a 2-functor

$$(2\text{-}\mathbf{CAT})^{2\text{-OP}} \longrightarrow 2\text{-}\mathbf{CAT}$$

which in fact is a "2-isomorphism".

2.4 DEFINITION A derivator in the sense of Heller is a 2-functor

$$D:(2\text{-}\underline{CAT})^{1\text{-OP}} \longrightarrow 2\text{-}\mathbf{CAT}.$$

2.5 EXAMPLE Fix a category $\underline{C}$ -- then there is a derivator $D_{\underline{C}}$ in the sense of Heller that sends $\underline{I} \in \text{Ob }\underline{CAT}$ to $[\underline{I},\underline{C}]$.

2.6 RAPPEL Let $\underline{C}$ be a locally small category and let $W \subset \text{Mor }\underline{C}$ be a class of morphisms -- then $(\underline{C},W)$ is a category pair if W is closed under composition and contains the identities of $\underline{C}$.

2.7 EXAMPLE Let $(\underline{C},W)$ be a category pair. Given $\underline{I} \in \text{Ob }\underline{CAT}$, let $W_{\underline{I}} \subset \text{Mor}[\underline{I},\underline{C}]$ be the class of morphisms that are levelwise in W -- then

$$([\underline{I},\underline{C}],W_{\underline{I}})$$

is a category pair, so it makes sense to form the localization of $[\underline{I},\underline{C}]$ at $W_{\underline{I}}$:

$$W_{\underline{I}}^{-1}[\underline{I},\underline{C}].$$

Define now a derivator $D_{(\underline{C},W)}$ in the sense of Heller by first specifying that

$$D_{(\underline{C},W)}\underline{I} = W_{\underline{I}}^{-1}[\underline{I},\underline{C}].$$

Next, given a functor $F:\underline{I} \to \underline{J}$, the precomposition functor

$$F^*:[\underline{J},\underline{C}] \to [\underline{I},\underline{C}]$$

is a morphism of category pairs (i.e., $F^*W_{\underline{J}} \subset W_{\underline{I}}$), thus there is a functor

$$\overline{F^*}:W_{\underline{J}}^{-1}[\underline{J},\underline{C}] \longrightarrow W_{\underline{I}}^{-1}[\underline{I},\underline{C}],$$

call it $D_{(\underline{C},W)}F$, hence

$$D_{(\underline{C},W)}F:D_{(\underline{C},W)}\underline{J} \longrightarrow D_{(\underline{C},W)}\underline{I}.$$

Finally, a natural transformation $\Xi:F \to G$ induces a natural transformation

$$D_{(\underline{C},W)}\Xi:D_{(\underline{C},W)}F \longrightarrow D_{(\underline{C},W)}G.$$

2.8 REMARK A <u>derivator</u> in the sense of Grothendieck is a 2-functor

$$D:(2\text{-}\underline{CAT})^{1,2\text{-}OP} \longrightarrow 2\text{-}\mathfrak{CAT}.$$

[Note: Using opposites, one can pass back and forth between the two notions.]

<u>N.B.</u> What I call a derivator (be it in the sense of Heller or Grothendieck) others call a prederivator and what I call a homotopy theory (definition omitted) others call a derivator.

2.9 CONSTRUCTION Suppose that $\mathfrak{C}$ is a 2-category, fix a 0-cell $X \in O$, and define a 2-functor

$$\Phi_X:\mathfrak{C} \longrightarrow 2\text{-}\mathfrak{CAT}$$

as follows.

- Given a 0-cell $Y \in O$, let

$$\Phi_X Y = \mathbf{C}(X,Y),$$

a 0-cell in 2-$\mathbf{CAT}$.

- Given an ordered pair $Y,Z \in O$, let

$$(\Phi_X)_{Y,Z}:\mathbf{C}(Y,Z) \longrightarrow 2\text{-}\mathbf{CAT}(\Phi_X Y,\Phi_X Z)$$

be the functor that sends a 1-cell $g:Y \to Z$ in $\mathbf{C}(Y,Z)$ to the 1-cell

$$(\Phi_X)_{Y,Z}g:\mathbf{C}(X,Y) \longrightarrow \mathbf{C}(X,Z)$$

in 2-$\mathbf{CAT}$ specified by the rule

$$\left[\begin{array}{l} ((\Phi_X)_{Y,Z}g)f = g \circ f \\ \\ ((\Phi_X)_{Y,Z}g)\alpha = \mathrm{id}_g * \alpha \end{array}\right.$$

and sends a 2-cell $\beta:g \Longrightarrow g'$ in $\mathbf{C}(Y,Z)$ to the 2-cell

$$(\Phi_X)_{Y,Z}\beta:(\Phi_X)_{Y,Z}g \Longrightarrow (\Phi_X)_{Y,Z}g'$$

specified by the rule

$$((\Phi_X)_{Y,Z}\beta)_f = \beta * \mathrm{id}_f.$$

2.10 EXAMPLE

- Take $\mathbf{C} = (2\text{-}\underline{\mathrm{CAT}})^{1\text{-OP}}$ -- then the construction assigns to each small category $\underline{I}$ a derivator

$$\Phi_{\underline{I}}:(2\text{-}\underline{\mathrm{CAT}})^{1\text{-OP}} \longrightarrow 2\text{-}\mathbf{CAT}$$

in the sense of Heller.

- Take $\mathbf{C} = (2\text{-}\underline{\mathrm{CAT}})^{1,2\text{-OP}}$ -- then the construction assigns to each small category $\underline{I}$ a derivator

$$\Phi_{\underline{I}}:(2\text{-}\underline{\mathrm{CAT}})^{1,2\text{-OP}} \longrightarrow 2\text{-}\mathbf{CAT}$$

in the sense of Grothendieck.

Let $\mathbf{C}$, $\mathbf{C}'$ be 2-categories and let $F,G:\mathbf{C} \to \mathbf{C}'$ be 2-functors -- then a <u>2-natural transformation</u> $\Xi:F \to G$ is a rule that assigns to each 0-cell $X \in O$ a 1-cell $\Xi_X:FX \to GX$ subject to the following assumptions.

(1) For any 1-cell $f:X \to Y$, the diagram

$$\begin{tikzcd} FX \arrow[r, "\Xi_X"] \arrow[d, "F_{X,Y}f"'] & GX \arrow[d, "G_{X,Y}f"] \\ FY \arrow[r, "\Xi_Y"'] & GY \end{tikzcd}$$

commutes.

(2) For any pair of 1-cells $f,g:X \to Y$ and for any 2-cell $\alpha:f \Longrightarrow g$,

$$\mathrm{id}_{\Xi_Y} * F_{X,Y}\alpha = G_{X,Y}\alpha * \mathrm{id}_{\Xi_X}.$$

[Note: Ξ is a <u>2-natural isomorphism</u> if $\forall\ X \in O$, Ξ_X is a 2-isomorphism (cf. 1.15).]

Points (1) and (2) can be rephrased.

2.11 NOTATION

- Define a functor

$$\Lambda_{F,G}:\mathbf{C}'(FX,FY) \longrightarrow \mathbf{C}'(FX,GY)$$

on objects by

$$\Lambda_{F,G}f' = \Xi_Y \circ f' \quad (f':FX \to FY)$$

and a morphism by

$$\Lambda_{F,G}\alpha' = \mathrm{id}_{\Xi_Y} * \alpha' \quad (\alpha':f' \Longrightarrow g').$$

• Define a functor

$$\Lambda_{G,F}:\mathbf{C}'(GX,GY) \longrightarrow \mathbf{C}'(FX,GY)$$

on objects by

$$\Lambda_{G,F}g' = g' \circ \Xi_X \quad (g':GX \to GY)$$

and on morphisms by

$$\Lambda_{G,F}\beta' = \beta' * \mathrm{id}_{\Xi_X} \quad (\beta':g' \Longrightarrow f').$$

Then it is clear that points (1) and (2) amount to the demand that the diagram

$$\begin{tikzcd} \mathbf{C}(X,Y) \arrow[r, "F_{X,Y}"] \arrow[d, "G_{X,Y}"'] & \mathbf{C}'(FX,FY) \arrow[d, "\Lambda_{F,G}"] \\ \mathbf{C}'(GX,GY) \arrow[r, "\Lambda_{G,F}"'] & \mathbf{C}'(FX,GY) \end{tikzcd}$$

commutes.

2.12 EXAMPLE Let $\underline{C}$ and $\underline{C}'$ be locally small finitely complete categories, let $\phi,\psi:\underline{C} \to \underline{C}'$ be functors that preserve finite limits, and let $\xi:\phi \to \psi$ be a natural transformation -- then there is an induced 2-natural transformation

$$\mathbf{CAT}(\xi):\mathbf{CAT}(\phi) \longrightarrow \mathbf{CAT}(\psi) \quad \text{(cf. 2.2)}.$$

2.13 EXAMPLE Suppose that $\mathbf{C}$ is a 2-category and let $f:X \to Y$ be a 1-cell -- then there are 2-functors

$$\left[\begin{array}{l} \Phi_X:\mathbf{C} \longrightarrow 2\text{-}\mathbf{CAT} \\ \\ \Phi_Y:\mathbf{C} \longrightarrow 2\text{-}\mathbf{CAT} \end{array}\right. \qquad \text{(cf. 2.9)}.$$

And there is a 2-natural transformation

$$\Phi_f:\Phi_Y \longrightarrow \Phi_X,$$

namely the rule that assigns to each 0-cell Z the 1-cell

$$(\Phi_f)_Z:\mathbf{C}(Y,Z) \longrightarrow \mathbf{C}(X,Z)$$

defined by

$$\begin{cases} (\Phi_f)_Z g = g \circ f \\ (\Phi_f)_Z \beta = \beta * \mathrm{id}_f. \end{cases}$$

2.14 DEFINITION Let $\mathbf{C}$, $\mathbf{C}'$ be 2-categories and let $F:\mathbf{C} \to \mathbf{C}'$ be a 2-functor -- then F is a <u>2-equivalence</u> if there is a 2-functor $F':\mathbf{C}' \to \mathbf{C}$ and 2-natural isomorphisms

$$\begin{cases} F' \circ F \longrightarrow \mathrm{id}_{\mathbf{C}} \\ F \circ F' \longrightarrow \mathrm{id}_{\mathbf{C}'}. \end{cases}$$

2.15 LEMMA A 2-functor $F:\mathbf{C} \to \mathbf{C}'$ is a 2-equivalence iff

(1) $\forall\ X,Y \in O$, the functor

$$F_{X,Y}:\mathbf{C}(X,Y) \longrightarrow \mathbf{C}'(FX,FY)$$

is an isomorphism of categories;

(2) $\forall\ X' \in O'$, $\exists\ X \in O$ such that FX is isomorphic to X' in $U\mathbf{C}'$.

Let $\mathbf{C}$, $\mathbf{C}'$ be 2-categories and let $F,G:\mathbf{C} \to \mathbf{C}'$ be 2-functors. Suppose that $\Xi,\Omega:F \to G$ are 2-natural transformations -- then a <u>2-modification</u>

$$Ч:\Xi \to \Omega$$

is a rule that assigns to each 0-cell $X \in O$ a 2-cell

$$\text{Ч}_X:\Xi_X \Longrightarrow \Omega_X$$

such that for any pair of 1-cells $f,g:X \to Y$ and for any 2-cell $\alpha:f \Longrightarrow g$,

$$\text{Ч}_Y * F_{X,Y}\alpha = G_{X,Y}\alpha * \text{Ч}_X.$$

Let $\mathbf{C}$, $\mathbf{C}'$ be 2-categories -- then there is a 2-metacategory 2-$[\mathbf{C},\mathbf{C}']$ whose 0-cells are the 2-functors from $\mathbf{C}$ to $\mathbf{C}'$, whose 1-cells are the 2-natural transformations, and whose 2-cells are the 2-modifications.

[To explicate matters:

- If $F,G:\mathbf{C} \to \mathbf{C}'$ are 2-functors, if $\Xi,\Omega,\Gamma:F \to G$ are 2-natural transformations, and if $\text{Ч}:\Xi \to \Omega$, $\text{И}:\Omega \to \Gamma$ are 2-modifications, then $\text{И} \bullet \text{Ч}:\Xi \to \Gamma$ is defined levelwise:

$$(\text{И} \bullet \text{Ч})_X = \text{И}_X \bullet \text{Ч}_X.$$

- If $F,G,H:\mathbf{C} \to \mathbf{C}'$ are 2-functors, if

$$\left[\begin{array}{l} \Xi,\Omega:F \to G \\ \Gamma,\Upsilon:G \to H \end{array}\right.$$

are 2-natural transformations, and if $\text{Ч}:\Xi \to \Omega$, $\text{И}:\Gamma \to \Upsilon$ are 2-modifications, then $\text{И} * \text{Ч}:\Gamma \bullet \Xi \to \Upsilon \bullet \Omega$ is defined levelwise:

$$(\text{И} * \text{Ч})_X = \text{И}_X * \text{Ч}_X.]$$

2.16 EXAMPLE Let $\mathbf{C}$ be a 2-category — then there is a 2-functor

$$\mathbf{C}^{1\text{-OP}} \xrightarrow{\Phi} 2\text{-}[\mathbf{C},2\text{-}\mathbf{CAT}].$$

To wit:

- Send X to Φ_X (cf. 2.9).
- Send $X \xrightarrow{f} Y$ to $\Phi_f:\Phi_Y \to \Phi_X$ (cf. 2.13).

- Send $\alpha:f \Longrightarrow g$ to $\Phi_\alpha:\Phi_f \to \Phi_g$, where $\forall\ Z \in O$,

$$(\Phi_\alpha)_Z:(\Phi_f)_Z \to (\Phi_g)_Z$$

is the 2-natural transformation defined by stipulating that at a 1-cell $h:Y \to Z$,

$$((\Phi_\alpha)_Z)_h = \mathrm{id}_h * \alpha.$$

[Note:

$$\left[\begin{array}{l} \alpha:f \Longrightarrow g \\ \\ \mathrm{id}_h:h \Longrightarrow h \end{array}\right. \quad \Rightarrow \mathrm{id}_h * \alpha:h \circ f \to h \circ g.$$

And

$$\left[\begin{array}{l} ((\Phi_f)_Z)_h = h \circ f \\ \\ ((\Phi_g)_Z)_h = h \circ g.] \end{array}\right.$$

2.17 EXAMPLE Let $\underline{2}$ be the category with two objects and one arrow not the identity -- then if $\underline{C}$ is a category, its arrow category $\underline{C}(\to)$ can be identified with the functor category $[\underline{2},\underline{C}]$. Now let 2 be the 2-category determined by $\underline{2}$ (cf. 1.14) -- then if $\mathbf{C}$ is a 2-category, we put

$$\mathbf{C}(\to) = 2\text{-}[2,\mathbf{C}].$$

Therefore the 0-cells of $\mathbf{C}(\to)$ "are" the 1-cells of $\mathbf{C}$, the 1-cells of $\mathbf{C}(\to)$ "are" the commutative squares of 1-cells of $\mathbf{C}$, and the 2-cells of $\mathbf{C}(\to)$ "are" the pairs

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "\phi'"'] & Y' \end{tikzcd} \quad , \quad \begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "\psi'"'] & Y' \end{tikzcd}$$

of commutative squares of 1-cells of $\mathbf{C}$ plus 2-cells

$$\left[\begin{array}{l} \alpha:\phi \Longrightarrow \psi \\ \alpha':\phi' \Longrightarrow \psi' \end{array}\right.$$

subject to

$$\mathrm{id}_g * \alpha = \alpha' * \mathrm{id}_f.$$

[Note: The categories $(U\mathbf{C})(\to)$, $U\mathbf{C}(\to)$ have the same objects but the first is a nonfull subcategory of the second.]

2.18 NOTATION $\mathbf{CAT}_2$ is the 2-metacategory whose 0-cells are the 2-categories, whose 1-cells are the 2-functors, and whose 2-cells are the 2-natural transformations.

[If $\Xi:F \to F'$ and $\Omega:G \to G'$ are 2-natural transformations, then

$$\Omega * \Xi:G \circ F \longrightarrow G' \circ F'$$

or still,

$$(\Omega * \Xi)_X:GFX \longrightarrow G'F'X,$$

which in turn is defined as the corner arrow in the commutative diagram

$$\begin{tikzcd} GFX \arrow[r, "\Omega_{FX}"] \arrow[d, "G_{FX,F'X}\Xi_X"'] & G'FX \arrow[d, "G'_{FX,F'X}\Xi_X"] \\ GF'X \arrow[r, "\Omega_{F'X}"'] & G'F'X \end{tikzcd}.$$

[Note: 2-functors are composed in the obvious way.]

§3. *PSEUDO FUNCTORS*

Suppose that $\mathbf{C}$ and $\mathbf{C}'$ are 2-categories with 0-cells O and O' — then a pseudo functor $F:\mathbf{C} \to \mathbf{C}'$ is the specification of a rule that assigns to each 0-cell $X \in O$ a 0-cell $FX \in O'$ and the specification of a rule that assigns to each ordered pair $X,Y \in O$ a functor

$$F_{X,Y}:\mathbf{C}(X,Y) \longrightarrow \mathbf{C}'(FX,FY)$$

plus natural isomorphisms

$$\gamma_{X,Y,Z}:C_{FX,FY,FZ} \circ (F_{X,Y} \times F_{Y,Z}) \longrightarrow F_{X,Z} \circ C_{X,Y,Z}$$

and

$$\delta_X:I_{FX} \longrightarrow F_{X,X} \circ I_X$$

satisfying the following conditions.

(coh_1) Given composable 1-cells f,g,h in $\mathbf{C}$, the diagram

$$\begin{array}{ccc} Fh \circ Fg \circ Ff & \xRightarrow{\ \mathrm{id}_{Fh} * \gamma_{f,g}\ } & Fh \circ F(g \circ f) \\ {\scriptstyle \gamma_{g,h} * \mathrm{id}_{Ff}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \gamma_{g \circ f,h}} \\ F(h \circ g) \circ Ff & \xRightarrow[\ \gamma_{f,h \circ g}\ ]{} & F(h \circ g \circ f) \end{array}$$

of 2-cells commutes:

$$\gamma_{g \circ f,h} \bullet (\mathrm{id}_{Fh} * \gamma_{f,g}) = \gamma_{f,h \circ g} \bullet (\gamma_{g,h} * \mathrm{id}_{Ff}).$$

(coh_2) Given a 1-cell $f:X \to Y$ in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
Ff \circ 1_{FX} \arrow[r, Rightarrow, "id_{Ff} * \delta_{X*}"] \arrow[d, equal] & Ff \circ F1_X \arrow[d, Rightarrow, "\gamma_{1_X,f}"] \\
Ff \arrow[r, equal] & F(f \circ 1_X)
\end{tikzcd}$$

of 2-cells commutes:

$$\gamma_{1_X,f} \bullet (id_{Ff} * \delta_{X*}) = id_{Ff},$$

and the diagram

$$\begin{tikzcd}
1_{FY} \circ Ff \arrow[r, Rightarrow, "\delta_{Y*} * id_{Ff}"] \arrow[d, equal] & F1_Y \circ Ff \arrow[d, Rightarrow, "\gamma_{f,1_Y}"] \\
Ff \arrow[r, equal] & F(1_Y \circ f)
\end{tikzcd}$$

of 2-cells commutes:

$$\gamma_{f,1_Y} \bullet (\delta_{Y*} * id_{Ff}) = id_{Ff}.$$

[Note: To ease the notational load, indices on F and γ have been suppressed, e.g., if $f:X \to Y$ and $g:Y \to Z$, then $\gamma_{f,g} = (\gamma_{X,Y,Z})_{f,g}$. Also,

$$\left[\begin{array}{l} \delta_{X*} \\ \\ \delta_{Y*} \end{array}\right.$$

stands for $\left[\begin{array}{l} \delta_X \\ \\ \delta_Y \end{array}\right.$ evaluated at the unique object of $\underline{1}$. Finally, when it is necessary to exhibit the implicit dependence on F, append a superscript, e.g., $\gamma^F_{f,g}$, δ^F_{X*} .]

N.B. In $\mathbf{C}$, if $f,f':X \longrightarrow Y$, if $\alpha:f \Longrightarrow f'$, if $g,g':Y \longrightarrow Z$, and if $\beta:g \Longrightarrow g'$, then by naturality, the diagram

$$\begin{tikzcd}
Fg \circ Ff \arrow[r, Rightarrow, "\gamma_{f,g}"] \arrow[d, Rightarrow, "F\beta * F\alpha"'] & F(g \circ f) \arrow[d, Rightarrow, "F(\beta * \alpha)"] \\
Fg' \circ Ff' \arrow[r, Rightarrow, "\gamma_{f',g'}"'] & F(g' \circ f')
\end{tikzcd}$$

of 2-cells commutes:

$$F(\beta * \alpha) \bullet \gamma_{f,g} = \gamma_{f',g'} \bullet (F\beta * F\alpha).$$

3.1 REMARK A pseudo functor is a 2-functor iff all the $\gamma_{X,Y,Z}$ and δ_X are identities.

3.2 NOTATION Let $\mathbf{MOD}$ stand for the 2-metacategory whose 0-cells are the combinatorial model categories, whose 1-cells are the model pairs (F,F') (F a left model functor, F' a right model functor), and whose 2-cells are the natural transformations of left model functors.

3.3 EXAMPLE Define a pseudo functor

$$\underline{H}:\mathbf{MOD} \longrightarrow 2\text{-}\mathbf{CAT}$$

as follows.

- Given a combinatorial model category $\underline{C}$, let

$$\underline{HC} = \mathcal{W}^{-1}\underline{C},$$

the localization of $\underline{C}$ at the weak equivalences $\mathcal{W}$.

- Given an ordered pair $\underline{C}$, $\underline{C}'$ of combinatorial model categories and a model pair (F,F'), thus

$$\underline{C} \underset{F'}{\overset{F}{\rightleftarrows}} \underline{C}',$$

send (F,F') to

$$\mathbb{L}F:\underline{HC} \longrightarrow \underline{HC}',$$

where $\mathbb{L}F$ is the absolute total left derived functor of F.

- Given a natural transformation $\Xi:F \rightarrow G$ of left model functors, let

$$\mathbb{L}\Xi:\mathbb{L}F \longrightarrow \mathbb{L}G$$

be the induced natural transformation of absolute total left derived functors.

3.4 NOTATION Let $2\text{-}\underline{GR}$ stand for the 2-category whose 0-cells are the groups, whose 1-cells are the group homomorphisms, and whose 2-cells are the inner automorphisms.

[Spelled out, if G and H are groups and if $f,g:G \rightarrow H$ are group homomorphisms, then a 2-cell $\alpha:f \Longrightarrow g$ is an element $\alpha \in H$ such that $\forall\ \sigma \in G$,

$$f(\sigma)\alpha = \alpha g(\sigma).]$$

3.5 EXAMPLE Fix a nonempty topological space B. Define a pseudo functor

$$\text{PRIN}_B:2\text{-}\underline{GR} \longrightarrow 2\text{-}\mathbf{CAT}$$

as follows.

- Given a group G, let $\text{PRIN}_{B,G}$ be the category of principal G-spaces X over B (cf. 9.3).

- Given a group homomorphism $f:G \to H$, let

$$\mathrm{PRIN}_{B,f}:\mathrm{PRIN}_{B,G} \longrightarrow \mathrm{PRIN}_{B,H}$$

be the functor that sends X to $X \times_f H$, where

$$X \times_f H = X \times H/\{(x \cdot \sigma,\tau)\sim(x,f(\sigma)\tau)\}.$$

- Given $\alpha:f \Longrightarrow g$, let

$$\mathrm{PRIN}_{B,\alpha}:\mathrm{PRIN}_{B,f} \longrightarrow \mathrm{PRIN}_{B,g}$$

be the natural transformation which at X is the arrow

$$X \times_f H \longrightarrow X \times_g H$$

that sends (x,τ) to $(x,\alpha^{-1}\tau)$.

[Note: If $f:G \to H$, $g:H \to K$, then $\gamma_{f,g}$ is the canonical isomorphism

$$(X \times_f H) \times_g K \longrightarrow X \times_{g \circ f} K.$$

And δ_{G*} is the canonical isomorphism

$$X \longrightarrow X \times_{\mathrm{id}_G} G.]$$

3.6 DEFINITION Let $\mathbf{C} \xrightarrow{F} \mathbf{C}' \xrightarrow{F'} \mathbf{C}''$ be pseudo functors -- then their <u>composition</u> $F' \circ F$ is the pseudo functor defined by

$$X \longrightarrow F'FX$$

and

$$(F' \circ F)_{X,Y} = F'_{FX,FY} \circ F_{X,Y}$$

plus

- Given 1-cells $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathbf{C}$, the 2-cell $\gamma_{f,g}^{F' \circ F}$ is the composition

$$F'Fg \circ F'Ff \xRightarrow{\gamma^{F'}_{Ff,Fg}} F'(Fg \circ Ff) \xRightarrow{F'\gamma^{F}_{f,g}} F'F(g \circ f)$$

and

- Given a 0-cell X in $\mathbf{C}$, the 2-cell $\delta^{F' \circ F}_{X*}$ is the composition

$$1_{F'FX} \xRightarrow{\delta^{F'}_{FX*}} F'1_{FX} \xRightarrow{F'\delta^{F}_{X*}} F'F1_X.$$

Let $\mathbf{C}$, $\mathbf{C}'$ be 2-categories and let $F,G:\mathbf{C} \to \mathbf{C}'$ be pseudo functors -- then a <u>pseudo natural transformation</u> $\Xi:F \to G$ is a rule that assigns to each 0-cell $X \in O$ a 1-cell $\Xi_X:FX \to GX$ plus a natural isomorphism

$$\tau_{X,Y}:\Lambda_{G,F} \circ G_{X,Y} \longrightarrow \Lambda_{F,G} \circ F_{X,Y}$$

satisfying the following conditions.

(coh$_1$) Given 1-cells $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
Gg \circ Gf \circ \Xi_X \arrow[r, Rightarrow, "id_{Gg} * \tau_f"] \arrow[d, Rightarrow, "\gamma^G_{f,g} * id_{\Xi_X}"'] & Gg \circ \Xi_Y \circ Ff \arrow[r, Rightarrow, "\tau_g * id_{Ff}"] & \Xi_Z \circ Fg \circ Ff \arrow[d, Rightarrow, "id_{\Xi_Z} * \gamma^F_{F,g}"] \\
G(g \circ f) \circ \Xi_X \arrow[rr, Rightarrow, "\tau_{g \circ f}"'] & & \Xi_Z \circ F(g \circ f)
\end{tikzcd}$$

of 2-cells commutes:

$$(id_{\Xi_Z} * \gamma^F_{f,g}) \bullet (\tau_g * id_{Ff}) \bullet (id_{Gg} * \tau_f) = \tau_{g \circ f} \bullet (\gamma^G_{f,g} * id_{\Xi_X}).$$

(coh_2) Given a 0-cell X in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
1_{GX} \circ \Xi_X \arrow[r, Rightarrow, "\delta^G_{X*} * id_{\Xi_X}"] \arrow[d, Rightarrow] & G1_X \circ \Xi_X \arrow[d, Rightarrow, "\tau_{1_X}"] \\
\Xi_X \circ 1_{FX} \arrow[r, Rightarrow, "id_{\Xi_X} * \delta^F_{X*}"'] & \Xi_X \circ F1_X
\end{tikzcd}$$

of 2-cells commutes:

$$\tau_{1_X} \bullet (\delta^G_{X*} * id_{\Xi_X}) = id_{\Xi_X} * \delta^F_{X*} .$$

(coh_3) Given 1-cells $f,g:X \to Y$ in $\mathbf{C}$ and a 2-cell $\alpha:f \Longrightarrow g$ in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
Gf \circ \Xi_X \arrow[r, Rightarrow, "\tau_f"] \arrow[d, Rightarrow, "G\alpha * id_{\Xi_X}"'] & \Xi_Y \circ Ff \arrow[d, Rightarrow, "id_{\Xi_Y} * F\alpha"] \\
Gg \circ \Xi_X \arrow[r, Rightarrow, "\tau_g"'] & \Xi_Y \circ Fg
\end{tikzcd}$$

of 2-cells commutes:

$$(id_{\Xi_Y} * F\alpha) \bullet \tau_f = \tau_g \bullet (G\alpha * id_{\Xi_X}).$$

[Note: Again, some of the indices have been omitted.]

3.7 REMARK If $F,G:\mathbf{C} \to \mathbf{C}'$ are 2-functors, then a pseudo natural transformation

Ξ is a 2-natural transformation iff all the $\tau_{X,Y}$ are identities.

3.8 DEFINITION Let $F,G,H:\mathbf{C} \to \mathbf{C}'$ be pseudo functors and let $\Xi:F \to G$, $\Omega:G \to H$ be pseudo natural transformations -- then their <u>composition</u> $\Omega \bullet \Xi$ is the pseudo natural transformation defined by letting

$$(\Omega \bullet \Xi)_X = \Omega_X \circ \Xi_X$$

and

$$\tau_f^{\Omega \bullet \Xi} = (\mathrm{id}_{\Omega_Y} * \tau_f^{\Xi}) \bullet (\tau_f^{\Omega} * \mathrm{id}_{\Xi_X}).$$

[Note: Here τ^{Ξ} and τ^{Ω} refer to the natural transformations belonging to the pseudo natural transformations Ξ and Ω.]

3.9 REMARK There is a metacategory whose objects are the pseudo functors from $\mathbf{C}$ to $\mathbf{C}'$ and whose morphisms are the pseudo natural transformations.

Let $\mathbf{C}$, $\mathbf{C}'$ be 2-categories and let $F,G:\mathbf{C} \to \mathbf{C}'$ be pseudo functors. Suppose that $\Xi,\Omega:F \to G$ are pseudo natural transformations -- then a <u>pseudo modification</u>

$$\Psi:\Xi \longrightarrow \Omega$$

is a rule that assigns to each 0-cell $X \in O$ a 2-cell

$$\Psi_X:\Xi_X \Longrightarrow \Omega_X$$

such that for any pair of 1-cells $f,g:X \to Y$ and for any 2-cell $\alpha:f \Longrightarrow g$,

$$(\Psi_Y * F_{X,Y}\alpha) \bullet (\tau_{X,Y}^{\Xi})_f = (\tau_{X,Y}^{\Omega})_g \bullet (G_{X,Y}\alpha * \Psi_X).$$

3.10 REMARK If $F,G:\mathbf{C} \to \mathbf{C}'$ are 2-functors and if $\Xi:F \to G$, $\Omega:F \to G$ are 2-natural transformations, then the τ^{Ξ}, τ^{Ω} are identities and a pseudo modification $\Psi:\Xi \to \Omega$ is a 2-modification.

Pseudo modifications are composed by exactly the same procedure as 2-modifications (recall the definition of 2-$[\mathbf{C},\mathbf{C}']$).

3.11 NOTATION PS-$[\mathbf{C},\mathbf{C}']$ is the 2-metacategory whose 0-cells are the pseudo functors from $\mathbf{C}$ to $\mathbf{C}'$, whose 1-cells are the pseudo natural transformations, and whose 2-cells are the pseudo modifications.

N.B. 2-$[\mathbf{C},\mathbf{C}']$ is a sub-2-metacategory of PS-$[\mathbf{C},\mathbf{C}']$.

3.12 REMARK The triple consisting of 2-categories, pseudo functors, and pseudo natural transformations is not a 2-metacategory.

[Note: There is a metacategory whose objects are the 2-categories and whose morphisms are the pseudo functors.]

§4. *FIBRATIONS*

Fix a category $\underline{B}$ -- then the objects of $\mathbf{CAT}/\underline{B}$ are the pairs $(\underline{E},P)$, where $P:\underline{E} \to \underline{B}$ is a functor, and the morphisms $(\underline{E},P) \to (\underline{E}',P')$ of $\mathbf{CAT}/\underline{B}$ are the functors $F:\underline{E} \to \underline{E}'$ such that $P' \circ F = P$.

[Note: $\mathbf{CAT}/\underline{B}$ can be regarded as a 2-metacategory, call it 2-$\mathbf{CAT}/\underline{B}$: Given 1-cells $F,G:(\underline{E},P) \to (\underline{E}',P')$, a 2-cell $F \Longrightarrow G$ is a natural transformation $\Xi:F \to G$ such that $\forall\ X \in \text{Ob}\ \underline{E}$, $P'\Xi_X = \text{id}_{PX}$. Another way to put it is this. There are commutative diagrams

$$\begin{array}{ccc} \underline{E} & \xrightarrow{F} & \underline{E}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \underline{B} & = & \underline{B} \end{array}, \qquad \begin{array}{ccc} \underline{E} & \xrightarrow{G} & \underline{E}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \underline{B} & = & \underline{B} \end{array}.$$

And a natural transformation $\Xi:F \to G$ is a 2-cell iff

$$\text{id}_{P'} * \Xi = \text{id}_P.$$

Here

$$\text{id}_P:P \to P \quad ((\text{id}_P)_X = \text{id}_{PX}).$$

Meanwhile,

$$\text{id}_{P'} * \Xi:P' \circ F \to P' \circ G$$

and

$$(\text{id}_{P'} * \Xi)_X = P'\Xi_X.]$$

4.1 DEFINITION Let $P:\underline{E} \to \underline{B}$ be a functor and let $B \in \text{Ob}\ \underline{B}$ -- then the <u>fiber</u> $\underline{E}_B$ of P over B is the subcategory of $\underline{E}$ whose objects are the $X \in \text{Ob}\ \underline{E}$ such that

$PX = B$ and whose morphisms are the arrows $f \in \text{Mor } \underline{E}$ such that $Pf = id_B$.

[Note: In general, $\underline{E}_B$ is not full and it may very well be the case that B and B' are isomorphic, yet $\underline{E}_B = \underline{0}$ and $\underline{E}_{B'} \neq \underline{0}$.]

N.B. There is a pullback square

$$\begin{tikzcd} \underline{E}_B \arrow[r] \arrow[d] & \underline{E} \arrow[d, "P"] \\ \underline{1} \arrow[r, "K_B"'] & \underline{B} \end{tikzcd}$$

in $\mathbf{CAT}$.

4.2 NOTATION Given $X, X' \in \text{Ob } \underline{E}_B$, let $\text{Mor}_B(X,X')$ stand for the morphisms $X \to X'$ in $\underline{E}_B$.

4.3 DEFINITION Let $X, X' \in \text{Ob } \underline{E}$ and let $u \in \text{Mor}(X,X')$ -- then u is <u>prehorizontal</u> if $\forall$ morphism $w: X_0 \to X'$ of $\underline{E}$ such that $Pw = Pu$, there exists a unique morphism $v \in \text{Mor}_{PX}(X_0,X)$ such that $u \circ v = w$:

$$\begin{tikzcd} X \arrow[r, "u"] & X' \\ X_0 \arrow[u, "v"] \arrow[r, equal] & X_0 \arrow[u, "w"'] \end{tikzcd}$$

[Note: Let

$$\text{Mor}_u(X_0,X') = \{w \in \text{Mor}(X_0,X') : Pw = Pu\}.$$

Then there is an arrow

$$\text{Mor}_{PX}(X_0,X) \to \text{Mor}_u(X_0,X'),$$

viz. $v \to u \circ v$ (in fact, $P(u \circ v) = Pu \circ Pv = Pu \circ id_{PX} = Pu$) and the condition that u be prehorizontal is that $\forall\ X_0 \in \underline{E}_{PX}$, this arrow is bijective.]

4.4 DEFINITION Let $X, X' \in \mathrm{Ob}\ \underline{E}$ and let $u \in \mathrm{Mor}(X,X')$ -- then u is <u>preophorizontal</u> if $\forall$ morphism $w: X \to X_0$ of $\underline{E}$ such that $Pw = Pu$, there exists a unique morphism $v \in \mathrm{Mor}_{PX'}(X',X_0)$ such that $v \circ u = w$:

$$\begin{array}{ccc} X & \xrightarrow{u} & X' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ X_0 & = & X_0. \end{array}$$

[Note: Let

$$\mathrm{Mor}_u(X,X_0) = \{w \in \mathrm{Mor}(X,X_0) : Pw = Pu\}.$$

Then there is an arrow

$$\mathrm{Mor}_{PX'}(X',X_0) \to \mathrm{Mor}_u(X,X_0),$$

viz. $v \to v \circ u$ (in fact, $P(v \circ u) = Pv \circ Pu = id_{PX'} \circ Pu = Pu$) and the condition that u be preophorizontal is that $\forall\ X_0 \in \underline{E}_{PX'}$, this arrow is bijective.]

4.5 LEMMA The isomorphisms in $\underline{E}$ are prehorizontal (preophorizontal).

4.6 REMARK The composite of two prehorizontal (preophorizontal) morphisms need not be prehorizontal (preophorizontal).

4.7 DEFINITION The functor $P: \underline{E} \to \underline{B}$ is a <u>prefibration</u> if for any object $X' \in \mathrm{Ob}\ \underline{E}$ and any morphism $g: B \to PX'$, there exists a prehorizontal morphism $u: X \to X'$

such that $Pu = g$.

4.8 DEFINITION The functor $P:\underline{E} \to \underline{B}$ is a <u>preopfibration</u> if for any object $X \in \mathrm{Ob}\,\underline{E}$ and any morphism $g:PX \to B$, there exists a preophorizontal morphism $u:X \to X'$ such that $Pu = g$.

4.9 LEMMA The functor $P:\underline{E} \to \underline{B}$ is a prefibration iff $\forall\, B \in \mathrm{Ob}\,\underline{B}$, the canonical functor

$$\underline{E}_B \longrightarrow B\backslash\underline{E} \quad (X \to (\mathrm{id}_B, X))$$

has a right adjoint.

4.10 LEMMA The functor $P:\underline{E} \to \underline{B}$ is a preopfibration iff $\forall\, B \in \mathrm{Ob}\,\underline{B}$, the canonical functor

$$\underline{E}_B \longrightarrow \underline{E}/B \quad (X \to (X, \mathrm{id}_B))$$

has a left adjoint.

4.11 DEFINITION Let $X, X' \in \mathrm{Ob}\,\underline{E}$ and let $u \in \mathrm{Mor}(X,X')$ -- then u is <u>horizontal</u> if $\forall$ morphism $w:X_0 \to X'$ of $\underline{E}$ and $\forall$ factorization

$$Pw = Pu \circ x \quad (x \in \mathrm{Mor}(PX_0, PX)),$$

there exists a unique morphism $v:X_0 \to X$ such that $Pv = x$ and $u \circ v = w$. Schematically:

$$\underbrace{\overbrace{X_0 \overset{}{\underset{v}{\cdots\cdots>}} X \xrightarrow[u]{} X'}^{w}}_{}\ ,\quad \overbrace{PX_0 \xrightarrow[x]{} PX \xrightarrow[Pu]{} PX'}^{Pw}.$$

<u>N.B.</u> If u is horizontal, then u is prehorizontal. Proof: For $Pw = Pu \Rightarrow PX_0 = PX$, so we can take $x = \mathrm{id}_{PX}$, hence $Pv = \mathrm{id}_{PX} \Rightarrow v \in \mathrm{Mor}_{PX}(X_0, X)$.

4.12 DEFINITION Let $X, X' \in \operatorname{Ob} \underline{E}$ and let $u \in \operatorname{Mor}(X, X')$ -- then u is <u>ophorizontal</u> if $\forall$ morphism $w: X \to X_0$ of $\underline{E}$ and $\forall$ factorization

$$Pw = x \circ Pu \quad (x \in \operatorname{Mor}(PX', PX_0)),$$

there exists a unique morphism $v: X' \to X_0$ such that $Pv = x$ and $v \circ u = w$. Schematically:

$$\begin{tikzcd} X \arrow[r, "u"'] \arrow[rr, bend left, "w"] & X' \arrow[r, dashed, "v"'] & X_0 \end{tikzcd}, \quad \begin{tikzcd} PX \arrow[r, "Pu"'] \arrow[rr, bend left, "Pw"] & PX' \arrow[r, "x"'] & PX_0 \end{tikzcd}.$$

<u>N.B.</u> If u is ophorizontal, then u is preophorizontal. Proof: For $Pw = Pu \Rightarrow PX_0 = PX'$, so we can take $x = \mathrm{id}_{PX'}$, hence $Pv = \mathrm{id}_{PX'} \Rightarrow v \in \operatorname{Mor}_{PX'}(X', X_0)$.

4.13 DEFINITION The functor $P: \underline{E} \to \underline{B}$ is a <u>fibration</u> if for any object $X' \in \operatorname{Ob} \underline{E}$ and any morphism $g: B \to PX'$, there exists a horizontal morphism $u: X \to X'$ such that $Pu = g$.

<u>N.B.</u> If $\tilde{u}: \tilde{X} \to X'$ is another horizontal morphism such that $P\tilde{u} = g$, then $\exists$ a unique isomorphism $f \in \operatorname{Mor} \underline{E}_B$ such that $\tilde{u} = u \circ f$.

[We have

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "v"'] \arrow[rr, bend left, "\tilde{u}"] & X \arrow[r, "u"'] & X' \end{tikzcd}, \quad \begin{tikzcd} P\tilde{X} \arrow[r, "\mathrm{id}_B"'] \arrow[rr, bend left, "P\tilde{u}"] & PX \arrow[r, "Pu"'] & PX' \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, dashed, "\tilde{v}"'] \arrow[rr, bend left, "u"] & \tilde{X} \arrow[r, "\tilde{u}"'] & X' \end{tikzcd}, \quad \begin{tikzcd} PX \arrow[r, "\mathrm{id}_B"'] \arrow[rr, bend left, "Pu"] & P\tilde{X} \arrow[r, "P\tilde{u}"'] & PX' \end{tikzcd}.$$

Here

$$\left[\begin{array}{l} Pv = \mathrm{id}_B \ \& \ u \circ v = \tilde{u} \\ P\tilde{v} = \mathrm{id}_B \ \& \ \tilde{u} \circ \tilde{v} = u. \end{array}\right.$$

Therefore

$$\begin{cases} \tilde{u} \circ \tilde{v} \circ v = u \circ v = \tilde{u} \\ u \circ v \circ \tilde{v} = \tilde{u} \circ \tilde{v} = u, \end{cases}$$

so

$$\begin{cases} \tilde{v} \circ v = \mathrm{id}_{\tilde{X}} \\ v \circ \tilde{v} = \mathrm{id}_{X}.] \end{cases}$$

4.14 DEFINITION The functor $P:\underline{E} \to \underline{B}$ is an <u>opfibration</u> if for any object $X \in \mathrm{Ob}\,\underline{E}$ and any morphism $g:PX \to B$, there exists an ophorizontal morphism $u:X \to X'$ such that $Pu = g$.

<u>N.B.</u> If $\tilde{u}:X \to \tilde{X}'$ is another ophorizontal morphism such that $P\tilde{u} = g$, then $\exists$ a unique isomorphism $f \in \mathrm{Mor}\,\underline{E}_B$ such that $\tilde{u} = f \circ u$ (cf. supra).

4.15 LEMMA The functor $P:\underline{E} \to \underline{B}$ is a fibration iff the functor $P^{OP}:\underline{E}^{OP} \to \underline{B}^{OP}$ is an opfibration.

Because of 4.15, in so far as the theory is concerned, it suffices to deal with fibrations. Still, opfibrations are pervasive.

4.16 EXAMPLE The functor $\underline{E} \to \underline{1}$ is a fibration.

[Note: The functor $\underline{0} \to \underline{B}$ is a fibration (all requirements are satisfied vacuously).]

4.17 EXAMPLE The functor $\mathrm{id}_{\underline{E}}:\underline{E} \to \underline{E}$ is a fibration.

4.18 EXAMPLE Given groups $\begin{cases} G \\ H \end{cases}$, denote by $\begin{cases} \underline{G} \\ \underline{H} \end{cases}$ the groupoids having a single object $*$ with $\begin{cases} \mathrm{Mor}_{\underline{G}}(*,*) = G \\ \mathrm{Mor}_{\underline{H}}(*,*) = H \end{cases}$ -- then a group homomorphism $\phi:G \to H$ can be regarded as a functor $\underline{\phi}:\underline{G} \to \underline{H}$ and, as such, $\underline{\phi}$ is a fibration iff ϕ is surjective.

[Note: The fiber $\underline{G}_*$ of $\underline{\phi}$ over $*$ "is" the kernel of ϕ.]

4.19 EXAMPLE Let $U:\underline{TOP} \to \underline{SET}$ be the forgetful functor -- then U is a fibration. To see this, consider a morphism $g:Y \to UX'$, where Y is a set and X' is a topological space. Denote by X the topological space that arises by equipping Y with the initial topology per g (i.e., with the smallest topology such that g is continuous when viewed as a function from Y to X') -- then for any topological space X_0, a function $X_0 \to X$ is continuous iff the composition $X_0 \to X \to X'$ is continuous, from which it follows that the arrow $X \to X'$ is horizontal.

[Note: The fiber $\underline{TOP}_Y$ of U over Y is the partially ordered set of topologies on Y thought of as a category.]

4.20 LEMMA The isomorphisms in $\underline{E}$ are horizontal.

4.21 LEMMA Let $u \in \mathrm{Mor}(X,X')$, $u' \in \mathrm{Mor}(X',X'')$. Assume: u' is horizontal -- then $u' \circ u$ is horizontal iff u is horizontal.

[Note: Therefore the class of horizontal morphisms is closed under composition (cf. 4.6).]

4.22 THEOREM Suppose that $P:\underline{E} \to \underline{B}$ is a fibration. Let $u \in \mathrm{Mor}(X,X')$ be horizontal. Assume: Pu is an isomorphism -- then u is an isomorphism.

PROOF In the definition of horizontal, take $X_0 = X'$, $w = \mathrm{id}_{X'}$, and consider the factorization

$$Pw = \mathrm{id}_{PX'} = Pu \circ (Pu)^{-1} \quad (x = (Pu)^{-1}).$$

Choose $v:X' \to X$ accordingly, thus $u \circ v = \mathrm{id}_{X'}$, so v is a right inverse for u. But thanks to 4.20 and 4.21, v is horizontal. Since $Pv = (Pu)^{-1}$, the argument can be repeated to get a right inverse for v. Therefore u is an isomorphism.

4.23 APPLICATION A fibration $P:\underline{E} \to \underline{B}$ has the isomorphism lifting property (cf. 1.23).

[Let $\psi:PX' \to B$ be an isomorphism in $\underline{B}$. Choose a horizontal morphism $u:X \to X'$ such that $Pu = \psi^{-1}$ -- then u is an isomorphism in $\underline{E}$ (cf. 4.22) and $Pu^{-1} = \psi$.]

4.24 LEMMA Suppose that $P:\underline{E} \to \underline{B}$ is a fibration. Consider any object $X' \in \mathrm{Ob}\,\underline{E}$ and any morphism $g:B \to PX'$. Assume: $\tilde{u}:\tilde{X} \to X'$ is prehorizontal and $P\tilde{u} = g$ -- then $\tilde{u}$ is horizontal.

PROOF Choose a horizontal $u:X \to X'$ such that $Pu = g$ -- then u is prehorizontal so $\exists$ a unique isomorphism $f \in \mathrm{Mor}\,\underline{E}_B$ such that $\tilde{u} = u \circ f$. Therefore $\tilde{u}$ is horizontal (cf. 4.20 and 4.21).

[Note: Here are the details. Consider the commutative diagrams

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] & X' \\ X \arrow[u, "\tilde{v}"] \arrow[r, equal] & X \arrow[u, "u"'] \end{tikzcd}, \qquad \begin{tikzcd} X \arrow[r, "u"] & X' \\ \tilde{X} \arrow[u, "v"] \arrow[r, equal] & \tilde{X} \arrow[u, "\tilde{u}"'] \end{tikzcd}.$$

Then

$$\begin{cases} \tilde{u} \circ \tilde{v} \circ v = u \circ v = \tilde{u} \\ u \circ v \circ \tilde{v} = \tilde{u} \circ \tilde{v} = u. \end{cases}$$

On the other hand, there are commutative diagrams

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] & X' \\ \tilde{X} \arrow[u, "id_{\tilde{X}}"] \arrow[r, equal] & \tilde{X} \arrow[u, "\tilde{u}"'] \end{tikzcd} , \qquad \begin{tikzcd} X \arrow[r, "u"] & X' \\ X \arrow[u, "id_X"] \arrow[r, equal] & X \arrow[u, "u"'] \end{tikzcd} .$$

Therefore by the uniqueness inherent in the definition of prehorizontal,

$$\begin{cases} \tilde{v} \circ v = id_{\tilde{X}} \\ v \circ \tilde{v} = id_X \end{cases} .]$$

4.25 THEOREM Let $P:\underline{E} \to \underline{B}$ be a functor -- then P is a fibration iff

1. $\forall X' \in \text{Ob } \underline{E}$ and $\forall g \in \text{Mor}(B,PX')$, $\exists$ a prehorizontal $\tilde{u} \in \text{Mor}(\tilde{X},X'):P\tilde{u} = g$ (cf. 4.7);

2. The composition of two prehorizontal morphisms is prehorizontal.

PROOF The conditions are clearly necessary (for point 2, cf. 4.24 and recall 4.21). Turning to the sufficiency, one has only to prove that the $\tilde{u}$ of point 1 is actually horizontal. Consider a morphism $w:X_0 \to X'$ of $\underline{E}$ and a factorization

$$Pw = P\tilde{u} \circ x \quad (x \in \text{Mor}(PX_0,P\tilde{X})).$$

Then there is a prehorizontal $\tilde{u}_0 \in \text{Mor}(\tilde{X}_0,\tilde{X}):P\tilde{u}_0 = x$ ($\Rightarrow P\tilde{X}_0 = PX_0$). Here

$$\tilde{X}_0 \xrightarrow{\tilde{u}_0} \tilde{X} \xrightarrow{\tilde{u}} X'$$

and

$$P(\tilde{u} \circ \tilde{u}_0) = P\tilde{u} \circ P\tilde{u}_0 = P\tilde{u} \circ x = Pw.$$

But $\tilde{u} \circ \tilde{u}_0$ is prehorizontal, thus there exists a unique morphism $\tilde{v}_0 \in \text{Mor}_{P\tilde{X}_0}(X_0,\tilde{X}_0)$ such that $\tilde{u} \circ \tilde{u}_0 \circ \tilde{v}_0 = w$:

$$\begin{tikzcd} \tilde{X}_0 \arrow[r, "\tilde{u} \circ \tilde{u}_0"] & X' \\ X_0 \arrow[u, "\tilde{v}_0"] \arrow[r, equal] & X_0 \arrow[u, "w"'] \end{tikzcd} .$$

Put $v = \tilde{u}_0 \circ \tilde{v}_0$ — then $Pv = P\tilde{u}_0 \circ P\tilde{v}_0 = P\tilde{u}_0 \circ \text{id}_{P\tilde{X}_0} = P\tilde{u}_0 = x$ and $\tilde{u} \circ v =$ $\tilde{u} \circ \tilde{u}_0 \circ \tilde{v}_0 = w$. To establish that v is unique, let $v':X_0 \to \tilde{X}$ be another morphism with $Pv' = x$ and $\tilde{u} \circ v' = w$. Since $\tilde{u}_0$ is prehorizontal and since $Pv' = x = P\tilde{u}_0$, the diagram

$$\begin{tikzcd} \tilde{X}_0 \arrow[r, "\tilde{u}_0"] & \tilde{X} \\ X_0 \arrow[u, dotted, "v''"] \arrow[r, equal] & X_0 \arrow[u, "v'"'] \end{tikzcd}$$

admits a unique filler $v'' \in \text{Mor}_{P\tilde{X}_0}(X_0,\tilde{X}_0):u_0 \circ v'' = v'$. Finally

$$\tilde{u} \circ \tilde{u}_0 \circ v'' = \tilde{u} \circ v' = w$$

$$\Rightarrow v'' = \tilde{v}_0 \Rightarrow v = \tilde{u}_0 \circ \tilde{v}_0 = \tilde{u}_0 \circ v'' = v'.$$

4.26 DEFINITION Let $P:\underline{E} \to \underline{B}$ be a functor -- then a morphism $f:X \to Y$ in $\underline{E}$ is <u>vertical</u> if Pf is the identity on $PX = PY$.

4.27 EXAMPLE $\forall\ B \in \mathrm{Ob}\ \underline{B}$, the elements of $\mathrm{Mor}\ \underline{E}_B$ are vertical.

4.28 LEMMA Suppose that $P:\underline{E} \to \underline{B}$ is a fibration -- then every morphism in $\underline{E}$ can be factored as a vertical morphism followed by a horizontal morphism.

PROOF Let $f:Y \to X'$ be a morphism in $\underline{E}$, thus $Pf:PY \to PX'$. Choose a horizontal $u:X \to X'$ such that $Pu = Pf$ ($\Rightarrow PX = PY$). Consider

$$\overbrace{Y \underset{v}{\dashrightarrow} X \xrightarrow[u]{\qquad} X'}^{f}, \quad PY = \overbrace{PX \xrightarrow[\mathrm{id}_{PX}]{\qquad} PX \xrightarrow[Pu]{\qquad} PX'}^{Pf},$$

where $Pv = \mathrm{id}_{PX}$ (so v is vertical) and $u \circ v = f$.

4.29 DEFINITION A morphism $F:(\underline{E},P) \to (\underline{E}',P')$ in $\mathbf{CAT}/\underline{B}$ is said to be <u>horizontal</u> if the functor $F:\underline{E} \to \underline{E}'$ sends horizontal arrows to horizontal arrows.

4.30 NOTATION $\mathbf{CAT}_h/\underline{B}$ is the wide submetacategory of $\mathbf{CAT}/\underline{B}$ whose morphisms are the horizontal morphisms.

4.31 NOTATION $\underline{\mathrm{FIB}}(\underline{B})$ is the full submetacategory of $\mathbf{CAT}_h/\underline{B}$ whose objects are the pairs $(\underline{E},P)$, where $P:\underline{E} \to \underline{B}$ is a fibration.

4.32 EXAMPLE Take $\underline{B} = \underline{1}$ -- then $\underline{\mathrm{FIB}}(\underline{1})$ is $\mathbf{CAT}$.

By definition, the 2-cells of $2\text{-}\mathbf{CAT}/\underline{B}$ are the <u>vertical</u> natural transformations, i.e., if $F,G:(\underline{E},P) \to (\underline{E}',P')$ are morphisms, then a 2-cell $F \Rightarrow G$ is a natural transformation $\Xi:F \to G$ such that $\forall\ X \in \text{Ob}\ \underline{E}$, $P'\Xi_X = id_{PX}$ or still, such that $\forall\ X \in \text{Ob}\ \underline{E}$, Ξ_X is a morphism in $\underline{E}'_{PX}$ $(P'FX = PX = P'GX)$, hence Ξ_X is vertical (per P').

4.33 NOTATION $2\text{-}\mathbf{CAT}_h/\underline{B}$ is the sub-2-metacategory of $2\text{-}\mathbf{CAT}/\underline{B}$ whose 0-cells are the objects of $\mathbf{CAT}/\underline{B}$, whose 1-cells are the horizontal morphisms, and whose 2-cells are the vertical natural transformations.

4.34 NOTATION $\mathcal{FIB}(\underline{B})$ is the 2-cell full sub-2-metacategory of $2\text{-}\mathbf{CAT}_h/\underline{B}$ whose underlying category is $\underline{\mathrm{FIB}}(\underline{B})$.

4.35 LEMMA Let $(\underline{E}_1,P_1)$, $(\underline{E}_2,P_2)$ be objects of $\mathbf{CAT}/\underline{B}$. Assume: $\underline{E}_1$ and $\underline{E}_2$ are equivalent as categories over $\underline{B}$, thus there are functors $F_1:\underline{E}_1 \to \underline{E}_2$ and $F_2:\underline{E}_2 \to \underline{E}_1$ over $\underline{B}$ and vertical natural isomorphisms

$$\left[\begin{array}{l} \Xi_{12}:F_1 \circ F_2 \longrightarrow id_{\underline{E}_2} \\[1em] \Xi_{21}:F_2 \circ F_1 \longrightarrow id_{\underline{E}_1}. \end{array}\right.$$

Then $\left[\begin{array}{l} F_1 \\[1em] F_2 \end{array}\right.$ send horizontal arrows to horizontal arrows.

PROOF It suffices to discuss F_1. So let $u_1:X_1 \to X_1'$ be a horizontal arrow in $\underline{E}_1$, the contention being that F_1u_1 is a horizontal arrow in $\underline{E}_2$. Suppose that $w_2:X_2 \to F_1X_1'$ is a morphism of $\underline{E}_2$ and consider a factorization

$$P_2w_2 = P_2F_1u_1 \circ x_2 \quad (x_2 \in \text{Mor}(P_2X_2,P_2F_1X_1)).$$

Put

$$i = (\Xi_{21})_{X_1'},$$

thus $i:F_2F_1X_1' \longrightarrow X_1'$ and $P_1 i = id_{P_1X_1'}$. Working with

$$i \circ F_2w_2 : F_2X_2 \longrightarrow X_1',$$

write

$$\begin{aligned} P_1(i \circ F_2w_2) &= P_1 i \circ P_1F_2w_2 \\ &= id_{P_1X_1'} \circ P_2w_2 \\ &= P_2w_2 \\ &= P_2F_1u_1 \circ x_2 \\ &= P_1u_1 \circ x_2. \end{aligned}$$

Since u_1 is horizontal, there exists a unique morphism $v_1:F_2X_2 \to X_1$ such that $P_1v_1 = x_2$ and $u_1 \circ v_1 = i \circ F_2w_2$. Put

$$j = ((\Xi_{12})_{X_2})^{-1},$$

thus $j:X_2 \longrightarrow F_1F_2X_2$ and $P_2 j = id_{P_2X_2}$. Let

$$v_2 = F_1v_1 \circ j.$$

Then

$$\begin{aligned} P_2v_2 &= P_2(F_1v_1 \circ j) \\ &= P_2F_1v_1 \circ P_2 j \\ &= P_1v_1 \circ id_{P_2X_2} \\ &= x_2. \end{aligned}$$

It remains to check that

$$F_1u_1 \circ v_2 = w_2.$$

To begin with,

$$\begin{aligned} F_1u_1 \circ v_2 &= F_1u_1 \circ F_1v_1 \circ j \\ &= F_1(u_1 \circ v_1) \circ j \\ &= F_1(i \circ F_2w_2) \circ j \\ &= F_1i \circ F_1F_2w_2 \circ j. \end{aligned}$$

On the other hand, by naturality, there is a commutative diagram

$$\begin{tikzcd} F_1F_2X_2 \arrow[d, "F_1F_2w_2"'] & X_2 \arrow[l, "j"'] \arrow[d, "w_2"] \\ F_1F_2F_1X_1' & F_1X_1' \arrow[l, "k"] \end{tikzcd} .$$

Therefore

$$\begin{aligned} F_1i \circ F_1F_2w_2 \circ j &= F_1i \circ k \circ w_2 \\ &= w_2. \end{aligned}$$

Here

$$F_1X_1' \xrightarrow{k} F_1F_2F_1X_1' \xrightarrow{F_1i} F_1X_1'$$

is the canonical arrow, hence is the identity.

[Note: The proof of uniqueness is left to the reader.]

4.36 APPLICATION $P_1:\underline{E}_1 \to \underline{B}$ is a fibration iff $P_2:\underline{E}_2 \to \underline{B}$ is a fibration.

[Suppose that P_1 is a fibration. Let $g:B \to P_2X_2'$ be a morphism in $\underline{B}$ -- then the claim is that $\exists$ a horizontal morphism $u_2:X_2 \to X_2'$ such that $P_2u_2 = g$.

• Assume first that $X_2' = F_1X_1'$, hence $P_2X_2' = P_2F_1X_1' = P_1X_1'$, hence $g:B \to P_1X_1'$. Choose a horizontal $u_1:X_1 \to X_1'$ such that $P_1u_1 = g$ ($\Rightarrow P_1X_1 = B$) -- then $F_1u_1:F_1X_1 \to F_1X_1'$ is horizontal and $P_2F_1u_1 = P_1u_1 = g$, so we can take $u_2 = F_1u_1$.

• In general, given an arbitrary X_2', there exists an X_1' and an isomorphism $\psi:F_1X_1' \to X_2'$, from which an isomorphism $P_2\psi:P_2F_1X_1' \to P_2X_2'$ or still, an isomorphism $P_2\psi:P_1X_1' \to P_2X_2'$. If now $g:B \to P_2X_2'$, then $(P_2\psi)^{-1}:P_2X_2' \to P_1X_1'$ and, in view of what has been said above, $\exists$ a horizontal morphism u_2 such that $P_2u_2 = (P_2\psi)^{-1} \circ g$ or still, $P_2\psi \circ P_2u_2 = g$ or still, $P_2(\psi \circ u_2) = g$. And $\psi \circ u_2$ is horizontal (cf. 4.20 and 4.21).]

4.37 DEFINITION Let $P:\underline{E} \to \underline{B}$, $P':\underline{E}' \to \underline{B}$ be fibrations -- then P, P' are <u>equivalent</u> if $\underline{E}$, $\underline{E}'$ are equivalent as categories over $\underline{B}$.

<u>N.B.</u> If $(\underline{E},P)$, $(\underline{E}',P')$ are objects of $\mathbf{CAT}/\underline{B}$ and if $F:(\underline{E},P) \to (\underline{E}',P')$ is a morphism, then $\forall\ B \in \text{Ob}\ \underline{B}$, F restricts to a functor $F_B:\underline{E}_B \to \underline{E}_B'$.

4.38 CRITERION Let $P:\underline{E} \to \underline{B}$, $P':\underline{E}' \to \underline{B}$ be fibrations, $F:(\underline{E},P) \to (\underline{E}',P')$ a horizontal functor -- then F is an equivalence of categories over $\underline{B}$ iff $\forall\ B \in \text{Ob}\ \underline{B}$, the functor $F_B:\underline{E}_B \to \underline{E}_B'$ is an equivalence of categories.

4.39 NOTATION Given objects $(\underline{E},P)$, $(\underline{E}',P')$ in $\underline{\text{FIB}}(\underline{B})$, let $[\underline{E},\underline{E}']_{\underline{B}}$ be the metacategory whose objects are the horizontal functors $F:(\underline{E},P) \to (\underline{E}',P')$ and whose morphisms are the vertical natural transformations.

4.40 EXAMPLE Take $\underline{B} = \underline{1}$ -- then

$$[\underline{E},\underline{E}']_{\underline{1}} = [\underline{E},\underline{E}'].$$

§5. *FIBRATIONS: EXAMPLES*

The ensuing compilation will amply illustrate the ubiquity of the theory.

5.1 EXAMPLE The functor

$$\mathrm{Ob}:\underline{\mathrm{CAT}} \to \underline{\mathrm{SET}}$$

that sends a small category $\underline{C}$ to its set of objects is a fibration.

[Suppose that $g:B \to \mathrm{Ob}\ \underline{C}'$, where B is a set. To construct a horizontal $u:\underline{C} \to \underline{C}'$ such that $\mathrm{Ob}\ u = g$, let $\underline{C}$ have objects B and given $x,y \in B$, let

$$\mathrm{Mor}(x,y) = \{x\} \times \mathrm{Mor}(g(x),g(y)) \times \{y\},$$

composition and identities being those of $\underline{C}'$. Define the functor $u:\underline{C} \to \underline{C}'$ by taking $u = g$ on objects and by taking

$$u:\mathrm{Mor}(x,y) \to \mathrm{Mor}(g(x),g(y))$$

to be the projection.]

5.2 EXAMPLE Let $\underline{C}$ be a category with pullbacks. Consider the arrow category $\underline{C}(\to)$ -- then the objects of $\underline{C}(\to)$ are the triples (X,f,Y), where $f:X \to Y$ is an arrow in $\underline{C}$, and a morphism

$$(X,f,Y) \to (X',f',Y')$$

is a pair

$$\begin{cases} \phi:X \to X' \\ \psi:Y \to Y' \end{cases}$$

of arrows in $\underline{C}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\psi"'] & Y' \end{tikzcd}$$

commutes. Define

$$\mathrm{cod}:\underline{C}(\to) \to \underline{C}$$

by

$$\mathrm{cod}(X \xrightarrow{f} Y) = Y,\ \mathrm{cod}(\phi,\psi) = \psi.$$

Then cod is a fibration and the fiber $\underline{C}(\to)_Y$ of cod over Y can be identified with $\underline{C}/Y$.

[A morphism (ϕ,ψ) is horizontal iff the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\psi"'] & Y' \end{tikzcd}$$

is a pullback square. This said, given a morphism $g:Z \to Y'$ in $\underline{C}$, to construct a horizontal

$$u:(X,f,Y) \to (X',f',Y')$$

such that cod $u = g$, form the pullback square

$$\begin{tikzcd} Z \times_{Y'} X' \arrow[r, "p_{X'}"] \arrow[d, "p_Z"'] & X' \arrow[d, "f'"] \\ Z \arrow[r, "g"'] & Y' \end{tikzcd}.$$

Then

$$(p_{X'},g):(Z \times_{Y'} X',p_Z,Z) \to (X',f',Y')$$

is horizontal and $\mathrm{cod}(p_{X'},g) = g$, so we can take $X = Z \times_{Y'} X'$, $f = p_Z$, $Y = Z$, $u = (p_{X'},g)$.]

5.3 EXAMPLE Let $\underline{C}$ be a locally small finitely complete category. Fix an internal group G in $\underline{C}$ — then the restriction of cod to G-$\underline{\text{BUN}}(\underline{C})$ is a fibration.

[Recall the definitions:

- An object of G-$\underline{\text{BUN}}(\underline{C})$ is an object $E \xrightarrow{p} B$ of $\underline{C}/B$ together with an arrow $E \times G \xrightarrow{\mu} E$ such that the diagram

$$\begin{tikzcd} E \times G \arrow[r, "\mu"] \arrow[d, "p_E"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

commutes.

- A morphism

$$(E \xrightarrow{p} B) \longrightarrow (E' \xrightarrow{p'} B')$$

of G-$\underline{\text{BUN}}(\underline{C})$ is a pair

$$\begin{cases} \phi: E \longrightarrow E' \\ \psi: B \longrightarrow B' \end{cases}$$

of arrows in $\underline{C}$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\mu"'] & E' \arrow[d, "\mu'"] \\ B \arrow[r, "\psi"'] & B' \end{tikzcd}$$

commutes and ϕ is G-equivariant, i.e., the diagram

$$\begin{tikzcd} E \times G \arrow[r, "\mu"] \arrow[d, "\phi \times id_G"'] & E \arrow[d, "\phi"] \\ E' \times G \arrow[r, "\mu'"'] & E' \end{tikzcd}$$

commutes.]

[Note: Given a morphism $g:\tilde{B} \to B$ in $\underset{\sim}{C}$, to construct a horizontal

$$u:(\tilde{E} \xrightarrow{\tilde{p}} \tilde{B}) \longrightarrow (E \xrightarrow{p} B)$$

such that cod $u = g$, form the pullback square

$$\begin{tikzcd} \tilde{E} = \tilde{B} \times_B E \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{p}"'] & E \arrow[d, "p"] \\ \tilde{B} \arrow[r, "g"'] & B . \end{tikzcd}$$

Then the universal property of pullback determines a unique arrow $\tilde{E} \times G \xrightarrow{\tilde{\mu}} \tilde{E}$ such that the diagram

$$\begin{tikzcd} \tilde{E} \times G \arrow[r, "\tilde{\mu}"] \arrow[d, "p_{\tilde{E}}"'] & \tilde{E} \arrow[d, "\tilde{p}"] \\ \tilde{E} \arrow[r, "\tilde{p}"'] & \tilde{B} \end{tikzcd}$$

commutes subject to

$$\tilde{g} \circ \tilde{\mu} = \mu \circ (\tilde{g} \times id_G).$$

Therefore $u = (\tilde{g},g)$ is a horizontal morphism $\tilde{p} \to p$ such that cod $u = g$.]

5.4 EXAMPLE Given a category $\underline{C}$, define a category fam $\underline{C}$ as follows.

- The objects of fam $\underline{C}$ are the families $\{X_i : i \in I\}$, where I is a set and $X_i \in \text{Ob } \underline{C}$.

- A morphism

$$\{X_i : i \in I\} \rightarrow \{Y_j : j \in J\}$$

of fam $\underline{C}$ is a pair $(\phi, \{f_i : i \in I\})$, where $\phi : I \rightarrow J$ is a function and $f_i : X_i \rightarrow Y_{\phi(i)}$ is a morphism in $\underline{C}$.

[Note: The composite

$$(\psi, \{g_j : j \in J\}) \circ (\phi, \{f_i : i \in I\})$$

is the pair

$$(\psi \circ \phi, \{g_{\phi(i)} \circ f_i : i \in I\}).]$$

Let $U : \text{fam } \underline{C} \rightarrow \underline{\text{SET}}$ be the functor that sends $\{X_i : i \in I\}$ to I and $(\phi, \{f_i : i \in I\})$ to ϕ -- then U is a fibration.

[Let $\phi : I \rightarrow J$ be a function, $\{Y_j : j \in J\}$ a family of objects of $\underline{C}$. Put $X_i = Y_{\phi(i)}$ and let $f_i : X_i \rightarrow Y_{\phi(i)}$ be the identity -- then the morphism $(\phi, \{f_i : i \in I\})$ is horizontal and its image under U is ϕ.]

[Note: The horizontal morphisms are the pairs $(\phi, \{f_i : i \in I\})$, where $\forall\ i \in I$, f_i is an isomorphism.]

<u>N.B.</u> Let $\left[\begin{array}{l} \underline{C} \\ \underline{D} \end{array}\right.$ be categories, let

$$\left[\begin{array}{l} U : \text{fam } \underline{C} \rightarrow \underline{\text{SET}} \\ V : \text{fam } \underline{D} \rightarrow \underline{\text{SET}} \end{array}\right.$$

be the associated fibrations, and let $F:\underline{C} \to \underline{D}$ be a functor -- then F induces a horizontal functor

$$\text{fam } F:\text{fam } \underline{C} \to \text{fam } \underline{D}$$

by setting

$$\text{fam } F\{X_i:i \in I\} = \{FX_i:i \in I\}$$

and

$$\text{fam } F(\phi,\{f_i:i \in I\}) = (\phi,\{Ff_i:i \in I\}).$$

5.5 REMARK Take $\underline{C} = \underline{SET}$ -- then the fibrations

$$U:\text{fam } \underline{SET} \to \underline{SET},\ \text{cod}:\underline{SET}(\to) \to \underline{SET}$$

are equivalent.

[Define a horizontal functor

$$\text{fam } \underline{SET} \to \underline{SET}(\to)$$

on objects by sending the family $\{X_i:i \in I\}$ to the triple

$$(\coprod_{i \in I} X_i,f,I),$$

where $f(X_i) = i$, and define a horizontal functor

$$\underline{SET}(\to) \to \text{fam } \underline{SET}$$

on objects by sending the triple (X,f,Y) to the family $\{f^{-1}(y):y \in Y\}$.]

5.6 EXAMPLE Let $\underline{C}$ be a locally small finitely complete category. Suppose that $\mathcal{M} = (M,O,s,t,e,c)$ is an internal category in $\underline{C}$, thus M is an object of $\underline{C}$, O is an object of $\underline{C}$, and there are morphisms $s:M \to O$, $t:M \to O$, $e:O \to M$, $c:M \times_O M \to M$ satisfying the usual category theoretic relations.

Here

$$\begin{tikzcd} M \times_O M \arrow[r, "\pi_s"] \arrow[d, "\pi_t"'] & M \arrow[d, "t"] \\ M \arrow[r, "s"'] & O . \end{tikzcd}$$

Define a category $\underset{\sim}{C}(M)$ as follows.

- The objects of $\underset{\sim}{C}(M)$ are the pairs (I,u), where I is an object of $\underset{\sim}{C}$ and $u:I \to O$ is a morphism of $\underset{\sim}{C}$.

- A morphism

$$(I,u) \to (J,v)$$

of $\underset{\sim}{C}(M)$ is a pair (ϕ,f), where $\phi:I \to J$ and $f:I \to M$ are morphisms of $\underset{\sim}{C}$ such that $s \circ f = u$, $t \circ f = v \circ \phi$.

[Note: To formulate the composition law, let

$$(\phi,f):(I,u) \to (J,v), \quad (\psi,g):(J,v) \to (K,w)$$

be morphisms. Consider the arrows

$$I \xrightarrow{f} M \xrightarrow{t} O, \quad I \xrightarrow{\phi} J \xrightarrow{g} M \xrightarrow{s} O.$$

Then

$$s \circ g \circ \phi = v \circ \phi = t \circ f,$$

from which an arrow $h:I \to M \times_O M$ such that

$$\begin{cases} \pi_s \circ h = f \\ \pi_t \circ h = g \circ \phi. \end{cases}$$

Now put

$$(\psi,g) \circ (\phi,f) = (\psi \circ \phi, c \circ h)$$

and observe that

$$\begin{cases} s \circ c \circ h = s \circ \pi_s \circ h = s \circ f = u \\ t \circ c \circ h = t \circ \pi_t \circ h = t \circ g \circ \phi = w \circ \psi \circ \phi.] \end{cases}$$

Let $U_M:\underline{C}(M) \to \underline{C}$ be the functor that sends (I,u) to I and (ϕ,f) to ϕ -- then U_M is a fibration.

[Let $\phi:I \to J$ be a morphism of $\underline{C}$, where (J,v) is an object of $\underline{C}(M)$ -- then the morphism

$$(\phi, e \circ v \circ \phi):(I, v \circ \phi) \to (J,v)$$

is horizontal and its image under U_M is ϕ.]

N.B. Let $\underline{C}$ be a locally small finitely complete category, let $\begin{cases} M \\ N \end{cases}$ be internal categories in $\underline{C}$, let

$$\begin{cases} U_M:\underline{C}(M) \to \underline{C} \\ U_N:\underline{C}(N) \to \underline{C} \end{cases}$$

be the associated fibrations, and let $F:M \to N$ be an internal functor (so $F = (F_0,F_1)$ is a pair of morphisms $F_0:O \to P$, $F_1:M \to N$ subject to ...) -- then F induces a horizontal functor

$$\underline{C}(F):\underline{C}(M) \to \underline{C}(N)$$

by setting

$$\underline{C}(F)(I,u) = (I,F_0 \circ u)$$

and

$$\underline{C}(F)(\phi,f) = (\phi,F_1 \circ f).$$

[Note: If $F,G:\mathcal{M} \to \mathcal{N}$ are internal functors and if $\Xi:F \to G$ is an internal natural transformation (thought of as a morphism $\Xi:O \to N$ subject to ...), then the prescription

$$\underline{C}(\Xi)_{(I,u)} = (\mathrm{id}_I, \Xi \circ u)$$

determines a vertical natural transformation

$$\underline{C}(\Xi):\underline{C}(F) \to \underline{C}(G).$$

Denote by $[\mathcal{M},\mathcal{N}]_{\mathrm{int}}$ the category whose objects are the internal functors from $\mathcal{M}$ to $\mathcal{N}$ and whose morphisms are the internal natural transformations -- then the association $F \to \underline{C}(F)$, $\Xi \to \underline{C}(\Xi)$ defines a functor

$$[\mathcal{M},\mathcal{N}]_{\mathrm{int}} \to [\underline{C}(\mathcal{M}),\underline{C}(\mathcal{N})]_{\underline{C}} \quad \text{(cf. 4.39)}$$

which is full and faithful. Therefore, from the 2-category perspective, $\mathbf{CAT}(\underline{C})$ (cf. 1.6) is 2-equivalent to a full sub-2-category of $\underline{\underline{\mathrm{FIB}}}(\underline{C})$.]

5.7 REMARK Let X be an object of $\underline{C}$. Put $O = X$, $M = X$, take $s = t = \mathrm{id}_X$, $e = \mathrm{id}_X$, $c = \mathrm{id}_X$, and let $\mathcal{X}$ be the internal category of $\underline{C}$ thereby determined -- then $\underline{C}(\mathcal{X})$ can be identified with $\underline{C}/X$ and $U_{\mathcal{X}}$ becomes the forgetful functor $U_X:\underline{C}/X \to \underline{C}$. Moreover, the functor

$$\underline{C} \to \underline{\underline{\mathrm{FIB}}}(\underline{C})$$

that sends X to $(\underline{C}(\mathcal{X}),U_{\mathcal{X}})$ is full and faithful.

[Note: The assumption that $\underline{C}$ is finitely complete is not needed for these considerations.]

Let $\underline{I}$ be a small category, $F:\underline{I} \to \underline{\underline{\mathrm{CAT}}}$ a functor.

5.8 DEFINITION The <u>integral of F over $\underline{I}$</u>, denoted $\underline{INT}_{\underline{I}}F$, is the category whose objects are the pairs (i,X), where $i \in \mathrm{Ob}\,\underline{I}$ and $X \in \mathrm{Ob}\,Fi$, and whose morphisms are the arrows $(\delta,f):(i,X) \to (j,Y)$, where $\delta \in \mathrm{Mor}(i,j)$ and $f \in \mathrm{Mor}((F\delta)X,Y)$ (composition is given by

$$(\delta',f') \circ (\delta,f) = (\delta' \circ \delta,\ f' \circ (F\delta')f)).$$

5.9 NOTATION Let

$$\Theta_F:\underline{INT}_{\underline{I}}F \to \underline{I}$$

be the functor that sends (i,X) to i and (δ,f) to δ.

[Note: The fiber of Θ_F over i is isomorphic to the category Fi.]

The relevant points then are these.

- The preophorizontal morphisms are the (δ,f), where f is an isomorphism.

[Note: The composition of two preophorizontal morphisms is therefore preophorizontal.]

- Θ_F is a preopfibration.

5.10 FACT Θ_F is an opfibration (quote 4.25 in its "op" rendition).

Let $F,G:\underline{I} \to \underline{CAT}$ be functors, $\Xi:F \to G$ a natural transformation.

5.11 DEFINITION The <u>integral of Ξ over $\underline{I}$</u>, denoted $\underline{INT}_{\underline{I}}\Xi$, is the functor

$$\underline{INT}_{\underline{I}}F \to \underline{INT}_{\underline{I}}G$$

defined by the prescription

$$\begin{cases} (\underline{INT}_{\underline{I}}\Xi)(i,X) = (i,\Xi_i X) \\ (\underline{INT}_{\underline{I}}\Xi)(\delta,f) = (\delta,\Xi_j f). \end{cases}$$

Obviously,

$$\Theta_G \circ \underline{\mathrm{INT}}_{\underline{I}}\Xi = \Theta_F,$$

and $\underline{\mathrm{INT}}_{\underline{I}}\Xi$ sends ophorizontal arrows to ophorizontal arrows. Therefore $\underline{\mathrm{INT}}_{\underline{I}}\Xi$ is an ophorizontal functor from $\underline{\mathrm{INT}}_{\underline{I}}F$ to $\underline{\mathrm{INT}}_{\underline{I}}G$.

N.B. The association

$$\begin{cases} F \to (\underline{\mathrm{INT}}_{\underline{I}}F, \Theta_F) \\ \Xi \to \underline{\mathrm{INT}}_{\underline{I}}\Xi \end{cases}$$

defines a functor

$$\underline{\mathrm{INT}}_{\underline{I}}:[\underline{I},\underline{\mathrm{CAT}}] \to \underline{\mathrm{CAT}}/\underline{I}.$$

5.12 EXAMPLE Let $\underline{I}$ be a small category -- then the twisted arrow category $\underline{I}(\sim>)$ of $\underline{I}$ is the category whose objects are the triples (i,δ,j), where $\delta:i \to j$ is an arrow in $\underline{I}$, and a morphism

$$(i,\delta,j) \to (i',\delta',j')$$

is a pair

$$\begin{cases} \phi:i' \to i \\ \psi:j \to j' \end{cases}$$

of arrows in $\underline{I}$ such that the diagram

$$\begin{tikzcd} i \arrow[d, "\delta"'] & i' \arrow[l, "\phi"'] \arrow[d, "\delta'"] \\ j \arrow[r, "\psi"'] & j' \end{tikzcd}$$

commutes. Denote by $\left[\begin{array}{l} s_{\underline{I}} \\ t_{\underline{I}} \end{array}\right.$ the canonical projections

$$\left[\begin{array}{l} \underline{I}(\sim>) \to \underline{I}^{OP} \\ \\ \underline{I}(\sim>) \to \underline{I}, \end{array}\right.$$

hence

$$\left[\begin{array}{ll} s_{\underline{I}}\delta = \mathrm{dom}\ \delta & s_{\underline{I}}(\phi,\psi) = \phi \\ \\ t_{\underline{I}}\delta = \mathrm{cod}\ \delta, & t_{\underline{I}}(\phi,\psi) = \psi, \end{array}\right.$$

and $\left[\begin{array}{l} s_{\underline{I}} \\ t_{\underline{I}} \end{array}\right.$ are opfibrations.

[Let

$$H_{\underline{I}}:\underline{I}^{OP} \times \underline{I} \to \underline{\underline{CAT}}$$

be the functor $(j,i) \to \mathrm{Mor}(j,i)$, where the set $\mathrm{Mor}(j,i)$ is regarded as a discrete category -- then

$$\underline{\underline{INT}}_{\underline{I}^{OP} \times \underline{I}} H_{\underline{I}}$$

can be identified with $I(\sim>)$, $\Theta_{H_{\underline{I}}}$ becoming the functor

$$(s_{\underline{I}},t_{\underline{I}}):\underline{I}(\sim>) \to \underline{I}^{OP} \times \underline{I}.$$

Therefore $\left[\begin{array}{l} s_{\underline{I}} \\ t_{\underline{I}} \end{array}\right.$ are opfibrations (the ambient projections are opfibrations and opfibrations are composition closed).]

The notion of pseudo pullback, as formulated in 1.22, can be extended from $\mathbf{CAT}$ to $\mathbf{CAT}/\underline{B}$.

5.13 CONSTRUCTION Fix a category $\underline{B}$. Let $\begin{bmatrix} (\underline{E}_1,P_1) \\ (\underline{E}_2,P_2) \end{bmatrix}$, $(\underline{E},P)$ be objects of $\mathbf{CAT}/\underline{B}$ and let

$$\begin{bmatrix} F_1:(\underline{E}_1,P_1) \to (\underline{E},P) \\ F_2:(\underline{E}_2,P_2) \to (\underline{E},P) \end{bmatrix}$$

be morphisms of $\mathbf{CAT}/\underline{B}$ -- then the <u>pseudo pullback</u> $\underline{E}_1 \times_{\underline{E}} \underline{E}_2$ of the 2-sink

$$(\underline{E}_1,P_1) \xrightarrow{F_1} (\underline{E},P) \xleftarrow{F_2} (\underline{E}_2,P_2)$$

is the following category.

- An object of $\underline{E}_1 \times_{\underline{E}} \underline{E}_2$ is a quadruple (B,X_1,X_2,ϕ), where $B \in \text{Ob } \underline{B}$, $X_1 \in \text{Ob}(\underline{E}_1)_B$, $X_2 \in \text{Ob}(\underline{E}_2)_B$, and $\phi:F_1X_1 \to F_2X_2$ is an isomorphism in $\underline{E}_B$.

- A morphism

$$(B,X_1,X_2,\phi) \longrightarrow (B',X_1',X_2',\phi')$$

is a pair (f_1,f_2), where $f_1:X_1 \to X_1'$ is a morphism in $\underline{E}_1$, $f_2:X_2 \to X_2'$ is a morphism in $\underline{E}_2$, subject to f_1 and f_2 induce the same morphism $B \to B'$ (i.e., $P_1f_1 = P_2f_2$)

and the diagram

$$\begin{tikzcd} F_1X_1 \arrow[r, "\phi"] \arrow[d, "F_1f_1"'] & F_2X_2 \arrow[d, "F_2f_2"] \\ F_1X_1' \arrow[r, "\phi'"'] & F_2X_2' \end{tikzcd}$$

commutes.

Define functors

$$\begin{cases} p_1 : \underline{E}_1 \times_{\underline{E}} \underline{E}_2 \to \underline{E}_1 \\ p_2 : \underline{E}_1 \times_{\underline{E}} \underline{E}_2 \to \underline{E}_2 \end{cases}$$

by

$$\begin{cases} p_1(B,X_1,X_2,\phi) = X_1 \quad (p_1(f_1,f_2) = f_1) \\ p_2(B,X_1,X_2,\phi) = X_2 \quad (p_2(f_1,f_2) = f_2) \end{cases}$$

and define a natural transformation

$$\Xi : F_1 \circ p_1 \to F_2 \circ p_2$$

by

$$\Xi_{(B,X_1,X_2,\phi)} : F_1X_1 \xrightarrow{\phi} F_2X_2.$$

Then the diagram

$$\begin{tikzcd} \underline{E}_1 \times_{\underline{E}} \underline{E}_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \underline{E}_2 \arrow[d, "F_2"] \\ \underline{E}_1 \arrow[r, "F_1"'] & \underline{E} \end{tikzcd}$$

of 0-cells in 2-$\mathbf{CAT}/\underline{B}$ is 2-commutative.

[Note: Let

$$\Pi:\underline{E}_1 \times_{\underline{E}} \underline{E}_2 \to \underline{B}$$

be the canonical projection -- then

$$\begin{cases} F_1 \circ p_1:(\underline{E}_1 \times_{\underline{E}} \underline{E}_2,\Pi) \to (\underline{E},P) \\ F_2 \circ p_2:(\underline{E}_1 \times_{\underline{E}} \underline{E}_2,\Pi) \to (\underline{E},P) \end{cases}$$

are morphisms in $\mathbf{CAT}/\underline{B}$. E.g.:

$$P \circ F_1 \circ p_1(B,X_1,X_2,\phi) = PF_1X_1 = P_1X_1 = B$$

while

$$\Pi(B,X_1,X_2,\phi) = B.$$

Moreover, Ξ is vertical. In fact,

$$P\Xi_{(B,X_1,X_2,\phi)} = P\phi = \mathrm{id}_B = \mathrm{id}_{\Pi(B,X_1,X_2,\phi)}.]$$

<u>N.B.</u> As regards the fibers, $\forall\ B \in \mathrm{Ob}\ \underline{B}$,

$$(\underline{E}_1 \times_{\underline{E}} \underline{E}_2)_B \approx (\underline{E}_1)_B \times_{\underline{E}_B} (\underline{E}_2)_B.$$

5.14 EXAMPLE If $\begin{cases} (\underline{E}_1,P_1) \\ (\underline{E}_2,P_2) \end{cases}$, $(\underline{E},P)$ are objects of $\underline{\mathrm{FIB}}(\underline{B})$ and if

$$\begin{cases} F_1:(\underline{E}_1,P_1) \to (\underline{E},P) \\ F_2:(\underline{E}_2,P_2) \to (\underline{E},P) \end{cases}$$

are morphisms of $\underline{\text{FIB}}(\underline{B})$, then the canonical projection

$$\Pi:\underline{E}_1 \times_{\underline{E}} \underline{E}_2 \to \underline{B}$$

is a fibration.

5.15 DEFINITION The functor $P:\underline{E} \to \underline{B}$ is a <u>bifibration</u> if it is both a fibration and an opfibration.

5.16 EXAMPLE The functor

$$\mathrm{Ob}:\underline{\mathrm{CAT}} \to \underline{\mathrm{SET}}$$

figuring in 5.1 is a bifibration.

5.17 EXAMPLE The functor

$$\mathrm{cod}:\underline{C}(\to) \to \underline{C}$$

figuring in 5.2 is a bifibration.

§6. *FIBRATIONS: SORITES*

6.1 LEMMA If $F:\underline{C} \to \underline{D}$ and $G:\underline{D} \to \underline{E}$ are fibrations, then so is their composition $G \circ F:\underline{C} \to \underline{E}$.

6.2 REMARK Display the data:

$$\begin{tikzcd} \underline{C} \arrow[r, "F"] \arrow[d, "G \circ F"'] & \underline{D} \arrow[d, "G"] \\ \underline{E} \arrow[r, equal] & \underline{E} \end{tikzcd}.$$

Then F defines a morphism

$$(\underline{C}, G \circ F) \to (\underline{D}, G)$$

in $\mathbf{CAT}/\underline{E}$ but more is true: F sends horizontal arrows to horizontal arrows. Therefore F defines a morphism

$$(\underline{C}, G \circ F) \to (\underline{D}, G)$$

in $\mathbf{CAT}_h/\underline{E}$ or still, F defines a morphism

$$(\underline{C}, G \circ F) \to (\underline{D}, G)$$

in $\underline{\text{FIB}}(\underline{E})$.

6.3 LEMMA The projection functor

$$\underline{C} \times \underline{D} \to \underline{D}$$

is a fibration.

6.4 LEMMA If $F:\underline{C} \to \underline{D}$ and $F':\underline{C}' \to \underline{D}'$ are fibrations, then the product functor

$$F \times F':\underline{C} \times \underline{C}' \to \underline{D} \times \underline{D}'$$

is a fibration.

6.5 LEMMA Let $F:\underline{C} \rightarrow \underline{D}$ be a fibration and let $\underline{I}$ be a small category -- then

$$F_*:[\underline{I},\underline{C}] \rightarrow [\underline{I},\underline{D}]$$

is a fibration.

6.6 RAPPEL Given a 2-sink $\underline{B}' \xrightarrow{\beta} \underline{B} \xleftarrow{P} \underline{E}$ in $\mathbf{CAT}$, its pullback $\underline{E}' = \underline{B}' \times_{\underline{B}} \underline{E}$ is the category whose objects are the pairs (B',X) ($B' \in \text{Ob } \underline{B}'$, $X \in \text{Ob } \underline{E}$) such that $\beta B' = PX$ and whose morphisms

$$(B_1',X_1) \rightarrow (B_2',X_2)$$

are the pairs (ϕ,f), where $\phi:B_1' \rightarrow B_2'$ is a morphism in $\underline{B}'$ and $f:X_1 \rightarrow X_2$ is a morphism in $\underline{E}$ such that $\beta\phi = Pf$, there being, then, a commutative diagram

$$\begin{tikzcd} \underline{E}' \arrow[r, "pr_{\underline{E}}"] \arrow[d, "P'"'] & \underline{E} \arrow[d, "P"] \\ \underline{B}' \arrow[r, "\beta"'] & \underline{B} \end{tikzcd}$$

6.7 LEMMA Suppose that the functor $P:\underline{E} \rightarrow \underline{B}$ is a fibration -- then for any functor $\beta:\underline{B}' \rightarrow \underline{B}$, the functor $P':\underline{E}' \rightarrow \underline{B}'$ is a fibration.

PROOF Let $g':B'' \rightarrow P'(B',X)$ ($= B'$) be a morphism in $\underline{B}'$. Choose a horizontal $u:Y \rightarrow X$ such that $Pu = \beta g'$, thus $PY = \beta B''$, $PX = \beta B'$, and

$$(g',u):(B'',Y) \rightarrow (B',X)$$

is a horizontal morphism in $\underline{E}'$ such that $P'(g',u) = g'$.

[Note: The opposite of a pullback square is a pullback square. So, if the functor $P:\underline{E} \to \underline{B}$ is an opfibration, then for any functor $\beta:\underline{B}' \to \underline{B}$, the functor $P':\underline{E}' \to \underline{B}'$ is an opfibration.]

N.B. The pair

$$\begin{bmatrix} (\underline{E}',P') \\ \\ (\underline{E},P) \end{bmatrix} \text{ is an object of } \begin{bmatrix} \underline{\mathrm{FIB}}(\underline{B}') \\ \\ \underline{\mathrm{FIB}}(\underline{B}). \end{bmatrix}$$

And the projection $pr_{\underline{E}}:\underline{E}' \to \underline{E}$ sends horizontal arrows to horizontal arrows.

6.8 APPLICATION Suppose that

$$\begin{bmatrix} P_1:\underline{E}_1 \to \underline{B} \\ P_2:\underline{E}_2 \to \underline{B} \end{bmatrix}$$

are fibrations. Form the pullback square

$$\begin{tikzcd} \underline{E}_1 \times_{\underline{B}} \underline{E}_2 \arrow[r] \arrow[d] & \underline{E}_2 \arrow[d, "P_2"] \\ \underline{E}_1 \arrow[r, "P_1"'] & \underline{B} \end{tikzcd} .$$

Then the corner arrow

$$\underline{E}_1 \times_{\underline{B}} \underline{E}_2 \to \underline{B}$$

is a fibration (recall 6.1).

6.9 REMARK The category $\underline{\underline{FIB}}(\underline{B})$ has finite products.

[The projections

$$\begin{cases} \underline{E}_1 \times_{\underline{B}} \underline{E}_2 \to \underline{E}_1 \\ \\ \underline{E}_1 \times_{\underline{B}} \underline{E}_2 \to \underline{E}_2 \end{cases}$$

are morphisms in $\underline{\underline{FIB}}(\underline{B})$ (cf. 6.2). Therefore $\underline{\underline{FIB}}(\underline{B})$ has binary products. And $\mathrm{id}_{\underline{B}}$ serves as a final object (cf. 4.17).]

Given a 2-sink $\underline{B}' \xrightarrow{\beta} \underline{B} \xleftarrow{P} \underline{E}$ in $\mathbf{CAT}$, one can form its pseudo pullback $\underline{B}' \underline{\times}_{\underline{B}} \underline{E}$ (cf. 1.22). Introduce the comparison functor

$$\Gamma : \underline{B}' \times_{\underline{B}} \underline{E} \to \underline{B}' \underline{\times}_{\underline{B}} \underline{E} \qquad \text{(cf. 1.23)}$$

and consider the diagram

$$\begin{tikzcd}
\underline{B}' \times_{\underline{B}} \underline{E} \arrow[r, "\Gamma"] \arrow[d, "P'"'] & \underline{B}' \underline{\times}_{\underline{B}} \underline{E} \arrow[r] \arrow[d] & \underline{E} \arrow[d, "P"] \\
\underline{B}' \arrow[r, equal] & \underline{B}' \arrow[r, "\beta"'] & \underline{B}
\end{tikzcd},$$

the square on the right being 2-commutative.

6.10 LEMMA Suppose that the functor $P : \underline{E} \to \underline{B}$ is a fibration -- then the projection $\underline{B}' \underline{\times}_{\underline{B}} \underline{E} \to \underline{B}'$ is a fibration.

PROOF If (B',X) is an object of $\underline{B}' \times_{\underline{B}} \underline{E}$, then

$$\Gamma(B',X) = (B',X,\mathrm{id}) \to B' = P'(B',X).$$

But P has the isomorphism lifting property (cf. 4.23), hence Γ is an equivalence over $\underline{B}'$ (cf. 1.23), from which the assertion (cf. 4.36).

6.11 DEFINITION Let $P_1:\underline{E}_1 \to \underline{B}$, $P_2:\underline{E}_2 \to \underline{B}$ be fibrations -- then a morphism $F:(\underline{E}_1,P_1) \to (\underline{E}_2,P_2)$ in $\underline{\mathrm{FIB}}(\underline{B})$ is said to be internal if given any vertical arrow $f_2 \in \mathrm{Mor}\ \underline{E}_2$ (thus $P_2f_2 = \mathrm{id}$ (cf. 4.26)), there exists a horizontal arrow $f_1 \in \mathrm{Mor}\ \underline{E}_1$ per F such that $Ff_1 = f_2$ ($\Rightarrow P_1f_1 = P_2Ff_1 = P_2f_2 = \mathrm{id}$).

[Note: In this context, there are three possibilities for the term "horizontal", viz. per P_1, per P_2, or per F.]

N.B. If F is a fibration, then F is internal (recall that F is necessarily a morphism in $\underline{\mathrm{FIB}}(\underline{B})$).

6.12 LEMMA Suppose that F is internal -- then $\forall\ B \in \mathrm{Ob}\ \underline{B}$,

$$F_B:(\underline{E}_1)_B \to (\underline{E}_2)_B$$

is a fibration.

6.13 LEMMA Suppose that F is internal -- then F is a fibration.

PROOF Given a morphism $g:X_2 \to FX_1'$, the claim is that there exists a horizontal morphism $u:X_1 \to X_1'$ per F such that $Fu = g$. To establish this, start by applying P_2, hence $P_2g:P_2X_2 \to P_2FX_1' = P_1X_1'$. Next, choose a horizontal morphism $\tilde{u}:\tilde{X}_1 \to X_1'$ per P_1 such that $P_1\tilde{u} = P_2g$ ($\Rightarrow P_1\tilde{X}_1 = P_2X_2$) -- then $F\tilde{u}$ is, by assumption, horizontal per P_2. Consider now the factorization

$$P_2F\tilde{X}_1 \xrightarrow[\mathrm{id}]{} P_2F\tilde{X}_1 \xrightarrow[P_2F\tilde{u}]{} P_2FX_1'$$

or, equivalently, the factorization

$$\begin{tikzcd} P_2X_2 \arrow[r, "\mathrm{id}"'] \arrow[rr, bend left, "P_2g"] & P_1\tilde{X}_1 \arrow[r, "P_2F\tilde{u}"'] & P_1X_1' \end{tikzcd} .$$

From the definitions, there is a unique morphism $v:X_2 \to F\tilde{X}_1$ such that $P_2v = \mathrm{id}$ and $F\tilde{u} \circ v = g$. Schematically:

$$\begin{tikzcd} X_2 \arrow[r, dotted, "v"'] \arrow[rr, bend left, "g"] & F\tilde{X}_1 \arrow[r, "F\tilde{u}"'] & FX_1' \end{tikzcd} .$$

But v is vertical, so, F being internal, one can find a horizontal arrow $\tilde{v}$ per F such that $F\tilde{v} = v$, where the codomain of $\tilde{v}$ is $\tilde{X}_1$. Put $u = \tilde{u} \circ \tilde{v}$ -- then $Fu = F\tilde{u} \circ F\tilde{v} = F\tilde{u} \circ v = g$ and u is horizontal per F (verification left to the reader).]

§7. *THE FUNDAMENTAL 2-EQUIVALENCE*

Let $\underline{B}$ be a category -- then $\underline{B}$ can be regarded as a 2-category $\mathbf{B}$ for which $U\mathbf{B} \approx \underline{B}$ (cf. 1.14), but we shall abuse notation and write $\underline{B}$ in place of $\mathbf{B}$ (no confusion will result in so doing).

N.B. Traditionally, $\underline{B}$ is replaced by $\underline{B}^{OP}$, the relevant 2-metacategories being

$$2\text{-}[\underline{B}^{OP}, 2\text{-}\mathbf{CAT}]$$

and

$$PS\text{-}[\underline{B}^{OP}, 2\text{-}\mathbf{CAT}].$$

[Note: The first is a sub-2-metacategory of the second.]

The 0-cells of

$$PS\text{-}[\underline{B}^{OP}, 2\text{-}\mathbf{CAT}]$$

are the pseudo functors from $\underline{B}^{OP}$ to 2-$\mathbf{CAT}$. If $F:\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$ is a pseudo functor, then $\forall\ B \in \text{Ob}\ \underline{B}$, FB is a category and $\forall\ B, B' \in \text{Ob}\ \underline{B}$ and $\forall\ \beta \in \text{Mor}(B,B')$, $F\beta:FB' \to FB$ is a functor.

7.1 EXAMPLE Take $\underline{B} = \underline{TOP}$ and let (X,τ_X) be a topological space -- then τ_X can be viewed as a category and a continuous function $f:(X,\tau_X) \to (Y,\tau_Y)$ induces a functor $f^{-1}:\tau_Y \to \tau_X$. Therefore this data determines a 2-functor

$$\underline{TOP}^{OP} \longrightarrow 2\text{-}\mathbf{CAT}.$$

7.2 EXAMPLE Take $\underline{B} = \underline{CAT}$ and fix a category $\underline{D}$ -- then for any small category $\underline{C}$, $[\underline{C},\underline{D}]$ is a category and a functor $F:\underline{C} \to \underline{C}'$ induces a functor $F^*:[\underline{C}',\underline{D}] \to [\underline{C},\underline{D}]$. Therefore this data determines a 2-functor

$$\underline{CAT}^{OP} \longrightarrow 2\text{-}\mathbf{CAT}.$$

7.3 EXAMPLE Take $\underline{B} = \underline{SCH}$ and given a scheme X, let $\underline{QCO}(X)$ be the category of quasi-coherent sheaves on X -- then a morphism $f:X \to Y$ induces a functor $f^*:\underline{QCO}(Y) \to \underline{QCO}(X)$. Therefore this data determines a pseudo functor

$$\underline{SCH}^{OP} \longrightarrow 2\text{-}\mathbf{CAT}.$$

[Note: Bear in mind that if $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)^*:\underline{QCO}(Z) \to \underline{QCO}(X)$ is not literally $f^* \circ g^*:\underline{QCO}(Z) \to \underline{QCO}(X)\ldots$.]

7.4 NOTATION Given pseudo functors $F,G:\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$, let $PS(F,G)$ stand for the metacategory whose objects are the pseudo natural transformations $\Xi:F \to G$ and whose morphisms are the pseudo modifications $\Psi:\Xi \to \Omega$.

Here is the main result.

7.5 THEOREM There is a 2-functor

$$\text{gro}_{\underline{B}}:PS\text{-}[\underline{B}^{OP}, 2\text{-}\mathbf{CAT}] \to \mathcal{FIB}(\underline{B})$$

with the following properties.

(1) $\forall$ ordered pair F,G of pseudo functors $\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$,

$$(\text{gro}_{\underline{B}})_{F,G}:PS(F,G) \to [\text{gro}_{\underline{B}}F, \text{gro}_{\underline{B}}G]_{\underline{B}}$$

is an isomorphism of metacategories.

(2) $\forall$ fibration $P:\underline{E} \to \underline{B}$, $\exists$ a pseudo functor $F:\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$ such that $\underline{E}$ is isomorphic to $\text{gro}_{\underline{B}}F$ in $\underline{FIB}(\underline{B})$.

7.6 REMARK Therefore $PS\text{-}[\underline{B}^{OP}, 2\text{-}\mathbf{CAT}]$ and $\mathcal{FIB}(\underline{B})$ are 2-equivalent (cf. 2.15).

The proof of 7.5, when taken in all detail, is lengthy.

7.7 GROTHENDIECK CONSTRUCTION Let $F:\underline{B}^{OP} \rightarrow 2\text{-}\mathbf{CAT}$ be a pseudo functor — then $\text{gro}_{\underline{B}}F$ is the category whose objects are the pairs (B,X), where $B \in \text{Ob } \underline{B}$ and $X \in \text{Ob } FB$, and whose morphisms are the arrows $(\beta,f):(B,X) \rightarrow (B',X')$, where $\beta \in \text{Mor}(B,B')$ and $f \in \text{Mor}(X,(F\beta)X')$.

[Note: Suppose that

$$\left[\begin{array}{l} (\beta,f):(B,X) \rightarrow (B',X') \\ (\beta',f'):(B',X') \rightarrow (B'',X''). \end{array}\right.$$

Then by definition

$$(\beta',f') \circ (\beta,f) = (\beta' \circ \beta, f' \circ_F f).$$

Here

$$f' \circ_F f \in \text{Mor}(X,F(\beta' \circ \beta)X'')$$

is the composition

$$X \xrightarrow{f} (F\beta)X' \xrightarrow{(F\beta)f'} (F\beta)(F\beta')X'' \approx F(\beta' \circ \beta)X'',$$

the isomorphism on the right being implicit in the definition of pseudo functor. Using the first axiom for a pseudo functor (cf. §3), one can check that this composition law is associative and using the second axiom for a pseudo functor (cf. §3), one can check that the identity in $\text{Mor}((B,X),(B,X))$ is the pair $(\text{id}_B, X \approx F(\text{id}_B)X)$.]

7.8 NOTATION Let

$$\Theta_F:\text{gro}_{\underline{B}}F \rightarrow \underline{B}$$

be the functor that sends (B,X) to B and (β,f) to β.

7.9 LEMMA Θ_F is a fibration and the fiber of Θ_F over B is isomorphic to the category FB.

To complete the definition of $\mathrm{gro}_{\underline{B}}$ so as to make it a 2-functor, one has to consider its action on the pseudo natural transformations and the pseudo modifications.

• Let $F,G:\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$ be pseudo functors, $\Xi:F \to G$ a pseudo natural transformation, the associated data thus being $\forall\ B \in \mathrm{Ob}\ \underline{B}$, a functor

$$\Xi_B:FB \to GB,$$

and $\forall\ \beta \in \mathrm{Mor}(B,B')$, a 2-commutative diagram

$$\begin{tikzcd} FB \arrow[r, "\Xi_B"] & GB \\ FB' \arrow[u, "F\beta"] \arrow[r, "\Xi_{B'}"'] & GB' \arrow[u, "G\beta"'] \end{tikzcd}$$

in $2\text{-}\mathbf{CAT}$, where

$$\tau_\beta:\Xi_B \circ F\beta \longrightarrow G\beta \circ \Xi_{B'}$$

is a natural isomorphism subject to the coherency conditions. We then define a horizontal functor

$$\mathrm{gro}_{\underline{B}}\Xi:\mathrm{gro}_{\underline{B}}F \longrightarrow \mathrm{gro}_{\underline{B}}G$$

by the prescription

$$\begin{cases} (\mathrm{gro}_{\underline{B}}\Xi)(B,X) = (B,\Xi_B X) \\ (\mathrm{gro}_{\underline{B}}\Xi)(\beta,f) = (\beta,g), \end{cases}$$

where $g \in \mathrm{Mor}(\Xi_B X, (G\beta)(\Xi_{B'}X'))$ is the composition

$$\Xi_B X \xrightarrow{\Xi_B f} \Xi_B(F\beta)(X') \xrightarrow{\tau_{\beta,X'}} (G\beta)(\Xi_{B'}X').$$

● Let $F,G:\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$ be pseudo functors, $\Xi,\Omega:F \to G$ pseudo natural transformations, and $\text{Ч}:\Xi \to \Omega$ a pseudo modification, the associated data thus being $\forall\ B \in \mathrm{Ob}\ \underline{B}$, a natural transformation $\text{Ч}_B:\Xi_B \to \Omega_B$ subject to the commutativity of the diagram

$$\begin{tikzcd}
\Xi_B(F\beta)(X') \arrow[r, "\tau^{\Xi}_{\beta,X'}"] \arrow[d, "\text{Ч}_{B,(F\beta)X'}"'] & (G\beta)(\Xi_{B'}X') \arrow[d, "(G\beta)(\text{Ч}_{B',X'})"] \\
\Omega_B(F\beta)(X') \arrow[r, "\tau^{\Omega}_{\beta,X'}"'] & (G\beta)(\Omega_{B'}X')
\end{tikzcd}.$$

We then define a vertical natural transformation

$$\mathrm{gro}_{\underline{B}}\text{Ч}:\mathrm{gro}_{\underline{B}}\Xi \to \mathrm{gro}_{\underline{B}}\Omega$$

by the prescription

$$(\mathrm{gro}_{\underline{B}}\text{Ч})_{(B,X)} = (\mathrm{id}_B, \text{Ч}_{B,X}).$$

[Note: To see that this makes sense, observe first that $\mathrm{gro}_{\underline{B}}\text{Ч}$ has to be indexed by the pairs (B,X) $(B \in \mathrm{Ob}\ \underline{B},\ X \in FB)$, so

$$(\mathrm{gro}_{\underline{B}}\text{Ч})_{(B,X)}:(\mathrm{gro}_{\underline{B}}\Xi)(B,X) \to (\mathrm{gro}_{\underline{B}}\Omega)(B,X)$$

or still,

$$(\mathrm{gro}_{\underline{B}}\text{Ч})_{(B,X)}:(B,\Xi_B X) \to (B,\Omega_B X).$$

But

$$\left[\begin{array}{l} X \in FB \Rightarrow \Xi_B X \in GB \\ X \in FB \Rightarrow \Omega_B X \in GB. \end{array}\right.$$

And $\forall\ X \in FB$,

$$Ч_{B,X} \in \operatorname{Mor}(\Xi_B X,\ \Omega_B X).$$

Therefore the pair $(\mathrm{id}_B,\ Ч_{B,X})$ belongs to

$$\operatorname{Mor}((\mathrm{gro}_{\underline{B}}\Xi)(B,X),\ (\mathrm{gro}_{\underline{B}}\Omega)(B,X))$$

per $\mathrm{gro}_{\underline{B}}G$. That $\mathrm{gro}_{\underline{B}}Ч$ is vertical is obvious:

$$\begin{aligned} \Theta_G(\mathrm{gro}_{\underline{B}}Ч)_{(B,X)} &= \Theta_G(\mathrm{id}_B,\ Ч_{B,X}) \\ &= \mathrm{id}_B = \mathrm{id}_{\Theta_F(B,X)}.] \end{aligned}$$

In summary: The Grothendieck construction provides us with a 2-functor

$$\mathrm{gro}_{\underline{B}}:\mathrm{PS}\text{-}[\underline{B}^{\mathrm{OP}},\ 2\text{-}\mathbf{CAT}] \to \mathcal{FIB}(\underline{B})$$

and it remains to address points (1) and (2) of 7.5. Since the verification of the first point is straightforward (albeit tedious), we shall focus on the second which requires some additional input.

Let $P:\underline{E} \to \underline{B}$ be a fibration and suppose that $g:B \to B'$ is an arrow in $\underline{B}$. Assuming that $\underline{E}_{B'} \neq \underline{0}$, for each $X' \in \operatorname{Ob} \underline{E}_{B'}$, choose a horizontal $u:X \to X'$ such that $Pu = g$ and define $g^*:\underline{E}_{B'} \to \underline{E}_B$ as follows.

- On an object X', let $g^*X' = X$.
- On a morphism $\phi:X' \to \tilde{X}'$, noting that $P(\phi \circ u) = P\phi \circ Pu = \mathrm{id}_{B'} \circ Pu =$

$g = P\tilde{u}$, let $g^*\phi$ be the unique filler in the fiber over B for the diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] & \tilde{X}' \\ X \arrow[u, dashed, "g^*\phi"] \arrow[r, equal] & X \arrow[u, "\phi \circ u"'] \end{tikzcd}$$

7.10 LEMMA $g^*:\underline{E}_{B'} \to \underline{E}_B$ is a functor.

[Note: Take g^* to be the canonical inclusion if $\underline{E}_{B'} = \underline{0}$.]

Needless to say, the definition of g^* hinges on the choice of the horizontal $u:X \to X'$.

7.11 DEFINITION A <u>cleavage</u> for P is a functor σ which assigns to each pair (g,X'), where $g:B \to PX'$, a horizontal morphism $u = \sigma(g,X')$ $(u:X \to X')$ such that $Pu = g$.

[Note: The axiom of choice for classes implies that every fibration has a cleavage.]

7.12 EXAMPLE Consider $\text{gro}_{\underline{B}}F$ -- then the <u>canonical cleavage</u> for Θ_F is the rule that sends $\beta:B \to B'$ $(= \Theta_F(B',X'))$ to the horizontal morphism

$$(\beta,\text{id}_{(F\beta)X'}):(B,(F\beta)X') \to (B',X').$$

Consider now a pair (P,σ), where σ is a cleavage for P -- then the association

$$B \longrightarrow \underline{E}_B,\ (B \xrightarrow{g} B') \longrightarrow (\underline{E}_{B'} \xrightarrow{g^*} \underline{E}_B)$$

defines a pseudo functor $\Sigma_{P,\sigma}$ from $\underline{B}^{\text{OP}}$ to 2-$\mathbf{CAT}$.

7.13 LEMMA If $P:\underline{E} \to \underline{B}$ is a fibration, then $\underline{E}$ is isomorphic to $\text{gro}_{\underline{B}}\Sigma_{P,\sigma}$ in $\underline{\text{FIB}}(\underline{B})$.

PROOF Define a horizontal functor $\Phi:\underline{E} \to \text{gro}_{\underline{B}}\Sigma_{P,\sigma}$ by the following procedure.

- Given $X \in \text{Ob } \underline{E}$, let

$$\Phi X = (PX,X) \quad (X \in \text{Ob } E_{PX} = \text{Ob } \Sigma_{P,\sigma}PX).$$

- Given a morphism $f:Y \to X$ in $\underline{E}$, Φf must send $\Phi Y = (PY,Y)$ to $\Phi X = (PX,X)$.

So let $\Phi f = (Pf,\phi_f)$, where

$$\phi_f \in \text{Mor}(Y,(\Sigma_{P,\sigma}Pf)X),$$

or still,

$$\phi_f \in \text{Mor}(Y,(Pf)^*X) \quad ((Pf)^*X \in E_{PY})$$

is defined to be the unique filler in the fiber over PY for the diagram

$$\begin{tikzcd}
(Pf)^*X \arrow[rr, "\sigma(Pf,X)"] & & X \\
Y \arrow[u, dotted, "\phi_f"] \arrow[rr, equal] & & Y \arrow[u, "f"']
\end{tikzcd}$$

Here, by definition, $P\sigma(Pf,X) = Pf$.

The claim then is that Φ is an isomorphism of categories. But it is clear that Φ is bijective on objects. As for the morphisms, the arrow

$$\text{Mor}(Y,X) \to \text{Mor}((PY,Y),(PX,X))$$

taking f to (Pf,ϕ_f) is manifestly injective:

$$(Pf,\phi_f) = (Pg,\phi_g)$$

$\Rightarrow$

$$f = \sigma(Pf,X) \circ \phi_f = \sigma(Pg,X) \circ \phi_g = g.$$

To establish that it is surjective, consider a pair (g,ψ), where $g:PY \to PX$ and $\psi:Y \to (\Sigma_{P,\sigma}g)X$ (so $P\psi = \mathrm{id}_{PY}$). Let $f = \sigma(g,PX) \circ \psi$ -- then

$$\begin{aligned} Pf &= P\sigma(g,PX) \circ P\psi \\ &= g \circ \mathrm{id}_{PY} = g. \end{aligned}$$

Schematically:

$$\overbrace{Y \overset{\psi}{\dashrightarrow} g^*X \xrightarrow[\sigma(g,PX)]{} X}^{f}, \quad \overbrace{PY \xrightarrow[\mathrm{id}_{PY}]{} PY \xrightarrow[g]{} PX}^{Pf}.$$

Because $\sigma(g,PX)$ is horizontal, ψ is characterized by the relations $P\psi = \mathrm{id}_{PY}$ and $\sigma(g,PX) \circ \psi = f$. Meanwhile

$$Y \xrightarrow{\phi_f} (Pf)^*X \xrightarrow{\sigma(Pf,X)} X$$

or still,

$$Y \xrightarrow{\phi_f} g^*X \xrightarrow{\sigma(g,PX)} X.$$

However $P\phi_f = \mathrm{id}_{PY}$ (ϕ_f is, by definition, a morphism in the fiber over PY) and $\sigma(g,PX) \circ \phi_f = f$. Accordingly, by uniqueness, $\phi_f = \psi$. Therefore

$$\Phi f = (Pf,\phi_f) = (g,\psi).$$

The proof of 7.5 is therefore complete.

§8. *SPLITTINGS*

Let $P:\underline{E} \rightarrow \underline{B}$ be a fibration.

8.1 DEFINITION A cleavage σ for P is said to be split if the following conditions are satisfied.

(1) $\sigma(\mathrm{id}_{PX'},X') = \mathrm{id}_{X'}$.

(2) $\sigma(g' \circ g,X'') = \sigma(g',X'') \circ \sigma(g,g'^*X'')$.

[Note: A fibration is split if it has a cleavage that splits or, in brief, has a splitting.]

8.2 EXAMPLE In the notation of 4.18, assume that $\phi:G \rightarrow H$ is surjective, hence that $\underline{\phi}:\underline{G} \rightarrow \underline{H}$ is a fibration -- then a cleavage σ for $\underline{\phi}$ is a subset K of G which maps bijectively onto H and $\underline{\phi}$ is split iff K is a subgroup of G. Therefore $\underline{\phi}$ is split iff ϕ is a retract, i.e., iff $\exists$ a homomorphism $\psi:H \rightarrow G$ such that $\phi \circ \psi = \mathrm{id}_H$.

8.3 REMARK The association

$$\Sigma_{P,\sigma}:\underline{B}^{OP} \rightarrow 2\text{-}\mathbf{CAT}$$

is a 2-functor iff P is split.

8.4 THEOREM Every fibration is equivalent to a split fibration.

[Note: The meaning of the term "equivalent" is that of 4.37.]

There are some preliminaries that have to be dealt with first. So suppose that $P:\underline{E} \rightarrow \underline{B}$ is a fibration -- then $\forall$ B $\in$ Ob $\underline{B}$, there is a fibration $U_B:\underline{B}/B \rightarrow \underline{B}$

(cf. 5.7) and a functor

$$F_{P,B}:[\underline{B}/B,\underline{E}]_{\underline{B}} \to \underline{E}_B,$$

namely:

(1) Given a horizontal functor

$$F:(\underline{B}/B,U_B) \to (\underline{E},P),$$

assign to F the object $F(id_B)$ in Ob $\underline{E}_B$.

(2) Given horizontal functors

$$F,G:(\underline{B}/B,U_B) \to (\underline{E},P)$$

and a vertical natural transformation $\Xi:F \to G$, assign to Ξ the arrow $\Xi_{id_B}:F(id_B) \to G(id_B)$ in Mor $\underline{E}_B$.

8.5 LEMMA The functor

$$F_{P,B}:[\underline{B}/B,\underline{E}]_{\underline{B}} \to \underline{E}_B$$

is an equivalence.

[It is not difficult to prove that $F_{P,B}$ is fully faithful. To see that $F_{P,B}$ has a representative image, fix an $X \in$ Ob $\underline{E}_B$ and define a horizontal functor $F_X:\underline{B}/B \to \underline{E}$ by the following procedure.

- Given an object $a:A \to B$ of $\underline{B}/B$, put

$$F_Xa = a^*X \quad (a^*:\underline{E}_B \to \underline{E}_A \quad (\text{cf. } 7.10)).$$

- Given a morphism

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & A' \arrow[d, "a'"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

of $\underline{B}/B$, there are horizontal arrows

$$\begin{cases} u{:}a^*X \longrightarrow X & (Pu = a) \\ u'{:}a'^*X \longrightarrow X & (Pu' = a') \end{cases}$$

with

$$Pu = a = a' \circ f = Pu' \circ f,$$

so there exists a unique morphism

$$a^*f{:}F_Xa = a^*X \longrightarrow a'^*X = F_Xa'$$

such that $Pa^*f = f$ and $u' \circ a^*f = u$. Schematically:

$$\overbrace{a^*X \overset{}{\underset{a^*f}{\cdots\cdots>}} a'^*X \xrightarrow[u']{} X}^{u}, \quad \overbrace{A \xrightarrow[f]{} A' \xrightarrow[a']{} B}^{Pu}.$$

The definitions then imply that

$$\begin{aligned} F_{P,B}F_X &= F_X(\mathrm{id}_B) \\ &= \mathrm{id}_B^*X \approx X.] \end{aligned}$$

Now introduce a 2-functor

$$\mathrm{sp}(P){:}\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$$

by stipulating that

$$\mathrm{sp}(P)(B) = [\underline{B}/B, \underline{E}]_{\underline{B}}$$

and letting

$$\mathrm{sp}(P)\beta{:}\mathrm{sp}(P)(B') \to \mathrm{sp}(P)(B) \quad (\beta{:}B \to B')$$

operate by precomposition via the horizontal arrow $\beta_*{:}\underline{B}/B \to \underline{B}/B'$ induced by β.

[Note: Strictly speaking, $[\underline{B}/B,\underline{E}]_{\underline{B}}$ is a metacategory rather than a category but this point can be safely ignored.]

Pass next to $\text{gro}_{\underline{B}}\text{sp}(P)$ -- then the canonical cleavage for $\Theta_{\text{sp}(P)}$ is split (cf. 7.12).

The final step in the proof of 8.4 is to define a horizontal functor

$$F_P:\text{gro}_{\underline{B}}\text{sp}(P) \to \underline{E}$$

with the property that $\forall\ B \in \text{Ob}\ \underline{B}$, $(F_P)_B = F_{P,B}$. This done, it then follows from 4.38 that F_P is an equivalence of categories over B (cf. 8.5).

Consider an object (B,X) of $\text{gro}_{\underline{B}}\text{sp}(P)$ -- then

$$X \in \text{Ob sp}(P)(B) = \text{Ob}\ [\underline{B}/B,\underline{E}]_{\underline{B}},$$

so $X:\underline{B}/B \to \underline{E}$ is a horizontal functor and we put

$$F_P(B,X) = X(\text{id}_B) \in \text{Ob}\ \underline{E}_B.$$

Turning to a morphism $(\beta,f):(B,X) \to (B',X')$ of $\text{gro}_{\underline{B}}\text{sp}(P)$, as usual, $\beta:B \to B'$, while

$$f:X \to (\text{sp}(P)\beta)X'$$

is a vertical natural transformation indexed by the objects $A \to B$ of $\underline{B}/B$. To define

$$F_P(\beta,f):X(\text{id}_B) \to X'(\text{id}_{B'}),$$

note first that

$$f_{\text{id}_B}:X(\text{id}_B) \to ((\text{sp}(P)\beta)X')(\text{id}_B).$$

Proceeding,

$$\text{sp}(P)\beta:[\underline{B}/B',\underline{E}]_{\underline{B}} \to [\underline{B}/B,\underline{E}]_{\underline{B}},$$

where

$$(\mathrm{sp}(P)\beta)X' = X' \circ \beta_*,$$

hence

$$((\mathrm{sp}(P)\beta)X')(\mathrm{id}_B) = (X' \circ \beta_*)(\mathrm{id}_B)$$

$$= X'(B \xrightarrow{\beta} B').$$

In the category $\underline{B}/B'$, $\mathrm{id}_{B'}:B' \to B'$ is a final object, thus there is an arrow

$$X'(B \xrightarrow{\beta} B') \longrightarrow X'(\mathrm{id}_{B'}).$$

Definition: $F_P(\beta,f)$ is the result of composing

$$f_{\mathrm{id}_B}:X(\mathrm{id}_B) \to X'(B \xrightarrow{\beta} B')$$

with the preceding arrow, thus

$$F_P(\beta,f):X(\mathrm{id}_B) \to X'(\mathrm{id}_B).$$

§9. *CATEGORIES FIBERED IN GROUPOIDS*

Let $P:\underline{E} \to \underline{B}$ be a fibration.

9.1 DEFINITION $\underline{E}$ is <u>fibered in groupoids by P</u> if $\forall\ B \in \text{Ob}\ \underline{B}$, $\underline{E}_B$ is a groupoid.

9.2 RAPPEL Let G be a topological group, X a topological space. Suppose that X is a free right G-space: $\begin{bmatrix} X \times G \to X \\ (x,g) \to x \cdot g \end{bmatrix}$ — then X is said to be <u>principal</u> provided that the continuous bijection $\theta:X \times G \to X \times_{X/G} X$ defined by $(x,g) \to (x,x \cdot g)$ is a homeomorphism.

Let G be a topological group — then an X in $\underline{\text{TOP}}/B$ is said to be a <u>principal G-space over B</u> if X is a principal G-space, B is a trivial G-space, the projection $X \to B$ is open, surjective, and equivariant, and G operates transitively on the fibers. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[d] \\ X/G \arrow[r] & B \end{tikzcd}$$

and the arrow $X/G \to B$ is a homeomorphism.

9.3 NOTATION Let

$$\underline{\text{PRIN}}_{B,G}$$

be the category whose objects are the principal G-spaces over B and whose morphisms are the equivariant continuous functions over B, thus

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X' \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd},$$

with ϕ equivariant.

9.4 FACT Every morphism in $\underline{\text{PRIN}}_{B,G}$ is an isomorphism.

[Note: The objects in $\underline{\text{PRIN}}_{B,G}$ which are isomorphic to $B \times G$ (product topology) are said to be trivial, thus the trivial objects are precisely those that admit a section.]

9.5 EXAMPLE Let G be a topological group -- then the classifying stack of G is the category $\underline{\text{PRIN}}(G)$ whose objects are the principal G-spaces $X \to B$ and whose morphisms $(\phi,f):(X \to B) \to (X' \to B')$ are the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B', \end{array}$$

where ϕ is equivariant. Define now a functor $P:\underline{\text{PRIN}}(G) \to \underline{\text{TOP}}$ by $P(X \to B) = B$ and $P(\phi,f) = f$ -- then P is a fibration. Moreover, $\underline{\text{PRIN}}(G)$ is fibered in groupoids by P:

$$\underline{\text{PRIN}}(G)_B = \underline{\text{PRIN}}_{B,G},$$

which is a groupoid by 9.4.

9.6 REMARK Suppose that $P:\underline{E} \to \underline{B}$ is a functor with the property that $\forall\ B \in \text{Ob}\ \underline{B}$, $\underline{E}_B$ is a groupoid -- then it is not true in general that P is a fibration.

[E.g.: In the notation of 4.18, consider a homomorphism $\phi:G \to H$ which is not surjective.]

9.7 LEMMA If $\underline{E}$ is fibered in groupoids by P, then every morphism in $\underline{E}$ is horizontal.

PROOF Let $f \in \text{Mor}(X,X')$ $(X,X' \in \text{Ob}\ \underline{E})$, thus $Pf:PX \rightarrow PX'$, so one can find a horizontal $u_0:X_0 \rightarrow X'$ such that $Pu_0 = Pf$. But u_0 is necessarily prehorizontal, hence there exists a unique morphism $v \in \text{Mor}_{PX_0}(X,X_0)$ such that $u \circ v = f$:

$$\begin{tikzcd} X_0 \arrow[r, "u"] & X' \\ X \arrow[u, "v"] \arrow[r, equal] & X. \arrow[u, "f"'] \end{tikzcd}$$

Since u is horizontal and v is an isomorphism, it follows that f is horizontal (cf. 4.20 and 4.11).

<u>N.B.</u> Suppose that

$$\begin{cases} \underline{E} \text{ is fibered in groupoids by } P \\ \underline{E}' \text{ is fibered in groupoids by } P'. \end{cases}$$

Then every functor $F:\underline{E} \rightarrow \underline{E}'$ such that $P' \circ F = P$ is automatically a horizontal functor from $\underline{E}$ to $\underline{E}'$ and $[\underline{E},\underline{E}']_{\underline{B}}$ is a groupoid.

9.8 LEMMA Let $P:\underline{E} \rightarrow \underline{B}$ be a functor. Assume: Every arrow in $\underline{E}$ is horizontal and for any morphism $g:B \rightarrow PX'$, there exists a morphism $u:X \rightarrow X'$ such that $Pu = g$ -- then P is a fibration and $\underline{E}$ is fibered in groupoids by P.

PROOF The conditions obviously imply that P is a fibration. Consider now an arrow $f:X \rightarrow X'$ of $\underline{E}_B$ for some $B \in \text{Ob}\ \underline{B}$ -- then f is horizontal, so there exists a unique morphism $v \in \mathrm{Mor}_B(X',X)$ $(PX = B = PX')$ such that $f \circ v = \mathrm{id}_{X'}$:

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ X' \arrow[u, "v"] \arrow[r, equal] & X' \arrow[u, "\mathrm{id}_{X'}"'] \end{tikzcd} .$$

Therefore every arrow in $\underline{E}_B$ has a right inverse. But this means in particular that v must have a right inverse, thus f is invertible.

9.9 LEMMA Suppose that

$$\begin{cases} \underline{E}_1 \text{ is fibered in groupoids by } P_1 \\ \underline{E}_2 \text{ is fibered in groupoids by } P_2 \end{cases}$$

and

$$\underline{E} \text{ is fibered in groupoids by } P.$$

Let

$$\begin{cases} F_1 : (\underline{E}_1, P_1) \to (\underline{E}, P) \\ F_2 : (\underline{E}_2, P_2) \to (\underline{E}, P) \end{cases}$$

be morphisms in $\underline{\underline{\mathrm{FIB}}}(\underline{B})$ -- then the canonical projection

$$\Pi : \underline{E}_1 \times_{\underline{E}} \underline{E}_2 \to \underline{B}$$

is a fibration (cf. 5.14) and $\underline{E}_1 \times_{\underline{E}} \underline{E}_2$ is fibered in groupoids by Π.

[Recall that

$$(\underline{E}_1 \times_{\underline{E}} \underline{E}_2)_B \approx (\underline{E}_1)_B \times_{\underline{E}_B} (\underline{E}_2)_B$$

and the pseudo pullback on the right is a groupoid (cf. 1.22).]

Let $P:\underline{E} \to \underline{B}$ be a fibration. Denote by $\underline{E}_{hor}$ the wide subcategory of $\underline{E}$ whose morphisms are the horizontal arrows of $\underline{E}$. Put

$$P_{hor} = P|\underline{E}_{hor}.$$

9.10 LEMMA $P_{hor}:\underline{E}_{hor} \to \underline{B}$ is a fibration and $\underline{E}_{hor}$ is fibered in groupoids by P_{hor}.

§10. *DISCRETE FIBRATIONS*

10.1 RAPPEL A category is said to be <u>discrete</u> if all its morphisms are identities.

[Note: Functors between discrete categories correspond to functions on their underlying classes.]

<u>N.B.</u> A discrete category is necessarily locally small.

10.2 EXAMPLE Every class is a discrete category and every set is a small discrete category.

10.3 LEMMA A category $\underline{C}$ is equivalent to a discrete category iff $\underline{C}$ is a groupoid with the property that $\forall\ X,X' \in \text{Ob}\ \underline{C}$, there is at most one morphism from X to X'.

Every discrete category is a groupoid. So, if $P:\underline{E} \to \underline{B}$ is a fibration, then the statement that $\underline{E}$ is "fibered in discrete categories by P" (or, in brief, that $\underline{E}$ is <u>discretely fibered</u> by P) is a special case of 9.1.

10.4 EXAMPLE Let $\underline{C}$ be a locally small category -- then $\forall\ X \in \text{Ob}\ \underline{C}$, the forgetful functor $U_X:\underline{C}/X \to \underline{C}$ is a fibration (cf. 5.7). Moreover, $\underline{C}/X$ is discretely fibered by U_X ($\forall\ Y \in \text{Ob}\ \underline{C}$, the fiber $(\underline{C}/X)_Y$ is the set $\text{Mor}(Y,X)$).

10.5 LEMMA Let $P:\underline{E} \to \underline{B}$ be a functor -- then $\underline{E}$ is discretely fibered by P iff for any morphism $g:B \to PX'$, there exists a unique morphism $u:X \to X'$ such that $Pu = g$.

PROOF Assume first that $\underline{E}$ is discretely fibered by P, choose $u:X \to X'$ per g and consider a second arrow $\tilde{u}:\tilde{X} \to X'$ per g -- then $P\tilde{u} = Pu$. Since u is horizontal (cf. 9.7), thus is prehorizontal, there exists a unique morphism $v \in \text{Mor}_{PX}(\tilde{X},X)$ such that $u \circ v = \tilde{u}$:

$$\begin{tikzcd} X \arrow[r, "u"] & X' \\ \tilde{X} \arrow[u, "v"] \arrow[r, equal] & \tilde{X} \arrow[u, "\tilde{u}"'] \end{tikzcd}$$

But the fiber $\underline{E}_{PX}$ is discrete, hence $X = \tilde{X}$ and v is the identity, so $\tilde{u} = u$. In the other direction, consider a setup

$$\begin{tikzcd} X_0 \arrow[r, dotted] \arrow[rr, bend left, "w"] & X \arrow[r, "u"'] & X' \end{tikzcd}, \qquad \begin{tikzcd} PX_0 \arrow[r, "x"'] \arrow[rr, bend left, "Pw"] & PX \arrow[r, "Pu"'] & PX'. \end{tikzcd}$$

With "x" playing the role of "g", let $v:X_0 \to X$ be the unique morphism such that $Pv = x$ -- then

$$\left[\begin{array}{l} u \circ v:X_0 \to X' \Rightarrow P(u \circ v):PX_0 \to PX' \\ \\ \quad w:X_0 \to X' \Rightarrow \qquad P(w):PX_0 \to PX'. \end{array}\right.$$

Accordingly, by uniqueness, $u \circ v = w$. Therefore every arrow in $\underline{E}$ is horizontal which implies that $\underline{E}$ is fibered in groupoids by P (cf. 9.8). That the fibers are discrete is clear.

Suppose that $P:\underline{E} \to \underline{B}$ is a fibration such that $\underline{E}$ is fibered in sets by P (so, $\forall\ B \in \text{Ob}\ \underline{B}$, $\underline{E}_B$ is a set). Let $g:B \to B'$ be an arrow in $\underline{B}$ -- then the data defining the functor $g^*:\underline{E}_{B'} \to \underline{E}_B$ of 7.10 is uniquely determined, as is the cleavage

$\sigma:P \to \Sigma_{P,\sigma}$, where in this context, $\Sigma_{P,\sigma}$ is to be viewed as a functor from $\underline{E}^{OP}$ to $\underline{SET}$.

10.6 NOTATION $\underline{FIB}_{\underline{SET}}(\underline{B})$ is the full subcategory of $\underline{FIB}(\underline{B})$ whose objects are the fibrations $P:\underline{E} \to \underline{B}$ which are fibered in sets by P.

If $F:(\underline{E},P) \to (\underline{E}',P')$ is a morphism in $\underline{FIB}_{\underline{SET}}(\underline{B})$, then there is an induced natural transformation

$$\Xi_F:\Sigma_{P,\sigma} \to \Sigma_{P',\sigma'}.$$

10.7 LEMMA The functor

$$\underline{FIB}_{\underline{SET}}(\underline{B}) \to [\underline{E}^{OP},\underline{SET}]$$

that sends $(\underline{E},P)$ to $\Sigma_{P,\sigma}$ is an equivalence of metacategories.

[To reverse matters, take an $F:\underline{E}^{OP} \to \underline{SET}$ and consider $\mathrm{gro}_{\underline{B}}F$ -- then here a morphism $(B,X) \to (B',X')$ is an arrow $\beta:B \to B'$ such that $X = (F\beta)X'$ and it is obvious that $\mathrm{gro}_{\underline{B}}F$ is fibered in sets by Θ_F (cf. 7.9).]

10.8 EXAMPLE Let $\underline{C}$ be a locally small category -- then an object of

$$\hat{\underline{C}} = [\underline{C}^{OP},\underline{SET}]$$

is called a <u>presheaf of sets</u> on $\underline{C}$. Given $X \in \mathrm{Ob}\,\underline{C}$, put

$$h_X = \mathrm{Mor}(\text{—},X).$$

Then

$$\mathrm{Mor}(X,Y) \approx \mathrm{Nat}(h_X,h_Y)$$

and in this notation the Yoneda embedding

$$Y_{\underline{C}}:\underline{C} \to \hat{\underline{C}}$$

sends X to h_X. Moreover, under the correspondence of 10.7,

$$\underline{C}/X \longleftrightarrow h_X.$$

Thus, symbolically,

$$\underline{C} \longrightarrow \hat{\underline{C}} \longrightarrow \underline{\mathrm{FIB}}_{\underline{\mathrm{SET}}}(\underline{C}) \longrightarrow \underline{\mathrm{FIB}}(\underline{C}).$$

§11. *COVERING FUNCTIONS*

Let $\underline{C}$ be a category.

11.1 DEFINITION Given an object $X \in \mathrm{Ob}\,\underline{C}$, a <u>covering</u> of X is a subclass $\mathcal{C}$ of $\mathrm{Ob}\,\underline{C}/X$.

11.2 DEFINITION If $\mathcal{C}$, $\mathcal{C}'$ are coverings of X, then $\mathcal{C}$ is a <u>refinement</u> of $\mathcal{C}'$ (or $\mathcal{C}$ refines $\mathcal{C}'$ or $\mathcal{C}'$ is refined by $\mathcal{C}$) if each arrow $g \in \mathcal{C}$ factors through an arrow $g' \in \mathcal{C}'$:

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "g"'] & Y' \arrow[d, "g'"] \\ X \arrow[r, equal] & X \end{tikzcd} .$$

[Note: If $\mathcal{C} \subset \mathcal{C}'$, then $\mathcal{C}$ is a refinement of $\mathcal{C}'$, the converse being false in general.]

11.3 EXAMPLE Take $\mathcal{C} = \{\mathrm{id}_X : X \to X\}$ and suppose that $\mathcal{C}$ is a refinement of $\mathcal{C}'$ -- then there is an element of $\mathcal{C}'$ which is a split epimorphism (a.k.a. retraction):

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d, "\mathrm{id}_X"'] & Y' \arrow[d, "g'"] \\ X \arrow[r, equal] & X \end{tikzcd} .$$

11.4 DEFINITION A <u>covering function</u> κ is a rule that assigns to each $X \in \mathrm{Ob}\,\underline{C}$ a conglomerate κ_X of coverings of X.

11.5 REMARK If the cardinality of Ob $\underline{C}/X$ is n, then there are 2^n subsets of Ob $\underline{C}/X$, thus there are 2^{2^n} possible choices for κ_X.

11.6 NOTATION Given covering functions κ and κ', write $\kappa' \leq \kappa$ (and term κ' <u>subordinate</u> to κ) if for each $X \in \mathrm{Ob}\,\underline{C}$, every covering $\mathcal{C}' \in \kappa'_X$ is refined by some covering $\mathcal{C} \in \kappa_X$.

11.7 EXAMPLE

- Define a covering function κ by setting $\kappa_X = \emptyset$ -- then κ is subordinate to all covering functions.

- Define a covering function κ by setting κ_X = all coverings of X -- then every covering function is subordinate to κ.

11.8 NOTATION Given covering functions κ and κ', write $\kappa \equiv \kappa'$ if $\kappa' \leq \kappa$ and $\kappa \leq \kappa'$, and when this is so, call κ and κ' <u>equivalent</u>.

11.9 DEFINITION Let κ be a covering function -- then its <u>saturation</u> is the covering function sat κ whose coverings are the coverings that have a refinement in κ.

11.10 EXAMPLE Assume that $\kappa_X \neq \emptyset$ and let $\phi: X' \to X$ be an isomorphism -- then $\{\phi\} \in (\mathrm{sat}\,\kappa)_X$. Indeed, every $\mathcal{C} \in \kappa_X$ refines $\{\phi\}$:

$$\begin{tikzcd} Y \arrow[r, "\phi^{-1} \circ g"] \arrow[d, "g"'] & X' \arrow[d, "\phi"] \\ X \arrow[r, equal] & X \end{tikzcd} .$$

11.11 LEMMA Suppose that κ is a covering function — then κ is equivalent to sat κ and sat κ is saturated. Moreover, κ is saturated iff κ = sat κ.

11.12 LEMMA Suppose that κ and κ' are covering functions — then κ and κ' are equivalent iff sat κ = sat κ'.

11.13 DEFINITION Let κ be a covering function — then κ is a <u>coverage</u> if $\forall\ X \in \text{Ob}\ \underline{C}$, $\forall\ C \in \kappa_X$, and $\forall\ f':X' \to X$, there is a $C_{f'} \in \kappa_{X'}$ such that

$$f' \circ C_{f'} = \{f' \circ g' : g' \in C_{f'}\}\ (Y' \xrightarrow{g'} X' \xrightarrow{f'} X)$$

is a refinement of C.

11.14 EXAMPLE Define a covering function κ by letting κ_X be comprised of all singletons $\{f\}$ ($f \in \text{Ob}\ \underline{C}/X$) — then κ is a coverage iff for each $X \in \text{Ob}\ \underline{C}$, every diagram of the form

$$\begin{tikzcd} & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

can be completed to a commutative square

$$\begin{tikzcd} Y \arrow[r, "g_2"] \arrow[d, "g_1"'] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & X \end{tikzcd} .$$

[Note: This condition is realized by the opposite of the category of finite sets and injective functions.]

11.15 LEMMA Suppose that κ and κ' are equivalent covering functions — then κ is a coverage iff κ' is a coverage.

N.B. Therefore κ is a coverage iff sat κ is a coverage (cf. 11.11).

11.16 DEFINITION Let κ be a covering function — then κ is a Grothendieck coverage if $\forall\ X \in \mathrm{Ob}\ \underline{C}$, $\forall\ C \in \kappa_X$, $\forall\ g:Y \to X$ in C, and $\forall\ f':X' \to X$, there is a pullback square

$$\begin{tikzcd} X' \times_X Y \arrow[r] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "f'"'] & X \end{tikzcd}$$

such that the covering

$$\{X' \times_X Y \xrightarrow{g'} X' : g \in C\}$$

belongs to $\kappa_{X'}$.

[Note: It is a question here of a specific choice for the pullback.]

11.17 REMARK By construction, $f' \circ g'$ factors through g, hence a Grothendieck coverage is a coverage.

11.18 EXAMPLE Given a topological space X, let O(X) be the set of open subsets of X, thus under the operations

$$U \leq V \iff U \subset V, \quad \begin{cases} U \wedge V = U \cap V \\ U \vee V = U \cup V \end{cases}, \quad 0 = \emptyset,\ 1 = X,$$

$O(X)$ is a bounded lattice. Let $\underline{O}(X)$ be the category underlying $O(X)$ and define a covering function κ by stipulating that κ_U is comprised of the collections $\{U_i\}$ of open subsets U_i of U whose union $\bigcup_i U_i$ is U -- then κ is a Grothendieck coverage.

[Given a 2-sink $U' \longrightarrow U \longleftarrow U_i$ in $\underline{O}(X)$, the commutative diagram

$$\begin{tikzcd} U' \cap U_i \arrow[r] \arrow[d] & U_i \arrow[d] \\ U' \arrow[r] & U \end{tikzcd}$$

is a pullback square and

$$\bigcup_i U' \cap U_i = U' \cap \bigcup_i U_i = U' \cap U = U'.]$$

11.19 EXAMPLE Take $\underline{C} = \underline{TOP}$ and fix $X \in \text{Ob } \underline{C}$. Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is an open map and the induced arrow $\coprod_i Y_i \to X$ is surjective -- then κ is a Grothendieck coverage, the <u>open map coverage</u>.

[Note: The pullback of an open map along a continuous function is an open map (in this context, "open" incorporates "continuous").]

11.20 EXAMPLE Take $\underline{C} = \underline{TOP}$ and fix $X \in \text{Ob } \underline{C}$.

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is an open inclusion and the induced arrow $\coprod_i Y_i \to X$ is surjective -- then κ is a Grothendieck coverage, the <u>open subset coverage</u>.

[Note: The pullback of an open inclusion along a continuous function is an open inclusion.]

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is an open embedding and the induced arrow $\coprod_i Y_i \to X$ is surjective -- then κ is a Grothendieck coverage, the <u>open embedding coverage</u>.

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is a local homeomorphism and the induced arrow $\coprod_i Y_i \to X$ is surjective -- then κ is a Grothendieck coverage, the <u>local homeomorphism coverage</u>.

[Note: A local homeomorphism is necessarily an open map and the pullback of a local homeomorphism along a continuous function is a local homeomorphism.]

FACT The open subset coverage, the open embedding coverage, and the local homeomorphism coverage are equivalent. Moreover, each of these is subordinate to the open map coverage.

11.21 EXAMPLE Let $\underline{C^\infty\text{-MAN}}$ be the category whose objects are the C^∞-manifolds and whose morphisms are the C^∞-functions -- then $\underline{C^\infty\text{-MAN}}$ does not have all pullbacks but it does have certain pullbacks, e.g., the pullback of a surjective submersion along a C^∞-function is again a surjective submersion. Since an open subset of a C^∞-manifold can be viewed as a C^∞-manifold, one can form the open submanifold coverage. On the other hand, there is a Grothendieck coverage κ in which κ_M is comprised of all singletons $\{f\}$, $f:N \to M$ a surjective submersion. E.g.: If $\{U_i\}$ is an open submanifold coverage of M, then the induced arrow $\coprod_i U_i \to M$ is a surjective submersion.

[Note: If $f:N \to M$ is a surjective submersion, then $\forall$ $y \in N$, there is an open subset $U_y \subset M$ with $f(y) \in U_y$ and a C^∞-function $s:U_y \to M$ such that $f \circ s = \mathrm{id}$

and $s(f(y)) = y$:

$$\begin{tikzcd} U_y \arrow[r, "s"] \arrow[d] & N \arrow[d, "f"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

Therefore the surjective submersion coverage is subordinate to the open submanifold coverage.]

11.22 EXAMPLE Suppose that $\underline{C}$ has pullbacks -- then there is a Grothendieck coverage κ in which κ_X is comprised of all singletons $\{f\}$ ($f \in \text{Ob } \underline{C}/X$), where f is a split epimorphism.

[Split epimorphisms are stable under pullback.]

11.23 RAPPEL A locally small, finitely complete category $\underline{C}$ fulfills the <u>standard conditions</u> if $\underline{C}$ has coequalizers and the epimorphisms that are coequalizers are pullback stable.

[Note: <u>SET</u> fulfills the standard conditions (as does every topos) but <u>TOP</u> does not fulfill the standard conditions (quotient maps are not pullback stable).]

11.24 EXAMPLE Suppose that $\underline{C}$ fulfills the standard conditions -- then there is a Grothendieck coverage κ in which κ_X is comprised of all singletons $\{f\}$ ($f \in \text{Ob } \underline{C}/X$), where f is an epimorphism that is a coequalizer.

11.25 DEFINITION Given an object $X \in \text{Ob } \underline{C}$, an <u>opcovering</u> of X is a covering of X in $\underline{C}^{OP}$.

11.26 EXAMPLE Let $\underline{\text{RNG}}$ be the category of commutative rings with unit. Define an opcovering function κ by letting κ_A be comprised of the collections $\{\pi_i:A \to A[a_i^{-1}]\}$, where $\forall$ i, $A[a_i^{-1}]$ is the localization of A at a_i and the ideal generated by the set $\{a_i:i \in I\}$ is all of A -- then κ is a Grothendieck opcoverage, the <u>Zariski opcoverage</u>.

[If $f:A \to B$ is a homomorphism, then $\forall$ i, there is a pushout square

$$\begin{array}{ccc} A & \longrightarrow & A[a_i^{-1}] \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & B[(f(a_i))^{-1}] \end{array} \quad .]$$

11.27 DEFINITION Suppose that κ is a coverage -- then κ is a <u>pretopology</u> if $\forall$ $X \in \text{Ob}\,\underline{C}$, $\forall$ $\mathcal{C} \in \kappa_X$, $\forall$ $g:Y \to X$ in $\mathcal{C}$, and $\forall$ $\mathcal{C}_g \in \kappa_Y$, there is a $\mathcal{C}_0 \in \kappa_X$ such that $\mathcal{C}_0$ is a refinement of

$$\bigcup_{g\in\mathcal{C}} g \circ \mathcal{C}_g = \{g \circ h:g \in \mathcal{C} \ \&\ h \in \mathcal{C}_g\} \ (Z \xrightarrow{h} Y \xrightarrow{g} X).$$

11.28 LEMMA If κ and κ' are equivalent coverages, then κ is a pretopology iff κ' is a pretopology.

11.29 LEMMA Suppose that κ is a pretopology. Fix $X \in \text{Ob}\,\underline{C}$ and let $\mathcal{C}_1,\mathcal{C}_2 \in \kappa_X$ -- then $\exists$ $\mathcal{C} \in \kappa_X:\mathcal{C}$ is a refinement of $\left[\begin{array}{l} \mathcal{C}_1 \\ \\ \mathcal{C}_2 \end{array}\right.$.

PROOF For each $f_2:X_2 \to X$ in $\mathcal{C}_2$, there is a $\mathcal{C}_{f_2} \in \kappa_{X_2}$ such that $f_2 \circ \mathcal{C}_{f_2}$

refines C_1 (cf. 11.13). On the other hand, there is a $C \in \kappa_X$ such that

$$\bigcup_{f_2 \in C_2} f_2 \circ C_{f_2}$$

is refined by C (cf. 11.27). But

$$\bigcup_{f_2 \in C_2} f_2 \circ C_{f_2}$$

refines both C_1 and C_2.

11.30 LEMMA Let κ be a covering function -- then κ is a pretopology iff κ_{sat} is a pretopology.

11.31 DEFINITION Suppose that κ is a coverage -- then κ is a Grothendieck pretopology if $\forall\ X \in \text{Ob}\ \underline{C}$, $\forall\ C \in \kappa_X$, $\forall\ g:Y \to X$ in C, and $\forall\ C_g \in \kappa_Y$,

$$\bigcup_{g \in C} g \circ C_g = \{g \circ h : g \in C\ \&\ h \in C_g\} \quad (Z \xrightarrow{h} Y \xrightarrow{g} X)$$

belongs to κ_X.

N.B. It is obvious that a Grothendieck pretopology is a pretopology.

11.32 REMARK The various examples of Grothendieck coverages set forth above are Grothendieck pretopologies.

[The morphisms appearing in 11.22 and 11.24 are composition stable, while the verification of the requisite property in 11.26 is mildly tedious pure algebra (the terminology in this situation would be Grothendieck preoptopology...).]

[Note: Take κ per 11.14 and impose on $\underline{C}$ the conditions therein (so that κ is a coverage) -- then κ is a pretopology but it need not be a Grothendieck pretopology.]

11.33 DEFINITION A pretopology (or a Grothendieck pretopology) κ is said to have identities if $\forall\ X \in \text{Ob}\ \underline{C}$, $\{id_X: X \rightarrow X\}$ refines some covering in κ_X (or belongs to κ_X).

[Note: This will be the case in all examples of interest.]

11.34 REMARK If $\phi: X' \rightarrow X$ is an isomorphism in $\underline{C}$, then $\{\phi\}$ might or might not belong to κ_X.

[Consider the open subset coverage of 11.20 -- then an arbitrary homeomorphism $\phi: X' \rightarrow X$ is certainly not admissible.]

11.35 LEMMA Let κ be a Grothendieck pretopology with the property that for any isomorphism $\phi: X' \rightarrow X$, the covering $\{\phi\}$ belongs to κ_X -- then the coverings $\mathcal{C} \in \kappa_X$ are closed under precomposition with isomorphisms, i.e., if $g: Y \rightarrow X$ is in $\mathcal{C}$ and if $\psi_g: Y' \rightarrow Y$ is an isomorphism, then $\{g \circ \psi_g : g \in \mathcal{C}\} \in \kappa_X$.

PROOF By hypothesis, $\{\psi_g\} \in \kappa_{\text{dom}\ g}$, so we can take $\mathcal{C}_g = \{\psi_g\}$, hence

$$\bigcup_{g \in \mathcal{C}} g \circ \mathcal{C}_g = \{g \circ \psi_g : g \in \mathcal{C}\} \in \kappa_X.$$

11.36 REMARK Suppose that $\mathcal{C}$ has pullbacks and the scenario in 11.35 is in force -- then the particular choice for the pullbacks figuring in 11.16 is immaterial.

Let κ be a covering function. Fix $X \in \text{Ob}\ \underline{C}$ -- then κ induces a covering function $\bar{\kappa}$ on $\underline{C}/X$ via the following procedure. Fix an object $f': X' \rightarrow X$ in $\underline{C}/X$ -- then a covering

$$\{(g: Y \longrightarrow X) \xrightarrow{g'} (F': X' \longrightarrow X)\}$$

of f' belongs to $\bar{\kappa}_{f'}$ iff the covering $\{g':Y \to X'\}$ belongs to $\kappa_{X'}$.

[Note: There is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g'"] \arrow[d, "g"'] & X' \arrow[d, "f'"] \\ X \arrow[r, equal] & X \end{tikzcd} .]$$

<u>N.B.</u> If κ is a pretopology, then so is $\bar{\kappa}$.

§12. *SIEVES*

Let $\underline{C}$ be a category.

12.1 DEFINITION Let $X \in \text{Ob } \underline{C}$ — then a <u>sieve</u> over X is a subclass $\mathcal{S}$ of $\text{Ob } \underline{C}/X$ such that the composition $Z \xrightarrow{g} Y \xrightarrow{f} X$ belongs to $\mathcal{S}$ if $Y \xrightarrow{f} X$ belongs to $\mathcal{S}$.

E.g.: The <u>minimal sieve</u> over X is $\mathcal{S}_{min} = \emptyset$.

12.2 LEMMA If $\mathcal{S}$ and $\mathcal{S}'$ are sieves over X, then $\mathcal{S}$ refines $\mathcal{S}'$ iff $\mathcal{S} \subset \mathcal{S}'$.

12.3 LEMMA Every covering $\mathcal{C}$ of X is contained in a sieve $\mathcal{S}(\mathcal{C})$ minimal w.r.t. inclusion (the sieve <u>generated</u> by $\mathcal{C}$).

[$\mathcal{S}(\mathcal{C})$ is comprised of all morphisms with codomain X which factor through some element of $\mathcal{C}$.]

12.4 EXAMPLE The sieve generated by $\{id_X : X \to X\}$ is

$$\mathcal{S}_{max} \equiv \text{Ob } \underline{C}/X,$$

the <u>maximal sieve</u> over X.

[Given $f:Y \to X$, consider

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "f"'] & X \arrow[d, equal] \\ X \arrow[r, equal] & X \end{tikzcd} .]$$

It follows from 12.3 that every covering function κ gives rise to a covering function $\mathcal{S}(\kappa)$ whose coverings at X are the $\mathcal{S}(\mathcal{C})$ $(\mathcal{C} \in \kappa_X)$.

[Note: $\mathcal{S}(\kappa)$ is equivalent to κ.]

12.5 DEFINITION A <u>sifted covering function</u> is a covering function all of whose coverings are sieves.

[Note: The term <u>sifted coverage</u> is to be assigned the obvious meaning.]

12.6 NOTATION Given a sieve $\mathcal{S}$ over X and a morphism $f:Y \to X$, put

$$f^*\mathcal{S} = \{g:\text{cod } g = Y \ \& \ f \circ g \in \mathcal{S}\}.$$

Then $f^*\mathcal{S}$ is a sieve over Y.

12.7 LEMMA Suppose that κ is a sifted covering function -- then κ is a sifted coverage iff $\forall\, X \in \text{Ob } \underline{C}$, $\forall\, \mathcal{S} \in \kappa_X$, and $\forall\, f':X' \to X$, $f'^*\mathcal{S}$ has a refinement $\mathcal{S}'$ in $\kappa_{X'}$.

PROOF Using the notation of 11.13, let us first prove the sufficiency of the condition. Thus put $\mathcal{C}_{f'} = \mathcal{S}'$, the claim being that $f' \circ \mathcal{S}'$ is a refinement of $\mathcal{S}$. But

$$g' \in \mathcal{S}' \Rightarrow g' \in f'^*\mathcal{S} \ (\text{cf. } 12.2) \Rightarrow f' \circ g' \in \mathcal{S}.$$

I.e.:

$$f' \circ \mathcal{S}' \subset \mathcal{S},$$

so $f' \circ \mathcal{S}'$ is a refinement of $\mathcal{S}$. As for the necessity, write $\mathcal{S}'$ in place of $\mathcal{C}_{f'}$, hence by assumption $f' \circ \mathcal{S}'$ is a refinement of $\mathcal{S}$, hence $f' \circ \mathcal{S}' \subset \mathcal{S}$ (cf. 12.2) ($f' \circ \mathcal{S}'$ is a sieve over X). To see that $\mathcal{S}' \subset f'^*\mathcal{S}$, let $g' \in \mathcal{S}'$ — then

$$f' \circ g' \in f' \circ \mathcal{S}' \subset \mathcal{S} \Rightarrow g' \in f'^*\mathcal{S}.$$

12.8 DEFINITION A sifted covering function κ is <u>sieve saturated</u> if $\mathcal{S} \in \kappa_X$ and $\mathcal{S} \subset \mathcal{S}' \Rightarrow \mathcal{S}' \in \kappa_X$.

12.9 LEMMA Suppose that κ is a sieve saturated sifted covering function -- then κ is a sifted coverage iff $\forall\ X \in \text{Ob}\ \underline{C}$, $\forall\ \mathcal{S} \in \kappa_X$, and $\forall\ f':X' \to X$, $f'^*\mathcal{S} \in \kappa_{X'}$.

12.10 LEMMA Suppose that κ is a sieve saturated sifted covering function -- then κ is a pretopology iff κ is a Grothendieck pretopology.

12.11 DEFINITION A sifted covering function κ is <u>locally closed</u> provided the following condition is satisfied: If $\mathcal{S} \in \kappa_X$ and if $\mathcal{S}'$ is a sieve over X such that $f^*\mathcal{S}' \in \kappa_Y$ for all $f:Y \to X$ in $\mathcal{S}$, then $\mathcal{S}' \in \kappa_X$.

12.12 LEMMA Suppose that κ is a sieve saturated sifted coverage -- then κ is a Grothendieck pretopology iff κ is locally closed.

PROOF Using the notation of 11.31 (with "g" replaced by "f"), to check that

"Grothendieck pretopology" => "locally closed",

take $\mathcal{S}_f = f^*\mathcal{S}' \in \kappa_Y$ -- then

$$\bigcup_{f \in \mathcal{S}} f \circ \mathcal{S}_f = \{f \circ h : f \in \mathcal{S}\ \&\ h \in f^*\mathcal{S}'\}$$

belongs to κ_X. But

$$h \in f^*\mathcal{S}' \Rightarrow f \circ h \in \mathcal{S}'$$

$$\Rightarrow \bigcup_{f \in \mathcal{S}} f \circ \mathcal{S}_f \subset \mathcal{S}'.$$

Therefore $\mathcal{S}' \in \kappa_X$ (κ being sieve saturated), so κ is locally closed. Turning to the converse, the data is the sieve

$$\mathcal{S}' = \{f \circ h : f \in \mathcal{S}\ \&\ h \in \mathcal{S}_f\}$$

and the claim is that it belongs to κ_X. But $\forall\ f \in \mathcal{S}$,

$$f^*\mathcal{S}' \supset \mathcal{S}_f \in \kappa_Y \Rightarrow f^*\mathcal{S}' \in \kappa_Y$$

$$\Rightarrow \mathcal{S}' \in \kappa_X.$$

12.13 LEMMA Let κ be a sifted covering function. Assume: κ is locally closed and $\forall\ X \in \text{Ob}\ \underline{C}$, $\mathcal{S}_{max} \in \kappa_X$ -- then κ is sieve saturated.

PROOF Fix $\mathcal{S} \in \kappa_X$, suppose that $\mathcal{S} \subset \mathcal{S}'$, and let $f:Y \to X$ be an element of $\mathcal{S}$ -- then

$$f^*\mathcal{S} \subset f^*\mathcal{S}'.$$

But

$$f^*\mathcal{S} = \text{Ob}\ \underline{C}/Y \in \kappa_Y \Rightarrow f^*\mathcal{S}' \in \kappa_Y$$

$$\Rightarrow \mathcal{S}' \in \kappa_X.$$

12.14 DEFINITION Suppose that κ is a sifted coverage -- then κ is a <u>Grothendieck topology</u> if it is locally closed and $\forall\ X \in \text{Ob}\ \underline{C}$, $\mathcal{S}_{max} \in \kappa_X$.

[Note: It follows from 12.13 that κ is sieve saturated. Therefore κ is a Grothendieck pretopology (cf. 12.12) and it is automatic that 12.9 is in force.]

12.15 LEMMA If κ is a Grothendieck topology and if $\mathcal{S},\mathcal{S}' \in \kappa_X$, then $\mathcal{S} \cap \mathcal{S}' \in \kappa_X$.

PROOF For any $f:Y \to X$ in $\mathcal{S}$,

$$f^*\mathcal{S}' = f^*(\mathcal{S} \cap \mathcal{S}').$$

However, thanks to 12.9 (applied to $\mathcal{S}'$), $f^*\mathcal{S}' \in \kappa_Y$, so

$$f^*(\mathcal{S} \cap \mathcal{S}') \in \kappa_Y \Rightarrow \mathcal{S} \cap \mathcal{S}' \in \kappa_X.$$

12.16 EXAMPLE Take $\underline{C} = \underline{O}(X)$, X a topological space (cf. 11.18). Given an open set $U \subset X$, a sieve $\mathcal{S}$ over U is a set of open subsets V of U which is hereditary in the sense that

$$V \in \mathcal{S} \ \& \ V' \subset V \Rightarrow V' \in \mathcal{S}.$$

One then says that $\mathcal{S}$ <u>covers</u> U if $\bigcup_{V \in \mathcal{S}} V = U$. Denoting by κ_U the set of all such $\mathcal{S}$, the assignment $U \to \kappa_U$ is a Grothendieck topology κ on $\underline{O}(X)$.

12.17 DEFINITION Let κ be a sifted covering function -- then its <u>sifted saturation</u> is the sifted covering function sif κ whose coverings are the sieves that contain a sieve in κ.

12.18 LEMMA For any covering function κ,

$$\text{sif } \mathcal{S}(\kappa) = \mathcal{S}(\text{sat } \kappa).$$

Denote this covering function by $J(\kappa)$ -- then $J(\kappa)$ is sifted and sieve saturated.

12.19 LEMMA Suppose that $\mathcal{S}$ is a sieve over X -- then $\mathcal{S} \in J(\kappa)_X$ iff $\mathcal{S}$ contains an element of κ_X.

12.20 THEOREM If κ is a pretopology with identities (cf. 11.33), then $J(\kappa)$ is a Grothendieck topology.

PROOF The assumption that κ is a pretopology implies that sat κ is a pretopology (cf. 11.28) (κ and sat κ are equivalent), hence that $\mathcal{S}(\text{sat } \kappa)$ is a pretopology (cf. 11.28) (sat κ and $\mathcal{S}(\text{sat } \kappa)$ are equivalent), in particular $J(\kappa) = \mathcal{S}(\text{sat } \kappa)$ is a coverage. Therefore $J(\kappa)$ is a Grothendieck pretopology (cf. 12.10) ($J(\kappa)$ is sieve saturated), thus $J(\kappa)$ is locally closed (cf. 12.12). Finally, if $\{id_X : X \to X\}$ refines $C \in \kappa_X$, then

$$\mathcal{S}(\{id_X : X \to X\}) \subset \mathcal{S}(C) \in \mathcal{S}(\kappa)_X \subset J(\kappa)_X.$$

But

$$\mathcal{S}(\{\mathrm{id}_X:X \to X\}) = \mathcal{S}_{max} \quad (\text{cf. } 12.4)$$

$$\Rightarrow$$

$$\mathcal{S}(C) = \mathcal{S}_{max} \Rightarrow \mathcal{S}_{max} \in J(\kappa)_X.$$

[Note: The two descriptions of $J(\kappa)$ supplied by 12.18 are used in the proof.]

12.21 REMARK In the literature, terminology varies. For example, some authorities would say that a "Grothendieck topology" is a covering function κ which is a Grothendieck pretopology with identities whose underlying coverage is a Grothendieck coverage. Such a κ generates a "Grothendieck topology" in our sense via passage to $J(\kappa)$ (cf. 12.20).

12.22 EXAMPLE Take for κ the coverage defined in 11.14 (assuming the relevant conditions on $\underline{C}$) -- then κ is a pretopology (cf. 11.32) with identities (...) and here $\mathcal{S} \in J(\kappa)_X$ iff $\mathcal{S}$ is nonempty (cf. 12.19).

§13. *SITES*

Let $\underline{C}$ be a small category.

13.1 DEFINITION A <u>Grothendieck topology</u> on $\underline{C}$ is a function τ that assigns to each $X \in \text{Ob } \underline{C}$ a set τ_X of sieves over X subject to the following assumptions.

(1) The maximal sieve $\mathcal{S}_{max} \in \tau_X$.

(2) If $\mathcal{S} \in \tau_X$ and if $f:Y \to X$ is a morphism, then $f^*\mathcal{S} \in \tau_Y$.

(3) If $\mathcal{S} \in \tau_X$ and if $\mathcal{S}'$ is a sieve over X such that $f^*\mathcal{S}' \in \tau_Y$ for all $f:Y \to X$ in $\mathcal{S}$, then $\mathcal{S}' \in \tau_X$.

[Note: Within the setting of a small category, this is just a rephrasing of the definition of "Grothendieck topology" as formulated in 12.14 (however, "κ" has been replaced by "τ" and τ_X is a set rather than a mere conglomerate).]

13.2 DEFINITION A <u>site</u> is a pair $(\underline{C},\tau)$, where $\underline{C}$ is a small category and τ is a Grothendieck topology on $\underline{C}$.

13.3 REMARK Suppose that we have an assignment $X \to \tau_X$ satisfying (1), (2) of 13.1 and for which

$$\mathcal{S} \in \tau_X \ \& \ \mathcal{S} \subset \mathcal{S}' \Rightarrow \mathcal{S}' \in \tau_X.$$

Then to check (3) of 13.1, it suffices to consider those $\mathcal{S}'$ such that $\mathcal{S}' \subset \mathcal{S}$.

13.4 DEFINITION

- The <u>minimal Grothendieck topology</u> on $\underline{C}$ is the assignment $X \to \{\mathcal{S}_{max}\}$.

- The <u>maximal Grothendieck topology</u> on $\underline{C}$ is the assignment $X \to \{\mathcal{S}\}$, where $\mathcal{S}$ runs through all the sieves over X.

13.5 NOTATION Let $\tau_{\underline{C}}$ stand for the set of Grothendieck topologies on $\underline{C}$.

13.6 EXAMPLE Take $\underline{C} = \underline{1}$ — then $\underline{C}$ has two Grothendieck topologies: $\{\$_{\max}\}$ and $\{\$_{\min}, \$_{\max}\}$.

Given $\tau, \tau' \in \tau_{\underline{C}}$, write $\tau \leq \tau'$ if $\forall\ X \in \text{Ob } \underline{C}$, $\tau_X \subset \tau'_X$.

13.7 LEMMA The poset $\tau_{\underline{C}}$ is a bounded lattice.

PROOF If $\tau, \tau' \in \tau_{\underline{C}}$, let $\tau \wedge \tau'$ be their set theoretical intersection and let $\tau \vee \tau'$ be the smallest Grothendieck topology containing their set theoretical union. As for 0 and 1, take 0 to be the minimal Grothendieck topology and 1 to be the maximal Grothendieck topology.

13.8 THEOREM The bounded lattice $\tau_{\underline{C}}$ is a complete Heyting algebra or, equivalently, the bounded lattice $\tau_{\underline{C}}$ is a locale.

§14. *SUBFUNCTORS*

Let $\underline{C}$ be a locally small category.

14.1 DEFINITION A <u>subfunctor</u> of a functor $F:\underline{C}^{OP} \to \underline{SET}$ is a functor $G:\underline{C}^{OP} \to \underline{SET}$ such that $\forall\ X \in \text{Ob } \underline{C}$, GX is a subset of FX and the corresponding inclusions constitute a natural transformation $G \to F$, so $\forall\ f:Y \to X$ there is a commutative diagram

$$\begin{tikzcd} GY \arrow[r, "i_Y"] & FY \\ GX \arrow[u, "Gf"] \arrow[r, "i_X"'] & FX \arrow[u, "Ff"'] \end{tikzcd} .$$

[Note: There is a one-to-one correspondence between the subobjects of F and the subfunctors of F.]

14.2 LEMMA Fix an object X in $\underline{C}$ -- then there is a one-to-one correspondence between the sieves over X and the subfunctors of h_X (cf. 10.8).

PROOF If $\mathcal{S}$ is a sieve over X, then the designation

$$GY = \{f:Y \to X \ \&\ f \in \mathcal{S}\}$$

defines a subfunctor of h_X (given $Z \xrightarrow{g} Y$, $Gg:GY \to GZ$ is the map $f \to f \circ g$). Conversely, if G is a subfunctor of h_X, then $GY \subset \text{Mor}(Y,X)$ and

$$\mathcal{S} = \bigcup_Y GY$$

is a sieve over X.

14.3 EXAMPLE The subfunctor corresponding to $\mathcal{S}_{max}$ is h_X and the subfunctor corresponding to $\$_{\min}$ is $\emptyset_{\hat{\underline{C}}}$ (the initial object of $\hat{\underline{C}}$).

Suppose now that $\underline{C}$ is a small category -- then in view of 14.2, the notion of Grothendieck topology can be reformulated.

14.4 NOTATION Given a subfunctor G of h_X and a morphism $f:Y \to X$, define f^*G by the pullback square

$$\begin{tikzcd} f^*G \arrow[r] \arrow[d, "i_{f^*G}"'] & G \arrow[d, "i_G"] \\ h_Y \arrow[r, "h_f"'] & h_X \end{tikzcd}$$

in $\hat{\underline{C}}$ — then f^*G is a subfunctor of h_Y.

14.5 DEFINITION A <u>Grothendieck topology</u> on $\underline{C}$ is a function τ that assigns to each $X \in \text{Ob } \underline{C}$ a set τ_X of subfunctors of h_X subject to the following assumptions.

(1) The subfunctor $h_X \in \tau_X$.

(2) If $G \in \tau_X$ and if $f:Y \to X$ is a morphism, then $f^*G \in \tau_Y$.

(3) If $G \in \tau_X$ and if G' is a subfunctor of h_X such that $f^*G' \in \tau_Y$ for all $f \in GY$, then $G' \in \tau_X$.

14.6 LEMMA Let τ be a Grothendieck topology on $\underline{C}$ -- then

$$G \in \tau_X \ \& \ G \subset G' \Rightarrow G' \in \tau_X.$$

14.7 LEMMA Let τ be a Grothendieck topology on $\underline{C}$ — then

$$G, G' \in \tau_X \Rightarrow G \cap G' \in \tau_X.$$

14.8 REMARK Suppose that we have an assignment $X \to \tau_X$ satisfying (1), (2) of 14.5 and for which

$$G \in \tau_X \ \& \ G \subset G' \Rightarrow G' \in \tau_X.$$

Then to check (3) of 14.5, it suffices to consider those G' such that $G' \subset G$.

§15. *SHEAVES*

In what follows, all categories are assumed to be locally small for the generalities and small for the sheaf specifics.

15.1 RAPPEL A full, isomorphism closed subcategory $\underline{D}$ of a category $\underline{C}$ is said to be a reflective subcategory of $\underline{C}$ if the inclusion $\iota:\underline{D} \to \underline{C}$ has a left adjoint R, a reflector for $\underline{D}$.

[Note: A reflective subcategory $\underline{D}$ of a category $\underline{C}$ is closed under the formation of limits in $\underline{C}$.]

Let $\underline{D}$ be a reflective subcategory of a category $\underline{C}$, R a reflector for $\underline{D}$ -- then one may attach to each $X \in \text{Ob } \underline{C}$ a morphism $r_X:X \to RX$ in $\underline{C}$ with the following property: Given any $Y \in \text{Ob } \underline{D}$ and any morphism $f:X \to Y$ in $\underline{C}$, there exists a unique morphism $g:RX \to Y$ in $\underline{D}$ such that $f = g \circ r_X$.

N.B. Matters can always be arranged in such a way as to ensure that $R \circ \iota = \text{id}_{\underline{D}}$.

Let $\underline{C}$ be a small category. Suppose that $\underline{S}$ is a reflective subcategory of $\hat{\underline{C}}$. Denote the reflector by $\underline{a}$ -- then there is an adjoint pair $(\underline{a},\iota)$, $\iota:\underline{S} \to \hat{\underline{C}}$ the inclusion.

Assume: $\underline{a}$ preserves finite limits.

[Note: It is automatic that $\underline{a}$ preserves colimits.]

15.2 THEOREM Given $X \in \text{Ob } \underline{C}$, let τ_X be the set of those subfunctors $G \xrightarrow{i_G} h_X$ such that $\underline{a}i_G$ is an isomorphism -- then the assignment $X \to \tau_X$ is a Grothendieck topology τ on $\underline{C}$ (in the sense of 14.5).

PROOF Since

$$\underline{a}(\mathrm{id}_{h_X}) = \mathrm{id}_{\underline{a}h_X},$$

it follows that $h_X \in \tau_X$, hence (1) is satisfied. As for (2), by assumption $\underline{a}$ preserves finite limits, so in particular $\underline{a}$ preserves pullbacks, thus

$$\begin{tikzcd} \underline{a}f^*G \arrow[r] \arrow[d, "\underline{a}i_{f^*G}"'] & \underline{a}G \arrow[d, "\underline{a}i_G"] \\ \underline{a}h_Y \arrow[r, "\underline{a}h_f"'] & \underline{a}h_X \end{tikzcd}$$

is a pullback square in $\underline{S}$. But $\underline{a}i_G$ is an isomorphism. Therefore $\underline{a}i_{f^*G}$ is an isomorphism, i.e., $f^*G \in \tau_Y$. The verification of (3), however, is more complicated.

- Suppose that $G \in \tau_X$ and G is a subfunctor of G':

$$\left[\begin{array}{l} i_G:G \to h_X \\ \\ i_{G'}:G' \to h_X \end{array}\right. , \ i:G \to G'.$$

Then

$$i_G = i_{G'} \circ i \Rightarrow \underline{a}i_G = \underline{a}i_{G'} \circ \underline{a}i.$$

But $\underline{a}i_G$ is an isomorphism, hence

$$\mathrm{id} = \underline{a}i_{G'} \circ \underline{a}i \circ (\underline{a}i_G)^{-1},$$

which implies that $\underline{a}i_{G'}$ is a split epimorphism. On the other hand, $\underline{a}$ preserves monomorphisms, hence $\underline{a}i_{G'}$ is a monomorphism. Therefore $\underline{a}i_{G'}$ is an isomorphism, i.e., $G' \in \tau_X$.

• It remains to establish (3) under the restriction that G' is a subfunctor of G (cf. 14.8). Using the Yoneda lemma, identify each $f \in GY$ with $f \in \mathrm{Nat}(h_Y, G)$ and display the data in the diagram

$$\begin{tikzcd} h_Y \times_G G' \arrow[r] \arrow[d, "i_f"'] & G' \arrow[r, equal] \arrow[d, "i"] & G' \arrow[d, "i_{G'}"] \\ h_Y \arrow[r, "f"'] & G \arrow[r, "i_G"'] & h_X \end{tikzcd} .$$

There is one such diagram for each Y and each $f \in GY$, so upon consolidation we have

$$\begin{tikzcd} \coprod\limits_Y \coprod\limits_f h_Y \times_G G' \arrow[rr, "\Pi_{G'}"] \arrow[d, "\coprod \coprod i_f"'] & & G' \arrow[d, "i"] \\ \coprod\limits_Y \coprod\limits_f h_Y \arrow[rr, "\Gamma_G"'] & & G \end{tikzcd} .$$

Now i is an equalizer (all monomorphisms in $\hat{\underline{C}}$ are equalizers), thus $\underline{a}i$ is an equalizer (by the assumption on $\underline{a}$). But the assumption on G' is that $\forall\, Y$ and $\forall\, f \in GY$, $\underline{a}i_f$ is an isomorphism, thus $\underline{a}i$ is an epimorphism (see 15.6 below). And this means that $\underline{a}i$ is an isomorphism (in any category, a morphism which is an equalizer and an epimorphism is an isomorphism). Finally,

$$i_{G'} = i_G \circ i \Rightarrow \underline{a}i_{G'} = \underline{a}i_G \circ \underline{a}i.$$

Therefore $\underline{a}i_{G'}$ is an isomorphism, i.e., $G' \in \tau_X$.

15.3 RAPPEL Given a category $\underline{C}$, a set $\mathcal{U}$ of objects in $\underline{C}$ is said to be a <u>separating set</u> if for every pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ of distinct morphisms, there exists a $U \in \mathcal{U}$ and a morphism $\sigma: U \to X$ such that $f \circ \sigma \neq g \circ \sigma$.

15.4 EXAMPLE Suppose that $\underline{C}$ is small -- then the h_Y ($Y \in \mathrm{Ob}\,\underline{C}$) are a separating set for $\hat{\underline{C}}$.

15.5 LEMMA Let $\underline{C}$ be a category with coproducts and let $\mathcal{U}$ be a separating set -- then $\forall\, X \in \mathrm{Ob}\,\underline{C}$, the unique morphism

$$\coprod_{U \in \mathcal{U}} \ \coprod_{f \in \mathrm{Mor}(U,X)} \mathrm{dom}\, f \xrightarrow{\Gamma_X} X$$

such that $\forall\, f$, $\Gamma_X \circ \mathrm{in}_f = f$ is an epimorphism.

15.6 APPLICATION Suppose that $\underline{C}$ is small. Working with $\hat{\underline{C}}$, take $X = G$ in 15.5 -- then

$$\coprod_Y \coprod_f h_Y \xrightarrow[\Gamma_G]{} G$$

is an epimorphism.

[Note: To finish the argument that $\underline{a}i$ is an epimorphism, start with the relation

$$\Gamma_G \circ \coprod \coprod i_f = i \circ \Pi_{G'}.$$

Then

$$\underline{a}\Gamma_G \circ \underline{a}(\coprod \coprod i_f) = \underline{a}i \circ \underline{a}\Pi_{G'}.$$

Since Γ_G is an epimorphism, the same is true of $\underline{a}\Gamma_G$ (left adjoints preserve epimorphisms). And

$$\underline{a}(\coprod \coprod i_f) = \coprod \coprod \underline{a}i_f$$

is an isomorphism, call it Φ, hence

$$\underline{a}\Gamma_G = \underline{a}i \circ (\underline{a}\Pi_{G'} \circ \Phi^{-1}).$$

Therefore $\underline{a}i$ is an epimorphism.]

15.7 DEFINITION Fix a Grothendieck topology $\tau \in \tau_{\underline{C}}$ -- then a presheaf $F \in \text{Ob}\ \hat{\underline{C}}$ is called a $\underline{\tau\text{-sheaf}}$ if $\forall\ X \in \text{Ob}\ \underline{C}$ and $\forall\ G \in \tau_X$, the precomposition map

$$i_G^*: \text{Nat}(h_X, F) \to \text{Nat}(G, F)$$

is bijective.

Write $\underline{\text{Sh}}_\tau(\underline{C})$ for the full subcategory of $\hat{\underline{C}}$ whose objects are the τ-sheaves.

15.8 EXAMPLE Take for τ the minimal Grothendieck topology on $\underline{C}$ -- then $\text{Sh}_\tau(\underline{C}) = \hat{\underline{C}}$.

[Note: In particular, $\underline{\text{Sh}}_\tau(\underline{1}) = \hat{\underline{1}} \approx \underline{\text{SET}}$.]

15.9 EXAMPLE Take for τ the maximal Grothendieck topology on $\underline{C}$ -- then the objects of $\underline{\text{Sh}}_\tau(\underline{C})$ are the final objects in $\hat{\underline{C}}$.

[First, $\forall\ X \in \text{Ob}\ \underline{C}$, $\emptyset_{\hat{\underline{C}}} \to h_X$. But $\emptyset_{\hat{\underline{C}}}$ is initial, thus the condition that F be a τ-sheaf amounts to the existence for each X of a unique morphism $h_X \to F$. Meanwhile, by Yoneda, $\text{Nat}(h_X,F) \approx FX$.]

15.10 THEOREM The inclusion $\iota_\tau:\underline{\text{Sh}}_\tau(\underline{C}) \to \hat{\underline{C}}$ admits a left adjoint $\underline{a}_\tau:\hat{\underline{C}} \to \underline{\text{Sh}}_\tau(\underline{C})$ that preserves finite limits.

[Note: We can and will assume that $\underline{a}_\tau \circ \iota_\tau$ is the identity.]

Various categorical generalities can then be specialized to the situation at hand.

15.11 DEFINITION A morphism $f:A \to B$ and an object X in a category $\underline{C}$ are said to be <u>orthogonal</u> ($f \perp X$) if the precomposition map $f^*:\text{Mor}(B,X) \to \text{Mor}(A,X)$ is bijective.

15.12 RAPPEL Let $\underline{D}$ be a reflective subcategory of a category $\underline{C}$, R a reflector for $\underline{D}$. Let $W_{\underline{D}}$ be the class of morphisms in $\underline{C}$ rendered invertible by R.

- Let $X \in \text{Ob } \underline{C}$ -- then $X \in \text{Ob } \underline{D}$ iff $\forall f \in W_{\underline{D}}$, $f \perp X$.
- Let $f \in \text{Mor } \underline{C}$ -- then $f \in W_{\underline{D}}$ iff $\forall X \in \text{Ob } \underline{D}$, $f \perp X$.

15.13 NOTATION Let W_τ be the class of morphisms in $\hat{\underline{C}}$ rendered invertible by $\underline{a}_\tau$.

15.14 EXAMPLE If $F \in \text{Ob } \hat{\underline{C}}$, then F is a τ-sheaf iff $\forall\ \Xi \in W_\tau$, $\Xi \perp F$.

15.15 EXAMPLE If $\Xi \in \text{Mor } \hat{\underline{C}}$, then $\Xi \in W_\tau$ iff for every τ-sheaf F, $\Xi \perp F$.

[Note: If $X \in \mathrm{Ob}\,\underline{C}$ and if $G \in \tau_X$, then for every τ-sheaf F, $i_G \perp F$, thus $i_G \in \mathcal{W}_\tau$.]

15.16 RAPPEL Let $\underline{D}$ be a reflective subcategory of a category $\underline{C}$, R a reflector for $\underline{D}$ -- then the localization $\mathcal{W}_{\underline{D}}^{-1}\underline{C}$ is equivalent to $\underline{D}$.

15.17 APPLICATION The localization $\mathcal{W}_\tau^{-1}\hat{\underline{C}}$ is equivalent to $\underline{\mathrm{Sh}}_\tau(\underline{C})$.

15.18 RAPPEL Let $\underline{D}$ be a reflective subcategory of a finitely complete category $\underline{C}$, R a reflector for $\underline{D}$ -- then R preserves finite limits iff $\mathcal{W}_{\underline{D}}$ is pullback stable.

15.19 APPLICATION Since $\underline{a}_\tau : \hat{\underline{C}} \to \underline{\mathrm{Sh}}_\tau(\underline{C})$ preserves finite limits, it follows that $\mathcal{W}_\tau$ is pullback stable.

15.20 EXAMPLE Take $\underline{C} = \underline{1}$, so $\hat{\underline{1}} \approx \underline{\mathrm{SET}}$ -- then $\#\tau_{\underline{1}} = 2$. On the other hand, $\underline{\mathrm{SET}}$ has precisely 3 reflective subcategories: $\underline{\mathrm{SET}}$ itself, the full subcategory of final objects, and the full subcategory of final objects plus the empty set ($\#RX = 1$ if $X \neq \emptyset$, $R\emptyset = \emptyset$). In terms of Grothendieck topologies, the first two are accounted for by 15.8 and 15.9. But the third cannot be a category of sheaves per a Grothendieck topology on $\underline{C} = \underline{1}$. To see this, note that the class of morphisms rendered invertible by R consists of all functions $f : X \to Y$ with $X \neq \emptyset$ as well as the function $\emptyset \to \emptyset$ (thus the arrows $\emptyset \to X$ $(X \neq \emptyset)$ are excluded). Suppose now that Z is a nonempty set and X, Y are nonempty subsets of Z with an empty intersection. Consider the pullback square

$$\begin{tikzcd} \emptyset = X \cap Y \arrow[r, "\tilde{i}_X"] \arrow[d, "\tilde{i}_Y"'] & Y \arrow[d, "i_Y"] \\ X \arrow[r, "i_X"'] & Z \end{tikzcd},$$

where i_X, i_Y are the inclusions -- then Ri_Y is an isomorphism but $R\tilde{i}_Y$ is not an isomorphism. Therefore the class of morphisms rendered invertible by R is not pullback stable.

15.21 NOTATION Let $F \in \text{Ob}\ \hat{\underline{C}}$ be a presheaf. Given $X \in \text{Ob}\ \underline{C}$, let $\tau_X(F)$ be the set of subfunctors $i_G:G \to h_X$ such that for any morphism $f:Y \to X$, the precomposition arrow

$$(i_{f^*G})^*:\text{Nat}(h_Y,F) \to \text{Nat}(f^*G,F)$$

is bijective.

15.22 LEMMA The assignment $X \to \tau_X(F)$ is a Grothendieck topology $\tau(F)$ on $\underline{C}$.

<u>N.B.</u> $\tau(F)$ is the largest Grothendieck topology in which F is a sheaf.

15.23 SCHOLIUM For any class $\mathcal{F}$ of presheaves, there exists a largest Grothendieck topology $\tau(\mathcal{F})$ on $\underline{C}$ in which the $F \in \mathcal{F}$ are sheaves.

15.24 DEFINITION The <u>canonical</u> Grothendieck topology τ_{can} on $\underline{C}$ is the largest Grothendieck topology on $\underline{C}$ in which the $h_X (X \in \text{Ob}\ \underline{C})$ are sheaves.

[Note: Let $\tau \in \tau_{\underline{C}}$ — then τ is said to be <u>subcanonical</u> if the $h_X (X \in \text{Ob}\ \underline{C})$ are τ-sheaves.]

15.25 EXAMPLE Take $\underline{C} = \underline{O}(X)$, X a topological space (cf. 11.18) — then the Grothendieck topology τ on $\underline{O}(X)$ per 12.16 is the canonical Grothendieck topology, $\underline{\text{Sh}}_\tau(\underline{O}(X))$ being the traditional sheaves of sets on X, i.e., $\underline{\text{Sh}}(X)$.

§16. *SHEAVES: SORITES*

The category $\underline{\mathrm{Sh}}_\tau(\underline{C})$ associated with a site $(\underline{C},\tau)$ has a number of properties that will be cataloged below.

16.1 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is complete and cocomplete.

[This is because $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is a reflective subcategory of $\hat{\underline{C}}$ which is both complete and cocomplete. Accordingly, limits in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are computed as in $\hat{\underline{C}}$ while colimits in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are computed by applying $\underline{a}_\tau$ to the corresponding colimts in $\hat{\underline{C}}$.]

16.2 EXAMPLE Given $\tau \in \tau_{\underline{C}}$, define 0_τ by the rule

$$0_\tau(X) = \begin{cases} \{0\} \text{ if } \emptyset_{\hat{\underline{C}}} \in \tau_X \\ \\ \emptyset \text{ if } \emptyset_{\hat{\underline{C}}} \notin \tau_X. \end{cases}$$

Then 0_τ is a τ-sheaf and, moreover, is an initial object in $\underline{\mathrm{Sh}}_\tau(\underline{C})$.

16.3 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is cartesian closed.

16.4 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ admits a subobject classifier.

16.5 REMARK Therefore $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is a topos.

16.6 LEMMA $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is balanced.

16.7 LEMMA Every monomorphism in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is an equalizer.

[Let $\Xi:F \to G$ be a monomorphism in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ -- then $\iota_\tau\Xi:\iota_\tau F \to \iota_\tau G$ is a monomorphism in $\hat{\underline{C}}$, hence is an equalizer. But $\underline{a}_\tau$ preserves equalizers (since it preserves finite limits).]

N.B. Monomorphisms in $\underline{Sh}_\tau(\underline{C})$ are pushout stable.

16.8 LEMMA Every epimorphism in $\underline{Sh}_\tau(\underline{C})$ is a coequalizer.

16.9 LEMMA $\underline{Sh}_\tau(\underline{C})$ fulfills the standard conditions (cf. 11.23).

[Epimorphisms in $\underline{Sh}_\tau(\underline{C})$ are pullback stable (cf. 17.16) and every epimorphism in $\underline{Sh}_\tau(\underline{C})$ is a coequalizer (cf. 16.8).]

16.10 LEMMA In $\underline{Sh}_\tau(\underline{C})$, filtered colimits commute with finite limits.

16.11 RAPPEL Coproducts in $\hat{\underline{C}}$ are disjoint.

[In other words, if $F = \coprod_{i \in I} F_i$ is a coproduct of a set of presheaves F_i, then $\forall\, i \in I$, $in_i : F_i \to F$ is a monomorphism and $\forall\, i,j \in I$ $(i \neq j)$, the pullback $F_i \times_F F_j$ is the initial object in $\hat{\underline{C}}$.]

16.12 LEMMA Coproducts in $\underline{Sh}_\tau(\underline{C})$ are disjoint.

16.13 RAPPEL Coproducts in $\hat{\underline{C}}$ are pullback stable.

[In other words, if $F = \coprod_{i \in I} F_i$ is a coproduct of a set of presheaves F_i, then for every arrow $F' \to F$,

$$\coprod_{i \in I} F' \times_F F_i \approx F'.]$$

16.14 LEMMA Coproducts in $\underline{Sh}_\tau(\underline{C})$ are pullback stable.

16.15 DEFINITION Let $\underline{C}$ be a category which fulfills the standard conditions. Suppose that $R \underset{v}{\overset{u}{\rightrightarrows}} X$ is an equivalence relation on an object X in $\underline{C}$. Consider the coequalizer diagram

$$R \underset{v}{\overset{u}{\rightrightarrows}} X \xrightarrow{\pi} X/R \equiv \mathrm{coeq}(u,v).$$

Then there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "v"] \arrow[d, "u"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & X/R \end{tikzcd}$$

and a pullback square

$$\begin{tikzcd} X \times_{X/R} X \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & X/R . \end{tikzcd}$$

One then says that R is <u>effective</u> if the canonical arrow

$$R \longrightarrow X \times_{X/R} X$$

is an isomorphism (it is always a monomorphism).

[Note: $\underline{C}$ has <u>effective equivalence relations</u> if every equivalence relation is effective.]

16.16 LEMMA Equivalence relations in $\underline{\mathrm{Sh}}_\tau(\underline{C})$ are effective.

[The usual methods apply: Equivalence relations in $\underline{SET}$ are effective, hence equivalence relations in $\underline{C}$ are effective etc.]

16.17 LEMMA The $\underline{a}_\tau h_X$ ($X \in \text{Ob } \underline{C}$) are a separating set for $\underline{Sh}_\tau(\underline{C})$.

PROOF Let $\Xi, \Xi' : F \to G$ be distinct arrows in $\underline{Sh}_\tau(\underline{C})$ -- then the claim is that $\exists\ X \in \text{Ob } \underline{C}$ and $\sigma : \underline{a}_\tau h_X \to F$ such that $\Xi \circ \sigma \neq \Xi' \circ \sigma$. But $\Xi \neq \Xi'$ implies that $\Xi_X \neq \Xi'_X$ ($\exists\ X \in \text{Ob } \underline{C}$) which implies that $\Xi_X x \neq \Xi'_X x$ ($\exists\ x \in FX$). Owing to the Yoneda lemma, $FX \approx \text{Nat}(h_X, F)$, so x corresponds to a $\sigma' \in \text{Nat}(h_X, F)$, thus $\Xi \circ \sigma' \neq \Xi' \circ \sigma'$. Determine $\sigma : \underline{a}_\tau h_X \to F$ by the diagram

$$\begin{tikzcd} h_X \arrow[r] \arrow[d, "\sigma'"'] & \underline{a}_\tau h_X \arrow[d, "\sigma"] \\ F \arrow[r, equal] & F \end{tikzcd} .$$

Then $\Xi \circ \sigma \neq \Xi' \circ \sigma$.

N.B. All epimorphisms in $\underline{Sh}_\tau(\underline{C})$ are coequalizers (cf. 16.8). So, for every τ-sheaf F, the epimorphism Γ_F of 15.5 is automatically a coequalizer. Therefore the $\underline{a}_\tau h_X$ ($X \in \text{Ob } \underline{C}$) are a "strong" separating set for $\underline{Sh}_\tau(\underline{C})$.

16.18 DEFINITION Let $\underline{C}$ be a cocomplete category and let κ be a regular cardinal -- then an object $X \in \text{Ob } \underline{C}$ is $\underline{\kappa\text{-definite}}$ if $\text{Mor}(X, —)$ preserves κ-filtered colimits.

16.19 LEMMA $\underline{Sh}_\tau(\underline{C})$ is presentable.

PROOF Fix a regular cardinal $\kappa > \#\text{Mor } \underline{C}$ -- then $\forall\ X \in \text{Ob } \underline{C}$, $h_X \in \text{Ob } \hat{\underline{C}}$ is κ-definite, the contention being that $\forall\ X \in \text{Ob } \underline{C}$, $\underline{a}_\tau h_X \in \text{Ob } \underline{Sh}_\tau(\underline{C})$ is κ-definite, which suffices. To see this, note first that a κ-filtered colimit of τ-sheaves can be computed levelwise, i.e., its κ-filtered colimit per $\hat{\underline{C}}$ is a τ-sheaf. Now fix a κ-filtered category $\underline{I}$ and let $\Delta:\underline{I} \to \underline{\mathrm{Sh}}_\tau(\underline{C})$ be a diagram -- then

$$\begin{aligned}
\mathrm{Nat}(\underline{a}_\tau h_X, \mathrm{colim}_{\underline{I}}\ \Delta_i) &\approx \mathrm{Nat}(\underline{a}_\tau h_X, \mathrm{colim}_{\underline{I}}\ \iota_\tau \Delta_i) \\
&\approx \mathrm{Nat}(h_X, \mathrm{colim}_{\underline{I}}\ \iota_\tau \Delta_i) \\
&\approx \mathrm{colim}_{\underline{I}}\ \mathrm{Nat}(h_X, \iota_\tau \Delta_i) \\
&\approx \mathrm{colim}_{\underline{I}}\ \mathrm{Nat}(\underline{a}_\tau h_X, \Delta_i).
\end{aligned}$$

16.20 REMARK A presentable category is necessarily wellpowered and cowellpowered.

16.21 DEFINITION Let $\underline{E}$ be a topos -- then $\underline{E}$ is said to be a <u>Grothendieck topos</u> if $\underline{E}$ is cocomplete and has a separating set.

[Note: In general, a cocomplete topos need not admit a separating set.]

It therefore follows from 16.17 that the cocomplete topos $\underline{\mathrm{Sh}}_\tau(\underline{C})$ is a Grothendieck topos.

§17. *LOCAL ISOMORPHISMS*

Let $\underline{C}$ be a locally small category.

17.1 DEFINITION Let $f:X \to Y$ be a morphism in $\underline{C}$ -- then a <u>decomposition</u> of f is a pair of arrows $X \xrightarrow{k} M \xrightarrow{m} Y$ such that $f = m \circ k$, where k is an epimorphism and m is a monomorphism. The decomposition (k,m) of f is said to be <u>minimal</u> (and M is said to be the <u>image</u> of f, denoted im f) if for any other factorization $X \xrightarrow{\ell} N \xrightarrow{n} Y$ of f with n a monomorphism, there is an $h:M \to N$ such that $h \circ k = \ell$ and $n \circ h = m$.

17.2 LEMMA Suppose that $\underline{C}$ fulfills the standard conditions (cf. 11.23) -- then every morphism $f:X \to Y$ in $\underline{C}$ admits a minimal decomposition $f = m \circ k$, where k is a coequalizer and m is a monomorphism, the data being unique up to isomorphism.

Let $\underline{C}$ be a small category.

17.3 RAPPEL $\hat{\underline{C}}$ fulfills the standard condtions (and is balanced).

Let $H,K \in \text{Ob } \hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat}(H,K)$. Form the pullback square

$$\begin{tikzcd} H \times_K H \arrow[r, "q"] \arrow[d, "p"'] & H \arrow[d, "\Xi"] \\ H \arrow[r, "\Xi"'] & K \end{tikzcd} .$$

Then p and q are epimorphisms.

17.4 NOTATION $\delta_H:H \to H \times_K H$ is the canonical arrow associated with id_H, thus $p \circ \delta_H = id_H = q \circ \delta_H$.

N.B. δ_H is a monomorphism.

17.5 LEMMA Ξ is a monomorphism iff δ_H is an epimorphism.

[Note: Consequently, if Ξ is a monomorphism, then δ_H is an isomorphism.]

Fix a Grothendieck topology $\tau \in \tau_{\underline{C}}$.

17.6 DEFINITION Let $H,K \in \text{Ob } \hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat}(H,K)$. Factor Ξ per 17.2:

$$H \xrightarrow{k} M \xrightarrow{m} K.$$

Then Ξ is a τ-local epimorphism if for any $f:h_Y \to K$, the subfunctor f^*M of h_Y defined by the pullback square

$$\begin{tikzcd} f^*M \arrow[r] \arrow[d, "i_{f^*M}"'] & M \arrow[d, "m"] \\ h_Y \arrow[r, "f"'] & K \end{tikzcd}$$

is in τ_Y.

17.7 LEMMA Every epimorphism in $\hat{\underline{C}}$ is a τ-local epimorphism.

17.8 DEFINITION Let $H,K \in \text{Ob } \hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat}(H,K)$ — then Ξ is a τ-local monomorphism if δ_H is a τ-local epimorphism (cf. 17.5).

17.9 LEMMA Every monomorphism in $\hat{\underline{C}}$ is a τ-local monomorphism.

17.10 DEFINITION Let $H,K \in \mathrm{Ob}\,\hat{\underline{C}}$ be presheaves and let $\Xi \in \mathrm{Nat}(H,K)$ -- then Ξ is a <u>τ-local isomorphism</u> if Ξ is both a τ-local epimorphism and a τ-local monomorphism.

17.11 EXAMPLE If $G \in \tau_X$, then $i_G:G \to h_X$ is a τ-local isomorphism.

[For any $f:Y \to X$, there is a pullback square

$$\begin{tikzcd} f^*G \arrow[r] \arrow[d, "i_{f^*G}"'] & G \arrow[d, "i_G"] \\ h_Y \arrow[r, "h_f"'] & h_X \end{tikzcd}$$

in $\hat{\underline{C}}$ and $f^*G \in \tau_Y$, thus i_G is a τ-local epimorphism. On the other hand, i_G is a monomorphism, hence i_G is a τ-local monomorphism (cf. 17.9).]

[Note: If G is a subfunctor of h_X and if $i_G:G \to h_X$ is a τ-local epimorphism, then $G \in \tau_X$. Proof: Take $f = \mathrm{id}_X$ and consider

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "i_G"'] & G \arrow[d, "i_G"] \\ h_X \arrow[r, equal] & h_X \end{tikzcd} .]$$

17.12 THEOREM $\mathcal{W}_\tau$ is the class of τ-local isomorphisms.

17.13 NOTATION Denote by $\underline{S}_{\underline{C}}$ the "set" of reflective subcategories $\underline{S}$ of $\hat{\underline{C}}$ with the property that the inclusion $\iota:\underline{S} \to \hat{\underline{C}}$ has a left adjoint $\underline{a}:\hat{\underline{C}} \to \underline{S}$ that preserves finite limits.

We shall now proceed to establish the "fundamental correspondence".

17.14 THEOREM The arrows

$$\left[\begin{array}{ll} \underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}} & \text{(cf. 15.2)} \\ \\ \tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}} & \text{(cf. 15.10)} \end{array}\right.$$

are mutually inverse.

To dispatch the second of these, consider the composite

$$\tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}}.$$

Take a $\tau \in \tau_{\underline{C}}$ and pass to $\underline{\mathrm{Sh}}_\tau(\underline{C})$ -- then the Grothendieck topology on $\underline{C}$ determined by $\underline{\mathrm{Sh}}_\tau(\underline{C})$ via 15.2 assigns to each $X \in \mathrm{Ob}\ \underline{C}$ the set of those subfunctors $i_G: G \to h_X$ such that $\underline{a}_\tau i_G$ is an isomorphism or, equivalently, those subfunctors $i_G: G \to h_X$ such that i_G is a τ-local isomorphism (cf. 17.12). But, as has been seen above, the subfunctors of h_X with this property are precisely the elements of τ_X (cf. 17.11). Therefore the composite

$$\tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}}$$

is the identity map.

It remains to prove that the composite

$$\underline{S}_{\underline{C}} \longrightarrow \tau_{\underline{C}} \longrightarrow \underline{S}_{\underline{C}}$$

is the identity map. So take an $\underline{S} \in \underline{S}_{\underline{C}}$, produce a Grothendieck topology τ on $\underline{C}$ per 15.2, and pass to $\underline{\mathrm{Sh}}_\tau(\underline{C})$ -- then $\underline{S} \subset \underline{\mathrm{Sh}}_\tau(\underline{C})$. Thus let $F \in \mathrm{Ob}\ \underline{S}$, the claim being that $F \in \underline{\mathrm{Ob}}\ \underline{\mathrm{Sh}}_\tau(\underline{C})$ or still, that F is a τ-sheaf, or still, that $\forall\ X \in \mathrm{Ob}\ \underline{C}$ and

$\forall\ G \in \tau_X$, $i_G \perp F$, which is clear since $i_G \in \mathcal{W}_\tau$ (cf. 15.15). To reverse matters and deduce that $\underline{\mathrm{Sh}}_\tau(\underline{C}) \subset \underline{S}$, one has only to show that if $\Xi:H \to K$ is a morphism in $\hat{\underline{C}}$ and if $\underline{a}\Xi$ is an isomorphism, then $\underline{a}_\tau\Xi$ is an isomorphism (cf. 17.17 infra). To this end, factor Ξ per 17.2:

$$H \xrightarrow{k} M \xrightarrow{m} K.$$

Then $\underline{a}\Xi = \underline{a}m \circ \underline{a}k$. But $\underline{a}\Xi$ is an isomorphism and $\underline{a}m$ is a monomorphism ($\underline{a}$ preserves finite limits). Therefore $\underline{a}k$ is a monomorphism. But $\underline{a}k$ is a coequalizer ($\underline{a}$ is a left adjoint), thus $\underline{a}k$ is an isomorphism (in any category, a morphism which is a monomorphism and a coequalizer is an isomorphism). And then $\underline{a}m$ is an isomorphism as well.

● Assume that $\underline{a}\Xi$ is an isomorphism, where Ξ is a monomorphism -- then $\underline{a}_\tau\Xi$ is an isomorphism.

[Bearing in mind that here $H = M$, consider a pullback square

$$\begin{tikzcd} f^*H \arrow[r] \arrow[d, "i_{f^*H}"'] & H \arrow[d, "\Xi"] \\ h_Y \arrow[r, "f"'] & K \end{tikzcd}.$$

Then the assumption that $\underline{a}\Xi$ is an isomorphism implies that $\underline{a}i_{f^*H}$ is an isomorphism which in turn implies that $i_{f^*H} \in \tau_Y$. Therefore Ξ is a τ-local epimorphism or still, Ξ is a τ-local isomorphism, hence $\Xi \in \mathcal{W}_\tau$ (cf. 17.12), so $\underline{a}_\tau\Xi$ is an isomorphism.

● Assume that $\underline{a}\Xi$ is an isomorphism, where Ξ is a coequalizer -- then $\underline{a}_\tau\Xi$ is an isomorphism.

[Because $\underline{a}_\tau \Xi$ is a coequalizer, to conclude that $\underline{a}_\tau \Xi$ is an isomorphism, it suffices to verify that $\underline{a}_\tau \Xi$ is a monomorphism. For this purpose, consider the pullback square

$$\begin{tikzcd} H \times_K H \arrow[r, "q"] \arrow[d, "p"'] & H \arrow[d, "\Xi"] \\ H \arrow[r, "\Xi"'] & K \end{tikzcd} .$$

Then δ_H is a monomorphism and there are pullback squares

$$\begin{tikzcd} \underline{a}H \times_{\underline{a}K} \underline{a}H \arrow[r, "\underline{a}q"] \arrow[d, "\underline{a}p"'] & \underline{a}H \arrow[d, "\underline{a}\Xi"] \\ \underline{a}H \arrow[r, "\underline{a}\Xi"'] & \underline{a}K \end{tikzcd} , \quad \begin{tikzcd} \underline{a}_\tau H \times_{\underline{a}_\tau K} \underline{a}_\tau H \arrow[r, "\underline{a}_\tau q"] \arrow[d, "\underline{a}_\tau p"'] & \underline{a}_\tau H \arrow[d, "\underline{a}_\tau \Xi"] \\ \underline{a}_\tau H \arrow[r, "\underline{a}_\tau \Xi"'] & \underline{a}_\tau K \end{tikzcd} .$$

But $\underline{a}\delta_H = \delta_{\underline{a}H}$ is an isomorphism, thus $\underline{a}_\tau \delta_H = \delta_{\underline{a}_\tau H}$ is an isomorphism (cf. supra), so $\underline{a}_\tau \Xi$ is a monomorphism.]

17.15 THEOREM Let $H,K \in \text{Ob}\, \hat{\underline{C}}$ be presheaves and let $\Xi \in \text{Nat}(H,K)$ -- then $\underline{a}_\tau \Xi : \underline{a}_\tau H \to \underline{a}_\tau K$ is an epimorphism in $\underline{\text{Sh}}_\tau(\underline{C})$ iff Ξ is a τ-local epimorphism.

17.16 APPLICATION The epimorphisms in $\underline{\text{Sh}}_\tau(\underline{C})$ are pullback stable.

[The class of τ-local epimorphisms is pullback stable.]

17.17 LEMMA Let $\underline{D}_1$, $\underline{D}_2$ be reflective subcategories of a category $\underline{C}$. Suppose that $W_{\underline{D}_2} \subset W_{\underline{D}_1}$ -- then $\underline{D}_1 \subset \underline{D}_2$.

PROOF Take $X_1 \in \mathrm{Ob}\, \underline{D}_1$. To conclude that $X_1 \in \mathrm{Ob}\, \underline{D}_2$, it need only be shown that $\forall\, f \in \mathcal{W}_{\underline{D}_2}$, $f \perp X_1$ (cf. 15.12). But

$$X_1 \in \mathrm{Ob}\, \underline{D}_1 \Rightarrow \mathcal{W}_{\underline{D}_1} \perp X_1$$

$$\Rightarrow \mathcal{W}_{\underline{D}_2} \perp X_1 = X_1 \in \mathrm{Ob}\, \underline{D}_2.$$

§18. κ-SHEAVES

Let $\underline{C}$ be a category.

18.1 DEFINITION Let $\mathcal{C}$ be a covering of $X \in \text{Ob } \underline{C}$ -- then a functor $F:\underline{C}^{OP} \rightarrow \underline{SET}$ has the sheaf property w.r.t. $\mathcal{C}$ if the following condition is satisfied: Given elements

$$x_g \in FY \quad (g:Y \rightarrow X \text{ in } \mathcal{C})$$

which are compatible in the sense that if

$$\text{(i)} \quad \left[\begin{array}{ll} h_1:Z \rightarrow \text{dom } g_1 & (g_1:Y_1 \rightarrow X \text{ in } \mathcal{C}) \\ h_2:Z \rightarrow \text{dom } g_2 & (g_2:Y_2 \rightarrow X \text{ in } \mathcal{C}) \end{array}\right.$$

and

$$\text{(ii)} \quad g_1 \circ h_1 = g_2 \circ h_2$$

imply

$$\text{(iii)} \quad (Fh_1(x_{g_1}) = (Fh_2(x_{g_2}),$$

then there exists a unique $x \in FX$ such that $\forall\ g:Y \rightarrow X$ in $\mathcal{C}$,

$$(Fg)x = x_g.$$

18.2 REMARK Suppose that $\mathcal{S}$ is a sieve -- then elements $x_f \in FY$ $(f:Y \rightarrow X$ in $\mathcal{S})$ are compatible iff whenever $Z \xrightarrow{g} Y \xrightarrow{f} X$, there follows

$$x_{f \circ g} = (Fg)(x_f).$$

[Note: If $\underline{C}$ is locally small, then

$$\text{sieves} \longleftrightarrow \text{subfunctors} \quad (\text{cf. } 14.2),$$

say

$$\mathcal{S} \longleftrightarrow G \subset h_X.$$

Accordingly, a compatible family corresponds to a natural transformation $G \rightarrow F$ and F has the sheaf property w.r.t. $\mathcal{S}$ iff every natural tranformation $G \rightarrow F$ extends uniquely to a natural transformation $h_X \rightarrow F$.]

18.3 EXAMPLE Take $C = \{id_X : X \rightarrow X\}$ -- then every functor $F:\underline{C}^{OP} \rightarrow \underline{SET}$ has the sheaf property w.r.t. C.

18.4 LEMMA A functor $F:\underline{C}^{OP} \rightarrow \underline{SET}$ has the sheaf property w.r.t. C iff it has the sheaf property w.r.t. $\mathcal{S}(C)$ (cf. 12.3).

18.5 EXAMPLE Fix $X \in \text{Ob } \underline{C}$ -- then every functor $F:\underline{C}^{OP} \rightarrow \underline{SET}$ has the sheaf property w.r.t. $\mathcal{S}_{max}$ (cf. 12.4).

18.6 DEFINITION Suppose that κ is a covering function -- then a functor $F:\underline{C}^{OP} \rightarrow \underline{SET}$ is a <u>κ-sheaf</u> if it has the sheaf property w.r.t. all the coverings in κ.

<u>N.B.</u> The κ-sheaves and the $\mathcal{S}(\kappa)$-sheaves are one and the same.

18.7 REMARK Let $\underline{C}$ be a small category and suppose that τ is a Grothendieck topology on $\underline{C}$ -- then τ can be defined as in 13.1 or as in 14.5, thus there are two possible interpretations of the phrase "τ-sheaf", viz. the one above or that of 15.7. Fortunately, however, there is no ambiguity: Both are descriptions of the same entity.

18.8 LEMMA If κ is a coverage and if $\kappa' \leq \kappa$, then every κ-sheaf is a κ'-sheaf.

[This is because if F is a κ-sheaf, then F has the sheaf property w.r.t. every covering that has a refinement in κ.]

18.9 APPLICATION Equivalent coverages have the same sheaves.

Write $\underline{\text{Sh}}_{\kappa}(\underline{C})$ for the full submetacategory of $[\underline{C}^{\text{OP}},\underline{\text{SET}}]$ whose objects are the κ-sheaves.

18.10 LEMMA Suppose that κ is a coverage -- then

$$\begin{aligned}\underline{\text{Sh}}_{\kappa}(\underline{C}) &= \underline{\text{Sh}}_{\text{sat}\ \kappa}(\underline{C})\\ &= \underline{\text{Sh}}_{\mathcal{S}(\text{sat}\ \kappa)}(\underline{C}).\end{aligned}$$

18.11 THEOREM Suppose that κ is a pretopology with identities -- then $J(\kappa)$ is a Grothendieck topology (cf. 12.20) and

$$\underline{\text{Sh}}_{J(\kappa)}(\underline{C}) = \underline{\text{Sh}}_{\kappa}(\underline{C}).$$

In the presence of a size restriction and pullbacks, there is another way to formulate the sheaf property. Thus let $\mathcal{C}$ be a covering of $X \in \text{Ob}\ \underline{C}$, say $\mathcal{C} = \{Y_i \xrightarrow{g_i} X: i \in I\}$, where I is set. Assume that the pullbacks

$$\begin{array}{ccc} Y_i \times_X Y_j & \xrightarrow{\pi_{ij}^1} & Y_i \\ {\scriptstyle \pi_{ij}^2}\downarrow & & \downarrow{\scriptstyle g_i} \\ Y_j & \xrightarrow[g_j]{} & X \end{array}$$

exist for all $i,j \in I$.

18.12 LEMMA Under the preceding conditions, a functor $F:\underset{\sim}{C}^{OP} \to \underline{SET}$ has the sheaf property w.r.t. C iff in the diagram

$$
\begin{tikzcd}
FX \arrow[r, "Fg_i"] \arrow[d, equal] & FY_i \arrow[r, "F\pi^1_{ij}"] & F(Y_i \times_X Y_j) \\
FX \arrow[r, "e"] \arrow[d, equal] & \prod\limits_k FY_k \arrow[u, "pr_i"'] \arrow[d, "pr_j"] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \prod\limits_{k,\ell} F(Y_k \times_X Y_\ell) \arrow[u, "pr_{ij}"'] \arrow[d, "pr_{ij}"] \\
FX \arrow[r, "Fg_j"'] & FY_j \arrow[r, "F\pi^2_{ij}"'] & F(Y_i \times_X Y_j),
\end{tikzcd}
$$

e is an equalizer of p_1 and p_2 in $\underline{SET}$.

18.13 DEFINITION Let $\underset{\sim}{C}$ be a locally small category, κ a covering function -- then κ is <u>subcanonical</u> if $\forall\ X \in \text{Ob}\ \underset{\sim}{C}$, h_X is a κ-sheaf.

18.14 EXAMPLE Assuming that $\underset{\sim}{C}$ has pullbacks, define κ by $\kappa_X = \{f\}$, where $f \in \text{Ob}\ \underset{\sim}{C}/X$ -- then κ is subcanonical iff the f are coequalizers.

18.15 EXAMPLE Take $\underset{\sim}{C} = \underline{TOP}$ -- then the open map coverage is subcanonical. But the open subset coverage, the open embedding coverage, and the local homeomorphism coverage are all subordinate to the open map coverage, hence they too are subcanonical (cf. 18.8).

18.16 EXAMPLE Take $\underline{C}$ = SCH (cf. 0.6) and fix $X \in \mathrm{Ob}\ \underline{C}$ ($\mathcal{O}_X$ being understood).

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is an open immersion and $\cup\, g_i(Y_i) = X$ -- then κ is a Grothendieck coverage, the Zariski coverage.

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is étale and $\cup\, g_i(Y_i) = X$ -- then κ is a Grothendieck coverage, the étale coverage.

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is smooth and $\cup\, g_i(Y_i) = X$ -- then κ is a Grothendieck coverage, the smooth coverage.

- Let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is flat + locally of finite presentation and $\cup\, g_i(Y_i) = X$ -- then κ is a Grothendieck coverage, the fppf coverage.

18.17 REMARK Each of these Grothendieck coverages is a Grothendieck pretopology with identities.

An open immersion is necessarily étale, an étale morphism is necessarily smooth, and a smooth morphism is necessarily flat + locally of finite presentation. Therefore the Zariski coverage is subordinate to the étale coverage which in turn is subordinate to the smooth coverage which in turn is subordinate to the fppf coverage.

[Note: If κ is the fppf coverage and if κ' is the Zariski coverage, then every κ-sheaf is a κ'-sheaf (cf. 18.8) but there are κ'-sheaves that are not κ-sheaves.]

18.18 THEOREM The fppf coverage is subcanonical.

Consequently, the Zariski coverage, the étale coverage, and the smooth coverage are all subcanonical (cf. 18.8).

It turns out that the fppf coverage is subordinate to the so-called "fpqc coverage" (see below).

18.19 DEFINITION Let $f:X \to Y$ be a surjective morphism of schemes -- then f is <u>locally quasi-compact</u> provided that every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

18.20 EXAMPLE Let $f:X \to Y$ be a surjective morphism of schemes.

(1) If f is quasi-compact, then f is locally quasi-compact.

(2) If f is open, then f is locally quasi-compact.

Given a scheme X, let κ_X be comprised of the collections $\{g_i:Y_i \to X\}$ such that $\forall\ i$, g_i is flat, $\cup\, g_i(Y_i) = X$, and $\coprod_i Y_i \to X$ is locally quasi-compact -- then κ is a Grothendieck coverage, the <u>fpqc coverage</u>.

[Note: Like its predecessors, the fpqc coverage is a Grothendieck pretopology with identities.]

18.21 LEMMA The fppf coverage is subordinate to the fpqc coverage.

[A flat morphism locally of finite presentation is open.]

18.22 THEOREM The fpqc coverage is subcanonical.

Therefore

$$18.22 \Rightarrow 18.18.$$

18.23 REMARK The coverage κ that assigns to each scheme X the collections

$\{g_i:Y_i \to X\}$ such that $\forall$ i, g_i is flat and $\cup\, g_i(Y_i) = X$ is not subcanonical.

Returning to the generalities, let again $\underline{C}$ be a locally small category.

18.24 **LEMMA** Suppose that κ is a subcanonical covering function -- then $\forall$ $X \in \text{Ob}\,\underline{C}$, the induced covering function $\bar{\kappa}$ on $\underline{C}/X$ is subcanonical.

18.25 EXAMPLE Take $\underline{C} = \underline{\text{TOP}}$, let κ be the open subset coverage, and fix $X \in \text{Ob}\,\underline{C}$ -- then

$$\underline{\text{Sh}}_{\kappa}(\underline{O}(X)) = \underline{\text{Sh}}(X)$$

and the inclusion $\underline{O}(X) \to \underline{\text{TOP}}/X$ induces an arrow

$$R:\underline{\text{Sh}}_{\bar{\kappa}}(\underline{\text{TOP}}/X) \to \underline{\text{Sh}}(X)$$

of restriction. On the other hand, there is also an arrow

$$P:\underline{\text{Sh}}(X) \to \underline{\text{Sh}}_{\bar{\kappa}}(\underline{\text{TOP}}/X)$$

of prolongment and (P,R) is an adjoint pair.

§19. *PRESITES*

19.1 DEFINITION A presite is a pair $(\underline{C},\kappa)$, where $\underline{C}$ is a small category and κ is a covering function which is a Grothendieck pretopology with identities whose underlying coverage is a Grothendieck coverage (cf. 12.21).

Explicated:

19.1 DEFINITION (bis) A presite is a pair $(\underline{C},\kappa)$, where $\underline{C}$ is a small category and κ is a covering function subject to the following assumptions.

(1) $\forall\ X \in \text{Ob}\ \underline{C},\ \{\text{id}_X:X \rightarrow X\} \in \kappa_X$.

(2) $\forall\ X \in \text{Ob}\ \underline{C},\ \forall\ \mathcal{C} \in \kappa_X, \forall\ g:Y \rightarrow X$ in $\mathcal{C}$, and $\forall\ f':X' \rightarrow X$, there is a pullback square

$$\begin{tikzcd} X' \times_X Y \arrow[r] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "f'"'] & X \end{tikzcd}$$

such that the covering

$$\{X' \times_X Y \xrightarrow{g'} X':g \in \mathcal{C}\}$$

belongs to $\kappa_{X'}$.

(3) $\forall\ X \in \text{Ob}\ \underline{C},\ \forall\ \mathcal{C} \in \kappa_X,\ \forall\ g:Y \rightarrow X$ in $\mathcal{C}$, and $\forall\ \mathcal{C}_g \in \kappa_Y$,

$$\bigcup_{g \in \mathcal{C}} g \circ \mathcal{C}_g = \{g \circ h:g \in \mathcal{C}\ \&\ h \in \mathcal{C}_g\}\ (Z \xrightarrow{h} Y \xrightarrow{g} X)$$

belongs to κ_X.

[Note: Here, of course, it is understood that $\forall\, X \in \text{Ob}\, \underline{C}$, κ_X is a set of subsets of $\text{Ob}\, \underline{C}/X$.]

19.2 THEOREM Suppose that $(\underline{C},\kappa)$ is a presite — then

$$\underline{\text{Sh}}_{J(\kappa)}(\underline{C}) = \underline{\text{Sh}}_{\kappa}(\underline{C}) \quad (\text{cf. } 18.11)$$

and the elements of $\underline{\text{Sh}}_{\kappa}(\underline{C})$ are characterized by the equalizer diagram figuring in 18.12.

19.3 EXAMPLE Take $\underline{C} = \underline{O}(X)$, X a topological space (cf. 11.18) and define the covering function κ as there -- then the pair $(\underline{C},\kappa)$ is a presite and $J(\kappa)$ is the Grothendieck topology τ on $\underline{O}(X)$ per 12.16. And a functor $F:\underline{C}^{\text{OP}} \to \underline{\text{SET}}$ is a κ-sheaf iff for any subset $U \subset X$, any open covering $U = \bigcup\limits_{i\in I} U_i$, and any collection $s_i \in FU_i$ $(i \in I)$ such that $\forall\, i,j \in I$,

$$s_i|U_i \cap U_j = s_j|U_i \cap U_j,$$

there exists a unique $s \in FU$ such that $s_i = s|U_i$ $\forall\, i \in I$, or, equivalently, the diagram

$$FU \longrightarrow \prod_i FU_i \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer diagram.

[Note: The empty covering of the empty set is admissible. Suppose that it is excluded (retaining, however, $\text{id}_\emptyset:\emptyset \to \emptyset$) -- then the result is another presite $(\underline{C},\kappa')$ but now $\underline{\text{Sh}}_{J(\kappa')}(\underline{C})$ is $\underline{\text{Sh}}(X \coprod \{*\})$, the open subsets of $X \coprod \{*\}$ being the empty set and any set of the form $U \cup \{*\}$ with $U \subset X$ open. For instance, consider the case when X is a singleton -- then $X \coprod \{*\}$ has two points, the underlying topological space is Sierpinski space, and $\underline{\mathrm{Sh}}_{J(\kappa')}(\underline{C})$ is equivalent to the arrow category $\underline{\mathrm{SET}}(\rightarrow)$.]

19.4 DEFINITION Let $(\underline{C},\kappa)$, $(\underline{C}',\kappa')$ be presites -- then a functor $\Phi:\underline{C} \rightarrow \underline{C}'$ is <u>geometric</u> provided the following conditions are satisfied.

(1) $\forall\ X \in \mathrm{Ob}\ \underline{C},\ \forall\ \mathcal{C} \in \kappa_X$,

$$\Phi \circ \mathcal{C} \in (\mathrm{sat}\ \kappa')_{\Phi X}.$$

(2) $\forall\ X \in \mathrm{Ob}\ \underline{C},\ \forall\ \mathcal{C} \in \kappa_X,\ \forall\ g:Y \rightarrow X$ in $\mathcal{C}$, and $\forall\ f':X' \rightarrow X$, the canonical arrow

$$\Phi(X' \times_X Y) \rightarrow \Phi X' \times_{\Phi X} \Phi Y$$

is an isomorphism.

<u>N.B.</u> The first condition is equivalent to requiring that $\Phi \circ \mathcal{C}$ has a refinement in κ' (cf. 11.9).

19.5 EXAMPLE Take $\underline{C} = \underline{C}'$ -- then $\mathrm{id}_{\underline{C}}$ is geometric iff $\kappa \leq \kappa'$ (cf. 11.6 (with the roles of κ and κ' reversed)).

19.6 NOTATION <u>PRESITE</u> is the locally small category whose objects are the presites and whose morphisms are the geometric functors.

[Note: <u>PRESITE</u> is a locally small large category.]

19.7 LEMMA Let $(\underline{C},\kappa)$, $(\underline{C}',\kappa')$ be presites and suppose that $\Phi:\underline{C} \rightarrow \underline{C}'$ is a geometric functor. Let F' be a κ'-sheaf -- then $F' \circ \Phi$ is a κ-sheaf.

PROOF Let $\mathcal{C}$ be a covering in κ -- then $\Phi \circ \mathcal{C}$ has a refinement in κ', hence F' has the sheaf property w.r.t. $\Phi \circ \mathcal{C}$ (cf. 18.8). Assuming that $\mathcal{C} = \{Y_i \xrightarrow{g_i} X:$

$i \in I\}$, where I is a set, this means that the diagram

$$F'\Phi X \longrightarrow \prod_i F'\Phi Y_i \rightrightarrows \prod_{i,j} F'(\Phi Y_i \times_{\Phi X} \Phi Y_j)$$

is an equalizer diagram in $\underline{SET}$. But

$$\Phi(Y_i \times_X Y_j) \approx \Phi Y_i \times_{\Phi X} \Phi Y_j$$

$\Rightarrow$

$$F' \circ \Phi(Y_i \times_X Y_j) \approx F'(\Phi Y_i \times_{\Phi X} \Phi Y_j),$$

thus it remains only to quote 18.12.

A functor $\Phi:\underline{C} \to \underline{C}'$ determines a functor $\Phi^{OP}:\underline{C}^{OP} \to (\underline{C}')^{OP}$, from which an induced functor

$$(\Phi^{OP})^*:[(\underline{C}')^{OP},\underline{SET}] \to [\underline{C}^{OP},\underline{SET}],$$

i.e.,

$$(\Phi^{OP})^*:\hat{\underline{C}}' \to \hat{\underline{C}}.$$

Assume now that $(\underline{C},\kappa)$, $(\underline{C}',\kappa')$ are presites and that $\Phi:\underline{C} \to \underline{C}'$ is a geometric functor -- then in 19.7, it is officially a question of

$$F' \circ \Phi^{OP} \equiv (\Phi^{OP})^*F'$$

rather than $F' \circ \Phi$. Agreeing to abbreviate $(\Phi^{OP})^*$ to Φ^*, there is an induced functor

$$\underline{Sh}\ \Phi^*:\underline{Sh}_{\kappa'}(\underline{C}') \to \underline{Sh}_{\kappa}(\underline{C})$$

and a commutative diagram

$$\begin{array}{ccc} \hat{\underline{C}}' & \xrightarrow{\Phi^*} & \hat{\underline{C}} \\ \uparrow{\scriptstyle \iota_{\kappa'}} & & \uparrow{\scriptstyle \iota_{\kappa}} \\ \underline{Sh}_{\kappa'}(\underline{C}') & \xrightarrow[\underline{Sh}\ \Phi^*]{} & \underline{Sh}_{\kappa}(\underline{C}). \end{array}$$

19.8 EXAMPLE Let X,Y be topological spaces and let $f:X \to Y$ be a continuous function. Define κ as in 11.18 (per X or Y) — then there are presites

$$\begin{cases} (\underline{O}(X),\kappa) \text{ with } \underline{Sh}_{\kappa}(\underline{O}(X)) = \underline{Sh}(X) \\ (\underline{O}(Y),\kappa) \text{ with } \underline{Sh}_{\kappa}(\underline{O}(Y)) = \underline{Sh}(Y). \end{cases} \quad \text{(cf. 15.25)}$$

In addition, the functor $f^{-1}:\underline{O}(Y) \to \underline{O}(X)$ is geometric and $\forall\ F \in \underline{Sh}(X)$,

$$F \circ (f^{-1})^{OP} = f_*F,$$

where

$$(f_*F)V = F(f^{-1}V).$$

19.9 NOTATION Given a presite $(\underline{C},\kappa)$, $J(\kappa)$ is a Grothendieck topology and

$$\underline{Sh}_{J(\kappa)}(\underline{C}) = \underline{Sh}_{\kappa}(\underline{C}) \quad \text{(cf. 18.11).}$$

Write ι_{κ} $(\equiv \iota_{J(\kappa)})$ for the inclusion $\underline{Sh}_{\kappa}(\underline{C}) \to \hat{\underline{C}}$ and denote its left adjoint by $\underline{a}_{\kappa}$ $(\equiv \underline{a}_{J(\kappa)})$ (cf. 15.10).

Let $(\underline{C},\kappa)$, $(\underline{C}',\kappa')$ be presites and suppose that $\Phi:\underline{C} \to \underline{C}'$ is a geometric functor — then by the theory of Kan extensions, Φ^* has a left adjoint $\Phi_!:\hat{\underline{C}} \to \hat{\underline{C}}'$.

19.10 LEMMA The composite

$$\underline{\mathrm{Sh}}_{\kappa}(\underline{C}) \xrightarrow{\iota_{\kappa}} \hat{\underline{C}} \xrightarrow{\Phi_!} \hat{\underline{C}}' \xrightarrow{\underline{a}_{\kappa'}} \underline{\mathrm{Sh}}_{\kappa'}(\underline{C}')$$

is a left adjoint for

$$\underline{\mathrm{Sh}}\ \Phi^*:\underline{\mathrm{Sh}}_{\kappa'}(\underline{C}') \to \underline{\mathrm{Sh}}_{\kappa}(\underline{C}).$$

PROOF If F is a κ-sheaf and F' is a κ'-sheaf, then

$$\mathrm{Mor}(\underline{a}_{\kappa'} \circ \Phi_! \circ \iota_{\kappa}F,F')$$

$$\approx \mathrm{Mor}(\iota_{\kappa'} \circ \underline{a}_{\kappa'} \circ \Phi_! \circ \iota_{\kappa}F,\iota_{\kappa'}F')$$

$$\approx \mathrm{Mor}(\underline{a}_{\kappa'} \circ \iota_{\kappa'} \circ \underline{a}_{\kappa'} \circ \Phi_! \circ \iota_{\kappa}F,F')$$

$$\approx \mathrm{Mor}(\underline{a}_{\kappa'} \circ \Phi_! \circ \iota_{\kappa}F,F')$$

$$\approx \mathrm{Mor}(\Phi_! \circ \iota_{\kappa}F,\iota_{\kappa'}F')$$

$$\approx \mathrm{Mor}(\iota_{\kappa}F,\Phi^* \circ \iota_{\kappa'}F')$$

$$\approx \mathrm{Mor}(\iota_{\kappa}F,\iota_{\kappa} \circ \underline{\mathrm{Sh}}\ \Phi^*F')$$

$$\approx \mathrm{Mor}(F,\underline{\mathrm{Sh}}\ \Phi^*F').$$

19.11 REMARK The pair

$$(\underline{a}_{\kappa'} \circ \Phi_! \circ \iota_{\kappa}, \underline{\mathrm{Sh}}\ \Phi^*)$$

defines a geometric morphism

$$\underline{\mathrm{Sh}}_{\kappa'}(\underline{C}') \to \underline{\mathrm{Sh}}_{\kappa}(\underline{C})$$

if in addition $\underline{a}_{\kappa'} \circ \Phi_! \circ \iota_{\kappa}$ preserves finite limits.

19.12 EXAMPLE Consider the setup of 19.8. Dictionary:

$$\begin{cases} f^{-1} \longleftrightarrow \Phi \\ f_* \longleftrightarrow \underline{\mathrm{Sh}}\ \Phi^* \\ f^* \longleftrightarrow \underline{a}_{\kappa'} \circ \Phi_! \circ \iota_\kappa. \end{cases}$$

In traditional terminology:

$$\begin{cases} f_* = \text{direct image} \\ f^* = \text{inverse image.} \end{cases}$$

[Note: The pair (f^*,f_*) defines a geometric morphism $\underline{\mathrm{Sh}}(X) \to \underline{\mathrm{Sh}}(Y)$.]

19.13 LEMMA There is a 2-functor

$$\underline{\mathrm{Sh}}:\underline{\mathrm{PRESITE}}^{\mathrm{OP}} \to 2\text{-}\mathbf{CAT}$$

which on objects sends $(\underline{C},\kappa)$ to $\underline{\mathrm{Sh}}_\kappa(\underline{C})$.

<u>N.B.</u> It then makes sense to form

$$\mathrm{gro}_{\underline{\mathrm{PRESITE}}}\ \underline{\mathrm{Sh}} \quad \text{(cf. 7.7).}$$

19.14 EXAMPLE Take the data as in 19.8 -- then there is a functor

$$\underline{\mathrm{TOP}}^{\mathrm{OP}} \to \underline{\mathrm{PRESITE}}$$

which on objects sends X to $(\underline{O}(X),\kappa)$. From here, pass to opposites and postcompose with $\underline{\mathrm{Sh}}$ to get a 2-functor

$$\underline{\mathrm{TOP}} \longrightarrow \underline{\mathrm{PRESITE}}^{\mathrm{OP}} \xrightarrow{\underline{\mathrm{Sh}}} 2\text{-}\mathbf{CAT}$$

which on objects sends X to $\underline{\mathrm{Sh}}(X)$. One may then consider its Grothendieck opconstruction ... .

§20. *INVERSE IMAGES*

Let $P:\underline{E} \to \underline{B}$ be a fibration. Suppose that κ is a covering function on $\underline{B}$ -- then its <u>inverse image</u> $P^{-1}\kappa$ is the covering function on $\underline{E}$ specified by the following procedure. Let $X' \in \mathrm{Ob}\,\underline{E}$ and let $\{g:B \to PX'\} \in \kappa_{PX'}$. For each g, choose a horizontal morphism $u:X \to X'$ such that $Pu = g$ -- then the class $\{u:X \to X'\}$ is a covering of X'. One then takes for $(P^{-1}\kappa)_{X'}$ the conglomerate of all such coverings of X'.

20.1 LEMMA If κ is a coverage, then $P^{-1}\kappa$ is a coverage.

20.2 LEMMA If κ is a Grothendieck coverage, then $P^{-1}\kappa$ is a Grothendieck coverage.

PROOF Referring to 11.16, take $X' \in \mathrm{Ob}\,\underline{E}$, let $C \in (P^{-1}\kappa)_{X'}$, take $u:X \to X'$ in C, and let $f:Y \to X'$ -- then the problem is to construct a pullback

$$\begin{tikzcd} Y \times_{X'} X \arrow[r] \arrow[d, "v"'] & X \arrow[d, "u"] \\ Y \arrow[r, "f"'] & X' \end{tikzcd}$$

of u along f such that the covering

$$\{Y \times_{X'} X \xrightarrow{v} Y : u \in C\}$$

belongs to $(P^{-1}\kappa)_Y$. To this end, pass to B and form $PY \times_{PX'} B$ per the assumption on κ:

$$\begin{tikzcd} PY \times_{PX'} B \arrow[r, "pr_B"] \arrow[d, "h"'] & B = PX \arrow[d, "g"] \\ PY \arrow[r, "Pf"'] & PX' \end{tikzcd} \quad .$$

Choose a horizontal $v:Z \to Y$ such that $Pv = h$, hence $PZ = PY \times_{PX'} B$, the claim being that Z is a pullback of u along f. The first step in the verification is to find a morphism $k:Z \to X$ rendering the diagram

$$
\begin{tikzcd}
Z \arrow[r, "k"] \arrow[d, "v"'] & X \arrow[d, "u"] \\
Y \arrow[r, "f"'] & X'
\end{tikzcd}
$$

commutative. So consider

$$
\overbrace{Z \dashrightarrow X \xrightarrow[u]{} X'}^{f \circ v}, \quad \overbrace{PZ \xrightarrow[pr_B]{} PX \xrightarrow[Pu = g]{} PX'}^{P(f \circ v)}.
$$

Then

$$
P(f \circ v) = Pf \circ Pv.
$$

On the other hand,

$$
Pu \circ pr_B = g \circ pr_B = Pf \circ h = Pf \circ Pv.
$$

Accordingly, since u is horizontal, there exists a unique morphism $k:Z \to X$ such that $Pk = pr_B$ and $u \circ k = f \circ v$. There remains the universality of Z: If

$$
\begin{bmatrix} \tilde{k}:\tilde{Z} \to X \\ \tilde{v}:\tilde{Z} \to Y \end{bmatrix}
$$

subject to $u \circ \tilde{k} = f \circ \tilde{v}$, then there is a unique $\phi:\tilde{Z} \to Z$ such that

$$
\begin{bmatrix} k \circ \phi = \tilde{k} \\ v \circ \phi = \tilde{v}. \end{bmatrix}
$$

<u>Existence of ϕ</u> Since $PZ = PY \times_{PX'} B$ is a pullback, there is a unique $\psi:P\tilde{Z} \to PZ$

such that

$$\left[\begin{array}{l} pr_B \circ \psi \ (= Pk \circ \psi) = P\tilde{k} \\ \quad h \circ \psi \ (= Pv \circ \psi) = P\tilde{v}. \end{array}\right.$$

Bearing in mind that v is horizontal, consider

$$\overbrace{\tilde{Z} \dashrightarrow Z \xrightarrow[v]{} Y}^{\tilde{v}}, \quad \overbrace{P\tilde{Z} \xrightarrow[\psi]{} PZ \xrightarrow[Pv]{} PY}^{P\tilde{v}}.$$

Then

$$P\tilde{v} = Pv \circ \psi,$$

which implies that there exists a unique morphism $\phi : \tilde{Z} \to Z$ such that $P\phi = \psi$ and $v \circ \phi = \tilde{v}$. To check that $k \circ \phi = \tilde{k}$, consider

$$\overbrace{\tilde{Z} \dashrightarrow X \xrightarrow[u]{} X'}^{u \circ \tilde{k}}, \quad \overbrace{P\tilde{Z} \xrightarrow[P\tilde{k}]{} PX \xrightarrow[Pu]{} PX'}^{P(u \circ \tilde{k})}.$$

Because u is horizontal, there is a unique morphism $\tilde{\ell} : \tilde{Z} \to X$ such that $P\tilde{\ell} = P\tilde{k}$ and $u \circ \tilde{\ell} = u \circ \tilde{k}$. Obviously, then, $\tilde{\ell} = \tilde{k}$. But meanwhile,

$$v \circ \phi = \tilde{v} \Rightarrow f \circ v \circ \phi = f \circ \tilde{v} = u \circ \tilde{k}.$$

I.e.:

$$u \circ k \circ \phi = u \circ \tilde{k}.$$

And

$$P(k \circ \phi) = Pk \circ P\phi = pr_B \circ \psi = P\tilde{k}.$$

Therefore $k \circ \phi = \tilde{k}$.

<u>Uniqueness of ϕ</u> If $\phi_1, \phi_2 : \tilde{Z} \to Z$ both satisfy the requisite conditions, then

$$\left[\begin{array}{l} P\phi_1 = \psi \\ \\ P\phi_2 = \psi \end{array}\right. \text{ and } \left[\begin{array}{l} v \circ \phi_1 = \tilde{v} \\ \\ v \circ \phi_2 = \tilde{v} \end{array}\right., \text{ thus } \phi_1 = \phi_2 \text{ (cf. supra).}$$

20.3 REMARK It is not assumed that $\underline{B}$ or $\underline{E}$ has pullbacks but merely certain pullbacks as per the definition of Grothendieck coverage.

20.4 LEMMA If κ is a pretopology, then $P^{-1}\kappa$ is a pretopology.

20.5 LEMMA If κ is a Grothendieck pretopology, then $P^{-1}\kappa$ is a Grothendieck pretopology.

20.6 LEMMA If κ is a pretopology (or a Grothendieck pretopology) with identities, then $P^{-1}\kappa$ is a pretopology (or a Grothendieck pretopology) with identities.

20.7 REMARK Ignoring issues of size, it follows that if $(\underline{B},\kappa)$ is a "presite", then $(\underline{E},P^{-1}\kappa)$ is a "presite" (cf. 19.1 and 19.1 (bis)).

§21. *ALGEBRAIC STRUCTURES*

Let $(\underline{C},\kappa)$ be a presite.

21.1 LEMMA Let $F:\underline{C}^{OP} \to \underline{SET}$ be a functor -- then F is a κ-sheaf iff $\forall\ S \in \text{Ob}\ \underline{SET}$, the presheaf $X \to \text{Mor}(S,FX)$ is a κ-sheaf.

21.2 DEFINITION Let $\underline{A}$ be a locally small category with products -- then a functor $F:\underline{C}^{OP} \to \underline{A}$ is a $\underline{\kappa\text{-sheaf with values in } \underline{A}}$ if $\forall\ A \in \text{Ob}\ \underline{A}$, the presheaf $X \to \text{Mor}(A,FX)$ is a κ-sheaf.

Write $\underline{Sh}_\kappa(\underline{C},\underline{A})$ for the full subcategory of $[\underline{C}^{OP},\underline{A}]$ whose objects are the κ-sheaves with values in $\underline{A}$ (thus

$$\underline{Sh}_\kappa(\underline{C}) \equiv \underline{Sh}_\kappa(\underline{C},\underline{SET})).$$

21.3 REMARK Let $\mathcal{C} = \{Y_i \xrightarrow{g} X : i \in I\} \in \kappa_X$, where I is a set -- then for any functor $F:\underline{C}^{OP} \to \underline{A}$, the diagram

$$FX \longrightarrow \prod_i FY_i \rightrightarrows \prod_{i,j} F(Y_i \times_X Y_j)$$

is an equalizer diagram in $\underline{A}$ iff $\forall\ A \in \text{Ob}\ \underline{A}$, the diagram

$$\text{Mor}(A,FX) \longrightarrow \prod_i \text{Mor}(A,FY_i) \rightrightarrows \prod_{i,j} \text{Mor}(A,F(Y_i \times_X Y_j))$$

is an equalizer diagram in $\underline{SET}$.

The central problem at this juncture is to find conditions on $\underline{A}$ which suffice to ensure that the inclusion

$$\iota_\kappa:\underline{Sh}_\kappa(\underline{C},\underline{A}) \to [\underline{C}^{OP},\underline{A}]$$

admits a left adjoint

$$\underline{a}_\kappa:[\underline{C}^{OP},\underline{A}] \to \underline{Sh}_\kappa(\underline{C},\underline{A})$$

that preserves finite limits (cf. 15.10 for the case $\underline{A} = \underline{SET}$).

• Assume: $\underline{A}$ is a construct, i.e., there is a faithful functor $U:\underline{A} \to \underline{SET}$ which, in addition, reflects isomorphisms.

21.4 EXAMPLE $\underline{HTOP}$ is not a construct. $\underline{TOP}$ is a construct but the forgetful functor $U:\underline{TOP} \to \underline{SET}$ does not reflect isomorphisms.

One then imposes the following conditions on the pair $(\underline{A},U)$.

(1) $\underline{A}$ is complete and U is limit preserving.

(2) $\underline{A}$ has filtered colimits and U is filtered colimit preserving.

21.5 EXAMPLE Taking for U the forgetful functor, these conditions are met by the category of abelian groups, groups, commutative rings, rings, modules over a fixed ring, vector spaces over a fixed field, ...

[Note: Neither coproducts nor coequalizers are preserved by U.]

21.6 LEMMA Let $F:\underline{C}^{OP} \to \underline{A}$ be a functor -- then F is a κ-sheaf with values in $\underline{A}$ iff $U \circ F$ is a κ-sheaf.

21.7 REMARK The forgetful functor $U:\underline{TOP} \to \underline{SET}$ preserves limits and colimits. On the other hand, it is not difficult to exhibit a presite $(\underline{O}(X),\kappa)$ (cf. 19.8) and a functor $F:\underline{O}(X)^{OP} \rightarrow \underline{TOP}$ such that $U \circ F$ is a κ-sheaf but F is not a κ-sheaf with values in $\underline{TOP}$.

[Note: This does not contradict 21.6 (cf. 21.4).]

21.8 THEOREM The inclusion

$$\iota_\kappa:\underline{Sh}_\kappa(\underline{C},\underline{A}) \rightarrow [\underline{C}^{OP},\underline{A}]$$

admits a left adjoint

$$\underline{a}_\kappa:[\underline{C}^{OP},\underline{A}] \rightarrow \underline{Sh}_\kappa(\underline{C},\underline{A})$$

that preserves finite limits.

Implicit in the proof is the fact that for any functor $F:\underline{C}^{OP} \rightarrow \underline{A}$,

$$\underline{a}_\tau(U \circ F) = U \circ \underline{a}_\tau F$$

thus there is a commutative diagram

$$\begin{tikzcd} \underline{Sh}_\kappa(\underline{C},A) \arrow[r, shift left, "\iota_\kappa"] \arrow[d, "U_*"'] & {[\underline{C}^{OP},\underline{A}]} \arrow[l, shift left, "\underline{a}_\kappa"] \arrow[d, "U_*"] \\ \underline{Sh}_\kappa(\underline{C}) \arrow[r, shift left, "\iota_\kappa"] & {[\underline{C}^{OP},\underline{SET}]} \arrow[l, shift left, "\underline{a}_\kappa"] \end{tikzcd}$$

Here U_* is given on objects by

$$U_*F = U \circ F$$

and on morphisms by

$$(U_*\Xi)_X = U\Xi_X.$$

APPENDIX

Let $\underline{C}$ be a category.

NOTATION $\underline{\text{SIC}}$ is the functor category $[\underline{\Delta}^{OP},\underline{C}]$ and a simplicial object in $\underline{C}$ is an object in $\underline{\text{SIC}}$.

In particular:

$$\underline{\text{SISET}} = \hat{\underline{\Delta}}$$

is the category of simplicial sets.

Let $\underline{C}$ be a small category -- then

$$\begin{aligned}\underline{\text{SI}\hat{\text{C}}} &= [\underline{\Delta}^{OP},[\underline{C}^{OP},\underline{\text{SET}}]] \\ &\approx [(\underline{C} \times \underline{\Delta})^{OP},\underline{\text{SET}}] \\ &\approx [\underline{C}^{OP},[\underline{\Delta}^{OP},\underline{\text{SET}}]] \\ &= [\underline{C}^{OP},\underline{\text{SISET}}],\end{aligned}$$

the objects of the latter being termed simplicial presheaves.

Suppose that $(\underline{C},\kappa)$ is a presite.

DEFINITION The objects of $\underline{\text{SISh}}_{\kappa}(\underline{C})$ are called simplicial κ-sheaves.

The product $\underline{C} \times \underline{\Delta}$ is a presite, viz.

$$\kappa_{X \times [n]} = i_n \kappa_X,$$

where

$$i_n:\underline{C} \to \underline{C} \times \underline{\Delta}$$

is the inclusion

$$\begin{cases} i_n X = X \times [n] \\ i_n f = f \times \mathrm{id}_{[n]}. \end{cases}$$

It thus makes sense to form $\underline{\mathrm{Sh}}_\kappa(\underline{C} \times \underline{\Delta})$.

LEMMA We have

$$\underline{\mathrm{SISh}}_\kappa(\underline{C}) \approx \underline{\mathrm{Sh}}_\kappa(\underline{C} \times \underline{\Delta}).$$

All the basic results on presheaves and κ-sheaves of sets extend without essential change to simplicial presheaves and simplicial κ-sheaves.

<u>N.B.</u> It is customary to use the same symbols $\begin{cases} \underline{a}_\kappa \\ \iota_\kappa \end{cases}$ for the induced adjoint pair

$$\begin{cases} \underline{\mathrm{SI}\hat{C}} \longrightarrow \underline{\mathrm{SISh}}_\kappa(\underline{C}) \\ \underline{\mathrm{SISh}}_\kappa(\underline{C}) \longrightarrow \underline{\mathrm{SI}\hat{C}} \quad . \end{cases}$$

LEMMA $\underline{\mathrm{Sh}}_\kappa(\underline{C},\underline{\mathrm{SISET}})$ can be identified with

$$\underline{\mathrm{SISh}}_\kappa(\underline{C}) \approx \underline{\mathrm{Sh}}_\kappa(\underline{C} \times \underline{\Delta}).$$

PROOF A simplicial presheaf $F:\underline{C}^{\mathrm{OP}} \to \underline{\mathrm{SISET}}$ determines a sequence $\{F_n\}$ of functors $F_n:\underline{C}^{\mathrm{OP}} \to \underline{\mathrm{SET}}$ via the prescription $F_n X = (FX)([n])$ and F is a simplicial κ-sheaf iff $\forall$ n, F_n is a κ-sheaf. Assume now that $F:\underline{C}^{\mathrm{OP}} \to \underline{\mathrm{SISET}}$ is a κ-sheaf with values in $\underline{\mathrm{SISET}}$ — then for every simplicial set S, the presheaf $X \to \mathrm{Mor}(S,FX)$

is a κ-sheaf. In particular: $\forall$ n, the presheaf

$$X \to \mathrm{Mor}(\Delta[n],FX)$$

is a κ-sheaf. But

$$\mathrm{Mor}(\Delta[n],FX) \approx (FX)([n]) = F_n X,$$

so $\forall$ n, F_n is a κ-sheaf, i.e., F is a simplicial κ-sheaf. Conversely, if F is a simplicial κ-sheaf, then F is a κ-sheaf with values in $\underline{\mathrm{SISET}}$. To see this, given a simplicial set S, write

$$S = \mathrm{colim}_i \ \Delta[n_i].$$

Then

$$\begin{aligned} \mathrm{Mor}(S,FX) &= \mathrm{Mor}(\mathrm{colim}_i \ \Delta[n_i],FX) \\ &\approx \lim_i \ \mathrm{Mor}(\Delta[n_i],FX) \\ &\approx \lim_i \ F_{n_i} X. \end{aligned}$$

And $\lim_i F_{n_i} \in \mathrm{Ob}\ \underline{\mathrm{Sh}}_\kappa(\underline{C})$ is computed levelwise.

§22. $\underline{A}$ SPACES

Let $\underline{A}$ be a locally small category with products.

22.1 NOTATION Given a topological space X, write $\underline{Sh}(X,\underline{A})$ for the category whose objects are the κ-sheaves with values in $\underline{A}$.

[Note: Here κ is taken per 11.18, so

$$\underline{Sh}(X,\underline{A}) = \underline{Sh}_{\kappa}(\underline{O}(X),\underline{A}).]$$

N.B. Therefore

$$\underline{Sh}(X) = \underline{Sh}(X,\underline{SET}).$$

22.2 EXAMPLE For any κ-sheaf F on X with values in $\underline{A}$, $F\emptyset$ is a final object in $\underline{A}$.

22.3 LEMMA Suppose that X is a one point space -- then the functor

$$\underline{Sh}(X,\underline{A}) \xrightarrow{\text{ev}} \underline{A}$$

that sends F to FX is an equivalence of categories.

22.4 REMARK If X is a one point space, $[\underline{O}(X)^{OP},\underline{A}]$ can be identified with the arrow category $\underline{A}(\rightarrow)$. Fix a final object $*_{\underline{A}}$ in $\underline{A}$ -- then the functor $\underline{A} \rightarrow \underline{A}(\rightarrow)$ which sends an object A to the arrow $\underline{A} \xrightarrow{!} *_{\underline{A}}$ has a left adjoint, viz. dom.

22.5 LEMMA Let X,Y be topological spaces and let $f:X \rightarrow Y$ be a continuous function -- then there is an induced functor

$$f_*:\underline{Sh}(X,\underline{A}) \rightarrow \underline{Sh}(Y,\underline{A}) \quad \text{(cf. 19.8)}.$$

22.6 EXAMPLE Assuming that X is not empty, fix a point $x \in X$ and let $i_x:\{x\} \rightarrow X$

be the inclusion -- then there is an induced functor

$$(i_x)_*:\underline{Sh}(\{x\},\underline{A}) \to \underline{Sh}(X,\underline{A}).$$

Now choose a final object $*_{\underline{A}}$ in $\underline{A}$, from which an induced functor

$$\mathrm{Sky}_x:\underline{A} \to \underline{Sh}(X,\underline{A}),$$

where

$$\mathrm{Sky}_x(A)(U) = \begin{cases} A & (x \in U) \\ *_{\underline{A}} & (x \notin U). \end{cases}$$

22.7 LEMMA If $\underline{A}$ is cocomplete, then Sky_x admits a left adjoint

$$\underline{Sh}(X,\underline{A}) \to \underline{A},$$

the <u>stalk functor</u>.

PROOF Let $\underline{O}(X)_x$ be the subcategory of $\underline{O}(X)$ whose objects are the open subsets of X containing x -- then the inclusion $\iota_x:\underline{O}(X)_x \to \underline{O}(X)$ is geometric, hence there is an induced functor

$$\iota_x^*:\underline{Sh}_\kappa(\underline{O}(X),\underline{A}) \to \underline{Sh}_\kappa(\underline{O}(X)_x,\underline{A}).$$

This said, consider the composite

$$\underline{Sh}(X,\underline{A}) = \underline{Sh}_\kappa(\underline{O}(X),\underline{A}) \xrightarrow{\iota_x^*} \underline{Sh}_\kappa(\underline{O}(X)_\kappa,\underline{A}) \xrightarrow{\mathrm{colim}} \underline{A}.$$

22.8 DEFINITION An <u>$\underline{A}$ space</u> is a pair $(X,\mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a κ-sheaf with values in $\underline{A}$.

[Note: If $\underline{A}$ is cocomplete, the stalk of $\mathcal{O}_X$ at $x \in X$ is denoted by the symbol $\mathcal{O}_{X,x}$.]

$\underline{\mathrm{TOP}}_{\underline{A}}$ is the category whose objects are the $\underline{A}$ spaces and whose morphisms are the pairs

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y),$$

where $f:X \to Y$ is a continuous function, $f^\#:\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism in $\underline{\mathrm{Sh}}(Y,\underline{A})$, and $f_*\mathcal{O}_X = \mathcal{O}_X \circ (f^{-1})^{\mathrm{OP}}$.

[Note: The composition

$$(g,g^\#)\circ(f,f^\#)$$

of

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y) \xrightarrow{(g,g^\#)} (Z,\mathcal{O}_Z)$$

has first component $g \circ f$ and second component $g_*(f^\#) \circ g^\#$ $((g \circ f)_* = g_* \circ f_*)$. And $\mathrm{id}_{(X,\mathcal{O}_X)}$ is the arrow

$$(X,\mathcal{O}_X) \xrightarrow{(\mathrm{id}_X,\mathrm{id}_{\mathcal{O}_X})} (X,\mathcal{O}_X).]$$

N.B. Define a 2-functor $F:\underline{\mathrm{TOP}} \to 2\text{-}\mathbf{CAT}$ by sending X to $\underline{\mathrm{Sh}}(X,\underline{A})$ and $f:X \to Y$ to f_*. One can then introduce $\mathrm{gro}_{\underline{\mathrm{TOP}}}\, F$, the Grothendieck opconstruction on F. Thus its objects are the pairs $(X,\mathcal{O}_X)$, where $\mathcal{O}_X$ is a κ-sheaf with values in $\underline{A}$, and its morphisms are the pairs

$$(X,\mathcal{O}_X) \xrightarrow{(f,{}^\#f)} (Y,\mathcal{O}_Y),$$

where $f:X \to Y$ is a continuous function, ${}^\#f:f_*\mathcal{O}_X \to \mathcal{O}_Y$ is a morphism in $\underline{\mathrm{Sh}}(Y,\underline{A})$, and $f_*\mathcal{O}_X = \mathcal{O}_X \circ (f^{-1})^{OP}$. Here

$$(g,{}^{\#}g)\circ(f,{}^{\#}f) = (g \circ f,{}^{\#}g \circ g_*({}^{\#}f))$$

and

$$\mathrm{id}_{(X,\mathcal{O}_X)} = (\mathrm{id}_X,\mathrm{id}_{\mathcal{O}_X}).$$

Conclusion: ...?

22.9 EXAMPLE Take $\underline{A} = \underline{RNG}$ (cf. 11.26) — then $\underline{TOP}_{\underline{RNG}}$ is the category of <u>ringed spaces</u>.

If U is an open subset of X and if $i_U:U \to X$ is the inclusion, then

$$(i_U)_*:\underline{Sh}(U,\underline{A}) \to \underline{Sh}(X,\underline{A})$$

admits a left adjoint

$$(i_U)^*:\underline{Sh}(X,\underline{A}) \to \underline{Sh}(U,\underline{A}).$$

This is true without any additional assumptions on $\underline{A}$. To proceed in general, however, we shall suppose that $\underline{A}$ is complete and cocomplete and impose on $\underline{A}$ the conditions set forth in §21, thereby ensuring that 21.8 is in force, hence that

$$f_*:\underline{Sh}(X,\underline{A}) \to \underline{Sh}(Y,\underline{A})$$

has a left adjoint

$$f^*:\underline{Sh}(Y,\underline{A}) \to \underline{Sh}(X,\underline{A}) \quad \text{(cf. 19.12)},$$

so

$$\mathrm{Mor}(f^*\mathcal{O}_Y,\mathcal{O}_X) \approx \mathrm{Mor}(\mathcal{O}_Y,f_*\mathcal{O}_X),$$

with arrows of adjunction

$$\begin{cases} \mu_{\mathcal{O}_Y}:\mathcal{O}_Y \longrightarrow f_*f^*\mathcal{O}_Y \\ \nu_{\mathcal{O}_X}:f^*f_*\mathcal{O}_X \longrightarrow \mathcal{O}_X. \end{cases}$$

22.10 NOTATION Let $P_{\underline{A}}:\underline{TOP}_{\underline{A}} \rightarrow \underline{TOP}$ be the functor that sends $(X,\mathcal{O}_X)$ to X and $(f,f^{\#})$ to f.

22.11 LEMMA $P_{\underline{A}}$ is a fibration.

PROOF Given $(Y,\mathcal{O}_Y)$ and $f:X \rightarrow Y$, the morphism

$$(f,\mu_{\mathcal{O}_Y}):(X,f^*\mathcal{O}_Y) \rightarrow (Y,\mathcal{O}_Y)$$

is horizontal.

22.12 EXAMPLE Take $X = U$, $Y = X$, $f = i_U$ -- then $i_U^*\mathcal{O}_X = \mathcal{O}_X|U$ and

$$(i_U,\mu_{\mathcal{O}_X}):(U,\mathcal{O}_X|U) \rightarrow (X,\mathcal{O}_X)$$

is horizontal. Here

$$\mu_{\mathcal{O}_X}:\mathcal{O}_X \rightarrow i_*(\mathcal{O}_X|U)$$

at an open subset $V \subset X$ is computed by

$$\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$$

per $U \cap V \rightarrow V$.

Let

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^{\#})} (Y,\mathcal{O}_Y)$$

be a morphism of $\underline{A}$ spaces -- then $f^{\#}:\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism in $\underline{Sh}(Y,\underline{A})$, thus corresponds to a morphism $f_{\#}:f^*\mathcal{O}_Y \rightarrow \mathcal{O}_X$ in $\underline{Sh}(X,\underline{A})$ under the identification

$$\mathrm{Mor}(f^*\mathcal{O}_Y,\mathcal{O}_X) \approx \mathrm{Mor}(\mathcal{O}_Y,f_*\mathcal{O}_X).$$

[Note: The composite

$$f^*\mathcal{O}_Y \xrightarrow{f^*(f^\#)} f^*f_*\mathcal{O}_X \xrightarrow{\nu_{\mathcal{O}_X}} \mathcal{O}_X$$

is $f_\#$. Observe too that

$$(\mathrm{id}_X, f_\#):(X,\mathcal{O}_X) \longrightarrow (X, f^*\mathcal{O}_Y)$$

is a morphism of $\underline{A}$ spaces: $f^*\mathcal{O}_Y \xrightarrow{f_\#} (\mathrm{id}_X)_*\mathcal{O}_X = \mathcal{O}_X$

and the diagram

$$\begin{tikzcd}
(X,\mathcal{O}_X) \arrow[r, "{(f,f^\#)}"] \arrow[d, "{(\mathrm{id}_X,f_\#)}"'] & (Y,\mathcal{O}_Y) \arrow[d, equal] \\
(X,f^*\mathcal{O}_Y) \arrow[r, "{(f,\mu_{\mathcal{O}_Y})}"'] & (Y,\mathcal{O}_Y)
\end{tikzcd}$$

in $\underline{\mathrm{TOP}}_{\underline{A}}$ commutes.]

Consequently, at the level of stalks, $\forall\ x \in X$, there is a morphism

$$(f_\#)_x:(f^*\mathcal{O}_Y)_x \to \mathcal{O}_{X,x}$$

in $\underline{A}$.

22.13 LEMMA Fix $x \in X$ -- then the stalk functor at $f(x)$ is the composition $(i_x)^* \circ f^*$.

[The functor $(i_x)^* \circ f^*$ is a left adjoint for $f_* \circ (i_x)_* = (f \circ i_x)_* = (i_{f(x)})_*$.]

[Note: Technically,

$$\underline{\mathrm{Sh}}(X,\underline{A}) \xrightarrow{(i_x)^*} \underline{\mathrm{Sh}}(\{x\},\underline{A})$$

so "taking the stalk at x" is really $(i_x)^*$ modulo the equivalence

$$\underline{\mathrm{Sh}}(\{x\},\underline{A}) \longrightarrow \underline{A} \quad \text{(cf. 22.3).]}$$

22.14 APPLICATION $\forall\ x \in X$,

$$\mathcal{O}_{Y,f(x)} = (i_x)^*(f^*\mathcal{O}_Y) = (f^*\mathcal{O}_Y)_x.$$

In particular:

$$(f_*\mathcal{O}_X)_{f(x)} = (f^*f_*\mathcal{O}_X)_x.$$

Fix a one point space $*$ and consider $X \xrightarrow{!} *$ -- then

$$!_*:\underline{\mathrm{Sh}}(X,\underline{A}) \to \underline{\mathrm{Sh}}(*,\underline{A}).$$

Now postcompose $!_*$ with the equivalence $\underline{\mathrm{Sh}}(*,\underline{A}) \xrightarrow{\mathrm{ev}} \underline{A}$ of 22.3 to get a functor

$$\Gamma:\underline{\mathrm{Sh}}(X,\underline{A}) \to \underline{A},$$

the <u>global section functor</u>:

$$\Gamma F = FX.$$

[Note: If

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y)$$

is a morphism of $\underline{A}$ spaces, then

$$\Gamma\mathcal{O}_Y = \mathcal{O}_Y(Y) \xrightarrow{f^\#_Y} (f_*\mathcal{O}_X)(Y)$$

$$= \mathcal{O}_X(f^{-1}Y)$$

$$= \mathcal{O}_X(X) = \Gamma\mathcal{O}_X$$

is a morphism in $\underline{A}$.]

22.15 LEMMA The global section functor Γ is the restriction to $\underline{Sh}(X,\underline{A})$ of

$$\lim:[\underline{O}(X)^{OP},\underline{A}].$$

22.16 RAPPEL The functor

$$\lim:[\underline{O}(X)^{OP},\underline{A}] \to \underline{A}$$

is a right adjoint for the constant diagram functor

$$K:\underline{A} \to [\underline{O}(X)^{OP},\underline{A}].$$

Display the data:

$$\underline{A} \underset{\lim}{\overset{K}{\rightleftarrows}} [\underline{O}(X)^{OP},\underline{A}] \underset{\iota_\kappa}{\overset{\underline{a}_\kappa}{\rightleftarrows}} \underline{Sh}(X,\underline{A}).$$

Then a left adjoint for

$$\Gamma = \lim \circ \iota_\kappa$$

is

$$\Delta = \underline{a}_\kappa \circ K.$$

22.17 EXAMPLE Let A be a commutative ring with unit. Consider the ringed space $(\text{Spec } A,\mathcal{O}_A)$ -- then

$$\Gamma\mathcal{O}_A = \mathcal{O}_A(\text{Spec } A) \approx A.$$

[Note: Here $\mathcal{O}_A \equiv \mathcal{O}_{\text{Spec } A}$ is the structure sheaf of Spec A.]

22.18 REMARK Spec A = ∅ iff A = {0} (a zero ring). Of course, {0} is a final object in $\underline{\text{RNG}}$ and

$$\mathcal{O}_{\{0\}} \text{ Spec } \{0\} = \mathcal{O}_{\{0\}} \emptyset = \{0\}$$

in agreement with 22.2.

22.19 LEMMA The diagram

$$\begin{tikzcd}
\underline{\text{Sh}}(*,\underline{A}) \arrow[r, "!^*"] \arrow[d, "\text{ev}"'] & \underline{\text{Sh}}(X,\underline{A}) \arrow[d, equal] \\
\underline{A} \arrow[r, "\Delta"'] & \underline{\text{Sh}}(X,\underline{A})
\end{tikzcd}$$

commutes up to isomorphism:

$$!^* \approx \Delta \circ \text{ev}.$$

PROOF For any $\mathcal{O}_*$ and for any $\mathcal{O}_X$,

$$\begin{aligned}
\text{Mor}(!^*\mathcal{O}_*,\mathcal{O}_X) &\approx \text{Mor}(\mathcal{O}_*,!_*\mathcal{O}_X) \\
&\approx \text{Mor}(\mathcal{O}_**,(!_*\mathcal{O}_X)(*)) \\
&\approx \text{Mor}(\text{ev } \mathcal{O}_*,\mathcal{O}_X(X)) \\
&\approx \text{Mor}(\text{ev } \mathcal{O}_*,\Gamma\mathcal{O}_X) \\
&\approx \text{Mor}(\Delta \circ \text{ev } \mathcal{O}_*,\mathcal{O}_X).
\end{aligned}$$

§23. LOCALLY RINGED SPACES

Let $\underline{C}$ be a category.

23.1 DEFINITION A subcategory $\underline{D}$ of $\underline{C}$ is said to be <u>replete</u> if for any object X in $\underline{D}$ and for any isomorphism $f:X \to Y$ in $\underline{C}$, both Y and f are in $\underline{D}$.

[Note: If $\underline{D}$ is a full subcategory of $\underline{C}$, then the term is <u>isomorphism closed</u>. E.g.: Reflective subcategories are isomorphism closed.]

23.2 EXAMPLE Let $\underline{\text{LOC-RNG}}$ be the subcategory of $\underline{\text{RNG}}$ whose objects are the local rings and whose morphisms are the local homomorphisms -- then $\underline{\text{LOC-RNG}}$ is a replete (nonfull) subcategory of $\underline{\text{RNG}}$.

23.3 DEFINITION Let $\underline{C},\underline{C}'$ be categories -- then a functor $F:\underline{C} \to \underline{C}'$ is said to be <u>replete</u> if it has the isomorphism lifting property (cf. 1.23), i.e., if $\forall$ isomorphism $\psi:FX \to X'$ in $\underline{C}'$, $\exists$ an isomorphism $\phi:X \to Y$ in $\underline{C}$ such that $F\phi = \psi$ (so $FY=X'$).

[Note: One can thus say that a subcategory $\underline{D}$ of $\underline{C}$ is replete provided the inclusion functor $\underline{D} \to \underline{C}$ is replete.]

23.4 EXAMPLE A fibration $P:\underline{E} \to \underline{B}$ is replete (cf. 4.23).

23.5 LEMMA Let $F:(\underline{E},P) \to (\underline{E}',P')$ be a morphism in $\mathbf{CAT}/\underline{B}$, where $P:\underline{E} \to \underline{B}$, $P':\underline{E}' \to \underline{B}$ are fibrations -- then F is replete iff $\forall\ B \in \text{Ob}\ \underline{B}$, the functor $F_B:\underline{E}_B \to \underline{E}'_B$ is replete.

23.6 REMARK The fiberwise condition on F amounts to the assertion that if $\psi:FX \to X'$ is a vertical isomorphism in $\underline{E}'$, then there exists a vertical isomorphism $\phi:X \to Y$ in $\underline{E}$ such that $F\phi = \psi$ (so $FY = X'$).

23.7 DEFINITION A ringed space $(X,\mathcal{O}_X)$ is a locally ringed space if each stalk $\mathcal{O}_{X,x}$ is a local ring.

[Note: $\mathfrak{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$ and $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ is the residue field of $\mathcal{O}_{X,x}$.]

23.8 REMARK Consider the pair $(\emptyset, \mathcal{O}_\emptyset)$, where $\mathcal{O}_\emptyset\emptyset = \{0\}$ (a zero ring)(cf. 22.18) -- then there is no stalk and the local ring condition is vacuous, so $(\emptyset, \mathcal{O}_\emptyset)$ is a locally ringed space.

[Note: Zero rings are not local rings.]

Let $(X,\mathcal{O}_X)$, $(Y,\mathcal{O}_Y)$ be locally ringed spaces. Suppose that

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y)$$

is a morphism of ringed spaces -- then $(f,f^\#)$ is a morphism of locally ringed spaces if $\forall\ x \in X$, the ring homomorphism

$$(f_\#)_x:\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is local.

23.9 NOTATION Let

$$\underline{\text{LOC-TOP}_{\text{RNG}}}$$

be the subcategory of $\underline{\text{TOP}_{\text{RNG}}}$ (cf. 22.9) whose objects are the locally ringed spaces and whose morphisms are the morphisms of locally ringed spaces.

[Note: To verify closure under composition, recall that

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y) \xrightarrow{(g,g^\#)} (Z,\mathcal{O}_Z)$$

has first component $g \circ f$ and second component $g_*(f^\#) \circ g^\#$. And here

$$(g \circ f)^* \approx f^* \circ g^* \ (\ldots)$$

while

$$(g_*(f^\#) \circ g^\#)_\# = f_\# \circ f^*(g_\#),$$

i.e.,

$$f^*g^*\mathcal{O}_Z \xrightarrow{f^*(g_\#)} f^*\mathcal{O}_Y \xrightarrow{f_\#} \mathcal{O}_X.$$

So, $\forall\ x \in X$, the stalk homomorphism

$$((g_*(f^\#) \circ g^\#)_\#)_x$$

is the arrow

$$(f_\# \circ f^*(g_\#))_x$$

which when explicated is the composition

$$\mathcal{O}_{Z,g \circ f(x)} \xrightarrow{(g_\#)_{f(x)}} \mathcal{O}_{Y,f(x)} \xrightarrow{(f_\#)_x} \mathcal{O}_{X,x}$$

of two local homomorphisms, thus is a local homomorphism.]

The functor

$$P_{\underline{RNG}}:\underline{TOP}_{\underline{RNG}} \longrightarrow \underline{TOP} \qquad \text{(cf. 22.10)}$$

restricts to

$$\underline{LOC\text{-}TOP}_{\underline{RNG}},$$

call it $\underline{LOC}\text{-}P_{\underline{RNG}}$.

23.10 LEMMA $\underline{LOC}\text{-}P_{\underline{RNG}}$ is a fibration.

PROOF In the notation of the proof of 22.11, if $(Y,\mathcal{O}_Y)$ is a locally ringed space, then so is $(X,f^*\mathcal{O}_Y)$ $(\forall\ x \in X,\ (f^*\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)})$. Moreover,

$$(f,\mu_{\mathcal{O}_Y}):(X,f^*\mathcal{O}_Y) \to (Y,\mathcal{O}_Y)$$

is a morphism of locally ringed spaces:

$$\mu_{\mathcal{O}_Y} \in \mathrm{Mor}(\mathcal{O}_Y,f_*f^*\mathcal{O}_Y)$$

$$\Rightarrow$$

$$(\mu_{\mathcal{O}_Y})_\# \in \mathrm{Mor}(f^*\mathcal{O}_Y,f^*\mathcal{O}_Y)$$

or still,

$$(\mu_{\mathcal{O}_Y})_\# = \mathrm{id}_{f^*\mathcal{O}_Y}.$$

In addition, it is horizontal when viewed from the perspective of $\underline{\underline{\mathrm{TOP}}}_{\underline{\mathrm{RNG}}}$. Consider now a setup

$$\begin{cases}(Z,\mathcal{O}_Z) \dashrightarrow (X,f^*\mathcal{O}_Y) \xrightarrow[(f,\mu_{\mathcal{O}_Y})]{} (Y,\mathcal{O}_Y) \quad \text{with } (Z,\mathcal{O}_Z) \xrightarrow{(h,h^\#)} (Y,\mathcal{O}_Y)\end{cases},\quad \begin{cases}Z \xrightarrow[g]{} X \xrightarrow[f]{} Y \quad \text{with } Z \to Y\end{cases} \quad (h = f \circ g),$$

where $(h,h^\#)$ is a morphism of locally ringed spaces -- then there is a unique filler

$$(g,g^\#):(Z,\mathcal{O}_Z) \to (X,f^*\mathcal{O}_Y)$$

in $\underline{\underline{\mathrm{TOP}}}_{\underline{\mathrm{RNG}}}$ such that

$$(f,\mu_{\mathcal{O}_Y}) \circ (g,g^\#) = (h,h^\#),$$

the claim being that $(g,g^\#)$ is a morphism of locally ringed spaces. To begin with

$$g^\#:f^*\mathcal{O}_Y \to g_*\mathcal{O}_Z.$$

On the other hand,

$$h^{\#}: \mathcal{O}_Y \to h_* \mathcal{O}_Z = (f \circ g)_* \mathcal{O}_Z$$

$$= f_* g_* \mathcal{O}_Z.$$

And

$$\mathrm{Mor}(f^* \mathcal{O}_Y, g_* \mathcal{O}_Z) \approx \mathrm{Mor}(\mathcal{O}_Y, f_* g_* \mathcal{O}_Z),$$

hence under this identification,

$$h^{\#} \in \mathrm{Mor}(\mathcal{O}_Y, f_* g_* \mathcal{O}_Z)$$

corresponds to an element

$$h_{\#f} \in \mathrm{Mor}(f^* \mathcal{O}_Y, g_* \mathcal{O}_Z)$$

which, in fact, is precisely $g^{\#}$ (since $f_*(h_{\#f}) \circ \mu_{\mathcal{O}_Y} = h^{\#}$). Accordingly, to ascertain that $\forall\, z \in Z$, $(g_{\#})_z$ is local, it suffices to consider $(h_{\#f,\#g})_z$:

$$h_{\#f} \in \mathrm{Mor}(f^* \mathcal{O}_Y, g_* \mathcal{O}_Z)$$

$$\longleftrightarrow h_{\#f,\#g} \in \mathrm{Mor}(g^* f^* \mathcal{O}_Y, \mathcal{O}_Z)$$

$$\approx \mathrm{Mor}((f \circ g)^* \mathcal{O}_Y, \mathcal{O}_Z)$$

$$\approx \mathrm{Mor}(h^* \mathcal{O}_Y, \mathcal{O}_Z).$$

But

$$\mathrm{Mor}(h^* \mathcal{O}_Y, \mathcal{O}_Z) \approx \mathrm{Mor}(\mathcal{O}_Y, h_* \mathcal{O}_Z).$$

Therefore

$$h_{\#f,\#g} \text{ " = " } h_{\#}.$$

And, $\forall\, z \in Z$, $(h_{\#})_z$ is, by hypothesis, local.

<u>N.B.</u> The pair

$$(\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}\ ,\ \underline{\text{LOC}}\text{-P}_{\underline{\text{RNG}}})$$

and the pair

$$(\underline{\text{TOP}}_{\underline{\text{RNG}}}\ ,\ \text{P}_{\underline{\text{RNG}}})$$

are objects of

$$\underline{\text{FIB}}(\underline{\text{TOP}})$$

and the inclusion functor

$$\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}} \rightarrow \underline{\text{TOP}}_{\underline{\text{RNG}}}$$

is horizontal.

[Suppose that

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y)$$

is horizontal in $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$. To see that it is horizontal in $\underline{\text{TOP}}_{\underline{\text{RNG}}}$, introduce

$$(f,\mu_{\mathcal{O}_Y}):(X,f^*\mathcal{O}_Y) \rightarrow (Y,\mathcal{O}_Y)$$

which is horizontal in $\underline{\text{TOP}}_{\underline{\text{RNG}}}$ -- then there is a vertical isomorphism

$$v:(X,\mathcal{O}_X) \rightarrow (X,f^*\mathcal{O}_Y)$$

and a commutative diagram

$$\begin{tikzcd}
(X,\mathcal{O}_X) \arrow[r, "(f,f^\#)"] \arrow[d, "v"'] & (Y,\mathcal{O}_Y) \arrow[d, equal] \\
(X,f^*\mathcal{O}_Y) \arrow[r, "(f,\mu_{\mathcal{O}_Y})"'] & (Y,\mathcal{O}_Y),
\end{tikzcd}$$

so

$$(f,f^{\#}) = (f,\mu_{\mathcal{O}_Y}) \circ v$$

is horizontal (cf. 4.20 and 4.21).]

23.11 LEMMA $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ is a replete (nonfull) subcategory of $\underline{\text{TOP}}_{\underline{\text{RNG}}}$.

[This is an application of 23.5 (and 23.6). Thus let

$$(\mathrm{id}_X,(\mathrm{id}_X)^{\#}):(X,\mathcal{O}_X) \to (X,\mathcal{O}_X')$$

be a vertical isomorphism in $\underline{\text{TOP}}_{\underline{\text{RNG}}}$, where $(X,\mathcal{O}_X)$ is in $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ -- then $(X,\mathcal{O}_X')$ is necessarily a locally ringed space and $(\mathrm{id}_X,(\mathrm{id}_X)^{\#})$ is a morphism of locally ringed spaces.]

[Note: It follows that the inclusion functor

$$\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}} \to \underline{\text{TOP}}_{\underline{\text{RNG}}}$$

reflects isomorphisms.]

23.12 REMARK Suppose that $(Y,\mathcal{O}_Y)$ is a locally ringed space. Let $f:X \to Y$ be a continuous function and let

$$(f,f^{\#}):(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$$

be a horizontal morphism in $\underline{\text{TOP}}_{\underline{\text{RNG}}}$ -- then $(X,\mathcal{O}_X)$ is a locally ringed space and $(f,f^{\#})$ is a morphism of locally ringed spaces.

[First choose a horizontal morphism

$$(\tilde{f},\tilde{f}^{\#}):(X,\tilde{\mathcal{O}}_X) \to (Y,\mathcal{O}_Y) \quad (\tilde{f} = f)$$

in $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ -- then $(\tilde{f},\tilde{f}^{\#})$ is a horizontal morphism in $\underline{\text{TOP}}_{\underline{\text{RNG}}}$, so there is a vertical isomorphism

$$v:(X,\tilde{\mathcal{O}}_X) \to (X,\mathcal{O}_X)$$

and a commutative diagram

$$\begin{tikzcd}
(X,\tilde{\mathcal{O}}_X) \arrow[r, "(\tilde{f},\tilde{f}^\#)"] \arrow[d, "v"'] & (Y,\mathcal{O}_Y) \arrow[d, equal] \\
(X,\mathcal{O}_X) \arrow[r, "(f,f^\#)"'] & (Y,\mathcal{O}_Y).
\end{tikzcd}$$

Since $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ is a replete subcategory of $\underline{\text{TOP}}_{\underline{\text{RNG}}}$, both $(X,\mathcal{O}_X)$ and v are in $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$. Finally,

$$(f,f^\#) \circ v = (\tilde{f},\tilde{f}^\#)$$

$$\Rightarrow (f,f^\#) = (\tilde{f},\tilde{f}^\#) \circ v^{-1},$$

hence $(f,f^\#)$ is a morphism of locally ringed spaces (and, as such, is horizontal).]

23.13 DEFINITION An <u>affine scheme</u> is a locally ringed space which is isomorphic as a locally ringed space to $(\text{Spec } A,\mathcal{O}_A)$ $(\mathcal{O}_A \equiv \mathcal{O}_{\text{Spec } A})$ for some $A \in \text{Ob } \underline{\text{RNG}}$ (cf. 22.17).

[Note: A ringed space which is isomorphic as a ringed space to a $(\text{Spec } A,\mathcal{O}_A)$ is automatically a locally ringed space and the isomorphism is one of locally ringed spaces.]

23.14 NOTATION <u>AFF-SCH</u> is the full subcategory of $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ whose objects are the affine schemes.

23.15 REMARK The category <u>AFF-SCH</u> has finite products and pullbacks, hence is finitely complete.

23.16 THEOREM The functor

$$(\mathrm{Spec},\mathcal{O}):\underline{\mathrm{RNG}}^{\mathrm{OP}} \to \underline{\mathrm{AFF\text{-}SCH}}$$

that sends A to $(\mathrm{Spec}\ A,\mathcal{O}_A)$ is an equivalence of categories.

N.B. We shall also view $(\mathrm{Spec},\mathcal{O})$ as a fully faithful functor

$$\underline{\mathrm{RNG}}^{\mathrm{OP}} \to \underline{\mathrm{LOC\text{-}TOP}}_{\underline{\mathrm{RNG}}}.$$

Let

$$\Gamma:\underline{\mathrm{LOC\text{-}TOP}}_{\underline{\mathrm{RNG}}} \to \underline{\mathrm{RNG}}^{\mathrm{OP}}$$

be the functor defined on objects $(X,\mathcal{O}_X)$ by

$$\Gamma(X,\mathcal{O}_X) = \mathcal{O}_X(X)$$

and on morphisms

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y)$$

by

$$f_Y^\#:\mathcal{O}_Y(Y) \to \mathcal{O}_X(X).$$

23.17 THEOREM The functor Γ is a left adjoint for the functor $(\mathrm{Spec},\mathcal{O})$:

$$\mathrm{Mor}(\Gamma(X,\mathcal{O}_X),A) \approx \mathrm{Mor}((X,\mathcal{O}_X),(\mathrm{Spec}\ A,\mathcal{O}_A)).$$

23.18 APPLICATION $(\mathrm{Spec}\ \mathbb{Z},\mathcal{O}_{\mathbb{Z}})$ is a final object in $\underline{\mathrm{LOC\text{-}TOP}}_{\underline{\mathrm{RNG}}}$.

[Indeed,

$$\mathrm{Mor}(\Gamma(X,\mathcal{O}_X),Z) \text{ in } \underline{\mathrm{RNG}}^{\mathrm{OP}}$$

is

$$\mathrm{Mor}(Z,\Gamma(X,\mathcal{O}_X)) \text{ in } \underline{\mathrm{RNG}}.]$$

23.19 DEFINITION A scheme is a locally ringed space with the property that every point has an open neighborhood which is an affine scheme.

23.20 NOTATION SCH is the full subcategory of $\underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ whose objects are the schemes (cf. 0.6).

[Note: AFF-SCH is a full subcategory of SCH.]

23.21 REMARK The category SCH has finite products and pullbacks, hence is finitely complete.

[Note: SCH does not have arbitrary products, hence is not complete. Consider, for example $\prod_1^\infty P^1_C$.]

N.B. If A is a zero ring, then Spec A is an initial object in SCH whereas Spec Z is a final object in SCH.

When dealing with schemes, one sometimes says "let X be a scheme" rather than "let $(X,\mathcal{O}_X)$ be a scheme."

23.22 DEFINITION Let X be a scheme -- then an open subset $U \subset X$ is an affine open subset of X if U is an affine scheme.

23.23 LEMMA The affine open subsets of a scheme X constitute a basis for the topology on X.

[Note: Therefore every open subset of X is a scheme.]

23.24 REMARK The intersection of two affine open subsets of X is open but it need not be affine open.

[Note: Let X be a scheme.

- X is <u>semi-separated</u> if for each pair $U,V \subset X$ of affine opens the intersection $U \cap V$ is affine open.

- X is <u>quasi-separated</u> if for each pair $U,V \subset X$ of affine opens the intersection $U \cap V$ is a finite union of affine opens.

One has

$$\text{separated} \Rightarrow \text{semi-separated} \Rightarrow \text{quasi-separated}.$$

Every affine scheme is separated.]

23.25 LEMMA The underlying topology on a scheme X is locally quasi-compact.

[Recall that $\forall$ A $\in$ Ob <u>RNG</u>, Spec A is quasi-compact (but rarely Hausdorff or even T_1). On the other hand, an open subset of Spec A is not necessarily quasi-compact (although this will be the case if, e.g., A is noetherian).]

23.26 DEFINITION Let I be a set.

- Given $i \in I$, let X_i be a scheme.

- Given $i,j \in I$, let $U_{ij} \subset X_i$ be an open subset and let

$$\phi_{ij}:U_{ij} \to U_{ji}$$

be an isomorphism of schemes (take $U_{ii} = X_i$ and $\phi_{ii} = \mathrm{id}_{X_i}$).

- Given $i,j,k \in I$, assume that

$$\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$

and that the diagram

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\phi_{ik}"] \arrow[dr, "\phi_{ij}"'] & & U_{ki} \cap U_{kj} \\ & U_{ji} \cap U_{jk} \arrow[ur, "\phi_{jk}"'] & \end{tikzcd}$$

commutes.

Then the collection

$$(I, (X_i : i \in I) \ , \ (U_{ij} : i,j \in I) \ , \ (\phi_{ij} : i,j \in I))$$

is called <u>glueing data</u>.

23.27 THEOREM Given glueing data, there exists a scheme X, open subschemes $U_i \subset X$, with $X = \bigcup_{i \in I} U_i$, and isomorphisms $\phi_i : X_i \to U_i$ of schemes such that

(1) $\phi_i(U_{ij}) = U_i \cap U_j$

and

(2) $\phi_{ij} = \phi_j^{-1} | U_i \cap U_j \circ \phi_i | U_{ij}$.

23.28 EXAMPLE Take $U_{ij} = \emptyset$ for all i,j -- then $X = \coprod_i X_i$.

[Note: If $A_1, \ldots, A_n$ are nonzero commutative rings with unit, then

$$\coprod_{i=1}^{n} \text{Spec } A_i \approx \text{Spec } (\prod_{i=1}^{n} A_i)$$

but for an infinite index set I, $\coprod_i \text{Spec } A_i$ is not an affine scheme (it is not quasi-compact).]

23.29 LEMMA Let S be a scheme and let X_i $(i \in I)$, Y_j $(j \in J)$ be objects of <u>SCH</u>/S -- then

$$(\coprod_i X_i) \times_S (\coprod_j Y_j) \approx \coprod_{i,j} (X_i \times_S Y_j).$$

§24. *MODULES*

Let $(X,\mathcal{O}_X)$ be a ringed space.

24.1 DEFINITION An $\underline{\mathcal{O}_X\text{-module}}$ is a sheaf F of abelian groups on X such that $\forall$ open subset $U \subset X$, the abelian group $F(U)$ is a left $\mathcal{O}_X(U)$-module and for each inclusion $V \subset U$ of open sets there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \times F(U) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ \mathcal{O}_X(V) \times F(V) \arrow[r] & F(V). \end{tikzcd}$$

24.2 NOTATION $\mathcal{O}_X\text{-}\underline{\text{MOD}}$ is the category whose objects are the $\mathcal{O}_X$-modules.

[Note: A morphism $F \to G$ of $\mathcal{O}_X$-modules is a morphism Ξ of sheaves of abelian groups such that $\forall$ open subset $U \subset X$, the arrow $\Xi_U: F(U) \to G(U)$ is a homomorphism of left $\mathcal{O}_X(U)$-modules. Denote the set of such by

$$\text{Hom}_{\mathcal{O}_X}(F,G).$$

Then this set is an abelian group which, moreover, is a left $\Gamma\mathcal{O}_X$-module: Given $s \in \Gamma\mathcal{O}_X$ and $\Xi: F \to G$, define $s\Xi$ by the prescription

$$(s\Xi)_U = (s|U)\Xi_U.$$

So, e.g., as left $\Gamma\mathcal{O}_X$-modules,

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,F) \approx \Gamma F.]$$

24.3 REMARK There is a standard list of operations that I shall not stop to rehearse (kernel, cokernel, image, coimage,...).

24.4 EXAMPLE Let $\mathcal{Z}$ be the sheaf associated with the constant presheaf $U \to \mathcal{Z}$ -- then a $\mathcal{Z}$-module is simply a sheaf of abelian groups on X.

24.5 THEOREM $\mathcal{O}_X$-MOD is an abelian category.

24.6 THEOREM $\mathcal{O}_X$-MOD has enough injectives.

24.7 THEOREM $\mathcal{O}_X$-MOD is complete and cocomplete.

[Any abelian category has equalizers and coequalizers.

- Given a set I and for each $i \in I$, an $\mathcal{O}_X$-module $\mathcal{F}_i$, the product

$$\prod_{i \in I} \mathcal{F}_i$$

is the sheaf that assigns to each open subset $U \subset X$, the product

$$\prod_{i \in I} \mathcal{F}_i(U)$$

of left $\mathcal{O}_X(U)$-modules. It is also the categorical product.

- Given a set I and for each $i \in I$, an $\mathcal{O}_X$-module $\mathcal{F}_i$, the direct sum

$$\bigoplus_{i \in I} \mathcal{F}_i$$

is the sheaf associated with the presheaf that assigns to each open subset $U \subset X$, the direct sum

$$\bigoplus_{i \in I} \mathcal{F}_i(U)$$

of left $\mathcal{O}_X(U)$-modules. It is also the categorical coproduct.]

24.8 DEFINITION Given $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, their <u>tensor product</u>

$$F \otimes_{\mathcal{O}_X} G$$

is the $\mathcal{O}_X$-module which is the sheaf associated with the presheaf that assigns to each open subset $U \subset X$, the tensor product

$$F(U) \otimes_{\mathcal{O}_X(U)} G(U)$$

of left $\mathcal{O}_X(U)$-modules.

24.9 DEFINITION Given $\mathcal{O}_X$-modules F and G, their <u>internal hom</u>

$$\mathcal{H}om_{\mathcal{O}_X}(F,G)$$

is the $\mathcal{O}_X$-module which is the sheaf that assigns to each open subset $U \subset X$, the left $\mathcal{O}_X(U)$-module

$$\mathrm{Hom}_{\mathcal{O}_X|U}(F|U,G|U).$$

24.10 LEMMA Let F,G,H be $\mathcal{O}_X$-modules -- then

$$\mathcal{H}om_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G,H) \approx \mathcal{H}om_{\mathcal{O}_X}(F,\mathcal{H}om_{\mathcal{O}_X}(G,H)).$$

[Note: As left $\Gamma\mathcal{O}_X$-modules,

$$\mathrm{Hom}_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G,H) \approx \mathrm{Hom}_{\mathcal{O}_X}(F,\mathcal{H}om_{\mathcal{O}_X}(G,H)).]$$

24.11 DEFINITION Suppose that

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y,\mathcal{O}_Y)$$

is a morphism of ringed spaces.

- Let F be an $\mathcal{O}_X$-module. Form f_*F (an object of $\underline{\mathrm{Sh}}(Y,\underline{\mathrm{AB}})$) -- then f_*F

is an $f_*\mathcal{O}_X$-module, hence is an $\mathcal{O}_Y$-module via the arrow $f^{\#}:\mathcal{O}_Y \to f_*\mathcal{O}_X$, call it $\mathrm{res}_f\ \mathcal{F}$.

• Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Form $f^*\mathcal{G}$ (an object of $\underline{\mathrm{Sh}}(X,\underline{\mathrm{AB}})$) -- then $f^*\mathcal{G}$ is an $f^*\mathcal{O}_Y$-module. On the other hand, $f_{\#}:f^*\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism in $\underline{\mathrm{Sh}}(X,\underline{\mathrm{RNG}})$, thus

$$\mathcal{O}_X \otimes_{f^*\mathcal{O}_Y} f^*\mathcal{G}$$

is an $\mathcal{O}_X$-module, call it $\mathrm{ext}_f\ \mathcal{G}$.

24.12 EXAMPLE Take $\mathcal{G} = \mathcal{O}_Y$ -- then

$$\mathrm{ext}_f\ \mathcal{O}_Y \approx \mathcal{O}_X.$$

24.13 LEMMA The functor

$$\mathrm{ext}_f:\mathcal{O}_Y\text{-}\underline{\mathrm{MOD}} \longrightarrow \mathcal{O}_X\text{-}\underline{\mathrm{MOD}}$$

is a left adjoint for the functor

$$\mathrm{res}_f:\mathcal{O}_X\text{-}\underline{\mathrm{MOD}} \longrightarrow \mathcal{O}_Y\text{-}\underline{\mathrm{MOD}}.$$

24.14 REMARK Let

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^{\#})} (Y,\mathcal{O}_Y)\ ,\ (Y,\mathcal{O}_Y) \xrightarrow{(g,g^{\#})} (Z,\mathcal{O}_Z)$$

be morphisms of ringed spaces -- then the functors $\mathrm{res}_g \circ \mathrm{res}_f$ and $\mathrm{res}_{g \circ f}$ are equal while the functors $\mathrm{ext}_f \circ \mathrm{ext}_g$ and $\mathrm{ext}_{g \circ f}$ are naturally isomorphic.

24.15 NOTATION $\mathcal{O}$-$\underline{\mathrm{MOD}}$ is the category whose objects are the triples $(X,\mathcal{O}_X,\mathcal{F})$, where $(X,\mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ is an $\mathcal{O}_X$-module, and whose morphisms are the triples

$$(f,f^{\#},\Xi):(X,\mathcal{O}_X,\mathcal{F}) \to (Y,\mathcal{O}_Y,\mathcal{G}),$$

where $f:X \to Y$ is a continuous function, $f^{\#}:\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism in $\underline{\mathrm{Sh}}(Y,\underline{\mathrm{RNG}})$, $\Xi:\mathcal{G} \to f_*\mathcal{F}$ is a morphism in $\underline{\mathrm{Sh}}(Y,\underline{\mathrm{AB}})$ such that $\forall$ open subset $U \subset X$, the diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \times \mathcal{G}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{O}_X(f^{-1}U) \times \mathcal{F}(f^{-1}U) \arrow[r] & \mathcal{F}(f^{-1}U)
\end{tikzcd}$$

commutes.

24.16 LEMMA The projection

$$(X,\mathcal{O}_X,\mathcal{F}) \to (X,\mathcal{O}_X)$$

is a fibration

$$P_{\underline{\mathrm{MOD}}}:\mathcal{O}\text{-}\underline{\mathrm{MOD}} \to \underline{\mathrm{TOP}}_{\underline{\mathrm{RNG}}}.$$

PROOF Given $(Y,\mathcal{O}_Y,\mathcal{G})$ and

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^{\#})} (Y,\mathcal{O}_Y),$$

the composition

$$(X,\mathcal{O}_X,\mathcal{O}_X \otimes_{f^*\mathcal{O}_Y} f^*\mathcal{G})$$

$$\longrightarrow (X,f^*\mathcal{O}_Y,f^*\mathcal{G}) \longrightarrow (Y,\mathcal{O}_Y,\mathcal{G})$$

is horizontal.

[Note: Recall that

$$(\mathrm{id}_X, f_\#):(X,\mathcal{O}_X) \to (X, f^*\mathcal{O}_Y)$$

is a morphism of ringed spaces and there are arrows

$$\begin{cases} \mathcal{O}_Y \longrightarrow f_* f^* \mathcal{O}_Y \\ \mathcal{G} \longrightarrow f_* f^* \mathcal{G} \end{cases}$$

of adjunction.]

24.17 REMARK The commutative diagram

$$\begin{tikzcd} \mathcal{O}\text{-}\underline{\mathrm{MOD}} \arrow[r, "P_{\underline{\mathrm{MOD}}}"] \arrow[d, "P_{\underline{\mathrm{RNG}}} \circ P_{\underline{\mathrm{MOD}}}"'] & \underline{\mathrm{TOP}}_{\underline{\mathrm{RNG}}} \arrow[d, "P_{\underline{\mathrm{RNG}}}"] \\ \underline{\mathrm{TOP}} \arrow[r, equal] & \underline{\mathrm{TOP}} \end{tikzcd}$$

is thus an instance of 6.2.

§25. QUASI-COHERENT MODULES

Let $(X,\mathcal{O}_X)$ be a ringed space.

25.1 NOTATION Given a set I and an $\mathcal{O}_X$-module F, write $F^{(I)}$ for the direct sum

$$\bigoplus_{i\in I} F_i \quad (\forall\ i,\ F_i = F).$$

25.2 DEFINITION An $\mathcal{O}_X$-module F is said to be quasi-coherent if $\forall\ x \in X$, there exists an open neighborhood U of x, sets I and J (depending on x), and an exact sequence

$$(\mathcal{O}_X|U)^{(I)} \longrightarrow (\mathcal{O}_X|U)^{(J)} \longrightarrow F|U \longrightarrow 0$$

of $\mathcal{O}_X|U$-modules.

25.3 NOTATION $\underline{\text{QCO}}(X)$ is the full subcategory of $\mathcal{O}_X$-$\underline{\text{MOD}}$ whose objects are the quasi-coherent $\mathcal{O}_X$-modules.

25.4 REMARK In general, $\underline{\text{QCO}}(X)$ is not an abelian category.

25.5 LEMMA Let F,G be quasi-coherent $\mathcal{O}_X$-modules -- then $F \oplus G$ is quasi- coherent.

[Note: An infinite direct sum of quasi-coherent $\mathcal{O}_X$-modules need not be quasi-coherent.]

25.6 LEMMA Let F,G be quasi-coherent $\mathcal{O}_X$-modules -- then $F \otimes_{\mathcal{O}_X} G$ is quasi-coherent.

[Note: On the other hand, $\mathcal{H}om_{\mathcal{O}_X}(F,G)$ need not be quasi-coherent.]

<u>N.B.</u> <u>QCO</u>(X) is a symmetric monoidal category under the tensor product (the unit is $\mathcal{O}_X$).

25.7 DEFINITION An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be <u>locally free</u> if $\forall\, x \in X$, there exists an open neighborhood U of x and a set I (depending on x) such that $\mathcal{F}|U$ is isomorphic to $(\mathcal{O}_X|U)^{(I)}$ as an $\mathcal{O}_X|U$-module.

25.8 LEMMA A locally free $\mathcal{O}_X$-module $\mathcal{F}$ is necessarily quasi-coherent.

25.9 LEMMA Suppose that

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^{\#})} (Y,\mathcal{O}_Y)$$

is a morphism of ringed spaces.

- Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module -- then $\text{res}_f\, \mathcal{F}$ is not necessarily a quasi-coherent $\mathcal{O}_Y$-module.

- Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module -- then $\text{ext}_f\, \mathcal{G}$ is necessarily a quasi-coherent $\mathcal{O}_X$-module.

25.10 CONSTRUCTION Let $(X,\mathcal{O}_X)$ be a ringed space. Suppose that $A \in \text{Ob}\, \underline{\text{RNG}}$ and $\phi: A \to \Gamma\mathcal{O}_X$ ($= \mathcal{O}_X(X)$) is a ring homomorphism. Let M be a left A-module. Consider the canonical arrow

$$(\pi,\pi^{\#}):(X,\mathcal{O}_X) \longrightarrow (*,\mathcal{O}_*),$$

where $\mathcal{O}_* * = A$ $(\pi^{\#} = \phi)$ -- then $\text{ext}_\pi\, M$ is quasi-coherent. In addition, the assignment

$$M \to \text{ext}_\pi\, M$$

defines a functor

$$\underline{A\text{-MOD}} \to \underline{QCO}(X)$$

and given any $\mathcal{O}_X$-module F,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathrm{ext}_\pi\, M, F) \approx \mathrm{Hom}_A(M, \Gamma F),$$

where the left A-module structure on ΓF comes from the left $\Gamma\mathcal{O}_X$-module structure via ϕ.

25.11 REMARK One can take $A = \Gamma\mathcal{O}_X$, $\phi = \mathrm{id}$, in which case it is customary to write F_M in place of $\mathrm{ext}_\pi\, M$.

Given $A \in \mathrm{Ob}\, \underline{RNG}$, we shall now recall the connection between A-$\underline{MOD}$ and $\underline{QCO}(\mathrm{Spec}\, A)$. So in 25.10, take $(X, \mathcal{O}_X) = (\mathrm{Spec}\, A, \mathcal{O}_A)$ (hence $\Gamma\mathcal{O}_A \approx A$) -- then for every left A-module M, the sheaf $\tilde{M}$ is canonically isomorphic to F_M (and this isomorphism is functorial in M). Therefore the $\tilde{M}$ are quasi-coherent and given any $\mathcal{O}_X$-module F,

$$\mathrm{Hom}_{\mathcal{O}_A}(\tilde{M}, F) \approx \mathrm{Hom}_A(M, \Gamma F).$$

25.12 LEMMA For all left A-modules M and N,

$$\mathrm{Hom}_{\mathcal{O}_A}(\tilde{M}, \tilde{N}) \approx \mathrm{Hom}_A(M, N).$$

[Bear in mind that

$$\left[\begin{array}{l} \Gamma\tilde{M} \approx M \\ \Gamma\tilde{N} \approx N.] \end{array}\right.$$

25.13 LEMMA For every quasi-coherent $\mathcal{O}_A$-module $\mathcal{F}$,

$$(\Gamma\mathcal{F})^{\sim} \approx \mathcal{F}.$$

25.14 THEOREM The functor

$$\sim : A\text{-}\underline{\mathrm{MOD}} \to \underline{\mathrm{QCO}}(\mathrm{Spec}\ A)$$

that sends M to $\tilde{M}$ is an equivalence of categories.

[In fact, ~ is fully faithful (cf. 25.12) and has a representative image (cf. 25.13).]

25.15 EXAMPLE The category of abelian groups is equivalent to $\underline{\mathrm{QCO}}(\mathrm{Spec}\ \underline{Z})$.

25.16 LEMMA Let A,B $\in$ Ob $\underline{\mathrm{RNG}}$, suppose that

$$(f,f^{\#}):(\mathrm{Spec}\ B,\mathcal{O}_B) \to (\mathrm{Spec}\ A,\mathcal{O}_A)$$

is a morphism of affine schemes, and let $\rho:A \to B$ be the associated ring homomorphism.

- For every left B-module N,

$$\mathrm{res}_f\ \tilde{N} \approx (\mathrm{res}_\rho\ N)^{\sim}$$

functorially in N.

- For every left A-module M,

$$\mathrm{ext}_f\ \tilde{M} \approx (\mathrm{ext}_\rho\ M)^{\sim}$$

functorially in M.

25.17 REMARK There is a functor

$$(\mathrm{Spec},\mathcal{O},\sim):\underline{\mathrm{MOD}}(\underline{\mathrm{AB}})^{\mathrm{OP}} \longrightarrow \mathcal{O}\text{-}\underline{\mathrm{MOD}}$$

which sends an object (A,M) to

$$(\text{Spec } A, \mathcal{O}_A, \tilde{M})$$

and which sends a morphism $(f,\phi):(A,M) \to (B,N)$ to

$$(\text{Spec } f, \mathcal{O}_f, \tilde{\phi}):(\text{Spec } B, \mathcal{O}_B, \tilde{N}) \longrightarrow (\text{Spec } A, \mathcal{O}_A, \tilde{M}).$$

[Note: On a principal open set $D(a)$ $(a \in A)$, $\tilde{M}(D(a)) = M_a$ and

$$((\text{Spec } f)_*\tilde{N})(D(a)) = \tilde{N}(D(f(a))) = N_{f(a)}.$$

Furthermore, there are arrows of localization

$$\begin{bmatrix} A \longrightarrow A_a \\ \\ B \longrightarrow B_{f(a)} \end{bmatrix}, \quad \begin{bmatrix} M \longrightarrow M_a \\ \\ N \longrightarrow N_{f(a)} \end{bmatrix}$$

and a commutative diagram

$$\begin{array}{ccc} (A,M) & \longrightarrow & (A_a,M_a) \\ \downarrow & & \downarrow \\ (B,N) & \longrightarrow & (B_{f(a)},N_{f(a)}) \end{array} .]$$

It remains to consider the pairs $(X,\mathcal{O}_X)$, where X is a scheme.

[Note: It has been shown by Rosenberg[†] that $(X,\mathcal{O}_X)$ can be reconstructed up to isomorphism from $\underline{\text{QCO}}(X)$.]

25.18 LEMMA Let $\mathcal{F}$ be an $\mathcal{O}_X$-module -- then $\mathcal{F}$ is quasi-coherent iff for every

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[†] *Lecture Notes in Pure and Applied Mathematics* 197 (1998), 257-274.

affine open $U \subset X$ ($U \approx \text{Spec } A$), the restriction $\mathcal{F}|U$ is of the form $\tilde{M}$ for some M in A-$\underline{\text{MOD}}$.

N.B. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then for all affine open U, V with $V \subset U$, the canonical arrow

$$\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to \mathcal{F}(V)$$

is an isomorphism of $\mathcal{O}_X(V)$-modules.

25.19 LEMMA Suppose that

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^{\#})} (Y,\mathcal{O}_Y)$$

is a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module -- then $\text{ext}_f\, \mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module (cf. 25.9).

25.20 REMARK The notation used in 7.3 is suggestive but misleading: Replace f^* by ext_f ... .

25.21 LEMMA Suppose that

$$(X,\mathcal{O}_X) \xrightarrow{(f,f^{\#})} (Y,\mathcal{O}_Y)$$

is a morphism of schemes, where f is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module -- then $\text{res}_f\, \mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module (cf. 25.9).

25.22 REMARK If U is an open subset of a scheme X, then in general, $\text{res}_{i_U}\, \mathcal{O}_X|U$ is not quasi-coherent.

25.23 THEOREM $\underline{\text{QCO}}(X)$ is an abelian category.

25.24 RAPPEL A <u>Grothendieck category</u> is a cocomplete abelian category in which filtered colimits commute with finite limits or, equivalently, in which filtered colimits of exact sequences are exact.

<u>N.B.</u> In a Grothendieck category, every filtered colimit of monomorphisms is a monomorphism, coproducts of monomorphisms are monomorphisms, and

$$t:\coprod_i X_i \to \prod_i X_i$$

is a monomorphism.

25.25 EXAMPLE Let A be a commutative ring with unit -- then A-$\underline{\mathrm{MOD}}$ is Grothendieck.

[Note: In particular, $\underline{\mathrm{AB}}$ is Grothendieck but its full subcategory whose objects are the finitely generated abelian groups is not Grothendieck.]

25.26 THEOREM $\underline{\mathrm{QCO}}(X)$ is a Grothendieck category.

25.27 DEFINITION Given a locally small category $\underline{C}$, an object U in $\underline{C}$ is said to be a <u>separator</u> for $\underline{C}$ if the functor $\mathrm{Mor}(U,—):\underline{C} \to \underline{\mathrm{SET}}$ is faithful, i.e., if for every pair $f,g:X \to Y$ of distinct morphisms, there exists a morphism $\sigma:U \to X$ such that $f \circ \sigma \neq g \circ \sigma$.

25.28 EXAMPLE Let A be a commutative ring with unit -- then A, viewed as a left A-module, is a separator for A-$\underline{\mathrm{MOD}}$.

25.29 THEOREM $\underline{\mathrm{QCO}}(X)$ admits a separator.

<u>N.B.</u> Every Grothendieck category with a separator is complete and has enough injectives.

25.30 REMARK It can be shown that $\underline{QCO}(X)$ is a coreflective subcategory of $\mathcal{O}_X$-$\underline{MOD}$, i.e., the inclusion functor

$$\underline{QCO}(X) \to \mathcal{O}_X\text{-}\underline{MOD}$$

has a right adjoint.

Fix a regular cardinal κ.

25.31 DEFINITION Let $\underline{C}$ be a locally small cocomplete category -- then an object $X \in \text{Ob } \underline{C}$ is <u>κ-definite</u> if $\text{Mor}(X,\text{—})$ preserves κ-filtered colimits.

25.32 EXAMPLE In $\underline{TOP}$, no nondiscrete X is κ-definite.

25.33 DEFINITION Let $\underline{C}$ be a locally small cocomplete category -- then $\underline{C}$ is <u>κ-presentable</u> if up to isomorphism, there exists a set of κ-definite objects and every object in $\underline{C}$ is a κ-filtered colimit of κ-definite objects.

25.34 EXAMPLE $\underline{SET}$ and $\underline{CAT}$ are $\aleph_0$-presentable but $\underline{TOP}$ is not κ-presentable for any κ.

25.35 DEFINITION Let $\underline{C}$ be a locally small cocomplete category -- then $\underline{C}$ is <u>presentable</u> if C is κ-presentable for some κ.

[Note: Every presentable category is cocomplete (by definition) and complete, wellpowered and cowellpowered.]

25.36 THEOREM (Beke[†]) Suppose that $\underline{C}$ is a Grothendieck category with a

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[†] *Math. Proc. Camb. Phil. Soc.* <u>129</u> (2000), 447-475.

separator -- then C is presentable.

25.37 APPLICATION QCO(X) is presentable.

§26. *LOCAL TRIVIALITY*

Let $\underline{C}$ be a category.

26.1 DEFINITION A subcategory of <u>trivial objects</u> is a replete subcategory of $\underline{C}$.

26.2 EXAMPLE If $\underline{C}$ has initial objects, then the associated full subcategory is isomorphism closed, hence is a subcategory of trivial objects.

26.3 EXAMPLE If $\underline{C}$ has final objects, then the asociated full subcategory is isomorphism closed, hence is a subcategory of trivial objects.

Let $\underline{A}$ be a category, $F:\underline{A} \to \underline{C}$ a functor.

26.4 DEFINITION The <u>replete full image</u> of F is the isomorphism closed full subcategory of $\underline{C}$ whose objects are those objects which are isomorphic to some FA ($A \in \text{Ob } \underline{A}$).

26.5 EXAMPLE Take $\underline{A} = \underline{\text{SET}}$, $\underline{C} = \underline{\text{GR}}$, $F:\underline{A} \to \underline{C}$ the left adjoint to the forgetful functor — then the replete full image of F is the category of free groups.

26.6 EXAMPLE Take $\underline{A} = \underline{\text{RNG}}^{\text{OP}}$, $\underline{C} = \underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$, $F:\underline{A} \to \underline{C}$ the functor that sends A to $(\text{Spec } A, \mathcal{O}_A)$ -- then the replete full image of F is the category of affine schemes.

Let $\underline{T} \subset \underline{C}$ be a subcategory of trivial objects.

26.7 DEFINITION Let $\mathcal{C}$ be a covering of an object X in $\underline{C}$ -- then X is <u>locally trivial</u> (w.r.t. $\underline{T}$) if the domain of each $g \in \mathcal{C}$ is in $\underline{T}$.

26.8 DEFINITION Let κ be a covering function on $\underline{C}$ -- then an object X in $\underline{C}$ is locally trivial (w.r.t. $\underline{T}$) if it is locally trivial (w.r.t. $\underline{T}$) for some $\mathcal{C} \in \kappa_X$.

<u>N.B.</u> To ensure that

"trivial" => "locally trivial",

it suffices to assume that $\forall\ T \in \text{Ob}\ \underline{T}$, $\{\text{id}_T:T \to T\} \in \kappa_T$.

26.9 REMARK Suppose that $\forall\ X \in \text{Ob}\ \underline{C}$, $\kappa_X = \{\text{id}_X:X \to X\}$ -- then for any $\underline{T}$, the locally trivial objects are the trivial objects.

26.10 EXAMPLE Take $\underline{C} = \underline{\text{SET}}$.

- Let $\underline{T}$ be the subcategory whose only object is the empty set $\emptyset$ and whose only morphism is $\text{id}_\emptyset:\emptyset \to \emptyset$. Define a covering function κ by setting $\kappa_X = \{\emptyset \to X\}$ -- then all objects are locally trivial.

- Let $\underline{T}$ be the subcategory whose objects are the singletons. Define a covering function κ by setting $\kappa_\emptyset = \text{id}_\emptyset$ and

$$\kappa_X = \{\{x\} \to X:x \in X\} \quad (X \neq \emptyset).$$

Then all objects are locally trivial.

26.11 EXAMPLE Take $\underline{C} = \underline{\text{TOP}}$, let κ be the open subset coverage (cf. 11.20), and take for $\underline{T}$ the euclidean spaces, i.e., the topological spaces which are homeomorphic to some open subset of some R^n -- then the locally trivial objects are the topological manifolds.

[Note: To say that X is a topological manifold means that X admits a covering by open sets $U_i \subset X$, where $\forall$ i, U_i is homeomorphic to an open subset of R^{n_i} (n_i depends on i).]

26.12 EXAMPLE Take $\underline{C} = \underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$, let κ be the open subset coverage, and take for $\underline{T}$ the affine schemes -- then the locally trivial objects are the schemes (cf. 23.19).

[Note: An open subset U of a locally ringed space $(X,\mathcal{O}_X)$ can be viewed as a locally ringed space (let $\mathcal{O}_U = \mathcal{O}_X|U$), thus it makes sense to consider the open subset coverage.]

26.13 EXAMPLE Take $\underline{C} = \underline{\text{TOP}}_{\underline{\text{RNG}}}$, let κ be the open subset coverage, and take $\underline{T} = \underline{\text{LOC-TOP}}_{\underline{\text{RNG}}}$ (which is replete (cf. 23.11)) -- then here, all locally trivial objects are trivial.

[Note: If $U \subset X$ is open, then the stalk of $\mathcal{O}_U$ at an $x \in U$ is $\mathcal{O}_{X,x}$.]

Consider a one point ringed space $(\{x\},\mathcal{O}_{\{x\}})$ -- then $\mathcal{O}_{\{x\}}\emptyset = \{0\}$ (a zero ring), $\mathcal{O}_{\{x\}}\{x\} = A$ (a ring). Abbreviate this setup to $(\{x\},A)$ -- then a morphism

$$(\{x\},A) \xrightarrow{(f,f^\#)} (\{y\},B)$$

of ringed spaces is simply a homomorphism $f^\#:B \to A$.

26.14 EXAMPLE Let $\underline{T}$ be the replete subcategory of $\underline{\text{TOP}}_{\underline{\text{RNG}}}$ whose objects are the pairs $(\{x\},A)$, where A is a local ring, and whose morphisms are the morphisms

$$(\{x\},A) \xrightarrow{(f,f^\#)} (\{y\},B)$$

of ringed spaces such that the momomorphism $f^\#:B \to A$ is a local homomorphism.

Define a covering function κ on $\underline{\mathrm{TOP}}_{\underline{\mathrm{RNG}}}$ by setting $\kappa_{(\emptyset,\mathcal{O}_\emptyset)} = \mathrm{id}_{(\emptyset,\mathcal{O}_\emptyset)}$ and

$$\kappa_{(X,\mathcal{O}_X)} = \{(\{x\},\mathcal{O}_{X,x}) \to (X,\mathcal{O}_X) : x \in X\} \quad (X \neq \emptyset).$$

Then the locally trivial objects are the locally ringed spaces.

Let $P:\underline{E} \to \underline{B}$ be a fibration. Suppose that $\underline{T} \subset \underline{E}$ is a subcategory of trivial objects and let κ be a covering function on $\underline{B}$.

26.15 DEFINITION An object $X \in \mathrm{Ob}\,\underline{E}$ is <u>locally trivial</u> (w.r.t. $\underline{T}$) if it is locally trivial (w.r.t. $\underline{T}$) for some $\mathcal{C} \in (P^{-1}\kappa)_X$.

[Note: This reduces to 26.8 if $\underline{E} = \underline{B}$, $P = \mathrm{id}$.]

Let $P:\underline{E} \to \underline{B}$ be a fibration. Suppose that $\underline{B}$ has a final object $*_{\underline{B}}$ and that $\underline{E}_{*_{\underline{B}}} \neq 0$. Let $\underline{C}$ be a subcategory of $\underline{E}_{*_{\underline{B}}}$. Denote by $\underline{T}_{\underline{C}}$ the full subcategory of $\underline{E}$ whose objects are the X for which there exists an object $C \in \mathrm{Ob}\,\underline{C}$ and a horizontal arrow $X \to C$.

26.16 LEMMA $\underline{T}_{\underline{C}}$ is a replete subcategory of $\underline{E}$.

26.17 REMARK There is an analogous statement involving opfibrations with trivial objects determined by a subcategory of the fiber over an initial object.

26.18 EXAMPLE Consider the fibration $P_{\underline{A}}:\underline{\mathrm{TOP}}_{\underline{A}} \to \underline{\mathrm{TOP}}$ of 22.11. Place on $\underline{\mathrm{TOP}}$ the open subset coverage κ and take for $\underline{C}$ the fiber over a singleton $*$, thus the objects of $\underline{T}_{\underline{C}}$ are the $\underline{A}$-spaces $(X,\mathcal{O}_X)$ which are the domain of a horizontal arrow $(X,\mathcal{O}_X) \to (*,\mathcal{O}_*)$ over $!:X \to *$ for some $\mathcal{O}_*$.

• The trivial objects are the $(X, \mathcal{O}_X)$ such that $\mathcal{O}_X \approx {!}^*\mathcal{O}_*$ ($\approx \Delta \circ \mathrm{ev}\, \mathcal{O}_*$ (cf. 22.19)).

[The point is that for any X, the arrow

$$(!, \mu_{\mathcal{O}_*}) : (X, {!}^*\mathcal{O}_*) \to (*, \mathcal{O}_*)$$

is horizontal (cf. 22.11).]

Observe next that if U is an open subset of X, then

$$i_U^* : \underline{\mathrm{Sh}}(X, \underline{A}) \to \underline{\mathrm{Sh}}(U, \underline{A})$$

and $\forall\ \mathcal{O}_X$, the arrow

$$(i_U, \mu_{\mathcal{O}_X}) : (U, \mathcal{O}_X | U) \to (X, \mathcal{O}_X)$$

is horizontal (cf. 22.12). So, if $X = \bigcup_{i \in I} U_i$, then

$$\{(i_{U_i}, \mu_{\mathcal{O}_X}) : (U_i, \mathcal{O}_X | U_i) \to (X, \mathcal{O}_X)\} \in (P_{\underline{A}}^{-1}\kappa)_{(X, \mathcal{O}_X)}.$$

• The locally trivial objects are the $(X, \mathcal{O}_X)$ such that X admits an open covering $\{U_i : i \in I\}$ with the following property: $\forall\ i$,

$$\mathcal{O}_X | U_i \approx {!}_i^*(\mathcal{O}_*)_i.$$

[Note: ${!}_i^*$ is calculated per U_i, hence

$${!}_i^* : \underline{\mathrm{Sh}}(*, \underline{A}) \to \underline{\mathrm{Sh}}(U_i, \underline{A})$$

and $(\mathcal{O}_*)_i$ is an object in $\underline{\mathrm{Sh}}(*, \underline{A})$ that depends on i.]

26.19 EXAMPLE Consider the fibration $\mathrm{Ob} : \underline{\mathrm{CAT}} \to \underline{\mathrm{SET}}$ of 5.1. Place on $\underline{\mathrm{SET}}$ the "inclusion of elements" coverage κ (cf. 26.10) and take for $\underline{C}$ the singleton $\{\underline{1}\}$ in the fiber over $*$, thus the objects of $\underline{T}_{\underline{C}}$ are the small categories $\underline{C}$ such that

$\underset{\sim}{C} \xrightarrow{!} \underset{\sim}{1}$ is horizontal.

• The trivial objects $\neq \underset{\sim}{0}$ are the small categories $\underset{\sim}{C}$ such that $\forall\ X,Y \in \text{Ob}\ \underset{\sim}{C}$, $\#\text{Mor}(X,Y) = 1$.

[Assume first that $\underset{\sim}{C}$ is trivial and pass to the arrow $\text{Ob}\ \underset{\sim}{C} \rightarrow *$. Proceeding as in 5.1, construct a category $\tilde{\underset{\sim}{C}}$ and a horizontal $\tilde{\underset{\sim}{C}} \rightarrow \underset{\sim}{1}$ such that Ob ! is $\text{Ob}\ \underset{\sim}{C} \xrightarrow{!} *$ -- then $\forall\ \tilde{X},\tilde{Y} \in \text{Ob}\ \tilde{\underset{\sim}{C}}$, $\#\text{Mor}(\tilde{X},\tilde{Y}) = 1$. But since $\underset{\sim}{C} \xrightarrow{!} \underset{\sim}{1}$ is horizontal, there is a vertical isomorphism $v:\tilde{\underset{\sim}{C}} \rightarrow \underset{\sim}{C}$ and a commutative diagram

$$\begin{tikzcd}
\tilde{\underset{\sim}{C}} \arrow[r, "!"] \arrow[d, "v"'] & \underset{\sim}{1} \arrow[d, equal] \\
\underset{\sim}{C} \arrow[r, "!"'] & \underset{\sim}{1}
\end{tikzcd},$$

so $\forall\ X,Y \in \text{Ob}\ \underset{\sim}{C}$, $\#\text{Mor}(X,Y) = 1$, which settles the necessity. Turning to the sufficiency, consider a setup

$$\begin{tikzcd}
\underset{\sim}{C}_0 \arrow[r, dashed] \arrow[rr, bend left, "w"] & \underset{\sim}{C} \arrow[r, "!"'] & \underset{\sim}{1}
\end{tikzcd}, \quad \begin{tikzcd}
\text{Ob}\ \underset{\sim}{C}_0 \arrow[r, "x"'] \arrow[rr, bend left, "\text{Ob}\ w"] & \text{Ob}\ \underset{\sim}{C} \arrow[r, "!"'] & *
\end{tikzcd},$$

the claim being that there exists a unique functor $v:\underset{\sim}{C}_0 \rightarrow \underset{\sim}{C}$ such that $\text{Ob}\ v = x$ and $! \circ v = w$. This, however, is obvious: Define v on an object X_0 by $vX_0 = xX_0$ and on a morphism $f_0:X_0 \rightarrow Y_0$ by $vf_0 = f$, the unique element of $\text{Mor}(xX_0,xY_0)$.]

<u>N.B.</u> The arrow $\underset{\sim}{0} \xrightarrow{!} \underset{\sim}{1}$ is horizontal. Therefore $\underset{\sim}{0}$ is trivial.

[In the foregoing, let $\underset{\sim}{C} = \underset{\sim}{0}$ -- then $\text{Ob}\ \underset{\sim}{C} = \emptyset$, hence $\text{Ob}\ \underset{\sim}{C}_0 = \emptyset$ and $x = \text{id}_\emptyset$. And this means that $\underset{\sim}{C}_0 = \underset{\sim}{0}$, so $v = \text{id}_{\underset{\sim}{0}}$.]

By definition, if $\underline{C} \neq \underline{0}$, then

$$\kappa_{\text{Ob}\,\underline{C}} = \{\{X\} \xrightarrow{i_X} \text{Ob}\,\underline{C}: X \in \text{Ob}\,\underline{C}\}.$$

Choose a horizontal $u_X:\underline{C}_X \to \underline{C}$ such that $\text{Ob}\,u_X = i_X$, thus $\text{Ob}\,\underline{C}_X = \{X\}$. And

$$\{\underline{C}_X \xrightarrow{u_X} \underline{C}: X \in \text{Ob}\,\underline{C}\} \in (\text{Ob}^{-1}\kappa)_{\underline{C}}.$$

• The locally trivial objects $\neq \underline{0}$ are the small categories $\underline{C}$ such that $\forall\, X \in \text{Ob}\,\underline{C}$, $\text{Mor}(X,X) = \{\text{id}_X\}$.

[Construct $\underline{C}_X$ as in 5.1, thus $\forall\, X \in \text{Ob}\,\underline{C}$,

$$\text{Mor}_{\underline{C}_X}(X,X) = \{X\} \times \text{Mor}(X,X) \times \{X\},$$

implying thereby that

$$\#\text{Mor}_{\underline{C}_X}(X,X) = 1 \iff \#\text{Mor}(X,X) = 1.]$$

E.g.: Every set viewed as a discrete category is locally trivial.

26.20 EXAMPLE Viewing $\mathbb{R}$ as a topological ring, given a topological space B, let

$$\theta_B = (B \times \mathbb{R} \to B).$$

Then θ_B is an internal ring in $\underline{\text{TOP}}/B$. This said, denote by $\underline{M}_B$ the category whose objects are the internal θ_B-modules.

(*) Take $B = *$ -- then $\underline{M}_B$ is the category of real topological vector spaces.

Define a pseudo functor $F:\underline{\text{TOP}}^{\text{OP}} \to 2\text{-}\mathbf{CAT}$ by sending B to $\underline{M}_B$ and $\beta:B \to B'$ to $F\beta:\underline{M}_{B'} \to \underline{M}_B$ ("pullback"). Use now the notation of 7.7 and form $\text{gro}_{\underline{\text{TOP}}}F$, the objects of which are the pairs (B,M), where $B \in \text{Ob}\,\underline{\text{TOP}}$ and $M \in \text{Ob}\,FB$, and whose morphisms are the arrows $(\beta,f):(B,M) \to (B',M')$, where $\beta \in \text{Mor}(B,B')$ and $f \in \text{Mor}(M,(F\beta)M')$.

Consider the fibration $\Theta_F:\text{gro}_{\underline{TOP}}F \to \underline{TOP}$ of 7.9. Place on $\underline{TOP}$ the open subset coverage κ and take for $\underline{C}$ the subcategory of the fiber over $*$ whose objects are the R^n, thus the objects of $\underline{T}_{\underline{C}}$ are the pairs (B,M) which are the domain of a horizontal arrow $(B,M) \to (*,R^n)$ over $!:B \to *$ for some R^n.

- The trivial objects are the (B,M) such that $M \approx B \times R^n$.

[The point is that for any B, the morphism

$$(!,\text{id}_{(F!)R^n}):(B,(F!)R^n) \to (*,R^n)$$

is horizontal (cf. 7.12) and $(F!)R^n \approx B \times R^n$.]

Observe next that if U is an open subset of B, then $Fi_U:\underline{M}_B \to \underline{M}_U$. Agreeing to write $M|U$ in place of $(Fi_U)M$, the arrow

$$(i_U,\text{id}_{M|U}):(U,M|U) \to (B,M)$$

is horizontal (cf. 7.12). So if $B = \bigcup_{i\in I} U_i$, then

$$\{(i_{U_i},\text{id}_{M|U_i}):(U_i,M|U_i) \to (B,M)\} \in (\Theta_F^{-1}\kappa)_{(B,M)}.$$

- The locally trivial objects are the (B,M) such that B admits an open covering $\{U_i:i \in I\}$ with the following property: $\forall\, i$,

$$M|U_i \approx U_i \times R^{n_i}.$$

[Note: Here n_i depends on i and the isomorphism is computed in $\underline{M}_{U_i}$.]

26.21 RAPPEL The triple $<\underline{AB},\otimes,Z>$ is a symmetric monoidal category and the commutative monoids therein are the commutative rings with unit.

26.22 NOTATION Given $A \in \mathrm{Ob}\ \underline{\mathrm{RNG}}$, let $A\text{-}\underline{\mathrm{MOD}}$ be the category of left A-modules.

Let A,B be commutative rings with unit and suppose that $f:A \to B$ is a ring homomorphism -- then there is a functor

$$B\text{-}\underline{\mathrm{MOD}} \xrightarrow{\mathrm{res}_f} A\text{-}\underline{\mathrm{MOD}} \qquad \text{(restriction of scalars)}$$

and a functor

$$A\text{-}\underline{\mathrm{MOD}} \xrightarrow{\mathrm{ext}_f} B\text{-}\underline{\mathrm{MOD}} \qquad \text{(extension of scalars).}$$

26.23 LEMMA The functor ext_f is a left adjoint for the functor res_f.

26.24 NOTATION $\underline{\mathrm{MOD}}(\underline{\mathrm{AB}})$ is the category whose objects are the pairs (A,M), where A is a commutative ring with unit and M is a left A-module, and whose morphisms are the arrows $(f,\phi):(A,M) \to (B,N)$, where $f:A \to B$ is a ring homomorphism and $\phi:M \to N$ is a morphism in $\underline{\mathrm{AB}}$ such that the diagram

$$\begin{tikzcd} A \otimes M \arrow[r, "f \otimes \phi"] \arrow[d] & B \otimes N \arrow[d] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

commutes, the vertical arrows being the actions of A and B on M and N.

26.25 REMARK There is a 2-functor

$$F:\underline{\mathrm{RNG}}^{\mathrm{OP}} \to 2\text{-}\mathbf{CAT}$$

that sends A to $A\text{-}\underline{\mathrm{MOD}}$ and $f:R \to S$ to $\mathrm{res}_f:B\text{-}\underline{\mathrm{MOD}} \to A\text{-}\underline{\mathrm{MOD}}$. Its Grothendieck construction $\mathrm{gro}_{\underline{\mathrm{RNG}}}F$ can be identified with $\underline{\mathrm{MOD}}(\underline{\mathrm{AB}})$.

[Note: There is a pseudo functor

$$F:\underline{\mathrm{RNG}} \to 2\text{-}\mathbf{CAT}$$

that sends A to A-$\underline{\mathrm{MOD}}$ and $f:A \to B$ to $\mathrm{ext}_f:A\text{-}\underline{\mathrm{MOD}} \to B\text{-}\underline{\mathrm{MOD}}$.]

26.26 LEMMA The projection $(A,M) \to A$ defines a fibration

$$P_{\underline{\mathrm{AB}}}:\underline{\mathrm{MOD}}(\underline{\mathrm{AB}}) \to \underline{\mathrm{RNG}}.$$

PROOF Given (B,N) and $f:A \to B$, the morphism

$$(A,\mathrm{res}_f N) \to (B,N)$$

is horizontal.

26.27 LEMMA The projection $(R,M) \to R$ defines an opfibration

$$P_{\underline{\mathrm{AB}}}:\underline{\mathrm{MOD}}(\underline{\mathrm{AB}}) \to \underline{\mathrm{RNG}}.$$

PROOF Given (A,M) and $f:A \to B$, the morphism

$$(A,M) \to (B,B \otimes_A M)$$

is ophorizontal.

26.28 REMARK Therefore $P_{\underline{\mathrm{AB}}}$ is a bifibration (cf. 5.15).

26.29 EXAMPLE Consider the opfibration $P_{\underline{\mathrm{AB}}}:\underline{\mathrm{MOD}}(\underline{\mathrm{AB}}) \to \underline{\mathrm{RNG}}$ of 26.27. Place on $\underline{\mathrm{RNG}}$ the Zariski coverage κ (cf. 11.16) and bearing in mind 26.17, take for $\underline{C}$ the subcategory of the fiber over Z whose objects are the Z^n, thus the objects of $\underline{T}_{\underline{C}}$ are the pairs (A,M) which are the codomain of an ophorizontal arrow $(Z,Z^n) \to (A,M)$ over $!:Z \to A$ for some Z^n.

• The trivial objects are the (A,M) such that M is a free left A-module of finite rank.

• The locally trivial objects are the (A,M) such that M is a finitely generated projective left A-module.

APPENDIX

Fix a topological group G and consider the fibration $G\text{-}\underline{BUN}(\underline{TOP}) \to \underline{TOP}$ of 5.3 -- then its fiber $G\text{-}\underline{BUN}(TOP)_*$ over $*$ is (isomorphic to) $\underline{MOD}_G$, the category of right G-modules over the monoid G in $\underline{TOP}$. Take for $\underline{C}$ the singleton subcategory $\{G \to *\}$, thus the objects of $\underline{T}_{\underline{C}}$ are the $X \to B$ which are isomorphic to a product $X \times G \to B$.

• Place on $\underline{TOP}$ the open subset coverage -- then the locally trivial objects over B are those objects $X \to B$ in $\underline{PRIN}_{B,G}$ for which there exists an open covering $\{U_i : i \in I\}$ of B such that $\forall$ i, $X|U_i \approx U_i \times G$ in $\underline{PRIN}_{U_i,G}$.

• Place on $\underline{TOP}$ the open map coverage (cf. 11.19) -- then the locally trivial objects over B are the objects $X \to B$ of $\underline{PRIN}_{B,G}$.

STACKS

Let $\underline{B}$ be a category equipped with a Grothendieck coverage κ such that $\forall\ B \in \text{Ob}\ \underline{B}$, $\{id_B:B \to B\} \in \kappa_B$.

<u>ST-1</u>: NOTATION Given $\{g_i:B_i \to B\} \in \kappa_B$, put

$$B_{ij} = B_i \times_B B_j$$

and define π^1_{ij}, π^2_{ij} per the pullback square

$$\begin{tikzcd} B_{ij} \arrow[r, "\pi^1_{ij}"] \arrow[d, "\pi^2_{ij}"'] & B_i \arrow[d, "g_i"] \\ B_j \arrow[r, "g_j"'] & B \end{tikzcd} .$$

<u>ST-2</u>: NOTATION Given $\{g_i:B_i \to B\} \in \kappa_B$, put

$$B_{ijk} = B_i \times_B B_j \times_B B_k$$

and define π^{12}_{ijk}, π^{13}_{ijk}, π^{23}_{ijk} by the pullback squares

$$\begin{tikzcd} B_{ijk} \arrow[r, "\pi^{23}_{ijk}"] \arrow[d, "\pi^{12}_{ijk}"'] & B_{jk} \arrow[d, "\pi^1_{jk}"] \\ B_{ij} \arrow[r, "\pi^2_{ij}"'] & B_j \end{tikzcd} , \qquad \begin{tikzcd} B_{ijk} \arrow[r, "\pi^{23}_{ijk}"] \arrow[d, "\pi^{13}_{ijk}"'] & B_{jk} \arrow[d, "\pi^2_{jk}"] \\ B_{ik} \arrow[r, "\pi^2_{ik}"'] & B_k \end{tikzcd} .$$

Let $F:\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$ be a pseudo functor (cf. §3).

<u>ST-3</u>: DEFINITION A set of <u>descent data</u> on $\{g_i:B_i \to B\} \in \kappa_B$ is a collection of objects $X_i \in FB_i$ and a collection of isomorphisms

$$\phi_{ij}:F(\pi_{ij}^2)X_j \to F(\pi_{ij}^1)X_i$$

in FB_{ij} which satisfy the <u>cocycle condition</u>

$$F(\pi_{ijk}^{13})\phi_{ik} = F(\pi_{ijk}^{12})\phi_{ij} \circ F(\pi_{ijk}^{23})\phi_{jk}$$

in FB_{ijk} modulo the "coherency" implicit in F.

[Spelled out, the demand is that the composition

$$\begin{aligned}
&F(\pi_{ijk}^{23})F(\pi_{jk}^{2})X_k \\
&\xrightarrow{\gamma_{\pi_{ijk}^{23},\pi_{jk}^{2}}X_k} F(\pi_{jk}^{2} \circ \pi_{ijk}^{23})X_k \\
&=\!=\!= F(\pi_{ik}^{2} \circ \pi_{ijk}^{13})X_k \\
&\xrightarrow{\gamma^{-1}_{\pi_{ijk}^{13},\pi_{ik}^{2}}X_k} F(\pi_{ijk}^{13}) \circ F(\pi_{ik}^{2})X_k \\
&\xrightarrow{F(\pi_{ijk}^{13})\phi_{ik}} F(\pi_{ijk}^{13}) \circ F(\pi_{ik}^{1})X_i
\end{aligned}$$

is the same as the composition

$$F(\pi_{ijk}^{23})F(\pi_{jk}^{2})X_k$$

$$\begin{aligned}
&\xrightarrow{F(\pi^{23}_{ijk})\phi_{jk}} F(\pi^{23}_{ijk})F(\pi^{1}_{jk})X_j \\
&\xrightarrow{\gamma_{\pi^{23}_{ijk},\pi^{1}_{jk}} X_j} F(\pi^{1}_{jk} \circ \pi^{23}_{ijk})X_j \\
&=\!=\!=\!=\!= F(\pi^{2}_{ij} \circ \pi^{12}_{ijk})X_j \\
&\xrightarrow{\gamma^{-1}_{\pi^{12}_{ijk},\pi^{2}_{ij}} X_j} F(\pi^{12}_{ijk}) \circ F(\pi^{2}_{ij})X_j \\
&\xrightarrow{F(\pi^{12}_{ijk})\phi_{ij}} F(\pi^{12}_{ijk}) \circ F(\pi^{1}_{ij})X_i \\
&\xrightarrow{\gamma_{\pi^{12}_{ijk},\pi^{1}_{ij}} X_i} F(\pi^{1}_{ij} \circ \pi^{12}_{ijk})X_i \\
&=\!=\!=\!=\!= F(\pi^{1}_{ik} \circ \pi^{13}_{ijk})X_i \\
&\xrightarrow{\gamma^{-1}_{\pi^{13}_{ijk},\pi^{1}_{ik}} X_i} F(\pi^{13}_{ijk}) \circ F(\pi^{1}_{ik})X_i.]
\end{aligned}$$

ST-4: DEFINITION If

$$\left[\begin{array}{l}(\{X_i\},\{\phi_{ij}\}) \\ \\ (\{X_i'\},\{\phi_{ij}'\})\end{array}\right.$$

are sets of descent data on $\{g_i:B_i \to B\} \in \kappa_B$, then a morphism

$$(\{X_i\},\{\phi_{ij}\}) \to (\{X_i'\},\{\phi_{ij}'\})$$

is a collection of arrows $\xi_i:X_i \to X_i'$ in FB_i such that the diagram

$$\begin{tikzcd}
F(\pi^2_{ij})X_j \arrow[r, "\phi_{ij}"] \arrow[d, "F(\pi^2_{ij})\xi_j"'] & F(\pi^1_{ij})X_i \arrow[d, "F(\pi^1_{ij})\xi_i"] \\
F(\pi^2_{ij})X_j' \arrow[r, "\phi_{ij}'"'] & F(\pi^1_{ij})X_i'
\end{tikzcd}$$

commutes in FB_{ij}.

ST-5: NOTATION Given $\{g_i:B_i \to B\} \in \kappa_B$, there is a category

$$F(\{g_i:B_i \to B\})$$

whose objects are the sets of descent data and whose morphisms are as above.

ST-6: LEMMA The assignment

$$FB \to F(\{g_i:B_i \to B\})$$

that sends $X \in FB$ to

$$(\{F(g_i)X\},\{\phi_{ij}\}),$$

where

$$\phi_{ij} = (\gamma^F_{\pi^1_{ij}, g_i} X)^{-1} \circ (\gamma^F_{\pi^2_{ij}, g_j} X),$$

is a functor.

ST-7: DEFINITION Suppose given $\underline{B}$ and κ -- then a pseudo functor $F:\underline{B}^{OP} \to 2\text{-}\mathbf{CAT}$ is said to be a stack if for all $B \in \text{Ob } \underline{B}$ and all $\{g_i:B_i \to B\} \in \kappa_B$, the functor

$$FB \longrightarrow F(\{g_i:B_i \to B\})$$

is an equivalence of categories.

ST-8: REMARK Consider the setup of 18.12 -- then $F:\underline{C}^{OP} \to \underline{SET}$ is a sheaf iff it is a stack.

[Note: As usual, $\underline{SET}$ is viewed as a sub-2-category of $2\text{-}\mathbf{CAT}$ whose only 2-cells are identities.]

ST-9: EXAMPLE The pseudo functor

$$\underline{TOP}^{OP} \to 2\text{-}\mathbf{CAT}$$

that sends X to $\underline{TOP}/X$ is a stack in the open subset coverage.

ST-10: EXAMPLE The pseudo functor

$$\underline{SCH}^{OP} \to 2\text{-}\mathbf{CAT}$$

that sends X to $\underline{QCO}(X)$ is a stack in the fpqc coverage (hence in the Zariski coverage, the étale coverage, the smooth coverage, and the fppf coverage).

ST:11: EXAMPLE Given a topological group G, the pseudo functor

$$\underline{TOP}^{OP} \to 2\text{-}\mathbf{CAT}$$

that sends B to $\underline{PRIN}_{B,G}$ is a stack in the open subset coverage.