ABSTRACT

These notes can serve as a mathematical supplement to the standard graduate level texts on general relativity and are suitable for selfstudy. The exposition is detailed and includes accounts of several topics of current interest, e.g., Lovelock theory and Ashtekar's variables.

MATHEMATICAL ASPECTS OF GENERAL RELATIVITY

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CONTENTS

Section 0: Introduction A preliminary version of these notes was distributed to the participants in a seminar on quantum gravity which I gave a couple of years ago. As they seemed to be rather well received, I decided that a revised and expanded account might be useful for a wider audience.

Like the original, the focus is on the formalism underlying general relativity, thus there is no physics and virtually no discussion of exact solutions. More seriously, the Cauchy problem is not considered. My only defense for such an omission is that certain choices have to be made and to do the matter justice would require another book.

The prerequisites are modest: Just some differential geometry, much of which is reviewed in the text anyway. As for what is covered, some of the topics are standard, others less so. Included among the latter is a proof of the Lovelock uniqueness theorem, a systematic discussion of the Palatini formalism, a complete global treatment of the Ashtekar variables, and an introduction to the asymptotic theory.

For the most part, the exposition is detail oriented and directed toward the beginner, not the expert. Frankly, I tire quickly of phrases like: "it follows readily" or "one shows without difficulty" or "a short calculation gives" or "it is easy to see that" ETC. To be sure I have left some things for the reader to work out but I have tried not to make a habit of it.

While I have yet to get around to compiling an index, the text is not too difficult to navigate given the number of section headings.

Naturally, I would like to hear about any typos or outright errors and comments and suggestions for improvement would be much appreciated.

<u>Section 1:</u> <u>Geometric Quantities</u> Let V be an n-dimensional real vector space and let V* be its dual.

Notation: B(V) is the set of ordered bases for V.

The general linear group $\underline{GL}(n,\underline{R})$ operates to the right on B(V):

$$\begin{cases} B(V) \times \underline{GL}(n,\underline{R}) \rightarrow B(V) \\ (E,g) \longrightarrow E\cdot g. \end{cases}$$

In detail: If $E = \{E_1,\ldots,E_n\} \in B(V)$, then $E\cdot g = \{E_i g^i{}_1,\ldots,E_i g^i{}_n\}$.

[Note: Therefore row vector conventions are in force: $E\cdot g$ is computed by inspection of

$$[E_1,\ldots,E_n] \cdot \begin{bmatrix} g^1{}_1 & \cdots & g^1{}_n \\ \vdots & & \vdots \\ g^n{}_1 & \cdots & g^n{}_n \end{bmatrix} .]$$

If B(V*) stands for the set of ordered bases in V*, then $\underline{GL}(n,\underline{R})$ operates to the right on B(V*) via duality, i.e., via multiplication by $(g^{-1})^T$.

Given a basis $E = \{E_1,\ldots,E_n\} \in B(V)$, its cobasis $\omega = \{\omega^1,\ldots,\omega^n\} \in B(V^*)$ is defined by $\omega^i(E_j) = \delta^i{}_j$.

Observation: Let $g \in \underline{GL}(n,\underline{R})$ -- then the cobasis corresponding to $E\cdot g$ is $\omega\cdot g$.

[Since

$$\begin{cases} (E\cdot g)_j = E_i g^i{}_j \\ (\omega\cdot g)^\ell = \omega^k (g^{-1})^\ell{}_k, \end{cases}$$

it follows that

$$(\omega\cdot g)^{\ell}\ ((E\cdot g)_j) = <\ (E\cdot g)_j,\ (\omega\cdot g)^{\ell}>$$

$$= <\ E_i g^i{}_j,\ \omega^k (g^{-1})^{\ell}{}_k>$$

$$= \omega^k(E_i)\ g^i{}_j (g^{-1})^{\ell}{}_k$$

$$= \delta^k{}_i g^i{}_j\ (g^{-1})^{\ell}{}_k$$

$$= g^i{}_j (g^{-1})^{\ell}{}_i$$

$$= (g^{-1})^{\ell}{}_i\ g^i{}_j$$

$$= \delta^{\ell}{}_j.]$$

[Note: From the definitions,

$$(\omega\cdot g)^{\ell} = \sum_k \omega^k ((g^{-1})^T)^k{}_{\ell}$$

$$= \sum_k \omega^k (g^{-1})^{\ell}{}_k \equiv \omega^k (g^{-1})^{\ell}{}_k,$$

which explains the flip in the indices.]

Let V^p_q stand for the vector space of tensors of type (p,q), thus an element $T\in V^p_q$ is a multilinear map

$$T : \overbrace{V^* \times\cdots\times V^*}^{p} \times \overbrace{V \times\cdots\times V}^{q} \to \underline{R},$$

hence admits an expansion

$$T = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} (E_{i_1} \otimes \cdots \otimes E_{i_p}) \otimes (\omega^{j_1} \otimes \cdots \otimes \omega^{j_q}),$$

where

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

$$= T(\omega^{i_1}, \ldots, \omega^{i_p}, E_{j_1}, \ldots, E_{j_q}).$$

If

$$\begin{cases} E \rightarrow E \cdot g \\ \omega \rightarrow \omega \cdot g, \end{cases}$$

then the components of T satisfy the <u>tensor transformation rule</u>:

$$T^{i_1' \cdots i_p'}{}_{j_1' \cdots j_q'}$$

$$= (g^{-1})^{i_1'}{}_{i_1} \cdots (g^{-1})^{i_p'}{}_{i_p} (g)^{j_1}{}_{j_1'} \cdots (g)^{j_q}{}_{j_q'} \; T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}.$$

[Note: Any map

$$T : B(V) \rightarrow \underline{R}^{n^{p+q}}$$

that assigns to each $E \in B(V)$ an n^{p+q}-tuple

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

which obeys the tensor transformation rule determines a unique tensor of type (p,q). So, for instance, the Kronecker delta $\delta^i{}_j$ is a tensor of type $(1,1)$.]

<u>Reality Check</u> Let $\begin{cases} X \in V \\ \Lambda \in V^* \end{cases}$, say

$$\left[\begin{array}{ll} X = X^i E_i & (X^i = X(\omega^i)) \\ \Lambda = \Lambda_j \omega^j & (\Lambda_j = \Lambda(E_j)). \end{array}\right.$$

Now change the basis: $E \to E\cdot g$ — then $X = X^{i'}(E\cdot g)_{i'}$, where

$$\begin{aligned} X^{i'} &= X((\omega\cdot g)^{i'}) \\ &= X(\omega^i (g^{-1})^{i'}{}_i) \\ &= (g^{-1})^{i'}{}_i \, X(\omega^i) \\ &= (g^{-1})^{i'}{}_i \, X^i, \end{aligned}$$

and $\Lambda = \Lambda_{j'} \, (\omega\cdot g)^{j'}$, where

$$\begin{aligned} \Lambda_{j'} &= \Lambda((E\cdot g)_{j'}) \\ &= \Lambda(E_j g^j{}_{j'}) \\ &= g^j{}_{j'} \, \Lambda(E_j) \\ &= g^j{}_{j'} \, \Lambda_j. \end{aligned}$$

<u>LEMMA</u> There is a canonical isomorphism

$$\iota : V^{q+p'}_{p+q'} \to \mathrm{Hom}(V^p_q, V^{p'}_{q'}).$$

[Given $T \in V^{q+p'}_{p+q'}$, put

$$(\iota T)(X_1 \otimes \cdots \otimes X_p \otimes \Lambda^1 \otimes \cdots \otimes \Lambda^q) \; (\Lambda^{1'},\ldots,\Lambda^{p'}, X_{1'},\ldots,X_{q'})$$

$$= T(\Lambda^1,\ldots,\Lambda^q, \Lambda^{1'},\ldots,\Lambda^{p'}, X_1,\ldots,X_p, X_{1'},\ldots,X_{q'})$$

and extend by linearity.]

[Note: Take p'=0, q'=0 to conclude that V_p^q is the dual of V_q^p.]

Products There is a map

$$\begin{bmatrix} V_q^p \times V_{q'}^{p'} & \rightarrow & V_{q+q'}^{p+p'} \\ (T,T') & \longrightarrow & T \otimes T' \end{bmatrix},$$

viz.

$$(T \otimes T')(\Lambda^1,\ldots,\Lambda^{p+p'}, X_1,\ldots,X_{q+q'})$$

$$= T(\Lambda^1,\ldots,\Lambda^p,X_1,\ldots,X_q)T'(\Lambda^{p+1},\ldots,\Lambda^{p+p'},X_{q+1},\ldots,X_{q+q'}).$$

In terms of components,

$$(T \otimes T')^{i_1\cdots i_{p+p'}}{}_{j_1\cdots j_{q+q'}}$$

$$= T^{i_1\cdots i_p}{}_{j_1\cdots j_q} T'^{i_{p+1}\cdots i_{p+p'}}{}_{j_{q+1}\cdots j_{q+q'}}.$$

Contractions $\forall\, k: 1 \leq k \leq p$ & $\forall\, \ell: 1 \leq \ell \leq q$, there is a map

$$C_\ell^k : V_q^p \rightarrow V_{q-1}^{p-1},$$

viz.

$$C_\ell^k(X_1 \otimes \cdots \otimes X_p \otimes \Lambda^1 \otimes \cdots \otimes \Lambda^q)$$

$$= \Lambda^\ell(X_k)(X_1 \otimes \cdots \otimes \hat{X}_k \otimes \cdots \otimes X_p \otimes \Lambda^1 \otimes \cdots \otimes \hat{\Lambda}^\ell \otimes \cdots \otimes \Lambda^q).$$

In terms of components,

$$(C^k_\ell T)^{i_1 \cdots \hat{i}_k \cdots i_p}{}_{j_1 \cdots \hat{j}_\ell \cdots j_q}$$

$$= T^{i_1 \cdots i_{k-1} a i_{k+1} \cdots i_p}{}_{j_1 \cdots j_{\ell-1} a j_{\ell+1} \cdots j_q}.$$

Definition: The Kronecker symbol of order p is the tensor of type (p,p) defined by

$$\delta^{i_1 \cdots i_p}{}_{j_1 \cdots j_p} = \begin{vmatrix} \delta^{i_1}{}_{j_1} & \cdots & \delta^{i_1}{}_{j_p} \\ \vdots & & \vdots \\ \delta^{i_p}{}_{j_1} & \cdots & \delta^{i_p}{}_{j_p} \end{vmatrix} .$$

Put

$$\begin{cases} I = \{i_1, \ldots, i_p\} \\ J = \{j_1, \ldots, j_p\} \end{cases} .$$

Then

$$\delta^{i_1 \cdots i_p}{}_{j_1 \cdots j_p}$$

vanishes if $I \neq J$ but is

$$\begin{cases} +1 \text{ if } I \text{ is an even permutation of } J \\ -1 \text{ if } I \text{ is an odd permutation of } J. \end{cases}$$

[Note: The Kronecker symbol of order p is antisymmetric under interchange of any two of the indices $i_1,\ldots,i_p$ or under interchange of any two of the indices $j_1,\ldots,j_p$. So, if any two of the indices $i_1,\ldots,i_p$ or $j_1,\ldots,j_p$ coincide, then

$$\delta^{i_1\cdots i_p}{}_{j_1\cdots j_p} = 0,$$

which is automatic if $p>n$.]

Example: Let $T \in V_p^0$:

$$T = T_{j_1\cdots j_p}\omega^{j_1} \otimes \cdots \otimes \omega^{j_p}.$$

Put

$$T_{[j_1\cdots j_p]} = \frac{1}{p!}\delta^{i_1\cdots i_p}{}_{j_1\cdots j_p} T_{i_1\cdots i_p}.$$

Then

$$\text{Alt } T = T_{[j_1\cdots j_p]}\omega^{j_1} \otimes \cdots \otimes \omega^{j_p}$$

belongs to $\Lambda^p V$.

Note: If $T \in \Lambda^p V$ to begin with, then Alt $T = T$, hence Alt$\circ$ Alt = Alt. As an element of V_p^0, the components of Alt T are given by

$$(\text{Alt } T)_{j_1\cdots j_p} = T_{[j_1\cdots j_p]}.]$$

<u>FACT</u> Suppose that $q<p$ -- then

$$\delta^{i_1\cdots i_q k_{q+1}\cdots k_p}{}_{j_1\cdots j_q k_{q+1}\cdots k_p}$$

$$= \frac{(n-q)!}{(n-p)!} \delta^{i_1 \cdots i_q}{}_{j_1 \cdots j_q}.$$

In particular:

$$\delta^{i_1 \cdots i_p}{}_{i_1 \cdots i_p} = \frac{n!}{(n-p)!}.$$

<u>Determinant Formula</u> Let $A = [a^i{}_j]$ be an n-by-n matrix -- then

$$\det A = \begin{cases} \delta^{i_1 \cdots i_n}{}_{1 \cdots n} a^1{}_{i_1} \cdots a^n{}_{i_n} \\ \\ \delta^{1 \cdots n}{}_{j_1 \cdots j_n} a^{j_1}{}_1 \cdots a^{j_n}{}_n \end{cases}.$$

Consider

$$\underline{R}^{n^{p+q}} = \underline{R}^{n^p} \otimes \underline{R}^{n^q}.$$

$\underline{R}^{n^p}$ $(p>0)$: View the elements of $\underline{R}^n$ as column vectors -- then $\underline{GL}(n,\underline{R})$ operates to the left on $\underline{R}^n$ via multiplication by g, hence by tensoring on

$$\underline{R}^{n^p} = \overbrace{\underline{R}^n \otimes \cdots \otimes \underline{R}^n}^{p}.$$

$\underline{R}^{n^q}$ $(q>0)$: View the elements of $\underline{R}^n$ as column vectors -- then $\underline{GL}(n,\underline{R})$ operates to the left on $\underline{R}^n$ via multiplication by $(g^{-1})^T$, hence by tensoring on

$$\underline{R}^{n^q} = \overbrace{\underline{R}^n \otimes \cdots \otimes \underline{R}^n}^{q}.$$

Combine these to get a left action of $\underline{GL}(n,\underline{R})$ on $\underline{R}^{n^{p+q}}$. We now claim that the tensors of type (p,q) can be identified with the equivariant maps $T:B(V) \rightarrow \underline{R}^{n^{p+q}}$, i.e., with the maps $T: B(V) \to \underline{R}^{n^{p+q}}$ such that $\forall\, g$,

$$T(E\cdot g) = g^{-1}\cdot T(E).$$

[Note: Incorporation of g^{-1} shifts the left action to a right action (bear in mind that $\underline{GL}(n,\underline{R})$ operates to the right on $B(V)$).]

To see this, it suffices to remark that the tensor transformation rule is equivalent to equivariance. Thus take a tensor T of type (p,q) and put

$$T(E) = T^{i_1\cdots i_p}{}_{j_1\cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q}.$$

Then

$$g^{-1}\cdot T(E)$$

$$= T^{i_1\cdots i_p}{}_{j_1\cdots j_q} g^{-1}e_{i_1} \otimes \cdots \otimes g^{-1}e_{i_p} \otimes g^{T}e^{j_1} \otimes \cdots \otimes g^{T}e^{j_q}$$

$$= T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$\times (g^{-1})^{i_1'}{}_{i_1} e_{i_1'} \otimes \cdots \otimes (g^{-1})^{i_p'}{}_{i_p} e_{i_p'} \otimes (g)^{j_1}{}_{j_1'} e^{j_1'} \otimes \cdots \otimes (g)^{j_q}{}_{j_q'} e^{j_q'}$$

$$= (g^{-1})^{i_1'}{}_{i_1} \cdots (g^{-1})^{i_p'}{}_{i_p} (g)^{j_1}{}_{j_1'} \cdots (g)^{j_q}{}_{j_q'} T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$\times e_{i_1'} \otimes \cdots \otimes e_{i_p'} \otimes e^{j_1'} \otimes \cdots \otimes e^{j_q'}$$

$$= T^{i_1'\cdots i_p'}{}_{j_1'\cdots j_q'} e_{i_1'} \otimes \cdots \otimes e_{i_p'} \otimes e^{j_1'} \otimes \cdots \otimes e^{j_q'}$$

$$= T(E\cdot g),$$

which is equivariance (the converse is also clear).

There remains one point of detail, namely when $p = q = 0$. In this situation, $\underline{R}^{n^{p+q}} = \underline{R}$ and we shall agree that $\underline{GL}(n,\underline{R})$ operates trivially on $\underline{R}: gx = x \ \forall \ x \in \underline{R}$. Consequently, the tensors of type $(0,0)$ are the constant maps $T:B(V) \to \underline{R}$, i.e., $V_0^0 = \underline{R}$ (the usual agreement).

Definition: Let $\chi:\underline{GL}(n,\underline{R}) \to \underline{R}^\times$ be a continuous homomorphism -- then a <u>tensor of type (p,q) and weight χ</u> is a map

$$T:B(V) \to \underline{R}^{n^{p+q}}$$

such that $\forall$ g,

$$T(E\cdot g) = \chi(g) \ g^{-1}\cdot T(E).$$

Special Cases:

1. Tensors of type (p,q) are obtained by taking $\chi(g) = |\det g|^0$;

2. Twisted tensors of type (p,q) are obtained by taking $\chi(g) = \operatorname{sgn} \det g$.

Rappel: The continuous homomorphisms $\chi:\underline{GL}(n,\underline{R}) \to \underline{R}^\times$ fall into two classes:

I: $g \to |\det g|^r \quad (r \in \underline{R})$;

II: $g \to \operatorname{sgn} \det g \cdot |\det g|^r \quad (r \in \underline{R})$.

- A <u>density</u> is a map

$$\lambda:B(V) \to \underline{R}$$

for which $\exists \ r \in \underline{R}: \forall$ g,

$$\lambda(E\cdot g) = |\det g|^r \ \lambda(E).$$

- A <u>twisted density</u> is a map

$$\lambda:B(V) \to \underline{R}$$

for which $\exists\ r \in \underline{R}: \forall\ g$,

$$\lambda(E \cdot g) = \text{sgn det } g \cdot |\det g|^r\ \lambda(E).$$

[Note: In either case, r is called the weight of λ.]

Trivially, tensors of type (0,0) are densities of weight 0.

Example: Suppose that T is a tensor of type (0,2) and weight χ, where $\chi(g) = |\det g|^r$. Define

$$\lambda_T : B(V) \to \underline{R}$$

by

$$\lambda_T(E) = \det T(E)$$

$$\equiv \det [T_{j_1 j_2}].$$

Then

$$\lambda_T(E \cdot g) = \det [T_{j_1' j_2'}]$$

$$= \det[\chi(g)(g)^{j_1}{}_{j_1'}(g)^{j_2}{}_{j_2'}\ T_{j_1 j_2}]$$

$$= |\det g|^{rn} \det[(g)^{j_1}{}_{j_1'}(g)^{j_2}{}_{j_2'}\ T_{j_1 j_2}]$$

$$= |\det g|^{rn} \det[(g)^{j_1}{}_{j_1'}\ T_{j_1 j_2}(g)^{j_2}{}_{j_2'}]$$

$$= |\det g|^{rn} \det[(g)^{j_1}{}_{j_1'}\ (T(E)g)^{j_1}{}_{j_2'}]$$

$$= |\det g|^{rn} \det[(g^T)^{j_1'}{}_{j_1}\ (T(E)g)^{j_1}{}_{j_2'}]$$

$$= |\det g|^{rn} \det(g^T T(E) g)$$

$$= |\det g|^{rn} \det g^{\mathsf{T}} \cdot \det T(E) \cdot \det g$$

$$= |\det g|^{rn} (\det g)^2 \det T(E)$$

$$= |\det g|^{rn+2} \lambda_T(E).$$

Therefore λ_T is a density of weight rn+2.

[Note: If T were instead a χ-tensor of type (2,0) or (1,1) (χ as above), then the corresponding λ_T is a density of weight rn-2 or rn.]

Example (The Orientation Map): In B(V), write $E' \sim E$ iff $\exists\ g \in \underline{GL}(n,\underline{R})$ $(\det g > 0)$: $E' = E \cdot g$. This is an equivalence relation in B(V) and it divides B(V) into two equivalence classes, say $B(V) = B^+(V) \amalg B^-(V)$. Define a map

$$Or: B(V) \to \underline{R}$$

by

$$\begin{cases} Or(B^+(V)) = \{+1\} \\ Or(B^-(V)) = \{-1\}. \end{cases}$$

Then $\forall$ g,

$$Or(E \cdot g) = \operatorname{sgn} \det g \cdot Or(E).$$

Therefore Or is a twisted density of weight 0.

[Note: Recall that two elements $E_1^+, E_2^+ \in B^+(V)$ or $E_1^-, E_2^- \in B^-(V)$ are said to have the <u>same orientation</u>, whereas two elements $E^+ \in B^+(V)$, $E^- \in B^-(V)$ are said to have the <u>opposite orientation</u>.]

Definition: A <u>scalar density</u> is a map

$$\lambda: B(V) \to \underline{R}$$

for which $\exists\ w \in \underline{Z}: \forall\ g$,

$$\lambda(E \cdot g) = (\det g)^{w}\ \lambda(E).$$

[Note: We have

$$(\det g)^{w} = \begin{cases} |\det g|^{w} & (w \text{ even}) \\ \operatorname{sgn} \det g \cdot |\det g|^{w} & (w \text{ odd}), \end{cases}$$

w being termed the <u>weight</u> of λ.]

<u>n-forms</u> Since $\Lambda^{n}V \subset V_{n}^{0}$, an element $T \in \Lambda^{n}V$ can be regarded as an equivariant map

$$B(V) \rightarrow \underline{R}^{n^{n}} \quad (p = 0,\ q = n).$$

We have

$$T = T_{j_1 \cdots j_n} \omega^{j_1} \otimes \cdots \otimes \omega^{j_n}$$

$$= T_{[j_1 \cdots j_n]} \omega^{j_1} \otimes \cdots \otimes \omega^{j_n}$$

$$= \frac{1}{n!} \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} T_{i_1 \cdots i_n} \omega^{j_1} \otimes \cdots \otimes \omega^{j_n}$$

$$= \frac{1}{n!} T_{j_1 \cdots j_n} \omega^{j_1} \wedge \cdots \wedge \omega^{j_n}$$

$$= T_{1 \ldots n} \omega^{1} \wedge \cdots \wedge \omega^{n}.$$

Therefore T also determines a map

$$B(V) \rightarrow \underline{R},$$

viz.

$$T(E) = T_{1 \ldots n}.$$

Consider the volume form

$$\omega^1 \wedge \cdots \wedge \omega^n .$$

Then $\forall$ g,

$$(\omega\cdot g)^{1'} \wedge \cdots \wedge (\omega\cdot g)^{n'}$$

$$= \omega^{j_1}(g^{-1})^{1'}{}_{j_1} \wedge \cdots \wedge \omega^{j_n}(g^{-1})^{n'}{}_{j_n}$$

$$= (g^{-1})^{1'}{}_{j_1}\cdots(g^{-1})^{n'}{}_{j_n}\, \omega^{j_1} \wedge \cdots \wedge \omega^{j_n}$$

$$= (g^{-1})^{1}{}_{j_1} \cdots (g^{-1})^{n}{}_{j_n}\, \delta^{j_1\cdots j_n}{}_{1\cdots n}\, \omega^1 \wedge \cdots \wedge \omega^n$$

$$= (\det g^{-1})\omega^1 \wedge \cdots \wedge \omega^n.$$

But

$$T = T_{1'\cdots n'}(\omega\cdot g)^{1'} \wedge \cdots \wedge (\omega\cdot g)^{n'}$$

$\Rightarrow$

$$T = T_{1'\cdots n'}(\det g^{-1})\omega^1 \wedge \cdots \wedge \omega^n$$

$\Rightarrow$

$$T_{1'\cdots n'}(\det g^{-1}) = T_{1\cdots n}$$

$\Rightarrow$

$$T_{1'\cdots n'} = (\det g)T_{1\cdots n}$$

$\Rightarrow$

$$T(E\cdot g) = (\det g)T(E).$$

Thus in this way one can attach to each $T \in \Lambda^n V$ a scalar density of weight 1.

[Note: Define

$$|T|:B(V) \rightarrow \underline{R}$$

by

$$|T|(E) = |T(E)|.$$

Then $\forall$ g,

$$\begin{aligned} |T|(E\cdot g) &= |T(E\cdot g)| \\ &= |(\det g)T(E)| \\ &= |\det g| \ |T(E)| \\ &= |\det g| \ |T|(E). \end{aligned}$$

I.e.: $|T|$ is a density of weight 1.]

Definition: The <u>upper Levi-Civita symbol of order n</u> is

$$\varepsilon^{i_1 \cdots i_n} = \delta^{i_1 \cdots i_n}{}_{1 \cdots n}$$

and the <u>lower Levi-Civita symbol of order n</u> is

$$\varepsilon_{j_1 \cdots j_n} = \delta^{1 \cdots n}{}_{j_1 \cdots j_n} .$$

<u>Determinant Formula</u> Let $A = [a^i{}_j]$ be an n-by-n matrix — then

$$\begin{cases} \varepsilon^{i_1' \cdots i_n'} \det A = \varepsilon^{i_1 \cdots i_n} a^{i_1'}{}_{i_1} \cdots a^{i_n'}{}_{i_n} \\ \varepsilon_{j_1' \cdots j_n'} \det A = \varepsilon_{j_1 \cdots j_n} a^{j_1}{}_{j_1'} \cdots a^{j_n}{}_{j_n'} . \end{cases}$$

Under a change of basis,

$$\varepsilon^{i_1'\cdots i_n'}$$

$$= \det g \; (g^{-1})^{i_1'}{}_{i_1} \cdots (g^{-1})^{i_n'}{}_{i_n} \; \varepsilon^{i_1\cdots i_n}$$

and

$$\varepsilon_{j_1'\cdots j_n'}$$

$$= \det g^{-1} (g)^{j_1}{}_{j_1'} \cdots (g)^{j_n}{}_{j_n'} \; \varepsilon_{j_1\cdots j_n} \; .$$

Therefore the upper (lower) Levi-Civita symbol is a tensor of type (n,0) (type (0,n)) and weight $\chi = \det$ ($\chi = \det^{-1}$).

Remark: The components of the Levi-Civita symbol (upper or lower) have the same numerical values w.r.t. all bases. They are +1, -1, or 0.

<u>Identities</u> We have

$$\varepsilon^{i_1\cdots i_n} \varepsilon_{j_1\cdots j_n} = \delta^{i_1\cdots i_n}{}_{j_1\cdots j_n}$$

and

$$\varepsilon^{i_1\cdots i_p k_1\cdots k_{n-p}} \varepsilon_{j_1\cdots j_p k_1\cdots k_{n-p}}$$

$$= (n-p)! \; \delta^{i_1\cdots i_p}{}_{j_1\cdots j_p} \; .$$

Example: Let $A = [a^i{}_j]$ be an n-by-n matrix -- then

$$\varepsilon_{j_1'\cdots j_n'} \det A = \varepsilon_{j_1\cdots j_n} \, a^{j_1}{}_{j_1'} \cdots a^{j_n}{}_{j_n'}$$

$\Rightarrow$

$$\varepsilon^{j_1'\cdots j_n'}\varepsilon_{j_1'\cdots j_n'} \det A$$

$$= \varepsilon^{j_1'\cdots j_n'}\varepsilon_{j_1\cdots j_n} a^{j_1}{}_{j_1'}\cdots a^{j_n}{}_{j_n'}$$

$\Rightarrow$

$$\delta^{j_1'\cdots j_n'}{}_{j_1'\cdots j_n'} \det A$$

$$= \delta^{j_1'\cdots j_n'}{}_{j_1\cdots j_n} a^{j_1}{}_{j_1'}\cdots a^{j_n}{}_{j_n'}$$

$\Rightarrow$

$$\det A = \frac{1}{n!}\, \delta^{j_1'\cdots j_n'}{}_{j_1\cdots j_n} a^{j_1}{}_{j_1'}\cdots a^{j_n}{}_{j_n'} .]$$

From its very definition,

$$\omega^{i_1} \wedge \cdots \wedge \omega^{i_n} = \varepsilon^{i_1\cdots i_n}\, \omega^1 \wedge \cdots \wedge \omega^n.$$

The interpretation of $\varepsilon_{j_1\cdots j_n}$ is, however, less direct.

Rappel: Each $X \in V$ defines an antiderivation $\iota_X : \Lambda^* V \to \Lambda^* V$ of degree -1, the <u>interior product</u> w.r.t. X. Explicitly: $\forall\ T \in \Lambda^p V$,

$$\iota_X T(X_1,\ldots,X_{p-1}) = T(X,X_1,\ldots,X_{p-1}).$$

One has

$$\iota_X(T_1 \wedge T_2) = \iota_X T_1 \wedge T_2 + (-1)^p T_1 \wedge \iota_X T_2.$$

Properties: (1) $\iota_X \circ \iota_X = 0$; (2) $\iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0$; (3) $\iota_{X+Y} = \iota_X + \iota_Y$; (4) $\iota_{rX} = r\iota_X$.

Example: By definition,

$$\iota_{E_j}(\omega^i) = \omega^i(E_j) = \delta^i{}_j.$$

<u>LEMMA</u> Let $T \in \Lambda^p V$, say

$$T = \frac{1}{p!} T_{j_1 \cdots j_p} \omega^{j_1} \wedge \cdots \wedge \omega^{j_p}.$$

Then

$$\iota_{E_i} T = \frac{1}{(p-1)!} T_{i j_2 \cdots j_p} \omega^{j_2} \wedge \cdots \wedge \omega^{j_p}.$$

Example: $\forall\ T \in \Lambda^p V$,

$$\begin{bmatrix} \omega^i \wedge \iota_{E_i} T = pT \\ \\ \iota_{E_i}(\omega^i \wedge T) = (n-p)T. \end{bmatrix}$$

Put

$$\text{vol}_E = \omega^1 \wedge \cdots \wedge \omega^n$$

and then set

$$\text{vol}_j = \iota_{E_j} \text{vol}_E.$$

Proceed from here by iteration:

$$\text{vol}_{j_1 j_2} = \iota_{E_{j_2}} \text{vol}_{j_1}$$

$$\vdots$$

$$\text{vol}_{j_1 \cdots j_n} = \iota_{E_{j_n}} \cdots \iota_{E_{j_1}} \text{vol}_E .$$

FACT We have

$$\mathrm{vol}_{j_1 \cdots j_n} = \varepsilon_{j_1 \cdots j_n} .$$

In the definition of density, twisted density, or scalar density, one can replace the target $\underline{R}$ by any finite dimensional real vector space W.

Example (The T-Construction): Let T be a symmetric tensor of type (0,2). Assume: T is nonsingular, hence det $T(E) \neq 0$ for all $E \in B(V)$. Define

$$\lambda_{|T|} : B(V) \to \underline{R}$$

by

$$\begin{aligned} \lambda_{|T|}(E) &= |\lambda_T(E)| \\ &= |\det T(E)|. \end{aligned}$$

Then $\forall$ g,

$$\begin{aligned} \lambda_{|T|}(E \cdot g) &= |\lambda_T(E \cdot g)| \\ &= |\det g|^2 \lambda_{|T|}(E). \end{aligned}$$

Given $E \in B(V)$, put

$$\mathrm{vol}_T(E) = (\lambda_{|T|}(E))^{1/2} \mathrm{vol}_E,$$

where, as before,

$$\mathrm{vol}_E = \omega^1 \wedge \cdots \wedge \omega^n.$$

Accordingly,

$$\mathrm{vol}_T : B(V) \to \Lambda^n V.$$

And $\forall$ g,

$$\begin{aligned} &\mathrm{vol}_T(E \cdot g) \\ &\quad = (\lambda_{|T|}(E \cdot g))^{1/2} (\omega \cdot g)^{1'} \wedge \cdots \wedge (\omega \cdot g)^{n'} \end{aligned}$$

$$= (\lambda_{|T|}(E\cdot g))^{1/2} (\det g^{-1})\omega^1 \wedge \cdots \wedge \omega^n$$

$$= |\det g| (\det g)^{-1} (\lambda_{|T|}(E))^{1/2} \mathrm{vol}_E$$

$$= \mathrm{sgn} \det g\cdot \mathrm{vol}_T(E).$$

Therefore vol_T is a $\Lambda^n V$-valued twisted density of weight 0.

[Note: It follows that the n-form $\mathrm{vol}_T(E)$ is an invariant of $E\in B^+(V)$ or $E\in B^-(V)$.]

Let $\varepsilon^\bullet$ stand for the upper Levi-Civita symbol -- then $\varepsilon^\bullet : B(V) \to \underline{R}^{n^n}$ is a tensor of type (n,0) and weight $\chi = \det$. On the other hand,

$$\frac{1}{(\lambda_{|T|})^{1/2}} : B(V) \to \underline{R}$$

is a density of weight -1 (T as above). Therefore the product

$$e^\bullet = \frac{1}{(\lambda_{|T|})^{1/2}} \cdot \varepsilon^\bullet$$

is a twisted tensor of type (n,0).

[Note: Analogous considerations apply to the lower Levi-Civita symbol $\varepsilon_\bullet$: The product

$$e_\bullet = (\lambda_{|T|})^{1/2} \cdot \varepsilon_\bullet$$

is a twisted tensor of type (0,n).]

Example: Consider

$$\mathrm{vol}_T(E) = (\lambda_{|T|}(E))^{1/2} \mathrm{vol}_E.$$

Then

$$\mathrm{vol}_E = \frac{1}{n!} \varepsilon_{j_1\cdots j_n} \omega^{j_1} \wedge \cdots \wedge \omega^{j_n}$$

$\Rightarrow$

$$\mathrm{vol}_T(E) = \frac{1}{n!}\, e_{j_1 \cdots j_n}\, \omega^{j_1} \wedge \cdots \wedge \omega^{j_n} .$$

Section 2: Scalar Products Fix a pair (k,n-k), where $0 \leq k \leq n$. Put

$$\eta = \begin{bmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} .$$

Then the prescription

$$< x,y >_k = \eta_{ij}\, x^i y^j \quad \left[\begin{array}{l} x = x^i e_i \\ \\ y = y^j e_j \end{array}\right.$$

defines a scalar product on $\underline{R}^n$.

Definition: The semiorthogonal group $\underline{O}(k,n-k)$ consists of those $A \in \underline{GL}(n,\underline{R})$ such that $\forall\, x,y \in \underline{R}^n$,

$$< Ax,Ay >_k = < x,y >_k .$$

[Note: This amounts to requiring that

$$A^T \eta A = \eta .]$$

In other words, if $\underline{R}^{k,n-k}$ stands for $\underline{R}^n$ equipped with the inner product $< \ , \ >_k$, then $\underline{O}(k,n-k)$ is the linear isometry group of $\underline{R}^{k,n-k}$.

FACT $\forall\, A \in \underline{O}(k,n-k)$, $\det A = \pm 1$.

It is not difficult to see that

$$\underline{O}(k,n-k) \approx \underline{O}(n-k,k).$$

If k=0 or k=n,

$$\underline{O}(0,n) \approx \underline{O}(n,0)$$

is the orthogonal group $\underline{O}(n)$. It has two components

$$\begin{cases} \underline{O}^+(n) = \{A \in \underline{O}(n) : \det A = +1\} \\ \underline{O}^-(n) = \{A \in \underline{O}(n) : \det A = -1\}. \end{cases}$$

Suppose that $0 < k < n$ -- then $\underline{O}(k,n-k)$ has four components

$$\underline{O}^{++}(k,n-k),\ \underline{O}^{+-}(k,n-k),\ \underline{O}^{-+}(k,n-k),\ \underline{O}^{--}(k,n-k)$$

indexed by the signs of $\det A_T$ and $\det A_S$. Here

$$A = \begin{bmatrix} A_T & B \\ C & A_S \end{bmatrix}$$

with

$$A_T \in \underline{GL}(k,\underline{R}),\ A_S \in \underline{GL}(n-k,\underline{R}).$$

Definition: The special semiorthogonal group $\underline{SO}(k,n-k)$ consists of those $A \in \underline{O}(k,n-k)$ such that $\det A = 1$.

Therefore

$$\underline{SO}(k,n-k) = \underline{O}^{++}(k,n-k) \cup \underline{O}^{--}(k,n-k)$$

is both open and closed in $\underline{O}(k,n-k)$. One has

$$\underline{so}(k,n-k) = \underline{o}(k,n-k) = \{A \in \underline{g\ell}(n,\underline{R}) : A^T = -\eta A \eta\}.$$

Remark: By construction, $\underline{SO}(k,n-k)$ is the group of orientation preserving linear isometries $\underline{R}^{k,n-k} \to \underline{R}^{k,n-k}$. On the other hand,

$$\begin{cases} \underline{O}^{++}(k,n-k) \cup \underline{O}^{+-}(k,n-k) \\ \underline{O}^{++}(k,n-k) \cup \underline{O}^{-+}(k,n-k) \end{cases}$$

consist of those linear isometries $\underline{R}^{k,n-k} \to \underline{R}^{k,n-k}$ that preserve the

$$\begin{cases}\text{time orientation}\\ \text{space orientation,}\end{cases}$$

respectively.

[Note: If $0 < k < n$, then each of the groups

$$\begin{cases}\underline{O}^{++}(k,n-k) \cup \underline{O}^{--}(k,n-k)\\ \underline{O}^{++}(k,n-k) \cup \underline{O}^{+-}(k,n-k)\\ \underline{O}^{++}(k,n-k) \cup \underline{O}^{-+}(k,n-k)\end{cases}$$

is of index 2 in $\underline{O}(k,n-k)$.]

Let V be an n-dimensional real vector space -- then a scalar product on V is a nondegenerate symmetric bilinear form

$$g:V \times V \to \underline{R}.$$

N.B. Nondegeneracy amounts to saying that the map $g^{\flat}:V \to V^*$ defined by

$$g^{\flat}X(Y) = g(X,Y)$$

is bijective.

[Note: The inverse to $g^{\flat}$ is denoted by $g^{\#}$.]

Therefore g is a symmetric tensor of type $(0,2):g \in V_2^0$. In terms of a basis $E = \{E_1,\ldots,E_n\} \in B(V)$ and its cobasis $\omega = \{\omega^1,\ldots,\omega^n\} \in B(V^*)$,

$$g = g_{ij}\omega^i \otimes \omega^j,$$

where

$$g_{ij} = g(E_i,E_j) = g(E_j,E_i) = g_{ji}.$$

Observation: The assignment

$$g^{-1}:V^* \times V^* \to \underline{R}$$

characterized by the condition

$$g^{-1}(g^{\flat}X, g^{\flat}Y) = g(X,Y)$$

is a scalar product on V*.

Therefore g^{-1} is a symmetric tensor of type $(2,0): g^{-1} \in V_0^2$. And here

$$(g^{-1})^{ij} = g^{ij},$$

where g^{ij} is the ij^{th} entry of the matrix inverse to $[g_{ij}]$, so

$$g^{-1} = g^{ij} E_i \otimes E_j.$$

<u>LEMMA</u> We have

$$\begin{cases} \varepsilon_{j_1 \cdots j_n} = \dfrac{1}{\det g(E)} g_{j_1 i_1} \cdots g_{j_n i_n} \varepsilon^{i_1 \cdots i_n} \\ \varepsilon^{i_1 \cdots i_n} = \det g(E)\, g^{i_1 j_1} \cdots g^{i_n j_n} \varepsilon_{j_1 \cdots j_n}. \end{cases}$$

[Note: In the jargon of the trade, this shows that $\varepsilon_{\bullet}$ and $\varepsilon^{\bullet}$ are <u>not</u> obtained from one another by the operations of lowering or raising indices.]

Notation: Given $E \in B(V)$, put

$$|g|(E) = |\det g(E)|.$$

In the T-construction, take $T = g$ -- then

$$\lambda_{|g|}(E) = |\det g(E)| = |g|(E)$$

and, by definition,

$$\text{vol}_g(E) = (|g|(E))^{1/2}\, \text{vol}_E,$$

an n-form that depends only on the orientation class of E. Moreover,

$$\begin{cases} e^{\bullet} = \dfrac{1}{|g|^{1/2}} \cdot \varepsilon^{\bullet} \\ \\ e_{\bullet} = |g|^{1/2} \cdot \varepsilon_{\bullet} \ , \end{cases}$$

these being twisted tensors of type $\begin{cases} (n,0) \\ \\ (0,n) \end{cases}$.

<u>LEMMA</u> We have

$$\begin{cases} e_{j_1 \cdots j_n} = \text{sgn det } g(E) g_{j_1 i_1} \cdots g_{j_n i_n} e^{i_1 \cdots i_n} \\ \\ e^{i_1 \cdots i_n} = \text{sgn det } g(E) g^{i_1 j_1} \cdots g^{i_n j_n} e_{j_1 \cdots j_n} \ . \end{cases}$$

Definition: An element $E \in B(V)$ is said to be <u>orthonormal</u> if

$$g(E) = \text{diag}(-1, \ldots, -1, 1, \ldots, 1).$$

It is well-known that g admits such a basis.

[Note: The pair (k, n-k), where k is the number of (-1)-entries and n-k is the number of (+1)-entries, is called the <u>signature</u> of g and $\iota \in \{0,1\}$: $\iota \equiv k \bmod 2$ ($\Rightarrow (-1)^{\iota} = \text{sgn det } g(E)$) is called the <u>index</u> of g. These entities are well-defined, i.e., independent of E. In fact, the orthonormal elements of B(V) per g are precisely the $E \cdot A$ ($A \in \underline{O}(k, n-k)$).]

Remark: If $E \in B(V)$ is arbitrary, then

$$\text{sgn det } g(E) = (-1)^{\iota}.$$

Let $\underline{M}_{k,n-k}$ be the set of scalar products on V of signature (k,n-k) -- then

$$\underline{M}_{k,n-k} \leftrightarrow B(V)/\underline{O}(k,n-k)$$

or still,

$$\underline{M}_{k,n-k} \leftrightarrow \underline{GL}(n,\underline{R})/\underline{O}(k,n-k).$$

[Note: If $E = \{E_1,\ldots,E_n\} \in B(V)$, then the prescription

$$g_E(X,Y) = \eta_{ij}X^iY^j \qquad \begin{cases} X = X^iE_i \\ Y = Y^jE_j \end{cases}$$

defines a scalar product $g_E \in \underline{M}_{k,n-k}$ having E as an orthonormal basis. And

$$g_E = g_{E\cdot A}$$

for all $A \in \underline{O}(k,n-k)$.]

Suppose that $g \in \underline{M}_{k,n-k}$ and $E \in B(V)$ is orthonormal. Put

$$\varepsilon_i = g(E_i,E_i).$$

Then

$$\varepsilon_i = \begin{cases} -1 & (1 \leq i \leq k) \\ +1 & (k+1 \leq i \leq n). \end{cases}$$

<u>LEMMA</u> We have

$$g^\flat E_i = \varepsilon_i\omega^i \quad \text{(no sum)}.$$

Remark: If $E \in B(V)$ is arbitrary, then

$$\begin{cases} g^\flat E_i = g_{ij}\omega^j \quad (\equiv \omega_i) \\ g^\# \omega^i = g^{ij}E_j \quad (\equiv E^i). \end{cases}$$

Initially, we started with a scalar product g on V and then saw how g induces a scalar product on V*. More is true: g induces a scalar product $g[{}^p_q]$ on each of the V^p_q.

[Note: Here, $g[{}^1_0] = g$ and $g[{}^0_1] = g^{-1}$.]

Notation: Given $T \in V^p_q$, define

$$T^{\flat} \in V_{p+q}$$

by

$$T^{\flat}(X_1,\ldots,X_{p+q})$$
$$= T(g^{\flat}X_1,\ldots,g^{\flat}X_p,X_{p+1},\ldots,X_{p+q})$$

and define

$$T^{\#} \in V^{p+q}$$

by

$$T^{\#}(\Lambda_1,\ldots,\Lambda_{p+q})$$
$$= T(\Lambda_1,\ldots,\Lambda_p,g^{\#}\Lambda_{p+1},\ldots,g^{\#}\Lambda_{p+q}).$$

Components of $T^{\flat}$:

$$T_{i_1\cdots i_p j_1\cdots j_q} = g_{i_1k_1}\cdots g_{i_pk_p}T^{k_1\cdots k_p}{}_{j_1\cdots j_q}.$$

Components of $T^{\#}$:

$$T^{i_1\cdots i_p j_1\cdots j_q} = g^{j_1\ell_1}\cdots g^{j_q\ell_q}T^{i_1\cdots i_p}{}_{\ell_1\cdots\ell_q}.$$

Remark: If $p = 0$, then $T^{\flat} = T$ and if $q = 0$, then $T^{\#} = T$.

Example: Take T = g -- then

$$g^{\#}(g^{\flat}X, g^{\flat}Y)$$

$$= g(g^{\#}g^{\flat}X, g^{\#}g^{\flat}Y)$$

$$= g(X,Y)$$

$$\Rightarrow \quad g^{\#} = g^{-1}.$$

<u>LEMMA</u> The bilinear form

$$g[{}^p_q] : V^p_q \times V^p_q \to \underline{R}$$

that sends (T,S) to the complete contraction

$$C^1_1 \cdots C^{p+q}_{p+q} (T^{\#} \otimes S^{\flat})$$

is a scalar product on V^p_q.

[Note: If g is positive definite, then so is $g[{}^p_q]$.]

From the definitions,

$$T^{\#} \otimes S^{\flat} \in V^{p+q}_{p+q} .$$

Therefore

$$(T^{\#} \otimes S^{\flat})^{i_1 \cdots i_{p+q}}{}_{j_1 \cdots j_{p+q}}$$

$$= g^{i_{p+1}\ell_{p+1}} \cdots g^{i_{p+q}\ell_{p+q}} \, T^{i_1 \cdots i_p}{}_{\ell_{p+1} \cdots \ell_{p+q}}$$

$$\times \, g_{j_1 k_1} \cdots g_{j_p k_p} \, S^{k_1 \cdots k_p}{}_{j_{p+1} \cdots j_{p+q}}$$

$$= T^{i_1 \cdots i_p \, i_{p+1} \cdots i_{p+q}} S_{j_1 \cdots j_p \, j_{p+1} \cdots j_{p+q}} .$$

To compute the complete contraction of $T^{\#} \otimes S^{\flat}$, one then sets $i_1 = j_1, \ldots, i_{p+q} = j_{p+q}$ and sums the result.

Example: Suppose that $T \in V_2^0$ & $S \in V_2^0$ -- then $T^{\#} \in V_0^2$ & $S^{\flat} = S$, so

$$g[{}^0_2](T,S) = C_1^1 C_2^2 (T^{\#} \otimes S)$$

$$= (T^{\#} \otimes S)^{i_1 i_2}{}_{i_1 i_2}$$

$$= g^{i_1 \ell_1} g^{i_2 \ell_2} T_{\ell_1 \ell_2} S_{i_1 i_2}$$

$$= T^{i_1 i_2} S_{i_1 i_2} .$$

[Note: Take $T = g$ -- then

$$g[{}^0_2](g,S) = g^{i_1 i_2} S_{i_1 i_2}$$

$$= g^{i_2 i_1} S_{i_1 i_2}$$

$$= S^{i_2}{}_{i_2} .]$$

Let $E \in \mathcal{B}(V)$ be orthonormal -- then $(\omega^i)_j = \delta^i{}_j$

$$\Rightarrow$$

$$(\omega^{i_1} \otimes \cdots \otimes \omega^{i_n})_{j_1 \cdots j_n} = \delta^{i_1}{}_{j_1} \cdots \delta^{i_n}{}_{j_n} .$$

Therefore

$$g[{}^n_0](\mathrm{vol}_E, \mathrm{vol}_E)$$

$$= g[^n_0](\varepsilon_{i_1\cdots i_n}\omega^{i_1}\otimes\cdots\otimes\omega^{i_n},\ \varepsilon_{j_1\cdots j_n}\omega^{j_1}\otimes\cdots\otimes\omega^{j_n})$$

$$= \varepsilon_{i_1\cdots i_n}\varepsilon_{j_1\cdots j_n}\ g[^n_0](\omega^{i_1}\otimes\cdots\otimes\omega^{i_n},\ \omega^{j_1}\otimes\cdots\otimes\omega^{j_n})$$

$$= \varepsilon_{i_1\cdots i_n}\varepsilon_{j_1\cdots j_n}$$

$$\times\ (\omega^{i_1}\otimes\cdots\otimes\omega^{i_n})^{k_1\cdots k_n}(\omega^{j_1}\otimes\cdots\otimes\omega^{j_n})_{k_1\cdots k_n}$$

$$= \varepsilon_{i_1\cdots i_n}\varepsilon_{j_1\cdots j_n}$$

$$\times\ g^{k_1\ell_1}\cdots g^{k_n\ell_n}\,\delta^{i_1}{}_{\ell_1}\cdots\delta^{i_n}{}_{\ell_n}\delta^{j_1}{}_{k_1}\cdots\delta^{j_n}{}_{k_n}$$

$$= \varepsilon_{i_1\cdots i_n}(\varepsilon_{j_1\cdots j_n}\,g^{j_1 i_1}\cdots g^{j_n i_n})$$

$$= \varepsilon_{i_1\cdots i_n}(\varepsilon^{i_1\cdots i_n}\det g(E)^{-1})$$

$$= n!\ (-1)^{\iota}.$$

<u>Section 3:</u> <u>Interior Multiplication</u> Let V be an n-dimensional real vector space. Fix $g \in \underline{M}_{k,n-k}$ -- then g can be extended to a scalar product on the $\Lambda^p V$ $(p = 0,1,\ldots,n)$. While a direct approach is possible, it is more instructive to proceed conceptually.

- On $\Lambda^0 V = \underline{R}$, put

$$g(\alpha,\beta) = \alpha\beta.$$

- On $\Lambda^1 V$ $(= V_1^0 = V^*)$, put

$$g(\alpha,\beta) = g(g^\# \alpha, g^\# \beta).$$

[Note: Fix $E \in B(V)$ -- then

$$\begin{cases} \alpha = \alpha_i \omega^i \\ \beta = \beta_j \omega^j \end{cases} \quad \& \quad \begin{cases} \alpha^i \equiv g^{ik}\alpha_k \\ \beta^j \equiv g^{j\ell}\beta_\ell \end{cases}$$

$\Rightarrow$

$$\begin{aligned} g(\alpha,\beta) &= g(g^\# \alpha, g^\# \beta) \\ &= g(\alpha^i E_i, \beta^j E_j) \\ &= g_{ij}\alpha^i \beta^j \\ &= \alpha^i \beta_i \ .] \end{aligned}$$

Remark: We have

$$\begin{aligned} g(\omega^i,\omega^j) &= g(g^\# \omega^i, g^\# \omega^j) \\ &= g(g^{ik}E_k, g^{j\ell}E_\ell) \\ &= g^{ik}g^{j\ell}g(E_k,E_\ell) \end{aligned}$$

$$= g^{ik}g^{j\ell}g_{k\ell}$$

$$= g^{ik}g^{j\ell}g_{\ell k}$$

$$= g^{ik}\delta^j{}_k$$

$$= g^{ij}.]$$

Let $q \leq p$ -- then there is a bilinear map

$$\begin{cases} \iota : \Lambda^q V \times \Lambda^p V \rightarrow \Lambda^{p-q} V \\ (\beta,\alpha) \longrightarrow \iota_\beta \alpha \end{cases}$$

which is characterized by the following properties:

$$\forall\ \alpha,\beta \in \Lambda^1 V,\ \iota_\beta \alpha = g(\alpha,\beta),$$

$$\iota_\beta(\alpha_1 \wedge \alpha_2) = \iota_\beta \alpha_1 \wedge \alpha_2 + (-1)^{p_1}\ \alpha_1 \wedge \iota_\beta \alpha_2 \quad (\alpha_i \in \Lambda^{p_i} V, \beta \in \Lambda^1 V),$$

$$\iota_{\beta_1 \wedge \beta_2} = \iota_{\beta_2} \circ \iota_{\beta_1}.$$

[Note: One calls ι the <u>interior product</u> on $\Lambda^p V$. If $\beta \in \Lambda^0 V = \underline{R}$, then ι_β is simply multiplication by β.]

Remark: $\forall\ X \in V$,

$$\iota_X = \iota_{g^\flat X}.$$

[Indeed,

$$\iota_{g^\flat X}(g^\flat Y) = g(g^\flat X, g^\flat Y)$$

$$= g(g^\# g^\flat X, g^\# g^\flat Y)$$

$= g(X,Y)$

$= g(Y,X)$

$= g^{\flat} Y(X)$

$= \iota_X(g^{\flat} Y).]$

Per $E \in B(V)$, write

$$\begin{cases} \alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \\ \beta = \frac{1}{q!} \beta_{j_1 \cdots j_q} \omega^{j_1} \wedge \cdots \wedge \omega^{j_q} . \end{cases}$$

Put

$$\beta^{j_1 \cdots j_q} = g^{j_1 \ell_1} \cdots g^{j_q \ell_q} \beta_{\ell_1 \cdots \ell_q} .$$

<u>LEMMA</u> Let $\alpha \in \Lambda^p V$, $\beta \in \Lambda^q V$, where $q \leq p$ -- then

$$\iota_\beta \alpha = \frac{1}{q!(p-q)!} \beta^{j_1 \cdots j_q} \alpha_{j_1 \cdots j_q i_1 \cdots i_{p-q}} \omega^{i_1} \wedge \cdots \wedge \omega^{i_{p-q}} .$$

Take $q = p$ -- then $\iota_\beta \alpha$ is a real number and we set, by definition,

$$g(\alpha,\beta) = \iota_\beta \alpha = \iota_\alpha \beta .$$

Consequently,

$$g(\alpha,\beta) = \frac{1}{p!} \alpha^{i_1 \cdots i_p} \beta_{i_1 \cdots i_p}$$

and g is a scalar product on $\Lambda^p V$.

Remark: Due to the way that the definitions have been arranged,

$$g(\alpha,\beta) \neq g[{}^0_p](\alpha,\beta).$$

To see this, consider the RHS:

$$g[{}^0_p](\alpha,\beta)$$

$$= g[{}^0_p](\alpha_{i_1\cdots i_p}\, \omega^{i_1} \otimes \cdots \otimes \omega^{i_p},\ \beta_{j_1\cdots j_p}\, \omega^{j_1} \otimes \cdots \otimes \omega^{j_p})$$

$$= \alpha_{i_1\cdots i_p}\, \beta_{j_1\cdots j_p}\, g[{}^0_p](\omega^{i_1} \otimes \cdots \otimes \omega^{i_p}, \omega^{j_1} \otimes \cdots \otimes \omega^{j_p})$$

$$= \alpha_{i_1\cdots i_p}\, \beta_{j_1\cdots j_p}\, g^{j_1 i_1} \cdots g^{j_p i_p}$$

$$= g^{i_1 j_1} \cdots g^{i_p j_p}\, \alpha_{j_1\cdots j_p}\, \beta_{i_1\cdots i_p}$$

$$= \alpha^{i_1\cdots i_p}\, \beta_{i_1\cdots i_p}\,.$$

Example: Let

$$\left.\begin{matrix} \alpha^1,\ldots,\alpha^p \\ \\ \beta^1,\ldots,\beta^p \end{matrix}\right] \in \Lambda^1 V.$$

Then

$$g(\alpha^1 \wedge \cdots \wedge \alpha^p, \beta^1 \wedge \cdots \wedge \beta^p) = \det\, [g(\alpha^i,\beta^j)].$$

<u>LEMMA</u> Let $\{E_1,\ldots,E_n\}$ be an orthonormal basis for g — then the collection

$$\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$$

is an orthonormal basis for the extension of g to $\Lambda^p V$ $(1 \leq p \leq n)$.

[Note: We have

$$g(\omega^i,\omega^j) = g(g^\# \omega^i, g^\# \omega^j)$$

$$= g(\frac{1}{\varepsilon_i} E_i, \frac{1}{\varepsilon_j} E_j) \quad \text{(no sum)}$$

$$= \frac{1}{\varepsilon_i \varepsilon_j} g(E_i, E_j) \qquad \text{(no sum)}$$

$$= \begin{cases} \varepsilon_i & i = j \\ 0 & i \neq j \end{cases} .$$

Therefore

$$g(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}, \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) = \varepsilon_{i_1} \cdots \varepsilon_{i_p} = (-1)^P,$$

where P is the number of indices among $\{i_1,\ldots,i_p\}$ for which $\varepsilon_i = -1$.]

Let $\alpha \in \Lambda^p V$, $\beta \in \Lambda^q V$ $(q < p)$ — then $\forall\ \gamma \in \Lambda^{p-q} V$,

$$g(\iota_\beta \alpha, \gamma) = \iota_\gamma \iota_\beta \alpha$$

$$= \iota_{\beta \wedge \gamma} \alpha$$

$$= g(\alpha, \beta \wedge \gamma).$$

In other words, the operations

$$\begin{cases} \iota_\beta : \Lambda^p V \to \Lambda^{p-q} V \\ \beta \wedge \text{---} : \Lambda^{p-q} V \to \Lambda^p V \end{cases}$$

are mutually adjoint.

Consider now

$$\text{vol}_g = |g|^{1/2} \text{vol}_E.$$

This n-form depends only on the orientation class of E. Thus there are but two possibilities. Pick one, call it an orientation of V, and freeze it for the ensuing discussion.

N.B. We have

$$\text{vol}_g = \frac{1}{n!} e_{j_1 \cdots j_n} \omega^{j_1} \wedge \cdots \wedge \omega^{j_n}.$$

Definition: The star operator is the isomorphism

$$* : \Lambda^p V \to \Lambda^{n-p} V$$

given by

$$*\alpha = \iota_\alpha \text{vol}_g.$$

Therefore

$$*\alpha = \frac{1}{p!(n-p)!} \alpha^{i_1 \cdots i_p} e_{i_1 \cdots i_p j_1 \cdots j_{n-p}} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{n-p}}.$$

LEMMA We have

$$**\alpha = (-1)^{\iota} (-1)^{p(n-p)} \alpha .$$

Example: $*1 = \mathrm{vol}_g$

$\Rightarrow$

$$*\mathrm{vol}_g = **1 = (-1)^{\iota}$$

$\Rightarrow$

$$\begin{aligned} g(\mathrm{vol}_g,\mathrm{vol}_g) &= \iota_{\mathrm{vol}_g}\mathrm{vol}_g \\ &= *\mathrm{vol}_g \\ &= (-1)^{\iota}. \end{aligned}$$

Observation: Let $\alpha \in \Lambda^p V$, $\beta \in \Lambda^{n-p} V$ — then

$$\begin{aligned} g(\alpha\wedge\beta,\mathrm{vol}_g) &= \iota_{\alpha\wedge\beta}\mathrm{vol}_g \\ &= \iota_\beta \iota_\alpha \mathrm{vol}_g \\ &= \iota_\beta *\alpha \\ &= g(*\alpha,\beta). \end{aligned}$$

Example: We have

$$g(*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}), \omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_n})$$

$$\begin{aligned} &= g(\omega^{i_1} \wedge \cdots \wedge \omega^{i_n}, \mathrm{vol}_g) \\ &= |g|^{1/2}\, g(\omega^{i_1} \wedge \cdots \wedge \omega^{i_n}, \omega^1 \wedge \cdots \wedge \omega^n) \\ &= |g|^{1/2}\, g(\varepsilon^{i_1 \cdots i_n}\, \omega^1 \wedge \cdots \wedge \omega^n, \omega^1 \wedge \cdots \wedge \omega^n) \\ &= |g|^{1/2}\, \varepsilon^{i_1 \cdots i_n}\, g(\mathrm{vol}_E,\mathrm{vol}_E) \end{aligned}$$

$$= \frac{1}{|g|^{1/2}} \varepsilon^{i_1 \cdots i_n} g(\mathrm{vol}_g, \mathrm{vol}_g)$$

$$= \frac{1}{|g|^{1/2}} \varepsilon^{i_1 \cdots i_n} (-1)^{\iota} .$$

In what follows, $\alpha \in \Lambda^p V$ and $\beta \in \Lambda^q V$ (subject to the obvious restrictions).

<u>Rules</u>

- $\iota_\beta * \alpha = *(\alpha \wedge \beta)$.

[In fact,

$$\iota_\beta * \alpha = \iota_\beta \iota_\alpha \mathrm{vol}_g$$

$$= \iota_{\alpha\wedge\beta} \mathrm{vol}_g$$

$$= *(\alpha \wedge \beta).]$$

- $*\iota_\beta \alpha = (-1)^{q(n-q)} * \alpha \wedge \beta$.

[In fact,

$$\iota_\beta ** \alpha = *(*\alpha \wedge \beta)$$

$\Rightarrow$

$$*\iota_\beta ** \alpha = **(*\alpha \wedge \beta)$$

$\Rightarrow$

$$(-1)^{\iota} (-1)^{p(n-p)} * \iota_\beta \alpha$$

$$= (-1)^{\iota} (-1)^{(n-p+q)(n-(n-p+q))} * \alpha \wedge \beta$$

$\Rightarrow$

$$*\iota_\beta \alpha = (-1)^{q(n-q)} *\alpha\wedge\beta.]$$

- $\alpha\wedge*\beta = g(\alpha,\beta)\text{vol}_g = \beta\wedge*\alpha.$

[In fact,

$$\alpha\wedge*\beta = (-1)^{p(n-p)} *\beta\wedge\alpha$$

$$= (-1)^{p(n-p)} (-1)^{p(n-p)} *\iota_\alpha\beta$$

$$= g(\alpha,\beta)*1$$

$$= g(\alpha,\beta)\text{vol}_g.]$$

- $g(*\alpha,*\beta) = (-1)^\iota g(\alpha,\beta).$

[In fact,

$$g(*\alpha,*\beta)\text{vol}_g = *\alpha\wedge**\beta$$

$$= (-1)^\iota (-1)^{n(n-p)} *\alpha\wedge\beta$$

$$= (-1)^\iota \beta\wedge*\alpha$$

$$= (-1)^\iota g(\alpha,\beta)\text{vol}_g.]$$

Example: Specialize the relation

$$*\iota_\beta\alpha = (-1)^{q(n-q)} *\alpha\wedge\beta$$

and take $\beta = g^\flat X$ — then

$$*\iota_X\alpha = (-1)^{n-1} *\alpha\wedge g^\flat X.$$

Example: Let $\alpha,\beta\in\Lambda^2 V$ -- then

$$\iota_{E_i}\alpha\wedge\iota_{E^i}*\beta = *(\iota_{E_i}\alpha\wedge\iota_{E^i}\beta).$$

[Write

$$\alpha = \frac{1}{2} A_{ij}\omega^i\wedge\omega^j \qquad (A_{ij} = -A_{ji}).$$

Then

$$\iota_{E_i}\alpha\wedge\iota_{E^i}*\beta$$

$$= A_{ij}\omega^j\wedge\iota_{E^i}*\beta$$

$$= A_{ij}\omega^j\wedge\iota_{g^\flat E^i}*\beta$$

$$= A_{ij}\omega^j\wedge\iota_{g^\flat g^\# \omega^i}*\beta$$

$$= A_{ij}\omega^j\wedge\iota_{\omega^i}*\beta$$

$$= A_{ij}\omega^j\wedge*(\beta\wedge\omega^i)$$

$$= A_{ij}(-1)^\iota\ (-1)^{(n-2)(n-(n-2))}**(\omega^j\wedge*(\beta\wedge\omega^i))$$

$$= (-1)^\iota\ A_{ij}*(*(\omega^j\wedge*(\beta\wedge\omega^i)))$$

$$= (-1)^\iota\ (-1)^{n-3}A_{ij}*(*(*(\beta\wedge\omega^i)\wedge\omega^j))$$

$$= (-1)^\iota\ (-1)^{n-3}A_{ij}*((-1)^{n-1}*(*\iota_{\omega^j}(\beta\wedge\omega^i)))$$

$$= (-1)^\iota\ A_{ij}*(**(\iota_{\omega^j}(\beta\wedge\omega^i)))$$

$$= (-1)^{\iota} A_{ij} * ((-1)^{\iota} (-1)^{2(n-2)} \iota_{\omega^j}(\beta \wedge \omega^i))$$

$$= A_{ij} * (\iota_{\omega^j}(\beta \wedge \omega^i))$$

$$= A_{ij} * ((\iota_{\omega^j}\beta) \wedge \omega^i + \beta \iota_{\omega^j}\omega^i)$$

$$= A_{ij} * (\iota_{\omega^j}\beta \wedge \omega^i) + *\beta A_{ij} g^{ij}.$$

But

$$A_{ij} g^{ij} = -A_{ji} g^{ij} = -A_{ji} g^{ji} = -A_{ij} g^{ij}$$

$\Rightarrow$

$$A_{ij} g^{ij} = 0.$$

Therefore

$$\iota_{E_i} \alpha \wedge \iota_{E^i} * \beta$$

$$= A_{ij} * (\iota_{\omega^j}\beta \wedge \omega^i)$$

$$= *(\iota_{\omega^j}\beta \wedge A_{ij}\omega^i)$$

$$= - *(A_{ij}\omega^i \wedge \iota_{\omega^j}\beta)$$

$$= *(A_{ji}\omega^i \wedge \iota_{\omega^j}\beta)$$

$$= *(A_{ij}\omega^j \wedge \iota_{\omega^i}\beta)$$

$$= *(\iota_{E_i}\alpha \wedge \iota_{E^i}\beta).]$$

<u>FACT</u> We have

$$*\iota_{E_i}(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p})$$

$$=(-1)^{p+1}g^{\flat}E_i\wedge*(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}).$$

[For

$$*\iota_{E_i}(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p})$$

$$=*\iota_{g^{\flat}E_i}(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p})$$

$$=(-1)^{n-1}*(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p})\wedge g^{\flat}E_i$$

$$=(-1)^{n-1}(-1)^{n-p}g^{\flat}E_i\wedge*(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p})$$

$$=(-1)^{p+1}g^{\flat}E_i\wedge*(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}).]$$

<u>LEMMA</u> We have

$$*(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p})$$

$$=\frac{|g|^{1/2}}{(n-p)!}\,g^{i_1j_1}\cdots g^{i_pj_p}\,\varepsilon_{j_1\cdots j_n}\omega^{j_{p+1}}\wedge\cdots\wedge\omega^{j_n}.$$

[To understand the procedure, start with the simplest case:

$$*\omega^{i_1}=\iota_{\omega^1}\mathrm{vol}_g$$

$$=\iota_{g^{\#}\omega^1}\mathrm{vol}_g$$

$$= g^{i_1k_1}\iota_{E_{k_1}}\mathrm{vol}_g$$

$$= g^{i_1k_1}\iota_{E_{k_1}}(\tfrac{1}{n!}\,|g|^{1/2}\varepsilon_{j_1\cdots j_n}\omega^{j_1}\wedge\cdots\wedge\omega^{j_n})$$

$$= |g|^{1/2}\, g^{i_1k_1}\iota_{E_{k_1}}(\tfrac{1}{n!}\,\varepsilon_{j_1\cdots j_n}\omega^{j_1}\wedge\cdots\wedge\omega^{j_n})$$

$$= |g|^{1/2}\, g^{i_1k_1}(\tfrac{1}{(n-1)!}\,\varepsilon_{k_1j_2\cdots j_n}\omega^{j_2}\wedge\cdots\wedge\omega^{j_n})$$

$$= \frac{|g|^{1/2}}{(n-1)!}\, g^{i_1j_1}\varepsilon_{j_1\cdots j_n}\omega^{j_2}\wedge\cdots\wedge\omega^{j_n}.$$

Now go from here by iteration:

$$*(\omega^{i_1}\wedge\omega^{i_2}) = \iota_{\omega^{i_1}\wedge\omega^{i_2}}\mathrm{vol}_g$$

$$= \iota_{\omega^2}\iota_{\omega^1}\mathrm{vol}_g$$

$$= \iota_{\omega^2}|g|^{1/2}g^{i_1k_1}(\tfrac{1}{(n-1)!}\,\varepsilon_{k_1j_2\cdots j_n}\omega^{j_2}\wedge\cdots\wedge\omega^{j_n})$$

$$= |g|^{1/2}g^{i_1k_1}\iota_{\omega^2}(\tfrac{1}{(n-1)!}\,\varepsilon_{k_1j_2\cdots j_n}\omega^{j_2}\wedge\cdots\wedge\omega^{j_n})$$

$$= |g|^{1/2}g^{i_1k_1}g^{i_2k_2}(\tfrac{1}{(n-2)!}\,\varepsilon_{k_1k_2j_3\cdots j_n}\omega^{j_3}\wedge\cdots\wedge\omega^{j_n})$$

$$= \frac{|g|^{1/2}}{(n-2)!}\, g^{i_1j_1}g^{i_2j_2}\varepsilon_{j_1\cdots j_n}\omega^{j_3}\wedge\cdots\wedge\omega^{j_n}.]$$

Remark: Since

$$e_{j_1\cdots j_n} = |g|^{1/2}\varepsilon_{j_1\cdots j_n},$$

it is tempting to write

$$e^{i_1\cdots i_p}{}_{j_{p+1}\cdots j_n} = g^{i_1j_1}\cdots g^{i_pj_p}e_{j_1\cdots j_n}.$$

But this is nonsense: Take p = n and recall that

$$e^{i_1\cdots i_n} = (-1)^{\iota}g^{i_1j_1}\cdots g^{i_nj_n}e_{j_1\cdots j_n}.$$

<u>LEMMA</u> $\forall\ \alpha\in\Lambda^pV$,

$$\iota_{E_i}\alpha\wedge\iota_{E^i}*\alpha = 0.$$

Application: Let $\alpha,\beta\in\Lambda^pV$ -- then

$$\iota_{E_i}\alpha\wedge\iota_{E^i}*\beta = -\iota_{E_i}\beta\wedge\iota_{E^i}*\alpha.$$

[Consider

$$\iota_{E_i}(\alpha + \beta)\wedge\iota_{E^i}*(\alpha + \beta).]$$

<u>Section 4</u>: <u>Tensor Analysis</u> Let M be a connected C^∞ manifold of dimension n,

$$\mathcal{D}(M) = \bigoplus_{p,q=0}^{\infty} \mathcal{D}_q^p(M)$$

its tensor algebra.

[Note: Here, $\mathcal{D}_0^0(M) = C^\infty(M)$, $\mathcal{D}_0^1(M) = \mathcal{D}^1(M)$, the derivations of $C^\infty(M)$ (a.k.a. the vector fields on M), and $\mathcal{D}_1^0(M) = \mathcal{D}_1(M)$, the linear forms on $\mathcal{D}^1(M)$ viewed as a module over $C^\infty(M)$).]

Remark: By definition, $\mathcal{D}_q^p(M)$ is the $C^\infty(M)$-module of all $C^\infty(M)$-multilinear maps

$$\overbrace{\mathcal{D}_1(M) \times \cdots \times \mathcal{D}_1(M)}^{p} \times \overbrace{\mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M)}^{q} \to C^\infty(M).$$

One can also interpret the elements of $\mathcal{D}_q^p(M)$ geometrically. To this end, consider the frame bundle

$$\begin{array}{ccc} \underline{GL}(n,\underline{R}) & \to & LM \\ & & \downarrow \pi \\ & & M. \end{array}$$

Thinking of $\underline{R}^n$ as merely a vector space (and not as a manifold), let $T_q^p(n)$ be the tensors of type (p,q) -- then $\underline{GL}(n,\underline{R})$ operates to the left on $T_q^p(n)$ (cf. Section 1). Now form the vector bundle

$$T_q^p(M) = LM \times_{\underline{GL}(n,\underline{R})} T_q^p(n).$$

Then, on general grounds, there is a one-to-one correspondence between the sections T of $T_q^p(M)$ and the equivariant maps $\Phi: LM \to T_q^p(n)$.

Of course, as a set

$$\underline{L}M = \coprod_{x \in M} B(T_xM),$$

hence $\Phi = \{\Phi_x : x \in M\}$, where

$$\Phi_x : B(T_xM) \to T_q^p(n).$$

And we have

$$\mathcal{D}_q^p(M) \leftrightarrow \sec(T_q^p(M))$$

or still,

$$\mathcal{D}_q^p(M) \leftrightarrow \text{map}_{\underline{GL}(n,\underline{R})}(\underline{L}M, T_q^p(n)).$$

[Note: One advantage of the geometric point of view is that it can be readily generalized, e.g., to tensors of type (p,q) and weight χ.]

<u>Details</u> Given $(x,E) \in \underline{L}M$ ($\Rightarrow E \in B(T_xM)$), define $\zeta_E : \underline{R}^n \to T_xM$ by

$$\zeta_E(e_i) = E_i \quad (i=1,\ldots,n).$$

Then $\forall\, g \in \underline{GL}(n,\underline{R})$, the composite $\underline{R}^n \overset{g}{\to} \underline{R}^n \overset{\zeta_E}{\to} T_xM$ is $\zeta_{E \cdot g}$.

<u>$T \to \Phi_T$:</u> This is the arrow

$$\sec(T_q^p(M)) \to \text{map}_{\underline{GL}(n,\underline{R})}(\underline{L}M, T_q^p(n)),$$

where

$$\Phi_T(x,E)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= T_x(\Lambda^1 \circ \zeta_E^{-1},\ldots,\Lambda^p \circ \zeta_E^{-1}, \zeta_E(X_1),\ldots,\zeta_E(X_q)).$$

<u>$\Phi \to T_\Phi$:</u> This is the arrow

$$\mathrm{map}_{\underline{GL}(n,\underline{R})}(LM, T_q^p(n)) \rightarrow \sec(T_q^p(M)),$$

where

$$T_\Phi|_x(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= \Phi(x,E)(\Lambda^1 \circ \zeta_E,\ldots,\Lambda^p \circ \zeta_E, \zeta_E^{-1}(X_1),\ldots,\zeta_E^{-1}(X_q)).$$

<u>FACT</u> These arrows are mutually inverse:

$$\begin{bmatrix} T \rightarrow \Phi_T \rightarrow T_{\Phi_T} = T \\ \\ \Phi \rightarrow T_\Phi \rightarrow \Phi_{T_\Phi} = \Phi. \end{bmatrix}$$

In what follows, all operations will be defined globally. However, for computational purposes, it is important to have at hand their local expression as well, meaning the form they take on a connected open set $U \subset M$ equipped with coordinates $x^1,\ldots,x^n$.

Let $T \in \mathcal{D}_q^p(M)$ -- then locally

$$T = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \left(\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}}\right) \otimes (dx^{j_1} \otimes \cdots \otimes dx^{j_q}),$$

where

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

$$= T(dx^{i_1},\ldots,dx^{i_p}, \frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_q}}) \in C^\infty(U)$$

are the <u>components</u> of T.

Under a change of coordinates, the components of T satisfy the <u>tensor transformation rule</u>:

$$T^{i_1' \cdots i_p'}{}_{j_1' \cdots j_q'}$$

$$= \frac{\partial x^{i_1'}}{\partial x^{i_1}} \cdots \frac{\partial x^{i_p'}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j_1'}} \cdots \frac{\partial x^{j_q}}{\partial x^{j_q'}} T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

[Note: There are maps

$$g, g^{-1} : U \cap U' \to \underline{GL}(n, \underline{R}),$$

viz.

$$g(x) = [\left.\frac{\partial x^{i}}{\partial x^{i'}}\right|_x] , \quad g^{-1}(x) = [\left.\frac{\partial x^{i'}}{\partial x^{i}}\right|_x].]$$

<u>FACT</u> Equip $T_q^p(n)$ with its standard basis -- then

$$\forall \ \Phi \in \text{map}_{\underline{GL}(n,\underline{R})}(LM, T_q^p(n)),$$

we have

$$\Phi(x, \{ \left.\frac{\partial}{\partial x^1}\right|_x, \ldots, \left.\frac{\partial}{\partial x^n}\right|_x \})$$

$$= T_\Phi|_x{}^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

Remark: Suppose there is assigned to each U in a coordinate atlas for M, functions

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \in C^\infty(U)$$

subject to the tensor transformation rule -- then there is a unique $T \in \mathcal{D}_q^p(M)$

whose components in U are the $T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$.

[It is simply a matter of manufacturing a global section of $T_q^p(M)$ by gluing together local sections.]

Example: The <u>Kronecker tensor</u> is the tensor K of type (1,1) defined by $K(\Lambda,X) = \Lambda(X)$, thus

$$K^i{}_j = K(dx^i, \frac{\partial}{\partial x^j}) = \delta^i{}_j.$$

<u>FACT</u> There is a tensor K(p) of type (p,p) with the property that in any coordinate system,

$$K(p)^{i_1 \cdots i_p}{}_{j_1 \cdots j_p} = \delta^{i_1 \cdots i_p}{}_{j_1 \cdots j_p} .$$

Notation: Given $f \in C^\infty(U)$, write

$$\frac{\partial f}{\partial x^i} = f_{,i} .$$

Example: Let $X,Y \in \mathcal{D}^1(M)$ — then locally

$$\begin{cases} X = X^i \dfrac{\partial}{\partial x^i} & (X^i = < X, dx^i >) \\ Y = Y^j \dfrac{\partial}{\partial x^j} & (Y^j = < Y, dx^j >) \end{cases}$$

$\Rightarrow$

$$[X,Y] = (X^i Y^j{}_{,i} - Y^i X^j{}_{,i}) \frac{\partial}{\partial x^j} .$$

[Note: The bracket

$$[\ ,\ ]: \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M)$$

is $\underline{R}$-bilinear but not $C^\infty(M)$-bilinear. In fact,

$$[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.]$$

Definition: A type preserving $\underline{R}$-linear map

$$D:\mathcal{D}(M) \to \mathcal{D}(M)$$

which commutes with contractions is said to be a <u>derivation</u> if $\forall\ T_1,T_2 \in \mathcal{D}(M)$,

$$D(T_1 \otimes T_2) = DT_1 \otimes T_2 + T_1 \otimes DT_2.$$

[Note: To say that D is type preserving means that $D\mathcal{D}^p_q(M) \subset \mathcal{D}^p_q(M)$.]

The set of all derivations of $\mathcal{D}(M)$ forms a Lie algebra over $\underline{R}$, the bracket operation being defined by

$$[D_1,D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

Remark: For any $f \in C^\infty(M)$ and any $T \in \mathcal{D}(M)$, $fT = f \otimes T$, so $D(fT) = f(DT) + (Df)T$. In particular: D is a derivation of $C^\infty(M)$, hence is represented on $C^\infty(M)$ by a vector field.

Construction: Let

$$A \in \mathcal{D}^1_1(M) \approx \text{Hom}_{C^\infty(M)}(\mathcal{D}^1(M),\mathcal{D}^1(M)).$$

Then $\forall\ x \in M$,

$$A_x:T_xM \to T_xM$$

is $\underline{R}$-linear, hence can be uniquely extended to a derivation D_{A_x} of the tensor algebra over T_xM. This said, define

$$D_A:\mathcal{D}(M) \to \mathcal{D}(M)$$

by

$$(D_AT)_x = D_{A_x}T_x.$$

Then D_A is a derivation of $\mathcal{D}(M)$ which is zero on $C^\infty(M)$.

FACT Any derivation of $\mathcal{D}(M)$ which is zero on $C^\infty(M)$ is induced by a tensor of type (1,1).

[Note: If D is a derivation of $\mathcal{D}(M)$ and if $A\in\mathcal{D}_1^1(M)$, then $[D,D_A]|C^\infty(M) = 0$, hence $[D,D_A] = D_B$ for some $B\in\mathcal{D}_1^1(M)$. Therefore $\mathcal{D}_1^1(M)$ is an ideal in the Lie algebra of derivations of $\mathcal{D}(M)$.]

Product Formula Let $D:\mathcal{D}(M) \rightarrow \mathcal{D}(M)$ be a derivation -- then $\forall\ T\in\mathcal{D}_q^p(M)$,

$$D[T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)]$$

$$= (DT)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$+ \sum_{i=1}^{p} T(\Lambda^1,\ldots,D\Lambda^i,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$+ \sum_{j=1}^{q} T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,DX_j,\ldots,X_q).$$

[Note: This shows that D is known as soon as it is known on $C^\infty(M)$, $\mathcal{D}^1(M)$, and $\mathcal{D}_1(M)$. But for $\omega\in\mathcal{D}_1(M)$,

$$(D\omega)(X) = D[\omega(X)] - \omega(DX),$$

thus functions and vector fields suffice.]

FACT Let D_1,D_2 be derivations of $\mathcal{D}(M)$. Assume: $D_1 = D_2$ on $C^\infty(M)$ and $\mathcal{D}^1(M)$ -- then $D_1 = D_2$.

<u>EXTENSION PRINCIPLE</u> Suppose given a vector field X and an $\underline{R}$-linear map $\delta:\mathcal{D}^1(M) \to \mathcal{D}^1(M)$ such that

$$\delta(fY) = (Xf)Y + f\delta(Y)$$

for all $f\in C^\infty(M)$, $Y\in\mathcal{D}^1(M)$ -- then there exists a unique derivation

$$D:\mathcal{D}(M) \to \mathcal{D}(M)$$

such that $D|C^\infty(M) = X$ and $D|\mathcal{D}^1(M) = \delta$.

[Define D on $\mathcal{D}_1(M)$ by

$$(D\omega)(Y) = X[\omega(Y)] - \omega(\delta Y)$$

and extend to all of $\mathcal{D}(M)$ via the product formula.]

The notion of a tensor T of type (p,q) and weight X is clear, there being two possibilities for the form that the tensor transformation rule takes.

Notation: Put

$$J = \det\left[\frac{\partial x^i}{\partial x^{i'}}\right].$$

I: For some $r\in\underline{R}$,

$$T^{i_1'\cdots i_p'}{}_{j_1'\cdots j_q'}$$

$$= |J|^r \frac{\partial x^{i_1'}}{\partial x^{i_1}}\cdots\frac{\partial x^{i_p'}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j_1'}}\cdots\frac{\partial x^{j_q}}{\partial x^{j_q'}}T^{i_1\cdots i_p}{}_{j_1\cdots j_q};$$

II: For some $r\in\underline{R}$,

$$T^{i_1'\cdots i_p'}{}_{j_1'\cdots j_q'}$$

$$= \operatorname{sgn} J\cdot|J|^r \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} T^{i_1\cdots i_p}{}_{j_1\cdots j_q} .$$

Accordingly, there are two kinds of tensors of type (p,q) and weight χ, which we shall refer to as class I and class II. It is also convenient to single out a particular combination of these by an integrality condition.

Definition: A tensor of type (p,q) and weight w is a tensor T of type (p,q) and weight $\chi = (\det)^w$ ($w\in\underline{Z}$), hence

$$T^{i'_1\cdots i'_p}{}_{j'_1\cdots j'_q}$$

$$= J^w \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} T^{i_1\cdots i_p}{}_{j_1\cdots j_q} .$$

[Note: Needless to say, the tensors of type (p,q) and weight 0 are precisely the elements of $\mathcal{D}^p_q(M)$.]

Remark: The product of a tensor T of type (p,q) and weight w with a tensor T' of type (p',q') and weight w' is a tensor $T \otimes T'$ of type (p + p', q + q') and weight w + w'.

Example: The upper Levi-Civita symbol is a tensor of type (n,0) and weight 1 and the lower Levi-Civita symbol is a tensor of type (0,n) and weight -1.

[To discuss the upper Levi-Civita symbol, write

$$\varepsilon^{i'_1\cdots i'_n} = \delta^{i'_1\cdots i'_n}{}_{i'\cdots n'}$$

$$= \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_n}}{\partial x^{i_n}} \frac{\partial x^{j_1}}{\partial x^{1'}} \cdots \frac{\partial x^{j_n}}{\partial x^{n'}} \delta^{i_1\cdots i_n}{}_{j_1\cdots j_n}$$

$$= \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_n}}{\partial x^{i_n}} \frac{\partial x^{j_1}}{\partial x^{1'}} \cdots \frac{\partial x^{j_n}}{\partial x^{n'}} \varepsilon^{i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_n}$$

$$= (\varepsilon_{j_1 \cdots j_n} \frac{\partial x^{j_1}}{\partial x^{1'}} \cdots \frac{\partial x^{j_n}}{\partial x^{n'}}) \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_n}}{\partial x^{i_n}} \varepsilon^{i_1 \cdots i_n}$$

$$= \varepsilon_{1' \cdots n'} J \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_n}}{\partial x^{i_n}} \varepsilon^{i_1 \cdots i_n}$$

$$= J \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_n}}{\partial x^{i_n}} \varepsilon^{i_1 \cdots i_n} .]$$

When $(p,q) = (0,0)$, the foregoing considerations specialize to that of density, twisted density, and scalar density.

Density A density of weight r is a section of the line bundle $L_I^r(M)$ whose transition functions are the

$$|\det[\frac{\partial x^{i'}}{\partial x^{i}}]|^r .$$

[Note: The sections of

$$T_q^p(M) \otimes L_I^r(M)$$

are the class I tensors of type (p,q).]

Twisted Density A twisted density of weight r is a section of the line bundle $L_{II}^r(M)$ whose transition functions are the

$$\operatorname{sgn} \det[\frac{\partial x^{i'}}{\partial x^{i}}] \cdot |\det[\frac{\partial x^{i'}}{\partial x^{i}}]|^r .$$

[Note: The sections of

$$T^p_q(M) \otimes L^r_{II}(M)$$

are the class II tensors of type (p,q).]

<u>Scalar Density</u> A scalar density of weight w is a section of the line bundle $L^w(M)$ whose transition functions are the

$$(\det[\frac{\partial x^{i'}}{\partial x^i}])^w .$$

[Note: The sections of

$$T^p_q(M) \otimes L^w(M)$$

are the tensors of type (p,q) and weight w.]

Example: The <u>density bundle</u> is the line bundle

$$L_{den}(M) \to M$$

whose transition functions are the

$$|\det[\frac{\partial x^{i'}}{\partial x^i}]| .$$

Therefore

$$L_{den}(M) = L^1_I(M).$$

Example: The <u>orientation bundle</u> is the line bundle

$$Or(M) \to M$$

whose transition functions are the

$$\text{sgn} \det[\frac{\partial x^{i'}}{\partial x^i}].$$

Therefore

$$\mathrm{Or}(M) = L^{0}_{II}(M).$$

Example: The <u>canonical bundle</u> is the line bundle

$$L_{can}(M) \to M$$

whose transition functions are the

$$\det[\frac{\partial x^{i'}}{\partial x^{i}}].$$

Therefore

$$L_{can}(M) = L^{1}(M).$$

Remark: The canonical bundle can be identified with $\Lambda^n T^*M$, where T^*M is the cotangent bundle. Since

$$\Lambda^n M = \sec(\Lambda^n T^*M),$$

it follows that the n-forms on M are scalar densities of weight 1.

[Note: The upper Levi-Civita symbol is a section of

$$T^n_0(M) \otimes \Lambda^n T^*M$$

and the lower Levi-Civita symbol is a section of

$$T^0_n(M) \otimes (\Lambda^n T^*M)^{-1}.]$$

Section 5: Lie Derivatives Let M be a connected C^∞ manifold of dimension n.

LEMMA One may attach to each $X \in \mathcal{D}^1(M)$ a derivation

$$L_X : \mathcal{D}(M) \to \mathcal{D}(M)$$

called the Lie derivative w.r.t. X. It is characterized by the properties

$$L_X f = Xf \quad , \quad L_X Y = [X,Y].$$

[In the notation of the Extension Principle, define $\delta : \mathcal{D}^1(M) \to \mathcal{D}^1(M)$ by

$$\delta(Y) = [X,Y].$$

Then

$$\begin{aligned} \delta(fY) &= [X,fY] \\ &= f[X,Y] + (Xf)Y \\ &= (Xf)Y + f[X,Y] \\ &= (Xf)Y + f\delta(Y).] \end{aligned}$$

Owing to the product formula, $\forall\ T \in \mathcal{D}_q^p(M)$,

$$X[T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)]$$

$$= (L_X T)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$+ \sum_{i=1}^{p} T(\Lambda^1,\ldots, L_X\Lambda^i,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$+ \sum_{j=1}^{q} T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots, L_X X_j,\ldots,X_q).$$

[Note: If $\omega \in \mathcal{D}_1(M)$, then

$$(L_X\omega)(Y) = X\omega(Y) - \omega([X,Y]).]$$

Locally,

$$(L_X T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

$$= X^a T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q, a}$$

$$- X^{i_1}{}_{,a} T^{a i_2 \cdots i_p}{}_{j_1 \cdots j_q} - \cdots$$

$$+ X^a{}_{,j_1} T^{i_1 \cdots i_p}{}_{a j_2 \cdots j_q} + \cdots .$$

[Note: From the definitions,

$$\begin{cases} L_X \dfrac{\partial}{\partial x^i} = - X^a{}_{,i} \dfrac{\partial}{\partial x^a} \\ \\ L_X \, dx^i = X^i{}_{,a} \, dx^a \quad .] \end{cases}$$

At a given $x \in M$, the expression

$$- X^{i_1}{}_{,a} T^{a i_2 \cdots i_p}{}_{j_1 \cdots j_q} - \cdots$$

$$+ X^a{}_{,j_1} T^{i_1 \cdots i_p}{}_{a j_2 \cdots j_q} + \cdots$$

can be explained in terms of the canonical representation ρ of $\underline{GL}(n,\underline{R})$ on $T^p_q(n)$ or, more precisely, its differential $d\rho$.

To see this, fix for the moment an element $T \in T_q^p(n)$ -- then $\forall\ g \in \underline{GL}(n,\underline{R})$,

$$(g\cdot T)^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$= (g)^{i_1}{}_{i_1'} \cdots (g)^{i_p}{}_{i_p'} (g^{-1})^{j_1'}{}_{j_1} \cdots (g^{-1})^{j_q'}{}_{j_q} T^{i_1'\cdots i_p'}{}_{j_1'\cdots j_q'} .$$

Now pass to the derived map of Lie algebras

$$d\rho : \underline{g\ell}(n,\underline{R}) \rightarrow \underline{g\ell}(T_q^p(n)).$$

So, $\forall\ A \in \underline{g\ell}(n,\underline{R})$,

$$A\cdot T = \frac{d}{dt}(\exp(tA)\cdot T)\Big|_{t=0}$$

and we have

$$(A\cdot T)^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$= A^{i_1}{}_{a} T^{a i_2\cdots i_p}{}_{j_1\cdots j_q} + \cdots$$

$$- A^{a}{}_{j_1} T^{i_1\cdots i_p}{}_{a j_2\cdots j_q} - \cdots .$$

Returning to M, use the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_x , \ldots , \frac{\partial}{\partial x^n}\Big|_x \right\}$ to identify $T_x M$ with $\underline{R}^n$, thence $T_q^p T_x M$ with $T_q^p(n)$. Put

$$A^i{}_j(x) = - X^i{}_{,j}(x).$$

Then at x,

$$- X^{i_1}_{,a} T^{a i_2 \cdots i_p}{}_{j_1 \cdots j_q} - \cdots$$

$$+ X^a_{,j_1} T^{i_1 \cdots i_p}{}_{a j_2 \cdots j_q} + \cdots$$

equals

$$(A(x) \cdot T_x)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

Remark: The symbol

$$(L_X T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

is usually abbreviated to

$$L_X T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

<u>Rules</u>

$$\begin{bmatrix} L_{X+Y} = L_X + L_Y \ , \ L_{rX} = rL_X \quad (r \in \underline{R}) \\ \\ L_{[X,Y]} = [L_X, L_Y] \ (= L_X \circ L_Y - L_Y \circ L_X). \end{bmatrix}$$

Example: Let K be the Kronecker tensor -- then

$$L_X K = 0.$$

Indeed,

$$\begin{aligned} L_X K^i{}_j &= X^a \delta^i{}_{j,a} - X^i_{,a} \delta^a{}_j + X^a_{,j} \delta^i{}_a \\ &= 0 - X^i_{,j} + X^i_{,j} \\ &= 0. \end{aligned}$$

[Note: In general, ∀ p≥1,

$$L_X K(p) = 0.]$$

FACT Let $D:\mathcal{D}(M) \to \mathcal{D}(M)$ be a derivation -- then there is a unique $X \in \mathcal{D}^1(M)$ and a unique $A \in \mathcal{D}_1^1(M)$ such that

$$D = L_X + D_A.$$

Consider now the exterior algebra Λ^*M -- then L_X induces a derivation of Λ^*M:

$$L_X(\alpha\wedge\beta) = L_X\alpha\wedge\beta + \alpha\wedge L_X\beta.$$

Notation: ι_X is the interior product w.r.t. X, so

$$\iota_X:\Lambda^*M \to \Lambda^*M$$

is an antiderivation of degree -1.

Explicitly, $\forall\ \alpha\in\Lambda^pM$,

$$\iota_X\alpha(X_1,\ldots,X_{p-1}) = \alpha(X,X_1,\ldots,X_{p-1}).$$

And one has

$$\iota_X(\alpha_1\wedge\alpha_2) = \iota_X\alpha_1\wedge\alpha_2 + (-1)^p\alpha_1\wedge\iota_X\alpha_2.$$

Properties: (1) $\iota_X\circ\iota_X = 0$; (2) $\iota_X\circ\iota_Y + \iota_Y\circ\iota_X = 0$; (3) $\iota_{X+Y} = \iota_X + \iota_Y$; (4) $\iota_{fX} = f\iota_X$.

- $L_X = \iota_X\circ d + d\circ\iota_X$.

- $\iota_{[X,Y]} = L_X\circ\iota_Y - \iota_Y\circ L_X$.

Therefore

$$\begin{cases} L_X \circ d = d \circ L_X \\ L_X \circ \iota_X = \iota_X \circ L_X. \end{cases}$$

<u>FACT</u> $\forall\ f \in C^\infty(M)$,

$$L_{fX}\alpha = fL_X\alpha + df \wedge \iota_X\alpha.$$

[For

$$L_{fX}\alpha = \iota_{fX}d\alpha + d\iota_{fX}\alpha$$

$$= f\iota_X d\alpha + d(f\iota_X\alpha)$$

$$= f\iota_X d\alpha + df \wedge \iota_X\alpha + fd\iota_X\alpha$$

$$= f(\iota_X d + d\iota_X)\alpha + df \wedge \iota_X\alpha$$

$$= fL_X\alpha + df \wedge \iota_X\alpha.]$$

If $\varphi: N \to M$ is a diffeomorphism, then

$$\begin{cases} \varphi^* L_X\alpha = L_{\varphi^* X}\varphi^*\alpha \\ \varphi^*\iota_X\alpha = \iota_{\varphi^* X}\varphi^*\alpha. \end{cases}$$

If $\Phi: N \to M$ is a map and if X is Φ-related to Y, then

$$\begin{cases} \Phi^* L_X\alpha = L_Y\Phi^*\alpha \\ \Phi^*\iota_X\alpha = \iota_Y\Phi^*\alpha. \end{cases}$$

[Note: Recall that

$$X \in \mathcal{D}^1(M) \ \& \ Y \in \mathcal{D}^1(N)$$

are said to be <u>Φ-related</u> if

$$d\Phi(Y_y) = X_{\Phi(y)} \quad \forall\, y \in Y$$

or, equivalently, if

$$Y(f \circ \Phi) = Xf \circ \Phi$$

for all $f \in C^\infty(M)$.]

Denote by $w\text{-}\mathcal{D}^p_q(M)$ the tensors of type (p,q) and weight w -- then

$$w\text{-}\mathcal{D}^p_q(M) \leftrightarrow \sec(T^p_q(M) \otimes L^w(M))$$

or still,

$$w\text{-}\mathcal{D}^p_q(M) \leftrightarrow \sec(T^p_q(M) \otimes (\Lambda^n T^*M)^{\otimes w}).$$

Put

$$w\text{-}\mathcal{D}(M) = \bigoplus_{p,q=0}^{\infty} w\text{-}\mathcal{D}^p_q(M).$$

<u>FACT</u> One may attach to each $X \in \mathcal{D}^1(M)$ a type preserving $\underline{R}$-linear map

$$L_X : w\text{-}\mathcal{D}(M) \to w\text{-}\mathcal{D}(M)$$

called the <u>Lie derivative</u> w.r.t. X. Locally, $L_X T$ has the same form as a tensor of type (p,q) except that there is one additional term, namely

$$w X^a_{\ ,a} T^{i_1 \cdots i_p}_{\quad j_1 \cdots j_q}.$$

[Note: If

$$\begin{cases} T \in w\text{-}\mathcal{D}^p_q(M) \\ T' \in w'\text{-}\mathcal{D}^{p'}_{q'}(M), \end{cases}$$

then

$$T \otimes T' \in (w+w') - \mathcal{D}^{p+p'}_{q+q'}(M)$$

and

$$L_X(T \otimes T') = L_X T \otimes T' + T \otimes L_X T'.]$$

To understand how this comes about, it suffices to consider the case when $w = 1$. So suppose that

$$T = S \otimes \omega,$$

where

$$\begin{cases} S \in \mathcal{D}^p_q(M) \\ \omega \in \Lambda^n M. \end{cases}$$

Then

$$L_X T = L_X S \otimes \omega + S \otimes L_X \omega.$$

Bearing in mind that $L_X\omega$ is a scalar density of weight 1, write

$$\omega = \omega_{1\ldots n} dx^1 \wedge \ldots \wedge dx^n.$$

Then

$$\begin{aligned} L_X\omega = &(L_X\omega_{1\ldots n}) dx^1 \wedge \ldots \wedge dx^n \\ &+ \omega_{1\ldots n} L_X dx^1 \wedge \ldots \wedge dx^n \\ &+ \cdots + \omega_{1\cdots n} dx^1 \wedge \ldots \wedge L_X dx^n \\ &= (L_X\omega_{1\cdots n}) dx^1 \wedge \ldots \wedge dx^n \\ &+ \omega_{1\cdots n}(X^1_{,1} + \cdots + X^n_{,n}) dx^1 \wedge \ldots \wedge dx^n \end{aligned}$$

$$= (X^a \omega_{1\cdots n,a} + X^a{}_{,a}\omega_{1\cdots n}) dx^1 \wedge \cdots \wedge dx^n.$$

Therefore

$$L_X T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$= L_X S^{i_1\cdots i_p}{}_{j_1\cdots j_q}\omega_{1\cdots n}$$

$$+ S^{i_1\cdots i_p}{}_{j_1\cdots j_q}(X^a \omega_{1\cdots n,a} + X^a{}_{,a}\omega_{1\cdots n})$$

$$= X^a S^{i_1\cdots i_p}{}_{j_1\cdots j_q,a}\omega_{1\cdots n}$$

$$- X^{i_1}{}_{,a} S^{a i_2\cdots i_p}{}_{j_1\cdots j_q}\omega_{1\cdots n} - \cdots$$

$$+ X^a{}_{,j_1} S^{i_1\cdots i_p}{}_{a j_2\cdots j_q}\omega_{1\cdots n} + \cdots$$

$$+ S^{i_1\cdots i_p}{}_{j_1\cdots j_q}(X^a \omega_{1\cdots n,a} + X^a{}_{,a}\omega_{1\cdots n})$$

$$= X^a (S^{i_1\cdots i_p}{}_{j_1\cdots j_q}\omega_{1\cdots n})_{,a}$$

$$- X^{i_1}{}_{,a} S^{a i_2\cdots i_p}{}_{j_1\cdots j_q}\omega_{1\cdots n} - \cdots$$

$$+ X^a{}_{,j_1} S^{i_1\cdots i_p}{}_{a j_2\cdots j_q}\omega_{1\cdots n} + \cdots$$

$$+ X^a{}_{,a} S^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \omega_{1 \cdots n}$$

$$= X^a T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ,a}$$

$$- X^{i_1}{}_{,a} T^{a i_2 \cdots i_p}{}_{j_1 \cdots j_q} - \cdots$$

$$+ X^a{}_{,j_1} T^{i_1 \cdots i_p}{}_{a j_2 \cdots j_q} + \cdots$$

$$+ X^a{}_{,a} T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

Example: Let T be the upper Levi-Civita symbol (a tensor of type (n,0) and weight 1) or the lower Levi-Civita symbol (a tensor of type (0,n) and weight -1) -- then $L_X T = 0$.

[To discuss the upper Levi-Civita symbol, note that

$$L_X \varepsilon^{i_1 \cdots i_n}$$

$$= X^a \varepsilon^{i_1 \cdots i_n}{}_{,a}$$

$$- X^{i_1}{}_{,a} \varepsilon^{a i_2 \cdots i_n} - \cdots - X^{i_n}{}_{,a} \varepsilon^{i_1 \cdots i_{n-1} a}$$

$$+ X^a{}_{,a} \varepsilon^{i_1 \cdots i_n}$$

$$= - X^{i_1}{}_{,i_1} \varepsilon^{i_1 \cdots i_n} - \cdots - X^{i_n}{}_{,i_n} \varepsilon^{i_1 \cdots i_n}$$

$$+ X^a{}_{,a} \varepsilon^{i_1 \cdots i_n}$$

$$= (-X^{i_1}_{,i_1} - \cdots - X^{i_n}_{,i_n} + X^a_{,a})\varepsilon^{i_1 \cdots i_n}$$

$$= 0.]$$

[Note: The terms involving three identical indices are not summed.]

Given $w \in \underline{Z}$, let $\rho_w = (\det)^{-w}\rho$ and consider the derived map of Lie algebras

$$d\rho_w : \underline{g\ell}(n, \underline{R}) \to \underline{g\ell}(T^p_q(n)).$$

Then $\forall\ A \in \underline{g\ell}(n, \underline{R})$,

$$d\rho_w(A) = \frac{d}{dt}\rho_w(e^{tA})\Big|_{t=0}$$

$$= \frac{d}{dt}(\det e^{tA})^{-w}\rho(e^{tA})\Big|_{t=0}$$

$$= \frac{d}{dt}(e^{t\, tr(A)})^{-w}\rho(e^{tA})\Big|_{t=0}$$

$$= -w\, tr(A) + d\rho(A).$$

Put

$$A^i{}_j(x) = -X^i_{,j}(x)$$

and let $T \in w\text{-}\mathcal{D}^p_q(M)$ -- then at x,

$$- X^{i_1}_{,a} T^{ai_2 \cdots i_p}{}_{j_1 \cdots j_q} - \cdots$$

$$+ X^a_{,j_1} T^{i_1 \cdots i_p}{}_{aj_2 \cdots j_q} + \cdots$$

$$+ wX^a_{,a} T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

equals

$$(A(x)\cdot T_x)^{i_1\cdots i_p}{}_{j_1\cdots j_q}\ .$$

<u>Section 6</u>: <u>Flows</u> Let M be a connected C^∞ manifold of dimension n. Fix an $X \in \mathcal{D}^1(M)$ — then the image of a maximal integral curve of X is called a <u>trajectory</u> of X. The trajectories of X are connected, immersed submanifolds of M. They form a partition of M and their dimension is either 0 or 1 (the trajectories of dimension 0 are the points of M where the vector field X vanishes).

Definition: A <u>first integral</u> for X is an $f \in C^\infty(M):Xf=0$.

In order that f be a first integral for X it is necessary and sufficient that f be constant on the trajectories of X.

Recall now that there exists an open subset $D(X) \subset \underline{R} \times M$ and a differentiable function $\phi_X:D(X) \to M$ such that for each $x \in M$, the map $t \to \phi_X(t,x)$ is the trajectory of X with $\phi_X(0,x)=x$.

(1) $\forall\ x \in M$,

$$I_x(X)=\{t \in \underline{R}:(t,x) \in D(X)\}$$

is an open interval containing the origin and is the domain of the trajectory which passes through x.

(2) $\forall\ t \in \underline{R}$,

$$D_t(X)=\{x \in M:(t,x) \in D(X)\}$$

is open in M and the map

$$\phi_t, x \to \phi_X(t,x)$$

is a diffeomorphism $D_t(X) \to D_{-t}(X)$ with inverse ϕ_{-t}.

(3) If (t,x) and $(s,\phi_X(t,x))$ are elements of $D(X)$, then $(s+t,x)$ is an element of $D(X)$ and

$$\phi_X(s,\phi_X(t,x)) = \phi_X(s+t,x),$$

i.e.,

$$\phi_s \circ \phi_t(x) = \phi_{s+t}(x).$$

One calls ϕ_X the <u>flow</u> of X and X its <u>infinitesimal generator</u>.

[Note: X is said to be <u>complete</u> if $D(X) = \underline{R} \times M$.]

<u>FACT</u> Suppose that $X_x \neq 0$ -- then $\exists$ a chart U containing x such that $X|U = \frac{\partial}{\partial x^1}$ and $\phi_t(x^1,\ldots,x^n) = (x^1+t,x^2,\ldots,x^n)$.

Let $Y \in \mathcal{D}^1(M)$ -- then Y is <u>invariant</u> under ϕ_X if $(\phi_t)_* Y_x = Y_{\phi_t(x)}$ for all $(t,x) \in D(X)$.

Example: X is invariant under ϕ_X.

[Fix $(t_0,x_0) \in D(X)$ and suppose that f is a C^∞ function defined in some neighborhood of $\phi_{t_0}(x_0)$ -- then

$$((\phi_{t_0})_* X_{x_0})f = X_{x_0}(f \circ \phi_{t_0})$$

$$= \frac{d}{dt} f \circ \phi_{t_0} \circ \phi_X(t,x_0)\Big|_{t=0}$$

$$= \frac{d}{dt} f(\phi_X(t_0,\phi_X(t,x_0)))\Big|_{t=0}$$

$$= \frac{d}{dt} f(\phi_X(t_0+t,x_0))\Big|_{t=0}$$

$$= X_{\phi_{t_0}(x_0)} f.]$$

<u>FACT</u> Y is invariant under ϕ_X iff $[X,Y] = 0$.

<u>Push and Pull</u> Let $\varphi: M \rightarrow M$ be a diffeomorphism -- then there is a vector bundle isomorphism $T_q^p\varphi: T_q^p(M) \rightarrow T_q^p(M)$ and a commutative diagram

$$\begin{array}{ccc} & T_q^p\varphi & \\ T_q^p(M) & \rightarrow & T_q^p(M) \\ \pi \downarrow & & \downarrow \pi \\ M & \underset{\varphi}{\rightarrow} & M \end{array}$$

(a) Given $T \in \mathcal{D}_q^p(M)$, put

$$\varphi_* T = T_q^p \varphi \circ T \circ \varphi^{-1},$$

the <u>pushforward</u> of T.

[Note: Thus

$$\varphi_* T(\Lambda^1, \ldots, \Lambda^p, X_1, \ldots, X_q)$$

$$= T(\varphi^* \Lambda^1, \ldots, \varphi^* \Lambda^p, \varphi_*^{-1} X_1, \ldots, \varphi_*^{-1} X_q).]$$

(b) Given $T \in \mathcal{D}_q^p(M)$, put

$$\varphi^* T = T_q^p \varphi^{-1} \circ T \circ \varphi,$$

the <u>pullback</u> of T.

[Note: Thus

$$\varphi^* T(\Lambda^1, \ldots, \Lambda^p, X_1, \ldots, X_q)$$

$$= T(\varphi^{-1*} \Lambda^1, \ldots, \varphi^{-1*} \Lambda^p, \varphi_* X_1, \ldots, \varphi_* X_q).]$$

Remark: Obviously,

$$\varphi^* = (\varphi^{-1})_* .$$

The standard fact that

$$[X,Y]_x = L_X Y \Big|_x$$

$$= \lim_{t \to 0} \frac{\phi_t^* Y_{\phi_t(x)} - Y_x}{t}$$

can be generalized: $\forall\ T \in \mathcal{D}_q^p(M)$,

$$L_X T\Big|_x = \lim_{t \to 0} \frac{\phi_t^* T_{\phi_t(x)} - T_x}{t}.$$

[Note: For $t \neq 0$ and small, the difference quotient on the right makes sense (both $\phi_t^* T_{\phi_t(x)}$ and T_x are elements of the vector space $T_q^p T_x M$).]

So, in brief,

$$L_X T = \frac{d}{dt}\, \phi_t^* T\Big|_{t=0},$$

hence $L_X T = 0$ iff T is constant on the trajectories of X.

Let $\varphi : M \to M$ be a diffeomorphism — then φ lifts to a diffeomorphism $\bar{\varphi} : LM \to LM$, where $\bar{\varphi}(x,E)$ is computed from

$$\begin{array}{ccc} T_x M & \xrightarrow{d\varphi_x} & T_{\varphi(x)} M \\ \zeta_E \uparrow & & \\ \underline{R}^n & & \end{array}.$$

N.B. The pair $(\bar{\varphi}, \varphi)$ is an automorphism of $(LM, M; \underline{GL}(n,\underline{R}))$, i.e., $\bar{\varphi}$ is equivariant and the diagram

$$\begin{array}{ccc} LM & \xrightarrow{\bar{\varphi}} & LM \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[\varphi]{} & M \end{array}$$

commutes.

Observation: We have

$$\Phi_{\varphi^*T} = \Phi_T \circ \bar{\varphi}.$$

[In fact,

$$\Phi_T \circ \bar{\varphi}(x,E)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= \Phi_T(\varphi(x),\varphi_*E)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= T_{\varphi(x)}(\Lambda^1 \circ \zeta_{\varphi_*E}^{-1},\ldots,\Lambda^p \circ \zeta_{\varphi_*E}^{-1}, \zeta_{\varphi_*E}(X_1),\ldots,\zeta_{\varphi_*E}(X_q))$$

$$= T_{\varphi(x)}(\Lambda^1 \circ \zeta_E^{-1} \circ d\varphi_x^{-1},\ldots,\Lambda^p \circ \zeta_E^{-1} \circ d\varphi_x^{-1}, d\varphi_x(\zeta_E(X_1)),\ldots,d\varphi_x(\zeta_E(X_q)))$$

$$= (\varphi^*T)_x(\Lambda^1 \circ \zeta_E^{-1},\ldots,\Lambda^p \circ \zeta_E^{-1}, \zeta_E(X_1),\ldots,\zeta_E(X_q))$$

$$= \Phi_{\varphi^*T}(x,E).]$$

Let $X \in \mathcal{D}^1(M)$ -- then ϕ_X lifts to a flow $\bar{\phi}_{\bar{X}}$ on LM.

<u>LEMMA</u> We have

$$\Phi_{L_X T} = L_{\bar{X}} \Phi_T .$$

[At $t = 0$,

$$L_{\bar{X}} \Phi_T(x,E) \ (\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= \frac{d}{dt} \Phi_T(\bar{\phi}_t(x,E))(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= \frac{d}{dt}\, \Phi_T(\phi_t(x), \phi_{t*}(E))(\Lambda^1, \ldots, \Lambda^p, X_1, \ldots, X_q)$$

$$= \frac{d}{dt}\, T_{\phi_t(x)}(\Lambda^1 \circ \zeta_E^{-1} \circ \phi_{t*}^{-1}, \ldots, \Lambda^p \circ \zeta_E \circ \phi_{t*}^{-1}, \phi_{t*}(\zeta_E(X_1)), \ldots, \phi_{t*}(\zeta_E(X_q)))$$

$$= \frac{d}{dt}\, (\phi_t^* T)_x(\Lambda^1 \circ \zeta_E^{-1}, \ldots, \Lambda^p \circ \zeta_E^{-1}, \zeta_E(X_1), \ldots, \zeta_E(X_q))$$

$$= L_X T\Big|_x\, (\Lambda^1 \circ \zeta_E^{-1}, \ldots, \Lambda^p \circ \zeta_E^{-1}, \zeta_E(X_1), \ldots, \zeta_E(X_q))$$

$$= \Phi_{L_X T}\,(x,E)(\Lambda^1, \ldots, \Lambda^p, X_1, \ldots, X_q).]$$

<u>Section 7</u>: <u>Covariant Differentiation</u> Let M be a connected C^∞ manifold of dimension n. Suppose that $E \to M$ is a vector bundle -- then a <u>connection</u> ∇ on E is a map

$$\nabla : \mathcal{D}^1(M) \to \text{Hom}_{\underline{R}}(\sec(E), \sec(E))$$

such that

(1) $\nabla_{X+Y}s = \nabla_X s + \nabla_Y s$;

(2) $\nabla_X(s+t) = \nabla_X s + \nabla_X t$;

(3) $\nabla_{fX}s = f\nabla_X s$;

(4) $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

[Note: By definition, $\nabla_X s$ is the <u>covariant derivative</u> of s w.r.t. X.]

Rappel: There is a one-to-one correspondence

$$\begin{cases} \Gamma \to \nabla^\Gamma \\ \nabla \to \Gamma^\nabla \end{cases}$$

between the connections Γ on the frame bundle

$$\begin{array}{ccc} \underline{GL}(n,\underline{R}) \to & LM \\ & \downarrow \pi \\ & M \end{array}$$

and the connections ∇ on the tangent bundle

$$TM = LM \times_{\underline{GL}(n,\underline{R})} \underline{R}^n.$$

Let con TM stand for the set of connections on TM.

• Let $\nabla \in$ con TM -- then the assignment

$$\begin{cases} \mathcal{D}_1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to C^\infty(M) \\ (\Lambda, X, Y) \to \Lambda(\nabla_X Y) \end{cases}$$

is not a tensor.

• Let $\nabla', \nabla'' \in$ con TM — then the assignment

$$\begin{cases} \mathcal{D}_1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to C^\infty(M) \\ (\Lambda, X, Y) \to \Lambda(\nabla'_X Y - \nabla''_X Y) \end{cases}$$

is $C^\infty(M)$-multilinear, hence is a tensor.

• Let $\nabla \in$ con TM — then $\forall\, \Psi \in \mathcal{D}^1_2(M)$, the assignment

$$\begin{cases} \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M) \\ (X,Y) \to \nabla_X Y + \Psi(X,Y) \end{cases}$$

is a connection.

Scholium: con TM is an affine space with translation group $\mathcal{D}^1_2(M)$.

[The action $\nabla \cdot \Psi = \nabla + \Psi$ is free and transitive.]

Remark: Write con LM for the set of connections on LM — then, on general grounds, con LM is an affine space (in the 1-form description, the translation group is $\Lambda^1_{Ad}(LM; \underline{g\ell}(n,\underline{R}))$).

Let ∇ be a connection on TM. Put $\nabla_X f = Xf$ and in the notation of the Extension Principle, take $\delta = \nabla_X$ (permissible, since $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$) — then there exists a unique derivation

$$\nabla_X : \mathcal{D}(M) \to \mathcal{D}(M)$$

such that $\nabla_X | C^\infty(M) = X$ and $\nabla_X | \mathcal{D}^1(M) = \delta$.

[Note: The difference $\nabla_X - L_X$ is $C^\infty(M)$-linear on $\mathcal{D}^1(M)$:

$$(\nabla_X - L_X)(fY)$$

$$= (Xf)Y + f\nabla_X Y - (Xf)Y - fL_X Y$$

$$= f(\nabla_X Y - L_X Y),$$

hence ∇_X as a derivation of $\mathcal{D}(M)$ admits the decomposition

$$\nabla_X = L_X + D_{\nabla_X - L_X}\ .]$$

Remark: Write $\nabla = \nabla^\Gamma$ — then Γ induces a connection ∇_q^p on

$$T_q^p(M) = LM \times_{\underline{GL}(n,\underline{R})} T_q^p(n)$$

and matters are consistent: $\forall\ T \in \mathcal{D}_q^p(M)$,

$$\nabla_X^\Gamma T = \nabla_q^p(X)T.$$

On general grounds, each $X \in \mathcal{D}^1(M)$ admits a unique lifting to a horizontal vector field X^h on LM such that $\pi_* X^h = X$.

<u>FACT</u> We have

$$\Phi_{\nabla_X^\Gamma T} = L_{X^h}\Phi_T\ .$$

Owing to the product formula, $\forall\ T \in \mathcal{D}_q^p(M)$,

$$X[T(\Lambda^1,\ldots,\Lambda^p,\ X_1,\ldots,X_q)]$$

$$= (\nabla_X T)(\Lambda^1,\ldots,\Lambda^p,\ X_1,\ldots,X_q)$$

$$+ \sum_{i=1}^{p} T(\Lambda^1,\ldots,\nabla_X\Lambda^i,\ldots,\Lambda^p,\ X_1,\ldots,X_q)$$

$$+ \sum_{j=1}^{q} T(\Lambda^1,\ldots,\Lambda^p,\ X_1,\ldots,\nabla_X X_j,\ldots,X_q).$$

[Note: If $\omega \in \mathcal{D}_1(M)$, then

$$(\nabla_X \omega)(Y) = X\omega(Y) - \omega(\nabla_X Y).]$$

Definition: Let ∇ be a connection on TM. Suppose that $(U,\{x^1,\ldots,x^n\})$ is a chart -- then the connection coefficients of ∇ w.r.t. the coordinates $x^1,\ldots,x^n$ are the C^∞ functions $\Gamma^k{}_{ij}$ on U defined by the prescription

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k{}_{ij} \frac{\partial}{\partial x^k} .$$

Observation: $\forall\ X \in \mathcal{D}^1(M)$,

$$\begin{cases} \nabla_X \dfrac{\partial}{\partial x^i} = X^a \Gamma^k{}_{ai} \dfrac{\partial}{\partial x^k} \\ \\ \nabla_X dx^i = - X^a \Gamma^i{}_{ak} dx^k . \end{cases}$$

So locally,

$$\begin{aligned} &(\nabla_X T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \\ &\quad = X^a T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q, a} \\ &\quad + X^a \Gamma^{i_1}{}_{ab} T^{b i_2 \cdots i_p}{}_{j_1 \cdots j_q} + \cdots \\ &\quad - X^a \Gamma^b{}_{a j_1} T^{i_1 \cdots i_p}{}_{b j_2 \cdots j_q} - \cdots . \end{aligned}$$

Remark: The symbol

$$(\nabla_X T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

is usually abbreviated to

$$\nabla_X T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

Example: Let K be the Kronecker tensor -- then

$$\nabla_X K = 0.$$

Indeed,

$$\nabla_X K^i{}_j = X^a \delta^i{}_{j,a} + X^a \Gamma^i{}_{ab} \delta^b{}_j - X^a \Gamma^b{}_{aj} \delta^i{}_b$$

$$= 0 + X^a \Gamma^i{}_{aj} - X^a \Gamma^i{}_{aj}$$

$$= 0.$$

[Note: In general, $\forall\ p \geq 1$,

$$\nabla_X K(p) = 0.]$$

<u>LEMMA</u> Let ∇ be a connection on TM -- then on $U \cap U'$,

$$\Gamma^{k'}{}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \Gamma^k{}_{ij} + \frac{\partial^2 x^a}{\partial x^{i'} \partial x^{j'}} \frac{\partial x^{k'}}{\partial x^a} .$$

[Note: This relation is called the <u>connection transformation rule</u>.]

Therefore the $\Gamma^k{}_{ij}$ are not the components of a tensor.

<u>FACT</u> Assume that there is assigned to each U in a coordinate atlas for M, functions

$$\Gamma^k{}_{ij} \in C^\infty(U)$$

subject to the connection transformation rule -- then there is a unique connection ∇ on TM whose connection coefficients w.r.t. the coordinates $x^1,\ldots,x^n$ are the Γ^k_{ij}.

Remark: Consider the contraction Γ^j_{ij}. To determine its transformation law, write

$$\Gamma^{j'}_{i'j'} = \frac{\partial x^{j'}}{\partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\Gamma^k_{ij} + \frac{\partial^2 x^a}{\partial x^{i'}\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^a}.$$

Then

$$\frac{\partial x^{j'}}{\partial x^k}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\Gamma^k_{ij}$$

$$= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^k}\Gamma^k_{ij}$$

$$= \frac{\partial x^i}{\partial x^{i'}}\delta^j_k\Gamma^k_{ij} = \frac{\partial x^i}{\partial x^{i'}}\Gamma^j_{ij}.$$

On the other hand, by determinant theory,

$$\frac{\partial^2 x^a}{\partial x^{i'}\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^a} = \frac{\partial}{\partial x^{i'}}\log|J|.$$

[Note: Analogously,

$$\Gamma^{i'}_{i'j'} = \frac{\partial x^j}{\partial x^{j'}}\Gamma^i_{ij} + \frac{\partial}{\partial x^{j'}}\log|J|.]$$

Let ∇ be a connection on TM -- then ∇ induces a map $\mathcal{D}^p_q(M) \to \mathcal{D}^p_{q+1}(M)$, viz.

$$\nabla T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q,X)$$

$$= (\nabla_X T)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q).$$

[Note: One calls ∇T the covariant derivative of T.]

Working locally, put

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ; a}$$

$$= \nabla_a T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} ,$$

where

$$\nabla_a = \nabla_{\frac{\partial}{\partial x^a}} .$$

Then in view of what has been said above,

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ; a}$$

$$= T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q , a}$$

$$+ \Gamma^{i_1}{}_{ab} T^{b i_2 \cdots i_p}{}_{j_1 \cdots j_q} + \cdots$$

$$- \Gamma^{b}{}_{a j_1} T^{i_1 \cdots i_p}{}_{b j_2 \cdots j_q} - \cdots .$$

[Note: The components of ∇T are the

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ; a} .$$

Thus

$$(\nabla T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q a}$$

$$= \nabla T(dx^{i_1},\ldots,dx^{i_p}, \frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_q}}, \frac{\partial}{\partial x^a})$$

$$= \nabla_a T(dx^{i_1},\ldots,dx^{i_p}, \frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_q}})$$

$$= \nabla_a T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$= T^{i_1\cdots i_p}{}_{j_1\cdots j_q;a}\ .]$$

Example: $\forall\ X\in\mathcal{D}^1(M)$, $\nabla X\in\mathcal{D}^1_1(M)$, so locally,

$$\nabla X = X^i_{;j}\frac{\partial}{\partial x^i}\otimes dx^j,$$

where

$$\nabla_j X^i = X^i_{;j} = X^i_{,j} + X^a\Gamma^i{}_{ja}.$$

Remark: Let $T\in\mathcal{D}^p_q(M)$ — then T is said to be <u>parallel</u> if $\nabla T = 0$, which is the case iff $\nabla_X T = 0$ for all $X\in\mathcal{D}^1(M)$.

Notation: Define $\nabla^k:\mathcal{D}^p_q(M)\to\mathcal{D}^p_{q+k}(M)$ by $\nabla^1 = \nabla$ and $\nabla^k = \nabla(\nabla^{k-1})$ $(k>1)$.

<u>LEMMA</u> Let $X,Y\in\mathcal{D}^1(M)$ — then $\forall\ T\in\mathcal{D}^p_q(M)$,

$$\nabla^2 T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q,X,Y)$$

$$= \nabla_Y(\nabla_X T)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$- \nabla_{\nabla_Y X}T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q).$$

[Thanks to the product formula, we have

$$\nabla^2 T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q,X,Y)$$

$$= (\nabla_Y \nabla T)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q,X)$$

$$= Y[\nabla T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q,X)]$$

$$- \sum_{i=1}^{p} \nabla T(\Lambda^1,\ldots,\nabla_Y \Lambda^i,\ldots,\Lambda^p, X_1,\ldots,X_q,X)$$

$$- \sum_{j=1}^{q} \nabla T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,\nabla_Y X_j,\ldots,X_q,X)$$

$$- \nabla T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q,\nabla_Y X)$$

$$= Y[\nabla_X T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)]$$

$$- \sum_{i=1}^{p} \nabla_X T(\Lambda^1,\ldots,\nabla_Y \Lambda^i,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$- \sum_{j=1}^{q} \nabla_X T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,\nabla_Y X_j,\ldots,X_q)$$

$$- \nabla_{\nabla_Y X} T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= \nabla_Y(\nabla_X T)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$- \nabla_{\nabla_Y X} T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q).]$$

[Note: $\nabla^2 T \in \mathcal{D}^p_{q+2}(M)$ and

$$(\nabla^2 T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ab}$$

is written as

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q ; a ; b}$$

or still,

$$\nabla_b \nabla_a T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .]$$

Definition: Let ∇ be a connection on TM -- then the torsion of ∇ is the map

$$T : \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M)$$

defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

[Note: ∇ is said to be torsion free if $T \equiv 0$.]

Example: Let $f \in C^\infty(M)$ -- then $\nabla^2 f \in \mathcal{D}_2^0(M)$ and

$$\begin{aligned} \nabla^2 f(X,Y) &= \nabla_Y(\nabla_X f) - \nabla_{\nabla_Y X} f \\ &= (YX - \nabla_Y X) f \\ &= (XY - \nabla_X Y + T(X,Y)) f \\ &= \nabla^2 f(Y,X) + T(X,Y) f. \end{aligned}$$

Thus $\nabla^2 f$ is symmetric whenever ∇ is torsion free.

Obviously,

$$T(X,Y) = - T(Y,X).$$

It is also easy to check that

$$T(fX,gY) = fgT(X,Y) \quad (f,g \in C^\infty(M)).$$

Therefore the assignment

$$\begin{cases} \mathcal{D}_1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to C^\infty(M) \\ (\Lambda,X,Y) \to \Lambda(T(X,Y)) \end{cases}$$

is a tensor, the <u>torsion tensor</u> attached to ∇.

Construction: Given $\nabla \in$ con TM, define $\nabla' \in$ con TM by

$$\nabla' = \nabla - T.$$

This makes sense (recall that con TM is an affine space with translation group $\mathcal{D}_2^1(M)$). To compute the torsion of ∇', note that

$$\nabla'_X Y - \nabla'_Y X - [X,Y]$$

$$= \nabla_Y X + [X,Y] - \nabla_X Y - [Y,X] - [X,Y]$$

$$= \nabla_Y X - \nabla_X Y - [Y,X]$$

$$= T(Y,X) = - T(X,Y).$$

Therefore the connection

$$\frac{1}{2}\nabla + \frac{1}{2}\nabla'$$

is torsion free and

$$\nabla = (\frac{1}{2}\nabla + \frac{1}{2}\nabla') + \frac{1}{2}T.$$

Finally, suppose that

$$\nabla = \tilde{\nabla} + \frac{1}{2} S,$$

where $\tilde{\nabla}$ is torsion free and

$$S: \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M)$$

subject to

$$S(X,Y) = - S(Y,X).$$

Then the torsion of ∇ is the torsion of $\tilde{\nabla}$ plus

$$\frac{1}{2} S(X,Y) - \frac{1}{2} S(Y,X) = S(X,Y).$$

I.e.:

$$T = S$$

$$\Rightarrow$$

$$\tilde{\nabla} = \frac{1}{2} \nabla + \frac{1}{2} \nabla'.$$

Working locally, write

$$T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = T^k{}_{ij} \frac{\partial}{\partial x^k}.$$

Then

$$T^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}.$$

[Note: Consider the decomposition

$$\nabla = (\frac{1}{2} \nabla + \frac{1}{2} \nabla') + \frac{1}{2} T.$$

Then, in terms of connection coefficients,

$$\Gamma^k_{ij} = \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji}) + \frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}).]$$

Example: Let $f \in C^\infty(M)$ -- then

$$\nabla^2 f(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \nabla^2 f(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}) + T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) f$$

$\Rightarrow$

$$f_{;i;j} = f_{;j;i} + T^k_{ij} f_{,k}$$

or still,

$$\nabla_j \nabla_i f = \nabla_i \nabla_j f + T^k_{ij} f_{,k} .$$

Let $T \in \mathcal{D}^0_q(M)$ -- then

$$(L_X T)(X_1,\ldots,X_q)$$

$$= X[T(X_1,\ldots,X_q)]$$

$$- \sum_{j=1}^{q} T(X_1,\ldots,L_X X_j,\ldots,X_q).$$

On the other hand,

$$(\nabla_X T)(X_1,\ldots,X_q)$$

$$= X[T(X_1,\ldots,X_q)]$$

$$- \sum_{j=1}^{q} T(X_1,\ldots,\nabla_X X_j,\ldots,X_q).$$

Assume: ∇ is torsion free -- then

$$L_X X_j = [X,X_j] = \nabla_X X_j - \nabla_{X_j} X.$$

Therefore

$$(L_X T)(X_1,\ldots,X_q)$$

$$= X[T(X_1,\ldots,X_q)]$$

$$- \sum_{j=1}^{q} T(X_1,\ldots,\nabla_X X_j,\ldots,X_q)$$

$$+ \sum_{j=1}^{q} T(X_1,\ldots,\nabla_{X_j} X,\ldots,X_q)$$

$$= (\nabla_X T)(X_1,\ldots,X_q)$$

$$+ \sum_{j=1}^{q} T(X_1,\ldots,\nabla_{X_j} X,\ldots,X_q).$$

[Note: If T is parallel, i.e., if $\nabla T = 0$, then

$$(L_X T)(X_1,\ldots,X_q)$$

$$= \sum_{j=1}^{q} T(X_1,\ldots,\nabla_{X_j} X,\ldots,X_q).]$$

Turning now to the exterior algebra Λ^*M, suppose that $\alpha\in\Lambda^p M$ -- then

$$(\nabla_X \alpha)(X_1,\ldots,X_p)$$

$$= X[\alpha(X_1,\dots,X_p)]$$

$$- \sum_{i=1}^{p} \alpha(X_1,\dots,\nabla_X X_i,\dots,X_p),$$

so $\nabla_X\alpha \in \Lambda^p M$.

Observation: The following diagram

$$\begin{array}{ccc} \mathcal{D}_p^0(M) & \xrightarrow{\nabla_X} & \mathcal{D}_p^0(M) \\ \text{Alt} \downarrow & & \downarrow \text{Alt} \\ \Lambda^p M & \xrightarrow[\nabla_X]{} & \Lambda^p M \end{array}$$

commutes. Consequently,

$$\nabla_X(\alpha\wedge\beta) = \nabla_X\alpha\wedge\beta + \alpha\wedge\nabla_X\beta.$$

Rappel: The exterior derivative

$$d:\Lambda^p M \rightarrow \Lambda^{p+1} M$$

is given by

$$d\alpha(X_1,\dots,X_{p+1})$$

$$= \sum_{1\le i\le p+1} (-1)^{i+1} X_i\alpha(X_1,\dots,\hat{X}_i,\dots,X_{p+1})$$

$$+ \sum_{i<j} (-1)^{i+j}\alpha([X_i,X_j],\ X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1}).$$

There is a triangle

$$\begin{array}{ccc} \Lambda^p M & & \\ d \downarrow & \searrow \nabla & \\ \Lambda^{p+1} M & \underset{\text{Alt}}{\leftarrow} & \mathcal{D}^0_{p+1}(M) \end{array}$$

but $d \neq \text{Alt} \circ \nabla$.

<u>LEMMA</u> Suppose that ∇ is torsion free -- then on $\Lambda^p M$,

$$\text{Alt} \circ \nabla = \frac{(-1)^p}{p+1} d.$$

[Note: Under the assumption that ∇ is torsion free, $\forall\ \alpha \in \Lambda^p M$, we have

$$d\alpha(X_1, \ldots, X_{p+1})$$

$$= \sum_{1 \leq i \leq p+1} (-1)^{i+1} (\nabla_{X_i} \alpha)(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}),$$

thus locally

$$(d\alpha)_{j_1 \cdots j_{p+1}} = \sum_{a=1}^{p+1} (-1)^{a+1} \nabla_a \alpha_{j_1 \cdots \hat{j}_a \cdots j_{p+1}}.$$

E.g., take $p = 1$ -- then

$$d\alpha(X,Y) = \nabla\alpha(Y,X) - \nabla\alpha(X,Y),$$

thus $\nabla\alpha$ is symmetric iff α is closed.]

<u>FACT</u> Let $X, Y \in \mathcal{D}^1(M)$ -- then

$$\nabla_X \circ \iota_Y - \iota_Y \circ \nabla_X = \iota_{\nabla_X Y} .$$

Let Γ be a connection on LM. Suppose that ρ is a representation of $\underline{GL}(n,\underline{R})$ on a finite dimensional vector space W. Form the vector bundle

$$E = LM \times_{\underline{GL}(n,\underline{R})} W.$$

Then Γ induces a connection on E.

Specialize and take $W = T_q^p(n)$, $\rho = \rho_w$ -- then one may attach to each $X \in \mathcal{D}^1(M)$ a covariant derivative

$$\nabla_X : w\text{-}\mathcal{D}_q^p(M) \rightarrow w\text{-}\mathcal{D}_q^p(M).$$

Locally, $\nabla_X T$ has the same form as a tensor of type (p,q) except that there is one additional term, namely

$$- wX^a \Gamma^b{}_{ab} T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} .$$

[Note: If

$$\begin{cases} T \in w\text{-}\mathcal{D}_q^p(M) \\ T' \in w'\text{-}\mathcal{D}_{q'}^{p'}(M), \end{cases}$$

then

$$T \otimes T' \in (w+w')\text{-}\mathcal{D}_{q+q'}^{p+p'}(M)$$

and

$$\nabla_X(T \otimes T') = \nabla_X T \otimes T' + T \otimes \nabla_X T'.]$$

Remark: Given $\omega \in \Lambda^n M$, write

$$\omega = \omega_{1 \cdots n} dx^1 \wedge \cdots \wedge dx^n.$$

Then

$$\nabla_X\omega = (\nabla_X\omega_{1\cdots n})dx^1 \wedge \cdots \wedge dx^n$$

$$+ \omega_{1\cdots n}\nabla_X dx^1 \wedge \cdots \wedge dx^n$$

$$+ \cdots + \omega_{1\cdots n}dx^1 \wedge \cdots \wedge \nabla_X dx^n$$

$$= (\nabla_X\omega_{1\cdots n})dx^1 \wedge \cdots \wedge dx^n$$

$$+ \omega_{1\cdots n}(- X^a\Gamma^1{}_{a1} - \cdots - X^a\Gamma^n{}_{an})dx^1 \wedge \cdots \wedge dx^n$$

$$= (X^a\omega_{1\cdots n,a} - X^a\Gamma^b{}_{ab}\omega_{1\cdots n})dx^1 \wedge \cdots \wedge dx^n.$$

Example: Let T be the upper Levi-Civita symbol (a tensor of type (n,0) and weight 1) or the lower Levi-Civita symbol (a tensor of type (0,n) and weight -1) — then $\nabla_X T = 0$.

[To discuss the upper Levi-Civita symbol, note that

$$\nabla_X\varepsilon^{i_1\cdots i_n}$$

$$= X^a\varepsilon^{i_1\cdots i_n}{}_{,a}$$

$$+ X^a\Gamma^{i_1}{}_{ab}\varepsilon^{bi_2\cdots i_n} + \cdots + X^a\Gamma^{i_n}{}_{ab}\varepsilon^{i_1\cdots i_{n-1}b}$$

$$- X^a\Gamma^b{}_{ab}\varepsilon^{i_1\cdots i_n}$$

$$= X^a\Gamma^{i_1}{}_{ai_1}\varepsilon^{i_1\cdots i_n} + \cdots + X^a\Gamma^{i_n}{}_{ai_n}\varepsilon^{i_1\cdots i_n}$$

$$- X^a\Gamma^b{}_{ab}\varepsilon^{i_1\cdots i_n}$$

$$= X^a(\Gamma^{i_1}{}_{ai_1} + \cdots + \Gamma^{i_n}{}_{ai_n} - \Gamma^b{}_{ab})\varepsilon^{i_1 \cdots i_n}$$

$$= 0.]$$

[Note: The terms involving three identical indices are not summed.]

Example: Let $T \in 1-\mathcal{D}^1_0(M)$ -- then

$$\nabla_a T^i = T^i{}_{,a} + \Gamma^i{}_{ab}T^b - \Gamma^b{}_{ab}T^i.$$

Now contract over the indices a and i to get

$$\nabla_a T^a = T^a{}_{,a} + \Gamma^a{}_{ab}T^b - \Gamma^b{}_{ab}T^a$$

$$= T^a{}_{,a} + (\Gamma^a{}_{ab} - \Gamma^a{}_{ba})T^b,$$

hence

$$\nabla_a T^a = T^a{}_{,a}$$

provided ∇ is torsion free.

There is no difficulty in extending the theory to densities of weight r or twisted densities of weight r, hence to tensors T of class I or II.

[Note: ∇_X respects the class of T.]

Locally, $\nabla_X T$ has the same form as a tensor of type (p,q) except that there is one additional term, namely

$$- rX^a\Gamma^b{}_{ab}T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}.$$

<u>Reality Check</u> If ϕ is a density of weight r and ψ is a density of weight -r, then $\phi\psi \in C^\infty(M)$ and we have

$$\nabla_a(\phi\psi) = (\nabla_a\phi)\psi + \phi(\nabla_a\psi)$$

$$= (\phi_{,a} - r\Gamma^b{}_{ab}\phi)\psi + \phi(\psi_{,a} + r\Gamma^b{}_{ab}\psi)$$

$$= \phi_{,a}\psi + \phi\psi_{,a}$$

$$= \partial_a(\phi\psi).$$

Example: If ϕ is a scalar density of weight 1 and ψ is a density of weight -1, then $\phi\psi$ is a twisted density of weight 0 and

$$\nabla_a(\phi\psi) = \partial_a(\phi\psi).$$

<u>Section 8:</u> <u>Parallel Transport</u> Let M be a connected C^∞ manifold of dimension n. Suppose that

$$\begin{array}{ccc} G & \rightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

is a principal bundle with structure group G (which we shall take to be a Lie group) and let Γ be a connection on P.

Convention: Curves are piecewise smooth.

<u>THEOREM</u> Let $\gamma:[0,1] \rightarrow M$ be a curve. Fix a point $p_0 \in \pi^{-1}(\gamma(0))$ -- then there is a unique curve $\gamma^\dagger:[0,1] \rightarrow P$ such that (i) $\gamma^\dagger(0) = p_0$, (ii) $\pi \circ \gamma^\dagger = \gamma$, (iii) $\dot{\gamma}^\dagger(t) \in T^h_{\gamma^\dagger(t)} P$ $(0 \leq t \leq 1)$.

It follows from the theorem that there is a diffeomorphism

$$\tau_\gamma:\pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(1))$$

called <u>parallel transport</u> from $\gamma(0)$ to $\gamma(1)$.

Let ρ be a representation of G on a finite dimensional vector space W. Put

$$E = P \times_G W.$$

Then E is a vector bundle and there is a commutative diagram

$$\begin{array}{ccc} P \times W & \xrightarrow{pr_P} & P \\ pro \downarrow & & \downarrow \pi \\ E & \xrightarrow[\pi_E]{} & M . \end{array}$$

Here

$$\pi_E([p,w]) = \pi(p).$$

Let $e_0 \in E$. Take any point $(p_0,w_0) \in \text{pro}^{-1}(e_0)$ and define

$$f_{w_0}: P \rightarrow E$$

by

$$f_{w_0}(p) = [p,w_0].$$

Set

$$T^h_{e_0} E = (f_{w_0})_* T^h_{p_0} P \subset T_{e_0} E.$$

Then $T^h_{e_0} E$ is independent of the choice of (p_0,w_0) and is called the <u>horizontal subspace</u> of $T_{e_0} E$ (per the choice of Γ).

<u>THEOREM</u> Let $\gamma: [0,1] \rightarrow M$ be a curve. Fix a point $e_0 \in \pi_E^{-1}(\gamma(0))$ — then there is a unique curve $\gamma^\dagger: [0,1] \rightarrow E$ such that (i) $\gamma^\dagger(0) = e_0$, (ii) $\pi_E \circ \gamma^\dagger = \gamma$, (iii) $\dot{\gamma}^\dagger(t) \in T^h_{\gamma^\dagger(t)} E$ $(0 \leq t \leq 1)$.

It follows from the theorem that there is an isomorphism

$$\tau_\gamma: \pi_E^{-1}(\gamma(0)) \rightarrow \pi_E^{-1}(\gamma(1))$$

called <u>parallel transport</u> from $\gamma(0)$ to $\gamma(1)$.

Denote by ∇^Γ the connection on E determined by Γ. Fix $x \in M$ and let $X \in \mathcal{D}^1(M)$. Choose any curve $\gamma: [-\varepsilon,\varepsilon] \rightarrow M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$. Modify the notation and write

$$\tau_h : \pi_E^{-1}(\gamma(0)) \rightarrow \pi_E^{-1}(\gamma(h))$$

for the parallel transport from $\gamma(0)$ to $\gamma(h)$.

FACT $\forall\ s \in \sec(E)$,

$$\nabla_X^{\Gamma} s|_x = \lim_{h \to 0} \frac{1}{h} [\tau_h^{-1}(s(\gamma(h))) - s(\gamma(0))].$$

Specialize to $P = LM$ and $W = T_q^p(n)$ -- then, with the obvious choice for ρ, these generalities are applicable to the sections of $T_q^p(M)$, i.e., to $\mathcal{D}_q^p(M)$, or, replacing ρ by ρ_w, to the sections of $T_q^p(M) \otimes L^w(M)$, i.e., to $w\text{-}\mathcal{D}_q^p(M)$.

<u>Section 9</u>: <u>Curvature</u> Let M be a connected C^∞ manifold of dimension n.

Definition: Let ∇ be a connection on TM -- then the <u>curvature</u> of ∇ is the map

$$R:\mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathrm{Hom}_{\underline{R}}(\mathcal{D}^1(M),\mathcal{D}^1(M))$$

defined by

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Obviously,

$$R(X,Y) = - R(Y,X).$$

It is also easy to check that

$$R(fX,gY)hZ = fghR(X,Y)Z \quad (f,g,h\in C^\infty(M)).$$

Therefore the assignment

$$\begin{cases} \mathcal{D}_1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to C^\infty(M) \\ (\Lambda,Z,X,Y) \to \Lambda(R(X,Y)Z) \end{cases}$$

is a tensor, the <u>curvature tensor</u> attached to ∇.

Remark: The <u>Lie derivative</u> $L_X\nabla$ of the connection ∇ is the $C^\infty(M)$-multilinear map

$$\begin{cases} \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M) \\ (Y,Z) \to (L_X\nabla)(Y,Z), \end{cases}$$

where

$$(L_X\nabla)(Y,Z) = L_X(\nabla_YZ) - \nabla_{L_XY}Z - \nabla_YL_XZ.$$

Operationally,

$$\begin{aligned}(L_X\nabla)(Y,\text{--}) &= [L_X,\nabla_Y] - \nabla_{[X,Y]} \\ &= R(X,Y) + [L_X - \nabla_X,\nabla_Y].\end{aligned}$$

[Note: A vector field X is said to be an infinitesimal affine transformation if $L_X\nabla = 0$.]

Let ∇ be a connection on TM -- then ∇ is flat provided each $x \in M$ admits a connected neighborhood U such that $\forall\ y \in M$, the parallel transport $\tau: T_xM \to T_yM$ is independent of the curve joining x and y.

FACT ∇ is flat iff its curvature tensor is identically zero.

Convention: Given a $C^\infty(M)$-multilinear map

$$K: \overbrace{\mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M)}^{q} \to \mathcal{D}^1(M),$$

define

$$\nabla_X K: \overbrace{\mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M)}^{q} \to \mathcal{D}^1(M)$$

by

$$(\nabla_X K)(X_1,\ldots,X_q) = \nabla_X(K(X_1,\ldots,X_q))$$

$$- \sum_{j=1}^{q} K(X_1,\ldots,\nabla_X X_j,\ldots,X_q).$$

Example: Suppose that ∇ is a torsion free connection on TM. Let X be a vector field -- then $\nabla X \in \mathcal{D}_1^1(M)$ or, equivalently,

$$\nabla X \in \text{Hom}_{C^\infty(M)}(\mathcal{D}^1(M), \mathcal{D}^1(M)),$$

where

$$\nabla X(Y) = \nabla_Y X.$$

Assume now that X is an infinitesimal affine transformation, thus $L_X\nabla = 0$, hence

$$R(X,Y)Z = [\nabla_X - L_X, \nabla_Y]Z$$

$$= (\nabla_X - L_X)\nabla_Y Z - \nabla_Y(\nabla_X - L_X)Z$$

$$= \nabla_{\nabla_Y Z}X - \nabla_Y\nabla_Z X.$$

On the other hand,

$$(\nabla_Y\nabla X)Z = \nabla_Y(\nabla X(Z)) - \nabla X(\nabla_Y Z)$$

$$= \nabla_Y\nabla_Z X - \nabla_{\nabla_Y Z}X.$$

Therefore

$$R(X,Y)Z + (\nabla_Y\nabla X)Z = 0.$$

In particular:

$$R(Y,X)X = - R(X,Y)X$$

$$= (\nabla_Y\nabla X)X = \nabla_Y\nabla_X X - \nabla X(\nabla_Y X)$$

$$= \nabla_Y\nabla_X X - (\nabla X)^2 Y,$$

$(\nabla X)^2$ being the composite $\nabla X \circ \nabla X$.

<u>FACT</u> Suppose that ∇ is a torsion free connection on TM. Let X be an infinitesimal affine transformation -- then $L_X\nabla^k R = 0$ $(k=1,2,\ldots)$.

<u>LEMMA</u> <u>(Bianchi's First Identity)</u> We have

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$$

$$= T(T(X,Y),Z) + (\nabla_X T)(Y,Z)$$

$$+ T(T(Y,Z),X) + (\nabla_Y T)(Z,X)$$

$$+ T(T(Z,X),Y) + (\nabla_Z T)(X,Y).$$

[Note: Consequently, if ∇ is torsion free, then

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.]$$

Write

$$(\nabla_Z R)(X,Y) = [\nabla_Z, R(X,Y)]$$

$$- R(\nabla_Z X,Y) - R(X,\nabla_Z Y),$$

the bracket standing for a commutator of operators on vector fields.

[Note: To see where this is coming from, think of R as an element of

$$\text{Hom}_{C^\infty(M)}(\mathcal{D}^1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M), \mathcal{D}^1(M)).$$

Then, in view of the foregoing convention,

$$(\nabla_Z R)(W,X,Y) = \nabla_Z(R(X,Y)W)$$

$$- R(\nabla_Z W,X,Y) - R(W,\nabla_Z X,Y) - R(W,X,\nabla_Z Y)$$

$$= [\nabla_Z, R(X,Y)](W)$$

$$- R(W,\nabla_Z X,Y) - R(W,X,\nabla_Z Y).]$$

<u>LEMMA</u> (<u>Bianchi's Second Identity</u>) We have

$$(\nabla_Z R)(X,Y) + R(T(X,Y),Z)$$

$$+ (\nabla_X R)(Y,Z) + R(T(Y,Z),X)$$

$$+ (\nabla_Y R)(Z,X) + R(T(Z,X),Y)$$

$$= 0.$$

[Note: Consequently, if ∇ is torsion free, then

$$(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) = 0.]$$

Since

$$R(X,Y) \in \text{Hom}_{C^\infty(M)}(\mathcal{D}^1(M), \mathcal{D}^1(M)) \approx \mathcal{D}_1^1(M),$$

there exists a unique derivation

$$D_{R(X,Y)}: \mathcal{D}(M) \to \mathcal{D}(M)$$

which is zero on $C^\infty(M)$ and equals $R(X,Y)$ on $\mathcal{D}^1(M)$.

<u>LEMMA</u> <u>(The Ricci Identity)</u> Let $T \in \mathcal{D}_q^p(M)$ -- then

$$\nabla^2 T(-,X,Y) - \nabla^2 T(-,Y,X)$$

$$= (-D_{R(X,Y)} + \nabla_{T(X,Y)})T,$$

where $\nabla_{T(X,Y)}$ is the covariant derivative at the torsion $T(X,Y)$ of ∇.

[We have

$$\begin{cases} \nabla^2 T(-,X,Y) = \nabla_Y(\nabla_X T) - \nabla_{\nabla_Y X} T \\ \nabla^2 T(-,Y,X) = \nabla_X(\nabla_Y T) - \nabla_{\nabla_X Y} T \end{cases}$$

$\Rightarrow$

$$\nabla^2 T(-,X,Y) - \nabla^2 T(-,Y,X)$$

$$= (\nabla_Y \nabla_X - \nabla_X \nabla_Y)T + (\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X})T$$

$$= (\nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]}) T$$

$$+ (\nabla_{\nabla_X Y - \nabla_Y X - [X,Y]}) T$$

$$= (- D_{R(X,Y)} + \nabla_{T(X,Y)}) T.]$$

Remark: Let $T \in \mathcal{D}_q^0(M)$ -- then

$$(- D_{R(X,Y)} T)(X_1, \ldots, X_q)$$

$$= \sum_{j=1}^{q} T(X_1, \ldots, R(X,Y)X_j, \ldots, X_q).$$

So, if ∇ is torsion free, then

$$\nabla^2 T(X_1, \ldots, X_q, X, Y) - \nabla^2 T(X_1, \ldots, X_q, Y, X)$$

$$= \sum_{j=1}^{q} T(X_1, \ldots, R(X,Y)X_j, \ldots, X_q).$$

Working locally, write

$$R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}) \frac{\partial}{\partial x^j} = R^i{}_{jk\ell} \frac{\partial}{\partial x^i},$$

thus

$$R^i{}_{jk\ell} = R(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell})$$

$$= dx^i (R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}) \frac{\partial}{\partial x^j})$$

$$= dx^i ((\nabla_k \nabla_\ell - \nabla_\ell \nabla_k) \frac{\partial}{\partial x^j})$$

$$= \Gamma^i{}_{\ell j,k} - \Gamma^i{}_{kj,\ell} + \Gamma^a{}_{\ell j}\Gamma^i{}_{ka} - \Gamma^a{}_{kj}\Gamma^i{}_{\ell a}.$$

And

$$R^i{}_{jk\ell} = - R^i{}_{j\ell k}\ .$$

<u>Curvature Formulas</u> Assume that ∇ is torsion free.

<u>Bianchi's First Identity</u>:

$$R^i{}_{jk\ell} + R^i{}_{k\ell j} + R^i{}_{\ell jk} = 0.$$

<u>Bianchi's Second Identity</u>:

$$R^i{}_{jk\ell;m} + R^i{}_{j\ell m;k} + R^i{}_{jmk;\ell} = 0.$$

One can also write down local expressions for the Ricci identity.

Example: Let $X \in \mathcal{D}^1(M)$, say $X = X^j \frac{\partial}{\partial x^j}$ -- then $\nabla^2 X \in \mathcal{D}^1_2(M)$ and

$$\nabla_b\nabla_a X^i - \nabla_a\nabla_b X^i$$

$$= X^i{}_{;a;b} - X^i{}_{;b;a}$$

$$= \nabla^2 X(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) - \nabla^2 X(dx^i, \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^a})$$

$$= - dx^i(R(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})X) + dx^i(\nabla_{T(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})} X)$$

$$= - dx^i(X^j R(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) \frac{\partial}{\partial x^j})$$

$$+ dx^i(\nabla_{T^k{}_{ab} \frac{\partial}{\partial x^k}} X)$$

$$= - dx^i(X^j R^k{}_{jab} \frac{\partial}{\partial x^k})$$

$$+ dx^i(T^k{}_{ab} \nabla_k X)$$

$$= - X^j R^i{}_{jab} + T^k{}_{ab} dx^i(\nabla_k X)$$

$$= R^i{}_{jba} X^j + T^k{}_{ab} X^i{}_{;k} .$$

Consider R as an element of $\mathcal{D}^1_3(M)$ -- then the Ricci tensor Ric is the image of R under the contraction $C^1_2 : \mathcal{D}^1_3(M) \to \mathcal{D}^0_2(M)$ of the second slot in the covariant index.

Agreeing to write $R_{j\ell}$ in place of $\text{Ric}(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^\ell})$, it follows that

$$R_{j\ell} = R^i{}_{ji\ell}$$

$$= \Gamma^i{}_{\ell j,i} - \Gamma^i{}_{ij,\ell} + \Gamma^a{}_{\ell j}\Gamma^i{}_{ia} - \Gamma^a{}_{ij}\Gamma^i{}_{\ell a}.$$

Example: Since covariant differentiation commutes with contraction, we have

$$R_{j\ell;i} = \nabla_i R_{j\ell} = \nabla_i (C^1_2 R)_{j\ell}$$

$$= (C^1_2 \nabla_i R)_{j\ell}$$

$$= R^a{}_{ja\ell;i}.$$

But

$$R^a{}_{ja\ell;i} + R^a{}_{j\ell i;a} + R^a{}_{jia;\ell} = 0$$

$$\Rightarrow$$

$$R^a{}_{j\ell i;a} = - R^a{}_{ja\ell;i} - R^a{}_{jia;\ell}$$

$$= R^a{}_{jai;\ell} - R^a{}_{ja\ell;i}$$

$$= \nabla_\ell R_{ji} - \nabla_i R_{j\ell}.$$

In general, the Ricci tensor is not symmetric:

$$\text{Ric}(X,Y) \neq \text{Ric}(Y,X).$$

Notation: Define $[\text{Ric}] \in \Lambda^2 M$ by

$$[\text{Ric}](X,Y) = \text{Ric}(X,Y) - \text{Ric}(Y,X).$$

Bearing in mind that $R(X,Y) \in \mathcal{D}^1_1(M)$, put

$$\text{tr}(R(X,Y)) = C^1_1 R(X,Y) \in C^\infty(M),$$

where

$$C^1_1 : \mathcal{D}^1_1(M) \to C^\infty(M)$$

is the contraction.

<u>LEMMA</u> If ∇ is torsion free, then

$$[\text{Ric}](X,Y) = \text{tr}(R(X,Y)).$$

[In fact,

$$[\text{Ric}]\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^\ell}\right)$$

$$= R_{j\ell} - R_{\ell j}$$

$$= R^i{}_{ji\ell} - R^i{}_{\ell ij}.$$

On the other hand,

$$R^i_{\ kj\ell} = dx^i(R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^\ell}) \frac{\partial}{\partial x^k})$$

$\Rightarrow$

$$\mathrm{tr}(R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^\ell})) = R^i_{\ ij\ell}$$

$$= - R^i_{\ j\ell i} - R^i_{\ \ell ij}$$

$$= R^i_{\ ji\ell} - R^i_{\ \ell ij}.]$$

Observation: We have

$$R^i_{\ ij\ell}$$

$$= \Gamma^i_{\ \ell i,j} - \Gamma^i_{\ ji,\ell} + \Gamma^a_{\ \ell i}\Gamma^i_{\ ja} - \Gamma^a_{\ ji}\Gamma^i_{\ \ell a}$$

$$= \Gamma^i_{\ \ell i,j} - \Gamma^i_{\ ji,\ell} + \Gamma^a_{\ \ell i}\Gamma^i_{\ ja} - \Gamma^i_{\ ja}\Gamma^a_{\ \ell i}$$

$$= \Gamma^i_{\ \ell i,j} - \Gamma^i_{\ ji,\ell}$$

$$= \frac{\partial\Gamma^i_{\ \ell i}}{\partial x^j} - \frac{\partial\Gamma^i_{\ ji}}{\partial x^\ell}.$$

So, if $\exists$ a C^∞ function f of the coordinates such that

$$\frac{\partial f}{\partial x^k} = \Gamma^i_{\ ki},$$

then

$$\left[\begin{array}{l} \dfrac{\partial^2 f}{\partial x^\ell \partial x^j} = \dfrac{\partial \Gamma^i{}_{\ell i}}{\partial x^j} \\[2em] \dfrac{\partial^2 f}{\partial x^j \partial x^\ell} = \dfrac{\partial \Gamma^i{}_{ji}}{\partial x^\ell} \end{array}\right.$$

$\Rightarrow$

$$R^i{}_{ij\ell} = 0.$$

Thus, on this chart, Ric is symmetric.

Maintaining the assumption that ∇ is torsion free, let us globalize these considerations.

<u>LEMMA</u> Suppose that ϕ is a strictly positive density of weight 1 such that $\nabla\phi = 0$ -- then Ric is symmetric.

[In fact,

$$0 = \nabla_a \phi = \phi_{,a} - \Gamma^b{}_{ab}\phi$$

$\Rightarrow$

$$\Gamma^b{}_{ab} = \frac{\partial}{\partial x^a} \log \phi.]$$

[Note: This can also be read the other way in that the relation

$$\Gamma^b{}_{ab} = \frac{\partial}{\partial x^a} \log \phi$$

obviously implies that $\nabla\phi = 0$.]

By way of notation, put

$$\Gamma_a = \Gamma^b{}_{ab}.$$

Then

$$\operatorname{tr}(R(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})) = \Gamma_{b,a} - \Gamma_{a,b}.$$

If now ϕ is a density of weight r, then

$$\nabla_a \phi = \partial_a \phi - r\Gamma_a \phi$$

$\Rightarrow$

$$\nabla_b \nabla_a \phi = \partial_b \nabla_a \phi - r\Gamma_b \nabla_a \phi$$

$$= \partial_b \partial_a \phi - r\partial_b (\Gamma_a \phi) - r\Gamma_b (\partial_a \phi - r\Gamma_a \phi)$$

$$= \partial_b \partial_a \phi - r\phi \Gamma_{a,b} - r\Gamma_a \partial_b \phi - r\Gamma_b \partial_a \phi + r^2 \Gamma_b \Gamma_a \phi.$$

Therefore

$$\nabla_b \nabla_a \phi - \nabla_a \nabla_b \phi$$

$$= r(\Gamma_{b,a} - \Gamma_{a,b})\phi$$

$$= r(\operatorname{tr}(R(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})))\phi.$$

Section 10: Semiriemannian Manifolds Let M be a connected C^∞ manifold of dimension n.

Definition: A semiriemannian structure on M is a symmetric tensor $g \in \mathcal{D}_2^0(M)$ such that $\forall$ x,

$$g_x : T_xM \times T_xM \to \underline{R}$$

is a scalar product.

[Note: A riemannian structure on M is a positive definite semiriemannian structure.]

Notation: $\underline{M}$ is the set of semiriemannian structures on M, thus

$$\underline{M} = \coprod_{0 \leq k \leq n} \underline{M}_{k,n-k},$$

where $\underline{M}_{k,n-k}$ is the set of semiriemannian structures on M of signature (k,n-k) (so $\underline{M}_{0,n}$ is the set of riemannian structures on M).

Let $g \in \underline{M}$ -- then one may attach to g its orthonormal frame bundle

$$\begin{array}{ccc} \underline{O}(k,n-k) & \to & LM(g) \\ & & \downarrow \pi \\ & & M \end{array}.$$

[Note: Therefore LM(g) is a reduction of LM and the set of reductions of LM per the inclusion $\underline{O}(k,n-k) \to \underline{GL}(n,\underline{R})$ is in a one-to-one correspondence with $\underline{M}_{k,n-k}$.]

Rappel: LM is either connected or has two components.

- M is nonorientable if LM is connected.
- M is orientable if LM has two components.

[Note: If M is orientable, then the components of LM are called orientations and to orient M is to make a choice of one of them, in which case M is said to be oriented. Agreeing to write

$$LM = LM^+ \coprod LM^-,$$

it follows that there are reductions

$$\begin{array}{ccc} \underline{GL}_0(n,\underline{R}) & \rightarrow & LM^{\pm} \\ & & \downarrow \pi \\ & & M.] \end{array}$$

Remark: Let $g \in \underline{M}_{k,n-k}$.

• If $k = 0$ or $k = n$, then $LM(g)$ has at most two components. In the presence of an orientation μ, $LM(g)$ admits a reduction

$$\begin{array}{ccc} \underline{SO}(n) & \rightarrow & \mu LM(g) \\ & & \downarrow \pi \\ & & M \end{array}$$

to the oriented orthonormal frame bundle.

• If $0 < k < n$, then $LM(g)$ has at most four components. In the presence of an orientation μ, $LM(g)$ admits a reduction

$$\begin{array}{ccc} \underline{SO}(k,n-k) & \rightarrow & \mu LM(g) \\ & & \downarrow \pi \\ & & M \end{array}$$

to the oriented, orthonormal frame bundle and in the presence of an orientation μ plus a time orientation T, $LM(g)$ admits a reduction

$$\begin{array}{ccc} \underline{SO}_0(k,n-k) & \rightarrow & \mu_T LM(g) \\ & & \downarrow \pi \\ & & M \end{array}$$

to the oriented, time oriented, orthonormal frame bundle.

Given $g \in \underline{M}$, a connection ∇ on TM is said to be a <u>g-connection</u> if $\nabla g = 0$, i.e., if $\forall\ X,Y,Z \in \mathcal{D}^1(M)$,

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z).$$

Among all g-connections, there is exactly one with zero torsion, the <u>metric connection</u>, its defining property being the relation

$$g(\nabla_X Y,Z)$$

$$= \frac{1}{2}\,[Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)$$

$$+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X).]$$

<u>FACT</u> Every connection on LM(g) extends uniquely to a connection on LM, these extensions being precisely the g-connections.

Let con_gTM stand for the set of g-connections on TM.

Denote by $\mathcal{D}_2^1(M)_g$ the subspace of $\mathcal{D}_2^1(M)$ consisting of those Ψ such that $\forall\ X,Y,Z \in \mathcal{D}^1(M)$,

$$g(\Psi(X,Y),Z) + g(Y,\Psi(X,Z)) = 0.$$

• Let $\nabla',\nabla'' \in \text{con}_g\text{TM}$ -- then the assignment

$$\begin{cases} \mathcal{D}_1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to C^\infty(M) \\ (\Lambda,X,Y) \to \Lambda(\nabla'_X Y - \nabla''_X Y) \end{cases}$$

defines an element of $\mathcal{D}_2^1(M)_g$.

[In fact,

$$g(\nabla'_X Y - \nabla''_X Y,Z) + g(Y,\nabla'_X Z - \nabla''_X Z)$$

$$= g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z)$$

$$- g(\nabla''_X Y, Z) - g(Y, \nabla''_X Z)$$

$$= Xg(Y,Z) - Xg(Y,Z)$$

$$= 0.]$$

•Let $\nabla \in \text{con}_g TM$ — then $\forall\ \Psi \in \mathcal{D}^1_2(M)_g$, the assignment

$$\mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \nabla_X Y + \Psi(X,Y)$$

is a g-connection.

[In fact,

$$g(\nabla_X Y + \Psi(X,Y), Z) + g(Y, \nabla_X Z + \Psi(X,Z))$$

$$= g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$+ g(\Psi(X,Y), Z) + g(Y, \Psi(X,Z))$$

$$= Xg(Y,Z).]$$

Scholium: $\text{con}_g TM$ is an affine space with translation group $\mathcal{D}^1_2(M)_g$.

[The action $\nabla \cdot \Psi = \nabla + \Psi$ is free and transitive.]

Notation: $g^\flat : \mathcal{D}^1(M) \to \mathcal{D}_1(M)$ is the arrow defined by the rule

$$g^\flat X(Y) = g(X,Y).$$

It is an isomorphism and one writes $g^\#$ in place of $(g^\flat)^{-1}$.

Example: The <u>gradient</u> grad f of a function $f \in C^\infty(M)$ is $g^\#(df)$. So, $\forall\ X \in \mathcal{D}^1(M)$,

$$g(\text{grad } f, X) = g(g^\#(df), X)$$

$$= g^\flat(g^\#(df))(X)$$

$$= df(X)$$

$$= Xf.$$

Example: Let ∇ be the metric connection -- then $\forall\ \omega\in\mathcal{D}_1(M)$,

$$\nabla\omega = \frac{1}{2}(L_{g^\#\omega}\, g - d\omega).$$

[Write

$$\nabla\omega(X,Y) = \frac{1}{2}(\nabla\omega(X,Y) + \nabla\omega(Y,X)) + \frac{1}{2}(\nabla\omega(X,Y) - \nabla\omega(Y,X)).$$

Then

$$\begin{cases} \nabla\omega(X,Y) = Y\omega(X) - \omega(\nabla_Y X) \\ \nabla\omega(Y,X) = X\omega(Y) - \omega(\nabla_X Y). \end{cases}$$

Therefore

$$\nabla\omega(X,Y) - \nabla\omega(Y,X) = - d\omega(X,Y).$$

To discuss the sum, let $K = g^\#\omega$ -- then $\forall\ Z\in\mathcal{D}^1(M)$,

$$\omega(Z) = g^\flat g^\#\omega(Z)$$

$$= g^\flat K(Z) = g(K,Z).$$

Therefore

$$\nabla\omega(X,Y) + \nabla\omega(Y,X)$$

$$= X\omega(Y) + Y\omega(X) - \omega(\nabla_X Y + \nabla_Y X)$$

$$= Xg(K,Y) + Yg(K,X) - g(K,\nabla_X Y + \nabla_Y X).$$

But

$$(L_K g)(X,Y) = Kg(X,Y) - g([K,X],Y) - g(X,[K,Y])$$

$$= Kg(X,Y) - g(\nabla_K X,Y) - g(X,\nabla_K Y)$$

$$+ g(\nabla_X K,Y) + g(X,\nabla_Y K)$$

$$= g(\nabla_X K,Y) + g(X,\nabla_Y K)$$

$$= Xg(K,Y) - g(K,\nabla_X Y) + Yg(X,K) - g(\nabla_Y X,K)$$

$$= Xg(K,Y) + Yg(K,X) - g(K,\nabla_X Y + \nabla_Y X).]$$

<u>FACT</u> Fix $\varphi \in C^\infty(M) : \varphi > 0$ and put $\tilde{g} = \varphi g$. Let

$$\begin{bmatrix} \nabla \\ \tilde{\nabla} \end{bmatrix} \text{ be the metric connection associated with } \begin{bmatrix} g \\ \tilde{g} \end{bmatrix}.$$

Then

$$\tilde{\nabla}_X Y = \nabla_X Y$$

$$+ \frac{1}{2} [X(\log \varphi)Y + Y(\log \varphi)X$$

$$- g(X,Y)\operatorname{grad}(\log \varphi).]$$

<u>LEMMA</u> Let ∇ be a g-connection -- then $\forall\ X \in \mathcal{D}^1(M)$, the diagram

$$\begin{array}{ccc} \mathcal{D}^1(M) & \xrightarrow{g^\flat} & \mathcal{D}_1(M) \\ \nabla_X \downarrow & & \downarrow \nabla_X \\ \mathcal{D}^1(M) & \xrightarrow[g^\flat]{} & \mathcal{D}_1(M) \end{array}$$

commutes.

[In fact,

$$g^\flat(\nabla_X Y)(Z)$$

$$= g(\nabla_X Y, Z)$$

$$= Xg(Y,Z) - g(Y,\nabla_X Z)$$

$$= X(g^\flat Y(Z)) - g^\flat Y(\nabla_X Z)$$

$$= (\nabla_X g^\flat Y)(Z).]$$

Notation: $g^{-1} \in \mathcal{D}_0^2(M)$ is characterized by the condition

$$g^{-1}(g^\flat X, g^\flat Y) = g(X,Y).$$

Therefore $g^{-1} \otimes g \in \mathcal{D}_2^2(M)$ and the contraction $C_1^2(g^{-1} \otimes g) \in \mathcal{D}_1^1(M)$ is the Kronecker tensor K.

Observation: Let ∇ be a g-connection -- then $\nabla g^{-1} = 0$.

[We have

$$(\nabla_X g^{-1})(g^\flat Y, g^\flat Z)$$

$$= Xg^{-1}(g^\flat Y, g^\flat Z)$$

$$- g^{-1}(\nabla_X g^\flat Y, g^\flat Z) - g^{-1}(g^\flat Y, \nabla_X g^\flat Z)$$

$$= Xg^{-1}(g^\flat Y, g^\flat Z)$$

$$- g^{-1}(g^\flat \nabla_X Y, g^\flat Z) - g^{-1}(g^\flat Y, g^\flat \nabla_X Z)$$

$$= Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

$$= (\nabla_X g)(Y,Z)$$

$$= 0.]$$

Locally,

$$\begin{cases} g = g_{ij}dx^i \otimes dx^j \\ g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \end{cases}$$

where $[g^{ij}]$ is the matrix inverse to $[g_{ij}]$.

Example: Given $f \in C^\infty(M)$,

$$df = f_{,i}dx^i \Rightarrow \text{grad } f = g^{ij}f_{,i} \frac{\partial}{\partial x^j}.$$

Example: Let ∇ be the metric connection -- then the <u>hessian</u> H_f of a function $f \in C^\infty(M)$ is $\nabla^2 f$, thus $H_f \in \mathcal{D}_2^0(M)$ is symmetric (the metric connection being torsion free). Locally,

$$(H_f)_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k{}_{ij} \frac{\partial f}{\partial x^k}.$$

[Note: Since

$$Xg(\text{grad } f, Y)$$

$$= g(\nabla_X \text{grad } f, Y) + g(\text{grad } f, \nabla_X Y),$$

it follows that

$$g(\nabla_X \text{grad } f, Y)$$

$$= Xg(\text{grad } f, Y) - g(\text{grad } f, \nabla_X Y)$$

$$= XYf - (\nabla_X Y)f$$

$$= H_f(X,Y).]$$

<u>FACT</u> Let ∇ be the metric connection. Fix $x \in M$, $X_x \in T_x M$, and let $t \to \gamma(t)$

be the geodesic such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$ -- then

$$H_f\big|_x(X_x,X_x) = \frac{d^2 f(\gamma(t))}{dt^2}\bigg|_{t=0} .$$

Let $\nabla \in \text{con}_g TM$ -- then ∇ commutes with the operations of lowering or raising indices.

[Note: The point is that

$$\begin{cases} \nabla_a g_{ij} = 0 \\ \nabla_a g^{ij} = 0. \end{cases}$$

So, e.g.,

$$\begin{cases} \nabla_a g_{ik} T^{kj} = g_{ik} \nabla_a T^{kj} \\ \nabla_a g^{ik} T_{kj} = g^{ik} \nabla_a T_{kj} \end{cases} .]$$

<u>LEMMA</u> The connection coefficients of the metric connection are given by

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{k\ell}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}).$$

Put

$$|g| = |\det(g_{ij})|.$$

Then $|g|$ is a density of weight 2, hence $|g|^{1/2}$ is a density of weight 1.

Returning to the lemma, contract over k and i to get

$$\Gamma^i{}_{ij} = \frac{1}{2} g^{i\ell}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}).$$

But

$$g^{i\ell}g_{\ell j,i} = g^{\ell i}g_{ij,\ell}$$

$$= g^{i\ell}g_{ij,\ell}.$$

Therefore

$$\Gamma^i{}_{ij} = \frac{1}{2}\, g^{\ell i}g_{\ell i,j}$$

$$= \frac{1}{2}\,(\det g)^{-1}(\operatorname{cof}\, g_{\ell i})g_{\ell i,j}$$

$$= \frac{1}{2}\,(\det g)^{-1}\,\frac{\partial \det g}{\partial x^j}$$

$$= \frac{1}{2}\,\frac{\partial}{\partial x^j}\,\log|g|$$

$$= \frac{\partial}{\partial x^j}\,\log|g|^{1/2}$$

$$= \frac{1}{|g|^{1/2}}\,\frac{\partial|g|^{1/2}}{\partial x^j}\,.$$

Example: Let ∇ be the metric connection. Suppose that $X\in\mathcal{D}^1(M)$ -- then $\nabla X\in\mathcal{D}^1_1(M)$ and, by definition, the <u>divergence</u> div X of X is

$$\operatorname{div} X = C^1_1\nabla X\ \ (= \operatorname{tr}\,\nabla X).$$

Locally,

$$\operatorname{div} X = X^i{}_{;i} = X^i{}_{,i} + X^j\Gamma^i{}_{ij}$$

or still,

$$\operatorname{div} X = \frac{1}{|g|^{1/2}}\,\partial_i(X^i|g|^{1/2}).$$

[Note: The <u>laplacian</u> Δf of $f\in C^\infty(M)$ is the divergence of its gradient:

$$\Delta f = \text{div}(\text{grad } f).$$

Locally,

$$\Delta f = \frac{1}{|g|^{1/2}} \partial_i (g^{ij} |g|^{1/2} \partial_j f)$$

or still,

$$\Delta f = g^{ij}\{\partial_i \partial_j f - \Gamma^k{}_{ij} \partial_k f\} \quad (\equiv g^{ij} (H_f)_{ij}).]$$

<u>FACT</u> Let $f \in C^\infty(M)$ — then

$$\frac{1}{2} \Delta(g(\text{grad } f, \text{ grad } f))$$

$$= g[{}^0_2](H_f, H_f) + g(\text{grad } f, \text{ grad } \Delta f) + \text{Ric}(\text{grad } f, \text{ grad } f).$$

Let ∇ be a connection on TM — then

$$\nabla_j |g|^{1/2} = |g|^{1/2}_{,j} - \Gamma_j |g|^{1/2}$$

$$= |g|^{1/2}_{,j} - \Gamma^i{}_{ji} |g|^{1/2} .$$

Now take for ∇ the metric connection:

$$\Gamma_j = \Gamma^i{}_{ji}$$

$$= \Gamma^i{}_{ij} = \frac{1}{|g|^{1/2}} \frac{\partial |g|^{1/2}}{\partial x^j}$$

$$= \frac{1}{|g|^{1/2}} |g|^{1/2}_{,j}$$

$\Rightarrow$

$$\nabla_j |g|^{1/2} = 0.$$

[Note: Write

$$1 = |g|^{1/2} |g|^{-1/2}.$$

Then

$$0 = (\nabla_j |g|^{1/2}) |g|^{-1/2} + |g|^{1/2}(\nabla_j |g|^{-1/2})$$

$$= |g|^{1/2}(\nabla_j |g|^{-1/2})$$

$\Rightarrow$

$$\nabla_j |g|^{-1/2} = 0.]$$

Remark: It follows that the Ricci tensor associated with the metric connection is necessarily symmetric (see the discussion at the end of the last section), hence $\forall\ X,Y \in \mathcal{D}^1(M)$,

$$\mathrm{tr}(R(X,Y)) = 0.$$

Put

$$e^{\bullet} = \frac{1}{|g|^{1/2}} \cdot \varepsilon^{\bullet},$$

where $\varepsilon^{\bullet}$ is the upper Levi-Civita symbol. Then $e^{\bullet}$ is a twisted tensor of type $(n,0)$.

[Note: Analogous considerations apply to the lower Levi-Civita symbol $\varepsilon_{\bullet}$: The product

$$e_{\bullet} = |g|^{1/2} \cdot \varepsilon_{\bullet}$$

is a twisted tensor of type $(0,n)$.]

<u>LEMMA</u> Let ∇ be the metric connection -- then we have

$$\begin{cases} \nabla e^{\bullet} = 0 \\ \nabla e_{\bullet} = 0. \end{cases}$$

[To discuss $e^{\bullet}$, simply note that

$$\nabla_j e^\bullet = \nabla_j \left(\frac{1}{|g|^{1/2}} \cdot \varepsilon^\bullet\right)$$

$$= \nabla_j |g|^{-1/2} \cdot e^\bullet + |g|^{-1/2} \cdot \nabla_j \varepsilon^\bullet$$

$$= 0.]$$

Let ∇ be a connection on TM — then the assignment

$$(W,Z,X,Y) \rightarrow g(R(X,Y)Z,W)$$

is a tensor of type (0,4).

[Note: Obviously,

$$g(R(X,Y)Z,W) = g(W,R(X,Y)Z)$$

$$= g^\flat W(R(X,Y)Z).]$$

Locally,

$$R_{ijk\ell} = g_{ia}R^a{}_{jk\ell}$$

$$= g(\partial_i,\partial_a)R^a{}_{jk\ell}$$

$$= g(\partial_i,R^a{}_{jk\ell}\partial_a)$$

$$= g(\partial_i,R(\partial_k,\partial_\ell)\partial_j).$$

Specialize again to the case when ∇ is the metric connection — then

$$g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$$

and

$$g(R(X,Y)Z,W) = - g(R(X,Y)W,Z).$$

<u>Symmetries of Curvature</u> Let ∇ be the metric connection:

$$R_{ijk\ell} = - R_{jik\ell}$$

$$R_{ijk\ell} = - R_{ij\ell k}$$

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0$$

$$R_{ijk\ell} = R_{k\ell ij} .$$

[Note: Recall too that

$$R_{ijk\ell;m} + R_{ij\ell m;k} + R_{ijmk;\ell} = 0.]$$

Example: The <u>Kretschmann curvature invariant</u> k_R is, by definition,

$$R^{ijk\ell}R_{ijk\ell}.$$

<u>THEOREM</u> Let ∇ be the metric connection. Fix a point $x_0 \in M$ and let $x^1,\ldots,x^n$ be normal coordinates at x_0 -- then

$$g_{ij}(x) = g_{ij}(x_0) + \frac{1}{2}\left[- \frac{1}{3}(R_{ikj\ell} + R_{jki\ell})(x_0)\right]x^k x^\ell + \cdots.$$

Let ∇ be a torsion free connection on TM -- then

$$(L_X g)(Y,Z) = (\nabla_X g)(Y,Z)$$
$$+ g(\nabla_Y X,Z) + g(Y,\nabla_Z X).$$

In particular, when ∇ is the metric connection,

$$(L_X g)(Y,Z) = g(\nabla_Y X,Z) + g(Y,\nabla_Z X).$$

Observation: Let ∇ be the metric connection -- then

$$\begin{cases} Zg(X,Y) = g(\nabla_Z X,Y) + g(X,\nabla_Z Y) \\ Yg(X,Z) = g(\nabla_Y X,Z) + g(X,\nabla_Y Z) \end{cases}$$

$\Rightarrow$

$$\begin{aligned}(L_X g)(Y,Z) &= Zg(X,Y) - g(X,\nabla_Z Y) \\ &\quad + Yg(X,Z) - g(X,\nabla_Y Z) \\ &= Zg^{\flat}X(Y) - g^{\flat}X(\nabla_Z Y) \\ &\quad + Yg^{\flat}X(Z) - g^{\flat}X(\nabla_Y Z) \\ &= (\nabla_Z g^{\flat}X)(Y) + (\nabla_Y g^{\flat}X)(Z) \\ &= \nabla g^{\flat}X(Y,Z) + \nabla g^{\flat}X(Z,Y).\end{aligned}$$

[Note: Locally,

$$L_X g_{ij} = X_{i;j} + X_{j;i} = \nabla_j X_i + \nabla_i X_j.]$$

<u>FACT</u> $\forall\ X,Y \in \mathcal{D}^1(M)$,

$$(\nabla_Y X)^{\flat} = \iota_Y(\tfrac{1}{2}\, L_X g + \tfrac{1}{2}\, dX^{\flat}).$$

Let $X \in \mathcal{D}^1(M)$ -- then X is said to be an <u>infinitesimal isometry</u> if $L_X g = 0$.

<u>FACT</u> An infinitesimal isometry is necessarily an infinitesimal affine transformation.

From the definitions,

$$(L_X g)(Y,Z) = Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]),$$

so X is an infinitesimal isometry iff

$$Xg(Y,Z) = g([X,Y],Z) + g(Y,[X,Z])$$

or still, iff

$$\nabla g^\flat X(Y,Z) + \nabla g^\flat X(Z,Y) = 0.$$

Therefore an infinitesimal isometry is divergence free:

$$0 = X_{i;j} + X_{j;i}$$

$$\Rightarrow$$

$$X^i_{;i} = \nabla_i X^i$$

$$= \nabla_i g^{ik} X_k$$

$$= g^{ik} \nabla_i X_k$$

$$= g^{ik} X_{k;i}$$

$$= - g^{ik} X_{i;k}$$

$$= - g^{ki} X_{k;i}$$

$$= - g^{ik} X_{k;i}$$

$$\Rightarrow$$

$$\text{div } X = 0.$$

Example: Let X be an infinitesimal isometry. Put $\omega_X = g^\flat X$ -- then

$$(\omega_X \wedge d\omega_X)(X,Y,Z) = \omega_X(X) d\omega_X(Y,Z)$$

$$+ \omega_X(Y)d\omega_X(Z,X) + \omega_X(Z)d\omega_X(X,Y)$$

$$= g(X,X)[Y\omega_X(Z) - Z\omega_X(Y) - \omega_X([Y,Z])]$$

$$+ g(X,Y)[Z\omega_X(X) - X\omega_X(Z) - \omega_X([Z,X])$$

$$+ g(X,Z)[X\omega_X(Y) - Y\omega_X(X) - \omega_X([X,Y])].$$

But

$$(L_Xg)(X,Y) = Xg(X,Y) - g(X,[X,Y])$$

$$= 0$$

$\Rightarrow$

$$X\omega_X(Y) = \omega_X([X,Y]).$$

Analogously

$$X\omega_X(Z) = \omega_X([X,Z]).$$

Therefore

$$(\omega_X \wedge d\omega_X)(X,Y,Z) = g(X,X)d\omega_X(Y,Z)$$

$$+ g(X,Y)Z\omega_X(X) - g(X,Z)Y\omega_X(X).$$

Assume now that

$$\omega_X \wedge d\omega_X = 0.$$

Let $\phi = g(X,X)$ $(= \omega_X(X))$ -- then

$$\phi d\omega_X(Y,Z) + g(X,Y)d\phi(Z) - g(X,Z)d\phi(Y) = 0.$$

I.e.:

$$\phi d\omega_X + \omega_X \wedge d\phi = 0$$

$\Rightarrow$

$$d(\omega_X/\phi) = 0.$$

It thus follows from the Poincaré lemma that locally,

$$\omega_X = g(X,X)df \quad (\exists\ f).$$

Section 11: The Einstein Equation Let M be a connected C^∞ manifold of dimension n. Fix a semiriemannian structure g on M and let $\nabla \in \operatorname{con}_g TM$ be the metric connection.

Since $|g|^{1/2}$ is a strictly positive density of weight 1 such that $\nabla |g|^{1/2} = 0$, the Ricci tensor Ric is symmetric.

[Note: To check this using indices, write

$$\begin{aligned} R_{j\ell} &= R^i{}_{ji\ell} \\ &= g^{ik} R_{kji\ell} \\ &= g^{ik} R_{i\ell kj} \\ &= g^{ki} R_{i\ell kj} \\ &= R^k{}_{\ell kj} \\ &= R_{\ell j}.] \end{aligned}$$

Notation: Given a symmetric tensor $T \in \mathcal{D}^0_2(M)$, define $\operatorname{tr}(T) \in C^\infty(M)$ by

$$\operatorname{tr}(T) = T^i{}_i = g^{ij} T_{ji}.$$

Example: tr(g) is the C^∞ function on M of constant value n.

[In fact,

$$\operatorname{tr}(g) = g^{ij} g_{ji} = \delta^i{}_i = n.]$$

Definition: The <u>scalar curvature</u> S is tr Ric, thus

$$S = \mathrm{Ric}^i{}_i$$

or still,

$$S = g^{ik}R^{j}{}_{ijk}$$

$$= g^{ki}R^{j}{}_{ijk}$$

$$= R^{jk}{}_{jk}.$$

Notation: Write

$$\nabla^a = g^{ab}\nabla_b.$$

<u>LEMMA</u> (<u>The Fundamental Identity</u>) We have

$$\nabla^i R_{ki} = \frac{1}{2}\nabla_k S.$$

[To begin with

$$0 = R_{ijk\ell;m} + R_{ij\ell m;k} + R_{ijmk;\ell}$$

$$= \nabla_m R_{ijk\ell} + \nabla_k R_{ij\ell m} + \nabla_\ell R_{ijmk}.$$

Therefore

$$0 = g^{j\ell}g^{mi}(\nabla_m R_{ijk\ell} + \nabla_k R_{ij\ell m} + \nabla_\ell R_{ijmk}).$$

Now examine each term in succession.

(1) $g^{j\ell}g^{mi}\nabla_m R_{ijk\ell}$

$$= g^{j\ell}g^{im}\nabla_m R_{ijk\ell}$$

$$= g^{j\ell}\nabla^i R_{ijk\ell}$$

$$= \nabla^i g^{j\ell} R_{ijk\ell}$$

$$= \nabla^i g^{j\ell} R_{k\ell ij}$$

$$= \nabla^i(-g^{j\ell}R_{\ell kij})$$

$$= \nabla^i(-R^j{}_{kij})$$

$$= \nabla^i((-)(-)R^j{}_{kji})$$

$$= \nabla^i R_{ki}.$$

(2) $g^{j\ell}g^{mi}\nabla_k R_{ij\ell m}$

$$= \nabla_k g^{j\ell}g^{mi}R_{ij\ell m}$$

$$= \nabla_k g^{j\ell}R^m{}_{j\ell m}$$

$$= -\nabla_k g^{j\ell}R^m{}_{jm\ell}$$

$$= -\nabla_k g^{j\ell}R_{j\ell}$$

$$= -\nabla_k S.$$

(3) $g^{j\ell}g^{mi}\nabla_\ell R_{ijmk}$

$$= g^{mi}g^{j\ell}\nabla_\ell R_{ijmk}$$

$$= g^{mi}\nabla^j R_{ijmk}$$

$$= \nabla^j g^{mi}R_{ijmk}$$

$$= \nabla^j R^m{}_{jmk}$$

$$= \nabla^j R_{jk}$$

$$= \nabla^i R_{ki}.$$

Combining (1), (2), and (3) then gives

$$\nabla^i R_{ki} = \frac{1}{2}\nabla_k S.]$$

Notation: Given a symmetric tensor $T \in \mathcal{D}_2^0(M)$, define div $T \in \mathcal{D}_1(M)$ by

$$(\text{div } T)_j = g^{k\ell}(\nabla T)_{kj\ell}$$

$$= g^{k\ell}\nabla_\ell T_{kj}$$

$$= g^{k\ell}T_{kj;\ell}.$$

Scholium: We have

$$dS = 2\text{div Ric}.$$

[In fact,

$$dS_k = \partial_k S = \nabla_k S.$$

On the other hand,

$$2(\text{div Ric})_k = 2g^{ij}\text{Ric}_{ik;j}$$

$$= 2g^{ij}R_{ik;j}$$

$$= 2g^{ij}\nabla_j R_{ik}$$

$$= 2\nabla^i R_{ik}$$

$$= 2\nabla^i R_{ki}.]$$

<u>LEMMA</u> Let $f \in C^\infty(M)$ -- then

$$\text{div}(fg) = df.$$

[For

$$\begin{aligned}\mathrm{div}(fg)_j &= g^{k\ell}(fg)_{kj;\ell}\\ &= g^{k\ell}\nabla_\ell(fg)_{kj}\\ &= \nabla_\ell g^{k\ell}(fg)_{kj}\\ &= \nabla_\ell f g^{\ell k}g_{kj}\\ &= \nabla_\ell(f\delta^\ell{}_j)\\ &= (\nabla_\ell f)\delta^\ell{}_j + f(\nabla_\ell\delta^\ell{}_j)\\ &= \nabla_j f\\ &= \partial_j f.]\end{aligned}$$

Application: Suppose that $\mathrm{Ric} = \phi g$ $(\phi \in C^\infty(M))$ -- then ϕ is a constant if $n > 2$.

[To see this, note first that

$$\mathrm{div}\,\mathrm{Ric} = \mathrm{div}(\phi g)$$

$$\Rightarrow$$

$$\frac{dS}{2} = d\phi$$

$$\Rightarrow$$

$$\phi = \frac{S}{2} + C.$$

On the other hand,

$$\mathrm{tr}\,\mathrm{Ric} = \mathrm{tr}(\phi g)$$

$$\Rightarrow$$

$$S = \phi n.$$

Therefore

$$(2-n)\phi = 2C.]$$

Definition: The <u>Einstein tensor</u> Ein is the combination

$$\mathrm{Ein} = \mathrm{Ric} - \frac{1}{2} Sg.$$

So, $\mathrm{Ein} \in \mathcal{D}_2^0(M)$ is symmetric and one has

$$\begin{aligned} \operatorname{div} \mathrm{Ein} &= \operatorname{div} \mathrm{Ric} - \frac{1}{2} \operatorname{div}(Sg) \\ &= \operatorname{div} \mathrm{Ric} - \frac{1}{2} dS \\ &= 0. \end{aligned}$$

In addition,

$$\begin{aligned} \operatorname{tr} \mathrm{Ein} &= \operatorname{tr} \mathrm{Ric} - \frac{1}{2} \operatorname{tr}(Sg) \\ &= S - \frac{n}{2} S \end{aligned}$$

$$\Rightarrow$$

$$\operatorname{tr} \mathrm{Ein} = \begin{cases} (1 - \frac{n}{2}) S & (n \neq 2) \\ 0 & (n = 2). \end{cases}$$

Therefore

$$\begin{aligned} \mathrm{Ric} &= \mathrm{Ein} + \frac{1}{2} Sg \\ &= \mathrm{Ein} + \frac{1}{2-n} (\operatorname{tr} \mathrm{Ein}) g \quad (n \neq 2). \end{aligned}$$

[Note: When n = 4,

$$\begin{cases} \mathrm{Ric} = \mathrm{Ein} - \frac{1}{2} (\operatorname{tr} \mathrm{Ein}) g \\ \mathrm{Ein} = \mathrm{Ric} - \frac{1}{2} (\operatorname{tr} \mathrm{Ric}) g. \end{cases}$$

Thus in this case, the Einstein tensor and the Ricci tensor each has the same formal expression in terms of the other.]

Remark: Using the symmetries of R, it is easy to show that Ein automatically vanishes if dim M = 2.

Assume that dim M > 2 — then M is said to be a vacuum if Ein = 0, the equation

$$\text{Ein} = 0$$

being the vacuum field equation of general relativity.

[Note: By the above, M is a vacuum iff M is Ricci flat, i.e., iff Ric = 0. If dim M = 3, then Ric = 0 ⇒ R = 0.]

Notation: In computations, the Einstein tensor is often denoted by G.

Definition: Suppose that n > 1 — then M is said to be an Einstein manifold if ∃ a constant λ such that Ric = λg.

[Note: Matters are trivial when n = 1: In this situation, all M are necessarily Einstein.]

If Ric = λg, then

$$\text{tr Ric} = S \Rightarrow S = \lambda n.$$

Therefore

$$\begin{aligned} \text{Ein} &= \text{Ric} - \frac{1}{2} Sg \\ &= \frac{1}{n} Sg - \frac{1}{2} Sg \\ &= (\frac{1}{n} - \frac{1}{2}) Sg. \end{aligned}$$

<u>Section 12</u>: <u>Decomposition Theory</u> Let V be an n-dimensional real vector space. Suppose that $A:V \to V$ is a linear transformation — then

$$A = S + T,$$

where

$$\begin{cases} S = A - \frac{\mathrm{tr}(A)}{n} I \\ T = \frac{\mathrm{tr}(A)}{n} I \end{cases}$$

and

$$\mathrm{tr}(S) = 0, \ \mathrm{tr}(T) = \mathrm{tr}(A).$$

Therefore

$$\mathrm{Hom}(V,V) = \mathrm{Ker}(\mathrm{tr}) \oplus \underline{R}I.$$

Notation: $\mathcal{R}$ is the set of multilinear maps

$$R:V \times V \times V \times V \to \underline{R}$$

such that

(a) $R(X_1,X_2,X_3,X_4) = - R(X_2,X_1,X_3,X_4)$;

(b) $R(X_1,X_2,X_3,X_4) = - R(X_1,X_2,X_4,X_3)$;

(c) $R(X_1,X_2,X_3,X_4) + R(X_1,X_3,X_4,X_2) + R(X_1,X_4,X_2,X_3) = 0$;

(d) $R(X_1,X_2,X_3,X_4) = R(X_3,X_4,X_1,X_2)$.

Example: Let M be a connected C^∞ manifold of dimension n. Fix $g \in \underline{M}$ and let ∇ be the metric connection — then at each $x \in M$, the tensor

$$(W,Z,X,Y) \to g(R(X,Y)Z,W)$$

induces a multilinear map

$$R_x:T_xM \times T_xM \times T_xM \times T_xM \to \underline{R}$$

satisfying (a) - (d).

LEMMA R is a real vector space of dimension $\frac{1}{12}n^2(n^2-1)$.

[Note: Therefore

$$n = 1 \Rightarrow \dim R = 0;$$

$$n = 2 \Rightarrow \dim R = 1;$$

$$n = 3 \Rightarrow \dim R = 6;$$

$$n = 4 \Rightarrow \dim R = 20.]$$

Definition: Let $P,Q:V \times V \rightarrow \underline{R}$ be symmetric bilinear forms -- then the curvature product of P,Q is the tensor $P \times_c Q$ of type $(0,4)$ defined by

$$P \times_c Q(X_1,X_2,X_3,X_4)$$

$$= P(X_1,X_3)Q(X_2,X_4) + P(X_2,X_4)Q(X_1,X_3)$$

$$- P(X_1,X_4)Q(X_2,X_3) - P(X_2,X_3)Q(X_1,X_4).$$

Obviously,

$$P \times_c Q = Q \times_c P$$

and it is not difficult to check that

$$P \times_c Q \in R.$$

Now fix $g \in \underline{M}$ -- then the prescription

$$G(X_1,X_2,X_3,X_4) = g(X_1,X_3)g(X_2,X_4) - g(X_1,X_4)g(X_2,X_3)$$

defines an element of R and

$$g \times_c g = 2G.$$

$\underline{r}$: This is the map

$$r:R \rightarrow \text{Sym } V_2^0$$

defined by

$$r_R(X,Y) = \varepsilon_1 R(E_1,X,E_1,Y) + \cdots + \varepsilon_n R(E_n,X,E_n,Y),$$

where $E \in B(V)$ is orthonormal.

[Note: r_R is independent of the choice of E.]

Notation: Given $T \in \text{Sym } V_2^0$, put

$$\text{tr}(T) = g[{}^0_2](g,T).$$

We shall then agree to write s_R in place of $\text{tr}(r_R)$, thus

$$s_R = \varepsilon_1 r_R(E_1,E_1) + \cdots + \varepsilon_n r_R(E_n,E_n).$$

Remark: Let M be a connected C^∞ manifold of dimension n. Fix $g \in \underline{M}$ and let ∇ be the metric connection — then at each $x \in M$,

$$\begin{cases} r_{R_x} = \text{Ric}_x \\ \\ s_{R_x} = S(x). \end{cases}$$

[By definition,

$$\text{Ric}_x : T_xM \times T_xM \to \underline{R},$$

where

$$\text{Ric}_x(X,Y) = \text{tr}(Z \to R(Z,X)Y).$$

So, if $\{E_1,\ldots,E_n\}$ is an orthonormal basis for T_xM per g_x, then

$$\begin{aligned} \text{Ric}_x(X,Y) &= \varepsilon_1 g_x(R(E_1,X)Y,E_1) + \cdots + \varepsilon_n g_x(R(E_n,X)Y,E_n) \\ &= \varepsilon_1 R_x(E_1,Y,E_1,X) + \cdots + \varepsilon_n R_x(E_n,Y,E_n,X) \\ &= \varepsilon_1 R_x(E_1,X,E_1,Y) + \cdots + \varepsilon_n R_x(E_n,X,E_n,Y) \end{aligned}$$

$$= r_{R_x}(X,Y).$$

And

$$S(x) = \text{tr Ric}_x = g_x[{}^0_2](g_x, r_{R_x}).]$$

<u>LEMMA</u> Let $T \in \text{Sym } V_2^0$ — then

$$r_{g \times_c T}(X,Y) = (n-2)T + \text{tr}(T)g.$$

[We have

$$r_{g \times_c T}(X,Y) = \sum_{k=1}^{n} \varepsilon_k (g \times_c T)(E_k, X, E_k, Y)$$

$$= \sum_{k=1}^{n} [\varepsilon_k g(E_k,E_k)T(X,Y) + \varepsilon_k g(X,Y)T(E_k,E_k)]$$

$$- \sum_{k=1}^{n} [\varepsilon_k g(E_k,Y)T(X,E_k) + \varepsilon_k g(X,E_k)T(E_k,Y)]$$

$$= \sum_{k=1}^{n} \varepsilon_k^2 T(X,Y) + g(X,Y) \sum_{k=1}^{n} \varepsilon_k T(E_k,E_k)$$

$$- T(X, \sum_{k=1}^{n} \varepsilon_k g(E_k,Y)E_k) - T(\sum_{k=1}^{n} \varepsilon_k g(X,E_k)E_k, Y)$$

$$= nT(X,Y) + \text{tr}(T)g(X,Y) - T(X,Y) - T(X,Y)$$

$$= (n-2)T(X,Y) + \text{tr}(T)g.]$$

[Note: In particular,

$$r_G = \frac{1}{2} r_{g \times_c g} = \frac{1}{2} [(n-2)g + ng]$$

$$= \frac{1}{2} [(2n-2)g]$$

$$= (n-1)g.]$$

Example: Suppose that $n = 2$ -- then $\dim \mathcal{R} = 1$, hence $\forall\ R \in \mathcal{R}$, $\exists\ C_R \in \underline{R}$:

$$R = C_R G$$

$\Rightarrow$

$$r_R = C_R r_G = C_R g$$

$\Rightarrow$

$$s_R = 2C_R.$$

Therefore

$$R = \frac{s_R}{2} G.$$

Assume that $n > 2$ and let $R \in \mathcal{R}$ -- then

$$R = \frac{s_R}{n(n-1)} G + \frac{1}{n-2} [r_R - \frac{s_R}{n} g] \times_c g + C,$$

where, by definition,

$$C = R - \frac{s_R}{n(n-1)} G - \frac{1}{n-2} [r_R - \frac{s_R}{n} g] \times_c g$$

or still,

$$C = R + \frac{s_R}{(n-1)(n-2)} G - \frac{1}{n-2} r_R \times_c g.$$

Write $\mathrm{Sym}_0\ V_2^0$ for the kernel of

$$\mathrm{tr}:\mathrm{Sym}\ V_2^0 \rightarrow \underline{R}.$$

Example: $\forall\ R \in \mathcal{R}$,

$$\frac{1}{n-2} [r_R - \frac{s_R}{n} g] \in \mathrm{Sym}_0\ V_2^0.$$

Write $\mathcal{C}$ for the kernel of

$$r:\mathcal{R} \to \mathrm{Sym}_0\, V_2^0.$$

Example: $\forall\, R\in\mathcal{R}$, $C\in\mathcal{C}$.

[In fact,

$$r_C = r_R + \frac{s_R}{(n-1)(n-2)}\, r_G - \frac{1}{n-2}\, r_{r_R \times_c g}$$

$$= r_R + \frac{s_R}{(n-1)(n-2)}\,(n-1)g - \frac{1}{n-2}\,((n-2)r_R + s_R g)$$

$$= 0.]$$

<u>LEMMA</u> There is a direct sum decomposition

$$\mathcal{R} = \underline{\mathbf{R}}(g \times_c g) \oplus \mathrm{Sym}_0\, V_2^0 \times_c g \oplus \mathcal{C}.$$

[Note: More is true in that the decomposition is orthogonal (per $g[{}^0_4]$).]

Remark: If $n = 3$, then

$$\dim(\underline{\mathbf{R}}(g \times_c g) \oplus \mathrm{Sym}_0\, V_2^0 \times_c g) = 6.$$

But

$$\dim \mathcal{R} = \frac{1}{12}\, 3^2(3^2-1) = 6.$$

Consequently, $\mathcal{C}$ is trivial, thus in this case

$$R = \frac{s_R}{12}\, g \times_c g + [r_R - \frac{s_R}{3}\, g] \times_c g.$$

Definition: The elements of $\mathcal{C}$ are called the <u>Weyl tensors</u>.

Let M be a connected C^∞ manifold of dimension n. Fix $g\in\underline{M}$ and let ∇ be the metric connection -- then the preceding considerations can be globalized in the obvious way, the key new ingredient being the <u>Weyl tensor</u> $(n > 3)$:

$$C(W,Z,X,Y) = g(R(X,Y)Z,W)$$

$$+ \frac{S}{(n-1)(n-2)} (g(W,X)g(Z,Y) - g(W,Y)g(Z,X))$$

$$- \frac{1}{n-2} (\mathrm{Ric}(W,X)g(Z,Y) + \mathrm{Ric}(Z,Y)g(W,X)$$

$$- \mathrm{Ric}(W,Y)g(Z,X) - \mathrm{Ric}(Z,X)g(W,Y)).$$

Locally,

$$C_{ijk\ell} = R_{ijk\ell} + \frac{S}{(n-1)(n-2)} (g_{ik}g_{j\ell} - g_{i\ell}g_{jk})$$

$$- \frac{1}{n-2} (R_{ik}g_{j\ell} + R_{j\ell}g_{ik} - R_{i\ell}g_{jk} - R_{jk}g_{i\ell}).$$

<u>FACT</u> We have

$$C^i{}_{ji\ell} = 0.$$

<u>LEMMA</u> Fix $\varphi \in C^\infty(M): \varphi > 0$ and put $\tilde{g} = \varphi g$. Let

$$\begin{bmatrix} \nabla \\ \\ \tilde{\nabla} \end{bmatrix} \text{ be the metric connection associated with } \begin{bmatrix} g \\ \\ \tilde{g} \end{bmatrix}.$$

Then

$$\tilde{C} = \varphi C.$$

[Note: Therefore the Weyl tensor, when viewed as an element of $\mathcal{D}^1_3(M)$, is a conformal invariant.]

Section 13: Bundle Valued Forms Let M be a connected C^∞ manifold of dimension n. Suppose that $E \to M$ is a vector bundle -- then the sections of $E \otimes \Lambda^p T^*M$ are the p-forms on M with values in E.

Notation: Put

$$\Lambda^p(M;E) = \sec(E \otimes \Lambda^p T^*M).$$

[Note: When $p = 0$,

$$\Lambda^0(M;E) = \sec(E).]$$

Structurally,

$$\Lambda^p(M;E) = \Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^p M,$$

thus the elements of $\Lambda^p(M;E)$ are the $C^\infty(M)$-multilinear antisymmetric maps

$$\overbrace{\mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M)}^{p} \to \sec(E).$$

Remark: If E is a trivial vector bundle with fiber V, then $\Lambda^p(M;E)$ is the space of p-forms on M with values in V and is denoted by $\Lambda^p(M;V)$.

Example: Let

$$\begin{array}{ccc} G & \to & P \\ & & \downarrow \pi \\ & & M \end{array}$$

be a principal bundle with structure group G (which we shall take to be a Lie group). Let ρ be a representation of G on a real finite dimensional vector space V -- then a p-form

$$\alpha \in \Lambda^p(P;V)$$

is said to be of type ρ if

$$(R_\sigma)^*\alpha = \rho(\sigma^{-1})\alpha \quad \forall\ \sigma \in G.$$

Write

$$\Lambda_\rho^p(P;V)$$

for the space of p-forms on P of type ρ and let E be the vector bundle

$$P \times_G V.$$

Then there is a canonical one-to-one correspondence

$$\Lambda_\rho^p(P;V) \leftrightarrow \Lambda^p(M;E).$$

Suppose that $E \to M$ is a vector bundle. Let ∇ be a connection on E -- then ∇ gives rise to an $\underline{R}$-linear map

$$\nabla:\Lambda^0(M;E) \to \Lambda^1(M;E)$$

such that

$$\nabla(fs) = s \otimes df + f\nabla s \quad (f \in C^\infty(M),\ s \in \Lambda^0(M;E)),$$

viz.

$$\nabla s(X) = \nabla_X s.$$

Conversely, every $\underline{R}$-linear map

$$\nabla:\Lambda^0(M;E) \to \Lambda^1(M;E)$$

such that

$$\nabla(fs) = s \otimes df + f\nabla s \quad (f \in C^\infty(M),\ s \in \sec(E))$$

determines a connection on E. Thus let $X \in \mathcal{D}^1(M)$ -- then X induces a $C^\infty(M)$-linear map $ev_X:\Lambda^1 M \to C^\infty(M)$, hence there is an arrow

$$\Lambda^1(M;E) = \Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^1 M$$

$$\downarrow \text{id} \otimes ev_X$$

$$\Lambda^0(M;E) \otimes_{C^\infty(M)} C^\infty(M) = \Lambda^0(M;E),$$

call it EV_X. This said, the definitions then imply that the composite

$$\begin{array}{ccc} \Lambda^0(M;E) & \xrightarrow{\nabla} & \Lambda^1(M;E) \\ & & \downarrow EV_X \\ & & \Lambda^0(M;E) \end{array}$$

defines an operator

$$\nabla_X : \sec(E) \to \sec(E)$$

with the properties required of a connection.

Let $f:M' \to M$ be a smooth map and suppose that $E \to M$ is a vector bundle -- then there is a pullback square

$$\begin{array}{ccc} E' & \to & E \\ \downarrow & & \downarrow \\ M' & \to & M \end{array} \quad (E' = f^*E)$$

and arrows

$$\begin{cases} \Lambda^0(M;E) \to \Lambda^0(M';E') \\ \\ \Lambda^1 M \to \Lambda^1 M' \end{cases}$$

which can be tensored to give an arrow

$$\Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^1 M \to \Lambda^0(M';E') \otimes_{C^\infty(M')} \Lambda^1 M'$$

or still, an arrow

$$\Lambda^1(M;E) \to \Lambda^1(M';E').$$

<u>LEMMA</u> Let ∇ be a connection on E -- then there exists a unique connection ∇' on E' such that the diagram

$$\begin{array}{ccc} \Lambda^0(M;E) & \xrightarrow{\nabla} & \Lambda^1(M;E) \\ \downarrow & & \downarrow \\ \Lambda^0(M';E') & \xrightarrow[\nabla']{} & \Lambda^1(M';E') \end{array}$$

commutes.

The constructions E^*, $E \otimes F$, and $\text{Hom}(E,F)$ can be extended to constructions on vector bundles equipped with a connection.

$\underline{\nabla^*}$: Let ∇ be a connection on E -- then ∇ induces a connection ∇^* on E^* with the property that $\forall\ s \in \Lambda^0(M;E)$ & $\forall\ s^* \in \Lambda^0(M;E^*)$,

$$d(s,s^*) = (\nabla s,s^*) + (s,\nabla^* s^*),$$

this being an equality of elements of $\Lambda^1 M$.

[Note: Since

$$\Lambda^0(M;E^*) = \text{Hom}_{C^\infty(M)}(\Lambda^0(M;E),\ C^\infty(M)),$$

it follows that there is a nonsingular pairing

$$(\ \ ,\ \ ) : \Lambda^0(M;E) \times \Lambda^0(M;E^*) \to C^\infty(M),$$

viz. evaluation. Analogously, there are nonsingular pairings

$$\begin{cases} \Lambda^1(M;E) \times \Lambda^0(M;E^*) \to \Lambda^1 M \\ \Lambda^0(M;E) \times \Lambda^1(M;E^*) \to \Lambda^1 M.] \end{cases}$$

$\underline{\nabla_E \otimes \nabla_F}$: If ∇_E is a connection on E and ∇_F is a connection on F, then $\nabla_E \otimes \nabla_F$ is the connection on $E \otimes F$ defined by

$$(\nabla_E \otimes \nabla_F)(s \otimes t) = \nabla_E s \otimes t + s \otimes \nabla_F t.$$

[Note: The tensor products on the right are elements of $\Lambda^1(M;E \otimes F)$. For example,

$$\Lambda^1(M;E) \otimes_{C^\infty(M)} \Lambda^0(M;F)$$

$$= \Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^1 M \otimes_{C^\infty(M)} \Lambda^0(M;F)$$

$$= \Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^0(M;F) \otimes_{C^\infty(M)} \Lambda^1 M$$

$$= \Lambda^0(M;E \otimes F) \otimes_{C^\infty(M)} \Lambda^1 M$$

$$= \Lambda^1(M;E \otimes F).]$$

<u>$\nabla_{\text{Hom}(E,F)}$</u>: Let ∇_E be a connection on E and let ∇_F be a connection on F -- then the pair (∇_E,∇_F) induces a connection $\nabla_{\text{Hom}(E,F)}$ on Hom(E,F) with the property that $\forall\ \phi\in\Lambda^0(M;\text{Hom}(E,F))$ & $\forall\ s\in\Lambda^0(M;E)$,

$$\nabla_F(\phi,s) = (\phi,\nabla_E s) + (\nabla_{\text{Hom}(E,F)}\phi,s),$$

this being an equality of elements of $\Lambda^1(M;F)$.

[Note: First, there is a nonsingular pairing

$$(\ ,\ ) : \Lambda^0(M;\text{Hom}(E,F)) \times \Lambda^0(M;E) \to \Lambda^0(M;F).$$

Second, there is a nonsingular pairing

$$(\ ,\ ) : \Lambda^0(M;\text{Hom}(E,F)) \times \Lambda^1(M;E) \to \Lambda^1(M;F).$$

Third, there is a nonsingular pairing

$$(\ ,\ ) : \Lambda^1(M;\text{Hom}(E,F)) \times \Lambda^0(M;E) \to \Lambda^1(M;F).]$$

Remark: Under the identification $E \leftrightarrow E^{**}$, we have $\nabla \leftrightarrow \nabla^{**}$, and under the identification $E^* \otimes F \leftrightarrow \text{Hom}(E,F)$, we have $\nabla_{E^* \otimes F} \leftrightarrow \nabla_{\text{Hom}(E,F)}$.

<u>FACT</u> A connection ∇ on E induces a connection $\nabla_{\Lambda^k E}$ on $\Lambda^k E$ such that

$\nabla_{\Lambda^1 E} = \nabla$ and

$$\nabla_X(s\wedge t) = \nabla_X s\wedge t + (-1)^k s\wedge\nabla_X t,$$

where $s\in\sec(\Lambda^k E)$, $t\in\sec(\Lambda^\ell E)$.

[Note: We have

$$\sec(\Lambda^k E) = \Lambda^k \sec(E).]$$

Let ∇_1,∇_2 be connections on E -- then $\forall\ f\in C^\infty(M)$ & $\forall\ s\in\Lambda^0(M;E)$,

$$(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 - \nabla_2)s.$$

Therefore

$$\nabla_1 - \nabla_2\in\mathrm{Hom}_{C^\infty(M)}(\Lambda^0(M;E), \Lambda^1(M;E)).$$

On the other hand,

$$\begin{aligned}
&\mathrm{Hom}_{C^\infty(M)}(\Lambda^0(M;E), \Lambda^1(M;E)) \\
&= \mathrm{Hom}_{C^\infty(M)}(\Lambda^0(M;E), \Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^1 M) \\
&= \mathrm{Hom}_{C^\infty(M)}(\Lambda^0(M;E), \Lambda^0(M;E)) \otimes_{C^\infty(M)} \Lambda^1 M \\
&= \Lambda^0(M;\mathrm{Hom}(E,E)) \otimes_{C^\infty(M)} \Lambda^1 M \\
&= \Lambda^1(M;\mathrm{Hom}(E,E)).
\end{aligned}$$

So, under this identification,

$$\nabla_1 - \nabla_2\in\Lambda^1(M;\mathrm{Hom}(E,E)).$$

Conversely, if $\Gamma\in\Lambda^1(M;\mathrm{Hom}(E,E))$, then for any connection ∇, $\nabla + \Gamma$ is again a connection.

Let con E stand for the set of connections on E.

Scholium: con E is an affine space with translation group $\Lambda^1(M;\mathrm{Hom}(E,E))$.

[The action $\nabla\cdot\Gamma = \nabla + \Gamma$ is free and transitive.]

<u>Reality Check</u> Take E = TM -- then

$$\begin{aligned}
&\Lambda^1(M;\mathrm{Hom}(TM,TM)) \\
&= \Lambda^0(M;\mathrm{Hom}(TM,TM)) \underset{C^\infty(M)}{\otimes} \Lambda^1 M \\
&= \mathrm{Hom}_{C^\infty(M)}(\mathcal{D}^1(M),\mathcal{D}^1(M)) \underset{C^\infty(M)}{\otimes} \mathcal{D}_1(M) \\
&= \mathcal{D}_1^1(M) \underset{C^\infty(M)}{\otimes} \mathcal{D}_1(M) \\
&= \Lambda^0(M;T_1^1(M)) \underset{C^\infty(M)}{\otimes} \Lambda^0(M;T_1(M)) \\
&= \Lambda^0(M;T_1^1(M) \otimes T_1(M)) \\
&= \Lambda^0(M;T_2^1(M)) \\
&= \mathcal{D}_2^1(M).
\end{aligned}$$

<u>Projection Principle</u> Suppose that $E = E_1 \oplus E_2$ -- then there are canonical arrows

$$\begin{cases}
\mathrm{con}\ E \to \mathrm{con}\ E_1 & (\nabla \to \nabla^1) \\
\mathrm{con}\ E \to \mathrm{con}\ E_2 & (\nabla \to \nabla^2),
\end{cases}$$

viz.

$$\begin{cases}
\nabla_X^1 s_1 = \mathrm{pr}_1(\nabla_X s_1) & (s_1 \in \sec(E_1)) \\
\nabla_X^2 s_2 = \mathrm{pr}_2(\nabla_X s_2) & (s_2 \in \sec(E_2)).
\end{cases}$$

Let $E \to M$, $F \to M$ be vector bundles -- then there is a $C^\infty(M)$-bilinear product

$$\wedge : \Lambda^p(M;E) \underset{C^\infty(M)}{\otimes} \Lambda^q(M;F) \to \Lambda^{p+q}(M;E \otimes F)$$

which is characterized by the condition

$$(s \otimes \alpha)\wedge(t \otimes \beta) = (s \otimes t) \otimes (\alpha\wedge\beta).$$

[Note: We have

$$\Lambda^{p+q}(M;E \otimes F) = \Lambda^0(M;E \otimes F) \underset{C^\infty(M)}{\otimes} \Lambda^{p+q}M$$

and

$$\Lambda^0(M;E \otimes F) = \Lambda^0(M;E) \underset{C^\infty(M)}{\otimes} \Lambda^0(M;F).$$

Therefore $s \otimes t$ is an element of $\Lambda^0(M;E \otimes F)$.]

Example: Take $F = \varepsilon = M \times \underline{R}$, the trivial line bundle -- then

$$\Lambda^p(M;\varepsilon) = \Lambda^p M.$$

Since

$$\wedge : \Lambda^0(M;E) \underset{C^\infty(M)}{\otimes} \Lambda^p(M;\varepsilon) \to \Lambda^p(M;E \otimes \varepsilon)$$

and $E \otimes \varepsilon = E$, it follows that

$$s\wedge\alpha = s \otimes \alpha$$

in $\Lambda^p(M;E)$.

Suppose that $E \to M$ is a vector bundle. Given $\nabla \in \text{con } E$, let

$$d^\nabla : \Lambda^p(M;E) \to \Lambda^{p+1}(M;E)$$

be the $\underline{R}$-linear operator defined by the rule

$$d^\nabla(s \otimes \alpha) = s \otimes d\alpha + \nabla s\wedge\alpha.$$

[Note: Recall that $\nabla s \in \Lambda^1(M;E)$. Now view $\alpha \in \Lambda^p M$ as an element of $\Lambda^p(M;\varepsilon)$ -- then

$$\nabla s \wedge \alpha \in \Lambda^{p+1}(M;E \otimes \varepsilon) = \Lambda^{p+1}(M;E).]$$

It is easy to check that $d^\nabla = \nabla$ when $p = 0$.

LEMMA Let $\alpha \in \Lambda^p M$, $\beta \in \Lambda^q M$ -- then

$$d^\nabla((s \otimes \alpha)\wedge\beta) = d^\nabla(s \otimes \alpha)\wedge\beta + (-1)^p (s \otimes \alpha)\wedge d\beta.$$

[We have

$$d^\nabla((s \otimes \alpha)\wedge\beta) = d^\nabla(s \otimes (\alpha\wedge\beta))$$

$$= s \otimes d(\alpha\wedge\beta) + \nabla s\wedge(\alpha\wedge\beta)$$

$$= s \otimes (d\alpha\wedge\beta + (-1)^p \alpha\wedge d\beta) + (\nabla s\wedge\alpha)\wedge\beta$$

$$= (s \otimes d\alpha + \nabla s\wedge\alpha)\wedge\beta + (-1)^p (s \otimes \alpha)\wedge d\beta$$

$$= d^\nabla(s \otimes \alpha)\wedge\beta + (-1)^p (s \otimes \alpha)\wedge d\beta.]$$

[Note: This, of course, is an equality of elements in $\Lambda^{p+q+1}(M;E)$.]

Example: Take $E = \varepsilon$, so $\forall$ p, $\Lambda^p(M;\varepsilon) = \Lambda^p M$. Consider the map

$$\begin{cases} C^\infty(M) \rightarrow \Lambda^1 M \\ f \longrightarrow df. \end{cases}$$

Then d is a connection ∇ and d^∇ is the usual exterior differentiation.

FACT Let $E \rightarrow M$, $F \rightarrow M$ be vector bundles -- then there is an $\underline{R}$-linear map

$$d^{\nabla_E \otimes \nabla_F}:\Lambda^p(M;E \otimes F) \rightarrow \Lambda^{p+1}(M;E \otimes F)$$

and

$$\forall \begin{cases} s \in \Lambda^p(M;E) \\ t \in \Lambda^q(M;F), \end{cases}$$

$$d^{\nabla_E \otimes \nabla_F}(s \wedge t) = d^{\nabla_E} s \wedge t + (-1)^p s \wedge d^{\nabla_F} t.$$

Suppose that E is a vector bundle and let ∇ be a connection on E -- then there is a sequence

$$0 \to \Lambda^0(M;E) \overset{\nabla}{\to} \Lambda^1(M;E) \overset{d^\nabla}{\to} \Lambda^2(M;E) \overset{d^\nabla}{\to} \cdots,$$

which, in general, is not a complex since it need not be true that $d^\nabla \circ \nabla = 0$ (likewise for $d^\nabla \circ d^\nabla$).

Put $F^\nabla = d^\nabla \circ \nabla$ -- then F^∇ is a map from $\Lambda^0(M;E)$ to $\Lambda^2(M;E)$ and is $C^\infty(M)$-linear. Indeed,

$$\begin{aligned} d^\nabla \circ \nabla(fs) &= d^\nabla(s \otimes df + f\nabla s) \\ &= d^\nabla(s \otimes df) + d^\nabla(f\nabla s) \\ &= s \otimes d^2 f + \nabla s \wedge df + df \wedge \nabla s + f \wedge d^\nabla(\nabla s) \\ &= f \wedge d^\nabla(\nabla s) \\ &= f(d^\nabla \circ \nabla(s)). \end{aligned}$$

On the other hand,

$$\text{Hom}_{C^\infty(M)}(\Lambda^0(M;E), \Lambda^2(M;E))$$

$$= \text{Hom}_{C^\infty(M)}(\Lambda^0(M;E), \Lambda^0(M;E) \otimes_{C^\infty(M)} \Lambda^2 M)$$

$$= \text{Hom}_{C^\infty(M)}(\Lambda^0(M;E), \Lambda^0(M;E)) \otimes_{C^\infty(M)} \Lambda^2 M$$

$$= \Lambda^0(M;\text{Hom}(E,E)) \otimes_{C^\infty(M)} \Lambda^2 M$$

$$= \Lambda^2(M;\text{Hom}(E,E)).$$

Definition: The <u>curvature</u> of ∇ is

$$F^\nabla \in \Lambda^2(M;\text{Hom}(E,E)).$$

Let $s \otimes \alpha \in \Lambda^p(M;E)$ -- then

$$d^\nabla \circ d^\nabla(s \otimes \alpha) = d^\nabla(s \otimes d\alpha + \nabla s \wedge \alpha)$$

$$= s \otimes d^2\alpha + \nabla s \wedge d\alpha + d^\nabla(\nabla s) \wedge \alpha - \nabla s \wedge d\alpha$$

$$= d^\nabla \circ \nabla(s) \wedge \alpha$$

$$= F^\nabla(s) \wedge \alpha.$$

Therefore

$$0 \rightarrow \Lambda^0(M;E) \overset{\nabla}{\rightarrow} \Lambda^1(M;E) \overset{d^\nabla}{\rightarrow} \Lambda^2(M;E) \overset{d^\nabla}{\rightarrow} \cdots$$

is a complex provided $F^\nabla = 0$.

<u>LEMMA</u> We have

$$d^\nabla F^\nabla = 0,$$

where d^∇ is associated with $\nabla_{\text{Hom}(E,E)}$.

[$\forall\ \phi \in \Lambda^2(M;\text{Hom}(E,E))$ & $\forall\ s \in \Lambda^0(M;E)$,

$$d^\nabla(\phi,s) = (\phi,\nabla s) + (d^\nabla\phi,s),$$

this being an equality of elements of $\Lambda^3(M;E)$. Take $\phi = F^\nabla$ — then

$$(d^\nabla F^\nabla,s) = d^\nabla(F^\nabla,s) - (F^\nabla,\nabla s)$$

$$= d^\nabla \circ (d^\nabla \circ \nabla s) - (d^\nabla \circ d^\nabla) \circ \nabla s$$

$$= 0.]$$

Given $X,Y \in \mathcal{D}^1(M)$, put

$$R(X,Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}.$$

Then

$$R(X,Y):\Lambda^0(M;E) \to \Lambda^0(M;E)$$

and $\forall\ f \in C^\infty(M)$ & $\forall\ s \in \Lambda^0(M;E)$,

$$fR(X,Y)s = R(X,Y)(fs)$$

$$= R(fX,Y)s = R(X,fY)s.$$

There is also an arrow $ev_{X,Y}:\Lambda^2 M \to C^\infty(M)$ which can be tensored over $C^\infty(M)$ with $\Lambda^0(M;\text{Hom}(E,E))$ to give an arrow

$$EV_{X,Y}:\Lambda^0(M;\text{Hom}(E,E)) \otimes_{C^\infty(M)} \Lambda^2 M \to \Lambda^0(M;\text{Hom}(E,E)),$$

i.e., an arrow

$$EV_{X,Y}:\Lambda^2(M;\text{Hom}(E,E)) \to \Lambda^0(M;\text{Hom}(E,E)).$$

Put

$$F^\nabla_{X,Y} = EV_{X,Y}(F^\nabla)$$

$$\in\Lambda^0(M;\mathrm{Hom}(E,E)) = \mathrm{Hom}_{C^\infty(M)}(\Lambda^0(M;E),\Lambda^0(M;E)).$$

<u>FACT</u> We have

$$F^\nabla_{X,Y} = R(X,Y).$$

Define ι_X on $\Lambda^p(M;E)\ (p > 0)$ by

$$\iota_X(s \otimes \alpha) = s \otimes \iota_X\alpha.$$

[Note: Take $\iota_X = 0$ on $\Lambda^0(M;E)$.]

<u>LEMMA</u> Let $X,Y\in\mathcal{D}^1(M)$ — then $\forall\ s\in\Lambda^0(M;E)$,

$$\iota_Y\iota_X d^\nabla d^\nabla s = \iota_X d^\nabla \iota_Y d^\nabla s - \iota_Y d^\nabla \iota_X d^\nabla s - \nabla_{[X,Y]}s.$$

<u>Reality Check</u> Take $E = \varepsilon$ — then $d^2 = 0$ and $\forall\ f\in C^\infty(M)$,

$$\iota_X d\iota_Y df - \iota_Y d\iota_X df - [X,Y]f$$

$$= \iota_X d(Yf) - \iota_Y d(Xf) - (XY - YX)f$$

$$= XYf - YXf - XYf + YXf$$

$$= 0.$$

Remark: The lemma is merely a restatement of the fact that

$$F^\nabla_{X,Y} = R(X,Y).$$

Rappel: In the exterior algebra Λ^*M,

$$L_X = \iota_X \circ d + d \circ \iota_X.$$

Motivated by this, given $\nabla \in \text{con } E$, put

$$L_X^\nabla = \iota_X \circ d^\nabla + d^\nabla \circ \iota_X,$$

thus

$$L_X^\nabla : \Lambda^p(M;E) \to \Lambda^p(M;E).$$

[Note: When $p = 0$,

$$L_X^\nabla s = \iota_X \circ d^\nabla s$$

$$= \iota_X \nabla s = \nabla s(X) = \nabla_X s.]$$

<u>FACT</u> We have

$$L_X(s \otimes \alpha) = \nabla_X s \otimes \alpha + s \otimes L_X \alpha.$$

Specialize now to the vector bundle

$$T_q^p(M) = LM \times_{\underline{GL}(n,\underline{R})} T_q^p(n).$$

Then the elements of

$$\Lambda^k(M;T_q^p(M))$$

are the $C^\infty(M)$-multilinear antisymmetric maps

$$\overbrace{\mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M)}^{k} \to \mathcal{D}_q^p(M).$$

[Note: Bear in mind that

$$\Lambda^0(M;T_q^p(M)) = \mathcal{D}_q^p(M).]$$

Remark: Working locally, each $\alpha \in \Lambda^k(M;T_q^p(M))$ defines a k-form

$$\alpha^{i_1 \cdots i_p}{}_{j_1 \cdots j_q},$$

namely

$$\alpha^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(X_1, \ldots, X_k)$$

$$= \alpha(X_1, \ldots, X_k)(dx^{i_1}, \ldots, dx^{i_p}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_q}}),$$

these being the components of α.

Let ∇ be a connection on TM -- then ∇ induces a connection on $T_q^p(M)$, which again will be denoted by ∇. Accordingly, there is an $\underline{R}$-linear operator

$$d^\nabla : \Lambda^k(M; T_q^p(M)) \to \Lambda^{k+1}(M; T_q^p(M))$$

with the property that

$$d^\nabla(\alpha \wedge \beta) = d^\nabla \alpha \wedge \beta + (-1)^k \alpha \wedge d^\nabla \beta.$$

[Note: Here,

$$(\alpha \wedge \beta)(X_1, \ldots, X_{k+\ell})$$

$$= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\text{sgn } \sigma) \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)})$$

$$\otimes \beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)}).]$$

Example: Take $p = q = 0$ — then $\mathcal{D}_0^0(M) = C^\infty(M)$ and

$$\Lambda^k(M; C^\infty(M)) = \Lambda^k M.$$

In this situation, $d^\nabla = d$, hence is the same for all ∇.

Example: Let $T \in \mathcal{D}_q^p(M)$ -- then

$$d^\nabla T \in \Lambda^1(M; T_q^p(M))$$

and $\forall\ X \in \mathcal{D}^1(M)$,

$$d^\nabla T(X)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= (\nabla_X T)(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q)$$

$$= \nabla T(\Lambda^1,\ldots,\Lambda^p, X_1,\ldots,X_q,X).$$

[Note: Recall that, in general, if $E \rightarrow M$ is a vector bundle, then for any $\nabla \in \text{con } E$, on $\Lambda^0(M;E)$, $\nabla = d^\nabla$.]

Section 14: The Structural Equations Let M be a connected C^∞ manifold of dimension n.

Assume: M is parallelizable, i.e., that the frame bundle LM is trivial.

[Note: Accordingly,

$$LM \approx M \times \underline{GL}(n,\underline{R}),$$

thus LM has two components, hence M is orientable.]

Therefore LM admits global sections, these being the frames.

[Note: A frame $E = \{E_1,\ldots,E_n\}$ is, by definition, a basis for $\mathcal{D}^1(M)$ (as a module over $C^\infty(M)$). The associated coframe is the set $\omega = \{\omega^1,\ldots,\omega^n\}$, where the 1-forms ω^i are characterized by $\omega^i(E_j) = \delta^i{}_j$. So, $\forall$ $X \in \mathcal{D}^1(M)$, we have $X = \omega^i(X)E_i$.]

Remark: The components of a tensor $T \in \mathcal{D}^p_q(M)$ relative to a frame arise in exactly the same way as for a coordinate system. I.e.:

$$T = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(E_{i_1} \otimes \cdots \otimes E_{i_p}) \otimes (\omega^{j_1} \otimes \cdots \otimes \omega^{j_q}),$$

where

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} = T(\omega^{i_1},\ldots,\omega^{i_p}, E_{j_1},\ldots,E_{j_q}).$$

Let ∇ be a connection on TM -- then its connection 1-forms $\omega^i{}_j$ are defined by the requirement

$$\nabla_X E_j = \omega^i{}_j(X)E_i.$$

Agreeing to let

$$\nabla_{E_i} E_j = \Gamma^k{}_{ij} E_k,$$

it follows that

$$\omega^i{}_j = \Gamma^i{}_{kj}\omega^k.$$

Given $X \in \mathcal{D}^1(M)$, write

$$X = X^i E_i.$$

Then

$$\nabla X = E_i \otimes (dX^i + X^k \omega^i{}_k).$$

Given $\alpha \in \mathcal{D}_1(M)$, write

$$\alpha = \alpha_i \omega^i.$$

Then

$$\nabla \alpha = \omega^i \otimes (d\alpha_i - \alpha_k \omega^k{}_i).$$

Definition: Let $\nabla \in \text{con}\ TM$.

(T) The <u>torsion forms</u> Θ^i of ∇ are defined by

$$T(X,Y) = \Theta^i(X,Y)E_i.$$

(R) The <u>curvature forms</u> $\Omega^i{}_j$ of ∇ are defined by

$$R(X,Y)E_j = \Omega^i{}_j(X,Y)E_i.$$

<u>THEOREM</u> (<u>The Structural Equations</u>) We have

$$\left[\begin{array}{l} \Theta^i = d\omega^i + \omega^i{}_j \wedge \omega^j \\ \\ \Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \end{array}\right.$$

[Consider the first relation. Thus

$$\Theta^i(X,Y)E_i = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$= \nabla_X(\omega^j(Y)E_j) - \nabla_Y(\omega^j(X)E_j) - \omega^j([X,Y])E_j$$

$$= \{X\omega^j(Y) - Y\omega^j(X) - \omega^j([X,Y])\}E_j$$

$$+ \{\omega^j(Y)\omega^i{}_j(X) - \omega^j(X)\omega^i{}_j(Y)\}E_i$$

$$= d\omega^i(X,Y)E_i + (\omega^i{}_j\wedge\omega^j)(X,Y)E_i.$$

Consider the second relation. Thus

$$\Omega^i{}_j(X,Y)E_i = \nabla_X\nabla_Y E_j - \nabla_Y\nabla_X E_j - \nabla_{[X,Y]}E_j$$

$$= \nabla_X(\omega^i{}_j(Y)E_i) - \nabla_Y(\omega^i{}_j(X)E_i) - \omega^i{}_j([X,Y])E_i$$

$$= \{X\omega^i{}_j(Y) - Y\omega^i{}_j(X) - \omega^i{}_j([X,Y])\}E_i$$

$$+ \{\omega^i{}_j(Y)\omega^k{}_i(X) - \omega^i{}_j(X)\omega^k{}_i(Y)\}E_k$$

$$= d\omega^i{}_j(X,Y)E_i + (\omega^i{}_k\wedge\omega^k{}_j)(X,Y)E_i.]$$

Remark: If ∇ is torsion free, then

$$d\omega^i = -\omega^i{}_j\wedge\omega^j.$$

[Note: Put

$$\omega^{k_1\cdots k_r} = \omega^{k_1}\wedge\ldots\wedge\omega^{k_r}.$$

Then in the presence of zero torsion,

$$d\omega^{i_1\cdots i_p} = -\,\omega^{i_1}{}_j\wedge\omega^{ji_2\cdots i_p} - \cdots - \omega^{i_p}{}_j\wedge\omega^{i_1\cdots i_{p-1}j}.]$$

<u>FACT</u> Suppose that ∇ is torsion free -- then $\forall\ \alpha\in\Lambda^pM$,

$$d\alpha = \omega^i\wedge\nabla_{E_i}\alpha.$$

Write

$$d\omega^i = \frac{1}{2}\,C^i{}_{jk}\omega^j\wedge\omega^k \quad (C^i{}_{jk} = -\,C^i{}_{kj}).$$

Then the $C^i{}_{jk}$ are the <u>objects of anholonomity</u>.

[Note: Their transformation behavior is nontensorial.]

Observation: We have

$$[E_j,E_k] = \omega^i([E_j,E_k])E_i$$

$$= -\,(d\omega^i(E_j,E_k) - E_j\omega^i(E_k) + E_k\omega^i(E_j))E_i$$

$$= -\,C^i{}_{jk}E_i.$$

- There is an expansion

$$\Theta^i = \frac{1}{2}\,T^i{}_{k\ell}\omega^k\wedge\omega^\ell \quad (T^i{}_{k\ell} = -\,T^i{}_{\ell k})$$

and

$$T^i{}_{k\ell} = \Gamma^i{}_{k\ell} - \Gamma^i{}_{\ell k} + C^i{}_{k\ell}.$$

[Note: By definition,

$$T^i{}_{k\ell} = T(\omega^i,E_k,E_\ell).]$$

• There is an expansion

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jk\ell}\omega^k \wedge \omega^\ell \qquad (R^i{}_{jk\ell} = - R^i{}_{j\ell k})$$

and

$$R^i{}_{jk\ell} = E_k \Gamma^i{}_{\ell j} - E_\ell \Gamma^i{}_{kj}$$

$$+ \Gamma^a{}_{\ell j}\Gamma^i{}_{ka} - \Gamma^a{}_{kj}\Gamma^i{}_{\ell a} + C^a{}_{k\ell}\Gamma^i{}_{aj}.$$

[Note: By definition,

$$R^i{}_{jk\ell} = R(\omega^i, E_j, E_k, E_\ell).]$$

Put

$$\mathrm{Ric}_j = \iota_{E_i}\Omega^i{}_j.$$

Then

$$\mathrm{Ric}_j = \iota_{E_i}\left[\frac{1}{2} R^i{}_{jk\ell}\omega^k \wedge \omega^\ell\right]$$

$$= \frac{1}{2}\left[R^i{}_{jk\ell}\omega^k(E_i)\omega^\ell - R^i{}_{jk\ell}\omega^\ell(E_i)\omega^k\right]$$

$$= \frac{1}{2}\left[R^i{}_{ji\ell}\omega^\ell - R^i{}_{jki}\omega^k\right]$$

$$= \frac{1}{2}\left[R^i{}_{ji\ell}\omega^\ell + R^i{}_{jik}\omega^k\right]$$

$$= \frac{1}{2}\left[R^i{}_{ji\ell}\omega^\ell + R^i{}_{ji\ell}\omega^\ell\right]$$

$$= R^i{}_{ji\ell}\omega^\ell$$

$$= R_{j\ell}\omega^\ell.$$

The Ric_j $(j = 1,\ldots,n)$ are called the Ricci 1-forms. Obviously,

$$\begin{cases} Ric_j(E_i) = R_{ji} \\ Ric_i(E_j) = R_{ij} \end{cases}$$

but, in general, $R_{ji} \neq R_{ij}$.

<u>Section 15</u>: <u>Transition Formalities</u> Let M be a connected C^∞ manifold of dimension n.

Rappel: There is a one-to-one correspondence

$$\begin{cases} \Gamma \to \nabla^\Gamma \\ \nabla \to \Gamma^\nabla \end{cases}$$

between the connections Γ on the frame bundle

$$\begin{array}{ccc} \underline{GL}(n,\underline{R}) & \to & LM \\ & & \downarrow \pi \\ & & M \end{array}$$

and the connections ∇ on the tangent bundle

$$TM = LM \times_{\underline{GL}(n,\underline{R})} \underline{R}^n.$$

Assume now that M is parallelizable. Fix a frame $E = \{E_1,\ldots,E_n\}$ and let $s:M \to LM$ be the section thereby determined, thus $\forall\ x \in M$,

$$s(x) = \{E_1|_x,\ldots,E_n|_x\}$$

is a basis for T_xM.

<u>FACT</u> Fix $x \in M$ and let $\zeta_x:\underline{R}^n \to T_xM$ be the nonsingular linear transformation

$$(a_1,\ldots,a_n) \to a_1E_1|_x + \cdots + a_nE_n|_x.$$

Suppose that $X,Y \in \mathcal{D}^1(M)$ -- then

$$\nabla^\Gamma_X Y|_x = \zeta_x\omega_\Gamma(ds_xX_x)\zeta_x^{-1}Y_x + (XY^i)(x)E_i|_x.$$

The correspondence $\Gamma \leftrightarrow \omega_\Gamma \to s^*\omega_\Gamma$ identifies con LM with $\Lambda^1(M;\underline{g\ell}(n,\underline{R}))$.

And each $\nabla \in \text{con } TM$ gives rise to an element $\omega_\nabla \in \Lambda^1(M;\underline{\mathfrak{gl}}(n,\underline{R}))$, viz.

$$\omega_\nabla = [\omega^i{}_j].$$

<u>LEMMA</u> $\forall\ \Gamma \in \text{con } LM$,

$$\omega_{\nabla^\Gamma} = s^*\omega_\Gamma.$$

[By definition,

$$\nabla^\Gamma_X E_j\big|_x = (\omega^i{}_j)^\Gamma(X_x)E_i\big|_x.$$

On the other hand,

$$\nabla^\Gamma_X E_j\big|_x = \zeta_x\omega_\Gamma(ds_x X_x)\zeta_x^{-1}E_j\big|_x.$$

Here

$$\zeta_x^{-1}E_j\big|_x = e_j$$

and

$$\omega_\Gamma(ds_x X_x) = s^*\omega_\Gamma(X_x).$$

But

$$s^*\omega_\Gamma(X_x)e_j = s^*\omega_\Gamma(X_x)^i{}_j e_i$$

$\Rightarrow$

$$\zeta_x s^*\omega_\Gamma(X_x)e_j = s^*\omega_\Gamma(X_x)^i{}_j E_i\big|_x$$

$\Rightarrow$

$$(\omega^i{}_j)^\Gamma(X_x) = s^*\omega_\Gamma(X_x)^i{}_j.$$

Therefore

$$\omega_{\nabla^\Gamma} = s^*\omega_\Gamma.]$$

Given $\nabla \in \text{con}\ TM$, put

$$\Omega_{\nabla} = [\Omega^i{}_j] \in \Lambda^2(M;\underline{gl}(n,\underline{R})).$$

Then $\forall\ \Gamma \in \text{con}\ LM$,

$$\Omega_{\nabla_\Gamma} = s^*\Omega_\Gamma$$

In fact,

$$\Omega_\Gamma = d\omega_\Gamma + \omega_\Gamma \wedge \omega_\Gamma$$

$$\Rightarrow$$

$$s^*\Omega_\Gamma = ds^*\omega_\Gamma + s^*\omega_\Gamma \wedge s^*\omega_\Gamma$$

$$= d\omega_{\nabla_\Gamma} + \omega_{\nabla_\Gamma} \wedge \omega_{\nabla_\Gamma}$$

$$= \Omega_{\nabla_\Gamma}.$$

Definition: A <u>gauge transformation</u> is a C^∞ map

$$g: M \to \underline{GL}(n,\underline{R}).$$

Notation: $\underline{GAU}$ is the set of gauge transformations.

With respect to pointwise operations, $\underline{GAU}$ is a group and there is a right action

$$\begin{cases} \text{sec}\ LM \times \underline{GAU} \to \text{sec}\ LM \\ (E,g) \longrightarrow E\cdot g, \end{cases}$$

where

$$(E\cdot g)_j = E_i g^i{}_j.$$

<u>LEMMA</u> Let $\nabla \in \text{con}\ TM$ — then under a change of frame

$$E \to E \cdot g \quad (g \in \underline{GAU}),$$

the matrix

$$\omega_\nabla \in \Lambda^1(M; \underline{g\ell}(n, \underline{R}))$$

of connection 1-forms becomes

$$g^{-1}\omega_\nabla g + g^{-1}dg.$$

[Note: The products are matrix products and dg is the entrywise exterior derivative of $g:M \to \underline{GL}(n,\underline{R})$.]

Remark: The transformation property of Ω_∇ is simpler, viz.

$$\Omega_\nabla \to g^{-1}\Omega_\nabla g \quad (g \in \underline{GAU}).$$

[Invoke the lemma and observe that

$$g^{-1}g = I$$

$$\Rightarrow$$

$$g^{-1}(dg) + (dg^{-1})g = 0$$

$$\Rightarrow$$

$$g^{-1}(dg) = -(dg^{-1})g$$

$$\Rightarrow$$

$$g^{-1}(dg)g^{-1} = -dg^{-1}.]$$

In matrix notation, the relation

$$\nabla_X E_j = \omega^i{}_j(X)E_i$$

can be written

$$\nabla_X E = E\omega_\nabla(X).$$

So, $\forall\ g \in \underline{\underline{GAU}}$,

$$\nabla_X E\cdot g = E\cdot g(g^{-1}\omega_\nabla(X)g + g^{-1}dg(X)).$$

Let ∇ be a connection on TM and consider the $\underline{R}$-linear operator

$$d^\nabla : \Lambda^k(M;T^p_q(M)) \to \Lambda^{k+1}(M;T^p_q(M)).$$

Then $\forall\ \alpha \in \Lambda^k(M;T^p_q(M))$, one has

$$(d^\nabla\alpha)^{i_1\cdots i_p}{}_{j_1\cdots j_q} = d\alpha^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$+ \omega^{i_1}{}_a \wedge \alpha^{a i_2\cdots i_p}{}_{j_1\cdots j_q} + \cdots$$

$$- \omega^b{}_{j_1} \wedge \alpha^{i_1\cdots i_p}{}_{b j_2\cdots j_q} - \cdots .$$

In what follows, use matrix notation.

Example:

(1) Take $p = 1$, $q = 0$ -- then

$$(d^\nabla\alpha)^i = d\alpha^i + \omega^i{}_j \wedge \alpha^j$$

$$\Rightarrow$$

$$d^\nabla\alpha = d\alpha + \omega_\nabla \wedge \alpha.$$

(2) Take $p = 1$, $q = 1$ -- then

$$(d^\nabla\alpha)^i{}_j = d\alpha^i{}_j + \omega^i{}_a \wedge \alpha^a{}_j - \omega^b{}_j \wedge \alpha^i{}_b$$

$$= d\alpha^i{}_j + \omega^i{}_a \wedge \alpha^a{}_j - (-1)^k \alpha^i{}_b \wedge \omega^b{}_j$$

$\Rightarrow$

$$d^{\nabla}\alpha = d\alpha + \omega_{\nabla}\wedge\alpha - (-1)^{k}\alpha\wedge\omega_{\nabla}.$$

The $\omega^{i}\in\Lambda^{1}M$ are the components of an element

$$\omega\in\Lambda^{1}(M;T_{0}^{1}(M)).$$

Explicated: $\forall\ X\in\mathcal{D}^{1}(M)$,

$$\begin{cases} \omega:\mathcal{D}^{1}(M) \rightarrow \mathcal{D}^{1}(M) \\ \omega(X) = X \end{cases}$$

$\Rightarrow$

$$\begin{aligned} \omega(X)^{i} &= \omega(X)(\omega^{i}) \\ &= \omega^{i}(X). \end{aligned}$$

Analogously, the

$$\begin{cases} \Theta^{i} \\ \Omega^{i}{}_{j} \end{cases} \in\Lambda^{2}M$$

are the components of an element

$$\begin{cases} \Theta_{\nabla}\in\Lambda^{2}(M;T_{0}^{1}(M)) \\ \Omega_{\nabla}\in\Lambda^{2}(M;T_{1}^{1}(M)). \end{cases}$$

Example: The $\omega^{i}{}_{j}\in\Lambda^{1}M$ are not the components of an element $\omega_{\nabla}\in\Lambda^{1}(M;T_{1}^{1}(M))$.
[Suppose that $T\in\Lambda^{1}(M;T_{1}^{1}(M))$ -- then

$$T = T^i{}_j E_i \otimes \omega^j.$$

Replacing E by E·g changes $T^i{}_j$ to

$$(g^{-1})^i{}_k T^k{}_\ell g^\ell{}_j.$$

But this tensor transformation rule is not satisfied by ω_∇ since

$$\omega_\nabla \rightarrow g^{-1}\omega_\nabla g + g^{-1}dg.]$$

<u>$d^\nabla\omega$</u>: We have

$$d^\nabla\omega = d\omega + \omega_\nabla\wedge\omega$$

$$= \Theta_\nabla.$$

<u>$d^\nabla\Theta_\nabla$</u>: We have

$$d^\nabla\Theta_\nabla = d\Theta_\nabla + \omega_\nabla\wedge\Theta_\nabla$$

$$= d(d^\nabla\omega) + \omega_\nabla\wedge d^\nabla\omega$$

$$= d(d\omega + \omega_\nabla\wedge\omega) + \omega_\nabla\wedge(d\omega + \omega_\nabla\wedge\omega)$$

$$= d\omega_\nabla\wedge\omega - \omega_\nabla\wedge d\omega + \omega_\nabla\wedge d\omega + \omega_\nabla\wedge\omega_\nabla\wedge\omega$$

$$= d\omega_\nabla\wedge\omega + \omega_\nabla\wedge\omega_\nabla\wedge\omega$$

$$= (d\omega_\nabla + \omega_\nabla\wedge\omega_\nabla)\wedge\omega$$

$$= \Omega_\nabla\wedge\omega.$$

I.e.:

$$d^\nabla\Theta_\nabla = \Omega_\nabla\wedge\omega.$$

$\underline{d^\nabla \Omega_\nabla}$: We have

$$
\begin{aligned}
d^\nabla \Omega_\nabla &= d\Omega_\nabla + \omega_\nabla \wedge \Omega_\nabla - \Omega_\nabla \wedge \omega_\nabla \\
&= d\Omega_\nabla + \omega_\nabla \wedge (d\omega_\nabla + \omega_\nabla \wedge \omega_\nabla) \\
&\qquad - (d\omega_\nabla + \omega_\nabla \wedge \omega_\nabla) \wedge \omega_\nabla \\
&= d\Omega_\nabla + \omega_\nabla \wedge d\omega_\nabla - d\omega_\nabla \wedge \omega_\nabla \\
&= d(d\omega_\nabla + \omega_\nabla \wedge \omega_\nabla) + \omega_\nabla \wedge d\omega_\nabla - d\omega_\nabla \wedge \omega_\nabla \\
&= d\omega_\nabla \wedge \omega_\nabla - \omega_\nabla \wedge d\omega_\nabla + \omega_\nabla \wedge d\omega_\nabla - d\omega_\nabla \wedge \omega_\nabla \\
&= 0.
\end{aligned}
$$

I.e.:

$$d^\nabla \Omega_\nabla = 0.$$

Remark: The symbol Ω_∇ has two meanings, namely as an element of

$$\Lambda^2(M;\underline{g\ell}(n,\underline{R}))$$

or as an element of

$$\Lambda^2(M;T_1^1(M)).$$

Of course, if Ω_∇ is viewed in the second sense, viz. as a map

$$\Omega_\nabla : \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}_1^1(M),$$

then, upon taking components, Ω_∇ reappears in the first sense as a matrix, viz. $\forall\ X,Y \in \mathcal{D}^1(M)$,

$$\Omega_\nabla(X,Y)(\omega^i,E_j) = \Omega_\nabla(X,Y)^i{}_j = \Omega^i{}_j(X,Y).$$

Summary:

- Unwound, the relation

$$d^\nabla\Theta_\nabla = \Omega_\nabla\wedge\omega$$

becomes

$$d\Theta^i + \omega^i{}_j\wedge\Theta^j = \Omega^i{}_j\wedge\omega^j.$$

- Unwound, the relation

$$d^\nabla\Omega_\nabla = 0$$

becomes

$$d\Omega^i{}_j + \omega^i{}_k\wedge\Omega^k{}_j - \Omega^i{}_k\wedge\omega^k{}_j = 0.$$

Let $\alpha\in\Lambda^k(M;T^p_q(M))$ — then

$$(d^\nabla d^\nabla\alpha)^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$= \Omega^{i_1}{}_a\wedge\alpha^{ai_2\cdots i_p}{}_{j_1\cdots j_q} + \cdots$$

$$- \Omega^b{}_{j_1}\wedge\alpha^{i_1\cdots i_p}{}_{bj_2\cdots j_q} - \cdots .$$

So, when $R = 0$,

$$d^\nabla \circ d^\nabla = 0.$$

<u>Section 16:</u> <u>Metric Considerations</u> Let M be a connected C^∞ manifold of dimension n. Fix a semiriemannian structure $g \in \underline{M}_{k,n-k}$.

Assume: The orthonormal frame bundle LM(g) is trivial.

Therefore LM(g) admits global sections, these being the <u>orthonormal frames</u>.

Example: If M is parallelizable and if $E = \{E_1,\ldots,E_n\}$ is a frame, then the prescription

$$g_E(X,Y) = \eta_{ij}X^iY^j \quad \begin{cases} X = X^iE_i \\ \\ Y = Y^jE_j \end{cases}$$

defines a semiriemannian structure $g_E \in \underline{M}_{k,n-k}$ having E as an orthonormal frame. And

$$g_E = g_{E\cdot A}$$

for all

$$A \in C^\infty(M;\underline{O}(k,n-k)).$$

Suppose that $E = \{E_1,\ldots,E_n\}$ is an orthonormal frame. Put

$$\varepsilon_i = g(E_i,E_i).$$

Then

$$\varepsilon_i = \begin{cases} -1 & (1 \le i \le k) \\ \\ +1 & (k+1 \le i \le n). \end{cases}$$

[Note: Let $\omega = \{\omega^1,\ldots,\omega^n\}$ be the associated coframe -- then

$$g = \sum_i \varepsilon_i\omega^i \otimes \omega^i.]$$

Example: Let $X \in \mathcal{D}^1(M)$ -- then $\forall\ \nabla \in \text{con}\ TM$,

$$C_1^1 \nabla X = \varepsilon_j g(\nabla_{E_j} X, E_j).$$

[To see this, recall that

$$\nabla X = E_i \otimes (dX^i + X^k \omega^i{}_k)$$

or still,

$$\nabla X = (E_j X^i) E_i \otimes \omega^j + (X^k \Gamma^i{}_{jk}) E_i \otimes \omega^j$$

$\Rightarrow$

$$C_1^1 \nabla X = E_i X^i + X^k \Gamma^i{}_{ik}$$

$$= E_i X^i + X^j \Gamma^i{}_{ij}.$$

On the other hand,

$$\varepsilon_j g(\nabla_{E_j} X, E_j) = \varepsilon_j g(\nabla_{E_j} (X^i E_i), E_j)$$

$$= \varepsilon_j g((E_j X^i) E_i + X^i \nabla_{E_j} E_i, E_j)$$

$$= \varepsilon_j g((E_j X^i) E_i + X^i \omega^k{}_i (E_j) E_k, E_j)$$

$$= (\varepsilon_i)^2 E_i X^i + (\varepsilon_j)^2 X^i \omega^j{}_i (E_j)$$

$$= E_i X^i + X^i \Gamma^j{}_{ji}$$

$$= E_i X^i + X^j \Gamma^i{}_{ij}.]$$

Remark: To lower or raise an index i of a component of a tensor $T \in \mathcal{D}_q^p(M)$, one has only to multiply by ε_i. E.g.: If $T \in \mathcal{D}_2^1(M)$, then

$$T_{ijk} = g_{ia} T^a{}_{jk} = \delta^i{}_a \varepsilon_a T^a{}_{jk} = \varepsilon_i T^i{}_{jk} \text{ (no sum)}.$$

Fix $\nabla \in \text{con}_g\, TM$.

<u>LEMMA</u> We have

$$\omega^i{}_j = -\varepsilon_i\varepsilon_j\omega^j{}_i \quad \text{(no sum)}.$$

[In fact, $\forall\, X \in \mathcal{D}^1(M)$,

$$\begin{aligned} 0 &= Xg(E_i,E_j) \\ &= g(\nabla_X E_i,E_j) + g(E_i,\nabla_X E_j) \\ &= g(\omega^k{}_i(X)E_k,E_j) + g(E_i,\omega^k{}_j(X)E_k) \\ &= \omega^k{}_i(X)g_{kj} + \omega^k{}_j(X)g_{ik} \\ &= g_{ik}\omega^k{}_j(X) + g_{jk}\omega^k{}_i(X) \\ &= \varepsilon_i\omega^i{}_j(X) + \varepsilon_j\omega^j{}_i(X) \quad \text{(no sum)}.] \end{aligned}$$

[Note: If $E = \{E_1,\ldots,E_n\}$ is an arbitrary frame, then

$$\omega_{ij} + \omega_{ji} = dg_{ij}.]$$

In particular:

$$\omega^i{}_i = 0.$$

<u>LEMMA</u> We have

$$\Omega^i{}_j = -\varepsilon_i\varepsilon_j\Omega^j{}_i \quad \text{(no sum)}.$$

[In fact,

$$-\varepsilon_i\varepsilon_j\Omega^j{}_i = -\varepsilon_i\varepsilon_j[d\omega^j{}_i + \omega^j{}_k\wedge\omega^k{}_i]$$

$$= - \varepsilon_i\varepsilon_j[d(- \varepsilon_i\varepsilon_j\omega^i{}_j) + (- \varepsilon_j\varepsilon_k)\omega^k{}_j\wedge(- \varepsilon_k\varepsilon_i)\omega^i{}_k]$$

$$= - \varepsilon_i\varepsilon_j[- \varepsilon_i\varepsilon_j d\omega^i{}_j - \varepsilon_i\varepsilon_j(\omega^i{}_k\wedge\omega^k{}_j)]$$

$$= (\varepsilon_i\varepsilon_j)^2[d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j] = \Omega^i{}_j.]$$

In particular:

$$\Omega^i{}_i = 0.$$

Scholium: Let $E = \{E_1,\ldots,E_n\}$ be an orthonormal frame. Suppose that ∇ is a g-connection -- then

$$\omega_\nabla \in \Lambda^1(M;\underline{so}(k,n-k))$$

and

$$\Omega_\nabla \in \Lambda^2(M;\underline{so}(k,n-k)).$$

[This is just a restatement of the fact that

$$\omega^i{}_j = - \varepsilon_i\varepsilon_j\omega^j{}_i \quad \text{(no sum)}$$

and

$$\Omega^i{}_j = - \varepsilon_i\varepsilon_j\Omega^j{}_i \quad \text{(no sum)}.]$$

Assume now that ∇ is the metric connection -- then, since ∇ has zero torsion,

$$d\omega^i + \omega^i{}_j\wedge\omega^j = 0.$$

<u>LEMMA</u> We have

$$\Gamma^i{}_{kj} = \frac{1}{2}\,\varepsilon_i(\varepsilon_i d\omega^i(E_j,E_k) + \varepsilon_j d\omega^j(E_k,E_i) - \varepsilon_k d\omega^k(E_i,E_j)).$$

[Obviously,

$$d\omega^i(E_j,E_k) = -\omega^i{}_k(E_j) + \omega^i{}_j(E_k)$$

$\Rightarrow$

$$\varepsilon_i d\omega^i(E_j,E_k) = -\varepsilon_i\omega^i{}_k(E_j) + \varepsilon_i\omega^i{}_j(E_k).$$

Next, cyclically permute i,j,k and use the relations developed above to get

$$\varepsilon_j d\omega^j(E_k,E_i) = \varepsilon_i\omega^i{}_j(E_k) + \varepsilon_j\omega^j{}_k(E_i).$$

Repeating the procedure then gives

$$\varepsilon_k d\omega^k(E_i,E_j) = \varepsilon_j\omega^j{}_k(E_i) - \varepsilon_i\omega^i{}_k(E_j).$$

Now subtract the last equation from the sum of the first two.]

[Note: It follows that the connection 1-forms $\omega^i{}_j$ are the unique 1-forms satisfying

$$d\omega^i + \omega^i{}_j\wedge\omega^j = 0$$

and

$$\omega^i{}_j = -\varepsilon_i\varepsilon_j\omega^j{}_i \quad \text{(no sum)}.]$$

Remark: In the RHS of this formula, the indices i,j,k are not summed!

Let $E = \{E_1,\ldots,E_n\}$ be an arbitrary frame.

Notation: Write

$$dg_{ij} = g_{ij,k}\omega^k,$$

where

$$g_{ij,k} = E_k g_{ij}.$$

Then

$$\omega_{ij} + \omega_{ji} = dg_{ij}$$

$$\Rightarrow$$

$$g_{ij,k} = g_{ia}\Gamma^a{}_{kj} + g_{ja}\Gamma^a{}_{ki}.$$

<u>FACT</u> We have

$$\Gamma^i{}_{kj} = \frac{1}{2} (- C^i{}_{kj} + g_{ja}g^{ib}C^a{}_{kb} + g_{ka}g^{ib}C^a{}_{jb})$$

$$+ \frac{1}{2} g^{ib}(g_{jb,k} + g_{bk,j} - g_{kj,b}).$$

<u>Reality Check</u> If the frame is orthonormal, then the second term vanishes leaving

$$\Gamma^i{}_{kj} = \frac{1}{2} (- C^i{}_{kj} + \varepsilon_j\varepsilon_i C^j{}_{ki} + \varepsilon_k\varepsilon_i C^k{}_{ji}).$$

But

$$\begin{cases} C^i{}_{kj} = d\omega^i(E_k,E_j) \\ C^j{}_{ki} = d\omega^j(E_k,E_i) \\ C^k{}_{ji} = d\omega^k(E_j,E_i). \end{cases}$$

Therefore

$$\Gamma^i{}_{kj} = \frac{1}{2} (d\omega^i(E_j,E_k) + \varepsilon_j\varepsilon_i d\omega^j(E_k,E_i) - \varepsilon_k\varepsilon_i d\omega^k(E_i,E_j))$$

or still,

$$\Gamma^i{}_{kj} = \frac{1}{2} \varepsilon_i(\varepsilon_i d\omega^i(E_j,E_k) + \varepsilon_j d\omega^j(E_k,E_i) - \varepsilon_k d\omega^k(E_i,E_j)),$$

as desired.

Remark:

- Take $i = k$ -- then

$$\Gamma^i{}_{ij} = \frac{1}{2}(-C^i{}_{ij} + \varepsilon_j\varepsilon_i C^j{}_{ii} + \varepsilon_i\varepsilon_i C^i{}_{ji})$$

$$= \frac{1}{2}(-C^i{}_{ij} + C^i{}_{ji})$$

$$= -C^i{}_{ij}.$$

- Take $i = j$ -- then

$$\Gamma^i{}_{ki} = \frac{1}{2}(-C^i{}_{ki} + \varepsilon_i\varepsilon_i C^i{}_{ki} + \varepsilon_k\varepsilon_i C^k{}_{ii})$$

$$= \frac{1}{2}(-C^i{}_{ki} + C^i{}_{ki})$$

$$= 0.$$

<u>Section 17:</u> <u>Submanifolds</u> Let M be a connected C^∞ manifold of dimension n, $\Sigma \subset M$ an embedded connected submanifold of dimension d, $i:\Sigma \to M$ the inclusion. Fix a semiriemannian structure g on M.

Assumption: $\overline{g} \equiv i^*g$ is a semiriemannian structure on Σ.

So, $\forall\ x \in \Sigma$, $g_x | T_x\Sigma$ is nondegenerate and

$$T_xM = T_x\Sigma \oplus T_x\Sigma^\perp.$$

In the category of vector bundles, there is a pullback square

$$\begin{array}{ccc} i^*TM & \rightarrow & TM \\ \downarrow & & \downarrow \pi_M \\ \Sigma & \underset{i}{\rightarrow} & M \end{array}$$

and a split short exact sequence

$$0 \to T\Sigma \to i^*TM \to T\Sigma^\perp \to 0,$$

where $T\Sigma^\perp$ is the normal bundle of Σ.

Definition: A <u>vector field along Σ</u> is a section of i^*TM, i.e., a smooth map $X:\Sigma \to TM$ such that the triangle

$$\begin{array}{ccc} & & TM \\ & \overset{X}{\nearrow} & \downarrow \pi_M \\ \Sigma & \underset{i}{\longrightarrow} & M \end{array}$$

commutes.

Notation: $\mathcal{D}^1(\Sigma:M)$ stands for the set of vector fields along Σ.

[Note: $\mathcal{D}^1(\Sigma:M)$ is a module over $C^\infty(\Sigma)$. Furthermore, there is an arrow of restriction

$$\mathcal{D}^1(M) \to \mathcal{D}^1(\Sigma:M)$$

and an arrow of insertion

$$\mathcal{D}^1(\Sigma) \to \mathcal{D}^1(\Sigma:M).]$$

Let

$$\begin{cases} \tan:\mathcal{D}^1(\Sigma:M) \to \mathcal{D}^1(\Sigma) \\ \text{nor}:\mathcal{D}^1(\Sigma:M) \to \mathcal{D}^1(\Sigma)^\perp \end{cases}$$

be the projections, so $\forall\ X \in \mathcal{D}^1(\Sigma:M)$,

$$X = \tan X + \text{nor } X.$$

[Note: Both tan and nor are $C^\infty(\Sigma)$-linear.]

Rappel: Let ∇ be a connection on TM -- then ∇ induces a connection $i^*\nabla$ on i^*TM, i.e., a map

$$\begin{cases} \mathcal{D}^1(\Sigma) \times \mathcal{D}^1(\Sigma:M) \to \mathcal{D}^1(\Sigma:M) \\ \qquad (V,X) \longrightarrow i^*\nabla_V X \end{cases}$$

with the usual properties.

<u>LEMMA</u> The assignment

$$\begin{cases} \mathcal{D}^1(\Sigma) \times \mathcal{D}^1(\Sigma) \to \mathcal{D}^1(\Sigma) \\ \qquad (V,W) \longrightarrow \tan i^*\nabla_V W \end{cases}$$

defines a connection $\overline{\nabla}$ on $T\Sigma$.

Definition: The function

$$\Pi_\nabla : \mathcal{D}^1(\Sigma) \times \mathcal{D}^1(\Sigma) \to \mathcal{D}^1(\Sigma)^\perp$$

given by the rule

$$\Pi_\nabla(V,W) = \text{nor } i^*\nabla_V W$$

is called the <u>shape tensor</u>.

[Note: Π_∇ is $C^\infty(\Sigma)$-bilinear. To see this, observe first that $i^*\nabla_V W$ is $C^\infty(\Sigma)$-linear in V, hence so is Π_∇. On the other hand,

$$i^*\nabla_V(fW) = (Vf)W + fi^*\nabla_V W,$$

thus

$$\begin{aligned}\Pi_\nabla(V,fW) &= \text{nor } i^*\nabla_V(fW) \\ &= \text{nor}(fi^*\nabla_V W) \\ &= f \text{ nor}(i^*\nabla_V W) = f\Pi_\nabla(V,W).]\end{aligned}$$

Summary: $\forall\ V,W\in\mathcal{D}^1(\Sigma)$,

$$i^*\nabla_V W = \overline{\nabla}_V W + \Pi_\nabla(V,W).$$

<u>LEMMA</u> If ∇ is torsion free, then $\forall\ V,W\in\mathcal{D}^1(\Sigma)$,

$$i^*\nabla_V W - i^*\nabla_W V = [V,W].$$

Since

$$\begin{aligned}&i^*\nabla_V W - i^*\nabla_W V - [V,W] \\ &= \overline{\nabla}_V W - \overline{\nabla}_W V - [V,W] + \Pi_\nabla(V,W) - \Pi_\nabla(W,V),\end{aligned}$$

it follows that if ∇ is torsion free, then $\overline{\nabla}$ is also torsion free and Π_∇ is symmetric.

<u>LEMMA</u> Suppose that $\nabla \in \text{con}_g TM$ -- then $\forall\ V \in \mathcal{D}^1(\Sigma)$, $\forall\ X,Y \in \mathcal{D}^1(\Sigma:M)$,

$$Vg(X,Y) = g(i^*\nabla_V X,Y) + g(X,i^*\nabla_V Y).$$

Application: We have

$$\nabla \in \text{con}_g TM \Rightarrow \overline{\nabla} \in \text{con}_{\overline{g}} T\Sigma.$$

Therefore, if ∇ is the metric connection associated with g, then $\overline{\nabla}$ is the metric connection associated with $\overline{g}$.

<u>LEMMA</u> The assignment

$$\begin{cases} \mathcal{D}^1(\Sigma) \times \mathcal{D}^1(\Sigma)^\perp \to \mathcal{D}^1(\Sigma)^\perp \\ \quad (V,N) \longrightarrow \text{nor } i^*\nabla_V N \end{cases}$$

defines a connection $\nabla^\perp$ on $T\Sigma^\perp$.

Given $N \in \mathcal{D}^1(\Sigma)^\perp$, write

$$i^*\nabla_V N = \tan i^*\nabla_V N + \text{nor } i^*\nabla_V N$$

or still,

$$i^*\nabla_V N = - S_N V + \nabla_V^\perp N,$$

where

$$\begin{cases} S_N:\mathcal{D}^1(\Sigma) \to \mathcal{D}^1(\Sigma) \\ \quad S_N V = - \tan i^*\nabla_V N. \end{cases}$$

<u>LEMMA</u> Suppose that $\nabla \in \text{con}_g TM$ — then

$$S_N V = \bar{g}^{\#}(g(N, \Pi_\nabla(V, \_\_))).$$

[$\forall\ W \in \mathcal{D}^1(\Sigma)$,

$$\bar{g}(S_N V, W) = - g(i^* \nabla_V N, W)$$

$$= - Vg(N,W) + g(N, i^* \nabla_V W)$$

$$= g(N, i^* \nabla_V W)$$

$$= g(N, \text{nor } i^* \nabla_V W)$$

$$= g(N, \Pi_\nabla(V,W)).$$

Therefore, as elements of $\mathcal{D}_1(\Sigma)$,

$$\bar{g}(S_N V, \_\_) = g(N, \Pi_\nabla(V, \_\_)).$$

Consequently,

$$S_N V = \bar{g}^{\#}(\bar{g}(S_N V, \_\_))$$

$$= \bar{g}^{\#}(g(N, \Pi_\nabla(V, \_\_))).]$$

Remark: If ∇ is the metric connection associated with g, then Π_∇ is symmetric, hence

$$\bar{g}(S_N V, W) = g(N, \Pi_\nabla(V,W))$$

$$= g(N, \Pi_\nabla(W,V))$$

$$= \bar{g}(S_N W, V)$$

$$= \bar{g}(V, S_N W).$$

I.e.: S_N is selfadjoint.

Let $\nabla \in \text{con } TM$ be arbitrary -- then $\forall\ V_1, V_2 \in \mathcal{D}^1(\Sigma)$ & $\forall\ W \in \mathcal{D}^1(\Sigma)$,

$$
\begin{aligned}
R(V_1,V_2)W
&= i^*\nabla_{V_1} i^*\nabla_{V_2} W - i^*\nabla_{V_2} i^*\nabla_{V_1} W - i^*\nabla_{[V_1,V_2]} W \\
&= i^*\nabla_{V_1}(\overline{\nabla}_{V_2} W + \Pi_\nabla(V_2,W)) \\
&\qquad - i^*\nabla_{V_2}(\overline{\nabla}_{V_1} W + \Pi_\nabla(V_1,W)) \\
&\qquad - \overline{\nabla}_{[V_1,V_2]} W - \Pi_\nabla([V_1,V_2],W) \\
&= \overline{\nabla}_{V_1}\overline{\nabla}_{V_2} W + \Pi_\nabla(V_1,\overline{\nabla}_{V_2} W) \\
&\qquad - S_{\Pi_\nabla(V_2,W)} V_1 + \nabla^{\perp}_{V_1}\Pi_\nabla(V_2,W) \\
&\quad - \overline{\nabla}_{V_2}\overline{\nabla}_{V_1} W - \Pi_\nabla(V_2,\overline{\nabla}_{V_1} W) \\
&\qquad + S_{\Pi_\nabla(V_1,W)} V_2 - \nabla^{\perp}_{V_2}\Pi_\nabla(V_1,W) \\
&\quad - \overline{\nabla}_{[V_1,V_2]} W - \Pi_\nabla([V_1,V_2],W) \\
&= \overline{R}(V_1,V_2)W - S_{\Pi_\nabla(V_2,W)} V_1 + S_{\Pi_\nabla(V_1,W)} V_2 \\
&\quad + \Pi_\nabla(V_1,\overline{\nabla}_{V_2} W) - \Pi_\nabla(V_2,\overline{\nabla}_{V_1} W) - \Pi_\nabla([V_1,V_2],W) \\
&\qquad + \nabla^{\perp}_{V_1}\Pi_\nabla(V_2,W) - \nabla^{\perp}_{V_2}\Pi_\nabla(V_1,W).
\end{aligned}
$$

Write

$$(\nabla^{\perp}_{V_1}\Pi_{\nabla})(V_2,W)$$

$$= \nabla^{\perp}_{V_1}\Pi_{\nabla}(V_2,W) - \Pi_{\nabla}(\overline{\nabla}_{V_1}V_2,W) - \Pi_{\nabla}(V_2,\overline{\nabla}_{V_1}W)$$

and

$$(\nabla^{\perp}_{V_2}\Pi_{\nabla})(V_1,W)$$

$$= \nabla^{\perp}_{V_2}\Pi_{\nabla}(V_1,W) - \Pi_{\nabla}(\overline{\nabla}_{V_2}V_1,W) - \Pi_{\nabla}(V_1,\overline{\nabla}_{V_2}W).$$

Then

$$(\nabla^{\perp}_{V_1}\Pi_{\nabla})(V_2,W) + \Pi_{\nabla}(\overline{\nabla}_{V_1}V_2,W)$$

$$= \nabla^{\perp}_{V_1}\Pi_{\nabla}(V_2,W) - \Pi_{\nabla}(V_2,\overline{\nabla}_{V_1}W)$$

and

$$-(\nabla^{\perp}_{V_2}\Pi_{\nabla})(V_1,W) - \Pi_{\nabla}(\overline{\nabla}_{V_2}V_1,W)$$

$$= -\nabla^{\perp}_{V_2}\Pi_{\nabla}(V_1,W) + \Pi_{\nabla}(V_1,\overline{\nabla}_{V_2}W).$$

Therefore

$$R(V_1,V_2)W$$

$$= \overline{R}(V_1,V_2)W - S_{\Pi_{\nabla}(V_2,W)}V_1 + S_{\Pi_{\nabla}(V_1,W)}V_2$$

$$+ (\nabla^{\perp}_{V_1}\Pi_{\nabla})(V_2,W) - (\nabla^{\perp}_{V_2}\Pi_{\nabla})(V_1,W)$$

$$+ \Pi_{\nabla}(\overline{\nabla}_{V_1}V_2,W) - \Pi_{\nabla}(\overline{\nabla}_{V_2}V_1,W) - \Pi_{\nabla}([V_1,V_2],W)$$

or still,

$$R(V_1,V_2)W$$

$$= \overline{R}(V_1,V_2)W - S_{\Pi_\nabla(V_2,W)}V_1 + S_{\Pi_\nabla(V_1,W)}V_2$$

$$+ (\nabla^\perp_{V_1}\Pi_\nabla)(V_2,W) - (\nabla^\perp_{V_2}\Pi_\nabla)(V_1,W)$$

$$+ \Pi_\nabla(\overline{T}(V_1,V_2),W).$$

Corollaries

- Suppose that $\nabla \in \text{con}_g TM$ -- then $\forall\ W_1,W_2 \in \mathcal{D}^1(\Sigma)$,

$$g(W_1,R(V_1,V_2)W_2)$$

$$= \overline{g}(W_1,\overline{R}(V_1,V_2)W_2)$$

$$+ g(\Pi_\nabla(V_1,W_2),\Pi_\nabla(V_2,W_1)) - g(\Pi_\nabla(V_1,W_1),\Pi_\nabla(V_2,W_2)).$$

- Suppose that $\nabla \in \text{con}_g TM$ -- then $\forall\ N \in \mathcal{D}^1(\Sigma)^\perp$,

$$g(N,R(V_1,V_2)W)$$

$$= g(N,(\nabla^\perp_{V_1}\Pi_\nabla)(V_2,W)) - g(N,(\nabla^\perp_{V_2}\Pi_\nabla)(V_1,W))$$

$$+ \overline{g}(S_N\overline{T}(V_1,V_2),W).$$

Let $\nabla \in \text{con}\ TM$ be arbitrary -- then $\forall\ V_1,V_2 \in \mathcal{D}^1(\Sigma)$ & $\forall\ N \in \mathcal{D}^1(\Sigma)^\perp$,

$$R(V_1,V_2)N$$

$$= i^*\nabla_{V_1} i^*\nabla_{V_2} N - i^*\nabla_{V_2} i^*\nabla_{V_1} N - i^*\nabla_{[V_1,V_2]} N$$

$$= i^*\nabla_{V_1}(- S_N V_2 + \nabla^\perp_{V_2} N)$$

$$- i^*\nabla_{V_2}(- S_N V_1 + \nabla^\perp_{V_1} N)$$

$$+ S_N[V_1,V_2] - \nabla^\perp_{[V_1,V_2]} N$$

$$= - \overline{\nabla}_{V_1} S_N V_2 - \Pi_\nabla (V_1, S_N V_2)$$

$$- S_{\nabla^\perp_{V_2} N} V_1 + \nabla^\perp_{V_1} \nabla^\perp_{V_2} N$$

$$+ \overline{\nabla}_{V_2} S_N V_1 + \Pi_\nabla (V_2, S_N V_1)$$

$$+ S_{\nabla^\perp_{V_1} N} V_2 - \nabla^\perp_{V_2} \nabla^\perp_{V_1} N$$

$$+ S_N[V_1 V_2] - \nabla^\perp_{[V_1,V_2]} N$$

$$= R^\perp(V_1,V_2)N - \overline{T}_{S_N}(V_1,V_2)$$

$$+ S_{\nabla^\perp_{V_1} N} V_2 - S_{\nabla^\perp_{V_2} N} V_1$$

$$+ \Pi_\nabla (V_2, S_N V_1) - \Pi_\nabla (V_1, S_N V_2),$$

where, by definition,

$$\overline{T}_{S_N}(V_1,V_2) = \overline{\nabla}_{V_1} S_N V_2 - \overline{\nabla}_{V_2} S_N V_1 - S_N[V_1,V_2].$$

<u>Corollaries</u>

- Suppose that $\nabla \in \text{con}_g TM$ -- then $\forall\ N_1, N_2 \in \mathcal{D}^1(\Sigma)^\perp$,

$$g(N_1, R(V_1,V_2)N_2)$$

$$= g(N_1, R^{\perp}(V_1,V_2)N_2)$$

$$+ \bar{g}(S_{N_1}V_2, S_{N_2}V_1) - \bar{g}(S_{N_1}V_1, S_{N_2}V_2).$$

• Suppose that $\nabla \in \text{con}_g TM$ -- then $\forall\ W \in \mathcal{D}^1(\Sigma)$,

$$g(W, R(V_1,V_2)N)$$

$$= g(\nabla^{\perp}_{V_1}N, \Pi_{\nabla}(V_2,W)) - g(\nabla^{\perp}_{V_2}N, \Pi_{\nabla}(V_1,W))$$

$$- \bar{g}(\bar{T}_{S_N}(V_1,V_2), W).$$

Section 18: Extrinsic Curvature Let M be a connected C^∞ manifold of dimension n. Maintaining the assumptions and notation of the previous section, specialize and take for Σ a hypersurface (thus d = n-1) -- then the fibers of $T\Sigma^\perp$ are 1-dimensional and there are just two possibilities:

$$\begin{cases} (+):g|T\Sigma^\perp > 0 \\ (-1):g|T\Sigma^\perp < 0. \end{cases}$$

Definition: A unit normal to Σ is a section $\underline{n}:\Sigma \to T\Sigma^\perp$ such that

$$\begin{cases} (+):g(\underline{n},\underline{n}) = +1 \\ (-):g(\underline{n},\underline{n}) = -1. \end{cases}$$

Assumption: Σ admits a unit normal.

[Note: $\underline{n}$ always exists locally but the Möbius strip in $\underline{R}^3$ shows that $\underline{n}$ need not exist globally.]

Criterion If M is orientable, then Σ is orientable iff Σ admits a unit normal.

Definition: Let $\nabla \in \text{con } TM$ -- then the extrinsic curvature of the pair (Σ,∇) is the tensor $\kappa_\nabla \in \mathcal{D}_2^0(\Sigma)$ given by the rule

$$\Pi_\nabla(V,W) = \kappa_\nabla(V,W)\underline{n}.$$

[Note: κ_∇ depends on $\underline{n}$ (replacing $\underline{n}$ by $-\underline{n}$ changes the sign of κ_∇).]

Remark: If ∇ is torsion free, then Π_∇ is symmetric, thus so is κ_∇.

LEMMA Suppose that $\nabla \in \text{con}_g TM$ -- then $\nabla_V^\perp \underline{n} = 0$, hence

$$i^*\nabla_V \underline{n} = -S_{\underline{n}} V.$$

[This is because

$$0 = Vg(\underline{n},\underline{n}) = 2g(\nabla_V^{\perp}\underline{n},\underline{n}).]$$

Let $\nabla \in \text{con}_g TM$ — then

$$g(\Pi_\nabla(V,W),\underline{n}) = \kappa_\nabla(V,W)g(\underline{n},\underline{n})$$

or still,

$$\overline{g}(S_{\underline{n}}V,W) = \kappa_\nabla(V,W)g(\underline{n},\underline{n})$$

$$\Rightarrow$$

$$\kappa_\nabla(V,W) = \overline{g}(S_{\underline{n}}V,W)g(\underline{n},\underline{n}).$$

To simplify, at this point we shall assume that $\exists$ an orthonormal frame $\{E_0,E_1,\ldots,E_{n-1}\}$ such that $\forall\ x \in \Sigma$, $\{E_1|_x,\ldots,E_{n-1}|_x\}$ is an orthonormal basis for $T_x\Sigma$ and $\underline{R}E_0|_x = T_x\Sigma^{\perp}$.

[Note: In what follows, $\underline{n} = E_0|\Sigma$.]

Notation: Indices a,b,c run from 1 to n-1.

Agreeing to use an overbar for pullback to Σ, let $\nabla \in \text{con}_g TM$ -- then $\forall\ V \in \mathcal{D}^1(\Sigma)$,

$$\begin{cases} \overline{\nabla}_V E_b = \overline{\omega}^a{}_b(V)E_a \\ \Pi_\nabla(V,E_b) = \overline{\omega}^0{}_b(V)E_0 \\ S_{E_0}V = -\overline{\omega}^a{}_0(V)E_a. \end{cases}$$

Put

$$\kappa_{ab} = \kappa_\nabla(E_a,E_b).$$

Then

$$\varkappa_{ab} = \bar{g}(S_{E_0}E_a,E_b)\varepsilon_0$$

$$= -\bar{g}(\bar{\omega}^c{}_0(E_a)E_c,E_b)\varepsilon_0$$

$$= -\varepsilon_0\varepsilon_b\bar{\omega}^b{}_0(E_a)$$

$\Rightarrow$

$$\bar{\omega}^b{}_0 = \bar{\omega}^b{}_0(E_a)\bar{\omega}^a$$

$$= -\varepsilon_0\varepsilon_b\varkappa_{ab}\bar{\omega}^a$$

$\Rightarrow$

$$\bar{\omega}^0{}_b = -\varepsilon_0\varepsilon_b\bar{\omega}^b{}_0 = \varkappa_{ab}\bar{\omega}^a.$$

Remark: Suppose that ∇ is the metric connection associated with g -- then

$$d\bar{\omega}^0 = -\bar{\omega}^0{}_a\wedge\bar{\omega}^a = 0$$

$\Rightarrow$

$$\bar{\omega}^0{}_a(V)\bar{\omega}^a(W) = \bar{\omega}^0{}_a(W)\bar{\omega}^a(V).$$

Therefore

$$\varkappa_\nabla(V,W) = \varepsilon_0\bar{g}(S_{E_0}V,W)$$

$$= -\varepsilon_0\bar{g}(\bar{\omega}^a{}_0(V)E_a,\bar{\omega}^b(W)E_b)$$

$$= -\varepsilon_0\varepsilon_a\bar{\omega}^a{}_0(V)\bar{\omega}^a(W)$$

$$= \bar{\omega}^0{}_a(V)\bar{\omega}^a(W)$$

$$= \bar{\omega}^0{}_a(W)\bar{\omega}^a(V)$$

$$= -\varepsilon_0\varepsilon_a\bar{\omega}^a{}_0(W)\bar{\omega}^a(V)$$

$$= \varkappa_{\bar{\nabla}}(W,V),$$

which confirms what we already know to be the case.

[Note: Similar considerations imply that the tensor

$$(V,W) \rightarrow \sum_a \bar{\omega}^a{}_0(V)\bar{\omega}^a{}_0(W)$$

is symmetric.]

In anticipation of later developments, assume henceforth that $g \in \underline{M}_{1,n-1}$ and $\bar{g} > 0$ (so $\varepsilon_0 = -1, \varepsilon_a = 1$).

Let $\nabla \in \text{con}_g TM$ -- then

- $\bar{\Omega}^a{}_b = {}^{(n-1)}\Omega^a{}_b + \bar{\omega}^a{}_0 \wedge \bar{\omega}^0{}_b;$

- $\bar{\Omega}^a{}_0 = d\bar{\omega}^a{}_0 + \bar{\omega}^a{}_b \wedge \bar{\omega}^b{}_0.$

[Note: The $\bar{\omega}^a{}_b$ are the connection 1-forms of $\bar{\nabla}$ but the $\bar{\Omega}^a{}_b$ are not the curvature forms of $\bar{\nabla}$, these being the ${}^{(n-1)}\Omega^a{}_b$.]

Suppose now that ∇ is the metric connection associated with g (thus $\bar{\nabla}$ is the metric connection associated with $\bar{g}$).

Let G be the Einstein tensor -- then

$$\begin{bmatrix} G_{00} = \frac{1}{2}\,\Omega^a{}_b(E_a,E_b) \\ \\ G_{0a} = \Omega^b{}_0(E_b,E_a). \end{bmatrix}$$

[The second relation is trivial. To check the first, note that

$$G_{00} = R_{00} - \frac{1}{2} g_{00} S$$

$$= R_{00} + \frac{1}{2} S$$

$$= R_{00} + \frac{1}{2} (g^{ij} R_{ij})$$

$$= R_{00} + \frac{1}{2} (-R_{00} + \sum_a R_{aa})$$

$$= \frac{1}{2} R_{00} + \frac{1}{2} \sum_a R_{aa}.$$

On the other hand,

$$\frac{1}{2} \Omega^a{}_b (E_a, E_b) = \frac{1}{2} R^a{}_{bab}.$$

And

$$R^a{}_{bab} + R^0{}_{b0b} - R^0{}_{b0b}$$

$$= R_{bb} - R^0{}_{b0b}$$

$\Rightarrow$

$$\frac{1}{2} R^a{}_{bab} = \frac{1}{2} [\sum_b R_{bb} - \sum_b R^0{}_{b0b}].$$

But

$$R^0{}_{b0b} = - \varepsilon_0 \varepsilon_b R^b{}_{00b}$$

$$= R^b{}_{00b}$$

$$= - R^b{}_{0b0}.$$

Therefore

$$-\sum_b R^0{}_{b0b} = \sum_b R^b{}_{0b0}$$

$$= R^0{}_{000} + \sum_b R^b{}_{0b0}$$

$$= R_{00}$$

$\Rightarrow$

$$\frac{1}{2}\,\Omega^a{}_b(E_a,E_b)$$

$$= \frac{1}{2}\,R_{00} + \frac{1}{2}\;\sum_a R_{aa}$$

$$= G_{00}.]$$

Set $q = \bar{g}$ and given symmetric tensors $T,S\in\mathcal{D}^0_2(\Sigma)$, put

$$[T,S]_q = q[{}^0_2](T,S) = T^{ab}S_{ab}.$$

In particular:

$$\mathrm{tr}_q(T) = T^a{}_a = q^{ab}T_{ab} = [q,T]_q.$$

Observation: If $T\in\mathcal{D}^0_2(\Sigma)$ is symmetric, then

$$[T,T]_q = T^a{}_bT^b{}_a$$

$$= \sum_{a,b}(T_{ab})^2.$$

Returning to G, in terms of the extrinsic curvature, we have

$$\left[\begin{array}{l} \bar{G}_{00} = \frac{1}{2}\,S(q) + \frac{1}{2}[(\mathrm{tr}_q(\kappa_\nabla))^2 - [\kappa_\nabla,\kappa_\nabla]_q] \\ \bar{G}_{0a} = \bar{\nabla}_b\kappa_{ab} - \bar{\nabla}_a\mathrm{tr}_q(\kappa_\nabla). \end{array}\right.$$

[Note: $S(q)$ is the scalar curvature of q.]

To begin with,

$$\bar{G}_{00} = \frac{1}{2}\,\bar{\Omega}^a{}_b(E_a,E_b)$$

or still,

$$\bar{G}_{00} = \frac{1}{2}\,{}^{(n-1)}\Omega^a{}_b(E_a,E_b) + \frac{1}{2}\,(\bar{\omega}^a{}_0\wedge\bar{\omega}^0{}_b)(E_a,E_b).$$

From the definitions,

$$\frac{1}{2}\,{}^{(n-1)}\Omega^a{}_b(E_a,E_b) = \frac{1}{2}\,S(q).$$

In addition,

$$(\bar{\omega}^a{}_0\wedge\bar{\omega}^0{}_b)(E_a,E_b)$$

$$= \bar{\omega}^a{}_0(E_a)\bar{\omega}^0{}_b(E_b) - \bar{\omega}^a{}_0(E_b)\bar{\omega}^0{}_b(E_a)$$

$$= \varkappa_{aa}\varkappa_{bb} - \varkappa_{ba}\varkappa_{ab}$$

$$= \varkappa_{aa}\varkappa_{bb} - (\varkappa_{ab})^2$$

$$= (\mathrm{tr}_q(\varkappa_\nabla))^2 - [\varkappa_\nabla,\varkappa_\nabla]_q.$$

Turning to the formula for $\bar{G}_{0a}$, write

$$\bar{\Omega}^b{}_0(E_b,E_a)$$

$$= d\bar{\omega}^b{}_0(E_b,E_a) + (\bar{\omega}^b{}_c\wedge\bar{\omega}^c{}_0)(E_b,E_a)$$

$$= d(\varkappa_{cb}\bar{\omega}^c)(E_b,E_a) + \varkappa_{c'c}(\bar{\omega}^b{}_c\wedge\bar{\omega}^{c'})(E_b,E_a)$$

$$= (d\varkappa_{cb}\wedge\bar{\omega}^c)(E_b,E_a) + \varkappa_{cb}d\bar{\omega}^c(E_b,E_a)$$

$$+ \varkappa_{c'c}(\bar{\omega}^b{}_c\wedge\bar{\omega}^{c'})(E_b,E_a).$$

• We have

$$(d\varkappa_{cb}\wedge\overline{\omega}^{c})(E_b,E_a)$$

$$= d\varkappa_{cb}(E_b)\overline{\omega}^{c}(E_a) - d\varkappa_{cb}(E_a)\overline{\omega}^{c}(E_b)$$

$$= d\varkappa_{ab}(E_b) - d\varkappa_{bb}(E_a)$$

$$= E_b\varkappa_{\nabla}(E_a,E_b) - E_a\varkappa_{\nabla}(E_b,E_b).$$

• We have

$$\varkappa_{cb}d\overline{\omega}^{c}(E_b,E_a)$$

$$= \varkappa_{cb}(E_b\overline{\omega}^{c}(E_a) - E_a\overline{\omega}^{c}(E_b) - \overline{\omega}^{c}([E_b,E_a]))$$

$$= -\varkappa_{cb}\overline{\omega}^{c}([E_b,E_a])$$

$$= -\varkappa_{cb}[E_b,E_a]^{c}$$

$$= \varkappa_{cb}[E_a,E_b]^{c}$$

$$= \varkappa_{\nabla}(E_c,E_b)[E_a,E_b]^{c}$$

$$= \varkappa_{\nabla}(E_b,E_c)[E_a,E_b]^{c}$$

$$= \varkappa_{\nabla}(E_b,[E_a,E_b]^{c}E_c)$$

$$= \varkappa_{\nabla}(E_b,[E_a,E_b])$$

$$= \varkappa_{\nabla}(E_b,\overline{\nabla}_{E_a}E_b - \overline{\nabla}_{E_b}E_a)$$

$$= \varkappa_{\nabla}(E_b,\overline{\nabla}_{E_a}E_b) - \varkappa_{\nabla}(E_b,\overline{\nabla}_{E_b}E_a).$$

• We have

$$\varkappa_{c'c}(\bar{\omega}^{b}{}_{c}\wedge\bar{\omega}^{c'})(E_b,E_a)$$

$$= \varkappa_{c'c}(\bar{\omega}^{b}{}_{c}(E_b)\bar{\omega}^{c'}(E_a) - \bar{\omega}^{b}{}_{c}(E_a)\bar{\omega}^{c'}(E_b))$$

$$= \varkappa_{ac}\bar{\omega}^{b}{}_{c}(E_b) - \varkappa_{bc}\bar{\omega}^{b}{}_{c}(E_a)$$

$$= \varkappa_{\nabla}(E_a,E_c)\bar{\omega}^{b}{}_{c}(E_b) - \varkappa_{\nabla}(E_b,E_c)\bar{\omega}^{b}{}_{c}(E_a)$$

$$= \varkappa_{\nabla}(E_a,\bar{\omega}^{b}{}_{c}(E_b)E_c) - \varkappa_{\nabla}(E_b,\bar{\omega}^{b}{}_{c}(E_a)E_c)$$

$$= \varkappa_{\nabla}(E_b,\bar{\omega}^{c}{}_{b}(E_a)E_c) - \varkappa_{\nabla}(E_a,\bar{\omega}^{c}{}_{b}(E_b)E_c)$$

$$= \varkappa_{\nabla}(E_b,\bar{\nabla}_{E_a}E_b) - \varkappa_{\nabla}(E_a,\bar{\nabla}_{E_b}E_b).$$

Therefore $\bar{\Omega}^{b}{}_{0}(E_b,E_a)$ equals

$$E_b\varkappa_{\nabla}(E_a,E_b) - \varkappa_{\nabla}(E_b,\bar{\nabla}_{E_b}E_a) - \varkappa_{\nabla}(E_a,\bar{\nabla}_{E_b}E_b)$$

minus

$$E_a\varkappa_{\nabla}(E_b,E_b) - \varkappa_{\nabla}(E_b,\bar{\nabla}_{E_a}E_b) - \varkappa_{\nabla}(E_b,\bar{\nabla}_{E_a}E_b).$$

Since $\varkappa_{\nabla}$ is symmetric,

$$\begin{cases} \varkappa_{\nabla}(E_b,\bar{\nabla}_{E_b}E_a) = \varkappa_{\nabla}(\bar{\nabla}_{E_b}E_a,E_b) \\ \varkappa_{\nabla}(E_b,\bar{\nabla}_{E_a}E_b) = \varkappa_{\nabla}(\bar{\nabla}_{E_a}E_b,E_b). \end{cases}$$

But

$$\begin{cases} (\bar{\nabla}_{E_b}\varkappa_{\nabla})(E_a,E_b) = E_b\varkappa_{\nabla}(E_a,E_b) - \varkappa_{\nabla}(\bar{\nabla}_{E_b}E_a,E_b) - \varkappa_{\nabla}(E_a,\bar{\nabla}_{E_b}E_b) \\ (\bar{\nabla}_{E_a}\varkappa_{\nabla})(E_b,E_b) = E_a\varkappa_{\nabla}(E_b,E_b) - \varkappa_{\nabla}(\bar{\nabla}_{E_a}E_b,E_b) - \varkappa_{\nabla}(E_b,\bar{\nabla}_{E_a}E_b). \end{cases}$$

Therefore $\bar{\Omega}^{b}{}_{0}(E_b, E_a)$ equals

$$(\bar{\nabla}_{E_b} \varkappa_{\bar{\nabla}})(E_a, E_b) - (\bar{\nabla}_{E_a} \varkappa_{\bar{\nabla}})(E_b, E_b)$$

or still,

$$\bar{\nabla}_b \varkappa_{ab} - \bar{\nabla}_a \varkappa_{bb}.$$

I.e.:

$$\bar{G}_{0a} = \bar{\nabla}_b \varkappa_{ab} - \bar{\nabla}_a \mathrm{tr}_q(\varkappa_{\bar{\nabla}}).$$

Section 19: Hodge Conventions Let M be a connected C^∞ manifold of dimension n.

Rappel: If φ is a density of weight 1, i.e., if φ is a section of the density line bundle $L_{den}(M) \to M$, then one can associate with φ a Radon measure m_φ:

$$\int_M f\,dm_\varphi = \int_M f\varphi \qquad (f \in C_c(M)).$$

Let $g \in \underline{M}$ -- then $|g|^{1/2}$ is a density of weight 1, from which $m_{|g|^{1/2}}$. Assume now that M is orientable with orientation μ -- then there is a unique n-form $\text{vol}_g \in \Lambda^n M$ such that $\forall\, x \in M$ and every oriented orthonormal basis for $T_x M$,

$$\text{vol}_g\big|_x(E_1, \ldots, E_n) = 1.$$

[Note: In a connected open set $U \subset M$ equipped with coordinates $x^1, \ldots, x^n$ consistent with μ, i.e., such that

$$\left[\ \frac{\partial}{\partial x^1}\Big|_x, \ldots, \frac{\partial}{\partial x^n}\Big|_x\ \right] \in \mu_x \ \forall\, x \in U,$$

we have

$$\text{vol}_g = |g|^{1/2} dx^1 \wedge \ldots \wedge dx^n.]$$

<u>FACT</u> $\forall\, f \in C_c(M)$,

$$\int_M f\,dm_{|g|^{1/2}} = \int_M f\,\text{vol}_g.$$

Remark: Let Σ be a hypersurface (subject to the standing assumption that $\bar{g}$ is a semiriemannian structure on Σ). Suppose that Σ admits a unit normal $\underline{n}$ -- then the pair $(\mu, \underline{n})$ determines an orientation $\bar{\mu}$ of Σ and

$$\text{vol}_{\bar{g}} = i^*(\iota_{\underline{n}} \text{vol}_g).$$

<u>FACT</u> $\forall\ X \in \mathcal{D}^1(\Sigma:M)$,

$$i^*(\iota_X \text{vol}_g) = g(\underline{n},\underline{n})\, g(X,\underline{n})\, \text{vol}_{\bar{g}}\ .$$

<u>LEMMA</u> Let $X \in \mathcal{D}^1(M)$ -- then

$$L_X \text{vol}_g = (\text{div } X)\text{vol}_g.$$

[Working locally, we have

$$L_X(|g|^{1/2} dx^1 \wedge \ldots \wedge dx^n)$$

$$= X|g|^{1/2} dx^1 \wedge \ldots \wedge dx^n + |g|^{1/2} \sum_i dx^1 \wedge \ldots \wedge d(Xx^i) \wedge \ldots \wedge dx^n$$

$$= X^i |g|^{1/2}_{,i} dx^1 \wedge \ldots \wedge dx^n$$

$$+ |g|^{1/2} \sum_i dx^1 \wedge \ldots \wedge d(X^j \frac{\partial}{\partial x^j} x^i) \wedge \ldots \wedge dx^n$$

$$= (X^i |g|^{1/2}_{,i} + |g|^{1/2} X^i_{,i}) dx^1 \wedge \ldots \wedge dx^n$$

$$= \frac{1}{|g|^{1/2}} (X^i |g|^{1/2})_{,i} \text{vol}_g$$

$$= (\text{div } X)\text{vol}_g.]$$

[Note: By contrast,

$$\nabla_X \text{vol}_g = 0$$

if ∇ is the metric connection. Proof:

$$\nabla_X(|g|^{1/2} dx^1 \wedge \ldots \wedge dx^n)$$

$$= (X^a|g|^{1/2}_{,a} - X^a\Gamma^b_{ab}|g|^{1/2})dx^1 \wedge \ldots \wedge dx^n$$

$$= (X^a|g|^{1/2}_{,a} - X^a(\frac{1}{|g|^{1/2}} |g|^{1/2}_{,a}) |g|^{1/2})dx^1 \wedge \ldots \wedge dx^n$$

$$= 0.]$$

Application: Suppose that X has compact support -- then

$$\int_M (\text{div } X)\text{vol}_g = 0.$$

[In fact,

$$\int_M (\text{div } X)\text{vol}_g = \int_M L_X\text{vol}_g$$

$$= \int_M (\iota_X \circ d + d \circ \iota_X)\text{vol}_g$$

$$= \int_M d(\iota_X\text{vol}_g).$$

But $\iota_X\text{vol}_g$ is a compactly supported (n-1)-form, hence

$$\int_M d(\iota_X\text{vol}_g) = 0$$

by Stokes' theorem.]

[Note: Let $f\in C_c^\infty(M)$ -- then $\forall\ X\in\mathcal{D}^1(M)$, fX has compact support, so

$$0 = \int_M \text{div}(fX)\text{vol}_g = \int_M (Xf + f(\text{div } X))\text{vol}_g,$$

or, in index notation,

$$0 = \int_M \nabla_i(fX^i)\text{vol}_g = \int_M ((\nabla_i f)X^i + f(\nabla_i X^i))\text{vol}_g.]$$

Example (Yano's Formula): Working with the metric connection, let

let $X \in \mathcal{D}^1(M)$ — then

$$\nabla_b \nabla_a X^i - \nabla_a \nabla_b X^i$$

$$= X^i{}_{;a;b} - X^i{}_{;b;a}$$

$$= R^i{}_{jba} X^j$$

$\Rightarrow$

$$X^i{}_{;a;i} - X^i{}_{;i;a}$$

$$= R^i{}_{jia} X^j$$

$$= R_{ja} X^j = R_{ia} X^i = R_{ai} X^i$$

$\Rightarrow$

$$X^a X^i{}_{;a;i} - X^a X^i{}_{;i;a}$$

$$= R_{ai} X^a X^i = \text{Ric}(X,X).$$

In the relation

$$\text{div}(fX) = f\text{div } X + Xf,$$

take $f = \text{div } X$ to get

$$\text{div}((\text{div } X)X) = (\text{div } X)^2 + X(\text{div } X)$$

$$= (\text{div } X)^2 + d(\text{div } X)(X).$$

Since $\text{div } X = X^i{}_{;i}$, it follows that

$$d(\text{div } X)(X) = X^a X^i{}_{;i;a}.$$

Therefore

$$\text{Ric}(X,X) = X^a X^i{}_{;a;i} - d(\text{div } X)(X)$$

or still,

$$\mathrm{Ric}(X,X) - (\mathrm{div}\ X)^2$$

$$= X^a X^i{}_{;a;i} - \mathrm{div}((\mathrm{div}\ X)X).$$

Write

$$\nabla_i(X^a(\nabla_a X^i)) - (\nabla_i X^a)(\nabla_a X^i)$$

$$= (\nabla_i X^a)(\nabla_a X^i) + X^a \nabla_i \nabla_a X^i - (\nabla_i X^a)(\nabla_a X^i)$$

$$= X^a X^i{}_{;a;i}$$

and then note that

$$\mathrm{div}(\nabla_X X) = \nabla_i((\nabla_X X)^i)$$

$$= \nabla_i((\nabla_{X^a \partial_a} X)^i)$$

$$= \nabla_i((X^a \nabla_{\partial_a} X)^i)$$

$$= \nabla_i(X^a(\nabla_{\partial_a} X)^i)$$

$$= \nabla_i(X^a(X^i{}_{;a}))$$

$$= \nabla_i(X^a(\nabla_a X^i)).$$

Therefore

$$\mathrm{Ric}(X,X) - (\mathrm{div}\ X)^2$$

$$= \mathrm{div}(\nabla_X X) - (\nabla_i X^a)(\nabla_a X^i) - \mathrm{div}((\mathrm{div}\ X)X).$$

I.e.:

$$\text{Ric}(X,X) - (\text{div } X)^2 + (\nabla_i X^a)(\nabla_a X^i)$$

$$= \text{div}(\nabla_X X) - \text{div}((\text{div } X)X).$$

To understand the term

$$(\nabla_i X^a)(\nabla_a X^i),$$

recall that $\nabla X \in \mathcal{D}_1^1(M)$ or, equivalently,

$$\begin{cases} \nabla X \in \text{Hom}_{C^\infty(M)}(\mathcal{D}^1(M), \mathcal{D}^1(M)) \\ \nabla X(Y) = \nabla_Y X, \end{cases}$$

thus

$$\begin{cases} \nabla X \circ \nabla X \in \text{Hom}_{C^\infty(M)}(\mathcal{D}^1(M), \mathcal{D}^1(M)) \\ (\nabla X \circ \nabla X)(Y) = \nabla_{\nabla_Y X} X. \end{cases}$$

Claim:

$$\text{tr}(\nabla X \circ \nabla X) \quad (\text{a.k.a. } C_1^1(\nabla X)^2)$$

equals

$$(\nabla_i X^a)(\nabla_a X^i).$$

Indeed

$$(\nabla X \circ \nabla X)(\partial_i) = \nabla_{\nabla_i X} X$$

$$= \nabla_{X^a{}_{;i}\partial_a} X$$

$$= X^a_{;i} \nabla_a X$$

$$= X^a_{;i} X^j_{;a} \partial_j$$

$\Rightarrow$

$$\text{tr}(\nabla X \circ \nabla X) = X^a_{;i} X^i_{;a} = (\nabla_i X^a)(\nabla_a X^i).$$

So, if X has compact support, then

$$\int_M [\text{Ric}(X,X) - (\text{div } X)^2 + \text{tr}(\nabla X \circ \nabla X)]\text{vol}_g = 0.$$

Remark: We have

$$\begin{cases} (L_X g)_{ij} = \nabla_j X_i + \nabla_i X_j \\ (L_X g)^{ij} = \nabla^j X^i + \nabla^i X^j. \end{cases}$$

Therefore

$$g[{}^0_2](L_X g, L_X g) = (L_X g)^{ij}(L_X g)_{ij}$$

$$= \sum_{i,j} (\nabla^j X^i + \nabla^i X^j)(\nabla_j X_i + \nabla_i X_j)$$

$$= \sum_{i,j} (\nabla^j X^i)(\nabla_j X_i) + \sum_{i,j} (\nabla^i X^j)(\nabla_i X_j)$$

$$+ \sum_{i,j} (\nabla^j X^i)(\nabla_i X_j) + \sum_{i,j} (\nabla^i X^j)(\nabla_j X_i)$$

$$= 2 \sum_{i,j} (\nabla^j X^i)(\nabla_j X_i) + 2 \sum_{i,j} (\nabla^j X^i)(\nabla_i X_j).$$

• From the definitions,

$$(\nabla X)^i{}_j = \nabla_j X^i$$

$\Rightarrow$

$$g[{}^1_1](\nabla X, \nabla X) = (\nabla X)^{ij}(\nabla X)_{ij}$$

$$= (\nabla^j X^i)(\nabla_j X_i).$$

• From the definitions,

$$(\nabla_i X^j)(\nabla_j X^i)$$

$$= \nabla_i g^{jk} X_k \nabla_j X^i$$

$$= \nabla_i X_k g^{jk} \nabla_j X^i$$

$$= \nabla_i X_k g^{kj} \nabla_j X^i$$

$$= \nabla_i X_k \nabla^k X^i$$

$$= (\nabla_i X_j)(\nabla^j X^i)$$

$\Rightarrow$

$$\mathrm{tr}(\nabla X \circ \nabla X) = (\nabla_i X_j)(\nabla^j X^i) = (\nabla^j X^i)(\nabla_i X_j).$$

Therefore

$$\mathrm{tr}(\nabla X \circ \nabla X) = \frac{1}{2}\, g[{}^0_2](L_X g, L_X g) - g[{}^1_1](\nabla X, \nabla X).$$

<u>FACT</u> We have

$$\mathrm{tr}(\nabla X \circ \nabla X) = g[{}^1_1](\nabla X, \nabla X) - \frac{1}{2}\, g[{}^0_2](dg^\flat X, dg^\flat X).$$

[Observe that

$$(dg^\flat X)_{ji} = \nabla_j X_i - \nabla_i X_j.]$$

The material in Section 3 can be applied to the triple (M,g,μ) pointwise, hence need not be repeated here.

This said, consider the star operator

$$*:\Lambda^p M \to \Lambda^{n-p} M.$$

Then

$$\begin{cases} \alpha\wedge*\beta = g(\alpha,\beta)\mathrm{vol}_g \\ **\alpha = (-1)^{\iota}(-1)^{p(n-p)}\alpha \end{cases}$$

and

$$\begin{cases} *f = f\mathrm{vol}_g \\ *(f\mathrm{vol}_g) = (-1)^{\iota} f. \end{cases}$$

Example: $\forall\ X\in\mathcal{D}^1(M)$,

$$*(\mathrm{div}\ X) = (\mathrm{div}\ X)\mathrm{vol}_g = L_X\mathrm{vol}_g.$$

<u>LEMMA</u> Let ∇ be the metric connection — then $\forall\ X\in\mathcal{D}^1(M)$, the diagram

$$\begin{array}{ccc} \Lambda^p M & \xrightarrow{*} & \Lambda^{n-p} M \\ \nabla_X \downarrow & & \downarrow \nabla_X \\ \Lambda^p M & \xrightarrow[*]{} & \Lambda^{n-p} M \end{array}$$

commutes.

[Fix $\beta\in\Lambda^p M$ — then $\forall\ \alpha\in\Lambda^p M$,

$$\alpha\wedge*\beta = g(\alpha,\beta)\mathrm{vol}_g$$

$$\Rightarrow$$

$$\nabla_X(\alpha\wedge*\beta) = \nabla_X(g(\alpha,\beta)\mathrm{vol}_g)$$

$$= \nabla_X(g(\alpha,\beta))\mathrm{vol}_g + g(\alpha,\beta)\nabla_X \mathrm{vol}_g$$

$$= \nabla_X(g(\alpha,\beta))\mathrm{vol}_g$$

$\Rightarrow$

$$\nabla_X \alpha \wedge *\beta + \alpha \wedge \nabla_X *\beta$$

$$= g(\nabla_X \alpha,\beta)\mathrm{vol}_g + g(\alpha,\nabla_X \beta)\mathrm{vol}_g$$

$$= \nabla_X \alpha \wedge *\beta + \alpha \wedge *\nabla_X \beta$$

$\Rightarrow$

$$\alpha \wedge \nabla_X *\beta = \alpha \wedge *\nabla_X \beta$$

$\Rightarrow$

$$\nabla_X *\beta = *\nabla_X \beta.]$$

Definition: The <u>interior derivative</u>

$$\delta : \Lambda^p M \to \Lambda^{p-1} M$$

is

$$\delta = (-1)^{\iota}(-1)^{np+n+1} * \circ d \circ *.$$

[Note: Therefore $\delta f = 0$ $(f \in C^\infty(M))$.]

Observation: $\delta \circ \delta = 0$.

[This is because $* \circ * = \pm 1$ and $d \circ d = 0$.]

Example: Take $M = \underline{R}^{1,3}$ — then

$$(-1)^{\iota}(-1)^{np+n+1} = (-1)^{1}(-1)^{4p+4+1} = 1,$$

so in this case,

$$\delta\alpha = *d*\alpha.$$

Remark: The exterior derivative d does not depend on g. By contrast, the interior derivative δ depends on g (and μ).

Notation: Write $\Lambda_c^p M$ for the space of compactly supported p-forms on M and put

$$< \alpha,\alpha' >_g = \int_M g(\alpha,\alpha')\text{vol}_g \quad (\alpha,\alpha' \in \Lambda_c^p M).$$

Definition: A linear operator $A:\Lambda_c^p M \to \Lambda_c^p M$ is said to <u>admit an adjoint</u> if $\exists$ a linear operator $A^*:\Lambda_c^p M \to \Lambda_c^p M$ such that $\forall\ \alpha,\alpha' \in \Lambda_c^p M$,

$$< A\alpha,\alpha' >_g = < \alpha,A^*\alpha' >_g.$$

Example: Let ∇ be the metric connection — then $\forall\ \alpha,\alpha' \in \Lambda_c^p M$,

$$Xg(\alpha,\alpha') = g(\nabla_X\alpha,\alpha') + g(\alpha,\nabla_X\alpha').$$

On the other hand,

$$0 = \int_M (Xg(\alpha,\alpha') + g(\alpha,\alpha')\text{div } X)\text{vol}_g$$

$\Rightarrow$

$$< \nabla_X\alpha,\alpha' >_g = \int_M g(\nabla_X\alpha,\alpha')\text{vol}_g$$

$$= \int_M (Xg(\alpha,\alpha') - g(\alpha,\nabla_X\alpha'))\text{vol}_g$$

$$= - \int_M [g(\alpha,\nabla_X\alpha') + g(\alpha,(\text{div } X)\alpha')]\text{vol}_g$$

$$= < \alpha, - \nabla_X \alpha' - (\text{div } X)\alpha' >_g.$$

Accordingly, ∇_X admits an adjoint, namely

$$\nabla_X^* = - \nabla_X - \text{div } X.$$

<u>LEMMA</u> Let $\alpha \in \Lambda_c^p M$, $\beta \in \Lambda_c^{p+1} M$ -- then

$$< d\alpha, \beta >_g = < \alpha, \delta\beta >_g.$$

[We have

$$g(\alpha, \delta\beta)\text{vol}_g = \alpha \wedge * \delta\beta$$

$$= - (-1)^{\iota}(-1)^{n(p+2)} \alpha \wedge ** d * \beta$$

$$= - (-1)^{\iota}(-1)^{np} \alpha \wedge (-1)^{\iota}(-1)^{(n-p)p} d * \beta$$

$$= - (-1)^{p^2} \alpha \wedge d * \beta$$

$$= - (-1)^{p} \alpha \wedge d * \beta.$$

Therefore

$$g(d\alpha, \beta)\text{vol}_g - g(\alpha, \delta\beta)\text{vol}_g$$

$$= d\alpha \wedge * \beta + (-1)^{p} \alpha \wedge d * \beta$$

$$= d(\alpha \wedge * \beta).$$

And, by Stokes' theorem,

$$\int_M d(\alpha \wedge * \beta) = 0,$$

from which the result.]

Example: The Lie derivative $L_X:\Lambda_c^p M \to \Lambda_c^p M$ admits an adjoint. Thus put

$$\varepsilon_X = g^\flat X \wedge \_\_.$$

Then

$$< L_X\alpha,\alpha' >_g = \int_M g(L_X\alpha,\alpha')\text{vol}_g$$

$$= \int_M g((\iota_X \circ d + d \circ \iota_X)\alpha,\alpha')\text{vol}_g$$

$$= \int_M g(\iota_X d\alpha,\alpha')\text{vol}_g + \int_M g(d\iota_X\alpha,\alpha')\text{vol}_g$$

$$= \int_M g(d\alpha,\varepsilon_X\alpha')\text{vol}_g + < d\iota_X\alpha,\alpha' >_g$$

$$= < d\alpha,\varepsilon_X\alpha' >_g + < \iota_X\alpha,\delta\alpha' >_g$$

$$= < \alpha,\delta\varepsilon_X\alpha' >_g + \int_M g(\iota_X\alpha,\delta\alpha')\text{vol}_g$$

$$= < \alpha,\delta\varepsilon_X\alpha' >_g + \int_M g(\alpha,\varepsilon_X\delta\alpha')\text{vol}_g$$

$$= < \alpha,\delta\varepsilon_X\alpha' >_g + < \alpha,\varepsilon_X\delta\alpha' >_g$$

$$= < \alpha,(\delta \circ \varepsilon_X + \varepsilon_X \circ \delta)\alpha' >_g$$

$\Rightarrow$

$$L_X^* = \delta \circ \varepsilon_X + \varepsilon_X \circ \delta.$$

[Note: Up to a sign, the composite

$$\Lambda^{p-1} \xrightarrow{*} \Lambda^{n-p+1} \xrightarrow{\iota_X} \Lambda^{n-p} \xrightarrow{*} \Lambda^p$$

is ε_X. To see this, let $\beta\in\Lambda^{p-1}M$ -- then

$$\iota_X*\beta = \iota_X\iota_\beta \mathrm{vol}_g$$

$$= \iota_{g^\flat X}\iota_\beta \mathrm{vol}_g$$

$$= \iota_{\beta\wedge g^\flat X}\mathrm{vol}_g$$

$\Rightarrow$

$$*\iota_X*\beta = *(\iota_{\beta\wedge g^\flat X}\mathrm{vol}_g)$$

$$= (-1)^{p(n-p)}*(\mathrm{vol}_g)\wedge\beta\wedge g^\flat X$$

$$= (-1)^{\iota}(-1)^{p(n-p)}\beta\wedge g^\flat X$$

$$= (-1)^{\iota}(-1)^{p(n-p)}(-1)^{p-1}g^\flat X\wedge\beta$$

$$= (-1)^{\iota}(-1)^{np-1}\varepsilon_X\beta.]$$

<u>LEMMA</u> Let $X\in\mathcal{D}^1(M)$ -- then

$$\mathrm{div}\, X = -\,\delta g^\flat X.$$

[In fact, $\forall\ f\in C_c^\infty(M)$,

$$< f,\delta g^\flat X >_g = < df, g^\flat X >_g$$

$$= \int_M g(df, g^\flat X)\mathrm{vol}_g$$

$$= \int_M g(g^\flat g^\# df, g^\flat X)\mathrm{vol}_g$$

$$= \int_M g(g^\flat \text{grad } f, g^\flat X)\text{vol}_g$$

$$= \int_M g(\text{grad } f, X)\text{vol}_g$$

$$= \int_M Xf\text{vol}_g$$

$$= - \int_M f(\text{div } X)\text{vol}_g$$

$$= - < f, \text{div } X >_g$$

$\Rightarrow$

$$\text{div } X = - \delta g^\flat X.]$$

Consequently, if $\alpha \in \mathcal{D}_1(M)$, then locally

$$\delta\alpha = - \nabla^i \alpha_i.$$

Thus write $\alpha = g^\flat X$ -- then

$$\delta\alpha = \delta g^\flat X = - \text{div } X$$

$$= - X^a_{;a} = - \nabla_a X^a = - \nabla_a g^{ai}\alpha_i$$

$$= - g^{ai}\nabla_a \alpha_i = - g^{ia}\nabla_a \alpha_i = - \nabla^i \alpha_i.$$

To generalize this, let $\alpha \in \Lambda^p M$ $(p > 1)$ -- then locally

$$(d\alpha)_{j_1 \cdots j_{p+1}} = \sum_{a=1}^{p+1} (-1)^{a+1} \nabla_{j_a} \alpha_{j_1 \cdots \hat{j}_a \cdots j_{p+1}},$$

hence

$$(d\alpha)^{i_1\cdots i_{p+1}}$$

$$= g^{i_1j_1}\cdots g^{i_{p+1}j_{p+1}}(d\alpha)_{j_1\cdots j_{p+1}}$$

$$= \sum_{a=1}^{p+1}(-1)^{a+1}\nabla^{i_a}\alpha^{i_1\cdots\hat{i}_a\cdots i_{p+1}}.$$

So, from the definitions, $\forall\ \beta\in\Lambda^{p+1}M$,

$$g(d\alpha,\beta) = \frac{1}{(p+1)!}(d\alpha)^{i_1\cdots i_{p+1}}\beta_{i_1\cdots i_{p+1}}$$

$$= \frac{1}{(p+1)!}\sum_{a=1}^{p+1}(-1)^{a+1}\nabla^{i_a}\alpha^{i_1\cdots\hat{i}_a\cdots i_{p+1}}\beta_{i_1\cdots i_{p+1}}$$

$$= \frac{1}{(p+1)!}\sum_{a=1}^{p+1}(-1)^{a+1}[-\alpha^{i_1\cdots\hat{i}_a\cdots i_{p+1}}\nabla^{i_a}\beta_{i_1\cdots i_{p+1}}$$

$$+\nabla^{i_a}(\alpha^{i_1\cdots\hat{i}_a\cdots i_{p+1}}\beta_{i_1\cdots i_{p+1}})]$$

$$= \frac{1}{p!}\alpha^{i_1\cdots i_p}\tilde{\beta}_{i_1\cdots i_p}$$

$$+\frac{1}{(p+1)!}\sum_{a=1}^{p+1}(-1)^{a+1}\nabla^{i_a}(\alpha^{i_1\cdots\hat{i}_a\cdots i_{p+1}}\beta_{i_1\cdots i_{p+1}}),$$

where

$$\tilde{\beta}_{i_1\cdots i_p} = -\nabla^a\beta_{ai_1\cdots i_p}.$$

I.e.:

$$g(d\alpha,\beta) = g(\alpha,\tilde{\beta})$$

$$+ \frac{1}{(p+1)!} \sum_{a=1}^{p+1} (-1)^{a+1} \nabla^{i_a}(\alpha^{i_1 \cdots \hat{i}_a \cdots i_{p+1}} \beta_{i_1 \cdots i_{p+1}}).$$

But

$$\nabla^{i_a}(\alpha^{i_1 \cdots \hat{i}_a \cdots i_{p+1}} \beta_{i_1 \cdots i_{p+1}})$$

is a divergence, hence integrates to zero. Therefore $\tilde{\beta} = \delta\beta$.

Restated, these considerations lead to the conclusion that locally,

$$(\delta\alpha)_{i_1 \cdots i_{p-1}} = - \nabla^a \alpha_{a i_1 \cdots i_{p-1}}.$$

<u>FACT</u> $\forall\ f \in C^\infty(M)$,

$$\delta(f\alpha) = - \iota_{df}\alpha + f\delta\alpha.$$

Recall now that

$$\begin{aligned}\Delta &= \text{div} \circ \text{grad} \\ &= \text{div} \circ g^\# \circ d.\end{aligned}$$

Therefore

$$\begin{aligned}\Delta &= - \delta \circ g^\flat \circ g^\# \circ d \\ &= - \delta \circ d\end{aligned}$$

or still,

$$\begin{aligned}\Delta &= - [(-1)^\iota (-1)^{n+n+1} * \circ d \circ *] \circ d \\ &= (-1)^\iota * \circ d \circ * \circ d.\end{aligned}$$

Definition: The <u>laplacian</u>

$$\Delta:\Lambda^p M \rightarrow \Lambda^p M$$

is

$$\Delta = -(d \circ \delta + \delta \circ d).$$

Properties: (1) $\Delta = \Delta^*$; (2) $d \circ \Delta = \Delta \circ d$; (3) $\delta \circ \Delta = \Delta \circ \delta$; (4) $* \circ \Delta = \Delta \circ *$.

<u>FACT</u> Let $f \in C^\infty(M)$, $\alpha \in \Lambda^p M$ -- then

$$\Delta(f\alpha) = (\Delta f)\alpha + f(\Delta\alpha) + 2\nabla_{\text{grad } f}\alpha.$$

[Note: On functions,

$$\Delta(f_1 f_2) = (\Delta f_1)f_2 + f_1(\Delta f_2) + 2g(\text{grad } f_1, \text{grad } f_2).]$$

Definition: The <u>connection laplacian</u>

$$\Delta_{\text{con}}:\mathcal{D}_q^0(M) \rightarrow \mathcal{D}_q^0(M)$$

is

$$\Delta_{\text{con}} = \nabla^a \nabla_a.$$

[Note: In other words,

$$(\Delta_{\text{con}} T)_{j_1 \cdots j_q} = \nabla^a \nabla_a T_{j_1 \cdots j_q},$$

which makes it clear that Δ_{con} is a metric contraction of $\nabla^2 T$.]

Let $f \in C^\infty(M)$ -- then

$$\begin{aligned} \Delta f &= g^{ij}(H_f)_{ij} \\ &= g^{ij}(\nabla^2 f)_{ij} \\ &= g^{ij}\nabla_j \nabla_i f \end{aligned}$$

$$= \nabla^i \nabla_i f$$

$$= \Delta_{con} f.$$

I.e.:

$$\Delta = \Delta_{con}$$

on $\Lambda^0 M$ but, in general, $\Delta \neq \Delta_{con}$ on $\Lambda^p M$ $(p > 0)$.

To understand this, let $\alpha \in \Lambda^p M$ $(p > 0)$ -- then

$$(d\delta\alpha)_{i_1 \cdots i_p} = \sum_{k=1}^{p} (-1)^k \nabla_{i_k} \nabla^a \alpha_{a i_1 \cdots \hat{i}_k \cdots i_p}$$

and

$$(\delta d\alpha)_{i_1 \cdots i_p} = - \nabla^a \nabla_a \alpha_{i_1 \cdots i_p}$$

$$- \sum_{k=1}^{p} (-1)^k \nabla^a \nabla_{i_k} \alpha_{a i_1 \cdots \hat{i}_k \cdots i_p}$$

$\Rightarrow$

$$(\Delta\alpha)_{i_1 \cdots i_p} = - [(d\delta\alpha)_{i_1 \cdots i_p} + (\delta d\alpha)_{i_1 \cdots i_p}]$$

$$= \nabla^a \nabla_a \alpha_{i_1 \cdots i_p}$$

$$+ \sum_{k=1}^{p} (-1)^k (\nabla^a \nabla_{i_k} - \nabla_{i_k} \nabla^a) \alpha_{a i_1 \cdots \hat{i}_k \cdots i_p}.$$

Rappel: Thanks to the Ricci identity,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \alpha_{j_1 \cdots j_p}$$

$$= \sum_{\ell=1}^{p} R^i{}_{j_\ell ba} \alpha_{j_1 \cdots j_{\ell-1} i j_{\ell+1} \cdots j_p}$$

or still,

$$(\nabla^a\nabla_b - \nabla_b\nabla^a)\alpha_{j_1\cdots j_p}$$

$$= \sum_{\ell=1}^{p} R^i{}_{j_\ell b}{}^a \alpha_{j_1\cdots j_{\ell-1} i j_{\ell+1}\cdots j_p}$$

$\Rightarrow$

$$(\nabla^a\nabla_b - \nabla_b\nabla^a)\alpha_{aj_2\cdots j_p}$$

$$= R^i{}_{ab}{}^a \alpha_{ij_2\cdots j_p}$$

$$+ \sum_{\ell=2}^{p} R^i{}_{j_\ell b}{}^a \alpha_{aj_2\cdots j_{\ell-1} i j_{\ell+1}\cdots j_p}$$

$$= R^i{}_{ab}{}^a \alpha_{ij_2\cdots j_p}$$

$$+ \sum_{\ell=2}^{p} (-1)^\ell R^i{}_{j_\ell b}{}^a \alpha_{aij_2\cdots \hat{j}_\ell\cdots j_p}.$$

Therefore

$$\sum_{k=1}^{p} (-1)^k (\nabla^a\nabla_{i_k} - \nabla_{i_k}\nabla^a)\alpha_{ai_1\cdots \hat{i}_k\cdots i_p}$$

$$= \sum_{k=1}^{p} (-1)^k R^i{}_{ai_k}{}^a \alpha_{ii_1\cdots \hat{i}_k\cdots i_p}$$

$$+ 2\sum_{k<\ell} (-1)^{\ell+k} R^i{}_{i_\ell i_k}{}^a \alpha_{aii_1\cdots \hat{i}_k\cdots \hat{i}_\ell\cdots i_p}.$$

[Note: This is the so-called Weitzenboeck formula.]

Example: Take p = 1 -- then

$$(\Delta\alpha)_j = \nabla^a\nabla_a\alpha_j - R^i{}_{aj}{}^a\alpha_i.$$

Since the Ricci tensor is given by

$$R_{j\ell} = R^a{}_{ja\ell},$$

we have

$$R_j{}^i = g^{i\ell}R_{j\ell}$$

$$= R^a{}_{ja}{}^i.$$

But

$$R^i{}_{aj}{}^a = g^{ik}g^{ab}R_{kajb}$$

$$= g^{ik}g^{ab}R_{jbka}$$

$$= g^{ik}g^{ab}R_{bjak}$$

$$= R^a{}_{ja}{}^i$$

$$= R_j{}^i.$$

Therefore

$$(\Delta\alpha)_j = \nabla^a\nabla_a\alpha_j - R_j{}^i\alpha_i.$$

FACT On forms of degree n, $\Delta = \Delta_{con}$.

<u>Section 20</u>: <u>Star Formulae</u> Let M be a connected C^∞ manifold of dimension n, which we shall take to be orientable with orientation μ. Fix a semiriemannian structure g on M.

Assume: The orthonormal frame bundle LM(g) is trivial.

Suppose that $E = \{E_1,\ldots,E_n\}$ is an oriented frame (not necessarily orthonormal). Let $\omega = \{\omega^1,\ldots,\omega^n\}$ be its associated coframe -- then

$$\mathrm{vol}_g = |g|^{1/2}\omega^1 \wedge \cdots \wedge \omega^n$$

or still,

$$\mathrm{vol}_g = \frac{1}{n!}\, e_{j_1\cdots j_n}\omega^{j_1} \wedge \cdots \wedge \omega^{j_n},$$

where

$$e_\bullet = |g|^{1/2}\cdot\varepsilon_\bullet\ .$$

Rappel: The star operator is the isomorphism

$$*:\Lambda^p M \to \Lambda^{n-p}M$$

given by

$$*\alpha = \iota_\alpha \mathrm{vol}_g.$$

Therefore

$$*\alpha = \frac{1}{p!(n-p)!}\,\alpha^{i_1\cdots i_p}\, e_{i_1\cdots i_p j_1\cdots j_{n-p}}\,\omega^{j_1} \wedge \cdots \wedge \omega^{j_{n-p}}.$$

Another point to bear in mind is that

$$*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p})$$

$$= \frac{|g|^{1/2}}{(n-p)!}\, g^{i_1 j_1}\cdots g^{i_p j_p}\,\varepsilon_{j_1\cdots j_n}\,\omega^{j_{p+1}} \wedge \cdots \wedge \omega^{j_n}.$$

[Note: If E is orthonormal, then $|g| = 1$ and

$$*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p})$$

$$= \frac{1}{(n-p)!} \varepsilon_{i_1} \cdots \varepsilon_{i_p} \varepsilon_{i_1 \cdots i_p j_{p+1} \cdots j_n} \omega^{j_{p+1}} \wedge \cdots \wedge \omega^{j_n}.]$$

<u>LEMMA</u> Assume: $p > 1$ -- then

$$(p-1)\delta(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p})$$

$$= \sum_{k=1}^{p} (-1)^k \omega^{i_k} \wedge \delta(\omega^{i_1} \wedge \cdots \wedge \hat{\omega}^{i_k} \wedge \cdots \wedge \omega^{i_p})$$

$$+ (-1)^{\iota}(-1)^{np+p} \omega^i \wedge *(d\omega_i \wedge *(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p})).$$

[We have

$$\iota_{E_i} \delta(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p})$$

$$= (-1)^{\iota}(-1)^{np+n+1} \iota_{E_i} *(d*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}))$$

$$= (-1)^{\iota}(-1)^{np+n+1} *(d*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) \wedge g^{\flat} E_i)$$

$$= (-1)^{\iota}(-1)^{np+n+1} *(d*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) \wedge \omega_i)$$

$$= (-1)^{\iota}(-1)^{np+n+1}(-1)^{n-p+1} *(\omega_i \wedge d*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}))$$

$$= (-1)^{\iota}(-1)^{np+p} *(\omega_i \wedge d*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}))$$

$$= (-1)^{\iota}(-1)^{np+p} *(-d(\omega_i \wedge *(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}))$$

$$+ d\omega_i \wedge *(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p})).$$

But

$$\omega_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= (-1)^{n-p} *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}) \wedge \omega_i$$

$$= (-1)^{n-p}(-1)^{n-1} * \iota_{\omega_i}(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= (-1)^{p+1} * \sum_{k=1}^{p} (-1)^{k+1} (\iota_{\omega_i}\omega^{i_k})\omega^{i_1} \wedge \ldots \wedge \hat{\omega^{i_k}} \wedge \ldots \wedge \omega^{i_p}.$$

Write $\omega_i = g_{ia}\omega^a$ -- then

$$\iota_{\omega_i}\omega^{i_k} = g(\omega_i, \omega^{i_k})$$

$$= g_{ia}g(\omega^a, \omega^{i_k})$$

$$= g_{ia}g^{ai_k} = g^{i_k a}g_{ai} = \delta^{i_k}{}_i.$$

Therefore

$$\iota_{E_i}\delta(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= (-1)^{\iota}(-1)^{np+p}(-1)^{p+1} *(-d)* \sum_{k=1}^{p} (-1)^{k+1} \delta^{i_k}{}_i \omega^{i_1} \wedge \ldots \wedge \hat{\omega^{i_k}} \wedge \ldots \wedge \omega^{i_p}$$

$$+ (-1)^{\iota}(-1)^{np+p} *(d\omega_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}))$$

$$= (-1)^{\iota}(-1)^{n(p-1)+n+1} *d* \sum_{k=1}^{p} (-1)^{k} \delta^{i_k}{}_i \omega^{i_1} \wedge \ldots \wedge \hat{\omega^{i_k}} \wedge \ldots \wedge \omega^{i_p}$$

$$+ (-1)^{\iota}(-1)^{np+p} *(d\omega_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}))$$

$$= \sum_{k=1}^{p} (-1)^k \delta^{i_k}{}_i \delta(\omega^{i_1} \wedge \ldots \wedge \hat{\omega^{i_k}} \wedge \ldots \wedge \omega^{i_p})$$

$$+ (-1)^{\iota}(-1)^{np+p} *(d\omega_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})).$$

Since in general

$$\omega^i \wedge \iota_{E_i} \alpha = (p-1)\alpha \qquad (\alpha \in \Lambda^{p-1}M),$$

it thus follows that

$$(p-1)\delta(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= \sum_{k=1}^{p} (-1)^k \omega^{i_k} \wedge \delta(\omega^{i_1} \wedge \ldots \wedge \hat{\omega^{i_k}} \wedge \ldots \wedge \omega^{i_p})$$

$$+ (-1)^{\iota}(-1)^{np+p} \omega^i \wedge *(d\omega_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})).]$$

Remark: In principle, the lemma allows one to compute

$$\delta(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

by iteration provided that $p > 1$. As for $p = 1$,

$$\begin{aligned} \delta\omega^i &= \delta g^{\flat} E^i \\ &= - \operatorname{div} E^i \\ &= - \operatorname{div} g^{ij} E_j \\ &= - g^{ij} \operatorname{div} E_j \\ &= - g^{ij} \sum_k \Gamma^k{}_{kj}. \end{aligned}$$

So, if E is orthonormal, then

$$\delta\omega^i = -\varepsilon_i \sum_j \Gamma^j{}_{ji}$$

$$= -\varepsilon_i \sum_j - C^j{}_{ji}$$

$$= -\varepsilon_i \sum_j C^j{}_{ij}$$

$$\equiv -\varepsilon_i C_i \quad \text{(no sum)}$$

$$\equiv -C^i.$$

Example: Take p = 2 and suppose that E is orthonormal — then

$$\delta(\omega^{i_1}\wedge\omega^{i_2}) = -\omega^{i_1}\wedge\delta\omega^{i_2} + \omega^{i_2}\wedge\delta\omega^{i_1}$$

$$+ (-1)^{\iota}\omega^i\wedge*(d\omega_i\wedge*(\omega^{i_1}\wedge\omega^{i_2}))$$

or still,

$$\delta(\omega^{i_1}\wedge\omega^{i_2}) = C^{i_2}\omega^{i_1} - C^{i_1}\omega^{i_2}$$

$$+ (-1)^{\iota}\omega^i\wedge*(d\omega_i\wedge*(\omega^{i_1}\wedge\omega^{i_2})).$$

Write

$$d\omega_i = \varepsilon_i d\omega^i$$

$$= \varepsilon_i \frac{1}{2} C^i{}_{jk}\omega^j\wedge\omega^k.$$

Then

$$*(d\omega_i\wedge*(\omega^{i_1}\wedge\omega^{i_2}))$$

$$= \frac{1}{2}\varepsilon_i C^i{}_{jk}*(\omega^j\wedge\omega^k\wedge*(\omega^{i_1}\wedge\omega^{i_2}))$$

$$= \frac{1}{2}\,\varepsilon_i C^i{}_{jk} * (g(\omega^j \wedge \omega^k, \omega^{i_1} \wedge \omega^{i_2})\,\mathrm{vol}_g)$$

$$= (-1)^{\iota}\,\frac{1}{2}\,\varepsilon_i C^i{}_{jk}\, g(\omega^j \wedge \omega^k, \omega^{i_1} \wedge \omega^{i_2})$$

$$= (-1)^{\iota}\,\frac{1}{2}\,\varepsilon_i C^i{}_{jk}$$

$$\times \ \det \begin{bmatrix} g(\omega^j, \omega^{i_1}) & g(\omega^j, \omega^{i_2}) \\ g(\omega^k, \omega^{i_1}) & g(\omega^k, \omega^{i_2}) \end{bmatrix}$$

$$= (-1)^{\iota}\,\frac{1}{2}\,\varepsilon_i C^i{}_{jk}$$

$$\times \ \det \begin{bmatrix} \eta^{ji_1} & \eta^{ji_2} \\ \eta^{ki_1} & \eta^{ki_2} \end{bmatrix}$$

$$= (-1)^{\iota}\,\frac{1}{2}\,\varepsilon_i C^i{}_{jk}(\eta^{ji_1}\eta^{ki_2} - \eta^{ji_2}\eta^{ki_1})$$

$$= (-1)^{\iota}\,\frac{1}{2}\,\varepsilon_i (C^i{}_{i_1 i_2}\varepsilon_{i_1}\varepsilon_{i_2} - C^i{}_{i_2 i_1}\varepsilon_{i_2}\varepsilon_{i_1})$$

$$= (-1)^{\iota}\varepsilon_i \varepsilon_{i_1}\varepsilon_{i_2} C^i{}_{i_1 i_2}$$

$$\equiv (-1)^{\iota} C_i{}^{i_1 i_2}.$$

Therefore

$$\delta(\omega^{i_1} \wedge \omega^{i_2}) = C^{i_2}\omega^{i_1} - C^{i_1}\omega^{i_2} + C_i{}^{i_1 i_2}\omega^i.$$

[Note: Define C^∞ functions $M^i{}_a$ by

$$\Delta\omega^i (= - (d \circ \delta + \delta \circ d)) = M^i{}_a \omega^a.$$

Then the preceding considerations enable one to express the $M^i{}_a$ in terms of the $C^i{}_{jk}$ and the $B^i{}_{jk\ell}$, where

$$dC^i{}_{jk} = B^i{}_{jk\ell}\omega^\ell.]$$

<u>LEMMA</u> Let ∇ be a connection on TM -- then

$$d{\star}(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= \theta^a \wedge {\star}(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a)$$

$$+ (\omega^{ai_1} + dg^{ai_1}) \wedge {\star}(\omega_a \wedge \omega^{i_2} \wedge \ldots \wedge \omega^{i_p})$$

$$+ \cdots + (\omega^{ai_p} + dg^{ai_p}) \wedge {\star}(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{p-1}} \wedge \omega_a)$$

$$- (\omega^a{}_a - \frac{1}{2} g^{ab} dg_{ab}) \wedge (\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}).$$

[Note: The connection 1-forms of ∇ per E are given by

$$\nabla_X E_j = \omega^i{}_j(X) E_i$$

and one writes

$$\omega^{ij} = g^{jk}\omega^i{}_k.$$

Recall too that

$$\omega_i = g_{ij}\omega^j \quad (= g^\flat E_i).]$$

To establish this result, it will be convenient to divide the analysis into two parts.

Suppose first that E is orthonormal -- then

$$d*(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= \frac{1}{(n-p)!} \varepsilon_{i_1} \cdots \varepsilon_{i_p} \varepsilon_{i_1 \cdots i_p j_{p+1} \cdots j_n} d(\omega^{j_{p+1}} \wedge \ldots \wedge \omega^{j_n})$$

$$= d\omega^a \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a)$$

$$= (\Theta^a - \omega^a{}_b \wedge \omega^b) \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a)$$

$$= \Theta^a \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a)$$

$$- \omega^a{}_b \wedge \omega^b \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a).$$

But

$$\omega^a{}_b \wedge \omega^b \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a)$$

$$= (-1)^{n-p-1} \omega^a{}_b \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a) \wedge \omega^b$$

$$= (-1)^{n-p-1} (-1)^{n-1} \omega^a{}_b \wedge * \iota_{\omega^b} (\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a)$$

$$= (-1)^p \omega^a{}_b \wedge * \iota_{\omega^b} (\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a).$$

Agreeing to write

$$\omega_a{}^b = \varepsilon_a \varepsilon_b \omega^a{}_b \quad \text{(no sum)},$$

it then follows that

$$d*(\omega^{i_1}\wedge\ldots\wedge\omega^{i_p})$$

$$= \theta^a\wedge*(\omega^{i_1}\wedge\ldots\wedge\omega^{i_p}\wedge\omega_a)$$

$$+ \omega_a{}^{i_1}\wedge*(\omega^a\wedge\omega^{i_2}\wedge\ldots\wedge\omega^{i_p}) + \cdots + \omega_a{}^{i_p}\wedge*(\omega^{i_1}\wedge\ldots\wedge\omega^{i_{p-1}}\wedge\omega^a)$$

$$- \omega^a{}_a\wedge*(\omega^{i_1}\wedge\ldots\wedge\omega^{i_p}).$$

Since $dg^{ab} = 0$ and

$$\begin{cases} \omega^{ai} = \varepsilon_a\omega_a{}^i & \text{(no sum)} \\ \omega_a = \varepsilon_a\omega^a & \text{(no sum)}, \end{cases}$$

this formula is equivalent to that of the lemma (when specialized to an oriented orthonormal frame).

[Note: If $\nabla\in\text{con}_g TM$, then

$$\omega^b{}_a = -\varepsilon_a\varepsilon_b\omega^a{}_b \quad \text{(no sum)}.$$

Therefore

$$\omega_a{}^{i_k} = \varepsilon_a\varepsilon_{i_k}\omega^a{}_{i_k}$$

$$= \varepsilon_a\varepsilon_{i_k}(-\varepsilon_a\varepsilon_{i_k})\omega^{i_k}{}_a$$

$$= -\omega^{i_k}{}_a.$$

In addition,

$$\omega^a{}_a = 0 \quad \text{(no sum)}.$$

So, in this case,

$$d*(\omega^{i_1} \wedge \dots \wedge \omega^{i_p})$$

$$= \Theta^a \wedge *(\omega^{i_1} \wedge \dots \wedge \omega^{i_p} \wedge \omega_a)$$

$$- \omega^{i_1}{}_a \wedge *(\omega^a \wedge \omega^{i_2} \wedge \dots \wedge \omega^{i_p}) - \dots - \omega^{i_p}{}_a \wedge *(\omega^{i_1} \wedge \dots \wedge \omega^{i_{p-1}} \wedge \omega^a).$$

Moreover, the torsion term drops out if ∇ is actually the metric connection.]

To handle an arbitrary oriented frame, it suffices to consider

$$\hat{E} = E \cdot A,$$

where, as above, $E = \{E_1, \dots, E_n\}$ is orthonormal and

$$A: M \to \underline{GL}_0(n, \underline{R})$$

is smooth, thus

$$\begin{cases} \hat{E}_j = (E \cdot A)_j = A^i{}_j E_i \\ \hat{\omega}^j = (\omega \cdot A)^j = (A^{-1})^j{}_i \omega^i. \end{cases}$$

Now write

$$d*(\hat{\omega}^{j_1} \wedge \dots \wedge \hat{\omega}^{j_p})$$

$$= d*((A^{-1})^{j_1}{}_{i_1} \omega^{i_1} \wedge \dots \wedge (A^{-1})^{j_p}{}_{i_p} \omega^{i_p})$$

$$= d((A^{-1})^{j_1}{}_{i_1} \dots (A^{-1})^{j_p}{}_{i_p} *(\omega^{i_1} \wedge \dots \wedge \omega^{i_p}))$$

$$= A^{i_1}{}_j d(A^{-1})^{j_1}{}_{i_1} \wedge *(\hat{\omega}^j \wedge \hat{\omega}^{j_2} \wedge \dots \wedge \hat{\omega}^{j_p})$$

$$+ \cdots + A^{i_p}{}_j d(A^{-1})^{j_p}{}_{i_p} \wedge *(\hat{\omega}^{j_1} \wedge \ldots \wedge \hat{\omega}^{j_{p-1}} \wedge \hat{\omega}^j)$$

$$+ (A^{-1})^{j_1}{}_{i_1} \ldots (A^{-1})^{j_p}{}_{i_p} d*(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}),$$

where

$$d*(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= \Theta^i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_i)$$

$$+ \omega_i{}^{i_1} \wedge *(\omega^i \wedge \omega^{i_2} \wedge \ldots \wedge \omega^{i_p}) + \cdots + \omega_i{}^{i_p} \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{p-1}} \wedge \omega^i)$$

$$- \omega^i{}_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}).$$

Rappel: Under a change of basis $E \to \hat{E}$, the connection 1-forms compute as

$$\hat{\omega}^i{}_j = (A^{-1})^i{}_k \omega^k{}_\ell A^\ell{}_j + (A^{-1})^i{}_a dA^a{}_j.$$

Consider the torsion term:

$$(A^{-1})^{j_1}{}_{i_1} \ldots (A^{-1})^{j_p}{}_{i_p} \Theta^i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_i)$$

$$= \Theta^i \wedge *(\hat{\omega}^{j_1} \wedge \ldots \wedge \hat{\omega}^{j_p} \wedge \omega_i).$$

From the definitions,

$$\left[\begin{array}{l} \omega_i = \eta_{ia} \omega^a \\ \\ \hat{\omega}_j = \hat{g}_{jb} \hat{\omega}^b. \end{array}\right.$$

And

$$\hat{g}_{jb} = A^i{}_j A^a{}_b \eta_{ia}.$$

Therefore

$$\omega_i = \eta_{ia}\omega^a = \eta_{ia}A^a{}_b\hat{\omega}^b$$

$$= (A^{-1})^j{}_i\hat{g}_{jb}\hat{\omega}^b = (A^{-1})^j{}_i\hat{\omega}_j.$$

<u>SUBLEMMA</u> We have

$$\hat{\Theta}^j = (A^{-1})^j{}_i\Theta^i.$$

[In fact,

$$\Theta^i = d\omega^i + \omega^i{}_j\wedge\omega^j$$

$$= d(A^i{}_a\hat{\omega}^a)$$

$$+ [A^i{}_k\hat{\omega}^k{}_\ell(A^{-1})^\ell{}_j + A^i{}_a d(A^{-1})^a{}_j]\wedge(A^j{}_b\hat{\omega}^b)$$

$$= dA^i{}_a\wedge\hat{\omega}^a + A^i{}_a d\hat{\omega}^a$$

$$+ A^i{}_k(A^{-1})^\ell{}_j A^j{}_b\hat{\omega}^k{}_\ell\wedge\hat{\omega}^b$$

$$- dA^i{}_a[(A^{-1})^a{}_j A^j{}_b]\wedge\hat{\omega}^b$$

$$= dA^i{}_a\wedge\hat{\omega}^a + A^i{}_a d\hat{\omega}^a$$

$$+ A^i{}_k\delta^\ell{}_b\hat{\omega}^k{}_\ell\wedge\hat{\omega}^b - dA^i{}_a\delta^a{}_b\wedge\hat{\omega}^b$$

$$= dA^i{}_a \wedge \hat\omega^a + A^i{}_a d\hat\omega^a + A^i{}_k \hat\omega^k{}_\ell \wedge \hat\omega^\ell - dA^i{}_a \wedge \hat\omega^a$$

$$= A^i{}_a d\hat\omega^a + A^i{}_k \hat\omega^k{}_\ell \wedge \hat\omega^\ell$$

$$= A^i{}_j (d\hat\omega^j + \hat\omega^j{}_\ell \wedge \hat\omega^\ell)$$

$$= A^i{}_j \hat\Theta^j.$$

I.e.:

$$\hat\Theta^j = (A^{-1})^j{}_i \Theta^i.]$$

Therefore

$$\Theta^i \wedge *(\hat\omega^{j_1} \wedge \ldots \wedge \hat\omega^{j_p} \wedge \omega_i)$$

$$= (A^{-1})^j{}_i \Theta^i \wedge *(\hat\omega^{j_1} \wedge \ldots \wedge \hat\omega^{j_p} \wedge \hat\omega_j)$$

$$= \hat\Theta^j \wedge *(\hat\omega^{j_1} \wedge \ldots \wedge \hat\omega^{j_p} \wedge \hat\omega_j).$$

Next, consider

$$(A^{-1})^{j_1}{}_{i_1} \cdots (A^{-1})^{j_p}{}_{i_p}$$

$$\times \,(\omega^{ii_1} \wedge *(\omega_i \wedge \omega^{i_2} \wedge \ldots \wedge \omega^{i_p}) + \cdots + \omega^{ii_p} \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{p-1}} \wedge \omega_i))].$$

Since the treatment of each term is the same (up to notation), it suffices to deal with

$$(A^{-1})^{j_1}{}_{i_1} \cdots (A^{-1})^{j_p}{}_{i_p}(\omega^{ii_1} \wedge *(\omega_i \wedge \omega^{i_2} \wedge \ldots \wedge \omega^{i_p}))$$

or still,

$$(A^{-1})^{j_1}{}_{i_1}\omega^{ii_1} \wedge *(\omega_i \wedge \hat{\omega}^{j_2} \wedge \ldots \wedge \hat{\omega}^{j_p})$$

or still,

$$(A^{-1})^{j_1}{}_{i_1}\eta^{ki_1}(A^{-1})^j{}_i\omega^i{}_k \wedge *(\hat{\omega}_j \wedge \hat{\omega}^{j_2} \wedge \ldots \wedge \hat{\omega}^{j_p}).$$

We have

$$(A^{-1})^j{}_i\omega^i{}_k$$

$$= (A^{-1})^j{}_i[A^i{}_a\hat{\omega}^a{}_b(A^{-1})^b{}_k + A^i{}_c d(A^{-1})^c{}_k]$$

$$= (A^{-1})^j{}_i A^i{}_a\hat{\omega}^a{}_b(A^{-1})^b{}_k + (A^{-1})^j{}_i A^i{}_c d(A^{-1})^c{}_k$$

$$= \delta^j{}_a\hat{\omega}^a{}_b(A^{-1})^b{}_k + \delta^j{}_c d(A^{-1})^c{}_k$$

$$= \hat{\omega}^j{}_\ell(A^{-1})^\ell{}_k + d(A^{-1})^j{}_k.$$

And

$$(A^{-1})^{j_1}{}_{i_1}\eta^{ki_1}(A^{-1})^\ell{}_k\hat{\omega}^j{}_\ell$$

$$= (A^{-1})^{j_1}{}_{i_1}(A^{-1})^\ell{}_k\eta^{i_1k}\hat{\omega}^j{}_\ell$$

$$= \hat{g}^{j_1\ell}\hat{\omega}^j{}_\ell$$

$$= \hat{\omega}^{jj_1}.$$

Finally

$$\hat{g}^{jj_1} = (A^{-1})^j{}_k (A^{-1})^{j_1}{}_{i_1} \eta^{ki_1}$$

$\Rightarrow$

$$d\hat{g}^{jj_1} = d(A^{-1})^j{}_k (A^{-1})^{j_1}{}_{i_1} \eta^{ki_1} + (A^{-1})^j{}_k \eta^{ki_1} d(A^{-1})^{j_1}{}_{i_1}$$

$\Rightarrow$

$$d(A^{-1})^j{}_k (A^{-1})^{j_1}{}_{i_1} \eta^{ki_1}$$

$$= d\hat{g}^{jj_1} - (A^{-1})^j{}_k \eta^{ki_1} d(A^{-1})^{j_1}{}_{i_1}.$$

Retaining the differential $d\hat{g}^{jj_1}$, the claim then is that

$$A^{i_1}{}_j d(A^{-1})^{j_1}{}_{i_1} \wedge * (\hat{\omega}^j \wedge \hat{\omega}^{j_2} \wedge \ldots \wedge \hat{\omega}^{j_p})$$

$$= (A^{-1})^j{}_k \eta^{ki_1} d(A^{-1})^{j_1}{}_{i_1} \wedge * (\hat{\omega}_j \wedge \hat{\omega}^{j_2} \wedge \ldots \wedge \hat{\omega}^{j_p}).$$

But this is clear:

$$(A^{-1})^j{}_k \eta^{ki_1} \hat{\omega}_j$$

$$= (A^{-1})^j{}_k \eta^{ki_1} \hat{g}_{jb} \hat{\omega}^b$$

$$= (A^{-1})^j{}_k \eta^{ki_1} A^i{}_j A^a{}_b \eta_{ia} \hat{\omega}^b$$

$$= A^{i}{}_{j}(A^{-1})^{j}{}_{k}\eta^{ki_1}\eta_{ia}A^{a}{}_{b}\hat{\omega}^{b}$$

$$= \delta^{i}{}_{k}\eta^{ki_1}\eta_{ia}A^{a}{}_{b}\hat{\omega}^{b}$$

$$= \eta^{ii_1}\eta_{ia}A^{a}{}_{b}\hat{\omega}^{b}$$

$$= \eta^{i_1 i}\eta_{ia}A^{a}{}_{b}\hat{\omega}^{b}$$

$$= \delta^{i_1}{}_{a}A^{a}{}_{b}\hat{\omega}^{b}$$

$$= A^{i_1}{}_{b}\hat{\omega}^{b}$$

$$= A^{i_1}{}_{j}\hat{\omega}^{j}.$$

It remains to consider

$$- (A^{-1})^{j_1}{}_{i_1} \cdots (A^{-1})^{j_p}{}_{i_p}\,\omega^{i}{}_{i}\wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

or still,

$$- \omega^{i}{}_{i}\wedge *(\hat{\omega}^{j_1} \wedge \ldots \wedge \hat{\omega}^{j_p}).$$

To proceed from here, simply observe that

$$\omega^{i}{}_{i} = \hat{\omega}^{j}{}_{j} + A^{i}{}_{j}d(A^{-1})^{j}{}_{i}$$

$$= \hat{\omega}^{j}{}_{j} - dA^{i}{}_{j}(A^{-1})^{j}{}_{i}$$

$$= \hat{\omega}^{j}{}_{j} - \frac{1}{2}\,\hat{g}^{ab}d\hat{g}_{ab}.$$

Remark: Let $\nabla \in \text{con}_g TM$ — then in an arbitrary oriented frame,

$$d*(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= \Theta^a \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \omega_a)$$

$$- \omega^{i_1}{}_a \wedge *(\omega^a \wedge \omega^{i_2} \wedge \ldots \wedge \omega^{i_p}) - \cdots - \omega^{i_p}{}_a \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{p-1}} \wedge \omega^a).$$

[To see this, recall that

$$g_{ik}\omega^k{}_j + g_{j\ell}\omega^\ell{}_i = dg_{ij}.$$

There are then two points:

- $(\omega^{ai_1} + dg^{ai_1})g_{ab} = -\omega^{i_1}{}_b.$

Proof: $\omega^{ai_1}g_{ab} = g_{ba}\omega^{ai_1} = \omega_b{}^{i_1}$

&

$$dg^{ai_1}g_{ab} = dg^{i_1 a}g_{ab}$$

$$= -g^{i_1 a}dg_{ab}$$

$$= -g^{i_1 a}[g_{ak}\omega^k{}_b + g_{b\ell}\omega^\ell{}_a]$$

$$= -g^{i_1 a}g_{ak}\omega^k{}_b - g_{b\ell}g^{i_1 a}\omega^\ell{}_a$$

$$= -\delta^{i_1}{}_k\omega^k{}_b - g_{b\ell}\omega^{\ell i_1}$$

$$= -\omega^{i_1}{}_b - \omega_b{}^{i_1}.$$

• $\omega^a{}_a - \frac{1}{2} g^{ab} dg_{ab} = 0.$

Proof: $-\frac{1}{2} g^{ab} dg_{ab}$

$$= -\frac{1}{2} g^{ab}[g_{ak}\omega^k{}_b + g_{b\ell}\omega^\ell{}_a]$$

$$= -\frac{1}{2} [g^{ba}g_{ak}\omega^k{}_b + g^{ab}g_{b\ell}\omega^\ell{}_a]$$

$$= -\frac{1}{2} [\delta^b{}_k\omega^k{}_b + \delta^a{}_\ell\omega^\ell{}_a]$$

$$= -\frac{1}{2} [\omega^b{}_b + \omega^a{}_a] = -\omega^a{}_a.]$$

Suppose that E is an oriented orthonormal frame and take for ∇ the metric connection.

<u>LEMMA</u> Put $\omega^{ij} = \varepsilon_j\omega^i{}_j$ (no sum) -- then

$$\omega^{ij} = (-1)^\iota *[*d\omega^i \wedge \omega^j - *d\omega^j \wedge \omega^i + \frac{1}{2} (-1)^n \sum_k \varepsilon_k *(d\omega^k \wedge \omega^k) \wedge \omega^i \wedge \omega^j].$$

Since

$$\Gamma^i{}_{kj} = \omega^i{}_j(E_k),$$

we have

$$\omega^i{}_j(E_k) = \frac{1}{2} \varepsilon_i(\varepsilon_i d\omega^i(E_j,E_k) + \varepsilon_j d\omega^j(E_k,E_i) - \varepsilon_k d\omega^k(E_i,E_j)).$$

So, if $X = \sum_k \omega^k(X)E_k \in \mathcal{D}^1(M)$, then

$$\omega^i{}_j(X) = \frac{1}{2}\,\varepsilon_i(\varepsilon_i d\omega^i(E_j,X) + \varepsilon_j d\omega^j(X,E_i) - \sum_k \varepsilon_k d\omega^k(E_i,E_j)\omega^k(X)).$$

Therefore, in terms of the interior product,

$$\omega^i{}_j = \frac{1}{2}\,\varepsilon_i(\varepsilon_i \iota_{E_j} d\omega^i - \varepsilon_j \iota_{E_i} d\omega^j - \sum_k \varepsilon_k(\iota_{E_j}\iota_{E_i} d\omega^k)\omega^k).$$

But

$$(d\omega^k \wedge \omega^k)(E_i,E_j,X)$$

$$= \omega^k(E_i)d\omega^k(E_j,X) - \omega^k(E_j)d\omega^k(E_i,X) + d\omega^k(E_i,E_j)\omega^k(X)$$

$\Rightarrow$

$$\varepsilon_k d\omega^k(E_i,E_j)\omega^k(X)$$

$$= -\,\varepsilon_k\omega^k(E_i)d\omega^k(E_j,X) + \varepsilon_k\omega^k(E_j)d\omega^k(E_i,X) + \varepsilon_k(d\omega^k\wedge\omega^k)(E_i,E_j,X)$$

$\Rightarrow$

$$\sum_k \varepsilon_k d\omega^k(E_i,E_j)\omega^k(X)$$

$$= -\,\varepsilon_i d\omega^i(E_j,X) + \varepsilon_j d\omega^j(E_i,X) + \sum_k \varepsilon_k(d\omega^k\wedge\omega^k)(E_i,E_j,X).$$

From this, it follows that

$$\omega^i{}_j = \varepsilon_i(\varepsilon_i \iota_{E_j} d\omega^i - \varepsilon_j \iota_{E_i} d\omega^j - \frac{1}{2}\sum_k \varepsilon_k \iota_{E_j}\iota_{E_i}(d\omega^k\wedge\omega^k)).$$

Rappel: $\forall\ X\in\mathcal{D}^1(M)$, $\forall\ \alpha\in\Lambda^p M$,

$$*\iota_X\alpha = (-1)^{n-1} *\alpha\wedge g^\flat X$$

$\Rightarrow$

$$**\iota_X\alpha = (-1)^{\iota}(-1)^{(p-1)(n-p+1)}\iota_X\alpha$$

$$= (-1)^{n-1} * (*\alpha \wedge g^{\flat} X)$$

$\Rightarrow$

$$\iota_X \alpha = (-1)^{\iota} (-1)^{(p-1)(n-p+1)} (-1)^{n-1} * (*\alpha \wedge g^{\flat} X).$$

Let

$$\alpha = \begin{cases} d\omega^i \\ d\omega^j. \end{cases}$$

Then

$$(2-1)(n-2+1) = n-1$$

$\Rightarrow$

$$\begin{cases} \iota_{E_j} d\omega^i = (-1)^{\iota} * (*d\omega^i \wedge \varepsilon_j \omega^j) \\ \iota_{E_i} d\omega^j = (-1)^{\iota} * (*d\omega^j \wedge \varepsilon_i \omega^i). \end{cases}$$

[Note: Because it is a question of an orthonormal basis,

$$\begin{cases} g^{\flat} E_i = \varepsilon_i \omega^i \\ g^{\flat} E_j = \varepsilon_j \omega^j. \end{cases} \qquad \text{(no sum)}$$
]

Let

$$\alpha = \iota_{E_i} (d\omega^k \wedge \omega^k).$$

Then

$$(3-1)(n-3+1) = 2(n-3+1)$$

$\Rightarrow$

$$\iota_{E_i}(d\omega^k\wedge\omega^k) = (-1)^{\iota}(-1)^{n-1}*(*(d\omega^k\wedge\omega^k)\wedge\varepsilon_i\omega^i).$$

Now put

$$\beta = *(*(d\omega^k\wedge\omega^k)\wedge\varepsilon_i\omega^i).$$

Since β is a 2-form,

$$\iota_{E_j}\beta = (-1)^{\iota}*(*\beta\wedge\varepsilon_j\omega^j).$$

However

$$*\beta = **(*(d\omega^k\wedge\omega^k)\wedge\varepsilon_i\omega^i)$$

$$= (-1)^{\iota}(-1)^{(n-2)(n-(n-2))}*(d\omega^k\wedge\omega^k)\wedge\varepsilon_i\omega^i$$

$$= (-1)^{\iota}*(d\omega^k\wedge\omega^k)\wedge\varepsilon_i\omega^i.$$

Thus

$$\iota_{E_j}\iota_{E_i}(d\omega^k\wedge\omega^k)$$

$$= (-1)^{\iota}(-1)^{n-1}\iota_{E_j}\beta$$

$$= (-1)^{\iota}(-1)^{n-1}(-1)^{\iota}*(*\beta\wedge\varepsilon_j\omega^j)$$

$$= (-1)^{\iota}(-1)^{n-1}(-1)^{\iota}(-1)^{\iota}*(*(d\omega^k\wedge\omega^k)\wedge\varepsilon_i\omega^i\wedge\varepsilon_j\omega^j)$$

$$= (-1)^{\iota}(-1)^{n-1}*(*(d\omega^k\wedge\omega^k)\wedge\varepsilon_i\omega^i\wedge\varepsilon_j\omega^j).$$

Putting everything together then gives

$$\omega^i{}_j = \varepsilon_i(-1)^{\iota}*[\varepsilon_i(*d\omega^i\wedge\varepsilon_j\omega^j) - \varepsilon_j(*d\omega^j\wedge\varepsilon_i\omega^i)$$

$$- \frac{1}{2} (-1)^{n-1} \sum_k \varepsilon_k *(d\omega^k \wedge \omega^k) \wedge \varepsilon_i \omega^i \wedge \varepsilon_j \omega^j].$$

Therefore

$$\omega^{ij} = (-1)^{\iota} *[*d\omega^i \wedge \omega^j - *d\omega^j \wedge \omega^i$$

$$+ \frac{1}{2} (-1)^n \sum_k \varepsilon_k *(d\omega^k \wedge \omega^k) \wedge \omega^i \wedge \omega^j],$$

as asserted.

<u>Section 21</u>: <u>Metric Concomitants</u> Let M be a connected C^∞ manifold of dimension n.

Notation:

$$\begin{cases} (U, \{x^1,\ldots,x^n\}) \\ (\bar{U}, \{\bar{x}^1,\ldots,\bar{x}^n\}) \end{cases}$$

are charts with $U \cap \bar{U} \neq \emptyset$ such that

$$\begin{cases} \bar{x}^k = \bar{f}^k(x^1,\ldots,x^n) \quad (\equiv \bar{x}^k(x^i)) \\ x^k = f^i(\bar{x}^1,\ldots,\bar{x}^n) \quad (\equiv x^i(\bar{x}^k)). \end{cases}$$

[Note: In this section (as well as some others to follow), it will be more convenient to use bars rather than primes to designate a generic coordinate change.]

Put

$$J^i{}_k = \frac{\partial x^i}{\partial \bar{x}^k}, \quad \bar{J}^k{}_i = \frac{\partial \bar{x}^k}{\partial x^i}.$$

Then

$$\begin{cases} J^i{}_k \bar{J}^k{}_j = \delta^i{}_j \\ \bar{J}^k{}_i J^i{}_\ell = \delta^k{}_\ell. \end{cases}$$

Assume now that M is orientable — then the set $\mathcal{C}$ of coordinate systems on subsets of M splits as a disjoint union $\mathcal{C}^+ \cup \mathcal{C}^-$ such that within $\mathcal{C}^+$ or $\mathcal{C}^-$ one always has

$$J = \det[J^i{}_k] > 0.$$

Let $T \in w\text{-}\mathcal{D}_q^p(M)$ -- then

$$\bar{T}^{k_1 \cdots k_p}{}_{\ell_1 \cdots \ell_q}$$

$$= J^w \bar{J}^{k_1}{}_{i_1} \cdots \bar{J}^{k_p}{}_{i_p} J^{j_1}{}_{\ell_1} \cdots J^{j_q}{}_{\ell_q} T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}.$$

Definition: A semitensor of type (p,q) and weight w is an entity satisfying this condition within either C^+ or C^-, i.e., for coordinate changes subject to $J > 0$.

If $w\text{-}s\mathcal{D}_q^p(M)$ is the set of such, then

$$w\text{-}\mathcal{D}_q^p(M) \subset w\text{-}s\mathcal{D}_q^p(M).$$

[Note: The tensor transformation rule for the sections of $T_q^p(M) \otimes L_I^w(M)$ involves $|J|^w$, while the tensor transformation rule for the sections of $T_q^p(M) \otimes L_{II}^w(M)$ involves $\operatorname{sgn} J \cdot |J|^w$. Thus, in either case, a generic section is a semitensor of type (p,q) and weight w. For example, if $g \in \underline{M}$, then $\begin{cases} e^\bullet \\ e_\bullet \end{cases}$ are semitensors of type $\begin{cases} (n,0) \\ (0,n) \end{cases}$ and weight 0 but, being twisted, are not tensors of type $\begin{cases} (n,0) \\ (0,n) \end{cases}$.]

Definition: A metric concomitant of type (p,q), weight w, and order m is a map

$$F: \underline{M} \to w\text{-}s\mathcal{D}_q^p(M)$$

for which $\exists$ real valued C^∞ functions $F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$ of real variables

$x_{ab}, x_{ab,c_1}, \ldots, x_{ab,c_1\cdots c_m}$ such that if $(U,\{x^1,\ldots,x^n\})$ is a chart, then the components of $F(g)$ are given by

$$F(g)^{i_1\cdots i_p}{}_{j_1\cdots j_q}$$

$$= F^{i_1\cdots i_p}{}_{j_1\cdots j_q}(g_{ab}, g_{ab,c_1}, \ldots, g_{ab,c_1\cdots c_m}),$$

where the comma stands for partial differentiation, i.e.,

$$g_{ab,c_1} = \frac{\partial g_{ab}}{\partial x^{c_1}}, \ldots, g_{ab,c_1\cdots c_m} = \frac{\partial^m g_{ab}}{\partial x^{c_1}\cdots\partial x^{c_m}}.$$

[Note: The functions $F^{i_1\cdots i_p}{}_{j_1\cdots j_q}$ are not unique, thus equality of two metric concomitants means their equality as maps from $\underline{M}$ to $w\text{-}s\mathcal{D}^p_q(M)$.]

Remark: The index scheme is not set in concrete and depends on the situation, e.g., to free up a,b,c one can use r,s,t:

$$g_{rs}, g_{rs,t_1}, \ldots, g_{rs,t_1\cdots t_m}.$$

Notation: $MC_n(p,q,w,m)$ is the set of metric concomitants of type (p,q), weight w, and order m. With respect to the obvious operations, $MC(p,q,w,m)$ is a real vector space.

[Note: In general, $MC_n(p,q,w,m)$ is infinite dimensional but, under certain interesting circumstances, is finite dimensional (or even trivial).]

Example: The assignment $g \to |g|^{1/2}$ defines an element of $MC_n(0,0,1,0)$.

[Note: If $F \in MC_n(p,q,w,m)$, then the assignment $g \to |g|^{W/2}F(g)$ $(W \in \underline{Z})$

defines an element of $MC_n(p,q,w + W,m)$.]

Example: Given $g \in \underline{M}$, view the curvature tensor $R(g)$ attached to its metric connection as an element of $\mathcal{D}_4^0(M)$ — then the assignment $g \rightarrow R(g)$ is a metric concomitant of type $(0,4)$, weight 0, and order 2. Indeed,

$$R_{ijk\ell} = \frac{1}{2} (g_{i\ell,jk} - g_{ik,j\ell} + g_{jk,i\ell} - g_{j\ell,ik})$$

$$+ \Gamma_{ajk}\Gamma^a{}_{i\ell} - \Gamma_{aj\ell}\Gamma^a{}_{ik}.$$

Therefore $R_{ijk\ell}$ is linear in the second derivatives of g (but nonlinear in the first derivatives of g).]

[Note: Recall that

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{k\ell}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}).$$

Accordingly,

$$\Gamma_{kij} = g_{ka}\Gamma^a{}_{ij}$$

$$= \frac{1}{2} g^{\ell a}g_{ak}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell})$$

$$= \frac{1}{2} \delta^\ell{}_k(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell})$$

$$= \frac{1}{2} (g_{ki,j} + g_{kj,i} - g_{ij,k}).]$$

Remark: Entities such as $|R^{ij}R_{ij}|^{1/2}$ are not metric concomitants.

To reflect the underlying symmetries of the situation (stemming from the equality $g_{ab} = g_{ba}$), one can assume without loss of generality that all internal indices have been symmetrized.

Example: Suppose that $n = 2$ and let $F(g) = \det g$, a scalar density of weight 2, thus locally

$$F(g) = g_{11}g_{22} - g_{12}g_{21}.$$

Here we can take

$$F(x_{ab}) = F(x_{11},x_{12},x_{21},x_{22})$$

$$= x_{11}x_{22} - x_{12}x_{21}$$

or, in accordance with the foregoing convention,

$$F(x_{ab}) = F(x_{11}, \frac{x_{12} + x_{21}}{2}, \frac{x_{21} + x_{12}}{2}, x_{22}).$$

In the first situation

$$\frac{\partial F}{\partial x_{12}} = - x_{21} \neq \frac{\partial F}{\partial x_{21}} = - x_{12}$$

but in the second situation,

$$\frac{\partial F}{\partial x_{12}} = - \frac{1}{2} (x_{12} + x_{21}) = \frac{\partial F}{\partial x_{21}} .$$

Let $F \in MC_n(p,q,w,m)$ -- then the barred and unbarred components of $F(g)$ are related by

$$F^{k_1 \cdots k_p}{}_{\ell_1 \cdots \ell_q}(\bar{g}_{ab},\bar{g}_{ab,c_1},\ldots, \bar{g}_{ab,c_1\cdots c_m})$$

$$= J^w \bar{J}^{k_1}{}_{i_1} \cdots \bar{J}^{k_p}{}_{i_p} J^{j_1}{}_{\ell_1} \cdots J^{j_q}{}_{\ell_q}$$

$$\times F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(g_{ab},g_{ab,c_1},\ldots, g_{ab,c_1\cdots c_m}).$$

[Note: Differentiation of the tensor transformation rule

$$\bar{g}_{ab} = J^r{}_a J^s{}_b g_{rs}$$

leads to the transformation law for the derivatives of g_{ab}. To start the process, let

$$J^i{}_{k\ell} = \frac{\partial^2 x^i}{\partial \bar{x}^k \partial \bar{x}^\ell} .$$

Then we have

$$\bar{g}_{ab,c} = (J^r{}_{ac} J^s{}_b + J^r{}_a J^s{}_{bc}) g_{rs}$$

$$+ J^r{}_a J^s{}_b J^t{}_c g_{rs,t} .$$

Next, let

$$J^i{}_{k\ell m} = \frac{\partial^3 x^i}{\partial \bar{x}^k \partial \bar{x}^\ell \partial \bar{x}^m} .$$

Then we have

$$\bar{g}_{ab,cd} = (J^r{}_{acd} J^s{}_b + J^r{}_{ac} J^s{}_{bd}$$

$$+ J^r{}_{ad} J^s{}_{bc} + J^r{}_a J^s{}_{bcd}) g_{rs}$$

$$+ (J^r{}_{ac} J^s{}_b J^t{}_d + J^r{}_a J^s{}_{bc} J^t{}_d + J^r{}_{ad} J^s{}_b J^t{}_c$$

$$+ J^r{}_a J^s{}_{bd} J^t{}_c + J^r{}_a J^s{}_b J^t{}_{cd}) g_{rs,t}$$

$$+ J^r{}_a J^s{}_b J^t{}_c J^u{}_d \, g_{rs,tu} .$$

And so forth... .]

Remark: Fix indices

$$\begin{bmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_q \end{bmatrix}$$

and suppress them from the notation. Let

$$\Lambda^{ab} = \partial F(g)^{i_1\cdots i_p}{}_{j_1\cdots j_q}/\partial g_{ab}$$

and

$$\Lambda^{ab,c_1\cdots c_k} = \partial F(g)^{i_1\cdots i_p}{}_{j_1\cdots j_q}/\partial g_{ab,c_1\cdots c_k} \quad (k=1,\ldots,m).$$

Then, in general, the derivatives

$$\Lambda^{ab}, \Lambda^{ab,c_1}, \ldots, \Lambda^{ab,c_1\cdots c_m}$$

are not tensorial. However, it is possible to construct tensorial entities

$$\Pi^{ab}, \Pi^{ab,c_1}, \ldots, \Pi^{ab,c_1\cdots c_m}$$

from certain combinations of the $\Lambda^{ab,\cdots}$ which turn out to be the components of metric concomitants

$$DF(g)/Dg_{ab}, DF(g)/Dg_{ab,c_1}, \ldots, DF(g)/Dg_{ab,c_1\cdots c_m},$$

these being the so-called <u>tensorial derivatives</u> of $F(g)$.

[Note: A particular case of the construction is detailed later on when we take up the theory of lagrangians but, in brief, the procedure is this.

Given a symmetric $h \in \mathcal{D}_2^0(M)$, let

$$PF(g,h) = \Lambda^{ab} h_{ab} + \sum_{k=1}^{m} \Lambda^{ab,c_1 \cdots c_k} h_{ab,c_1 \cdots c_k} .$$

With the understanding that covariant differentiation is per the metric connection of g, the difference

$$h_{ab,c_1 \cdots c_k} - h_{ab;c_1 \cdots c_k}$$

involves the connection coefficients $\Gamma^{\cdot}_{\cdot\cdot}$, their derivatives, and the h_{ab}, $h_{ab,c_1 \cdots c_\ell}$ $(\ell < k)$. Successive substitution of these formulas (beginning with $k = m$ and ending with $k = 1$) then enables one to write

$$PF(g,h) = \Pi^{ab} h_{ab} + \sum_{k=1}^{m} \Pi^{ab,c_1 \cdots c_k} h_{ab;c_1 \cdots c_k} .$$

- Π^{ab}: This coefficient represents an element

$$DF(g)/Dg_{ab} \in MC_n(p + 2,q,w,m)$$

and

$$\Pi^{ab} = \Lambda^{ab} + \{\ldots\},$$

where ... involves the connection coefficients $\Gamma^{\cdot}_{\cdot\cdot}$, their derivatives, and the $\Lambda^{ab,c_1 \cdots c_k}$ $(k = 1,\ldots,m)$.

- $\Pi^{ab,c_1 \cdots c_k}$: This coefficient represents an element

$$DF(g)/Dg_{ab,c_1 \cdots c_k} \in MC_n(p + k + 2,q,w,m)$$

and

$$\Pi^{ab,c_1\cdots c_k} = \Lambda^{ab,c_1\cdots c_k} + \{\ldots\},$$

where ... involves the connection coefficients $\Gamma^{\cdot}_{\cdot\cdot}$, their derivatives, and the $\Lambda^{ab,c_1\cdots c_\ell}$ $(\ell > k)$.

[Note: If k = m, then

$$\Pi^{ab,c_1\cdots c_m} = \Lambda^{ab,c_1\cdots c_m}.]$$

It is not difficult to compute the tensorial derivatives when m = 1 or 2 but matters are more complicated when m = 3.]

To avoid trivialities, in what follows we shall assume that $n > 1$.

<u>LEMMA</u> Let $F \in MC_n(0,0,0,0)$ -- then $\exists$ a constant λ such that $F = \lambda$.

[To begin with,

$$F(\bar{g}_{ab}) = F(g_{ab})$$

or still,

$$F(J^r{}_a J^s{}_b g_{ab}) = F(g_{ab}).$$

Now differentiate this relation w.r.t. $J^i{}_k$:

$$\frac{\partial}{\partial J^i{}_k} F(J^r{}_a J^s{}_b g_{rs}) = \frac{\partial}{\partial J^i{}_k} F(g_{ab}) = 0$$

$\Rightarrow$

$$\frac{\partial F}{\partial \bar{g}_{ab}} \frac{\partial \bar{g}_{ab}}{\partial J^i{}_k} = 0$$

$\Rightarrow$

$$\frac{\partial F}{\partial \bar{g}_{ab}} (\delta^r{}_i \delta^k{}_a J^s{}_b g_{rs} + J^r{}_a \delta^s{}_i \delta^k{}_b g_{rs}) = 0$$

$\Rightarrow$

$$\frac{\partial F}{\partial \bar{g}_{ab}} (\delta^k{}_a J^s{}_b g_{is} + J^r{}_a \delta^k{}_b g_{ri}) = 0$$

$\Rightarrow$

$$\frac{\partial F}{\partial \bar{g}_{ab}} (\delta^k{}_a J^r{}_b g_{ir} + J^r{}_a \delta^k{}_b g_{ri}) = 0$$

$\Rightarrow$

$$\frac{\partial F}{\partial \bar{g}_{ab}} g_{ri} (\delta^k{}_a J^r{}_b + \delta^k{}_b J^r{}_a) = 0$$

$\Rightarrow$

$$2 \frac{\partial F}{\partial \bar{g}_{kb}} g_{ri} J^r{}_b = 0.$$

Specialize and take $\bar{x}^i = x^i$ -- then $J^r{}_b = \delta^r{}_b$, hence

$$\frac{\partial F}{\partial g_{kb}} g_{bi} = 0$$

$\Rightarrow$

$$\frac{\partial F}{\partial g_{k\ell}} = 0.$$

Therefore F is a constant, as claimed.]

Application: If $F \in MC_n(0,0,1,0)$, then $\exists$ a constant λ such that

$F(g) = \lambda|g|^{1/2}$.

[Consider the quotient $F(g)/|g|^{1/2}$.]

<u>LEMMA</u> If n is even and p + q is odd, then

$$MC_n(p,q,w,0) = \{0\}.$$

[Let $F \in MC_n(p,q,w,0)$ -- then

$$F^{k_1 \cdots k_p}{}_{\ell_1 \cdots \ell_q}(\bar{g}_{ab})$$

$$= J^w \bar{J}^{k_1}{}_{i_1} \cdots \bar{J}^{k_p}{}_{i_p} J^{j_1}{}_{\ell_1} \cdots J^{j_q}{}_{\ell_q} F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(g_{ab}).$$

Since n is even, we can take $\bar{x}^i = - x^i$. This gives

$$F^{k_1 \cdots k_p}{}_{\ell_1 \cdots \ell_q}(\bar{g}_{ab})$$

$$= [(-1)^n]^w \, (-1)^{p+q} F^{k_1 \cdots k_p}{}_{\ell_1 \cdots \ell_q}(g_{ab})$$

$$= (-1)^{p+q} \, F^{k_1 \cdots k_p}{}_{\ell_1 \cdots \ell_q}(g_{ab}).$$

But

$$\bar{g}_{ab} = J^r{}_a J^s{}_b g_{rs}$$

$$= (-\delta^r{}_a)(-\delta^s{}_b) g_{rs}$$

$$= g_{ab}.$$

Therefore F = 0.]

<u>FACT</u> If p + q is odd and less than n, then

$$MC_n(p,q,w,0) = \{0\}.$$

<u>Structural Considerations</u>

- Let $F \in MC_n(0,q,0,0)$. Assume: q is odd and q < n -- then F = 0.

- Let $F \in MC_n(0,n,0,0)$. Assume: n is odd -- then

$$F(g)_{j_1 \cdots j_n} = L|g|^{1/2} \varepsilon_{j_1 \cdots j_n},$$

where L is a constant.

- Let $F \in MC_n(0,q,0,0)$. Assume: q is even and q < n -- then

$$F(g)_{j_1 \cdots j_q} = \sum_{\sigma \in S_q} K_\sigma g_{\sigma(j_1)\sigma(j_2)} \cdots g_{\sigma(j_{q-1})\sigma(j_q)},$$

where the K_σ are constants.

- Let $F \in MC_n(0,n,0,0)$. Assume: n is even -- then

$$F(g)_{j_1 \cdots j_n} = \sum_{\sigma \in S_n} K_\sigma g_{\sigma(j_1)\sigma(j_2)} \cdots g_{\sigma(j_{n-1})\sigma(j_n)}$$

$$+ L|g|^{1/2} \varepsilon_{j_1 \cdots j_n},$$

where the K_σ and L are constants.

Remark: Due to the symmetry of g and the commutativity of multiplication, the decomposition

$$\sum_{\sigma \in S_q} K_\sigma g_{\sigma(j_1)\sigma(j_2)} \cdots g_{\sigma(j_{q-1})\sigma(j_q)}$$

contains redundancies, there being

$$\frac{q!}{(q/2)!2^{q/2}}$$

distinct terms (after combination of the constants).

Example: Let $F \in MC_4(0,4,0,0)$ -- then

$$F(g)_{j_1j_2j_3j_4} = K_1 g_{j_1j_2} g_{j_3j_4} + K_2 g_{j_1j_3} g_{j_2j_4}$$

$$+ K_3 g_{j_1j_4} g_{j_2j_3} + L|g|^{1/2} \varepsilon_{j_1j_2j_3j_4} ,$$

where K_1, K_2, K_3, L are constants.

Example: Let $F \in MC_4(0,4,0,0)$. Suppose that

$$F(g)_{j_1j_2j_3j_4} = - F(g)_{j_2j_1j_3j_4} .$$

Then

$$F(g)_{j_1j_2j_3j_4} = K(g_{j_1j_3} g_{j_2j_4} - g_{j_1j_4} g_{j_2j_3}) + L|g|^{1/2} \varepsilon_{j_1j_2j_3j_4} ,$$

where K and L are constants.

Remark: The situation when $q > n$ is more involved. To illustrate, $MC_2(0,4,0,0)$ contains elements of the form

$$g_{j_1j_2} |g|^{1/2} \varepsilon_{j_3j_4} \text{ and } |g|^{1/2} \varepsilon_{j_1j_2} \cdot |g|^{1/2} \varepsilon_{j_3} \varepsilon_{j_4} .$$

[Note: Using classical invariant theory, one can express an arbitrary element of $MC_n(0,q,0,0)$ $(q > n)$ in terms of products of the g_{ij} and lower Levi-Civita symbols.]

While formulated covariantly, all of the preceding results admit contravariant counterparts.

Example: Let $F \in MC_n(2,0,0,0)$ $(n > 2)$ -- then

$$F(g)^{ij} = Kg^{ij},$$

where K is a constant.

[Differentiate the identity

$$J^k{}_j J^\ell{}_i F^{ij}(J^r{}_s J^c{}_d g_{rc}) = F^{ij}(g_{sd})$$

w.r.t. $J^a{}_b$ and then set $\bar{x}^i = x^i$. This gives

$$(\delta^k{}_a \delta^b{}_j \delta^\ell{}_i + \delta^k{}_j \delta^\ell{}_a \delta^b{}_i) F^{ij}$$

$$+ \delta^k{}_j \delta^\ell{}_i (\delta^r{}_a \delta^b{}_s \delta^c{}_d + \delta^r{}_s \delta^c{}_a \delta^b{}_d) g_{rc} \frac{\partial F^{ij}}{\partial g_{sd}}$$

$$= 0$$

or still,

$$\delta^k{}_a F^{\ell b} + \delta^\ell{}_a F^{bk} + 2g_{ac} \frac{\partial F^{\ell k}}{\partial g_{bc}} = 0,$$

from which (upon multiplying by g^{ad}),

$$g^{kd} F^{\ell b} + g^{\ell d} F^{bk} = - 2 \frac{\partial F^{\ell k}}{\partial g_{bd}} .$$

But the RHS is symmetric in b & d, hence

$$g^{kd} F^{\ell b} + g^{\ell d} F^{bk} = g^{kb} F^{\ell d} + g^{\ell b} F^{dk}.$$

Now multiply through by g_{kd} -- then

$$nF^{\ell b} + F^{b\ell} = F^{\ell b} + \varkappa g^{\ell b},$$

where

$$\varkappa = g_{kd} F^{dk},$$

or still,

$$(n-1)F^{\ell b} + F^{b\ell} = \varkappa g^{\ell b}.$$

To solve for $F^{\ell b}$, note that

$$(n-1)^2F^{\ell b} + (n-1)F^{b\ell} = (n-1)\varkappa g^{\ell b}$$

$$\Rightarrow$$

$$(n^2-2n+1)F^{\ell b} + (\varkappa g^{\ell b}-F^{\ell b}) = (n-1)\varkappa g^{\ell b}$$

$$\Rightarrow$$

$$n(n-2)F^{\ell b} = (n-2)\varkappa g^{\ell b}$$

$$\Rightarrow$$

$$F^{\ell b} = \frac{1}{n}\varkappa g^{\ell b}.$$

To see that $\varkappa$ is a constant, substitute back into the differential equation, thus

$$g^{kd}\varkappa g^{\ell b} + g^{\ell d}\varkappa g^{bk} = -2\frac{\partial(\varkappa g^{\ell k})}{\partial g_{bd}}$$

$$= -2\left[\frac{\partial\varkappa}{\partial g_{bd}} g^{\ell k} + \varkappa \frac{\partial g^{\ell k}}{\partial g_{bd}}\right]$$

$$= -2\left[\frac{\partial\varkappa}{\partial g_{bd}} g^{\ell k} + \varkappa\left(-\frac{g^{\ell b}g^{dk}+g^{\ell d}g^{bk}}{2}\right)\right]$$

$$\Rightarrow$$

$$\frac{\partial\varkappa}{\partial g_{bd}} = 0.$$

I.e.: $\varkappa$ is a constant.]

[Note: If $F \in MC_n(2,0,1,0)$ $(n > 2)$, then $\exists$ a constant K such that $F(g)^{ij} = K|g|^{1/2} g^{ij}$ (apply the above analysis to the quotient $F(g)/|g|^{1/2}$).]

Example: Let $F \in MC_2(2,0,0,0)$ — then

$$F(g)^{ij} = Kg^{ij} + L \frac{\varepsilon^{ij}}{|g|^{1/2}},$$

where K and L are constants.

[From the preceding example, we have

$$F^{\ell b} + F^{b\ell} = \varkappa g^{\ell b},$$

thus

$$F^{\ell b} = \frac{\varkappa}{2} g^{\ell b} + \frac{1}{2} (F^{\ell b} - F^{b\ell}),$$

and so $(n = 2)$,

$$F^{\ell b} = \frac{\varkappa}{2} g^{\ell b} + \frac{\lambda}{2} \frac{\varepsilon^{\ell b}}{|g|^{1/2}}.$$

Therefore

$$g^{kd}\left(\frac{\varkappa}{2} g^{\ell b} + \frac{\lambda}{2} \frac{\varepsilon^{\ell b}}{|g|^{1/2}}\right) + g^{\ell d}\left(\frac{\varkappa}{2} g^{bk} + \frac{\lambda}{2} \frac{\varepsilon^{bk}}{|g|^{1/2}}\right)$$

$$= -2 \frac{\partial}{\partial g_{bd}} \left(\frac{\varkappa}{2} g^{\ell k} + \frac{\lambda}{2} \frac{\varepsilon^{\ell k}}{|g|^{1/2}}\right)$$

$\Rightarrow$

$$\frac{\lambda}{2|g|^{1/2}} (g^{kd} \varepsilon^{\ell b} + g^{\ell d} \varepsilon^{bk})$$

$$= -\frac{\partial \varkappa}{\partial g_{bd}} g^{\ell k} - 2 \frac{\partial}{\partial g_{bd}} \left(\frac{\lambda}{2} \frac{\varepsilon^{\ell k}}{|g|^{1/2}}\right).$$

Since the LHS of this relation is skew symmetric in k & ℓ, it follows that

$$\frac{\partial \varkappa}{\partial g_{bd}} = 0,$$

hence $\varkappa$ is a constant. Now take $k = 1$, $\ell = 2$ -- then

$$\frac{\lambda}{2|g|^{1/2}} (g^{1d} \varepsilon^{2b} + g^{2d} \varepsilon^{b1})$$

$$= \frac{\partial}{\partial g_{bd}} (\frac{\lambda}{|g|^{1/2}}).$$

Suppose that $b = 1$:

$$\frac{\lambda}{2|g|^{1/2}} (- g^{1d}) = \frac{\partial}{\partial g_{1d}} (\frac{\lambda}{|g|^{1/2}})$$

$$= \frac{\partial \lambda}{\partial g_{1d}} \frac{1}{|g|^{1/2}} + \lambda \frac{\partial |g|^{-1/2}}{\partial g_{1d}} .$$

But

$$\frac{\partial |g|^{-1/2}}{\partial g_{1d}} = - \frac{1}{2} |g|^{-3/2} \frac{\partial |g|}{\partial g_{1d}}$$

$$= - \frac{1}{2} |g|^{-3/2} |g| g^{1d}$$

$$= - \frac{1}{2} \frac{g^{1d}}{|g|^{1/2}} .$$

Therefore

$$- \frac{\lambda}{2|g|^{1/2}} g^{1d} = \frac{\partial \lambda}{\partial g_{1d}} \frac{1}{|g|^{1/2}} - \frac{\lambda}{2|g|^{1/2}} g^{1d}$$

$\Rightarrow$

$$\frac{\partial \lambda}{\partial g_{1d}} = 0.$$

The same argument shows that

$$\frac{\partial\lambda}{\partial g_{2d}} = 0.$$

Conclusion: λ is a constant.]

[Note: If $F \in MC_2(2,0,1,0)$, then $\exists$ constants K and L such that $F(g)^{ij} = K|g|^{1/2}g^{ij} + L\epsilon^{ij}$ (apply the above analysis to the quotient $F(g)/|g|^{1/2}$).]

Example: Let $F \in MC_4(6,0,1,0)$. Suppose that

$$F(g)^{abcrst} = F(g)^{rstabc}.$$

Then

$$\begin{aligned} F(g)^{abcrst} = |g|^{1/2}[&K_1(g^{ab}g^{cr}g^{st} + g^{at}g^{bc}g^{rs}) \\ &+ K_2(g^{ab}g^{cs}g^{rt} + g^{ac}g^{bt}g^{rs}) \\ &+ K_3(g^{ac}g^{br}g^{st} + g^{as}g^{cb}g^{rt}) \\ &+ K_4(g^{as}g^{cr}g^{bt} + g^{at}g^{br}g^{cs}) \\ &+ K_5g^{ab}g^{ct}g^{rs} + K_6g^{ac}g^{bs}g^{rt} \\ &+ K_7g^{ar}g^{bc}g^{st} + K_8g^{ar}g^{bs}g^{ct} \\ &+ K_9g^{ar}g^{bt}g^{cs} + K_{10}g^{as}g^{ct}g^{br} \\ &+ K_{11}g^{at}g^{bs}g^{cr}] \end{aligned}$$

$$+ L_1(g^{ab}{}_{\varepsilon}{}^{crst} + g^{rs}{}_{\varepsilon}{}^{tabc})$$

$$+ L_2(g^{ac}{}_{\varepsilon}{}^{brst} + g^{rt}{}_{\varepsilon}{}^{sabc})$$

$$+ L_3(g^{as}{}_{\varepsilon}{}^{crbt} + g^{rb}{}_{\varepsilon}{}^{tasc})$$

$$+ L_4(g^{at}{}_{\varepsilon}{}^{crsb} + g^{rc}{}_{\varepsilon}{}^{tabs})$$

$$+ L_5(g^{bc}{}_{\varepsilon}{}^{arst} + g^{st}{}_{\varepsilon}{}^{rabc})$$

$$+ L_6(g^{bt}{}_{\varepsilon}{}^{crsa} + g^{sc}{}_{\varepsilon}{}^{tabr}),$$

whre K_k $(k = 1,\ldots,11)$ and L_ℓ $(\ell = 1,\ldots,6)$ are constants.

[Note: The quantity $g^{ra}{}_{\varepsilon}{}^{cbst}$ has the required symmetry but there is no contradiction since

$$2g^{ra}{}_{\varepsilon}{}^{cbst} = -\ (g^{at}{}_{\varepsilon}{}^{crsb} + g^{rc}{}_{\varepsilon}{}^{tabs})$$

$$-\ (g^{as}{}_{\varepsilon}{}^{crbt} + g^{rb}{}_{\varepsilon}{}^{tasc})$$

$$-\ g^{ab}{}_{\varepsilon}{}^{rcst} + g^{rs}{}_{\varepsilon}{}^{atbc})$$

$$-\ (g^{ac}{}_{\varepsilon}{}^{brst} + g^{rt}{}_{\varepsilon}{}^{sabc}).]$$

<u>INDEPENDENCE THEOREM</u> Let $F \in MC_n(p,q,w,1)$, so that

$$F(g)^{i_1\cdots i_p}{}_{j_1\cdots j_q} = F^{i_1\cdots i_p}{}_{j_1\cdots j_q}(g_{ab}, g_{ab,c}).$$

Then

$$\partial F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(g_{ab}, g_{ab,c})/\partial g_{ab,c} = 0.$$

Therefore the components

$$F(g)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

do not depend on the $g_{ab,c}$ explicitly, thus are independent of the first partial derivatives.

Rappel: Let $g \in \underline{M}$. Fix a point $x_0 \in M$ and let $x^1, \ldots, x^n$ be normal coordinates at x_0 -- then there is a Taylor expansion

$$g_{ab}(x) = g_{ab}(x_0) + \frac{1}{2!} G_{abc_1c_2}(x_0)\ x^{c_1} x^{c_2} + \frac{1}{3!} G_{abc_1c_2c_3}(x_0)\ x^{c_1} x^{c_2} x^{c_3} + \cdots ,$$

where the coefficients

$$G_{abc_1 \cdots c_k} \in MC_n(0, 2 + k, 0, k)$$

possess the following symmetries:

(1) $G_{abc_1 \cdots c_k} = G_{bac_1 \cdots c_k}$;

(2) $G_{abc_1 \cdots c_k} = G_{ab(c_1 \cdots c_k)}$;

(3) $G_{a(bc_1 \cdots c_k)} = 0.$

[Note: By construction, $G_{abc_1 \cdots c_k}$ is a function of the curvature tensor of g (viewed as an element of $\mathcal{D}^0_4(M)$) and its repeated covariant derivatives.

So, e.g.,

$$G_{abc_1c_2} = -\frac{1}{3}(R_{ac_1bc_2} + R_{bc_1ac_2})$$

and

$$G_{abc_1c_2c_3} = -\frac{1}{6}(R_{ac_1bc_2;c_3} + R_{ac_2bc_3;c_1} + R_{ac_3bc_1;c_2}$$

$$+ R_{bc_1ac_2;c_3} + R_{bc_2ac_3;c_1} + R_{bc_3ac_1;c_2}).]$$

<u>REPLACEMENT THEOREM</u> Let $F \in MC_n(p,q,w,m)$ -- then

$$F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(g_{ab}, g_{ab,c_1}, \ldots, g_{ab,c_1 \cdots c_m})$$

$$= F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(g_{ab}, 0, G_{abc_1c_2}, \ldots, G_{abc_1 \cdots c_m}).$$

Example: If n is even and q is odd, then

$$MC_n(0,q,w,2) = \{0\}.$$

[Let $F \in MC_n(0,q,w,2)$ — then

$$F_{\ell_1 \cdots \ell_q}(\bar{g}_{ab}, \bar{G}_{abc_1c_2})$$

$$= J^w J^{j_1}{}_{\ell_1} \cdots J^{j_q}{}_{\ell_q} F_{j_1 \cdots j_q}(g_{ab}, G_{abc_1c_2}).$$

Since n is even, we can take $\bar{x}^i = -x^i$. This gives

$$F_{\ell_1 \cdots \ell_q}(\bar{g}_{ab}, \bar{G}_{abc_1c_2})$$

$$= [(-1)^n]^w \, (-1)^q \, F_{\ell_1 \cdots \ell_q}(g_{ab}, G_{abc_1c_2})$$

$$= (-1)^q \, F_{\ell_1 \cdots \ell_q}(g_{ab}, G_{abc_1c_2}).$$

But

$$\bar{g}_{ab} = g_{ab}$$

and

$$\bar{G}_{abc_1c_2} = G_{abc_1c_2}.$$

Therefore $F = 0$.]

Section 22: Lagrangians Let M be a connected C^∞ manifold of dimension n, which we shall assume is orientable.

Definition: A lagrangian of order m is an element

$$L \in MC_n(0,0,1,m).$$

In what follows, our primary concern will be with the case m = 2, thus

$$L(\bar{g}_{ab},\bar{g}_{ab,c},\bar{g}_{ab,cd}) = JL(g_{ab},g_{ab,c},g_{ab,cd}),$$

the basic identity.

[Note: Recall that the elements of $MC_n(0,0,1,0)$ are simply the constant multiples of $|g|^{1/2}$. As for the elements of $MC_n(0,0,1,1)$, say

$$L(g) = L(g_{ab},g_{ab,c}),$$

the Independence Theorem implies that

$$\frac{\partial L}{\partial g_{ab,c}} = 0,$$

hence L(g) depends solely on the g_{ab} and not their first derivatives.]

Given an

$$L \in MC_n(0,0,1,2),$$

put

$$\Lambda^{ab} = \frac{\partial L}{\partial g_{ab}}, \ \Lambda^{ab,c} = \frac{\partial L}{\partial g_{ab,c}}, \ \Lambda^{ab,cd} = \frac{\partial L}{\partial g_{ab,cd}}.$$

Then

$$\Lambda^{ab} = \Lambda^{ba}, \ \Lambda^{ab,c} = \Lambda^{ba,c}, \ \Lambda^{ab,cd} = \Lambda^{ba,cd} = \Lambda^{ab,dc}.$$

<u>Transformation Laws</u>

(1) $J\Lambda^{ab,cd}$

$$= \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab,cd}}$$

$$= \bar{\Lambda}^{ij,k\ell} J^{a}{}_{i} J^{b}{}_{j} J^{c}{}_{k} J^{d}{}_{\ell}.$$

(2) $J\Lambda^{ab,c}$

$$= \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab,c}} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}}.$$

(3) $J\Lambda^{ab}$

$$= \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial g_{ab}} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab}} + \bar{\Lambda}^{ij} \frac{\partial \bar{g}_{ij}}{\partial g_{ab}}.$$

[Note: Therefore $\Lambda^{ab,cd}$ is tensorial but $\Lambda^{ab,c}$ and Λ^{ab} are not tensorial.]

Denote by $S_2(M)$ the set of symmetric elements in $\mathcal{D}_2^0(M)$.

Definition: Let $L \in MC_n(0,0,1,2)$ -- then its <u>principal form</u> is the map

$$PL: \underline{M} \times S_2(M) \to 1\text{-}s\mathcal{D}_0^0(M)$$

defined by the prescription

$$PL(g,h) = \frac{d}{d\varepsilon} L(g+\varepsilon h) \Big|_{\varepsilon=0}.$$

Locally,

$$\frac{d}{d\varepsilon} L(g_{ab} + \varepsilon h_{ab}, g_{ab,c} + \varepsilon h_{ab,c}, g_{ab,cd} + \varepsilon h_{ab,cd}) \Big|_{\varepsilon=0}$$

$$= \Lambda^{ab}h_{ab} + \Lambda^{ab,c}h_{ab,c} + \Lambda^{ab,cd}h_{ab,cd}.$$

[Note: To check that PL(g,h) is an element of $1\text{-}s\mathcal{D}_0^0(M)$, use the foregoing transformation laws:

$$J\Lambda^{ab}h_{ab} + J\Lambda^{ab,c}h_{ab,c} + J\Lambda^{ab,cd}h_{ab,cd}$$

$$= \bar{\Lambda}^{ij,k\ell}\ [\frac{\partial\bar{g}_{ij,k\ell}}{\partial g_{ab,cd}}\, h_{ab,cd} + \frac{\partial\bar{g}_{ij,k\ell}}{\partial g_{ab,c}}\, h_{ab,c} + \frac{\partial\bar{g}_{ij,k\ell}}{\partial g_{ab}}\, h_{ab}]$$

$$+ \bar{\Lambda}^{ij,k}\ [\frac{\partial\bar{g}_{ij,k}}{\partial g_{ab,c}}\, h_{ab,c} + \frac{\partial\bar{g}_{ij,k}}{\partial g_{ab}}\, h_{ab}]$$

$$+ \bar{\Lambda}^{ij}\ [\frac{\partial\bar{g}_{ij}}{\partial g_{ab}}\, h_{ab}]$$

$$= \bar{\Lambda}^{ij,k\ell}\bar{h}_{ij,k\ell} + \bar{\Lambda}^{ij,k}\bar{h}_{ij,k} + \bar{\Lambda}^{ij}\bar{h}_{ij}.$$

Here it is necessary to keep in mind that the terms figuring in the transformation laws for g_{ab} and its derivatives are precisely the terms figuring in the transformation laws for h_{ab} and its derivatives. For instance,

$$\frac{\partial\bar{g}_{ij,k}}{\partial g_{ab}} = J^a{}_{ik}J^b{}_j + J^a{}_iJ^b{}_{jk}.]$$

Remark: In reality,

$$\frac{d}{d\varepsilon}\, L(g+\varepsilon h)\Big|_{\varepsilon=0}$$

is meaningful only if h is compactly supported, the difficulty being that, in general, $g+\varepsilon h \notin \underline{M}$ no matter the choice of $\varepsilon \neq 0$. E.g.: Take $M = \underline{R}$, let g be the usual metric, and consider $g+\varepsilon h$, where $h_x = xg_x$ -- then at $x = -1/\varepsilon$,

$$g_{-1/\varepsilon} + \varepsilon(-1/\varepsilon)g_{-1/\varepsilon} = 0.$$

Thus, strictly speaking, the introduction of

$$\frac{d}{d\varepsilon} L(g+\varepsilon h)\Big|_{\varepsilon=0}$$

serves merely to motivate the definition of $PL(g,h)$.

Let $\mathrm{con}_0 TM$ stand for the set of torsion free connections on TM.

Rappel: If $\nabla \in \mathrm{con}_0 TM$ and $h \in S_2(M)$, then

$$h_{ij;k} = h_{ij,k} - \Gamma^a{}_{ik} h_{aj} - \Gamma^a{}_{jk} h_{ia}$$

and

$$\begin{aligned} h_{ij;k\ell} = h_{ij,k\ell} &- \Gamma^a{}_{ik} h_{aj,\ell} - \Gamma^a{}_{jk} h_{ia,\ell} - \Gamma^a{}_{ik,\ell} h_{aj} - \Gamma^a{}_{jk,\ell} h_{ia} \\ &- \Gamma^b{}_{i\ell} h_{bj,k} + \Gamma^b{}_{i\ell}(\Gamma^c{}_{bk} h_{cj} + \Gamma^c{}_{jk} h_{bc}) \\ &- \Gamma^b{}_{j\ell} h_{ib,k} + \Gamma^b{}_{j\ell}(\Gamma^c{}_{ik} h_{cb} + \Gamma^c{}_{bk} h_{ic}) \\ &- \Gamma^b{}_{k\ell} h_{ij,b} + \Gamma^b{}_{k\ell}(\Gamma^c{}_{ib} h_{cj} + \Gamma^c{}_{jb} h_{ic}). \end{aligned}$$

Given $\nabla \in \mathrm{con}_0 TM$, define $\Pi_\nabla^{ij,k}$ by

$$\Pi_\nabla^{ij,k} = \Lambda^{ij,k} + 2\Gamma^i{}_{a\ell}\Lambda^{aj,k\ell} + 2\Gamma^j{}_{a\ell}\Lambda^{ai,k\ell} + \Gamma^k{}_{b\ell}\Lambda^{ij,b\ell}$$

and define Π_∇^{ij} by

$$\Pi_\nabla^{ij} = \Lambda^{ij} + \Gamma^i{}_{ak,\ell}\Lambda^{aj,k\ell} + \Gamma^j{}_{ak,\ell}\Lambda^{ai,k\ell}$$

$$- \Gamma^{b}{}_{a\ell}\Gamma^{i}{}_{bk}\Lambda^{aj,k\ell} - \Gamma^{b}{}_{c\ell}\Gamma^{j}{}_{bk}\Lambda^{ci,k\ell}$$

$$- \Gamma^{i}{}_{b\ell}\Gamma^{j}{}_{ck}\Lambda^{bc,k\ell} - \Gamma^{j}{}_{b\ell}\Gamma^{i}{}_{ck}\Lambda^{bc,k\ell}$$

$$- \Gamma^{b}{}_{k\ell}\Gamma^{i}{}_{cb}\Lambda^{cj,k\ell} - \Gamma^{b}{}_{k\ell}\Gamma^{j}{}_{cb}\Lambda^{ci,k\ell}$$

$$+ \Gamma^{i}{}_{ak}\Pi_{\nabla}^{aj,k} + \Gamma^{j}{}_{ak}\Pi_{\nabla}^{ia,k}.$$

Complete the picture and set

$$\Pi_{\nabla}^{ij,k\ell} = \Lambda^{ij,k\ell}.$$

<u>LEMMA</u> $\forall\ h \in S_2(M)$, we have

$$\Lambda^{ij}h_{ij} + \Lambda^{ij,k}h_{ij,k} + \Lambda^{ij,k\ell}h_{ij,k\ell}$$

$$= \Pi_{\nabla}^{ij}h_{ij} + \Pi_{\nabla}^{ij,k}h_{ij;k} + \Pi_{\nabla}^{ij,k\ell}h_{ij;k\ell}.$$

[That these expressions are equal is simply a computational consequence of the definitions.]

<u>FACT</u> $\forall\ \nabla \in \mathrm{con}_0 TM$, Π_{∇}^{ij}, $\Pi_{\nabla}^{ij,k}$, and $\Pi_{\nabla}^{ij,k\ell}$ are tensorial.

<u>D_0L:</u> This is the map

$$\begin{bmatrix} M \times \mathrm{con}_0 TM & \rightarrow & 1\text{-}s\mathcal{D}_0^2(M) \\ (g,\nabla) & \longrightarrow & D_0L(g,\nabla) \end{bmatrix}$$

with components Π_{∇}^{ab}.

<u>D_1L</u>: This is the map

$$\begin{cases} M \times \text{con}_0 TM & \rightarrow & 1\text{-}s\mathcal{D}_0^3(M) \\ (g,\nabla) & \longrightarrow & D_1L(g,\nabla) \end{cases}$$

with components $\Pi_\nabla^{ab,c}$.

<u>D_2L</u>: This is the map

$$\begin{cases} M \times \text{con}_0 TM & \rightarrow & 1\text{-}s\mathcal{D}_0^4(M) \\ (g,\nabla) & \longrightarrow & D_2L(g,\nabla) \end{cases}$$

with components $\Pi_\nabla^{ab,cd}$.

Rappel: Let $g \in \underline{M}$ -- then the connection coefficients of the metric connection ∇^g are

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{k\ell}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}).$$

• The <u>tensorial derivative</u> of L w.r.t. g_{ab} is the element

$$DL(g)/Dg_{ab} \in MC_n(2,0,1,2)$$

defined by

$$g \rightarrow D_0L(g,\nabla^g).$$

• The <u>tensorial derivative</u> of L w.r.t. $g_{ab,c}$ is the element

$$DL(g)/Dg_{ab,c} \in MC_n(3,0,1,2)$$

defined by

$$g \rightarrow D_1L(g,\nabla^g).$$

• The <u>tensorial derivative</u> of L w.r.t. $g_{ab,cd}$ is the element

$$DL(g)/Dg_{ab,cd} \in MC_n(4,0,1,2)$$

defined by

$$g \rightarrow D_2 L(g,\nabla^g).$$

When working locally, the tensorial derivatives of L w.r.t. g_{ab}, $g_{ab,c}$, $g_{ab,cd}$ will be denoted by Π^{ab}, $\Pi^{ab,c}$, $\Pi^{ab,cd}$.

On the basis of the definitions,

$$\Pi^{ab} = \Pi^{ba},\ \Pi^{ab,c} = \Pi^{ba,c},\ \Pi^{ab,cd} = \Pi^{ba,cd} = \Pi^{ab,dc}.$$

In addition to these elementary symmetries, there are two others which lie deeper, viz.

$$\left[\begin{array}{l} \Pi^{ab,cd} = \Pi^{cd,ab} \\ \Pi^{ab,c} = 0. \end{array}\right.$$

<u>LEMMA</u> We have

$$\Pi^{ab,cd} + \Pi^{ac,db} + \Pi^{ad,bc} = 0.$$

[Consider the basic identity

$$L(\bar{g}_{ij},\bar{g}_{ij,k},\bar{g}_{ij,k\ell}) = JL(g_{ij},g_{ij,k},g_{ij,k\ell}).$$

Using the transformation laws, express $\bar{g}_{ij}$, $\bar{g}_{ij,k}$, $\bar{g}_{ij,k\ell}$ in terms of g_{ab}, $g_{ab,c}$, $g_{ab,cd}$, the result being an expression on the LHS involving $J^r{}_s$, $J^r{}_{st}$, $J^r{}_{stu}$ (the RHS is, of course, independent of these variables). Now differentiate w.r.t. $J^r{}_{stu}$ -- then

$$\bar{\Lambda}^{ij} \frac{\partial \bar{g}_{ij}}{\partial J^r{}_{stu}} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial J^r{}_{stu}} + \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial J^r{}_{stu}} = 0$$

or still,

$$\bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial J^r{}_{stu}} = 0,$$

where

$$\frac{\partial \bar{g}_{ij,k\ell}}{\partial J^r{}_{stu}} = \frac{\partial}{\partial J^r{}_{stu}} (J^a{}_{ik\ell} J^b{}_j + J^a{}_i J^b{}_{jk\ell}) g_{ab}.$$

But

$$\bar{\Lambda}^{ij,k\ell} \frac{\partial}{\partial J^r{}_{stu}} (J^a{}_i J^b{}_{jk\ell} g_{ab})$$

$$= \bar{\Lambda}^{ji,k\ell} \frac{\partial}{\partial J^r{}_{stu}} (J^a{}_i J^b{}_{jk\ell} g_{ab})$$

$$= \bar{\Lambda}^{ij,k\ell} \frac{\partial}{\partial J^r{}_{stu}} (J^a{}_j J^b{}_{ik\ell} g_{ab})$$

$$= \bar{\Lambda}^{ij,k\ell} \frac{\partial}{\partial J^r{}_{stu}} (J^b{}_j J^a{}_{ik\ell} g_{ba})$$

$$= \bar{\Lambda}^{ij,k\ell} \frac{\partial}{\partial J^r{}_{stu}} (J^b{}_j J^a{}_{ik\ell} g_{ab})$$

$$= \bar{\Lambda}^{ij,k\ell} \frac{\partial}{\partial J^r{}_{stu}} (J^a{}_{ik\ell} J^b{}_j g_{ab}).$$

Therefore

$$\bar{\Lambda}^{ij,k\ell} \frac{\partial}{\partial J^r{}_{stu}} (J^a{}_{ik\ell} J^b{}_j g_{ab}) = 0.$$

Since

$$J^r{}_{stu} = J^r{}_{sut} = J^r{}_{tus} = J^r{}_{tsu} = J^r{}_{ust} = J^r{}_{uts},$$

it follows that

$$\bar{\Lambda}^{ij,k\ell} J^b{}_j \delta^a{}_r (\delta^s{}_i \delta^t{}_k \delta^u{}_\ell + \delta^s{}_i \delta^u{}_k \delta^t{}_\ell + \delta^t{}_i \delta^u{}_k \delta^s{}_\ell$$

$$+ \delta^t{}_i \delta^s{}_k \delta^u{}_\ell + \delta^u{}_i \delta^s{}_k \delta^t{}_\ell + \delta^u{}_i \delta^t{}_k \delta^s{}_\ell) g_{ab} = 0.$$

Specialize and take $\bar{x}^i = x^i$, thus $J^b{}_j = \delta^b{}_j$ and matters reduce to

$$(\Lambda^{sb,tu} + \Lambda^{tb,us} + \Lambda^{ub,st}) g_{rb} = 0,$$

from which

$$\Lambda^{sb,tu} + \Lambda^{tb,us} + \Lambda^{ub,st} = 0$$

or still,

$$\Lambda^{bs,tu} + \Lambda^{bt,us} + \Lambda^{bu,st} = 0.$$

Put

$$b = a, \ s = b, \ t = c, \ u = d$$

to get

$$\Lambda^{ab,cd} + \Lambda^{ac,db} + \Lambda^{ad,bc} = 0$$

or still,

$$\Pi^{ab,cd} + \Pi^{ac,db} + \Pi^{ad,bc} = 0,$$

as desired.]

Application: $\Pi^{ab,cd} = \Pi^{cd,ab}$.

[To see this, write

$$\Pi^{ab,cd} = - \Pi^{ac,db} - \Pi^{ad,bc}$$

$$= - \Pi^{ca,db} - \Pi^{da,bc}$$

$$= \Pi^{cd,ba} + \Pi^{cb,ad} + \Pi^{db,ca} + \Pi^{dc,ab}$$

$$= \Pi^{cd,ab} + \Pi^{cd,ab} + \Pi^{cb,ad} + \Pi^{db,ca}$$

$$= 2\Pi^{cd,ab} + \Pi^{cb,ad} + \Pi^{db,ca}.$$

But

$$\Pi^{ab,cd} = \Pi^{ba,dc}$$

$$= - \Pi^{bd,ca} - \Pi^{bc,ad}$$

$$= - \Pi^{cb,ad} - \Pi^{db,ca}.$$

Therefore

$$\Pi^{ab,cd} = 2\Pi^{cd,ab} - \Pi^{ab,cd}$$

$$\Rightarrow$$

$$\Pi^{ab,cd} = \Pi^{cd,ab}.]$$

As a preliminary to the proof of the relation

$$\Pi^{ab,c} = 0,$$

differentiate the basic identity

$$L(\bar{g}_{ij},\bar{g}_{ij,k},\bar{g}_{ij,k\ell}) = JL(g_{ij},g_{ij,k},g_{ij,k\ell})$$

w.r.t. $J^r{}_{st}$, thus

$$\bar{\Lambda}^{ij} \frac{\partial \bar{g}_{ij}}{\partial J^r{}_{st}} + \bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial J^r{}_{st}} + \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial J^r{}_{st}} = 0$$

or still,

$$\bar{\Lambda}^{ij,k} \frac{\partial \bar{g}_{ij,k}}{\partial J^r{}_{st}} + \bar{\Lambda}^{ij,k\ell} \frac{\partial \bar{g}_{ij,k\ell}}{\partial J^r{}_{st}} = 0.$$

Here

$$\frac{\partial \bar{g}_{ij,k}}{\partial J^r{}_{st}} = \frac{\partial}{\partial J^r{}_{st}} (J^a{}_{ik}J^b{}_j + J^a{}_i J^b{}_{jk}) g_{ab}$$

and

$$\frac{\partial \bar{g}_{ij,k\ell}}{\partial J^r{}_{st}} = \frac{\partial}{\partial J^r{}_{st}} (J^a{}_{ik}J^b{}_{j\ell} + J^a{}_{i\ell}J^b{}_{jk}) g_{ab}$$

$$+ \frac{\partial}{\partial J^r{}_{st}} (J^a{}_{ik}J^b{}_j J^c{}_\ell + J^a{}_i J^b{}_{jk} J^c{}_\ell + J^a{}_{i\ell} J^b{}_j J^c{}_k$$

$$+ J^a{}_i J^b{}_{j\ell} J^c{}_k + J^a{}_i J^b{}_j J^c{}_{k\ell}) g_{ab,c}.$$

Now do the math and then take $\bar{x}^i = x^i$ to get

$$2\Lambda^{sb,tc} g_{rb,c} + 2\Lambda^{tb,sc} g_{rb,c} + \Lambda^{ab,st} g_{ab,r}$$

$$+ \Lambda^{sb,t} g_{rb} + \Lambda^{tb,s} g_{rb} = 0.$$

Fix a point $x_0 \in M$ and introduce normal coordinates at x_0 — then $g_{ij,k}\big|_{x_0} = 0$ and $\Gamma^k{}_{ij}\big|_{x_0} = 0$, hence at x_0,

$$\Pi^{rs,t} = \Lambda^{rs,t},$$

so from the above,

$$\Pi^{sb,t} g_{rb} + \Pi^{tb,s} g_{rb} = 0$$

or still,

$$\Pi^{sb,t} + \Pi^{tb,s} = 0.$$

Replace b by r — then

$$\Pi^{sr,t} = - \Pi^{tr,s}$$

$$= - \Pi^{rt,s}$$

$$= \Pi^{st,r}$$

$$= \Pi^{ts,r}$$

$$= - \Pi^{rs,t}$$

$$= - \Pi^{sr,t}$$

$\Rightarrow$

$$\Pi^{sr,t} = 0$$

$\Rightarrow$

$$\Pi^{rs,t} = 0.$$

Since $\Pi^{rs,t}$ is tensorial and x_0 is arbitrary, it follows that

$$\Pi^{rs,t} = 0$$

throughout all of M.

Remark: Suppose that

$$L(g) = L(g_{ab}, g_{ab,c}).$$

Then $\Lambda^{ab,cd} = 0$, thus $\Pi^{ab,c} = \Lambda^{ab,c}$. But

$$\Pi^{ab,c} = 0.$$

Therefore

$$\frac{\partial L}{\partial g_{ab,c}} = 0,$$

which, as has been noted earlier, is a particular case of the Independence Theorem.

LEMMA We have

$$2\Lambda^{sb,cd}g_{rb,cd} + 2\Lambda^{ab,sd}g_{ab,rd}$$

$$+ 2\Lambda^{sb,c}g_{rb,c} + \Lambda^{ab,s}g_{ab,r} + 2\Lambda^{sb}g_{rb} = \delta^{s}{}_{r}L.$$

[Differentiate the basic identity w.r.t. $J^{r}{}_{s}$ and then take $\bar{x}^{i} = x^{i}$.]

Claim:

$$\Pi^{rs} = \frac{1}{2}\, g^{rs}L - \frac{2}{3}\,\Lambda^{k\ell,ra}R^{s}{}_{k\ell a}.$$

To see this, fix a point x_0 in M and introduce normal coordinates at x_0.

SUBLEMMA If H_{ij} is any quantity which is symmetric in i & j, then

$$H_{ij}\Lambda^{ai,bj} = -\frac{1}{2}\,H_{ij}\Lambda^{ab,ij}.$$

[In fact,

$$\Lambda^{ai,bj} + \Lambda^{ab,ji} + \Lambda^{aj,ib} = 0$$

$$\Rightarrow$$

$$\Lambda^{ai,bj} + \Lambda^{aj,bi} = -\,\Lambda^{ab,ij}.$$

Therefore

$$H_{ij}\Lambda^{ai,bj} = \frac{1}{2} H_{ij}(\Lambda^{ai,bj} + \Lambda^{aj,bi})$$

$$= -\frac{1}{2} H_{ij}\Lambda^{ab,ij}.]$$

At x_0,

$$g_{ab,cd} = g_{cd,ab}.$$

And at x_0,

$$R_{ijk\ell} = \frac{1}{2}(g_{i\ell,jk} - g_{ik,j\ell} + g_{jk,i\ell} - g_{j\ell,ik})$$

or still,

$$R_{ijk\ell} = g_{i\ell,jk} - g_{ik,j\ell}.$$

<u>Step 1</u>: At x_0,

$$\Lambda^{ab,sd}R_{brad}$$

$$= \Lambda^{ab,sd}(g_{bd,ra} - g_{ba,rd})$$

$$= \Lambda^{ab,sd}g_{bd,ra} - \Lambda^{ab,sd}g_{ba,rd}$$

$$= -\frac{1}{2}\Lambda^{as,bd}g_{bd,ra} - \Lambda^{ab,sd}g_{ba,rd}$$

$$= -\frac{1}{2}\Lambda^{bd,as}g_{bd,ra} - \Lambda^{ab,sd}g_{ba,rd}$$

$$= -\frac{1}{2}\Lambda^{ba,ds}g_{ba,rd} - \Lambda^{ab,sd}g_{ba,rd}$$

$$= -\frac{1}{2}\Lambda^{ab,sd}g_{ab,rd} - \Lambda^{ab,sd}g_{ab,rd}$$

$$= -\frac{3}{2}\Lambda^{ab,sd}g_{ab,rd}.$$

<u>Step 2</u>: At x_0,

$$\Lambda^{rs} = \Pi^{rs} - \Gamma^{r}{}_{ak,\ell}\Lambda^{as,k\ell} - \Gamma^{s}{}_{ak,\ell}\Lambda^{ar,k\ell}.$$

But

$$\begin{cases} \Gamma^{r}{}_{ak,\ell} = \frac{1}{2} g^{rb}(g_{ab,k\ell} + g_{bk,a\ell} - g_{ak,b\ell}) \\ \Gamma^{s}{}_{ak,\ell} = \frac{1}{2} g^{sb}(g_{ab,k\ell} + g_{bk,a\ell} - g_{ak,b\ell}). \end{cases}$$

Therefore

$$\begin{aligned} &- \Gamma^{r}{}_{ak,\ell}\Lambda^{as,k\ell} \\ &\quad = - \frac{1}{2} g^{rb} g_{ab,k\ell}\Lambda^{as,k\ell} \\ &- \frac{1}{2} g^{rb} g_{bk,a\ell}\Lambda^{as,k\ell} + \frac{1}{2} g^{rb} g_{ak,b\ell}\Lambda^{as,k\ell}. \end{aligned}$$

- Write

$$\begin{aligned} &- \frac{1}{2} g^{rb} g_{bk,a\ell}\Lambda^{as,k\ell} \\ &\quad = - \frac{1}{2} g^{rb} g_{bk,a\ell}\Lambda^{sa,k\ell} \\ &\quad = \frac{1}{4} g^{rb} g_{bk,a\ell}\Lambda^{sk,a\ell} \\ &\quad = \frac{1}{4} g^{rb} g_{ba,k\ell}\Lambda^{sa,k\ell} \\ &\quad = \frac{1}{4} g^{rb} g_{ab,k\ell}\Lambda^{as,k\ell}. \end{aligned}$$

- Write

$$\frac{1}{2} g^{rb} g_{ak,b\ell}\Lambda^{as,k\ell}$$

$$= \frac{1}{2} g^{rb} g_{ak,b\ell} \Lambda^{sa,k\ell}$$

$$= - \frac{1}{4} g^{rb} g_{ak,b\ell} \Lambda^{s\ell,ak}$$

$$= - \frac{1}{4} g^{rb} g_{\ell k,ba} \Lambda^{sa,\ell k}$$

$$= - \frac{1}{4} g^{rb} g_{k\ell,ab} \Lambda^{as,k\ell}$$

$$= - \frac{1}{4} g^{rb} g_{ab,k\ell} \Lambda^{as,k\ell}.$$

Therefore

$$- \Gamma^{r}{}_{ak,\ell} \Lambda^{as,k\ell} = - \frac{1}{2} g^{rb} g_{ab,k\ell} \Lambda^{as,k\ell}.$$

Interchanging r and s, we thus conclude that at x_0,

$$\Lambda^{rs} = \Pi^{rs} - \frac{1}{2} g^{rb} g_{ab,k\ell} \Lambda^{as,k\ell} - \frac{1}{2} g^{sb} g_{ab,k\ell} \Lambda^{ar,k\ell}.$$

Step 3: At x_0,

$$g_{ab,k\ell} \Lambda^{as,k\ell} = \Lambda^{k\ell,sa} g_{k\ell,ba}$$

$$= - \frac{2}{3} \Lambda^{k\ell,sa} R_{\ell bka}$$

and

$$g_{ab,k\ell} \Lambda^{ar,k\ell} = \Lambda^{k\ell,ra} g_{k\ell,ba}$$

$$= - \frac{2}{3} \Lambda^{k\ell,ra} R_{\ell bka}.$$

Therefore

$$\Lambda^{rs} = \Pi^{rs} + \frac{1}{3} g^{rb} \Lambda^{k\ell,sa} R_{\ell bka} + \frac{1}{3} g^{sb} \Lambda^{k\ell,ra} R_{\ell bka}.$$

Step 4: At x_0,

$$\Lambda^{sk,cd} g_{rk,cd} + \Lambda^{k\ell,sd} g_{k\ell,rd} + \Lambda^{sk} g_{rk} = \frac{1}{2}\,\delta^s{}_r L.$$

Here

$$\left[\begin{array}{l} \Lambda^{sk,cd} g_{rk,cd} = -\frac{2}{3}\,\Lambda^{cd,sk} R_{drck} \\[1em] \Lambda^{k\ell,sd} g_{k\ell,rd} = -\frac{2}{3}\,\Lambda^{k\ell,sd} R_{kr\ell d}\ . \end{array}\right.$$

In the first relation, replace c by ℓ, d by k, and k by d to get

$$\Lambda^{\ell k,sd} R_{kr\ell d}.$$

The net contribution is thus

$$-\frac{4}{3}\,\Lambda^{k\ell,sd} R_{kr\ell d}.$$

On the other hand,

$$\Lambda^{sk} g_{rk} = \varepsilon_r \Lambda^{rs} \quad \text{(no sum)}.$$

Therefore

$$\varepsilon_r \Lambda^{rs} = \frac{1}{2}\,\delta^s{}_r L + \frac{4}{3}\,\Lambda^{k\ell,sd} R_{kr\ell d}.$$

With this preparation, we are finally in a position to show that

$$\Pi^{rs} = \frac{1}{2}\, g^{rs} L - \frac{2}{3}\,\Lambda^{k\ell,ra} R^s{}_{k\ell a}.$$

Continuing to work at x_0,

$$\begin{aligned} \varepsilon_r \Lambda^{rs} &= \varepsilon_r [\Pi^{rs} + \frac{1}{3}\, g^{rb} \Lambda^{k\ell,sa} R_{\ell bka} + \frac{1}{3}\, g^{sb} \Lambda^{k\ell,ra} R_{\ell bka}] \\ &= \varepsilon_r [\Pi^{rs} + \frac{1}{3}\, \varepsilon_r \Lambda^{k\ell,sa} R_{\ell rka} + \frac{1}{3}\, \varepsilon_s \Lambda^{k\ell,ra} R_{\ell ska}] \end{aligned}$$

$$= \varepsilon_r \Pi^{rs} + \frac{1}{3} \Lambda^{k\ell,sa} R_{\ell rka} + \frac{1}{3} \varepsilon_r \varepsilon_s \Lambda^{k\ell,ra} R_{\ell ska}.$$

But

$$\Lambda^{k\ell,sa} R_{\ell rka} = \Lambda^{k\ell,sd} R_{\ell rkd}$$

$$= \Lambda^{\ell k,sd} R_{kr\ell d}$$

$$= \Lambda^{k\ell,sd} R_{kr\ell d}.$$

Therefore

$$\varepsilon_r \Pi^{rs} - \Lambda^{k\ell,sd} R_{kr\ell d} + \frac{1}{3} \varepsilon_r \varepsilon_s \Lambda^{k\ell,ra} R_{\ell ska} = \frac{1}{2} \delta^r{}_s L$$

or still,

$$\Pi^{rs} - \varepsilon_r \Lambda^{k\ell,sd} R_{kr\ell d} + \frac{1}{3} \varepsilon_s \Lambda^{k\ell,ra} R_{\ell ska} = \frac{1}{2} g^{rs} L$$

or still,

$$\Pi^{rs} + \Lambda^{k\ell,sd} R^r{}_{k\ell d} - \frac{1}{3} \Lambda^{k\ell,ra} R^s{}_{\ell ka} = \frac{1}{2} g^{rs} L$$

or still,

$$\Pi^{rs} + \Lambda^{k\ell,sa} R^r{}_{k\ell a} - \frac{1}{3} \Lambda^{k\ell,ra} R^s{}_{k\ell a} = \frac{1}{2} g^{rs} L.$$

Since $\Pi^{rs} = \Pi^{sr}$, we also have

$$\Pi^{rs} + \Lambda^{k\ell,ra} R^s{}_{k\ell a} - \frac{1}{3} \Lambda^{k\ell,sa} R^r{}_{k\ell a} = \frac{1}{2} g^{rs} L.$$

So, by subtraction,

$$\frac{4}{3} \Lambda^{k\ell,sa} R^r{}_{k\ell a} - \frac{4}{3} \Lambda^{k\ell,ra} R^s{}_{k\ell a} = 0$$

$$\Rightarrow$$

$$\Lambda^{k\ell,ra} R^s{}_{k\ell a} = \Lambda^{k\ell,sa} R^r{}_{k\ell a}.$$

And then, by addition,

$$2\,\Pi^{rs} + \frac{4}{3}\,\Lambda^{k\ell,ra}R^{s}{}_{k\ell a} = g^{rs}L$$

or still,

$$\Pi^{rs} = \frac{1}{2}\,g^{rs}L - \frac{2}{3}\,\Lambda^{k\ell,ra}R^{s}{}_{k\ell a}.$$

Since the issue is that of an equality of tensors, this relation is valid throughout all of M.

Summary (<u>The Invariance Identities</u>):

$$\Pi^{ij,k\ell} = \Pi^{k\ell,ij},\ \Pi^{ab,c} = 0,$$

$$\Pi^{rs} = \frac{1}{2}\,g^{rs}L - \frac{2}{3}\,\Lambda^{k\ell,ra}R^{s}{}_{k\ell a}.$$

<u>FACT</u> Let $L \in MC_n(0,0,1,2)$ -- then

$$\nabla_a L = \frac{2}{3}\,R_{ijk\ell;a}\Lambda^{i\ell,jk}.$$

Section 23: The Euler-Lagrange Equations Let M be a connected C^∞ manifold of dimension n, which we shall assume is orientable.

Definition: The Euler-Lagrange derivative is the map

$$E:MC_n(0,0,1,m) \to MC_n(2,0,1,2m)$$

given locally by the expression

$$E^{ij}(L) = -\frac{\partial L}{\partial g_{ij}}$$

$$+ \sum_{p=1}^{m} (-1)^{p+1} \frac{\partial^p}{\partial x^{k_1} \cdots \partial x^{k_p}} \left(\frac{\partial L}{\partial g_{ij,k_1 \cdots k_p}}\right).$$

[Note: It is clear that $E^{ij}(L)$ is symmetric. However, since the definition involves nontensorial quantities, it is not completely obvious that $E^{ij}(L)$ is actually tensorial. In the case of interest, viz. when m = 2, this will be verified below.]

One then says that L satisfies the Euler-Lagrange equations provided $E(L) = 0$.

Example: Let $L = |g|^{1/2}S$ (S the scalar curvature of g) -- then (cf. infra)

$$E^{ij}(L) = |g|^{1/2}[R^{ij} - \frac{1}{2} Sg^{ij}].$$

But

$$R^{ij} - \frac{1}{2} Sg^{ij} = G^{ij}$$

$$\Rightarrow \qquad\qquad (G = \text{Ein})$$

$$E(L) = |g|^{1/2}G^{\#}.$$

Therefore $E(L) = 0$ iff the Einstein tensor of g vanishes identically.

[Note: Here, $E^{ij}(L)$ is of the second order in the g_{ij} and not of the fourth order (as might be expected).]

Take m = 2 -- then

$$E^{ij}(L) = - \Lambda^{ij} + \Lambda^{ij,k}_{\ \ ,k} - \Lambda^{ij,k\ell}_{\ \ ,k\ell} ,$$

where

$$\begin{cases} \Lambda^{ij,k}_{\ \ ,k} = \dfrac{\partial}{\partial x^k} \Lambda^{ij,k} \\ \\ \Lambda^{ij,k\ell}_{\ \ ,k\ell} = \dfrac{\partial}{\partial x^k \partial x^\ell} \Lambda^{ij,k\ell}. \end{cases}$$

<u>LEMMA</u> Let $L \in MC_n(0,0,1,2)$ -- then

$$E^{ij}(L) = - \Pi^{ij} + \Pi^{ij,k}_{\ \ ;k} - \Pi^{ij,k\ell}_{\ \ ;k\ell} .$$

[Note: This establishes that the $E^{ij}(L)$ are the components of a symmetric element $E(L) \in MC_n(2,0,1,4)$.]

To prove the lemma, it suffices to show that $\forall\ h \in S_2(M)$,

$$h_{ij}[E^{ij}(L) - (- \Pi^{ij} + \Pi^{ij,k}_{\ \ ;k} - \Pi^{ij,k\ell}_{\ \ ;k\ell})] = 0.$$

Rappel:

$$PL(g,h) = \Lambda^{ij}h_{ij} + \Lambda^{ij,k}h_{ij,k} + \Lambda^{ij,k\ell}h_{ij,k\ell}.$$

To recast this, observe that

$$h_{ij,k}\Lambda^{ij,k} = (h_{ij}\Lambda^{ij,k})_{,k} - h_{ij}\Lambda^{ij,k}_{\ \ ,k}$$

and

$$h_{ij,k\ell}\Lambda^{ij,k\ell} = (h_{ij,k}\Lambda^{ij,k\ell})_{,\ell} - h_{ij,k}\Lambda^{ij,k\ell}_{\ \ ,\ell}$$

$$= (h_{ij,\ell}\Lambda^{ij,k\ell})_{,k} - (h_{ij}\Lambda^{ij,k\ell}_{\ \ \ ,\ell})_{,k} + h_{ij}\Lambda^{ij,k\ell}_{\ \ \ ,\ell k}.$$

Therefore

$$PL(g,h) = - h_{ij}E^{ij}(L)$$

$$+ [h_{ij}\Lambda^{ij,k} + h_{ij,\ell}\Lambda^{ij,k\ell} - h_{ij}\Lambda^{ij,k\ell}_{\ \ \ ,\ell}]_{,k} .$$

Rappel:

$$PL(g,h) = \Pi^{ij}h_{ij} + \Pi^{ij,k}h_{ij;k} + \Pi^{ij,k\ell}h_{ij;k\ell}.$$

Straightforward manipulations now lead to

$$PL(g,h) = - h_{ij}(- \Pi^{ij} + \Pi^{ij,k}_{\ \ \ ;k} - \Pi^{ij,k\ell}_{\ \ \ ;k\ell})$$

$$+ [h_{ij}\Pi^{ij,k} + h_{ij;\ell}\Pi^{ij,k\ell} - h_{ij}\Pi^{ij,k\ell}_{\ \ \ ;\ell}]_{;k}$$

or still,

$$PL(g,h) = - h_{ij}(- \Pi^{ij} + \Pi^{ij,k}_{\ \ \ ;k} - \Pi^{ij,k\ell}_{\ \ \ ;k\ell})$$

$$+ [h_{ij}\Pi^{ij,k} + h_{ij;\ell}\Pi^{ij,k\ell} - h_{ij}\Pi^{ij,k\ell}_{\ \ \ ;\ell}]_{,k} .$$

[Note: The terms inside the brackets are the components of an element $1\text{-}s\mathcal{D}^1_0(M)$. Since the indices are contracted over k, the covariant derivative equals the partial derivative.]

From the definitions,

$$h_{ij}\Pi^{ij,k} + h_{ij;\ell}\Pi^{ij,k\ell} - h_{ij}\Pi^{ij,k\ell}_{\ \ \ ;\ell}$$

$$= h_{ij}(\Lambda^{ij,k} + 2\Gamma^i_{\ a\ell}\Lambda^{aj,k\ell} + 2\Gamma^j_{\ a\ell}\Lambda^{ai,k\ell} + \Gamma^k_{\ b\ell}\Lambda^{ij,b\ell})$$

$$+ (h_{ij,\ell} - \Gamma^a{}_{\ell i}h_{aj} - \Gamma^a{}_{\ell j}h_{ia})\Lambda^{ij,k\ell}$$

$$- h_{ij}(\Lambda^{ij,k\ell}{}_{,\ell} + \Gamma^i{}_{\ell a}\Lambda^{aj,k\ell} + \Gamma^j{}_{\ell a}\Lambda^{ia,k\ell}$$

$$+ \Gamma^k{}_{\ell a}\Lambda^{ij,a\ell} + \Gamma^\ell{}_{\ell a}\Lambda^{ij,ka} - \Gamma^b{}_{\ell b}\Lambda^{ij,k\ell}).$$

But

- $h_{ij}\Gamma^j{}_{a\ell}\Lambda^{ai,k\ell}$

$$= h_{ji}\Gamma^i{}_{a\ell}\Lambda^{aj,k\ell}$$

$$= h_{ij}\Gamma^i{}_{a\ell}\Lambda^{aj,k\ell}.$$

- $\Gamma^a{}_{\ell j}h_{ia}\Lambda^{ij,k\ell}$

$$= \Gamma^a{}_{\ell i}h_{ja}\Lambda^{ji,k\ell}$$

$$= \Gamma^a{}_{\ell i}h_{aj}\Lambda^{ij,k\ell}.$$

- $h_{ij}\Gamma^j{}_{\ell a}\Lambda^{ia,k\ell}$

$$= h_{ji}\Gamma^i{}_{\ell a}\Lambda^{ja,k\ell}$$

$$= h_{ij}\Gamma^i{}_{\ell a}\Lambda^{aj,k\ell}.$$

- $\Gamma^b{}_{\ell b}\Lambda^{ij,k\ell}$

$$= \Gamma^{\ell}{}_{b\ell}\Lambda^{ij,kb}$$

$$= \Gamma^{\ell}{}_{a\ell}\Lambda^{ij,ka}$$

$$= \Gamma^{\ell}{}_{\ell a}\Lambda^{ij,ka}.$$

Therefore

$$h_{ij}\Pi^{ij,k} + h_{ij;\ell}\Pi^{ij,k\ell} - h_{ij}\Pi^{ij,k\ell}{}_{;\ell}$$

$$= h_{ij}(\Lambda^{ij,k} + 4\Gamma^{i}{}_{a\ell}\Lambda^{aj,k\ell} + \Gamma^{k}{}_{b\ell}\Lambda^{ij,b\ell})$$

$$+ (h_{ij,\ell}\Lambda^{ij,k\ell} - 2\Gamma^{a}{}_{\ell i}h_{aj}\Lambda^{ij,k\ell})$$

$$- h_{ij}(\Lambda^{ij,k\ell}{}_{,\ell} + 2\Gamma^{i}{}_{\ell a}\Lambda^{aj,k\ell} + \Gamma^{k}{}_{\ell a}\Lambda^{ij,a\ell})$$

$$= h_{ij}\Lambda^{ij,k} + h_{ij,\ell}\Lambda^{ij,k\ell} - h_{ij}\Lambda^{ij,k\ell}{}_{,\ell}.$$

Finally, then,

$$0 = - PL(g,h) + PL(g,h)$$

$$= h_{ij}E^{ij}(L) - [h_{ij}\Lambda^{ij,k} + h_{ij,\ell}\Lambda^{ij,k\ell} - h_{ij}\Lambda^{ij,k\ell}{}_{,\ell}]_{,k}$$

$$- h_{ij}(- \Pi^{ij} + \Pi^{ij,k}{}_{;k} - \Pi^{ij,k\ell}{}_{;k\ell})$$

$$+ [h_{ij}\Pi^{ij,k} + h_{ij;\ell}\Pi^{ij,k\ell} - h_{ij}\Pi^{ij,k\ell}{}_{;\ell}]_{,k}$$

$$= h_{ij}E^{ij}(L) - h_{ij}(- \Pi^{ij} + \Pi^{ij,k}{}_{;k} - \Pi^{ij,k\ell}{}_{;k\ell})$$

$$= h_{ij}[E^{ij}(L) - (-\Pi^{ij} + \Pi^{ij,k}_{\ ;k} - \Pi^{ij,k\ell}_{\ ;k\ell})].$$

Since $\Pi^{ij,k} = 0$, it follows that

$$E^{ij}(L) = -\Pi^{ij} - \Pi^{ij,k\ell}_{\ ;k\ell}$$

or still,

$$E^{ij}(L) = -\Pi^{ij} - \Lambda^{ij,k\ell}_{\ ;k\ell}$$

or still,

$$E^{ij}(L) = -\frac{1}{2} g^{ij}L + \frac{2}{3} \Lambda^{k\ell,ia}R^{j}_{\ k\ell a} - \Lambda^{ij,k\ell}_{\ ;k\ell} .$$

[Note: Recall that

$$\Lambda^{k\ell,ia} = \Lambda^{ia,k\ell}.]$$

<u>The Canonical Example</u> Let

$$L = |g|^{1/2} S - 2\lambda|g|^{1/2},$$

where λ is a constant. Locally,

$$S = g^{ac}g^{bd}R_{abcd}$$

and

$$\frac{\partial R_{abcd}}{\partial g_{ij,k\ell}} = \frac{\partial}{\partial g_{ij,k\ell}} \left(\frac{1}{2} (g_{ad,bc} - g_{ac,bd} + g_{bc,ad} - g_{bd,ac})\right).$$

In accordance with our symmetrization convention, write

$$\begin{cases} g_{ad,bc} = \frac{1}{4} (g_{ad,bc} + g_{da,bc} + g_{ad,cb} + g_{da,cb}) \\ g_{ac,bd} = \frac{1}{4} (g_{ac,bd} + g_{ca,bd} + g_{ac,db} + g_{ca,db}) \\ g_{bc,ad} = \frac{1}{4} (g_{bc,ad} + g_{cb,ad} + g_{bc,da} + g_{cb,da}) \\ g_{bd,ac} = \frac{1}{4} (g_{bd,ac} + g_{db,ac} + g_{bd,ca} + g_{db,ca}). \end{cases}$$

Then

$$\frac{\partial R_{abcd}}{\partial g_{ij,k\ell}} = I - II + III - IV$$

with

$$I = \frac{1}{8}\,(\delta^i{}_a\delta^j{}_d\delta^k{}_b\delta^\ell{}_c + \delta^i{}_d\delta^j{}_a\delta^k{}_b\delta^\ell{}_c + \delta^i{}_a\delta^j{}_d\delta^k{}_c\delta^\ell{}_b + \delta^i{}_d\delta^j{}_a\delta^k{}_c\delta^\ell{}_b),$$

$$II = \frac{1}{8}\,(\delta^i{}_a\delta^j{}_c\delta^k{}_b\delta^\ell{}_d + \delta^i{}_c\delta^j{}_a\delta^k{}_b\delta^\ell{}_d + \delta^i{}_a\delta^j{}_c\delta^k{}_d\delta^\ell{}_b + \delta^i{}_c\delta^j{}_a\delta^k{}_d\delta^\ell{}_b),$$

$$III = \frac{1}{8}\,(\delta^i{}_b\delta^j{}_c\delta^k{}_a\delta^\ell{}_d + \delta^i{}_c\delta^j{}_b\delta^k{}_a\delta^\ell{}_d + \delta^i{}_b\delta^j{}_c\delta^k{}_d\delta^\ell{}_a + \delta^i{}_c\delta^j{}_b\delta^k{}_d\delta^\ell{}_a),$$

$$IV = \frac{1}{8}\,(\delta^i{}_b\delta^j{}_d\delta^k{}_a\delta^\ell{}_c + \delta^i{}_d\delta^j{}_b\delta^k{}_a\delta^\ell{}_c + \delta^i{}_b\delta^j{}_d\delta^k{}_c\delta^\ell{}_a + \delta^i{}_d\delta^j{}_b\delta^k{}_c\delta^\ell{}_a).$$

Therefore

$$\Lambda^{ij,k\ell} = \frac{\partial L}{\partial g_{ij,k\ell}}$$

$$= |g|^{1/2}\, g^{ac}g^{bd}\,\frac{\partial R_{abcd}}{\partial g_{ij,k\ell}}$$

$$= |g|^{1/2}\, g^{ac}g^{bd}\,(I - II + III - IV).$$

• $g^{ac}g^{bd}\ I$

$$= \frac{1}{8}\,(g^{i\ell}g^{kj} + g^{j\ell}g^{ki} + g^{ik}g^{\ell j} + g^{jk}g^{\ell i})$$

$$= \frac{1}{8}\,(2g^{ik}g^{j\ell} + 2g^{i\ell}g^{jk})$$

$$= \frac{1}{4}\,(g^{ik}g^{j\ell} + g^{i\ell}g^{jk}).$$

• $g^{ac}g^{bd}$ II

$$= \frac{1}{8}\,(g^{ij}g^{k\ell} + g^{ji}g^{k\ell} + g^{ij}g^{\ell k} + g^{ji}g^{\ell k})$$

$$= \frac{1}{8}\,(4g^{ij}g^{k\ell}) = \frac{1}{2}\,g^{ij}g^{k\ell}.$$

• $g^{ac}g^{bd}$ III

$$= \frac{1}{8}\,(g^{kj}g^{i\ell} + g^{ki}g^{j\ell} + g^{\ell j}g^{ik} + g^{\ell i}g^{jk})$$

$$= \frac{1}{8}\,(2g^{ik}g^{j\ell} + 2g^{i\ell}g^{jk})$$

$$= \frac{1}{4}\,(g^{ik}g^{j\ell} + g^{i\ell}g^{jk}).$$

• $g^{ac}g^{bd}$ IV

$$= \frac{1}{8}\,(g^{k\ell}g^{ij} + g^{k\ell}g^{ji} + g^{\ell k}g^{ij} + g^{\ell k}g^{ji})$$

$$= \frac{1}{8}\,(4g^{ij}g^{k\ell}) = \frac{1}{2}\,g^{ij}g^{k\ell}.$$

Combining terms thus gives

$$\Lambda^{ij,k\ell} = -\,|g|^{1/2}[g^{ij}g^{k\ell} - \frac{1}{2}\,(g^{ik}g^{j\ell} + g^{i\ell}g^{jk})].$$

But then

$$\Lambda^{ij,k\ell}{}_{;k} = 0 \quad\Rightarrow\quad \Lambda^{ij,k\ell}{}_{;k\ell} = 0.$$

Consequently,

$$E^{ij}(L) = -\,\frac{1}{2}\,g^{ij}L + \frac{2}{3}\,\Lambda^{ia,k\ell}R^{j}{}_{k\ell a}$$

$$= -\frac{1}{2} g^{ij}[|g|^{1/2}S - 2\lambda|g|^{1/2}]$$

$$+ \frac{2}{3} (-|g|^{1/2}[g^{ia}g^{k\ell} - \frac{1}{2} (g^{ik}g^{a\ell} + g^{i\ell}g^{ak})])R^{j}{}_{k\ell a}.$$

It remains to analyze

$$[g^{ia}g^{k\ell} - \frac{1}{2} (g^{ik}g^{a\ell} + g^{i\ell}g^{ak})]R^{j}{}_{k\ell a}.$$

(1) $g^{ia}g^{k\ell}R^{j}{}_{k\ell a}$

$$= g^{ia}g^{k\ell}g^{jb}R_{bk\ell a}$$

$$= - g^{ia}g^{jb}g^{k\ell}R_{bka\ell}$$

$$= - g^{ia}g^{jb}g^{k\ell}R_{kb\ell a}$$

$$= - g^{ia}g^{jb}R_{ba}$$

$$= - g^{ia}g^{jb}R_{ab}$$

$$= - R^{ij}.$$

(2) $R_{bk\ell a} + R_{b\ell ak} + R_{bak\ell} = 0$

$\Rightarrow$

$$0 = g^{ik}g^{a\ell}g^{jb}R_{bk\ell a} + g^{ik}g^{a\ell}g^{jb}R_{b\ell ak} + g^{ik}g^{a\ell}g^{jb}R_{bak\ell}$$

$$= g^{ik}g^{a\ell}g^{jb}R_{bk\ell a} + g^{ik}g^{a\ell}g^{jb}R_{ba\ell k} + g^{ik}g^{a\ell}g^{jb}R_{bak\ell}$$

$$= g^{ik}g^{a\ell}g^{jb}R_{bk\ell a} - g^{ik}g^{a\ell}g^{jb}R_{bak\ell} + g^{ik}g^{a\ell}g^{jb}R_{bak\ell}$$

$\Rightarrow$

$$g^{ik}g^{a\ell}g^{jb}R_{bk\ell a} = 0$$

$$\Rightarrow$$

$$g^{ik}g^{a\ell}R^{j}{}_{k\ell a} = 0.$$

(3) $g^{i\ell}g^{ak}R^{j}{}_{k\ell a}$

$$= g^{i\ell}g^{ak}g^{jb}R_{bk\ell a}$$

$$= g^{i\ell}g^{jb}g^{ak}R_{bk\ell a}$$

$$= g^{i\ell}g^{jb}g^{ka}R_{kba\ell}$$

$$= g^{i\ell}g^{jb}R_{b\ell}$$

$$= g^{i\ell}g^{jb}R_{\ell b}$$

$$= R^{ij}.$$

Therefore

$$E^{ij}(L) = -\frac{1}{2}g^{ij}[|g|^{1/2}S - 2\lambda|g|^{1/2}]$$

$$- |g|^{1/2}\frac{2}{3}[- R^{ij} - 0 - \frac{R^{ij}}{2}]$$

$$= |g|^{1/2}[R^{ij} - \frac{1}{2}Sg^{ij}] + \lambda|g|^{1/2}g^{ij}.$$

So, in this case, the Euler-Lagrange equations reduce to

$$[R^{ij} - \frac{1}{2}Sg^{ij}] + \lambda g^{ij} = 0,$$

the vacuum field equations of general relativity (with cosmological constant λ).

[Note: In this situation, the Euler-Lagrange equations $E(L) = 0$ are second order.]

FACT Let $F \in MC_n(4,0,0,0)$ $(n > 1)$. Suppose that

$$F^{ijk\ell} = F^{jik\ell} = F^{ij\ell k}$$

and

$$F^{ijk\ell} + F^{ik\ell j} + F^{i\ell jk} = 0.$$

Then

$$F^{ijk\ell} = K[g^{ij}g^{k\ell} - \frac{1}{2}(g^{ik}g^{j\ell} + g^{i\ell}j^{jk})],$$

where K is a constant.

Example: Let

$$L = |g|^{1/2}S^2.$$

Then

$$L = \frac{(|g|^{1/2}S)^2}{|g|^{1/2}}$$

$\Rightarrow$

$$\Lambda^{ij,k\ell} = \frac{2|g|^{1/2}S}{|g|^{1/2}} \cdot \frac{\partial(|g|^{1/2}S)}{\partial g_{ij,k\ell}}$$

$$= -2|g|^{1/2}S[g^{ij}g^{k\ell} - \frac{1}{2}(g^{ik}g^{j\ell} + g^{i\ell}g^{jk})]$$

$\Rightarrow$

$$\Lambda^{ij,k\ell}_{;k\ell} = -2|g|^{1/2}[g^{ij}g^{k\ell} - \frac{1}{2}(g^{ik}g^{j\ell} + g^{i\ell}g^{jk})]\nabla_\ell\nabla_k S.$$

Therefore

$$E^{ij}(L) = -\frac{1}{2} g^{ij}L + \frac{2}{3}\Lambda^{k\ell,ia}R^{j}{}_{k\ell a} - \Lambda^{ij,k\ell}{}_{;k\ell}$$

$$= -\frac{1}{2}|g|^{1/2}g^{ij}S^2 + 2|g|^{1/2}R^{ij}S$$

$$+ 2|g|^{1/2}g^{ij}g^{k\ell}\nabla_\ell\nabla_k S$$

$$- |g|^{1/2}(g^{ik}g^{j\ell} + g^{i\ell}g^{jk})\nabla_\ell\nabla_k S$$

$$= |g|^{1/2}S(2R^{ij} - \frac{1}{2}Sg^{ij})$$

$$+ 2|g|^{1/2}g^{ij}\nabla^k\nabla_k S - 2|g|^{1/2}\nabla^i\nabla^j S.$$

[Note: In this situation, the Euler-Lagrange equations $E(L) = 0$ are fourth order.]

There are two other "quadratic" lagrangians that are sometimes considered but their introduction increases the level of complexity.

• Let

$$L = |g|^{1/2}g[{}^0_2](\mathrm{Ric},\mathrm{Ric}).$$

Locally,

$$L = |g|^{1/2}R^{ij}R_{ij}$$

and

$$E^{ij}(L) = \frac{|g|^{1/2}}{2}\; g^{ij}[\nabla^a\nabla_a S - R^{k\ell}R_{k\ell}]$$

$$+ |g|^{1/2}\nabla^a\nabla_a R^{ij} - |g|^{1/2}\nabla^i\nabla^j S + 2|g|^{1/2}R^{ikj\ell}R_{k\ell}.$$

• Let

$$L = |g|^{1/2}g[{}^0_4](R,R).$$

Locally,

$$L = |g|^{1/2} R^{ijk\ell} R_{ijk\ell}$$

and

$$E^{ij}(L) = |g|^{1/2}[4\nabla^a\nabla_a R^{ij} - 2\nabla^i\nabla^j S]$$

$$+ |g|^{1/2}[2R^i{}_{abc}R^{jabc} + 4R^{ikj\ell}R_{k\ell} - 4R^{ia}R^j{}_a]$$

$$- \frac{|g|^{1/2}}{2}(R^{abcd}R_{abcd})g^{ij}.$$

Observation: We have

$$E^{ij}(|g|^{1/2}[S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}])$$

$$= |g|^{1/2}[2SR^{ij} - 4R^{ikj\ell}R_{k\ell} + 2R^i{}_{abc}R^{jabc} - 4R^{ia}R^j{}_a]$$

$$- \frac{|g|^{1/2}}{2}[S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}]g^{ij}.$$

<u>FACT</u> Take n = 4 — then

$$2SR^{ij} - 4R^{ikj\ell}R_{k\ell} + 2R^i{}_{abc}R^{jabc} - 4R^{ia}R^j{}_a$$

$$= \frac{1}{2}[S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}]g^{ij}.$$

I.e.:

$$E^{ij}(|g|^{1/2}[S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}]) = 0.$$

Remark: Take $n > 3$ and let

$$L = |g|^{1/2} g[{}^0_4](C,C).$$

Locally,

$$L = |g|^{1/2} C^{ijk\ell} C_{ijk\ell}$$

and

$$g[{}^0_4](C,C)$$

$$= g[{}^0_4](R,R) - \frac{4}{n-2} g[{}^0_2](\mathrm{Ric},\mathrm{Ric}) + \frac{2}{(n-1)(n-2)} S^2.$$

<u>THEOREM</u> Let $L \in MC_n(0,0,1,m)$ -- then the divergence of $E(L)$ is zero, i.e.,

$$\nabla_j E^{ij}(L) = 0.$$

While the result is valid for any m, we shall settle for a proof when m = 2.

Fix a point $x_0 \in M$ and introduce normal coordinates at x_0 -- then

$$\text{"covariant derivative}\Big|_{x_0}\text{"} = \text{"partial derivative}\Big|_{x_0}\text{"}.$$

Therefore

$$\nabla_j E^{ij}(L)\Big|_{x_0} = -\Lambda^{ij}{}_{,j}\Big|_{x_0} + \Lambda^{ij,k}{}_{,kj}\Big|_{x_0} - \Lambda^{ij,k\ell}{}_{,k\ell j}\Big|_{x_0}.$$

<u>$\Lambda^{ij,k\ell}{}_{,k\ell j}$:</u> Differentiation of the relation

$$\Lambda^{ij,k\ell} + \Lambda^{ik,\ell j} + \Lambda^{i\ell,jk} = 0$$

gives

$$\Lambda^{ij,k\ell}{}_{,k\ell j} + \Lambda^{ik,\ell j}{}_{,k\ell j} + \Lambda^{i\ell,jk}{}_{,k\ell j} = 0.$$

- $$\Lambda^{ik,\ell j}{}_{,k\ell j} = \Lambda^{i\ell,kj}{}_{,\ell kj}$$
  $$= \Lambda^{i\ell,kj}{}_{,k\ell j}$$
  $$= \Lambda^{i\ell,jk}{}_{,k\ell j}\ .$$

- $$\Lambda^{i\ell,jk}{}_{,k\ell j} = \Lambda^{ij,k\ell}{}_{,kj\ell}$$
  $$= \Lambda^{ij,k\ell}{}_{,k\ell j}\ .$$

But this implies that

$$\Lambda^{ij,k\ell}{}_{,k\ell j} = 0.$$

[Note: Normal coordinates play no role in this argument.]

Consequently,

$$\nabla_j E^{ij}(L)\Big|_{x_0} = -\Lambda^{ij}{}_{,j}\Big|_{x_0} + \Lambda^{ij,k}{}_{,kj}\Big|_{x_0}\ .$$

We shall now discuss the two terms on the RHS, beginning with

$$\Lambda^{ij,k}{}_{,kj}\Big|_{x_0}\ .$$

By definition,

$$\Pi^{ij,k} = \Lambda^{ij,k} + 2\Gamma^i{}_{a\ell}\Lambda^{aj,k\ell} + 2\Gamma^j{}_{a\ell}\Lambda^{ai,k\ell} + \Gamma^k{}_{b\ell}\Lambda^{ij,b\ell}.$$

But $\Pi^{ij,k} = 0$, hence

$$\Lambda^{ij,k} = -2\Gamma^i{}_{a\ell}\Lambda^{aj,k\ell} - 2\Gamma^j{}_{a\ell}\Lambda^{ai,k\ell} - \Gamma^k{}_{b\ell}\Lambda^{ij,b\ell}.$$

Since $\left[\begin{array}{l}\Gamma^{i}{}_{a\ell} \\ \\ \Gamma^{j}{}_{a\ell}\end{array}\right.$ is symmetric in a & ℓ, it follows that

$$\Lambda^{ij,k} = \Gamma^{i}{}_{a\ell}\Lambda^{jk,a\ell} + \Gamma^{j}{}_{a\ell}\Lambda^{ik,a\ell} - \Gamma^{k}{}_{b\ell}\Lambda^{ij,b\ell}.$$

Therefore

$$\Lambda^{ij,k}{}_{,kj} = (\Gamma^{i}{}_{a\ell}\Lambda^{jk,a\ell})_{,kj}$$

$$+ (\Gamma^{j}{}_{a\ell}\Lambda^{ik,a\ell})_{,kj} - (\Gamma^{k}{}_{b\ell}\Lambda^{ij,b\ell})_{,kj}$$

or still,

$$\Lambda^{ij,k}{}_{,kj} = (\Gamma^{i}{}_{a\ell}\Lambda^{jk,a\ell})_{,kj} ,$$

as

$$(\Gamma^{j}{}_{a\ell}\Lambda^{ik,a\ell})_{,kj} = (\Gamma^{j}{}_{b\ell}\Lambda^{ik,b\ell})_{,kj}$$

$$= (\Gamma^{k}{}_{b\ell}\Lambda^{ij,b\ell})_{,jk}$$

$$= (\Gamma^{k}{}_{b\ell}\Lambda^{ij,b\ell})_{,kj} .$$

However $\Lambda^{jk,a\ell} = \Lambda^{kj,a\ell} = \Lambda^{a\ell,kj}$, so

$$\Lambda^{ij,k}{}_{,kj} = (\Gamma^{i}{}_{a\ell}\Lambda^{a\ell,kj})_{,kj}$$

$$= (\Gamma^{i}{}_{a\ell,k}\Lambda^{a\ell,kj} + \Gamma^{i}{}_{a\ell}\Lambda^{a\ell,kj}{}_{,k})_{,j}$$

$$= \Gamma^{i}{}_{a\ell,kj}\Lambda^{a\ell,kj} + \Gamma^{i}{}_{a\ell,k}\Lambda^{a\ell,kj}{}_{,j} + \Gamma^{i}{}_{a\ell,j}\Lambda^{a\ell,kj}{}_{,k} + \Gamma^{i}{}_{a\ell}\Lambda^{a\ell,kj}{}_{,kj}$$

$\Rightarrow$

$$\Lambda^{ij,k}{}_{,kj}\Big|_{x_0} = \Gamma^i{}_{a\ell,kj}\Lambda^{a\ell,kj}\Big|_{x_0} + 2\Gamma^i{}_{a\ell,k}\Lambda^{a\ell,kj}{}_{,j}\Big|_{x_0} .$$

Rappel:

$$\Gamma^i{}_{a\ell} = \frac{1}{2} g^{ib}(g_{ba,\ell} + g_{b\ell,a} - g_{a\ell,b}).$$

Thus at x_0,

$$\Gamma^i{}_{a\ell,kj}\Lambda^{a\ell,kj}$$

$$= \frac{1}{2} g^{ib}(g_{ba,\ell kj} + g_{b\ell,akj} - g_{a\ell,bkj})\Lambda^{a\ell,kj}$$

$$= g^{ib}g_{ba,\ell kj}\Lambda^{a\ell,kj} - \frac{1}{2} g^{ib}g_{a\ell,bkj}\Lambda^{a\ell,kj}$$

and

$$2\Gamma^i{}_{a\ell,k}\Lambda^{a\ell,kj}{}_{,j}$$

$$= g^{ib}(g_{ba,\ell k} + g_{b\ell,ak} - g_{a\ell,bk})\Lambda^{a\ell,kj}{}_{,j} .$$

Claim: We have

$$g_{ba,\ell kj}\Lambda^{a\ell,kj} = 0.$$

[Multiply the identity

$$\Lambda^{a\ell,kj} + \Lambda^{ak,j\ell} + \Lambda^{aj,\ell k} = 0$$

by $g_{ba,\ell kj}$ -- then

• $g_{ba,\ell kj}\Lambda^{ak,j\ell}$

$$= g_{ba,k\ell j}\Lambda^{a\ell,jk}$$

$$= g_{ba,\ell kj}\Lambda^{a\ell,kj} .$$

• $g_{ba,\ell kj}\Lambda^{aj,\ell k}$

$$= g_{ba,jk\ell}\Lambda^{a\ell,jk}$$

$$= g_{ba,\ell kj}\Lambda^{a\ell,kj} .$$

Therefore

$$3g_{ba,\ell kj}\Lambda^{a\ell,kj} = 0.]$$

Accordingly,

$$\Gamma^{i}{}_{a\ell,kj}\Lambda^{a\ell,kj}\Big|_{x_0} = -\frac{1}{2}\, g^{ib} g_{a\ell,bkj}\Lambda^{a\ell,kj} .$$

Next

$$g^{ib}g_{b\ell,ak}\Lambda^{a\ell,kj}{}_{,j}$$

$$= g^{ib}g_{ba,\ell k}\,\Lambda^{\ell a,kj}{}_{,j}$$

$$= g^{ib}g_{ba,\ell k}\Lambda^{a\ell,kj}{}_{,j} .$$

Thus

$$2\Gamma^{i}{}_{a\ell,k}\Lambda^{a\ell,kj}{}_{,j}\Big|_{x_0}$$

$$= 2g^{ib}g_{ba,\ell k}\Lambda^{a\ell,kj}{}_{,j} - g^{ib}g_{a\ell,bk}\Lambda^{a\ell,kj}{}_{,j} .$$

Taking into account that $g_{ba,\ell k}$ is symmetric in ℓ & k,

$$2g^{ib}g_{ba,\ell k}\Lambda^{a\ell,kj}_{\ \ \ ,j}$$

$$= 2(-\tfrac{1}{2})\ g^{ib}g_{ba,\ell k}\Lambda^{aj,k\ell}_{\ \ \ ,j}$$

$$= -\ g^{ib}g_{ba,\ell k}\Lambda^{aj,k\ell}_{\ \ \ ,j}$$

$$= -\ g^{ib}g_{bk,\ell a}\Lambda^{kj,a\ell}_{\ \ \ ,j}$$

$$= -\ g^{ib}g_{bk,a\ell}\Lambda^{a\ell,kj}_{\ \ \ ,j}$$

or still, since we are working at x_0,

$$-\ g^{ib}g_{a\ell,bk}\Lambda^{a\ell,kj}_{\ \ \ ,j}\ .$$

Summary: At x_0,

$$\Lambda^{ij,k}_{\ \ \ ,kj}$$

$$= -\frac{1}{2}\ g^{ib}g_{a\ell,bkj}\Lambda^{a\ell,kj} - 2g^{ib}g_{a\ell,bk}\Lambda^{a\ell,kj}_{\ \ \ ,j}\ .$$

It remains to explicate

$$-\ \Lambda^{ij}_{\ \ ,j}\Big|_{x_0}\ .$$

To begin with,

$$\Lambda^{ij} = \frac{1}{2}\ g^{ij}L - g^{i\ell}g_{\ell b,c}\Lambda^{jb,c}$$

$$-\frac{1}{2}\ g^{i\ell}g_{ab,\ell}\Lambda^{ab,j} - g^{i\ell}(g_{ab,c\ell} + g_{c\ell,ab})\Lambda^{ab,jc}.$$

So, in view of the fact that

$$L_{,j} = \Lambda^{ab}g_{ab,j} + \Lambda^{ab,c}g_{ab,cj} + \Lambda^{ab,cd}g_{ab,cdj},$$

at x_0 we have,

$$\Lambda^{ij}{}_{,j} = \frac{1}{2} g^{ij}\Lambda^{ab} g_{ab,j} + \frac{1}{2} g^{ij}\Lambda^{ab,c} g_{ab,cj}$$

$$+ \frac{1}{2} g^{ij}\Lambda^{ab,cd} g_{ab,cdj}$$

$$- g^{i\ell} g_{\ell b,cj}\Lambda^{jb,c} - g^{i\ell} g_{\ell b,c}\Lambda^{jb,c}{}_{,j}$$

$$- \frac{1}{2} g^{i\ell} g_{ab,\ell j}\Lambda^{ab,j} - \frac{1}{2} g^{i\ell} g_{ab,\ell}\Lambda^{ab,j}{}_{,j}$$

$$- g^{i\ell} g_{ab,c\ell j}\Lambda^{ab,jc} - g^{i\ell} g_{ab,c\ell}\Lambda^{ab,jc}{}_{,j}$$

$$- g^{i\ell} g_{c\ell,abj}\Lambda^{ab,jc} - g^{i\ell} g_{c\ell,ab}\Lambda^{ab,jc}{}_{,j} .$$

But at x_0,

$$g_{ab,j} = 0, \; g_{\ell b,c} = 0, \; g_{ab,\ell} = 0$$

and

$$\Lambda^{ab,c} = 0, \; \Lambda^{jb,c} = 0, \; \Lambda^{ab,j} = 0.$$

Thus at x_0,

$$\Lambda^{ij}{}_{,j} = \frac{1}{2} g^{ij}\Lambda^{ab,cd} g_{ab,cdj}$$

$$- g^{i\ell} g_{ab,c\ell j}\Lambda^{ab,jc} - g^{i\ell} g_{ab,c\ell}\Lambda^{ab,jc}{}_{,j}$$

$$- g^{i\ell} g_{c\ell,abj}\Lambda^{ab,jc} - g^{i\ell} g_{c\ell,ab}\Lambda^{ab,jc}{}_{,j} .$$

Claim: We have

$$g_{c\ell,abj}\Lambda^{ab,jc} = 0.$$

[Multiply the identity

$$\Lambda^{ab,jc} + \Lambda^{aj,cb} + \Lambda^{ac,bj} = 0$$

by $g_{c\ell,abj}$ -- then

- $g_{c\ell,abj}\Lambda^{aj,cb}$

$$= g_{c\ell,jba}\Lambda^{ab,cj}$$

$$= g_{c\ell,abj}\Lambda^{ab,jc} .$$

- $g_{c\ell,abj}\Lambda^{ac,bj}$

$$= g_{c\ell,jba}\Lambda^{jc,ba}$$

$$= g_{c\ell,abj}\Lambda^{ab,jc} .$$

Therefore

$$3g_{c\ell,abj}\Lambda^{ab,jc} = 0.]$$

Summary: At x_0,

$$\Lambda^{ij}{}_{,j} = \frac{1}{2} g^{ij}g_{ab,cdj}\Lambda^{ab,cd}$$

$$- g^{i\ell}g_{ab,c\ell j}\Lambda^{ab,jc} - 2g^{i\ell}g_{ab,c\ell}\Lambda^{ab,jc}{}_{,j} .$$

Recall now that

$$\nabla_j E^{ij}(L)\Big|_{x_0} = - \Lambda^{ij}{}_{,j}\Big|_{x_0} + \Lambda^{ij,k}{}_{,kj}\Big|_{x_0} .$$

Obviously,

$$g^{i\ell} g_{ab,c\ell} \Lambda^{ab,jc}_{\quad ,j}$$

$$= g^{ib} g_{a\ell,cb} \Lambda^{a\ell,jc}_{\quad ,j}$$

$$= g^{ib} g_{a\ell,kb} \Lambda^{a\ell,jk}_{\quad ,j}$$

$$= g^{ib} g_{a\ell,bk} \Lambda^{a\ell,kj}_{\quad ,j} .$$

This leaves

$$- \frac{1}{2} g^{ij} g_{ab,cdj} \Lambda^{ab,cd} + g^{i\ell} g_{ab,c\ell j} \Lambda^{ab,jc}$$

$$- \frac{1}{2} g^{ib} g_{a\ell,bkj} \Lambda^{a\ell,kj} .$$

But

- $g^{i\ell} g_{ab,c\ell j} \Lambda^{ab,jc}$

$$= g^{i\ell} g_{ab,d\ell j} \Lambda^{ab,jd}$$

$$= g^{i\ell} g_{ab,d\ell c} \Lambda^{ab,cd}$$

$$= g^{ij} g_{ab,djc} \Lambda^{ab,cd}$$

$$= g^{ij} g_{ab,cdj} \Lambda^{ab,cd} .$$

- $g^{ib} g_{a\ell,bkj} \Lambda^{a\ell,kj}$

$$= g^{ij} g_{a\ell,jkb} \Lambda^{a\ell,kb}$$

$$= g^{ij} g_{ab,jk\ell} \Lambda^{ab,k\ell}$$

$$= g^{ij} g_{ab,jc\ell} \Lambda^{ab,c\ell}$$

$$= g^{ij} g_{ab,jcd} \Lambda^{ab,cd}$$

$$= g^{ij} g_{ab,cdj} \Lambda^{ab,cd} .$$

It therefore follows that

$$\nabla_j E^{ij}(L) \Big|_{x_0} = 0,$$

which completes the proof of the theorem.

Example: Take $L = |g|^{1/2} S$ -- then $E(L) = |g|^{1/2} G^{\#}$, hence

$$0 = \nabla_j (|g|^{1/2} G^{ij})$$

$$= (\nabla_j |g|^{1/2}) G^{ij} + |g|^{1/2} \nabla_j G^{ij}$$

$$= |g|^{1/2} \nabla_j G^{ij}$$

$\Rightarrow$

$$\text{div } G^{\#} = 0 \Rightarrow \text{div } G = 0.$$

Thus the vanishing of the divergence of the Einstein tensor is just a particular case of the theorem.

[Note: Officially, div $G \in \mathcal{D}_1^0(M)$, and

$$\text{div } G^{\#} = g^{\#} \text{div } G.$$

In fact,

$$(g^{\#} \text{div } G)^i = g^{ik} (\text{div } G)_k$$

$$= g^{ik} g^{\ell j} \nabla_j G_{k\ell}$$

$$= \nabla_j g^{ik} g^{j\ell} G_{k\ell}$$

$$= \nabla_j G^{ij}.]$$

<u>FACT</u> $\forall\ X \in \mathcal{D}^1(M)$,

$$2\nabla_j (X_i E^{ij}(L)) = (L_X g)_{ij} E^{ij}(L).$$

<u>Section 24:</u> <u>The Helmholtz Condition</u> Let M be a connected C^∞ manifold of dimension n, which we shall assume is orientable.

Fix a chart $(U,\{x^1,\ldots,x^n\})$ -- then a <u>field function</u> on U is a C^∞ function of the form

$$F(g_{ab},g_{ab,i_1},\ldots,g_{ab,i_1\cdots i_m}).$$

Notation: $\mathcal{F}(U)$ is the set of field functions on U.

[Note: Every field function on U is of finite order in the derivatives of the g_{ab} (but the order itself is not fixed).]

Example: Let $F \in MC_n(p,q,w,m)$ — then its components

$$F^{i_1\cdots i_p}{}_{j_1\cdots j_q}(g_{ab},g_{ab,c_1},\ldots,g_{ab,c_1\cdots c_m})$$

are field functions on U.

[Note: In general, however, field functions are definitely not tensorial.]

Given $F \in \mathcal{F}(U)$, abbreviate

$$\frac{\partial F}{\partial g_{ab,i_1\cdots i_k}}$$

to

$$F^{ab,i_1\cdots i_k}$$

with the understanding that

$$F^{ab,i_1\cdots i_0} = \frac{\partial F}{\partial g_{ab}},$$

and for each $i = 1,\ldots,n$, define a differential operator D_i on $\mathcal{F}(U)$ by

$$D_i F = \sum_{k=0}^{\infty} F^{ab,i_1\cdots i_k} g_{ab,i_1\cdots i_k i},$$

thus $D_iF = F_{,i}$.

[Note: Needless to say, the sum terminates at the order of F.]

Definition: The <u>Euler-Lagrange derivative</u> E^{ab} is the map $F(U) \rightarrow F(U)$ defined by the rule

$$E^{ab}(F) = \sum_{k=0}^{\infty} (-1)^{k+1} D_{i_1 \cdots i_k} F^{ab,i_1 \cdots i_k},$$

where $D_{i_1 \cdots i_0} F = F$ and $D_{i_1 \cdots i_k} = D_{i_1} \circ \cdots \circ D_{i_k}$ $(k \geq 1)$.

[Note: $\forall\ F \in F(U)$,

$$F^{ab,i_1 \cdots i_k} = F^{ba,i_1 \cdots i_k}$$

$\Rightarrow$

$$E^{ab}(F) = E^{ba}(F).]$$

<u>LEMMA</u> Suppose that $F^1, \ldots, F^n$ are elements of $F(U)$ — then

$$E^{ab}(D_iF^i) = 0.$$

[In fact,

$$E^{ab}(D_iF^i) = \sum_{k=0}^{\infty} (-1)^{k+1} D_{ii_1 \cdots i_k} (F^i)^{ab,i_1 \cdots i_k}$$

$$+ \sum_{k=1}^{\infty} (-1)^{k+1} D_{i_1 \cdots i_k} (F^{i_k})^{ab,i_1 \cdots i_{k-1}}$$

$$= 0.]$$

Any field function of the form

$$D_iF^i$$

is said to be an <u>ordinary divergence</u>. Therefore the lemma states that the Euler-Lagrange derivative annihilates all ordinary divergences.

[Note: In practice, when working locally, this means that one can add a possibly nontensorial ordinary divergence to a lagrangian without affecting the Euler-Lagrange derivative.]

Example: Let $L = |g|^{1/2}S$ -- then

$$E^{ij}(L) = |g|^{1/2}[R^{ij} - \frac{1}{2} Sg^{ij}].$$

Locally, there is a decomposition

$$L = A + B^{i}_{,i} ,$$

where the field functions A, B^i are given by

$$\begin{cases} A = |g|^{1/2}g^{ij}(\Gamma^{k}_{i\ell}\Gamma^{\ell}_{jk} - \Gamma^{k}_{ij}\Gamma^{\ell}_{k\ell}) \\ B^{i} = |g|^{1/2}(g^{k\ell}\Gamma^{i}_{k\ell} - g^{ik}\Gamma^{\ell}_{k\ell}). \end{cases}$$

Therefore

$$E^{ij}(L) = E^{ij}(A).$$

[Note: Neither A nor B^i is tensorial. On the other hand,

$$\begin{cases} A = A(g_{ab}, g_{ab,c}) \\ B^{i} = B^{i}(g_{ab}, g_{ab,c}). \end{cases}$$

Since A is independent of $g_{ab,cd}$, it follows that $E^{ij}(L)$ contains no third or fourth derivatives of g_{ab}.]

Details Working locally, write

$$|g|^{1/2}S$$

$$= |g|^{1/2}g^{ab}R_{ab}$$

$$= |g|^{1/2}g^{ab}[\Gamma^{c}{}_{ab,c} - \Gamma^{c}{}_{ac,b} + \Gamma^{c}{}_{ab}\Gamma^{d}{}_{cd} - \Gamma^{d}{}_{ac}\Gamma^{c}{}_{bd}]$$

$$= (|g|^{1/2}g^{ab}\Gamma^{c}{}_{ab})_{,c} - (|g|^{1/2}g^{ab})_{,c}\Gamma^{c}{}_{ab}$$

$$- (|g|^{1/2}g^{ab}\Gamma^{c}{}_{ac})_{,b} + (|g|^{1/2}g^{ab})_{,b}\Gamma^{c}{}_{ac}$$

$$+ |g|^{1/2}g^{ab}\Gamma^{c}{}_{ab}\Gamma^{d}{}_{cd} - |g|^{1/2}g^{ab}\Gamma^{d}{}_{ac}\Gamma^{c}{}_{bd}$$

$$= - (|g|^{1/2}g^{ab})_{,c}\Gamma^{c}{}_{ab} + (|g|^{1/2}g^{ab})_{,b}\Gamma^{c}{}_{ac}$$

$$+ (|g|^{1/2}g^{ab}\Gamma^{c}{}_{ab})_{,c} - (|g|^{1/2}g^{ab}\Gamma^{c}{}_{ac})_{,b}$$

$$+ |g|^{1/2}g^{ab}\Gamma^{c}{}_{ab}\Gamma^{d}{}_{cd} - |g|^{1/2}g^{ab}\Gamma^{d}{}_{ac}\Gamma^{c}{}_{bd}.$$

- $(|g|^{1/2}g^{ab}\Gamma^{c}{}_{ab})_{,c}$

$$= (|g|^{1/2}g^{cb}\Gamma^{a}{}_{cb})_{,a}$$

$$= (|g|^{1/2}g^{bc}\Gamma^{a}{}_{bc})_{,a}.$$

- $-(|g|^{1/2}g^{ab}\Gamma^{c}{}_{ac})_{,b}$

$$= - (|g|^{1/2}g^{ba}\Gamma^{c}{}_{bc})_{,a}$$

$$= - (|g|^{1/2}g^{ab}\Gamma^{c}{}_{bc})_{,a}.$$

Combining terms thus gives

$$(|g|^{1/2}g^{ab}\Gamma^{c}{}_{ab})_{,c} - (|g|^{1/2}g^{ab}\Gamma^{c}{}_{ac})_{,b}$$

$$= (|g|^{1/2}(g^{bc}\Gamma^{a}{}_{bc} - g^{ab}\Gamma^{c}{}_{bc}))_{,a}$$

$$= B^{a}{}_{,a} \ .$$

Next

$$\nabla_i(|g|^{1/2}g^{ab}) = 0$$

$$\Rightarrow$$

$$\left[\begin{array}{l} (|g|^{1/2}g^{ab})_{,c} = |g|^{1/2}g^{ab}\Gamma^{d}{}_{cd} - |g|^{1/2}g^{db}\Gamma^{a}{}_{cd} - |g|^{1/2}g^{ad}\Gamma^{b}{}_{cd} \\ (|g|^{1/2}g^{ab})_{,b} = |g|^{1/2}g^{ab}\Gamma^{d}{}_{bd} - |g|^{1/2}g^{db}\Gamma^{a}{}_{bd} - |g|^{1/2}g^{ad}\Gamma^{b}{}_{bd} \ . \end{array}\right.$$

$$\bullet - (|g|^{1/2}g^{ab})_{,c}\Gamma^{c}{}_{ab}$$

$$= - |g|^{1/2}g^{ab}\Gamma^{c}{}_{ab}\Gamma^{d}{}_{cd} + |g|^{1/2}g^{db}\Gamma^{c}{}_{ab}\Gamma^{a}{}_{cd}$$

$$+ |g|^{1/2}g^{ad}\Gamma^{c}{}_{ab}\Gamma^{b}{}_{cd}$$

$$= |g|^{1/2}[g^{db}\Gamma^{c}{}_{ab}\Gamma^{a}{}_{cd} - g^{ab}\Gamma^{c}{}_{ab}\Gamma^{d}{}_{cd}]$$

$$+ |g|^{1/2}g^{ad}\Gamma^{c}{}_{ab}\Gamma^{b}{}_{cd}$$

$$= |g|^{1/2}g^{ab}[\Gamma^{c}{}_{db}\Gamma^{d}{}_{ca} - \Gamma^{c}{}_{ab}\Gamma^{d}{}_{cd}]$$

$$+ |g|^{1/2}g^{ad}\Gamma^{c}{}_{ab}\Gamma^{b}{}_{cd}$$

$$= |g|^{1/2} g^{ab} [\Gamma^{c}_{\cdot ad} \Gamma^{d}_{\cdot bc} - \Gamma^{c}_{\cdot ab} \Gamma^{d}_{\cdot cd}]$$

$$+ |g|^{1/2} g^{ad} \Gamma^{c}_{\cdot ab} \Gamma^{b}_{\cdot cd}$$

$$= A + |g|^{1/2} g^{ad} \Gamma^{c}_{\cdot ab} \Gamma^{b}_{\cdot cd}$$

$$= A + |g|^{1/2} g^{ab} \Gamma^{c}_{\cdot ad} \Gamma^{d}_{\cdot cb}$$

$$= A + |g|^{1/2} g^{ab} \Gamma^{d}_{\cdot ac} \Gamma^{c}_{\cdot db}$$

$$= A + |g|^{1/2} g^{ab} \Gamma^{d}_{\cdot ac} \Gamma^{c}_{\cdot bd} .$$

So far then

$$|g|^{1/2} S = A + B^{a}_{,a}$$

$$+ (|g|^{1/2} g^{ab})_{,b} \Gamma^{c}_{\cdot ac} + |g|^{1/2} g^{ab} \Gamma^{c}_{\cdot ab} \Gamma^{d}_{\cdot cd} .$$

• $(|g|^{1/2} g^{ab})_{,b} \Gamma^{c}_{\cdot ac}$

$$= |g|^{1/2} g^{ab} \Gamma^{c}_{\cdot ac} \Gamma^{d}_{\cdot bd} - |g|^{1/2} g^{ad} \Gamma^{c}_{\cdot ac} \Gamma^{b}_{\cdot bd}$$

$$- |g|^{1/2} g^{db} \Gamma^{c}_{\cdot ac} \Gamma^{a}_{\cdot bd}$$

$$= |g|^{1/2} g^{ab} \Gamma^{c}_{\cdot ac} \Gamma^{d}_{\cdot bd} - |g|^{1/2} g^{ab} \Gamma^{c}_{\cdot ac} \Gamma^{d}_{\cdot db}$$

$$- |g|^{1/2} g^{db} \Gamma^{c}_{\cdot ac} \Gamma^{a}_{\cdot bd}$$

$$= - |g|^{1/2} g^{db} \Gamma^{c}_{\cdot ac} \Gamma^{a}_{\cdot bd}$$

$$= - |g|^{1/2} g^{ab} \Gamma^{c}_{\cdot dc} \Gamma^{d}_{\cdot ba}$$

$$= - |g|^{1/2} g^{ab} \Gamma^{d}_{cd} \Gamma^{c}_{ab} .$$

Therefore

$$|g|^{1/2} S = A + B^{a}_{,a} .$$

Example: Take $n = 4$ -- then

$$\varepsilon^{ijk\ell} R^{a}_{bij} R^{b}_{ak\ell}$$

$$= - [2\varepsilon^{ijk\ell} \Gamma^{b}_{ak} R^{a}_{bij} + \frac{4}{3} \varepsilon^{ijk\ell} \Gamma^{c}_{bi} \Gamma^{b}_{aj} \Gamma^{a}_{ck}]_{,\ell} .$$

[Note: It is clear that there is a lagrangian $L \in MC_4(0,0,1,2)$ which is given locally by

$$\varepsilon^{ijk\ell} R^{a}_{bij} R^{b}_{ak\ell} ,$$

so, being an ordinary divergence, $E^{rs}(L) = 0$. But $E(L)$ is tensorial, hence $E(L) = 0$.]

<u>FACT</u> Suppose that $L \in MC_n(0,0,1,m)$. Put

$$\mathcal{L} = g_{ij} E^{ij}(L) \quad \text{(a.k.a. } tr(E(L))\text{)}.$$

Then $\mathcal{L} \in MC_n(0,0,1,2m)$ and the order of $E^{ab}(\mathcal{L})$ is at most $2m$ (not $4m$).

[On general grounds,

$$PL(g,g) = \frac{\partial L}{\partial g_{ab}} g_{ab} + \sum_{k=1}^{m} L^{ab,i_1 \cdots i_k} g_{ab,i_1 \cdots i_k}$$

is an element of $MC_n(0,0,1,m)$, call it $\mathcal{L}_0$ -- then $\mathcal{L} + \mathcal{L}_0$ is an ordinary divergence, hence

$$E^{ab}(\mathcal{L}+\mathcal{L}_0) = E^{ab}(\mathcal{L}) + E^{ab}(\mathcal{L}_0) = 0.$$

But the order of $E^{ab}(L_0)$ is $\leq 2m$, thus the same holds for the order of $E^{ab}(L)$.]

Remark: The notion of ordinary divergence is local and involves partial derivatives rather than covariant derivatives. In this connection, recall that there is one important circumstance when the two notions coincide, viz. let $X \in 1\text{-}s\mathcal{D}_0^1(M)$ — then

$$\nabla_i X^i = X^i_{\ ,i}$$

provided ∇ is torsion free.

[Note: Put

$$\omega_X = \frac{X^i}{(n-1)!}\,\epsilon_{ij_1\cdots j_{n-1}}\,dx^{j_1}\wedge\ldots\wedge dx^{j_{n-1}}$$

or still,

$$\omega_X = \sum_{i=1}^{n}(-1)^{i+1}X^i dx^1\wedge\ldots\wedge\widehat{dx^i}\wedge\ldots dx^n.$$

Then

$$X^i_{\ ,i} = 0 \Rightarrow d\omega_X = 0.]$$

Notation: Given $F\in\mathcal{F}(U)$ and $k \geq 1$, put

$$E^{ab,i_1\cdots i_k}(F) = \sum_{\ell=k}^{\infty}(-1)^{\ell+1}\binom{\ell}{k}D_{i_{k+1}\cdots i_\ell}F^{ab,i_1\cdots i_\ell},$$

where $D_{i_{k+1}\cdots i_k}F = F$.

[Note: Extend this to $k = 0$ by the agreement $E^{ab,i_1\cdots i_0}(F) = E^{ab}(F)$.]

<u>LEMMA</u> We have

$$E^{ab,i_1\cdots i_k}(E^{a'b'}(F)) + E^{ab}(F)^{a'b',i_1\cdots i_k} = 0.$$

[Note: When k = 0, the equation reads

$$E^{ab}(E^{a'b'}(F)) + \frac{\partial E^{ab}(F)}{\partial g_{a'b'}} = 0.]$$

While we shall make no attempt at precise characterizations, it is of interest to at least say something about the kernel and the range of the

$$E^{ab}:\mathcal{F}(U) \to \mathcal{F}(U).$$

So, e.g., as has been shown above, all ordinary divergences are in their kernel.

Definition: Let $T^{ab} \in \mathcal{F}(U)$ $(a,b=1,\ldots,n)$ -- then the collection $\{T^{ab}\}$ is said to satisfy the Helmholtz condition if $\forall$ $k \geq 0$,

$$E^{ab,i_1\cdots i_k}(T^{a'b'}) + (T^{ab})^{a'b',i_1\cdots i_k} = 0.$$

Accordingly, in view of the lemma, any collection $\{T^{ab}\}$ in the range of the Euler-Lagrange derivative must satisfy the Helmholtz condition.

Example: We have

$$\begin{cases} E^{ij}(|g|^{1/2}S) = |g|^{1/2}G^{ij} \\ E^{ij}(-2\lambda|g|^{1/2}) = \lambda|g|^{1/2}g^{ij}, \end{cases}$$

thus the entities on the RHS satisfy the Helmholtz condition.

Example: Take n = 3 and put

$$C^{ij} = \varepsilon^{iab}R^{j}{}_{a;b} + \varepsilon^{jab}R^{i}{}_{a;b}.$$

Then the C^{ij} are the components of a symmetric element of $MC_3(2,0,1,3)$, the

Cotton tensor. We have

$$\left[\begin{array}{l} \nabla_j C^{ij} = 0 \\ \\ g_{ij} C^{ij} = 0. \end{array}\right.$$

In addition, it can be checked by computation that the Cotton tensor satisfies the Helmholtz condition, although it will be seen in the next section that there does not exist a lagrangian

$$L \in MC_3(0,0,1,m)$$

such that

$$E^{ij}(L) = C^{ij}.$$

Nevertheless, there are field functions F such that $E^{ij}(F) = C^{ij}$, one such being

$$F = -\varepsilon^{ak\ell}[\tfrac{1}{2}\Gamma^j{}_{ia}\Gamma^i{}_{jk,\ell} + \Gamma^j{}_{ia}\Gamma^b{}_{jk}\Gamma^i{}_{b\ell}].$$

Product Rule Let $F,G \in \mathcal{F}(U)$ — then

$$E^{ab,i_1\cdots i_k}(FG)$$

$$= \sum_{\ell=k}^{\infty} \binom{\ell}{k} [D_{i_{k+1}\cdots i_\ell}(F)E^{ab,i_1\cdots i_\ell}(G)$$

$$+ D_{i_{k+1}\cdots i_\ell}(G)E^{ab,i_1\cdots i_\ell}(F)].$$

In particular:

$$E^{ab}(FG)$$

$$= \sum_{\ell=0}^{\infty} [D_{i_1\cdots i_\ell}(F)E^{ab,i_1\cdots i_\ell}(G)$$

$$+ D_{i_1 \cdots i_\ell}(G) E^{ab, i_1 \cdots i_\ell}(F).$$

Suppose that $\{T^{ab}\}$ is a collection which satisfies the Helmholtz condition -- then

$$E^{ab}(g_{a'b'}, T^{a'b'})$$

$$= \sum_{\ell=0}^{\infty} [D_{i_1 \cdots i_\ell}(g_{a'b'}) E^{ab, i_1 \cdots i_\ell}(T^{a'b'})$$

$$+ D_{i_1 \cdots i_\ell}(T^{a'b'}) E^{ab, i_1 \cdots i_\ell}(g_{a'b'})$$

$$= - \sum_{\ell=0}^{\infty} [g_{a'b', i_1 \cdots i_\ell} (T^{ab})^{a'b', i_1 \cdots i_\ell} + T^{ab}].$$

Notation: Given a field function F, let

$$F_t = F(tg_{ab}, tg_{ab, i_1}, \ldots, tg_{ab, i_1 \cdots i_m}) \quad (t > 0).$$

We have

$$\frac{d}{dt} (t^2 T^{ab}_{t^2})$$

$$= 2tT^{ab}_{t^2} + t^2(2t) \sum_{\ell=0}^{\infty} (T^{ab})^{a'b', i_1 \cdots i_\ell}_{t^2} g_{a'b', i_1 \cdots i_\ell} .$$

Therefore

$$E^{ab}(g_{a'b'}, T^{a'b'}) = - \frac{1}{2} \cdot \frac{d}{dt} (t^2 T^{ab}_{t^2}) \Big|_{t = 1} .$$

[Note: In general,

$$E^{ab}(tg_{a'b'}, T^{a'b'}_{t^2}) = - \frac{1}{2} \cdot \frac{d}{dt} (t^2 T^{ab}_{t^2}).$$

To see this, let E_t^{ab} be the Euler-Lagrange derivative per $t^2 g$ -- then

$$E_t^{ab}(t^2 g_{a'b'}, T_{t^2}^{a'b'})$$

$$= - \sum_{\ell=0}^{\infty} [t^2 g_{a'b',i_1 \cdots i_\ell} (T^{ab})^{a'b',i_1 \cdots i_\ell} + T_{t^2}^{ab}]$$

$$= - \frac{1}{2t} \cdot \frac{d}{dt} (t^2 T_{t^2}^{ab}).$$

On the other hand,

$$E^{ab}(tg_{a'b'}, T_{t^2}^{a'b'})$$

$$= \frac{1}{t} E^{ab}(t^2 g_{a'b'}, T_{t^2}^{a'b'})$$

$$= \frac{1}{t} (t^2 E_t^{ab}(t^2 g_{a'b'}, T_{t^2}^{a'b'}))$$

$$= t(- \frac{1}{2t} \cdot \frac{d}{dt} (t^2 T_{t^2}^{ab}))$$

$$= - \frac{1}{2} \frac{d}{dt} (t^2 T_{t^2}^{ab}).]$$

<u>Section 25</u>: <u>Applications of Homogeneity</u> Let M be a connected C^∞ manifold of dimension n, which we shall assume is orientable.

Definition: Let $F \in \mathcal{F}(U)$ -- then F is said to be <u>homogeneous of degree $\varkappa$</u> if $\forall\ t > 0$,

$$F(tg_{ab}, tg_{ab,i_1}, \ldots, tg_{ab,i_1 \cdots i_m})$$

$$= t^{\varkappa} F(g_{ab}, g_{ab,i_1}, \ldots, g_{ab,i_1 \cdots i_m}),$$

i.e., if $\forall\ t > 0$,

$$F_t = t^{\varkappa} F.$$

For the record,

$$\begin{cases} R_{ijk\ell} \text{ is homogeneous of degree } 1 \\ R_{j\ell} \text{ is homogeneous of degree } 0 \\ g^{ik} g^{j\ell} R_{ijk\ell} \text{ is homogeneous of degree } -1. \end{cases}$$

[Note: $|g|^{1/2}$ is homogeneous of degree n/2.]

<u>LEMMA</u> Let $T^{ab} \in \mathcal{F}(U)$ be homogeneous of degree $\varkappa \neq -1$ $(a,b = 1,\ldots,n)$. Assume: The collection $\{T^{ab}\}$ satisfies the Helmholtz condition -- then

$$T^{ab} = E^{ab}(L),$$

where

$$L = -\frac{1}{\varkappa+1} g_{a'b'} T^{a'b'}.$$

[In fact,

$$\frac{d}{dt}\,(t^2 T^{ab}_{t^2})$$

$$= \frac{d}{dt}\,(t^2 t^{2\varkappa} T^{ab})$$

$$= 2(\varkappa+1)t^{2\varkappa+1}T^{ab}.$$

But

$$E^{ab}(tg_{a'b'},T^{a'b'}_{t^2}) = -\frac{1}{2}\cdot\frac{d}{dt}\,(t^2 T^{ab}_{t^2})$$

$\Rightarrow$

$$t^{2\varkappa+1}E^{ab}(g_{a'b'},T^{a'b'}) = -\frac{1}{2}\cdot 2(\varkappa+1)t^{2\varkappa+1}T^{ab}$$

$\Rightarrow$

$$E^{ab}(g_{a'b'},T^{a'b'}) = -(\varkappa+1)T^{ab}.]$$

Example: Take n = 4 -- then $|g|^{1/2}G^{ij}$ is homogeneous of degree 0 and, as can be checked by computation, the collection $\{|g|^{1/2}G^{ij}\}$ satisfies the Helmholtz condition. Therefore

$$|g|^{1/2}G^{ij} = E^{ij}(-L),$$

where

$$L = |g|^{1/2}g_{ij}G^{ij} = |g|^{1/2}(S - (\tfrac{4}{2})S) = -|g|^{1/2}S.$$

I.e.:

$$|g|^{1/2}G^{ij} = E^{ij}(|g|^{1/2}S),$$

in agreement with the general theory.

Example: Take n = 3 and let

$$C^{ij} = \varepsilon^{iab} R^{j}{}_{a;b} + \varepsilon^{jab} R^{i}{}_{a;b} .$$

Then the collection $\{C^{ij}\}$ satisfies the Helmholtz condition. Still, C^{ij} is homogeneous of degree -1, thus the foregoing construction is not applicable.

Remark: Let $A \in MC_n(2,0,1,m)$ $(n > 1)$ be homogeneous of degree -1 -- then it can be shown that $m \leq 3$ if n is odd and $m \leq n$ if n is even.

Symbol Pushing In the literature, one will find the following assertion. Suppose that $\{T^{ab}\}$ is a collection which satisfies the Helmholtz condition -- then

$$T^{ab} = E^{ab}(L),$$

where

$$L = - \int_0^1 2tg_{a'b'} T^{a'b'}_{t^2} dt.$$

[Formally,

$$\begin{aligned} E^{ab}(L) &= - \int_0^1 E^{ab}(2tg_{a'b'} T^{a'b'}_{t^2}) dt \\ &= \int_0^1 \frac{d}{dt} (t^2 T^{ab}_{t^2}) dt \\ &= T^{ab}. \end{aligned}$$

However there is a tacit assumption, namely that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 T^{ab}_{\varepsilon^2} = 0.$$

And this is not true in general.]

Let $F \in MC_n(p,q,w,m)$. Take $\bar{x}^i = \frac{1}{t} x$ $(t > 0)$ -- then

$$F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(t^2 g_{ab}, t^3 g_{ab,c_1}, \ldots, t^{2+m} g_{ab,c_1 \cdots c_m})$$

$$= [t^n]^w t^{q-p} F^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(g_{ab}, g_{ab,c_1}, \ldots, g_{ab,c_1 \cdots c_m}).$$

Specialize to the case when $q = 0$, $w = 0$ -- then if $F^{i_1 \cdots i_p}$ is homogeneous of degree $\varkappa$, we have

$$t^{-(2\varkappa+p)} F^{i_1 \cdots i_p}(g_{ab}, g_{ab,c_1}, \ldots, g_{ab,c_1 \cdots c_m})$$

$$= F^{i_1 \cdots i_p}(g_{ab}, t g_{ab,c_1}, \ldots, t^m g_{ab,c_1 \cdots c_m})$$

or still, in view of the Replacement Theorem,

$$t^{-(2\varkappa+p)} F^{i_1 \cdots i_p}(g_{ab}, g_{ab,c_1}, \ldots, g_{ab,c_1 \cdots c_m})$$

$$= F^{i_1 \cdots i_p}(g_{ab}, 0, t^2 G_{abc_1 c_2}, \ldots, t^m G_{abc_1 \cdots c_m}).$$

<u>Classification</u>

- If $2\varkappa+p = 0$, then $F^{i_1 \cdots i_p}$ is a function of g_{ab} alone.
- If $2\varkappa+p$ is positive, then $F^{i_1 \cdots i_p} = 0$.
- If $2\varkappa+p$ is negative and not an integer, then $F^{i_1 \cdots i_p} = 0$.
- If $2\varkappa+p = -1$, then $F^{i_1 \cdots i_p} = 0$.
- If $2\varkappa+p = -2, -3, \ldots$, then $F^{i_1 \cdots i_p}$ is a polynomial in the $G_{abc_1 \cdots c_k}$.

Remark: If $F \in MC_n(p,0,w,m)$ is homogeneous of degree $\varkappa$, then

$$|g|^{-w/2} F \in MC_n(p,0,1,m)$$

is homogeneous of degree $\varkappa - (\frac{n}{2})w$. Therefore the structure of F can be ascertained from the structure of $|g|^{-w/2}F$.

LEMMA If $n > 1$ is odd and if $L \in MC_n(0,0,1,m)$ is homogeneous of degree 0, then $L = 0$.

Example: The preceding lemma breaks down if n is even. For instance, when $n = 4$,

$$|g|^{1/2} C^{ijk\ell} C_{ijk\ell}$$

is a second order lagrangian which is homogeneous of degree 0, as is

$$\varepsilon^{abcd} C^i{}_{jab} C^j{}_{icd} .$$

FACT Suppose that $L \in MC_n(0,0,1,2)$ is homogeneous of degree 0 — then

$$g_{ij} E^{ij}(L) = 0.$$

[Recall that

$$E^{ij}(L) = -\Pi^{ij} - \Pi^{ij,k\ell}{}_{;k\ell} .$$

But here

$$\begin{cases} g_{ij}\Pi^{ij} = 0 \\ g_{ij}\Pi^{ij,k\ell} = 0. \end{cases}$$]

Notation: Given a field function F, let

$$F_{[t]} = F(g_{ab}, t g_{ab,i_1}, \ldots, t^m g_{ab,i_1 \cdots i_m}) \quad (t > 0).$$

So, if $F \in MC_n(p,q,w,m)$, then

$$F_{[t]} = t^{nw+q-p} F_{1/t^2}.$$

Example: Let $L \in MC_n(0,0,1,m)$ -- then

$$L_{[t]} = t^n L_{1/t^2}$$

$\Rightarrow$

$$\begin{aligned} L_{[t]}(s^2 g) &= t^n L((\tfrac{s}{t})^2 g) \\ &= t^n (\tfrac{s}{t})^n L_{[t/s]}(g) \\ &= s^n L_{[t/s]}(g). \end{aligned}$$

<u>LEMMA</u> $\forall\, L \in MC_n(0,0,1,m)$, we have

$$E(L_{[t]}) = E(L)_{[t]}.$$

Application: Suppose that $L \in MC_n(0,0,1,m)$ is homogeneous of degree 0 — then $E(L) \in MC_n(2,0,1,2m)$ is homogeneous of degree -1.

[In fact, $L_t = L$, hence

$$L_{[t]} = t^n L.$$

Therefore

$$E(L_{[t]}) = E(t^n L) = t^n E(L).$$

On the other hand,

$$E(L)_{[t]} = t^{n-2} E(L)_{1/t^2}.$$

Therefore

$$t^n E(L) = t^{n-2} E(L)_{1/t^2}$$

$$\Rightarrow$$

$$t^2 E(L) = E(L)_{1/t^2}$$

$$\Rightarrow$$

$$E(L)_t = t^{-1} E(L).]$$

Example: Take $n = 4$ -- then $L = |g|^{1/2} S^2$ is homogeneous of degree 0, hence

$$E^{ij}(L) = |g|^{1/2} S(2R^{ij} - \frac{1}{2} g^{ij} S)$$

$$+ 2|g|^{1/2} g^{ij} \nabla^k \nabla_k S - 2|g|^{1/2} \nabla^i \nabla^j S$$

is homogeneous of degree -1.

<u>LEMMA</u> Let $L \in MC_n(0,0,1,m)$, where $n > 1$ is odd. Assume: $E(L) \neq 0$ -- then $E(L)$ can not be homogeneous of degree -1.

[If $E(L)$ were homogeneous of degree -1, then the relation

$$E(L)_{[t]} = t^{n-2} E_{1/t^2}$$

reduces to

$$E(L)_{[t]} = t^n E(L),$$

hence

$$t^n E(L) = E(L_{[t]}).$$

But

$$\frac{d}{dt} E(L_{[t]}) = E(\frac{d}{dt} L_{[t]}).$$

Consequently,

$$E(L) = E(L'),$$

where

$$L' = \lim_{t \downarrow 0} \frac{1}{n!} \left(\frac{d^n}{dt^n} L_{[t]}\right)$$

is homogeneous of degree 0:

$$L'(s^2 g) = \lim_{t \downarrow 0} \frac{1}{n!} \left(\frac{d^n}{dt^n} L_{[t]}(s^2 g)\right)$$

$$= \lim_{t \downarrow 0} \frac{1}{n!} \left(\frac{d^n}{dt^n} s^n L_{[t/s]}(g)\right)$$

$$= \lim_{t \downarrow 0} \frac{1}{n!} \left(\frac{d^n}{dt^n} L_{[t]}(g)\right)$$

$$= L'(g).$$

Therefore, since $n > 1$ is odd,

$$L' = 0 \Rightarrow E(L') = 0 \Rightarrow E(L) = 0,$$

a contradiction.]

By way of a corollary, there does not exist a lagrangian

$$L \in MC_3(0,0,1,m)$$

such that

$$E^{ij}(L) = C^{ij}.$$

Remark: Let $A \in MC_n(2,0,1,m)$ -- then in order that $A = E(L)$ for some lagrangian L, it is necessary that $A^{ij} = A^{ji}$ & $\nabla_j A^{ij} = 0$. In addition, the collection $\{A^{ij}\}$ must satisfy the Helmholtz condition. But, as the Cotton tensor shows, these requirements are not sufficient.

FACT Let $A \in MC_n(2,0,1,m)$. Suppose that the collection $\{A^{ij}\}$ satisfies the Helmholtz condition -- then

$$\nabla_j A^{ij} = 0.$$

[Note: It is not assumed that $A^{ij} = A^{ji}$, thus the condition appears to be asymmetric. Still,

$$\nabla_j A^{ij} = 0 \quad \Leftrightarrow \quad \nabla_i A^{ij} = 0.]$$

Section 26: Questions of Uniqueness Let M be a connected C^∞ manifold of dimension n, which we shall assume is orientable.

Suppose that $L \in MC_n(0,0,1,2)$ -- then

$$\begin{cases} E^{ij}(L) = E^{ji}(L) \\ \nabla_j E^{ij}(L) = 0 \end{cases}$$

and, in general,

$$E^{ij}(L) = E^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}, g_{ab,cdr}, g_{ab,cdrs}).$$

However, for certain L (e.g., $L = |g|^{1/2}$ or $L = |g|^{1/2}S$), $E^{ij}(L)$ is of the second order, i.e., $E^{ij}(L)$ depends only on g_{ab}, $g_{ab,c}$, and $g_{ab,cd}$.

Problem: Find all elements

$$A \in MC_n(2,0,1,2)$$

subject to

$$\begin{cases} A^{ij} = A^{ji} \\ \nabla_j A^{ij} = 0. \end{cases}$$

Remarkably, this problem turns out to be tractable and a complete solution was obtained in the early 1970s by Lovelock.

Put

$$N = \begin{cases} n/2 \text{ if } n \text{ is even} \\ (n+1)/2 \text{ if } n \text{ is odd.} \end{cases}$$

THEOREM Suppose that $A \in MC_n(2,0,1,2)$ satisfies the conditions

$$\begin{cases} A^{ij} = A^{ji} \\ \nabla_j A^{ij} = 0. \end{cases}$$

Then $\exists$ constants C_p $(p = 1,\ldots,N-1)$, λ such that

$$A^{ij} = |g|^{1/2} \sum_{p=1}^{N-1} C_p g^{ik} \delta^{j\ell_1\cdots\ell_{2p}}_{kk_1\cdots k_{2p}} R^{k_1k_2}{}_{\ell_1\ell_2} \cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}} + \lambda|g|^{1/2} g^{ij}.$$

[Note: We shall also see that

$$\exists\ L \in MC_n(0,0,1,2)$$

for which

$$E^{ij}(L) = A^{ij}.]$$

Example: If $n = 1$ or $n = 2$, then

$$A^{ij} = \lambda|g|^{1/2} g^{ij}.$$

<u>The Fundamental Consequence</u> If $\dim M = 4$, then a symmetric $A \in MC_4(2,0,1,2)$ of zero divergence has components

$$A^{ij} = C|g|^{1/2}[R^{ij} - \frac{1}{2} S g^{ij}] + \lambda|g|^{1/2} g^{ij},$$

where C and λ are constants.

[It is a question of reducing

$$C_1|g|^{1/2}[g^{ik}\delta^{j\ell_1\ell_2}_{kk_1k_2} R^{k_1k_2}{}_{\ell_1\ell_2}] + \lambda|g|^{1/2} g^{ij}$$

to the stated form. By definition,

$$\delta^{j\ell_1\ell_2}{}_{kk_1k_2} = \begin{vmatrix} \delta^j{}_k & \delta^j{}_{k_1} & \delta^j{}_{k_2} \\ \delta^{\ell_1}{}_k & \delta^{\ell_1}{}_{k_1} & \delta^{\ell_1}{}_{k_2} \\ \delta^{\ell_2}{}_k & \delta^{\ell_2}{}_{k_1} & \delta^{\ell_2}{}_{k_1} \end{vmatrix}$$

$$= \delta^j{}_k \delta^{\ell_1\ell_2}{}_{k_1k_2} - \delta^j{}_{k_1}\delta^{\ell_1\ell_2}{}_{kk_2} + \delta^j{}_{k_2}\delta^{\ell_1\ell_2}{}_{kk_1}\ .$$

- $g^{ik}\delta^j{}_k\delta^{\ell_1\ell_2}{}_{k_1k_2}R^{k_1k_2}{}_{\ell_1\ell_2}$

$$= g^{ik}\delta^j{}_k(\delta^{\ell_1}{}_{k_1}\delta^{\ell_2}{}_{k_2} - \delta^{\ell_1}{}_{k_2}\delta^{\ell_2}{}_{k_1})R^{k_1k_2}{}_{\ell_1\ell_2}$$

$$= g^{ik}\delta^j{}_kR^{k_1k_2}{}_{k_1k_2} - g^{ik}\delta^j{}_kR^{k_1k_2}{}_{k_2k_1}$$

$$= 2g^{ik}\delta^j{}_kR^{k_1k_2}{}_{k_1k_2}$$

$$= 2g^{ij}R^{k_1k_2}{}_{k_1k_2} = 2g^{ij}S.$$

- $- g^{ik}\delta^j{}_{k_1}\delta^{\ell_1\ell_2}{}_{kk_2}R^{k_1k_2}{}_{\ell_1\ell_2}$

$$= - g^{ik}\delta^{\ell_1\ell_2}{}_{kk_2}R^{jk_2}{}_{\ell_1\ell_2}$$

$$= - g^{ik}\delta^{\ell_1\ell_2}{}_{k\ell}R^{j\ell}{}_{\ell_1\ell_2}\ .$$

$$\bullet\ g^{ik}\delta^{j}{}_{k_2}\delta^{\ell_1\ell_2}{}_{kk_1}R^{k_1k_2}{}_{\ell_1\ell_2}$$

$$= g^{ik}\delta^{\ell_1\ell_2}{}_{kk_1}R^{k_1j}{}_{\ell_1\ell_2}$$

$$= -\ g^{ik}\delta^{\ell_1\ell_2}{}_{kk_1}R^{jk_1}{}_{\ell_1\ell_2}$$

$$= -\ g^{ik}\delta^{\ell_1\ell_2}{}_{k\ell}R^{j\ell}{}_{\ell_1\ell_2}\ .$$

And

$$g^{ik}\delta^{\ell_1\ell_2}{}_{k\ell}R^{j\ell}{}_{\ell_1\ell_2}$$

$$= g^{ik}\begin{vmatrix} \delta^{\ell_1}{}_{k} & \delta^{\ell_1}{}_{\ell} \\ \delta^{\ell_2}{}_{k} & \delta^{\ell_2}{}_{\ell} \end{vmatrix} R^{j\ell}{}_{\ell_1\ell_2}$$

$$= g^{ik}\delta^{\ell_1}{}_{k}\delta^{\ell_2}{}_{\ell}R^{j\ell}{}_{\ell_1\ell_2} - g^{ik}\delta^{\ell_1}{}_{\ell}\delta^{\ell_2}{}_{k}R^{j\ell}{}_{\ell_1\ell_2}$$

$$= g^{ik}R^{j\ell}{}_{k\ell} - g^{ik}R^{j\ell}{}_{\ell k}$$

$$= 2g^{ik}R^{j\ell}{}_{k\ell}$$

$$= 2R^{j\ell i}{}_{\ell}$$

$$= 2R^{ji} = 2R^{ij}.$$

Therefore

$$g^{ik}\delta^{j\ell_1\ell_2}_{kk_1k_2}R^{k_1k_2}{}_{\ell_1\ell_2}$$

$$= 2g^{ij}S - 4R^{ij}.$$

But

$$G^{ij} = R^{ij} - \frac{1}{2}Sg^{ij}$$

$$\Rightarrow$$

$$g^{ik}\delta^{j\ell_1\ell_2}_{kk_1k_2}R^{k_1k_2}{}_{\ell_1\ell_2} = -4G^{ij}.$$

The desired reduction is thus achieved by taking $C = -4C_1$.]

Scholium: If $L \in MC_4(0,0,1,m)$ and if $E(L) \in MC_4(2,0,1,2)$, then $E^{ij}(L)$ necessarily has the form

$$C|g|^{1/2}[R^{ij} - \frac{1}{2}Sg^{ij}] + \lambda|g|^{1/2}g^{ij},$$

where C and λ are constants.

Remark: Let $L \in MC_4(0,0,1,2)$. Assume: $E(L) \in MC_4(2,0,1,2)$ -- then it can be shown that

$$L = C|g|^{1/2}S - 2\lambda|g|^{1/2}$$

$$+ C'\epsilon^{ijk\ell}R^a{}_{bij}R^b{}_{ak\ell}$$

$$+ C''|g|^{1/2}[S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}],$$

where C, C', C", and λ are constants. However, as we have seen earlier, the terms multiplying C' and C" are annihilated by the Euler-Lagrange derivative, hence per prediction, make no contribution to the Euler-Lagrange expression

$$C|g|^{1/2}[R^{ij} - \frac{1}{2} Sg^{ij}] + \lambda|g|^{1/2}g^{ij}.$$

[Note: The analysis that gives the structure of A^{ij} when dim M = 4 applies verbatim when dim M = 3. On the other hand, there is a simplification since the only $L \in MC_3(0,0,1,2)$ for which $E(L) \in MC_3(2,0,1,2)$ are the

$$C|g|^{1/2}S - 2\lambda|g|^{1/2}.]$$

Example: If n = 5 or n = 6, then

$$A^{ij} = C|g|^{1/2}[R^{ij} - \frac{1}{2} Sg^{ij}] + \lambda|g|^{1/2}g^{ij}$$

$$+ D|g|^{1/2}[2SR^{ij} - 4R^{ikj\ell}R_{k\ell} + 2R^{i}{}_{abc}R^{jabc} - 4R^{ia}R^{j}{}_{a}$$

$$- \frac{1}{2}(S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd})g^{ij}],$$

where C, D, and λ are constants.

We shall now turn to the proof of the theorem. So let

$$A \in MC_n(2,0,1,2)$$

subject to

$$\begin{cases} A^{ij} = A^{ji} \\ \nabla_j A^{ij} = 0. \end{cases}$$

Put

$$\begin{cases} A^{ij;ab} = \dfrac{\partial A^{ij}}{\partial g_{ab}} \\ A^{ij;ab,c} = \dfrac{\partial A^{ij}}{\partial g_{ab,c}} \\ A^{ij;ab,cd} = \dfrac{\partial A^{ij}}{\partial g_{ab,cd}}. \end{cases}$$

Then

$$A^{ij;ab,cd} = A^{ij;ba,cd} = A^{ij;ab,dc}.$$

Identities

- $A^{ij;ab,cd} + A^{ij;ac,db} + A^{ij;ad,bc} = 0.$

- $A^{ij;ab,cd} = A^{ij;cd,ab}.$

N.B. We have

$$\nabla_j A^{ij} = \frac{\partial A^{ij}}{\partial x_j} + \Gamma^i{}_{ja}A^{aj} + \Gamma^j{}_{ja}A^{ia} - \Gamma^a{}_{ja}A^{ij}$$

$\Rightarrow$

$$\nabla_j A^{ij} = A^{ij;ab,cd}g_{ab,cdj} + A^{ij;ab,c}g_{ab,cj} + A^{ij;ab}g_{ab,j} + \Gamma^i{}_{ja}A^{aj}$$

$\Rightarrow$

$$\frac{\partial(\nabla_j A^{ij})}{\partial g_{ab,rst}} = \frac{1}{3}\,(A^{it;ab,rs} + A^{is;ab,tr} + A^{ir;ab,st}).$$

Identities

- $A^{it;ab,rs} + A^{is;ab,tr} + A^{ir;ab,st} = 0.$

- $A^{it;ab,rs} = A^{rs;ab,it}.$

Therefore

$$\begin{aligned} A^{it;ab,rs} &= A^{it;rs,ab} \\ &= A^{ab;rs,it} \\ &= A^{ab;it,rs} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} &A^{it;ab,rs} + A^{ia;bt,rs} + A^{ib;ta,rs} \\ &= A^{ab;it,rs} + A^{ia;bt,rs} + A^{ib;ta,rs} \end{aligned}$$

$$= A^{ab;rs,it} + A^{ia;rs,bt} + A^{ib;rs,ta}$$

$$= A^{it;rs,ab} + A^{bt;rs,ia} + A^{ta;rs,ib}$$

$$= A^{ti;rs,ab} + A^{tb;rs,ia} + A^{ta;rs,ib}$$

$$= A^{ti;rs,ab} + A^{tb;rs,ia} + A^{ta;rs,bi}$$

$$= 0.$$

Notation: For $p = 1,2,\ldots,$ write

$$A^{ab;i_1 i_2,i_3 i_4;\ldots;i_{4p-3} i_{4p-2},i_{4p-1} i_{4p}}$$

$$= \partial A^{ab;i_1 \text{---} i_{4p-4}} / \partial g_{i_{4p-3} i_{4p-2},i_{4p-1} i_{4p}} .$$

[Note: This prescription defines an element of $MC_n(2+4p,0,1,2)$.]

Special Case Take $p = 2$ -- then

$$A^{ab;i_1 i_2,i_3 i_4;i_5 i_6,i_7 i_8}$$

$$= \partial A^{ab;i_1 i_2,i_3 i_4} / \partial g_{i_5 i_6,i_7 i_8}$$

$$= \partial^2 A^{ab} / \partial g_{i_1 i_2,i_3 i_4} \partial g_{i_5 i_6,i_7 i_8} .$$

Properties of $A^{ab;i_1 \text{---} i_{4p}}$

(1) It is symmetric in ab and $i_{2k-1} i_{2k}$ $(k = 1,\ldots,2p)$.

(2) It is symmetric under the interchange of ab and $i_{2k-1} i_{2k}$ $(k = 1,\ldots,2p)$.

(3) It satisfies the cyclic identity involving any three of the four indices $(ab)(i_{2k-1} i_{2k})$ $(k = 1,\ldots,2p)$.

[Note: To illustrate (3), take p = 2 — then, e.g.,

$$A^{ab;i_1i_2,i_3i_4;i_5i_6,i_7i_8}$$

$$+ A^{bi_1;ai_2,i_3i_4;i_5i_6,i_7i_8}$$

$$+ A^{i_1a;bi_2,i_3i_4;i_5i_6,i_7i_8}$$

$$= 0.]$$

Definition: An indexed entity

$$B^{j_1j_2\cdots j_{2q-1}j_{2q}} \quad (q > 1)$$

is said to have <u>property S</u> if:

(S_1) It is symmetric in $j_{2\ell-1}j_{2\ell}$ ($\ell = 1,\ldots,q$);

(S_2) It is symmetric under the interchange of j_1j_2 and $j_{2\ell-1}j_{2\ell}$ ($\ell = 2,\ldots,q$);

(S_3) It satisfies the cyclic identity involving any three of the four indices $(j_1j_2)(j_{2\ell-1}j_{2\ell})$ ($\ell = 2,\ldots,q$).

In particular:

$$A^{ab;i_1 \text{ --- } i_{4p}}$$

has property S.

<u>LEMMA</u> If an indexed entity has property S, then it vanishes whenever three (or more) indices coincide.

Recall that

$$N = \begin{cases} n/2 \text{ if } n \text{ is even} \\ (n+1)/2 \text{ if } n \text{ is odd.} \end{cases}$$

<u>LEMMA</u> If M is any integer $\geq$ N and if

$$B^{j_1 j_2 \cdots j_{4M+1} j_{4M+2}}$$

is an indexed entity with property S, then

$$B^{j_1 j_2 \cdots j_{4M+1} j_{4M+2}} = 0.$$

[In fact,

$$4M + 2 \geq 4N + 2 > 2n,$$

thus at least three of the indices

$$j_1, j_2, \ldots, j_{4M+1}, j_{4M+2}$$

coincide.]

So, as a corollary,

$$A^{ab;i_1 \cdots i_{4N}} = 0.$$

Consequently,

$$A^{ab;i_1 \cdots i_{4(N-1)}}$$

$$= \Phi^{ab;i_1 \cdots i_{4(N-1)}}(g_{rs}, g_{rs,t}).$$

Here

$$\Phi^{ab;i_1 \cdots i_{4(N-1)}} \in MC_n(4N-2,0,1,1)$$

has property S. But, thanks to the Independence Theorem,

$$\Phi^{ab;i_1 - i_{4(N-1)}}(g_{rs}, g_{rs,t}) = \Phi^{ab;i_1 - i_{4(N-1)}}(g_{rs}).$$

Therefore

$$A^{ab;i_1 - i_{4(N-2)}}$$

$$= \Phi^{ab;i_1 - i_{4(N-1)}} g_{i_{4(N-1)-3} i_{4(N-1)-2}, i_{4(N-1)-1} i_{4(N-1)}}$$

$$+ \phi^{ab}(g_{rs}, g_{rs,t}).$$

Rappel: We have

$$R_{jki\ell} = \frac{1}{2}(g_{j\ell,ki} - g_{ji,k\ell} + g_{ki,j\ell} - g_{k\ell,ji}) + \Gamma_{jki\ell},$$

where

$$\Gamma_{jki\ell} = \Gamma_{aki}\Gamma^a{}_{j\ell} - \Gamma_{ak\ell}\Gamma^a{}_{ji}.$$

Put

$$\begin{cases} i = i_{4(N-1)-3} \\ j = i_{4(N-1)-2} \\ k = i_{4(N-1)-1} \\ \ell = i_{4(N-1)}. \end{cases}$$

Then

$$\Phi^{ab;i_1 - i_{4(N-1)}} R_{jki\ell}$$

$$= -\frac{3}{2}\Phi^{ab;i_1 \cdots i_{4(N-1)}} g_{ij,k\ell}$$

$$+ \Phi^{ab;i_1 \cdots i_{4(N-1)}} \Gamma_{jki\ell}$$

$\Rightarrow$

$$A^{ab;i_1 \cdots i_{4(N-2)}}$$

$$= -\frac{2}{3}\Phi^{ab;i_1 \cdots i_{4(N-1)}} R_{jki\ell} + \psi^{ab} ,$$

where the metric concomitant

$$\psi^{ab} = \varphi^{ab} + \frac{2}{3}\Phi^{ab;i_1 \cdots i_{4(N-1)}} \Gamma_{jki\ell}$$

is at most a function of g_{rs} and $g_{rs,t}$, hence is a function of g_{rs} alone. Now iterate the procedure... .

Summary: We have

$$A^{ab} = \sum_{p=1}^{N-1} C_p \Phi^{ab;i_1 \cdots i_{4p}} \prod_{q=1}^{p} R_{i_{4q-2} i_{4q-1} i_{4q-3} i_{4q}} + \Psi^{ab} .$$

Here, the C_p are constants,

$$\Phi^{ab;i_1 \cdots i_{4p}} \in MC_n(2+4p,0,1,0)$$

has property S, and

$$\Psi^{ab} \in MC_n(2,0,1,0)$$

is symmetric, thus has the form

$$\lambda|g|^{1/2}g^{ab}$$

for some constant λ.

It remains to explicate the

$$\Phi^{ab;i_1 \cdots i_{4p}}.$$

LEMMA Fix $p: 1 \leq p \leq N-1$. Denote by $S^{ab}(n,4p)$ the subspace of $MC_n(2+4p,0,1,0)$ consisting of those entities with property S -- then

$$\dim S^{ab}(n,4p) = 1.$$

Notation: Put

$$D^{ijk\ell}{}_{abcd} = \frac{1}{2}(\delta^i{}_a\delta^j{}_d + \delta^i{}_d\delta^j{}_a)(\delta^k{}_b\delta^\ell{}_c + \delta^k{}_c\delta^\ell{}_b).$$

Maintaining the assumption that $1 \leq p \leq N-1$, define

$$D^{ab;i_1 \cdots i_{4p}} \in MC_n(2+4p,0,1,0)$$

by

$$D^{ab;i_1 \cdots i_{4p}}$$

$$= |g|^{1/2}(\delta^{aj_1\cdots j_{2p}}{}_{rr_1\cdots r_{2p}}g^{br} + \delta^{bj_1\cdots j_{2p}}{}_{rr_1\cdots r_{2p}}g^{ar})$$

$$\times g^{r_1s_1}\cdots g^{r_{2p}s_{2p}}$$

$$\times D^{i_1i_2i_3i_4}{}_{j_1j_2s_1s_2}\cdots D^{i_{4p-3}i_{4p-2}i_{4p-1}i_{4p}}{}_{j_{2p-1}j_{2p}s_{2p-1}s_{2p}}.$$

Then

$$D^{ab;i_1 \cdots i_{4p}} \in S^{ab}(n,4p).$$

Moreover,

$$D^{ab;i_1 \cdots i_{4p}} \neq 0,$$

as can be seen by noting that

$$D^{ab;i_1 \cdots i_{4p}} g_{ab} g_{i_1 i_2} \cdots g_{i_{4p-1} i_{4p}}$$

$$= (-1)^p\, 2^{p+1} \delta^{k\ell_1 \cdots \ell_{2p}}_{k\ell_1 \cdots \ell_{2p}}$$

$$= (-1)^p\, 2^{p+1} \frac{n!}{(n-2p-1)!} \qquad (1 \le p \le N-1 \Rightarrow n \ge 2p+1).$$

Accordingly, $\Phi^{ab;i_1 \cdots i_{4p}}$ is a constant multiple of $D^{ab;i_1 \cdots i_{4p}}$.

Therefore

$$A^{ab} = \sum_{p=1}^{N-1} C_p D^{ab;i_1 \cdots i_{4p}} \prod_{q=1}^{p} R_{i_{4q-2} i_{4q-1} i_{4q-3} i_{4q}} + \lambda |g|^{1/2} g^{ab}$$

after possible redefinition of the C_p.

Observation:

$$D^{ab;i_1 \cdots i_{4p}}$$

$$= |g|^{1/2} (2^{-p}) (\delta^{aj_1 \cdots j_{2p}}_{rr_1 \cdots r_{2p}} g^{br} + \delta^{bj_1 \cdots j_{2p}}_{rr_1 \cdots r_{2p}} g^{ar})$$

$$\times g^{r_1 s_1} \cdots g^{r_{2p} s_{2p}}$$

$$\times (D^{i_1 i_2 i_3 i_4}{}_{j_1 j_2 s_1 s_2} - D^{i_1 i_2 i_3 i_4}{}_{j_2 j_1 s_1 s_2})$$

$$\ldots \times \left(D^{i_{4p-3}i_{4p-2}i_{4p-1}i_{4p}}{}_{j_{2p-1}j_{2p}s_{2p-1}s_{2p}} - D^{i_{4p-3}i_{4p-2}i_{4p-1}i_{4p}}{}_{j_{2p}j_{2p-1}s_{2p-1}s_{2p}}\right).$$

To exploit this, note that

$$D^{i_{4q-3}i_{4q-2}i_{4q-1}i_{4q}}{}_{j_{2q-1}j_{2q}s_{2q-1}s_{2q}} R_{i_{4q-2}i_{4q-1}i_{4q-3}i_{4q}}$$

$$= - D^{i_{4q-3}i_{4q-2}i_{4q-1}i_{4q}}{}_{j_{2q-1}j_{2q}s_{2q-1}s_{2q}} R_{i_{4q-1}i_{4q-3}i_{4q-2}i_{4q}}$$

$$= - \left(R_{j_{2q-1}j_{2q}s_{2q-1}s_{2q}} + R_{j_{2q-1}s_{2q-1}j_{2q}s_{2q}}\right)$$

$\Rightarrow$

$$\left(D^{i_{4q-3}i_{4q-2}i_{4q-1}i_{4q}}{}_{j_{2q-1}j_{2q}s_{2q-1}s_{2q}} - D^{i_{4q-3}i_{4q-2}i_{4q-1}i_{4q}}{}_{j_{2q}j_{2q-1}s_{2q-1}s_{2q}}\right)$$

$$\times R_{i_{4q-2}i_{4q-1}i_{4q-3}i_{4q}}$$

$$= - 3R_{j_{2q-1}j_{2q}s_{2q-1}s_{2q}}.$$

Now bring in

$$g^{r_{2q-1}s_{2q-1}} g^{r_{2q}s_{2q}}$$

and write

$$g^{r_{2q-1}s_{2q-1}} g^{r_{2q}s_{2q}} R_{j_{2q-1}j_{2q}s_{2q-1}s_{2q}}$$

$$= R_{j_{2q-1}j_{2q}}{}^{r_{2q-1}r_{2q}}$$

$$= R^{r_{2q-1}r_{2q}}{}_{j_{2q-1}j_{2q}} .$$

Thus, after adjusting the constants, we conclude that

$$A^{ab} = |g|^{1/2} \sum_{p=1}^{N-1} C_p(\delta^{aj_1\cdots j_{2p}}{}_{rr_1\cdots r_{2p}} g^{br} + \delta^{bj_1\cdots j_{2p}}{}_{rr_1\cdots r_{2p}} g^{ar})$$

$$\times \prod_{q=1}^{p} R^{r_{2q-1}r_{2q}}{}_{j_{2q-1}j_{2q}} + \lambda|g|^{1/2}g^{ab} .$$

But

$$\delta^{aj_1\cdots j_{2p}}{}_{rr_1\cdots r_{2p}} g^{br} R^{r_1r_2}{}_{j_1j_2} \cdots R^{r_{2p-1}r_{2p}}{}_{j_{2p-1}j_{2p}}$$

$$= \delta^{bj_1\cdots j_{2p}}{}_{rr_1\cdots r_{2p}} g^{ar} R^{r_1r_2}{}_{j_1j_2} \cdots R^{r_{2p-1}r_{2p}}{}_{j_{2p-1}j_{2p}} .$$

So, modulo obvious notational changes, the proof of the theorem is complete.

Remark: The expression

$$\sum_{p=1}^{N-1} C_p g^{ik} \delta^{j\ell_1\cdots \ell_{2p}}{}_{kk_1\cdots k_{2p}} R^{k_1k_2}{}_{\ell_1\ell_2} \cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}$$

is a polynomial of degree N-1 in the $R^{ab}{}_{cd}$. Therefore, if A^{ij} is linear in the second derivatives of the g_{ab}, then $C_p = 0$ for $p > 1$, hence the A^{ij} must have the form

$$C|g|^{1/2}[R^{ij} - \frac{1}{2} Sg^{ij}] + \lambda|g|^{1/2}g^{ij}.$$

Rappel: Let

$$L \in MC_n(0,0,1,2).$$

Then

$$E^{ij}(L) = -\frac{1}{2}g^{ij}L + \frac{2}{3}\Lambda^{k\ell,ia}R^{j}{}_{k\ell a} - \Lambda^{ij,k\ell}{}_{;k\ell}\,.$$

<u>THEOREM</u> Let

$$L = -|g|^{1/2}\sum_{p=1}^{N-1} 2C_p\delta^{\ell_1\cdots\ell_{2p}}{}_{k_1\cdots k_{2p}}R^{k_1k_2}{}_{\ell_1\ell_2}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}} - 2\lambda|g|^{1/2}.$$

Then

$$E^{ij}(L) = A^{ij}.$$

To begin with

$$\Lambda^{ab,cd}$$

$$= -|g|^{1/2}\sum_{p=1}^{N-1} 2pC_p\delta^{\ell_1\cdots\ell_{2p}}{}_{k_1\cdots k_{2p}}R^{k_3k_4}{}_{\ell_3\ell_4}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}$$

$$\times\, g^{k_1s}g^{k_2r}D^{abcd}{}_{\ell_1\ell_2sr}\,,$$

from which

$$\Lambda^{ab,cd}{}_{;c}$$

$$= -|g|^{1/2}\sum_{p=1}^{N-1} 2p(p-1)C_p\delta^{\ell_1\cdots\ell_{2p}}{}_{k_1\cdots k_{2p}}\nabla_c R^{k_3k_4}{}_{\ell_3\ell_4}$$

$$\times\, R^{k_5k_6}{}_{\ell_5\ell_6}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}g^{k_1s}g^{k_2r}D^{abcd}{}_{\ell_1\ell_2sr}\,.$$

Standard manipulations involving the Bianchi identities then imply that

$$\Lambda^{ab,cd}{}_{;c} = 0.$$

Matters thus reduce to consideration of

$$-\frac{1}{2}g^{ij}L+\frac{2}{3}\Lambda^{k\ell,ia}R^{j}{}_{k\ell a}$$

or still, to consideration of

$$-\frac{1}{2}\delta^{i}{}_{j}L+\frac{2}{3}\Lambda^{k\ell,ia}R_{jk\ell a},$$

the claim being that this expression is equal to $A^{i}{}_{j}$, i.e., to

$$|g|^{1/2}\sum_{p=1}^{N-1}C_p\delta^{i\ell_1\cdots\ell_{2p}}{}_{jk_1\cdots k_{2p}}R^{k_1k_2}{}_{\ell_1\ell_2}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}+\lambda|g|^{1/2}\delta^{i}{}_{j}.$$

But

$$\frac{2}{3}\Lambda^{k\ell,ia}R_{jk\ell a}$$

$$=-\frac{2}{3}|g|^{1/2}\sum_{p=1}^{N-1}2pC_p(\tfrac{3}{2})\delta^{i\ell_2\cdots\ell_{2p}}{}_{k_1\cdots k_{2p}}R^{k_1k_2}{}_{j\ell_2}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}.$$

Therefore

$$-\frac{1}{2}\delta^{i}{}_{j}L+\frac{2}{3}\Lambda^{k\ell,ia}R_{jk\ell a}$$

$$=|g|^{1/2}\sum_{p=1}^{N-1}C_p(\delta^{i}{}_{j}\delta^{\ell_1\cdots\ell_{2p}}{}_{k_1\cdots k_{2p}}R^{k_1k_2}{}_{\ell_1\ell_2}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}$$

$$-2p\delta^{i\ell_2\cdots\ell_{2p}}{}_{k_1\cdots k_{2p}}R^{k_1k_2}{}_{j\ell_2}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}})+\lambda|g|^{1/2}\delta^{i}{}_{j}$$

$$=A^{i}{}_{j},$$

as claimed.

Remark: Let $A \in MC_3(2,0,1,3)$ be symmetric and divergence free -- then it can be shown that

$$A^{ij} = C|g|^{1/2}G^{ij} + cC^{ij} + \lambda|g|^{1/2}g^{ij},$$

where C, c, and λ are constants. But, as we know, there does not exist a lagrangian

$$L \in MC_3(0,0,1,m)$$

such that $E^{ij}(L) = C^{ij}$.

Given $p \geq 1$, put

$$L_p = -\ |g|^{1/2} 2\delta^{\ell_1 \cdots \ell_{2p}}{}_{k_1 \cdots k_{2p}} R^{k_1 k_2}{}_{\ell_1 \ell_2} \cdots R^{k_{2p-1} k_{2p}}{}_{\ell_{2p-1} \ell_{2p}}.$$

Then

$$E^{ij}(L_p) = |g|^{1/2} g^{ik} \delta^{j \ell_1 \cdots \ell_{2p}}{}_{k k_1 \cdots k_{2p}} R^{k_1 k_2}{}_{\ell_1 \ell_2} \cdots R^{k_{2p-1} k_{2p}}{}_{\ell_{2p-1} \ell_{2p}}.$$

<u>Reality Check</u> Take $p = 1$ -- then

$$L_1 = -\ |g|^{1/2} 2\delta^{\ell_1 \ell_2}{}_{k_1 k_2} R^{k_1 k_2}{}_{\ell_1 \ell_2}$$

$$= -\ |g|^{1/2} 4S$$

and

$$E^{ij}(-\ |g|^{1/2} 4S)$$

$$= |g|^{1/2}[g^{ik} \delta^{j \ell_1 \ell_2}{}_{k k_1 k_2} R^{k_1 k_2}{}_{\ell_1 \ell_2}]$$

$$= |g|^{1/2}(-\ 4G^{ij}).$$

I.e.:

$$E^{ij}(|g|^{1/2}S) = |g|^{1/2}G^{ij}.$$

Example: Take p = 2 -- then

$$L_2 = - |g|^{1/2} 8[S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}]$$

and

$$E^{ij}(L_2) = |g|^{1/2} g^{ik} \delta^{j\ell_1\ell_2\ell_3\ell_4}_{kk_1k_2k_3k_4} R^{k_1k_2}{}_{\ell_1\ell_2} R^{k_3k_4}{}_{\ell_3\ell_4}.$$

Now take n = 4 -- then

$$\delta^{j\ell_1\ell_2\ell_3\ell_4}_{kk_1k_2k_3k_4} = 0.$$

Therefore in this case

$$E^{ij}(|g|^{1/2}[S^2 - 4R^{ab}R_{ab} + R^{abcd}{}_{abcd}]) = 0.$$

<u>FACT</u> Suppose that n = 2p -- then locally, L_p is an ordinary divergence.

Foreshadowing considerations to follow, it will be convenient to redefine L_p as

$$|g|^{1/2} \frac{1}{2^p} (\delta^{\ell_1\cdots\ell_{2p}}_{k_1\cdots k_{2p}} R^{k_1k_2}{}_{\ell_1\ell_2} \cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}),$$

so that

$$E^i{}_j(L_p)$$
$$= - |g|^{1/2} \frac{1}{2^{p+1}} (\delta^{i\ell_1\cdots\ell_{2p}}_{jk_1\cdots k_{2p}} R^{k_1k_2}{}_{\ell_1\ell_2} \cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}).$$

Let

$$R^i{}_j(p) = \delta^{i\ell_2\cdots\ell_{2p}}_{k_1\cdots k_{2p}} R^{k_1k_2}{}_{j\ell_2} \cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}.$$

Then

$$g^{ij}R_{ij}(p) = S(p),$$

where

$$S(p) = \delta^{\ell_1 \cdots \ell_{2p}}_{k_1 \cdots k_{2p}} R^{k_1 k_2}{}_{\ell_1 \ell_2} \cdots R^{k_{2p-1} k_{2p}}{}_{\ell_{2p-1} \ell_{2p}}.$$

In addition,

$$E^i{}_j(L_p) = - |g|^{1/2} \frac{1}{2^{p+1}} (\delta^i{}_j S(p) - 2pR^i{}_j(p)).$$

But

$$\nabla_j E^{ij}(L_p) = 0.$$

Therefore

$$- |g|^{1/2} \frac{1}{2^{p+1}} \nabla_j (g^{ij} S(p) - 2pR^{ij}(p)) = 0$$

$$\Rightarrow$$

$$\nabla^i S(p) = 2p\nabla_j R^{ij}(p)$$

or still,

$$(dS(p))_i = \nabla_i S(p) = 2p\nabla^j R_{ij}(p).$$

[Note: We have

$$\begin{aligned} \nabla_i S(p) &= g_{ia}\nabla^a S(p) \\ &= 2pg_{ia}\nabla_j R^{aj}(p) \\ &= 2p\nabla_j g_{ia} R^{aj}(p) \\ &= 2p\nabla_j R_i{}^j(p) \end{aligned}$$

$$= 2p\nabla_j g^{jb} R_{ib}(p)$$

$$= 2p g^{bj} \nabla_j R_{ib}(p)$$

$$= 2p\nabla^b R_{ib}(p)$$

$$= 2p\nabla^j R_{ij}(p).]$$

Remark: The higher order version of Ric is Ric(p):

$$\text{Ric}(p)_{ij} = R_{ij}(p).$$

Ric(p) is symmetric and

$$\text{tr}_g \text{ Ric}(p) = S(p),$$

the higher order version of the scalar curvature.

Reality Check Take $p = 1$ -- then $L_1 = |g|^{1/2} S$. Moreover

$$R^i{}_j(1) = \delta^{i\ell_2}{}_{k_1k_2} R^{k_1k_2}{}_{j\ell_2} = 2R^i{}_j$$

and

$$S(1) = \delta^{\ell_1\ell_2}{}_{k_1k_2} R^{k_1k_2}{}_{\ell_1\ell_2} = 2S.$$

Therefore

$$E^i{}_j(L_1) = - |g|^{1/2} \frac{1}{4} (\delta^i{}_j S(1) - 2R^i{}_j(1))$$

$$= - |g|^{1/2} \frac{1}{4} (\delta^i{}_j 2S - 4R^i{}_j)$$

$$= |g|^{1/2} (R^i{}_j - \frac{1}{2} \delta^i{}_j S)$$

$$= |g|^{1/2} G^i{}_j.$$

[Note: The relation

$$(dS(1))_i = 2\nabla^j R_{ij}(1)$$

reduces to

$$(dS)_i = 2\nabla^j R_{ij},$$

in agreement with the earlier theory.]

Section 27: Globalization Let M be a connected C^∞ manifold of dimension n, which we shall take to be orientable with orientation μ. Fix a semiriemannian structure g on M.

Rappel: Given $x_0 \in M$, there exists a connected open set $U \subset M$ containing x_0 and vector fields $E_1, \ldots, E_n$ on U such that $\forall$ $x \in U$,

$$\begin{cases} g_x(E_i|_x, E_j|_x) = \eta_{ij} \\ \{E_1|_x, \ldots, E_n|_x\} \in \mu_x. \end{cases}$$

Because of this, there is no real loss of generality in assuming outright that the orthonormal frame bundle LM(g) is trivial.

[Note: As a matter of convenience, in what follows we shall work with oriented orthonormal frames but all the results in this section can be formulated in terms of an arbitrary oriented frame.]

So fix an oriented orthonormal frame $E = \{E_1, \ldots, E_n\}$. Denoting by $\omega = \{\omega^1, \ldots, \omega^n\}$ its associated coframe, put

$$\theta_{i_1 \cdots i_p} = \frac{1}{(n-p)!} \varepsilon_{i_1 \cdots i_p j_{p+1} \cdots j_n} \omega^{j_{p+1}} \wedge \ldots \wedge \omega^{j_n} .$$

Then

$$\begin{aligned} \theta^{i_1 \cdots i_p} &= \varepsilon_{i_1} \cdots \varepsilon_{i_p} \theta_{i_1 \cdots i_p} \quad \text{(no sum)} \\ &= *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}). \end{aligned}$$

Observation: View the $\theta_{i_1 \cdots i_p}$ as the components of an element θ_p of $\Lambda^{n-p}(M; T_p^0(M))$. Let ∇ be the metric connection -- then

$$d^{\nabla}\theta_p = 0.$$

Example: Suppose that p = 2 — then

$$d\theta_{ij} - \omega^k{}_i\wedge\theta_{kj} - \omega^k{}_j\wedge\theta_{ik} = 0$$

or, multiplying through by ε_j,

$$d\theta_i{}^j - \omega^k{}_i\wedge\theta_k{}^j - \varepsilon_j\omega^k{}_j\wedge\theta_{ik} = 0.$$

But

$$- \varepsilon_j\omega^k{}_j\wedge\theta_{ik}$$

$$= - \varepsilon_j\omega^k{}_j\wedge\varepsilon_k\varepsilon_k\theta_{ik}$$

$$= - \varepsilon_j\varepsilon_k\omega^k{}_j\wedge\varepsilon_k\theta_{ik}$$

$$= \omega^j{}_k\wedge\theta_i{}^k.$$

Therefore

$$d\theta_i{}^j - \omega^k{}_i\wedge\theta_k{}^j + \omega^j{}_k\wedge\theta_i{}^k = 0.$$

<u>LEMMA</u> We have

$$\Omega^{ij}\wedge\theta_{ij} = \text{Svol}_g \quad (= *S).$$

[In fact,

$$\Omega^{ij}\wedge\theta_{ij} = \frac{1}{2} R^{ij}{}_{k\ell}(\omega^k\wedge\omega^\ell)\wedge\theta_{ij}$$

$$= \frac{1}{2} \varepsilon_i\varepsilon_j R_{ijk\ell}(\omega^k\wedge\omega^\ell)\wedge\theta_{ij}$$

$$= \frac{1}{2} R_{ijk\ell}(\omega^k \wedge \omega^\ell) \wedge *(\omega^i \wedge \omega^j)$$

$$= \frac{1}{2} R_{ijk\ell} g(\omega^k \wedge \omega^\ell, \omega^i \wedge \omega^j) \mathrm{vol}_g$$

$$= \frac{1}{2} R_{ijk\ell} g(\omega^i \wedge \omega^j, \omega^k \wedge \omega^\ell) \mathrm{vol}_g$$

$$= \frac{1}{2} R_{ijk\ell} \det \begin{bmatrix} g(\omega^i,\omega^k) & g(\omega^i,\omega^\ell) \\ g(\omega^j,\omega^k) & g(\omega^j,\omega^\ell) \end{bmatrix} \mathrm{vol}_g$$

$$= \frac{1}{2} R_{ijk\ell}(g^{ik}g^{j\ell} - g^{i\ell}g^{jk}) \mathrm{vol}_g.$$

On the other hand,

$$S = g^{j\ell} R^k{}_{jk\ell}$$

$$= g^{j\ell} g^{ki} R_{ijk\ell}$$

$$= g^{ik} g^{j\ell} R_{ijk\ell}$$

and

$$S = g^{i\ell} R^k{}_{ik\ell}$$

$$= g^{i\ell} g^{kj} R_{jik\ell}$$

$$= - g^{i\ell} g^{jk} R_{ijk\ell}.]$$

<u>Splitting Principle</u> Start by writing

$$*S = \Omega^{ij}\wedge\theta_{ij}$$

$$= \varepsilon_j\Omega^{ij}\wedge\varepsilon_j\theta_{ij}$$

$$= \Omega^i{}_j\wedge\theta_i{}^j$$

$$= (d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j)\wedge\theta_i{}^j$$

$$= d(\omega^i{}_j\wedge\theta_i{}^j) + \omega^i{}_j\wedge d\theta_i{}^j + \omega^i{}_k\wedge\omega^k{}_j\wedge\theta_i{}^j.$$

From the above

$$d\theta_i{}^j - \omega^k{}_i\wedge\theta_k{}^j + \omega^j{}_k\wedge\theta_i{}^k = 0$$

$$\Rightarrow$$

$$\omega^i{}_j\wedge d\theta_i{}^j = \omega^i{}_j\wedge\omega^k{}_i\wedge\theta_k{}^j - \omega^i{}_j\wedge\omega^j{}_k\wedge\theta_i{}^k$$

$$= \omega^i{}_j\wedge\omega^k{}_i\wedge\theta_k{}^j - \omega^i{}_k\wedge\omega^k{}_j\wedge\theta_i{}^j.$$

Therefore

$$*S = \omega^i{}_j\wedge\omega^k{}_i\wedge\theta_k{}^j + d(\omega^i{}_j\wedge\theta_i{}^j).$$

[Note: This is the analog of the decomposition

$$|g|^{1/2}S = A + B^i{}_{,i} ,$$

where the field functions A, B^i are given by

$$\left[\begin{array}{l} A = |g|^{1/2} g^{ij} (\Gamma^k{}_{i\ell}\Gamma^\ell{}_{jk} - \Gamma^k{}_{ij}\Gamma^\ell{}_{k\ell}) \\ \\ B^i = |g|^{1/2} (g^{k\ell}\Gamma^i{}_{k\ell} - g^{ik}\Gamma^\ell{}_{k\ell}).] \end{array}\right.$$

<u>LEMMA</u> We have

$$\Omega^{ij}\wedge\theta_{ij} = -\,2d(\omega_i\wedge{*}d\omega^i)$$

$$-\,(d\omega^i\wedge\omega^j)\wedge{*}(d\omega_j\wedge\omega_i) + \frac{1}{2}\,(d\omega^i\wedge\omega_i)\wedge{*}(d\omega^j\wedge\omega_j).$$

[First

$$\Omega^{ij}\wedge\theta_{ij} = \Omega_{ij}\wedge\theta^{ij}$$

$$= (d\omega_{ij} + \omega_{ik}\wedge\omega^k{}_j)\wedge{*}(\omega^i\wedge\omega^j)$$

$$= (d\omega_{ij} + \omega_i{}^k\wedge\omega_{kj})\wedge{*}(\omega^i\wedge\omega^j).$$

But

$$d(\omega_{ij}\wedge{*}(\omega^i\wedge\omega^j))$$

$$= d\omega_{ij}\wedge{*}(\omega^i\wedge\omega^j) - \omega_{ij}\wedge d{*}(\omega^i\wedge\omega^j)$$

$$= d\omega_{ij}\wedge{*}(\omega^i\wedge\omega^j)$$

$$-\,\omega_{ij}\wedge[-\,\omega^i{}_a\wedge{*}(\omega^a\wedge\omega^j) - \omega^j{}_a\wedge{*}(\omega^i\wedge\omega^a)]$$

$$= d\omega_{ij}\wedge{*}(\omega^i\wedge\omega^j) + 2\omega_{ij}\wedge\omega^i{}_a\wedge{*}(\omega^a\wedge\omega^j)$$

$$= d\omega_{ij}\wedge *(\omega^i\wedge\omega^j) - 2\omega_{ij}\wedge\omega_a{}^i\wedge *(\omega^a\wedge\omega^j)$$

$$= d\omega_{ij}\wedge *(\omega^i\wedge\omega^j) + 2\omega_a{}^i\wedge\omega_{ij}\wedge *(\omega^a\wedge\omega^j)$$

$$= d\omega_{ij}\wedge *(\omega^i\wedge\omega^j) + 2\omega_i{}^a\wedge\omega_{aj}\wedge *(\omega^i\wedge\omega^j)$$

$$= d\omega_{ij}\wedge *(\omega^i\wedge\omega^j) + 2\omega_i{}^k\wedge\omega_{kj}\wedge *(\omega^i\wedge\omega^j).$$

Then

$$\Omega^{ij}\wedge\theta_{ij} = d(\omega_{ij}\wedge *(\omega^i\wedge\omega^j)) - \omega_i{}^k\wedge\omega_{kj}\wedge *(\omega^i\wedge\omega^j).$$

1. Consider

$$d(\omega_{ij}\wedge *(\omega^i\wedge\omega^j)).$$

Thus, as the metric connection is torsion free,

$$d\omega^i = -\omega^i{}_j\wedge\omega^j$$

$$\Rightarrow$$

$$*d\omega^i = -*(\omega^i{}_j\wedge\omega^j)$$

$$= *(\omega^j\wedge\omega^i{}_j)$$

$$= \varepsilon_i *(\omega^j\wedge\omega_{ij})$$

$$= \varepsilon_i \iota_{\omega_{ij}} *\omega^j$$

$$\Rightarrow$$

$$\omega^i\wedge *d\omega^i = \varepsilon_i\omega^i\wedge(\iota_{\omega_{ij}} *\omega^j).$$

Next

$$\omega^i\wedge *\omega^j = g(\omega^i,\omega^j)\mathrm{vol}_g,$$

where

$$g(\omega^i,\omega^j) = \begin{cases} \varepsilon_i & i = j \\ 0 & i \neq j \end{cases} .$$

Since $\iota_{\omega_{ii}} = 0$, it follows that

$$0 = \iota_{\omega_{ij}}(\omega^i \wedge *\omega^j) = (\iota_{\omega_{ij}}\omega^i)\wedge *\omega^j - \omega^i \wedge (\iota_{\omega_{ij}} *\omega^j).$$

Therefore

$$\omega^i \wedge *d\omega^i = \varepsilon_i(\iota_{\omega_{ij}}\omega^i)\wedge *\omega^j$$

$$= \varepsilon_i g(\omega_{ij},\omega^i) *\omega^j.$$

But

$$*(\omega^i \wedge \omega^j)\wedge\omega_{ij}$$

$$= (-1)^{\iota}(-1)^{n-1}*(*(*(\omega^i \wedge \omega^j)\wedge\omega_{ij}))$$

$$= (-1)^{\iota}(-1)^{n-1}*(\iota_{\omega_{ij}} **(\omega^i \wedge \omega^j))$$

$$= (-1)^{n-1}*(\iota_{\omega_{ij}}(\omega^i \wedge \omega^j))$$

$$= (-1)^{n-1}*[(\iota_{\omega_{ij}}\omega^i)\wedge\omega^j - \omega^i \wedge (\iota_{\omega_{ij}}\omega^j)]$$

$$= (-1)^{n-1}*[g(\omega_{ij},\omega^i)\omega^j - g(\omega_{ij},\omega^j)\omega^i]$$

$$= (-1)^{n-1}*[g(\omega_{ij},\omega^i)\omega^j - g(\omega_{ji},\omega^i)\omega^j]$$

$$= 2(-1)^{n-1} g(\omega_{ij}, \omega^i) * \omega^j$$

$$= 2(-1)^{n-1} \varepsilon_i (\omega^i \wedge * d\omega^i)$$

$$= 2(-1)^{n-1} (\omega_i \wedge * d\omega^i)$$

$\Rightarrow$

$$d(\omega_{ij} \wedge *(\omega^i \wedge \omega^j))$$

$$= (-1)^{n-2} d(*(\omega^i \wedge \omega^j) \wedge \omega_{ij})$$

$$= 2(-1)^{n-2}(-1)^{n-1} d(\omega_i \wedge * d\omega^i)$$

$$= - 2d(\omega_i \wedge * d\omega^i).$$

2. Consider

$$- \omega_i{}^k \wedge \omega_{kj} \wedge *(\omega^i \wedge \omega^j)$$

$$= \omega^k{}_i \wedge \omega_{kj} \wedge *(\omega^i \wedge \omega^j)$$

$$= \varepsilon_k \omega_{ki} \wedge \omega_{kj} \wedge *(\omega^i \wedge \omega^j)$$

or still,

$$\varepsilon_k [g(\omega_{ki}, \omega^i) g(\omega_{kj}, \omega^j) - g(\omega_{ki}, \omega^j) g(\omega_{kj}, \omega^i)] \mathrm{vol}_g.$$

Rappel: Let $a,b = 1,\ldots,n$ -- then

$$\omega_{ab} = \varepsilon_a \iota_{E_b} d\omega^a - \varepsilon_b \iota_{E_a} d\omega^b - \frac{1}{2} \sum_c \varepsilon_c \iota_{E_b} \iota_{E_a} (d\omega^c \wedge \omega^c).$$

$$\longrightarrow \quad g(\omega_{ki},\omega^i) = \iota_{\omega^i}\omega_{ki}$$

$$= \iota_{\omega^i}(\varepsilon_k \iota_{E_i} d\omega^k - \varepsilon_i \iota_{E_k} d\omega^i - \frac{1}{2} \sum_c \varepsilon_c \iota_{E_i} \iota_{E_k}(d\omega^c \wedge \omega^c))$$

$$= \iota_{\omega^i}(\varepsilon_k \iota_{g^\flat E_i} d\omega^k - \varepsilon_i \iota_{g^\flat E_k} d\omega^i - \frac{1}{2} \sum_c \varepsilon_c \iota_{g^\flat E_i} \iota_{g^\flat E_k}(d\omega^c \wedge \omega^c))$$

$$= \iota_{\omega^i}(\varepsilon_k \iota_{\varepsilon_i \omega^i} d\omega^k - \varepsilon_i \iota_{\varepsilon_k \omega^k} d\omega^i - \frac{1}{2} \sum_c \varepsilon_c \iota_{\varepsilon_i \omega^i} \iota_{\varepsilon_k \omega^k}(d\omega^c \wedge \omega^c))$$

$$= - \varepsilon_i \varepsilon_k \iota_{\omega^i} \iota_{\omega^k} d\omega^i$$

$$= - \varepsilon_i \varepsilon_k \iota_{\omega^k \wedge \omega^i} d\omega^i$$

$$= \varepsilon_i \varepsilon_k \iota_{\omega^i \wedge \omega^k} d\omega^i$$

$$= \varepsilon_i \varepsilon_k g(\omega^i \wedge \omega^k, d\omega^i)$$

$$= \varepsilon_i \varepsilon_k g(\omega^k, \iota_{\omega^i} d\omega^i).$$

Analogously

$$g(\omega_{kj},\omega^j) = \varepsilon_j \varepsilon_k g(\omega^k, \iota_{\omega^j} d\omega^j).$$

Therefore

$$\varepsilon_k g(\omega_{ki},\omega^i) g(\omega_{kj},\omega^j)$$

$$= \varepsilon_k \varepsilon_i \varepsilon_j g(\omega^k, \iota_{\omega^i} d\omega^i) g(\omega^k, \iota_{\omega^j} d\omega^j).$$

Write

$$\left[\begin{array}{l} \iota_{\omega^i} d\omega^i = g(\omega^k, \iota_{\omega^i} d\omega^i)\varepsilon_k \omega^k \\ \\ \iota_{\omega^j} d\omega^j = g(\omega^\ell, \iota_{\omega^j} d\omega^j)\varepsilon_\ell \omega^\ell. \end{array}\right.$$

Then

$$g(\iota_{\omega^i} d\omega^i, \iota_{\omega^j} d\omega^j)$$

$$= g(g(\omega^k, \iota_{\omega^i} d\omega^i)\varepsilon_k \omega^k, g(\omega^\ell, \iota_{\omega^j} d\omega^j)\varepsilon_\ell \omega^\ell)$$

$$= g(\omega^k, \iota_{\omega^i} d\omega^i) g(\omega^\ell, \iota_{\omega^j} d\omega^j)\varepsilon_k \varepsilon_\ell g(\omega^k, \omega^\ell)$$

$$= \varepsilon_k g(\omega^k, \iota_{\omega^i} d\omega^i) g(\omega^k, \iota_{\omega^j} d\omega^j)$$

$\Rightarrow$

$$\varepsilon_k g(\omega_{ki}, \omega^i) g(\omega_{kj}, \omega^j)$$

$$= \varepsilon_i \varepsilon_j g(\iota_{\omega^i} d\omega^i, \iota_{\omega^j} d\omega^j)$$

$$= \varepsilon_i \varepsilon_j g(d\omega^i, \omega^i \wedge \iota_{\omega^j} d\omega^j)$$

$$= \varepsilon_i \varepsilon_j g(d\omega^i, \iota_{\omega^j} \omega^i \wedge d\omega^j - \iota_{\omega^j}(\omega^i \wedge d\omega^j))$$

$$= \varepsilon_i \varepsilon_j g(d\omega^i, \iota_{\omega^j} \omega^i \wedge d\omega^j) - \varepsilon_i \varepsilon_j g(d\omega^i, \iota_{\omega^j}(\omega^i \wedge d\omega^j))$$

$$= \varepsilon_i g(d\omega^i, d\omega^i) - \varepsilon_i \varepsilon_j g(\omega^j \wedge d\omega^i, \omega^i \wedge d\omega^j)$$

$$= \varepsilon_i g(d\omega^i, d\omega^i) - \varepsilon_i \varepsilon_j g(\omega^i \wedge d\omega^j, \omega^j \wedge d\omega^i).$$

$$\longrightarrow \quad g(\omega_{ki}, \omega^j)$$

$$= \iota_{\omega^j}(\varepsilon_k \iota_{\varepsilon_i \omega^i} d\omega^k - \varepsilon_i \iota_{\varepsilon_k \omega^k} d\omega^i - \frac{1}{2} \sum_c \varepsilon_c \iota_{\varepsilon_i \omega^i} \iota_{\varepsilon_k \omega^k}(d\omega^c \wedge \omega^c))$$

$$= \varepsilon_i \varepsilon_k (g(d\omega^k, \omega^i \wedge \omega^j) - g(d\omega^i, \omega^k \wedge \omega^j) - \frac{1}{2} \sum_c \varepsilon_c g(d\omega^c \wedge \omega^c, \omega^k \wedge \omega^i \wedge \omega^j)).$$

The term involving $\sum_c$ can be simplified:

$$-\frac{1}{2} \sum_c \varepsilon_c g(d\omega^c \wedge \omega^c, \omega^k \wedge \omega^i \wedge \omega^j)$$

$$= -\frac{1}{2} \sum_c \varepsilon_c g(d\omega^c, \iota_{\omega^c}(\omega^k \wedge \omega^i \wedge \omega^j))$$

$$= -\frac{1}{2} \sum_c \varepsilon_c g(d\omega^c, \iota_{\omega^c}\omega^k \wedge \omega^i \wedge \omega^j - \omega^k \wedge \iota_{\omega^c}\omega^i \wedge \omega^j + \omega^k \wedge \omega^i \wedge \iota_{\omega^c}\omega^j)$$

$$= -\frac{1}{2} (g(d\omega^k, \omega^i \wedge \omega^j) - g(d\omega^i, \omega^k \wedge \omega^j) + g(d\omega^j, \omega^k \wedge \omega^i)).$$

Therefore

$$g(\omega_{ki}, \omega^j)$$

$$= \varepsilon_i \varepsilon_k \frac{1}{2} (g(d\omega^k, \omega^i \wedge \omega^j) - g(d\omega^i, \omega^k \wedge \omega^j) - g(d\omega^j, \omega^k \wedge \omega^i)).$$

Analogously

$$g(\omega_{kj}, \omega^i)$$

$$= \varepsilon_j\varepsilon_k \frac{1}{2} (g(d\omega^k,\omega^j\wedge\omega^i) - g(d\omega^j,\omega^k\wedge\omega^i) - g(d\omega^i,\omega^k\wedge\omega^j)).$$

The product

$$\varepsilon_k g(\omega_{ki},\omega^j) g(\omega_{kj},\omega^i)$$

thus equals $\varepsilon_k\varepsilon_i\varepsilon_j$ times #1 + #2 + ⋯ + #9, where

#1: $\frac{1}{4} g(d\omega^k,\omega^i\wedge\omega^j) g(d\omega^k,\omega^j\wedge\omega^i)$.

#2: $-\frac{1}{4} g(d\omega^k,\omega^i\wedge\omega^j) g(d\omega^j,\omega^k\wedge\omega^i)$.

#3: $-\frac{1}{4} g(d\omega^k,\omega^i\wedge\omega^j) g(d\omega^i,\omega^k\wedge\omega^j)$.

#4: $-\frac{1}{4} g(d\omega^i,\omega^k\wedge\omega^j) g(d\omega^k,\omega^j\wedge\omega^i)$.

#5: $\frac{1}{4} g(d\omega^i,\omega^k\wedge\omega^j) g(d\omega^j,\omega^k\wedge\omega^i)$.

#6: $\frac{1}{4} g(d\omega^i,\omega^k\wedge\omega^j) g(d\omega^i,\omega^k\wedge\omega^j)$.

#7: $-\frac{1}{4} g(d\omega^j,\omega^k\wedge\omega^i) g(d\omega^k,\omega^j\wedge\omega^i)$.

#8: $\frac{1}{4} g(d\omega^j,\omega^k\wedge\omega^i) g(d\omega^j,\omega^k\wedge\omega^i)$.

#9: $\frac{1}{4} g(d\omega^j,\omega^k\wedge\omega^i) g(d\omega^i,\omega^k\wedge\omega^j)$.

Six of the terms cancel out:

$$\left[\begin{array}{l} \varepsilon_k\varepsilon_i\varepsilon_j \times (\#1 + \#8) = 0 \\ \varepsilon_k\varepsilon_i\varepsilon_j \times (\#2 + \#7) = 0 \\ \varepsilon_k\varepsilon_i\varepsilon_j \times (\#3 + \#9) = 0. \end{array}\right.$$

E.g.: Take #8 and write

$$\varepsilon_k\varepsilon_i\varepsilon_j \frac{1}{4} g(d\omega^j,\omega^k\wedge\omega^i)g(d\omega^j,\omega^k\wedge\omega^i)$$

$$= \varepsilon_j\varepsilon_i\varepsilon_k \frac{1}{4} g(d\omega^k,\omega^j\wedge\omega^i)g(d\omega^k,\omega^j\wedge\omega^i)$$

$$= - \varepsilon_j\varepsilon_i\varepsilon_k \frac{1}{4} g(d\omega^k,\omega^i\wedge\omega^j)g(d\omega^k,\omega^j\wedge\omega^i),$$

which is $- \varepsilon_k\varepsilon_i\varepsilon_j \times$ (#1). Observe too that

$$\varepsilon_k\varepsilon_i\varepsilon_j \times (\#4) = \varepsilon_k\varepsilon_i\varepsilon_j \times (\#5).$$

It remains to discuss

$$\varepsilon_k\varepsilon_i\varepsilon_j \times (\#4 + \#5 + \#6).$$

To this end, note that

$$- \frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i\wedge\omega^i,d\omega^j\wedge\omega^j)$$

$$= - \frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i,\iota_{\omega^i}(d\omega^j\wedge\omega^j))$$

$$= - \frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i,\iota_{\omega^i}d\omega^j\wedge\omega^j + (\iota_{\omega^i}\omega^j)d\omega^j)$$

$$= - \frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i,\iota_{\omega^i}d\omega^j\wedge\omega^j) - \frac{1}{2} \varepsilon_i g(d\omega^i,d\omega^i)$$

$$= - \frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i,g(\omega^k,\iota_{\omega^i}d\omega^j)\varepsilon_k\omega^k\wedge\omega^j)$$

$$- \frac{1}{2} \varepsilon_i g(d\omega^i,d\omega^i)$$

$$= -\frac{1}{2}\,\varepsilon_k\varepsilon_i\varepsilon_j g(\omega^k, \iota_{\omega^i} d\omega^j)\, g(d\omega^i, \omega^k \wedge \omega^j)$$

$$- \frac{1}{2}\,\varepsilon_i g(d\omega^i, d\omega^i)$$

$$= -\frac{1}{2}\,\varepsilon_k\varepsilon_i\varepsilon_j g(\omega^i \wedge \omega^k, d\omega^j)\, g(d\omega^i, \omega^k \wedge \omega^j)$$

$$- \frac{1}{2}\,\varepsilon_i g(d\omega^i, d\omega^i)$$

$$= \frac{1}{2}\,\varepsilon_k\varepsilon_i\varepsilon_j g(\omega^k \wedge \omega^i, d\omega^j)\, g(d\omega^i, \omega^k \wedge \omega^j)$$

$$- \frac{1}{2}\,\varepsilon_i g(d\omega^i, d\omega^i)$$

$$= \varepsilon_k\varepsilon_i\varepsilon_j \times (\#4 + \#5) - \frac{1}{2}\,\varepsilon_i g(d\omega^i, d\omega^i).$$

Therefore

$$\varepsilon_k g(\omega_{ki}, \omega^j)\, g(\omega_{kj}, \omega^i)$$

$$= \varepsilon_k\varepsilon_i\varepsilon_j \times (\#4 + \#5) + \varepsilon_k\varepsilon_i\varepsilon_j \times (\#6)$$

$$= -\frac{1}{2}\,\varepsilon_i\varepsilon_j g(d\omega^i \wedge \omega^i, d\omega^j \wedge \omega^j) + \frac{1}{2}\,\varepsilon_i g(d\omega^i, d\omega^i)$$

$$+ \varepsilon_k\varepsilon_i\varepsilon_j \times (\#6).$$

The last step is to study #6. In terms of the objects of anholonomity, there is an expansion

$$d\omega^i = \frac{1}{2}\, C^i{}_{ab}\,\omega^a \wedge \omega^b \quad (C^i{}_{ab} = -\, C^i{}_{ba}).$$

So

$$g(d\omega^i, \omega^k \wedge \omega^j) = \frac{1}{2}\, C^i{}_{ab}\, g(\omega^a \wedge \omega^b, \omega^k \wedge \omega^j)$$

$$= \frac{1}{2} C^i{}_{ab}(g(\omega^a,\omega^k)g(\omega^b,\omega^j) - g(\omega^a,\omega^j)g(\omega^b,\omega^k))$$

$$= \frac{1}{2} C^i{}_{kj}\varepsilon_k\varepsilon_j - \frac{1}{2} C^i{}_{jk}\varepsilon_j\varepsilon_k$$

$$= \varepsilon_k\varepsilon_j C^i{}_{kj}.$$

On the other hand,

$$g(d\omega^i,d\omega^i) = \frac{1}{2} \varepsilon_k\varepsilon_j (C^i{}_{kj})^2.$$

Combining these facts then gives

$$\varepsilon_k\varepsilon_i\varepsilon_j \times (\#6) = \frac{1}{4} \varepsilon_k\varepsilon_i\varepsilon_j (C^i{}_{kj})^2$$

$$= (\frac{1}{2} \varepsilon_i)(\frac{1}{2} \varepsilon_k\varepsilon_j (C^i{}_{kj})^2)$$

$$= \frac{1}{2} \varepsilon_i g(d\omega^i,d\omega^i),$$

from which we conclude that

$$\varepsilon_k g(\omega_{ki},\omega^j)g(\omega_{kj},\omega^i)$$

$$= -\frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i\wedge\omega^i,d\omega^j\wedge\omega^j) + \varepsilon_i g(d\omega^i,d\omega^i).$$

In summary:

$$\varepsilon_k[g(\omega_{ki},\omega^i)g(\omega_{kj},\omega^j) - g(\omega_{ki},\omega^j)g(\omega_{kj},\omega^i)]\mathrm{vol}_g$$

$$= [\varepsilon_i g(d\omega^i,d\omega^i) - \varepsilon_i\varepsilon_j g(\omega^i\wedge d\omega^j,\omega^j\wedge d\omega^i)$$

$$+ \frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i\wedge\omega^i, d\omega^j\wedge\omega^j) - \varepsilon_i g(d\omega^i, d\omega^i)]\mathrm{vol}_g$$

$$= [- \varepsilon_i\varepsilon_j g(\omega^i\wedge d\omega^j, \omega^j\wedge d\omega^i) + \frac{1}{2} \varepsilon_i\varepsilon_j g(d\omega^i\wedge\omega^i, d\omega^j\wedge\omega^j)]\mathrm{vol}_g$$

$$= - \varepsilon_i\varepsilon_j (d\omega^i\wedge\omega^j)\wedge*(d\omega^j\wedge\omega^i) + \frac{1}{2} \varepsilon_i\varepsilon_j (d\omega^i\wedge\omega^i)\wedge*(d\omega^j\wedge\omega^j)$$

$$= - (d\omega^i\wedge\omega^j)\wedge*(d\omega_j\wedge\omega_i) + \frac{1}{2} (d\omega^i\wedge\omega_i)\wedge*(d\omega^j\wedge\omega_j).$$

All the terms appearing in the statement of the lemma are now accounted for.]

Put

$$L_0 = \mathrm{vol}_g.$$

Given $p \geq 1$, put

$$L_p = \Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_p j_p} \wedge \theta_{i_1 j_1 \cdots i_p j_p} \quad (2p \leq n).$$

Then

$$L_p \in \Lambda^n M.$$

[Note: These definitions are independent of the choice of E.]

Examples:

- $L_1 = S\mathrm{vol}_g$.

- $L_2 = (S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd})\mathrm{vol}_g$.

<u>LEMMA</u> We have

$$L_p = \frac{1}{2^p} (\delta^{\ell_1 \cdots \ell_{2p}}{}_{k_1 \cdots k_{2p}} R^{k_1 k_2}{}_{\ell_1 \ell_2} \cdots R^{k_{2p-1} k_{2p}}{}_{\ell_{2p-1} \ell_{2p}})\mathrm{vol}_g.$$

[In fact,

$$\Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_p j_p} \wedge \theta_{i_1 j_1 \cdots i_p j_p}$$

$$= \frac{1}{2} R^{i_1 j_1}{}_{k_1 \ell_1} \omega^{k_1} \wedge \omega^{\ell_1} \wedge \ldots \wedge \frac{1}{2} R^{i_p j_p}{}_{k_p \ell_p} \omega^{k_p} \wedge \omega^{\ell_p}$$

$$\times \frac{1}{(n-2p)!} \varepsilon_{i_1 j_1 \cdots i_p j_p a_{2p+1} \cdots a_n} \omega^{a_{2p+1}} \wedge \ldots \wedge \omega^{a_n}$$

$$= \frac{1}{2^p} (R^{i_1 j_1}{}_{k_1 \ell_1} \cdots R^{i_p j_p}{}_{k_p \ell_p})$$

$$\times \frac{1}{(n-2p)!} \varepsilon_{i_1 j_1 \cdots i_p j_p a_{2p+1} \cdots a_n}$$

$$\times \omega^{k_1} \wedge \omega^{\ell_1} \wedge \ldots \wedge \omega^{k_p} \wedge \omega^{\ell_p} \wedge \omega^{a_{2p+1}} \wedge \ldots \wedge \omega^{a_n}$$

$$= \frac{1}{2^p} (R^{i_1 j_1}{}_{k_1 \ell_1} \cdots R^{i_p j_p}{}_{k_p \ell_p})$$

$$\times \frac{1}{(n-2p)!} \varepsilon^{k_1 \ell_1 \cdots k_p \ell_p a_{2p+1} \cdots a_n} \varepsilon_{i_1 j_1 \cdots i_p j_p a_{2p+1} \cdots a_n}$$

$$\times (\omega^1 \wedge \ldots \wedge \omega^n)$$

$$= \frac{1}{2^p} (R^{i_1 j_1}{}_{k_1 \ell_1} \cdots R^{i_p j_p}{}_{k_p \ell_p})$$

$$\times \delta^{k_1 \ell_1 \cdots k_p \ell_p}_{i_1 j_1 \cdots i_p j_p} \mathrm{vol}_g,$$

which, upon relabling, is equivalent to the assertion.]

Remark: If M is compact and riemannian and if $n = 2p$, then by the Gauss-Bonnet theorem,

$$\chi(M) = \frac{1}{(4\pi)^p p!} \int_M L_p,$$

the LHS being the Euler characteristic of M.

[Note: Take $n = 2$ -- then $p = 1$ and $L_1 = S\mathrm{vol}_g$. Moreover, the scalar curvature S is twice the sectional curvature K and the Gauss-Bonnet theorem in this case says that

$$\chi(M) = \frac{1}{2\pi} \int_M K\mathrm{vol}_g.]$$

<u>FACT</u> Suppose that $n = 2p$. Fix $i \in \{1,\ldots,n\}$ and let

$$\Pi_p = \sum_{k=0}^{p-1} a_{k,p}\Phi_{k,p},$$

where

$$a_{k,p} = -\frac{\epsilon_i^{p-k}}{2^{k+p}\pi^p k!\,[1\cdot 3\cdots(2p-2k-1)]}$$

and

$$\Phi_{k,p} = \epsilon_{i_1\cdots i_{2p}}\delta^{i_1}{}_i\Omega^{i_2 i_3}\wedge\ldots\wedge\Omega^{i_{2k}i_{2k+1}}\wedge\omega^{i_{2k+2}}{}_i\wedge\ldots\wedge\omega^{i_{2p}}{}_i\,.$$

Then

$$d\Pi_p = \frac{1}{(4\pi)^p p!} L_p.$$

[Note: Therefore L_p $(n = 2p)$ is exact if the orthonormal frame bundle $LM(g)$ is trivial, hence is locally exact in general.]

<u>Reality Check</u> Take $n = 2$ and $i = 1$ -- then $p = 1$ and

$$\Pi_1 = a_{0,1}\Phi_{0,1}$$

$$= -\frac{\varepsilon_1}{2\pi}\,\varepsilon_{i_1 i_2}\delta^{i_1}{}_1\omega^{i_2}{}_1$$

$$= -\frac{\varepsilon_1}{2\pi}\,(\varepsilon_{12}\delta^1{}_1\omega^2{}_1 + \varepsilon_{21}\delta^2{}_1\omega^1{}_1)$$

$$= -\frac{\varepsilon_1}{2\pi}\,\omega^2{}_1$$

$$= -\frac{1}{2\pi}\,\omega^{21}$$

$$= \frac{1}{2\pi}\,\omega^{12}.$$

On the other hand,

$$\frac{1}{4\pi}\,L_1 = \frac{1}{4\pi}\,\varepsilon_{ij}\Omega^{ij}$$

$$= \frac{1}{4\pi}\,(\varepsilon_{12}\Omega^{12} + \varepsilon_{21}\Omega^{21})$$

$$= \frac{1}{4\pi}\,(\Omega^{12} - \Omega^{21})$$

$$= \frac{1}{2\pi}\,\Omega^{12}.$$

And

$$\Omega^1{}_2 = d\omega^1{}_2 + \omega^1{}_1\wedge\omega^1{}_2 + \omega^1{}_2\wedge\omega^2{}_2$$

$$= d\omega^1{}_2$$

$\Rightarrow$

$$\Omega^{12} = d\omega^{12}$$

$$\Rightarrow$$

$$d\Pi_1 = \frac{1}{4\pi} L_1.$$

Put

$$(G_0)^i{}_j = -\frac{1}{2}\delta^i{}_j.$$

Given $p \geq 1$, put

$$(G_p)^i{}_j = -\frac{1}{2^{p+1}}\delta^{i\ell_1\cdots\ell_{2p}}{}_{jk_1\cdots k_{2p}} R^{k_1k_2}{}_{\ell_1\ell_2}\cdots R^{k_{2p-1}k_{2p}}{}_{\ell_{2p-1}\ell_{2p}}.$$

Then

$$(G_p)^i{}_j = (G_p)^j{}_i$$

and

$$\nabla_j (G_p)^{ij} = 0.$$

Examples:

- $(G_1)^{ij} = R^{ij} - \frac{1}{2} S g^{ij}.$

- $$(G_2)^{ij} = [2SR^{ij} - 4R^{ikj\ell}R_{k\ell} + 2R^i{}_{abc}R^{jabc} - 4R^{ia}R^j{}_a] - \frac{1}{2}[S^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}]g^{ij}.$$

<u>SUBLEMMA</u> The wedge product

$$\omega^i \wedge \theta_{i_1\cdots i_p}$$

can be written as

$$(-1)^p \sum_{r=1}^{p} (-1)^r \delta^i{}_{i_r} \theta_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_p} .$$

[Recall that

$$g^\flat E_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}) = (-1)^{p+1} * \iota_{E_i}(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p}),$$

where

$$g^\flat E_i = \omega_i = \varepsilon_i \omega^i .$$

Therefore

$$\omega_i \wedge *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= (-1)^{p+1} \sum_{r=1}^{p} (-1)^{r+1} (\iota_{E_i} \omega^{i_r}) *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{r-1}} \wedge \omega^{i_{r+1}} \wedge \ldots \wedge \omega^{i_p})$$

$$= (-1)^p \sum_{r=1}^{p} (-1)^r g(\omega_i, \omega^{i_r}) *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{r-1}} \wedge \omega^{i_{r+1}} \wedge \ldots \wedge \omega^{i_p})$$

$$= (-1)^p \sum_{r=1}^{p} (-1)^r \varepsilon_i g(\omega^i, \omega^{i_r}) *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{r-1}} \wedge \omega^{i_{r+1}} \wedge \ldots \wedge \omega^{i_p})$$

$$= (-1)^p \sum_{r=1}^{p} (-1)^r \delta^i{}_{i_r} *(\omega^{i_1} \wedge \ldots \wedge \omega^{i_{r-1}} \wedge \omega^{i_{r+1}} \wedge \ldots \wedge \omega^{i_p})$$

$\Rightarrow$

$$\varepsilon_i \omega^i \wedge \varepsilon_{i_1} \cdots \varepsilon_{i_p} \theta_{i_1 \cdots i_p}$$

$$= (-1)^p \sum_{r=1}^{p} (-1)^r \delta^i{}_{i_r} \varepsilon_{i_1} \cdots \varepsilon_{i_{r-1}} \varepsilon_{i_{r+1}} \cdots \varepsilon_{i_p} \theta_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_p}$$

$\Rightarrow$

$$\omega^i \wedge \theta_{i_1 \cdots i_p}$$

$$= (-1)^p \sum_{r=1}^{p} (-1)^r \delta^i{}_{i_r} \varepsilon_{i_r} \varepsilon_i \theta_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_p} .$$

But $\delta^i{}_{i_r} \varepsilon_{i_r} \varepsilon_i = 0$ if $i_r \neq i$ while $\delta^i{}_{i_r} \varepsilon_{i_r} \varepsilon_i = \varepsilon_i \varepsilon_i = 1$ if $i_r = i$. Therefore

$$\omega^i \wedge \theta_{i_1 \cdots i_p}$$

$$= (-1)^p \sum_{r=1}^{p} (-1)^r \delta^i{}_{i_r} \theta_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_p} .]$$

Example: Suppose that p = 1 -- then

$$\omega_i \wedge {*\omega^j} = (-1)^{1+1} {*\iota_{E_i}} \omega^j$$

$$= *g(\omega_i, \omega^j)$$

$$= *\varepsilon_i g(\omega^i, \omega^j)$$

$$= *\delta^i{}_j = \delta^i{}_j \text{vol}_g$$

$\Rightarrow$

$$\varepsilon_i \omega^i \wedge \varepsilon_j \theta_j = \delta^i{}_j \text{vol}_g$$

$\Rightarrow$

$$\omega^i \wedge \theta_j = \varepsilon_i \varepsilon_j \delta^i{}_j \mathrm{vol}_g$$

$$= \delta^i{}_j \mathrm{vol}_g.$$

Now define the <u>Lovelock (n-1)-forms</u> by

$$\Xi(p)_a = \Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_p j_p} \wedge \theta_{i_1 j_1 \cdots i_p j_p a} \quad (a = 1,\ldots,n).$$

Example: Take p = 0 -- then

$$\Xi(0)_a = \theta_a$$

$$= \delta^i{}_a \theta_i$$

$$= -2(G_0)^i{}_a \theta_i$$

$$= -2(G_0)^i{}_a \varepsilon_i \varepsilon_i \theta_i$$

$$= -2(G_0)_{ai} {*\omega^i}.$$

Example: Take p = 1 -- then

$$\Xi(1)_a = \Omega^{ij} \wedge \theta_{ija}$$

$$= \frac{1}{2} R^{ij}{}_{k\ell} \omega^k \wedge \omega^\ell \wedge \theta_{aij}$$

$$= \frac{1}{2} R^{ij}{}_{k\ell} \omega^k \wedge (-1)^3 [- \delta^\ell{}_a \theta_{ij} + \delta^\ell{}_i \theta_{aj} - \delta^\ell{}_j \theta_{ai}]$$

$$= \frac{1}{2} R^{ij}{}_{k\ell} \omega^k \wedge [\delta^\ell{}_a \theta_{ij} - \delta^\ell{}_i \theta_{aj} + \delta^\ell{}_j \theta_{ai}].$$

• $\frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{a}\omega^{k}\wedge\theta_{ij}$

$$= \frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{a}[- \delta^{k}{}_{i}\theta_{j} + \delta^{k}{}_{j}\theta_{i}].$$

• $- \frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{i}\omega^{k}\wedge\theta_{aj}$

$$= - \frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{i}[- \delta^{k}{}_{a}\theta_{j} + \delta^{k}{}_{j}\theta_{a}].$$

• $\frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{j}\omega^{k}\wedge\theta_{ai}$

$$= \frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{j}[- \delta^{k}{}_{a}\theta_{i} + \delta^{k}{}_{i}\theta_{a}].$$

Collect the coefficients of θ_a:

$$(\frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{j}\delta^{k}{}_{i} - \frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{i}\delta^{k}{}_{j})\theta_{a}$$

$$= (\frac{1}{2} R^{ij}{}_{ij} - \frac{1}{2} R^{ij}{}_{ji})\theta_{a}$$

$$= \frac{1}{2} (R^{ij}{}_{ij} + R^{ij}{}_{ij})\theta_{a}$$

$$= S\theta_{a}.$$

Collect the coefficients of θ_i:

$$(\frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{a}\delta^{k}{}_{j} - \frac{1}{2} R^{ij}{}_{k\ell}\delta^{\ell}{}_{j}\delta^{k}{}_{a})\theta_{i}$$

$$= (\frac{1}{2} R^{ij}{}_{ja} - \frac{1}{2} R^{ij}{}_{aj})\theta_{i}$$

$$= \frac{1}{2}\,(-\,R^{ji}{}_{ja} - R^{ji}{}_{ja})\,\theta_i$$

$$= -\,R^{i}{}_{a}\theta_i.$$

Collect the coefficients of θ_j:

$$(\tfrac{1}{2}\,R^{ij}{}_{k\ell}\delta^{\ell}{}_{i}\delta^{k}{}_{a} - \tfrac{1}{2}\,R^{ij}{}_{k\ell}\delta^{\ell}{}_{a}\delta^{k}{}_{i})\,\theta_j$$

$$= (\tfrac{1}{2}\,R^{ij}{}_{ai} - \tfrac{1}{2}\,R^{ij}{}_{ia})\,\theta_j$$

$$= \frac{1}{2}\,(-\,R^{ij}{}_{ia} - R^{ij}{}_{ia})\,\theta_j$$

$$= -\,R^{j}{}_{a}\theta_j.$$

Therefore

$$\Xi(1)_a = S\theta_a - R^{i}{}_{a}\theta_i - R^{j}{}_{a}\theta_j$$

$$= S\delta^{i}{}_{a}\theta_i - 2R^{i}{}_{a}\theta_i$$

$$= (S\delta^{i}{}_{a} - 2R^{i}{}_{a})\,\theta_i$$

$$= -2(G_1)^{i}{}_{a}\theta_i$$

$$= -2(G_1)^{i}{}_{a}\varepsilon_i\varepsilon_i\theta_i$$

$$= -2(G_1)_{ai}{*\omega}^{i}.$$

<u>LEMMA</u> We have

$$\Xi(p)_a = -2(G_p)_{ai} * \omega^i.$$

Suppose that $n = 4p$ $(p = 1,2,\ldots)$. Given $(a_1,\ldots,a_{2k})$, put

$$\underline{\Omega}_{2k} = \Omega^{a_1}{}_{a_2} \wedge \cdots \wedge \Omega^{a_{2k}}{}_{a_1}$$

and set

$$\underline{\Pi}_{\underline{K}} = \underline{\Omega}_{2k_1} \wedge \cdots \wedge \underline{\Omega}_{2k_r} \quad (\underline{K} = (2k_1,\ldots,2k_r)),$$

where

$$2(2k_1 + \cdots + 2k_r) = n.$$

Then

$$\underline{\Pi}_{\underline{K}} \in \Lambda^n M.$$

[Note: The $\underline{\Pi}_{\underline{K}}$ are called <u>Pontryagin forms</u>. In view of the definition, their number is precisely $P(n/4)$ (P the partition function).]

Examples:

- $n = 4$:

$$\underline{\Pi}_{(2)} = \Omega^a{}_b \wedge \Omega^b{}_a.$$

- $n = 8$:

$$\left[\begin{array}{l} \underline{\Pi}_{(4)} = \Omega^a{}_b \wedge \Omega^b{}_c \wedge \Omega^c{}_d \wedge \Omega^d{}_a \\ \\ \underline{\Pi}_{(2,2)} = (\Omega^a{}_b \wedge \Omega^b{}_a) \wedge (\Omega^c{}_d \wedge \Omega^d{}_c). \end{array}\right.$$

Observation: $\underline{\Omega}_{2k}$ is closed, i.e.,

$$d\underline{\Omega}_{2k} = 0.$$

[This follows from the fact that

$$d\Omega^i{}_j + \omega^i{}_k \wedge \Omega^k{}_j - \Omega^i{}_k \wedge \omega^k{}_j = 0.]$$

Example: Consider $\Omega^a{}_b \wedge \Omega^b{}_a$. Thus

$$\Omega^a{}_b \wedge \Omega^b{}_a = (d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge \Omega^b{}_a$$

$$= d\omega^a{}_b \wedge \Omega^b{}_a + \omega^a{}_c \wedge \omega^c{}_b \wedge \Omega^b{}_a$$

$$= d(\omega^a{}_b \wedge \Omega^b{}_a) + \omega^a{}_b \wedge d\Omega^b{}_a + \omega^a{}_c \wedge \omega^c{}_b \wedge \Omega^b{}_a$$

$$= d(\omega^a{}_b \wedge \Omega^b{}_a)$$

$$+ \omega^a{}_b \wedge (- \omega^b{}_c \wedge \Omega^c{}_a + \Omega^b{}_c \wedge \omega^c{}_a)$$

$$+ \omega^a{}_c \wedge \omega^c{}_b \wedge \Omega^b{}_a$$

$$= d(\omega^a{}_b \wedge \Omega^b{}_a) - \omega^a{}_b \wedge \omega^b{}_c \wedge \Omega^c{}_a$$

$$+ \omega^a{}_b \wedge \omega^c{}_a \wedge \Omega^b{}_c + \omega^a{}_c \wedge \omega^c{}_b \wedge \Omega^b{}_a.$$

But

$$\omega^a{}_c \wedge \omega^c{}_b \wedge \Omega^b{}_a$$

$$= \omega^c{}_a \wedge \omega^a{}_b \wedge \Omega^b{}_c$$

$$= - \omega^a{}_b \wedge \omega^c{}_a \wedge \Omega^b{}_c.$$

Therefore

$$\Omega^a{}_b\wedge\Omega^b{}_a = d(\omega^a{}_b\wedge\Omega^b{}_a) - \omega^a{}_b\wedge\omega^b{}_c\wedge\Omega^c{}_a$$

$$= d(\omega^a{}_b\wedge(d\omega^b{}_a + \omega^b{}_c\wedge\omega^c{}_a)) - \omega^a{}_b\wedge\omega^b{}_c\wedge\Omega^c{}_a$$

$$= d[\omega^a{}_b\wedge d\omega^b{}_a + \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a]$$

$$- \omega^a{}_b\wedge\omega^b{}_c\wedge\Omega^c{}_a$$

$$= d[\omega^a{}_b\wedge d\omega^b{}_a + \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a]$$

$$- \omega^a{}_b\wedge\omega^b{}_c\wedge(d\omega^c{}_a + \omega^c{}_d\wedge\omega^d{}_a)$$

$$= d[\omega^a{}_b\wedge d\omega^b{}_a + \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a]$$

$$- d\omega^c{}_a\wedge\omega^a{}_b\wedge\omega^b{}_c - \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a$$

$$= d[\omega^a{}_b\wedge d\omega^b{}_a + \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a]$$

$$- d\omega^a{}_c\wedge\omega^c{}_b\wedge\omega^b{}_a - \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a$$

$$= d[\omega^a{}_b\wedge d\omega^b{}_a + \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a]$$

$$- d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a - \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a.$$

• $\frac{1}{3}\, d(\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a)$

$$= \frac{1}{3}\,[d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a - \omega^a{}_b\wedge d\omega^b{}_c\wedge\omega^c{}_a + \omega^a{}_b\wedge\omega^b{}_c\wedge d\omega^c{}_a]$$

$$= \frac{1}{3}\,[d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a - d\omega^b{}_c\wedge\omega^a{}_b\wedge\omega^c{}_a + d\omega^c{}_a\wedge\omega^a{}_b\wedge\omega^b{}_c]$$

$$= \frac{1}{3}\,[d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a - d\omega^a{}_c\wedge\omega^b{}_a\wedge\omega^c{}_b + d\omega^a{}_c\wedge\omega^c{}_b\wedge\omega^b{}_a]$$

$$= \frac{1}{3}\,[d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a - d\omega^a{}_b\wedge\omega^c{}_a\wedge\omega^b{}_c + d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a]$$

$$= \frac{1}{3}\,[d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a + d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a + d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a]$$

$$= d\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a.$$

Therefore

$$\Omega^a{}_b\wedge\Omega^b{}_a = d(\omega^a{}_b\wedge d\omega^b{}_a)$$

$$+ d(\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a) - \frac{1}{3}\,d(\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a)$$

$$- \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a$$

$$= d[\omega^a{}_b\wedge d\omega^b{}_a + \frac{2}{3}\,(\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a)]$$

$$- \omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a.$$

However the last term vanishes, so

$$\Omega^a{}_b\wedge\Omega^b{}_a = d[\omega^a{}_b\wedge d\omega^b{}_a + \frac{2}{3}\,(\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a)].$$

[Note: To check that

$$\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a = 0,$$

write

$$\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a$$

$$= -\,\omega^d{}_a\wedge\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d$$

$$= -\,\omega^a{}_d\wedge\omega^d{}_b\wedge\omega^b{}_c\wedge\omega^c{}_a$$

$$= -\,\omega^a{}_b\wedge\omega^b{}_d\wedge\omega^d{}_c\wedge\omega^c{}_a$$

$$= -\,\omega^a{}_b\wedge\omega^b{}_c\wedge\omega^c{}_d\wedge\omega^d{}_a.]$$

<u>FACT</u> We have

$$\underline{\Omega}_{2k} = dC_{2k},$$

where

$$C_{2k} = 2k \cdot \sum_{i=0}^{2k-1} \binom{2k-1}{i} \frac{1}{2k+i}\,\mathrm{tr}((\omega_\nabla)^{2i+1}\wedge(d\omega_\nabla)^{2k-1-i}).$$

[Note: To explain the notation, recall that

$$\omega_\nabla = [\omega^i{}_j]$$

is an element of $\Lambda^1(M;\underline{g\ell}(n,\underline{R}))$ (here, of course, ∇ is the metric connection). Accordingly,

$$(\omega_\nabla)^{2i+1} = \overbrace{\omega_\nabla \wedge \cdots \wedge \omega_\nabla}^{2i+1}.$$

Similar comments apply to

$$d\omega_\nabla = [d\omega^i{}_j].]$$

Reality Check Take k = 1 — then

$$\underline{\Omega}_2 = \Omega^a{}_b \wedge \Omega^b{}_a.$$

And

$$C_2 = 2[\frac{1}{2}\,\mathrm{tr}(\omega_\nabla \wedge d\omega_\nabla) + \frac{1}{3}\,\mathrm{tr}(\omega_\nabla \wedge \omega_\nabla \wedge \omega_\nabla)]$$

$$= \mathrm{tr}(\omega_\nabla \wedge d\omega_\nabla) + \frac{2}{3}\,\mathrm{tr}(\omega_\nabla \wedge \omega_\nabla \wedge \omega_\nabla)$$

$$= \omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\,(\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a),$$

which agrees with what was said above.

Remark: The C_{2k} are called Chern-Simons forms.

[Note: One can represent C_{2k} as an integral:

$$C_{2k} = 2k \cdot \int_0^1 \mathrm{tr}(\omega_\nabla \wedge (t^2(\omega_\nabla)^2 + t d\omega_\nabla)^{2k-1})\,dt.$$

To see this, use the binomial theorem and expand the RHS to get

$$2k \cdot \int_0^1 \mathrm{tr}(\omega_\nabla \wedge \sum_{i=0}^{2k-1} \binom{2k-1}{i} t^{2i} (\omega_\nabla)^{2i} \wedge t^{2k-1-i} (d\omega_\nabla)^{2k-1-i})\,dt$$

$$= 2k \cdot \sum_{i=0}^{2k-1} \binom{2k-1}{i} \mathrm{tr}((\omega_\nabla)^{2i+1} \wedge (d\omega_\nabla)^{2k-1-i}) \cdot \int_0^1 t^{2k-1+i}\,dt$$

$$= 2k \cdot \sum_{i=0}^{2k-1} \binom{2k-1}{i} \frac{1}{2k+i}\,\mathrm{tr}((\omega_\nabla)^{2i+1} \wedge (d\omega_\nabla)^{2k-1-i}).$$

E.g.: Take k = 1 and put

$$\Omega_\nabla(t) = td\omega_\nabla + t^2(\omega_\nabla \wedge \omega_\nabla).$$

Then

$$\underline{\Omega}_2 = \int_0^1 \frac{d}{dt}\, \mathrm{tr}(\Omega_\nabla(t) \wedge \Omega_\nabla(t))dt$$

$$= 2\int_0^1 \mathrm{tr}(\frac{d\Omega_\nabla(t)}{dt} \wedge \Omega_\nabla(t))dt$$

$$= 2d\int_0^1 \mathrm{tr}(\omega_\nabla \wedge \Omega_\nabla(t))dt$$

$$= 2d\int_0^1 \mathrm{tr}(t\omega_\nabla \wedge d\omega_\nabla + t^2\omega_\nabla \wedge \omega_\nabla \wedge \omega_\nabla)dt$$

$$= d\mathrm{tr}(\omega_\nabla \wedge d\omega_\nabla + \frac{2}{3}\, (\omega_\nabla \wedge \omega_\nabla \wedge \omega_\nabla)).]$$

Since $d\underline{\Omega}_{2k} = 0$, it follows that

$$\underline{\Pi}_{\underline{K}} = \underline{\Pi}(2k_1,\ldots,2k_r)$$

$$= d(C_{2k_1} \wedge \underline{\Omega}_{2k_2} \wedge \cdots \wedge \underline{\Omega}_{2k_r})$$

$$= d(\underline{\Omega}_{2k_1} \wedge C_{2k_2} \wedge \cdots \wedge \underline{\Omega}_{2k_r})$$

$$\vdots$$

$$= d(\underline{\Omega}_{2k_1} \wedge \cdots \wedge \underline{\Omega}_{2k_{r-1}} \wedge C_{2k_r}).$$

[Note: Suppose that $i < j$ — then the difference

$$\underline{\Omega}_{2k_1} \wedge \cdots \wedge C_{2k_i} \wedge \cdots \wedge \underline{\Omega}_{2k_r} - \underline{\Omega}_{2k_1} \wedge \cdots \wedge C_{2k_j} \wedge \cdots \wedge \underline{\Omega}_{2k_r}$$

equals

$$d(\underline{\Omega}_{2k_1} \wedge \cdots \wedge C_{2k_i} \wedge \cdots \wedge C_{2k_j} \wedge \cdots \wedge \underline{\Omega}_{2k_r}),$$

thus is exact.]

<u>Section 28</u>: <u>Functional Derivatives</u> Let U and V be linear spaces equipped with a bilinear functional $< , >: U \times V \rightarrow \underline{R}$.

Definition: $< , >$ is <u>nondegenerate</u> if

$$\begin{cases} < u,v > = 0 \ \forall \ v \in V \Rightarrow u = 0 \\ < u,v > = 0 \ \forall \ u \in U \Rightarrow v = 0. \end{cases}$$

Suppose that $< , >$ is nondegenerate -- then the arrows

$$\begin{cases} U \rightarrow V^* & (u \rightarrow \ < u, \ >) \\ V \rightarrow U^* & (v \rightarrow \ < \ ,v >) \end{cases}$$

are one-to-one (but, in general, are not onto).

(ϕ) Let $\phi: U \rightarrow \underline{R}$ -- then the <u>functional derivative</u> $\frac{\delta\phi}{\delta u}$ of ϕ w.r.t. $u \in U$ is the unique element of V (if it exists) such that $\forall \ u' \in U$,

$$\frac{d}{d\varepsilon} \phi(u + \varepsilon u')\Big|_{\varepsilon=0} = \ < u', \frac{\delta\phi}{\delta u} > .$$

(ψ) Let $\psi: V \rightarrow \underline{R}$ -- then the <u>functional derivative</u> $\frac{\delta\psi}{\delta v}$ of ψ w.r.t. $v \in V$ is the unique element of U (if it exists) such that $\forall \ v' \in V$,

$$\frac{d}{d\varepsilon} \psi(v + \varepsilon v')\Big|_{\varepsilon=0} = \ < \frac{\delta\psi}{\delta v}, v' > .$$

Remark: Functional derivatives give rise to maps

$$\begin{cases} D\phi: U \rightarrow V & (D\phi(u) = \frac{\delta\phi}{\delta u}) \\ D\psi: V \rightarrow U & (D\psi(v) = \frac{\delta\psi}{\delta v}). \end{cases}$$

Example: Take $U = V = \underline{R}^n$ and let $< \, , \, >: \underline{R}^n \times \underline{R}^n \to \underline{R}$ be the usual inner product: $< x,y > = x \cdot y$. Suppose that $f:\underline{R}^n \to \underline{R}$ is a C^∞ function -- then $\forall \, x,y \in \underline{R}^n$,

$$\frac{d}{d\varepsilon} f(x + \varepsilon y) \Big|_{\varepsilon=0} = \nabla f \Big|_x \cdot y$$

$\Rightarrow$

$$\frac{\delta f}{\delta x} = \nabla f \Big|_x .$$

Example: Let $U = V = C_c^\infty(\underline{R}^n)$ and put

$$< f,g > = \int_{\underline{R}^n} f(x) g(x) dx.$$

Define

$$I_k : C_c^\infty(\underline{R}^n) \to \underline{R}$$

by the rule

$$I_k(f) = \int_{\underline{R}^n} (f(x))^k dx \quad (k = 1,2,\ldots).$$

Then $\forall \, g$,

$$\frac{d}{d\varepsilon} I_k(f + \varepsilon g) \Big|_{\varepsilon=0} = \int_{\underline{R}^n} \frac{d}{d\varepsilon} (f(x) + \varepsilon g(x))^k \Big|_{\varepsilon=0} dx$$

$$= \int_{\underline{R}^n} k(f(x))^{k-1} g(x) dx$$

$$= < kf^{k-1}, g >$$

$\Rightarrow$

$$\frac{\delta I_k}{\delta f} = kf^{k-1}.$$

Let M be a connected C^∞ manifold of dimension n, which we shall assume is orientable.

Example: Take

$$\begin{cases} U = \Lambda_c^p M \\ V = \Lambda_c^{n-p} M \end{cases}$$

and let

$$< \alpha, \beta > = \int_M \alpha \wedge \beta.$$

$\underline{\Lambda_c^p}$: Suppose that $\phi:\Lambda_c^p M \to \underline{R}$ -- then

$$\frac{\delta\phi}{\delta\alpha} \in \Lambda_c^{n-p} M$$

is characterized by the relation

$$\frac{d}{d\varepsilon} \phi(\alpha + \varepsilon\alpha')\Big|_{\varepsilon=0} = \int_M \alpha' \wedge \frac{\delta\phi}{\delta\alpha}.$$

$\underline{\Lambda_c^{n-p}}$: Suppose that $\psi:\Lambda_c^{n-p} M \to \underline{R}$ -- then

$$\frac{\delta\psi}{\delta\beta} \in \Lambda_c^p M$$

is characterized by the relation

$$\frac{d}{d\varepsilon} \psi(\beta + \varepsilon\beta')\Big|_{\varepsilon=0} = \int_M \frac{\delta\psi}{\delta\beta} \wedge \beta'.$$

In practice, the following situation can arise:

1. There are linear spaces U and V but no assumption is made regarding a bilinear functional $< , >: U \times V \to \underline{R}$.

2. There is a linear subspace $U_c \subset U$ and a nondegenerate bilinear functional $< \, , \, >: U_c \times V \rightarrow \underline{R}$.

3. There is a subset $U_0 \subset U$ such that $\forall \ u_0 \in U_0$ & $\forall \ u_c \in U_c$, $u_0 + \varepsilon u_c \in U_0$ provided ε is sufficiently small.

Under these conditions, if $\phi: U_0 \rightarrow \underline{R}$, then it makes sense to consider $\frac{\delta \phi}{\delta u_0} \in V$:

$$\frac{d}{d\varepsilon} \phi(u_0 + \varepsilon u_c) \Big|_{\varepsilon=0} = < u_c, \frac{\delta \phi}{\delta u_0} > .$$

We shall now consider a realization of this setup.

Write $C_d^\infty(M)$ for $\sec(L_{den})$, a module over $C^\infty(M)$ -- then for any vector bundle $E \rightarrow M$, there is an arrow of evaluation

$$\text{ev}: \sec(E) \times \sec(E^* \otimes L_{den}) \rightarrow C_d^\infty(M).$$

Let

$$\begin{bmatrix} \text{Sym}^2 \ TM \\ \text{Sym}^2 \ T^*M \end{bmatrix}$$

be the second symmetric power of $\begin{bmatrix} TM \\ T^*M \end{bmatrix}$ -- then

$$\begin{bmatrix} S^2(M) = \sec(\text{Sym}^2 \ TM) \\ S_2(M) = \sec(\text{Sym}^2 \ T^*M). \end{bmatrix}$$

Put

$$S_d^2(M) = \sec(\text{Sym}^2 \ TM \otimes L_{den}).$$

Denote by $S_{2,c}(M)$ the set of compactly supported elements of $S_2(M)$ -- then there is a nondegenerate bilinear functional

$$<\ ,\ >: \mathcal{S}_{2,c}(M) \times \mathcal{S}_d^2(M) \to \underline{R},$$

viz.

$$< s, \lambda \otimes \varphi > = \int_M \lambda(s)\, dm_\varphi \ .$$

Scholium: The preceding considerations are realized by taking

$$\begin{bmatrix} U = \mathcal{S}_2(M),\ V = \mathcal{S}_d^2(M) \\ \\ U_c = \mathcal{S}_{2,c}(M),\ U_0 = \underline{M}. \end{bmatrix}$$

Example: Let

$$L \in MC_n(0,0,1,2)$$

be a lagrangian of the form

$$L(g) = |g|^{1/2} F(g),$$

where $F \in MC_n(0,0,0,2)$ (e.g. $|g|^{1/2} S$). Then, by definition,

$$PL(g,h) = \frac{d}{d\varepsilon} L(g + \varepsilon h) \Big|_{\varepsilon=0} \quad (h \in \mathcal{S}_{2,c}(M))$$

and we have

$$PL(g,h) = - ev(h, E(L)) + div\ X(g,h).$$

Here

$$X(g,h) \in sec(TM \otimes L_{den})$$

is compactly supported. If M is compact, then

$$\mathcal{L}(g) = \int_M L(g)$$

exists and

$$\frac{d}{d\varepsilon} L(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \int_M PL(g,h)$$

$$= \int_M - ev(h,E(L)) = < h, - E(L) >$$

$\Rightarrow$

$$\frac{\delta L}{\delta g} = - E(L).$$

On the other hand, if M is not compact, then the integral $\int_M L(g)$ need not exist but for any open, relatively compact subset $K \subset M$,

$$L_K(g) = \int_K L(g)$$

does exist and

$$\frac{\delta L_K}{\delta g} = - E(L)|K.$$

Notation: Put

$$\Lambda_d^1(M) = \sec(T^*M \otimes L_{den}).$$

Let $\mathcal{D}_c^1(M)$ stand for the set of compactly supported elements of $\mathcal{D}^1(M)$ -- then there is a nondegenerate bilinear functional

$$< , >:\mathcal{D}_c^1(M) \times \Lambda_d^1(M) \rightarrow \underline{R},$$

viz.

$$< X,\alpha \otimes \varphi > = \int_M \alpha(X)dm_\varphi.$$

Observation: Fix $g \in \underline{M}$ -- then $\forall\ X \in \mathcal{D}^1(M)$, $L_X g \in S_2(M)$. Indeed,

$$(L_X g)(Y,Z) = \nabla g^\flat X(Y,Z) + \nabla g^\flat X(Z,Y),$$

where ∇ is the metric connection attached to g (bear in mind that $\nabla g^\flat X \in \mathcal{D}_2^0(M)$).

[Note: Locally,

$$L_X g_{ij} = X_{i;j} + X_{j;i} = \nabla_j X_i + \nabla_i X_j.]$$

<u>LEMMA</u> Fix $g \in \underline{M}$ -- then $\forall\ X \in \mathcal{D}_c^1(M)$ & $\forall\ s \in S_2(M)$,

$$< L_X g,\ s^\# \otimes |g|^{1/2} > = -\ 2 < X,\ \text{div}_g\ s \otimes |g|^{1/2} >.$$

[Start with the LHS, thus

$$< L_X g,\ s^\# \otimes |g|^{1/2} >$$

$$= \int_M s^\# (L_X g) \text{vol}_g$$

$$= \int_M (X_{i;j} + X_{j;i}) s^{ij} \text{vol}_g$$

$$= -\ 2 \int_M X_i \nabla_j s^{ij} \text{vol}_g.$$

By definition, $\text{div}_g\ s$ is a 1-form:

$$(\text{div}_g\ s)_i = g^{kj} \nabla_j s_{ki} = g^{jk} \nabla_j s_{ik} = \nabla_j s_i^{\ j}.$$

Therefore

$$X_i \nabla_j s^{ij} = g_{ik} X^k \nabla_j s^{ij}$$

$$= X^k g_{ik} \nabla_j s^{ij}$$

$$= X^k g_{ki} \nabla_j s^{ij}$$

$$= X^k \nabla_j s_k{}^j$$

$$= X^i \nabla_j s_i{}^j$$

$$= X^i (\mathrm{div}_g\ s)_i$$

$\Rightarrow$

$$\int_M X_i \nabla_j s^{ij} \mathrm{vol}_g = \int_M X^i (\mathrm{div}_g\ s)_i \mathrm{vol}_g$$

$$= \int_M (\mathrm{div}_g\ s)(X) \mathrm{vol}_g$$

$$= \langle X, \mathrm{div}_g\ s \otimes |g|^{1/2} \rangle.]$$

[Note: There is an integration by parts implicit in the passage from

$$\int_M (X_{i;j} + X_{j;i}) s^{ij} \mathrm{vol}_g$$

to

$$- 2 \int_M X_i \nabla_j s^{ij} \mathrm{vol}_g.$$

In fact,

$$\nabla_j (X_i s^{ij}) = X_{i;j} s^{ij} + X_i \nabla_j s^{ij}$$

$\Rightarrow$

$$\int_M X_{i;j} s^{ij} \mathrm{vol}_g = - \int_M X_i \nabla_j s^{ij} \mathrm{vol}_g + \int_M \nabla_j (X_i s^{ij}) \mathrm{vol}_g.$$

Claim: $\exists\ Y \in \mathcal{D}_c^1(M)$ such that

$$Y^j = X_i s^{ij}.$$

To see this, observe that

$$s^\# \otimes g^\flat X \in \mathcal{D}_1^2(M)$$

has components

$$(s^\# \otimes g^\flat X)^{ij}{}_k = s^{ij} X_k.$$

Now apply the contraction

$$C_1^1 : \mathcal{D}_1^2(M) \to \mathcal{D}_0^1(M) \quad (= \mathcal{D}^1(M)).$$

Then

$$Y = C_1^1(s^\# \otimes g^\flat X) \in \mathcal{D}^1(M)$$

has components

$$s^{ij} X_i$$

and is compactly supported. Consequently,

$$\nabla_j (X_i s^{ij}) = Y^j_{;j}$$

$$\Rightarrow$$

$$\int_M \nabla_j (X_i s^{ij}) \mathrm{vol}_g = \int_M (\mathrm{div}_g\ Y) \mathrm{vol}_g = 0.]$$

Each $g \in \underline{M}$ determines a map

$$\begin{cases} S_d^2(M) \to \Lambda_d^1(M) \\ \quad \Lambda \to \mathrm{div}_g\ \Lambda. \end{cases}$$

Thus write $\Lambda = s^{\#} \otimes |g|^{1/2}$ $(s \in S_2(M))$ and set

$$\operatorname{div}_g \Lambda = \operatorname{div}_g s \otimes |g|^{1/2}.$$

The lemma then implies that $\forall\ X \in \mathcal{D}_c^1(M)$,

$$- 2 \int_M \operatorname{div}_g \Lambda(X)$$

$$= - 2 \int_M (\operatorname{div}_g s)(X) \mathrm{vol}_g$$

$$= \int_M s^{\#}(L_X g) \mathrm{vol}_g$$

$$= \int_M \Lambda(L_X g).$$

Example: Suppose that $X \in \mathcal{D}_c^1(M)$. Fix $g \in \underline{M}$ and define

$$I_{X,g}: S_d^2(M) \to \underline{R}$$

by

$$I_{X,g}(\Lambda) = \int_M \Lambda(L_X g).$$

Then

$$\frac{d}{d\varepsilon} I_{X,g}(\Lambda + \varepsilon\Lambda')\Big|_{\varepsilon=0}$$

$$= \int_M \frac{d}{d\varepsilon} (\Lambda + \varepsilon\Lambda')(L_X g)\Big|_{\varepsilon=0}$$

$$= \int_M \Lambda'(L_X g)$$

$$= \langle L_X g, \Lambda' \rangle$$

$\Rightarrow$

$$\frac{\delta I_{X,g}}{\delta \Lambda} = L_X g.$$

Example: Suppose that $X \in \mathcal{D}_c^1(M)$. Fix $\Lambda \in S_d^2(M)$ and define

$$I_{X,\Lambda} : \underline{M} \to \underline{R}$$

by

$$I_{X,\Lambda}(g) = \int_M \Lambda(L_X g).$$

Then

$$\frac{d}{d\varepsilon} I_{X,\Lambda}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \int_M \frac{d}{d\varepsilon} \Lambda(L_X g + \varepsilon L_X h)\Big|_{\varepsilon=0}$$

$$= \int_M \Lambda(L_X h)$$

$$= \int_M s^{\#}(L_X h)\,\mathrm{vol}_g$$

or still (cf. infra),

$$= \int_M - [(L_X s^{\#})(h) + s^{\#}(h)\,\mathrm{div}_g\, X]\,\mathrm{vol}_g$$

$$= \langle h, -L_X \Lambda \rangle$$

$\Rightarrow$

$$\frac{\delta I_{X,\Lambda}}{\delta g} = - L_X \Lambda.$$

Here

$$L_X \Lambda = L_X s^{\#} \otimes |g|^{1/2} + s^{\#} \otimes (\mathrm{div}_g\, X)\,|g|^{1/2}.$$

[Note: To justify the not so obvious step in the manipulation, recall that L_X commutes with contractions, hence

$$L_X(s^\#(h)) = L_X(C_1^1C_2^2(s^\# \otimes h))$$

$$= C_1^1C_2^2L_X(s^\# \otimes h)$$

$$= C_1^1C_2^2((L_Xs^\#) \otimes h + s^\# \otimes (L_Xh))$$

$$= (L_Xs^\#)(h) + s^\#(L_Xh).$$

Therefore

$$\int_M s^\#(L_Xh)\mathrm{vol}_g$$

$$= \int_M L_X(s^\#(h))\mathrm{vol}_g - \int_M (L_Xs^\#)(h)\mathrm{vol}_g$$

$$= - \int_M (L_Xs^\#)(h)\mathrm{vol}_g - \int_M s^\#(h)(\mathrm{div}_g\ X)\mathrm{vol}_g$$

$$= \int_M - [(L_Xs^\#)(h) + s^\#(h)\mathrm{div}_g\ X]\mathrm{vol}_g.]$$

Remark: Let $T \in \mathcal{D}_2^0(M)$. Suppose that T is symmetric -- then $\forall\ X \in \mathcal{D}^1(M)$, L_XT is symmetric.

[Recall that $\forall\ Y,Z \in \mathcal{D}^1(M)$,

$$(L_XT)(Y,Z) = XT(Y,Z)$$

$$- T([X,Y],Z) - T(Y,[X,Z]).]$$

Notation: Let $g \in \underline{M}$.

- Given $s \in S_2(M)$, put

$$\mathrm{tr}_g(s) = g[{}^0_2](g,s) = g^{ij}s_{ij}.$$

- Given $u,v \in S_2(M)$, put

$$[u,v]_g = g[{}^0_2](u,v) = u^{ij}v_{ij} \quad (= u_{ij}v^{ij}).$$

- Given $s \in S_2(M)$, put

$$(s*s)_{ij} = s_{ik}s^k{}_j.$$

[Note:

(1) $s*s \in S_2(M)$. Proof:

$$\begin{aligned} s_{ik}s^k{}_j &= s_{ik}g^{k\ell}s_{\ell j} \\ &= s_{j\ell}g^{\ell k}s_{ki} \\ &= s_{j\ell}s^\ell{}_i. \end{aligned}$$

(2) $\mathrm{tr}_g(s*s) = [s,s]_g$. Proof:

$$\begin{aligned} \mathrm{tr}_g(s*s) &= g^{ij}(s*s)_{ij} \\ &= g^{ij}s_{ik}s^k{}_j \\ &= s_{ki}g^{ij}s^k{}_j \\ &= s_{ki}s^{ki}.] \end{aligned}$$

Suppose given a function

$$\Phi:\underline{M} \to C_d^\infty(M).$$

Then $\forall\ g \in \underline{M}$, the prescription

$$D_g\Phi(h) = \frac{d}{d\varepsilon}\,\Phi(g + \varepsilon h)\Big|_{\varepsilon=0}$$

defines a function

$$D_g\Phi:S_{2,c}(M) \to C_d^\infty(M).$$

[Note: If M is compact and if

$$\phi(g) = \int_M \Phi(g),$$

then in the applications, $\frac{\delta\phi}{\delta g} \in S_d^2(M)$ exists, so

$$\frac{d}{d\varepsilon}\,\phi(g + \varepsilon h)\Big|_{\varepsilon=0} = \int_M D_g\Phi(h) = \langle h, \frac{\delta\phi}{\delta g}\rangle.]$$

Examples:

(1) Put $\Phi(g) = |g|^{1/2}$ — then

$$D_g\Phi(h) = \frac{1}{2}\,\mathrm{tr}_g(h)\,|g|^{1/2}.$$

Therefore

$$\frac{\delta\phi}{\delta g} = \frac{1}{2}\,g^\# \otimes |g|^{1/2}$$

provided M is compact.

[We have

$$\frac{d}{d\varepsilon}\,\det((g_{ij}) + \varepsilon(h_{ij}))\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} \det(g_{ij})\det(1 + \varepsilon(g^{ij})(h_{ij}))\Big|_{\varepsilon=0}$$

$$= \det(g_{ij})\operatorname{tr}((g^{ij})(h_{ij}))$$

$$= \operatorname{tr}_g(h)\det(g_{ij}).$$

Consequently,

$$\frac{d}{d\varepsilon}\ |g + \varepsilon h|^{1/2}\Big|_{\varepsilon=0} = \frac{1}{2}\ \frac{1}{|g|^{1/2}} \times \frac{d}{d\varepsilon} \pm \det((g_{ij}) + \varepsilon(h_{ij}))\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\operatorname{tr}_g(h)\ \frac{1}{|g|^{1/2}} \times \pm \det(g_{ij})$$

$$= \frac{1}{2}\operatorname{tr}_g(h)\,|g|^{1/2}.]$$

(2) Put $\Phi(g) = \dfrac{1}{|g|^{1/2}}$ -- then

$$D_g\Phi(h) = -\frac{1}{2}\operatorname{tr}_g(h)\,|g|^{-1/2}.$$

Therefore

$$\frac{\delta\phi}{\delta g} = -\frac{1}{2}\, g^{\#} \otimes |g|^{-1/2}$$

provided M is compact.

[In fact,

$$\frac{d}{d\varepsilon}\ \frac{1}{|g + \varepsilon h|^{1/2}}\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\ (|g + \varepsilon h|^{1/2})^{-1}\Big|_{\varepsilon=0}$$

$$= -\ (|g + \varepsilon h|^{1/2})^{-2}\Big|_{\varepsilon=0}\ \frac{d}{d\varepsilon}\ |g + \varepsilon h|^{1/2}\Big|_{\varepsilon=0}$$

$$= - (|g|^{1/2})^{-2} \ \frac{1}{2} \operatorname{tr}_g(h)\,|g|^{1/2}$$

$$= - \frac{1}{2} \operatorname{tr}_g(h)\,|g|^{-1/2}.]$$

(3) Fix $s \in S_2(M)$ and put

$$\Phi_s(g) = [s,s]_g |g|^{1/2}.$$

Then

$$D_g\Phi_s(h) = - 2[h, s*s]_g |g|^{1/2} + \frac{1}{2} [s,s]_g \operatorname{tr}_g(h)\,|g|^{1/2}.$$

Therefore

$$\frac{\delta\phi_s}{\delta g} = - 2(s*s)^{\#} \otimes |g|^{1/2} + \frac{1}{2} [s,s]_g g^{\#} \otimes |g|^{1/2}$$

provided M is compact.

[To begin with,

$$D_g\Phi_s(h) = \frac{d}{d\varepsilon} [s,s]_{g + \varepsilon h}\Big|_{\varepsilon=0} |g|^{1/2} + [s,s]_g \frac{d}{d\varepsilon} |g + \varepsilon h|^{1/2}\Big|_{\varepsilon=0}.$$

But

$$\frac{d}{d\varepsilon} (g + \varepsilon h)^{ij}\Big|_{\varepsilon=0} = - g^{ik}g^{j\ell}h_{k\ell}.$$

Accordingly,

$$\frac{d}{d\varepsilon} [s,s]_{g + \varepsilon h}\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} (g + \varepsilon h)^{ia}(g + \varepsilon h)^{jb}\Big|_{\varepsilon=0} s_{ab}s_{ij}$$

$$= - g^{ik}g^{a\ell}g^{jb}h_{k\ell}s_{ab}s_{ij} - g^{ia}g^{jk}g^{b\ell}h_{k\ell}s_{ab}s_{ij}$$

$$= - h^{ia}g^{jb}s_{ab}s_{ij} - h^{jb}g^{ia}s_{ab}s_{ij}$$

$$= - h^{ai}s_{ab}g^{bj}s_{ji} - h^{bj}s_{ba}g^{ai}s_{ij}$$

$$= - h^{ai}s_{ab}s^{b}{}_{i} - h^{bj}s_{ba}s^{a}{}_{j}$$

$$= - h^{ai}(s*s)_{ai} - h^{bj}(s*s)_{bj}$$

$$= - [h,s*s]_g - [h,s*s]_g$$

$$= - 2[h,s*s]_g.]$$

(4) Fix $s \in S_2(M)$ and put

$$\Phi_s(g) = tr_g(s)|g|^{1/2}.$$

Then

$$D_g\Phi_s(h) = - [h,s]_g|g|^{1/2} + \frac{1}{2} tr_g(s)tr_g(h)|g|^{1/2}.$$

Therefore

$$\frac{\delta\phi_g}{\delta g} = - s^{\#} \otimes |g|^{1/2} + \frac{1}{2} tr_g(s)g^{\#} \otimes |g|^{1/2}$$

provided M is compact.

[Simply note that

$$\frac{d}{d\varepsilon} tr_{g + \varepsilon h}(s)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} (g + \varepsilon h)^{ij} s_{ij} \Big|_{\varepsilon=0}$$

$$= - g^{ik} g^{j\ell} h_{k\ell} s_{ij}$$

$$= - h^{ij} s_{ij}$$

$$= - [h,s]_g .]$$

<u>Section 29</u>: <u>Variational Principles</u> Let M be a connected C^∞ manifold of dimension n, which we shall assume is orientable.

Let

$$\nabla:\underline{M} \to \text{con } TM$$

be the map that assigns to each $g \in \underline{M}$ its metric connection ∇^g -- then

$$D_g\nabla(h) = \frac{d}{d\varepsilon} \nabla^{g + \varepsilon h}\Big|_{\varepsilon=0}$$

is an element of $\mathcal{D}_2^1(M)$. Viewing $D_g\nabla(h)$ as a map $\mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M)$, we have

$$g(D_g\nabla(h)(X,Y),Z)$$

$$= \frac{1}{2} [\nabla_X h(Y,Z) + \nabla_Y h(X,Z) - \nabla_Z h(X,Y)].$$

Locally,

$$(D_g\nabla(h))^k{}_{ij} = \frac{1}{2} g^{k\ell}(\nabla_i h_{\ell j} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}),$$

which shows that $D_g\nabla(h)$ is symmetric in its covariant indices.

[Note: Let $\Gamma^k{}_{ij}(g + \varepsilon h)$ be the connection coefficients of $\nabla^{g + \varepsilon h}$ -- then

$$\frac{d}{d\varepsilon} \Gamma^k{}_{ij}(g + \varepsilon h)\Big|_{\varepsilon=0} = (D_g\nabla(h))^k{}_{ij}.]$$

<u>FACT</u> Take $h = L_X g$ $(X \in \mathcal{D}_c^1(M))$ -- then

$$D_g\nabla(L_X g) = L_X \nabla^g.$$

Example: Consider the interior derivative

$$\delta_g:\Lambda^1 M \to C^\infty(M),$$

so locally

$$\delta_g \alpha = - \nabla^i \alpha_i = - g^{ij}\nabla_j \alpha_i.$$

Then

$$\delta'_{g,h}(\alpha) = \frac{d}{d\varepsilon}\, \delta_{g\,+\,\varepsilon h}\alpha \Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\left(-(g + \varepsilon h)^{ij} \nabla_j^{g\,+\,\varepsilon h} \alpha_i \right) \Big|_{\varepsilon=0}$$

$$= h^{ij} \nabla_j \alpha_i - g^{ij} \frac{d}{d\varepsilon} \left(\nabla_j^{g\,+\,\varepsilon h} \alpha_i\right) \Big|_{\varepsilon=0}$$

$$= h^{ij} (\nabla\alpha)_{ij} - g^{ij} \frac{d}{d\varepsilon} \left(\alpha_{i,j} - \Gamma^k{}_{ij}(g + \varepsilon h)\alpha_k\right) \Big|_{\varepsilon=0}$$

$$= h^{ij} (\nabla\alpha)_{ij} + g^{ij} \frac{d}{d\varepsilon} \Gamma^k{}_{ij}(g + \varepsilon h) \Big|_{\varepsilon=0} \alpha_k .$$

But

$$g^{ij} \frac{d}{d\varepsilon} \Gamma^k{}_{ij}(g + \varepsilon h) \Big|_{\varepsilon=0} \alpha_k$$

$$= g^{ij}\, \frac{1}{2}\, g^{k\ell} (\nabla_i h_{\ell j} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) \alpha_k .$$

And

$$\left[\begin{array}{l} g^{ij} \nabla_i h_{\ell j} = (\mathrm{div}_g\, h)_\ell \\[2ex] g^{ij} \nabla_j h_{i\ell} = (\mathrm{div}_g\, h)_\ell \end{array}\right.$$

$\Rightarrow$

$$g^{ij}\, \frac{1}{2}\, g^{k\ell} (\nabla_i h_{\ell j} + \nabla_j h_{i\ell}) \alpha_k$$

$$= g^{\ell k} \alpha_k\, \frac{1}{2}\, g^{ij} (\nabla_i h_{\ell j} + \nabla_j h_{i\ell})$$

$$= \alpha^\ell (\mathrm{div}_g\, h)_\ell .$$

In addition,

$$\nabla_\ell (g^{ij} h_{ij}) = g^{ij} \nabla_\ell h_{ij}$$

$\Rightarrow$

$$g^{ij} \frac{1}{2} g^{k\ell} (-\nabla_\ell h_{ij}) \alpha_k$$

$$= - \frac{1}{2} \alpha^\ell \nabla_\ell (g^{ij} h_{ij})$$

$$= - \frac{1}{2} \alpha^\ell \partial_\ell (g^{ij} h_{ij}).$$

Therefore

$$\delta'_{g,h}(\alpha) = g[{}^0_2](h, \nabla\alpha)$$

$$+ g(\alpha, \mathrm{div}_g\, h) - \frac{1}{2} g(\alpha, d(\mathrm{tr}_g(h))).$$

[Note: On $C^\infty(M)$,

$$\Delta_g = - \delta_g \circ d.$$

Consequently,

$$\frac{d}{d\varepsilon} \Delta_{g + \varepsilon h} f \Big|_{\varepsilon=0} = - \frac{d}{d\varepsilon} \delta_{g + \varepsilon h} df \Big|_{\varepsilon=0}.]$$

Let $R^i{}_{jk\ell}(g + \varepsilon h)$ be the curvature components of $\nabla^{g + \varepsilon h}$.

<u>LEMMA</u> We have

$$\frac{d}{d\varepsilon} R^i{}_{jk\ell}(g + \varepsilon h) \Big|_{\varepsilon=0} = \nabla_k (D_g \nabla(h))^i{}_{j\ell} - \nabla_\ell (D_g \nabla(h))^i{}_{jk}.$$

[Put

$$\begin{cases} T^i{}_{j\ell} = (D_g\nabla(h))^i{}_{j\ell} = \frac{d}{d\varepsilon}\,\Gamma^i{}_{j\ell}(g + \varepsilon h)\Big|_{\varepsilon=0} \\ T^i{}_{jk} = (D_g\nabla(h))^i{}_{jk} = \frac{d}{d\varepsilon}\,\Gamma^i{}_{jk}(g + \varepsilon h)\Big|_{\varepsilon=0}. \end{cases}$$

Then

$$\frac{d}{d\varepsilon}\,R^i{}_{jk\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \partial_k T^i{}_{\ell j} - \partial_\ell T^i{}_{kj}$$

$$+ T^a{}_{\ell j}\Gamma^i{}_{ka} + \Gamma^a{}_{\ell j}T^i{}_{ka} - T^a{}_{kj}\Gamma^i{}_{\ell a} - \Gamma^a{}_{kj}T^i{}_{\ell a}.$$

On the other hand,

- $$\nabla_k T^i{}_{j\ell} = \partial_k T^i{}_{j\ell} + \Gamma^i{}_{ka}T^a{}_{j\ell}$$

  $$- \Gamma^a{}_{kj}T^i{}_{a\ell} - \Gamma^a{}_{k\ell}T^i{}_{ja}.$$

- $$- \nabla_\ell T^i{}_{jk} = - \partial_\ell T^i{}_{jk} - \Gamma^i{}_{\ell a}T^a{}_{jk}$$

  $$+ \Gamma^a{}_{\ell j}T^i{}_{ak} + \Gamma^a{}_{\ell k}T^i{}_{ja}.$$

But

$$\begin{cases} \Gamma^r{}_{st} = \Gamma^r{}_{ts} \\ T^r{}_{st} = T^r{}_{ts}, \end{cases}$$

so the equality of the two expressions is obvious.]

Therefore

$$\frac{d}{d\varepsilon} R^i{}_{jk\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} g^{ia}(h_{a\ell;j;k} + h_{ja;\ell;k} - h_{j\ell;a;k})$$

$$- \frac{1}{2} g^{ia}(h_{ak;j;\ell} + h_{ja;k;\ell} - h_{jk;a;\ell})$$

or still,

$$\frac{d}{d\varepsilon} R^i{}_{jk\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} g^{ia}(h_{aj;\ell;k} - h_{aj;k;\ell}$$

$$+ h_{a\ell;j;k} - h_{j\ell;a;k} + h_{jk;a;\ell} - h_{ak;j;\ell})$$

or still,

$$\frac{d}{d\varepsilon} R^i{}_{jk\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} g^{ia}(-R^b{}_{jk\ell}h_{ab} - R^b{}_{ak\ell}h_{jb}$$

$$+ h_{a\ell;j;k} - h_{j\ell;a;k} + h_{jk;a;\ell} - h_{ak;j;\ell}).$$

Application: We have

$$\frac{d}{d\varepsilon} \mathrm{Ric}(g + \varepsilon h)_{j\ell}\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} R^i{}_{ji\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} g^{ia}(-R^{b}{}_{ji\ell}h_{ab} - R^{b}{}_{ai\ell}h_{jb}$$

$$+ h_{a\ell;j;i} - h_{j\ell;a;i} + h_{ji;a;\ell} - h_{ai;j;\ell}).$$

Hidden within this formula (itself perfectly respectable) are certain conceptual features that are not immediately apparent.

Notation:

- Given $s \in S_2(M)$, define

$$R(s) \in S_2(M)$$

by

$$R(s)_{ij} = R^{a}{}_{i}{}^{b}{}_{j} s_{ab}.$$

- Given $u, v \in S_2(M)$, define

$$u*v \in \mathcal{D}_2^0(M)$$

by

$$(u*v)_{ij} = u_i{}^k v_{kj}.$$

Then

$$u*v + v*u \in S_2(M).$$

Definition: The <u>Lichnerowicz laplacian</u> is the map

$$\Delta_L : S_2(M) \to S_2(M)$$

defined by the prescription

$$\Delta_L s = - \Delta_{con} s + Ric*s + s*Ric - 2R(s).$$

[Note: Locally,

$$(\Delta_L s)_{ij} = - g^{ab} s_{ij;a;b} + R_i^{\ k} s_{kj} + R_j^{\ k} s_{ki} - 2R^a{}_i{}^b{}_j s_{ab}.]$$

<u>FACT</u> Suppose that $g \in \underline{M}$ is an Einstein metric:

$$\text{Ric}(g) = \frac{S(g)}{n} g.$$

Then

$$\text{div}_g \circ \Delta_L = - \Delta_{con} \circ \text{div}_g + \frac{S(g)}{n} \text{div}_g.$$

Given $\alpha \in \Lambda^1 M$, put

$$\Gamma_g \alpha = L_{\alpha^\#} g \in S_2(M),$$

thus locally,

$$(\Gamma_g \alpha)_{ij} = \alpha_{i;j} + \alpha_{j;i}.$$

<u>LEMMA</u> View Ric as a map

$$\text{Ric}: \underline{M} \to S_2(M).$$

Then

$$(D_g \text{Ric})(h) = \frac{d}{d\varepsilon} \text{Ric}(g + \varepsilon h) \Big|_{\varepsilon=0}$$

$$= \frac{1}{2} [\Delta_L h + \Gamma_{\text{div}_g h} - H_{\text{tr}_g(h)}].$$

[It is a question of comparing components. For this purpose, start with

$$\frac{1}{2} g^{ia}(h_{a\ell;j;i} - h_{j\ell;a;i} + h_{ji;a;\ell} - h_{ai;j;\ell}).$$

First

$$g^{ia} h_{a\ell;j;i} = g^{ia} h_{a\ell;i;j}$$

$$+ g^{ia}(-R^{b}{}_{\ell ij}h_{ab} - R^{b}{}_{aij}h_{\ell b}).$$

But

$$(\Gamma_{\mathrm{div}_g\, h})_{j\ell}$$

$$= (\mathrm{div}_g\, h)_{j;\ell} + (\mathrm{div}_g\, h)_{\ell;j}$$

$$= h_j{}^a{}_{;a;\ell} + h_\ell{}^a{}_{;a;j}.$$

And

$$g^{ia}h_{a\ell;i;j} + g^{ia}h_{ji;a;\ell}$$

$$= g^{ia}h_{\ell a;i;j} + g^{ai}h_{ji;a;\ell}$$

$$= h_\ell{}^i{}_{;i;j} + h_j{}^a{}_{;a;\ell}$$

$$= h_\ell{}^a{}_{;a;\ell} + h_j{}^a{}_{;a;\ell}.$$

So $\Gamma_{\mathrm{div}_g\, h}$ is accounted for. Next

$$- (\Delta_{\mathrm{con}}h)_{j\ell} = - g^{ab}h_{j\ell;a;b}$$

$$= - g^{ai}h_{j\ell;a;i}$$

$$= - g^{ia}h_{j\ell;a;i},$$

which takes care of one of the terms in $(\Delta_L h)_{j\ell}$. Finally

$$- g^{ia}h_{ai;j;\ell} = - g^{ai}h_{ai;j;\ell}$$

$$= - \mathrm{tr}_g(h)_{;j;\ell}$$

$$= (-H_{\mathrm{tr}_g(h)})_{j\ell},$$

thereby dispatching the hessian. What remains from $(\Delta_L h)_{j\ell}$ is

$$R_j^{\ k} h_{k\ell} + R_\ell^{\ k} h_{kj} - 2R^a{}_j{}^b{}_\ell h_{ab},$$

the claim being that this must equal

$$g^{ia}(-R^b{}_{ji\ell} h_{ab} - R^b{}_{ai\ell} h_{jb})$$

$$+ g^{ia}(-R^b{}_{\ell ij} h_{ab} - R^b{}_{aij} h_{\ell b}).$$

- $-g^{ia} R^b{}_{ji\ell} h_{ab}$

$$= -g^{ai} R^b{}_{ji\ell} h_{ab}$$

$$= -R^b{}_j{}^a{}_\ell h_{ab}$$

$$= -R^a{}_j{}^b{}_\ell h_{ab}.$$

- $-g^{ia} R^b{}_{\ell ij} h_{ab}$

$$= -g^{ai} R^b{}_{\ell ij} h_{ab}$$

$$= -R^b{}_\ell{}^a{}_j h_{ab}$$

$$= -R^a{}_j{}^b{}_\ell h_{ab}.$$

• $- g^{ia}R^{b}{}_{ai\ell}h_{jb}$

$= - R^{bi}{}_{i\ell}h_{bj}$

$= R^{ib}{}_{i\ell}h_{bj}$

$= R^{b}{}_{\ell}h_{bj}$

$= R_{\ell}{}^{b}h_{bj}.$

• $- g^{ia}R^{b}{}_{aij}h_{\ell b}$

$= - R^{bi}{}_{ij}h_{b\ell}$

$= R^{ib}{}_{ij}h_{b\ell}$

$= R^{b}{}_{j}h_{b\ell}$

$= R_{j}{}^{b}h_{b\ell}.$

The bookkeeping is therefore complete.]

FACT Take $h = L_X g$ $(X \in \mathcal{D}_c^1(M))$ -- then

$$(D_g \text{Ric})(L_X g) = L_X(\text{Ric}(g)).$$

Identities We have

$$\begin{cases} \text{tr}_g(\Delta_L h) = - \Delta_g \text{tr}_g(h) \\ \text{tr}_g(H_{\text{tr}_g(h)}) = \Delta_g \text{tr}_g(h) \\ \text{tr}_g(\Gamma_{\text{div}_g h}) = - 2\delta_g \text{div}_g h. \end{cases}$$

Consider now

$$\frac{d}{d\varepsilon} R^{ij}{}_{k\ell}(g + \varepsilon h)\Big|_{\varepsilon=0},$$

i.e.,

$$\frac{d}{d\varepsilon} [(g + \varepsilon h)^{jr} R^{i}{}_{rk\ell}(g + \varepsilon h)]\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} (g + \varepsilon h)^{jr}\Big|_{\varepsilon=0} R^{i}{}_{rk\ell} + g^{jr} \frac{d}{d\varepsilon} R^{i}{}_{rk\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}.$$

- $\frac{d}{d\varepsilon} (g + \varepsilon h)^{jr}\Big|_{\varepsilon=0} R^{i}{}_{rk\ell}$

$$= - g^{js}g^{rt}h_{st}R^{i}{}_{rk\ell}$$

$$= - g^{tr}R^{i}{}_{rk\ell}g^{js}h_{st}$$

$$= - R^{it}{}_{k\ell}h^{j}{}_{t}$$

$$= - R^{ia}{}_{k\ell}h^{j}{}_{a}.$$

- $g^{jr} \frac{d}{d\varepsilon} R^{i}{}_{rk\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$

$$= g^{jr} \frac{1}{2} g^{ia}(-R^{b}{}_{rk\ell}h_{ab} - R^{b}{}_{ak\ell}h_{rb}$$

$$+ h_{a\ell;r;k} - h_{r\ell;a;k} + h_{rk;a;\ell} - h_{ak;r;\ell})$$

$$= \frac{1}{2} (-R^{bj}{}_{k\ell}h^{i}{}_{b} - R^{bi}{}_{k\ell}h^{j}{}_{b})$$

$$+ \frac{1}{2} g^{jr}g^{ia}((\nabla\nabla h)_{a\ell rk} - (\nabla\nabla h)_{r\ell ak} + (\nabla\nabla h)_{rka\ell} - (\nabla\nabla h)_{akr\ell})$$

$$= \frac{1}{2}\left(R^{ia}{}_{k\ell}h^{j}{}_{a} - R^{aj}{}_{k\ell}h^{i}{}_{a}\right)$$

$$+ \frac{1}{2}\left((\nabla\nabla h)^{i}{}_{\ell}{}^{j}{}_{k} - (\nabla\nabla h)^{j}{}_{\ell}{}^{i}{}_{k}\right.$$

$$\left. + (\nabla\nabla h)^{j}{}_{k}{}^{i}{}_{\ell} - (\nabla\nabla h)^{i}{}_{k}{}^{j}{}_{\ell}\right).$$

$\Rightarrow$

$$\frac{d}{d\varepsilon} R^{ij}{}_{k\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\left(-R^{ia}{}_{k\ell}h^{j}{}_{a} - R^{aj}{}_{k\ell}h^{i}{}_{a}\right)$$

$$+ \frac{1}{2}\left((\nabla\nabla h)^{i}{}_{\ell}{}^{j}{}_{k} - (\nabla\nabla h)^{j}{}_{\ell}{}^{i}{}_{k}\right.$$

$$\left. + (\nabla\nabla h)^{j}{}_{k}{}^{i}{}_{\ell} - (\nabla\nabla h)^{i}{}_{k}{}^{j}{}_{\ell}\right).$$

Special Case

$$\frac{d}{d\varepsilon} \mathrm{Ric}(g + \varepsilon h)^{j}{}_{\ell}\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} R^{ij}{}_{i\ell}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\left(-R^{ia}{}_{i\ell}h^{j}{}_{a} - R^{aj}{}_{i\ell}h^{i}{}_{a}\right)$$

$$+ \frac{1}{2}\left((\nabla\nabla h)^{i}{}_{\ell}{}^{j}{}_{i} - (\nabla\nabla h)^{j}{}_{\ell}{}^{i}{}_{i}\right.$$

$$\left. + (\nabla\nabla h)^{j}{}_{i}{}^{i}{}_{\ell} - (\nabla\nabla h)^{i}{}_{i}{}^{j}{}_{\ell}\right).$$

So, as a corollary,

$$\frac{d}{d\varepsilon} S(g + \varepsilon h)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \mathrm{Ric}(g + \varepsilon h)^j{}_j\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} (-R^{ia}{}_{ij}h^j{}_a - R^{aj}{}_{ij}h^i{}_a)$$

$$+ \frac{1}{2} ((\nabla\nabla h)^i{}_j{}^j{}_i - (\nabla\nabla h)^j{}_j{}^i{}_i$$

$$+ (\nabla\nabla h)^j{}_i{}^i{}_j - (\nabla\nabla h)^i{}_i{}^j{}_j).$$

[Note: Each of the terms involving $\nabla\nabla h$ is a divergence. For example,

$$(\nabla\nabla h)^i{}_j{}^j{}_i = \nabla_i X^i = X^i{}_{;i},$$

where

$$X^i = (\nabla\nabla h)^i{}_j{}^j.]$$

Example: Given an open, relatively compact subset $K \subset M$, put

$$L_K(g) = \int_K S(g)\mathrm{vol}_g.$$

Then an element $g \in \underline{M}$ is said to be <u>critical</u> if $\forall$ K & $\forall$ $h \in S_{2,c}(M)$ (spt $h \subset K$),

$$\frac{d}{d\varepsilon} L_K(g + \varepsilon h)\Big|_{\varepsilon=0} = 0$$

or still,

$$\int_K \frac{d}{d\varepsilon} S(g + \varepsilon h)\Big|_{\varepsilon=0} \mathrm{vol}_g + \int_K S(g) \frac{d}{d\varepsilon} \mathrm{vol}_{g + \varepsilon h}\Big|_{\varepsilon=0} = 0.$$

But

$$\int_K S(g) \frac{d}{d\varepsilon} \mathrm{vol}_{g + \varepsilon h}\Big|_{\varepsilon=0} = \frac{1}{2} \int_K S(g)\mathrm{tr}_g(h)\mathrm{vol}_g$$

$$= \frac{1}{2} \int_K \mathrm{tr}_g(S(g)h)\mathrm{vol}_g$$

$$= \frac{1}{2} \int_K g[{}^0_2](S(g)g,h)\mathrm{vol}_g.$$

On the other hand,

$$\frac{d}{d\varepsilon} S(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} (-R^{ia}{}_{ij}h^j{}_a - R^{aj}{}_{ij}h^i{}_a + \ldots),$$

where each of the omitted terms is the divergence of a vector field whose support is compact and contained in K. But

$$\begin{bmatrix} R^{ia}{}_{ij}h^j{}_a = R^a{}_j h^j{}_a = R^{aj}h_{ja} = g[{}^0_2](\mathrm{Ric}(g),h) \\ R^{aj}{}_{ij}h^i{}_a = R^a{}_i h^i{}_a = R^{ai}h_{ia} = g[{}^0_2](\mathrm{Ric}(g),h). \end{bmatrix}$$

Therefore

$$\int_K \frac{d}{d\varepsilon} S(g + \varepsilon h)\Big|_{\varepsilon=0} \mathrm{vol}_g$$

$$= \int_K g[{}^0_2](-\mathrm{Ric}(g),h)\mathrm{vol}_g.$$

Since K is arbitrary, it follows that g is critical iff

$$g[{}^0_2](-\mathrm{Ric}(g) + \frac{1}{2} S(g)g,h) = 0$$

for all $h \in S_{2,c}(M)$, i.e., g is critical iff

$$\mathrm{Ric}(g) - \frac{1}{2} S(g)g = 0,$$

the vacuum field equation of general relativity.

<u>LEMMA</u> View S as a map

$$S:\underline{M} \to C^{\infty}(M).$$

Then

$$(D_g S)(h) = \frac{d}{d\varepsilon} S(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= - \Delta_g \mathrm{tr}_g(h) - \delta_g \mathrm{div}_g\, h - g[{}^0_2](\mathrm{Ric}(g), h).$$

[The third term has been identified above, so it is a question of explicating the other two.

<u>Ad $-\Delta_g \mathrm{tr}_g(h)$:</u> We have

$$- \Delta_g \mathrm{tr}_g(h) = - g^{ij}(H_{\mathrm{tr}_g(h)})_{ij}$$

$$= - g^{ij}\mathrm{tr}_g(h)_{;i;j}$$

$$= - g^{ij}(g^{ab}h_{ab})_{;i;j}$$

$$= g^{ij}\nabla_j\nabla_i(g^{ab}h_{ab})$$

$$= - g^{ij}g^{ab}h_{ab;i;j}$$

$$= - g^{ij}g^{ab}(\nabla\nabla h)_{abij}.$$

Compare this with

$$\frac{1}{2}\left(-(\nabla\nabla h)^{j\ i}_{\ j\ i} - (\nabla\nabla h)^{i\ j}_{\ i\ j}\right).$$

Thus

$$- (\nabla\nabla h)^{j\ i}_{\ j\ i}$$

$$= - g^{ja}g^{ib}(\nabla\nabla h)_{ajbi}$$

$$= - g^{ji}g^{ab}(\nabla\nabla h)_{ijba}$$

$$= - g^{ij}g^{ab}(\nabla\nabla h)_{ijab}$$

$$= - g^{ij}g^{ab}(\nabla\nabla h)_{abij}$$

and

$$- (\nabla\nabla h)^{i\ j}_{\ i\ j}$$

$$= - g^{ia}g^{jb}(\nabla\nabla h)_{aibj}$$

$$= - g^{ij}g^{ab}(\nabla\nabla h)_{jiba}$$

$$= - g^{ij}g^{ab}(\nabla\nabla h)_{ijab}$$

$$= - g^{ij}g^{ab}(\nabla\nabla h)_{abij}.$$

<u>Ad $- \delta_g \mathrm{div}_g\, h$:</u> We have

$$- \delta_g \mathrm{div}_g\, h = \nabla^i(\mathrm{div}_g\, h)_i$$

$$= \nabla^i g^{jk}\nabla_k h_{ji}$$

$$= \nabla^i\nabla^j h_{ji}$$

$$= \nabla^i\nabla^j h_{ij}.$$

Compare this with

$$\frac{1}{2}\,((\nabla\nabla h)^{i\ j}_{\ j\ i} + (\nabla\nabla h)^{j\ i}_{\ i\ j}).$$

Thus

$$(\nabla\nabla h)^{i\ j}_{\ j\ i}$$

$$= g^{ia}g^{jb}(\nabla\nabla h)_{ajbi}$$

$$= g^{ia}g^{jb}\nabla_i\nabla_b h_{aj}$$

$$= \nabla^a\nabla^j h_{aj}$$

$$= \nabla^i\nabla^j h_{ij}$$

and

$$(\nabla\nabla h)^{j\ i}_{\ i\ j}$$

$$= g^{ja}g^{ib}(\nabla\nabla h)_{aibj}$$

$$= g^{ja}g^{ib}\nabla_j\nabla_b h_{ai}$$

$$= \nabla^a\nabla^i h_{ai}$$

$$= \nabla^i\nabla^j h_{ij}.]$$

Example: Take M compact and let h = Ric(g) — then $\text{tr}_g(\text{Ric}(g)) = S(g)$ and

$$\text{div}_g \text{Ric}(g) = \frac{1}{2} dS(g)$$

$\Rightarrow$

$$-\delta_g \text{div}_g \text{Ric}(g) = \frac{1}{2}(-\delta_g dS(g))$$

$$= \frac{1}{2}\Delta_g S(g).$$

Therefore

$$(D_g S)(\mathrm{Ric}(g)) = -\frac{1}{2}\Delta_g S(g) - g[{}^0_2](\mathrm{Ric}(g),\mathrm{Ric}(g)).$$

FACT Take $h = L_X g$ $(X \in \mathcal{D}^1_c(M))$ -- then

$$(D_g S)(L_X g) = L_X(S(g)).$$

Remark: For later use, note that the preceding considerations imply that

$$\int_M (\Delta_g \mathrm{tr}_g(h) + \delta_g \mathrm{div}_g\, h)\mathrm{vol}_g = 0.$$

Define

$$\gamma_g : S_{2,c}(M) \to C^\infty_c(M)$$

by

$$\gamma_g(h) = -\Delta_g \mathrm{tr}_g(h) - \delta_g \mathrm{div}_g\, h - g[{}^0_2](\mathrm{Ric}(g),h)$$

and define

$$\gamma^*_g : C^\infty_c(M) \to S_{2,c}(M)$$

by

$$\gamma^*_g(f) = -(\Delta_g f)g + H_f - f\mathrm{Ric}(g).$$

Then

$$< \gamma_g(h), f > = < h, \gamma^*_g(f) >.$$

I.e.:

$$\int_M \gamma_g(h) f \mathrm{vol}_g = \int_M g[{}^0_2](h, \gamma^*_g(f))\mathrm{vol}_g.$$

Notation: Given $f \in C^\infty(M)$, put

$$(df \cdot \mathrm{Ric}(g))_i = (df)_j R^j{}_i.$$

<u>SUBLEMMA</u> Let $f \in C^\infty(M)$ -- then

$$\mathrm{div}_g\, H_f - d\Delta_g f - df \cdot \mathrm{Ric}(g) = 0.$$

[By definition,

$$(\mathrm{div}_g\, H_f)_i = \nabla^j (H_f)_{ij}$$

$$= \nabla^j \nabla_j (df)_i$$

$$= \Delta_{\mathrm{con}} (df)_i.$$

But, in view of the Weitzenboeck formula,

$$\Delta_{\mathrm{con}} (df)_i = (\Delta_g df)_i + (df)_j R^j{}_i.$$

And

$$(\Delta_g df)_i = (-(d \circ \delta_g + \delta_g \circ d) df)_i$$

$$= (d \circ -(\delta_g \circ d) f)_i$$

$$= (d\Delta_g f)_i.]$$

Suppose that $\gamma_g^*(f) = 0$, thus

$$-(\Delta_g f) g + H_f - f\mathrm{Ric}(g) = 0$$

and so, upon application of div_g,

$$-d\Delta_g f + \mathrm{div}_g\, H_f - df \cdot \mathrm{Ric}(g) - f\mathrm{div}_g\, \mathrm{Ric}(g) = 0.$$

Therefore

$$f\operatorname{div}_g \operatorname{Ric}(g) = 0$$

or still,

$$\frac{1}{2} f dS(g) = 0.$$

Consequently, if f is never zero, then $dS(g) = 0$, which implies that $S(g)$ is a constant, say $S(g) = \lambda$.

Example: Take M compact and $n > 1$. Fix $\varphi \in C_d^\infty(M) : \varphi > 0$. Given $g \in \underline{M}_{0,n}$ (the set of riemannian structures on M), put

$$L_\varphi(g) = \int_M S(g)\varphi.$$

Then g is stationary for L_φ, i.e.,

$$\frac{d}{d\varepsilon} L_\varphi(g + \varepsilon h)\Big|_{\varepsilon=0} = 0$$

for all $h \in S_2(M)$ iff $\operatorname{Ric}(g) = 0$ and $\varphi = C|g|^{1/2}$ (C a positive constant).

[Fix $f > 0$ in $C^\infty(M) : \varphi = f|g|^{1/2}$ -- then

$$\frac{d}{d\varepsilon} L_\varphi(g + \varepsilon h)\Big|_{\varepsilon=0} = \int_M \frac{d}{d\varepsilon} S(g + \varepsilon h)\Big|_{\varepsilon=0} f|g|^{1/2}$$

$$= \int_M \gamma_g(h) f \operatorname{vol}_g$$

$$= \int_M g[{}^0_2](h, \gamma_g^*(f)) \operatorname{vol}_g.$$

Accordingly, g is stationary for L_φ iff $\gamma_g^*(f) = 0$. Since

$$\gamma_g^*(f) = -(\Delta_g f)g + H_f - f\operatorname{Ric}(g),$$

the conditions

$$\begin{cases} f = C \\ \mathrm{Ric}(g) = 0 \end{cases}$$

are obviously sufficient. To see that they are also necessary, note that

$$0 = \gamma_g^*(f)$$

$$\Rightarrow$$

$$0 = \mathrm{tr}_g(\gamma_g^*(h))$$

$$= -(\Delta_g f)\mathrm{tr}_g(g) + \mathrm{tr}_g(H_f) - f\mathrm{tr}_g(\mathrm{Ric}(g))$$

$$= (1-n)\Delta_g f - f\lambda$$

$$\Rightarrow$$

$$\lambda f = (1-n)\Delta_g f$$

$$\Rightarrow$$

$$\lambda \int_M f\mathrm{vol}_g = (1-n)\int_M \Delta_g f\mathrm{vol}_g$$

$$= (1-n)\int_M f(\Delta_g 1)\mathrm{vol}_g$$

$$= 0.$$

But

$$\Delta_g f = \frac{\lambda}{1-n} f$$

$$\Rightarrow$$

$$\frac{\lambda}{1-n} \leq 0 \Rightarrow \lambda \geq 0.$$

If $\lambda > 0$, then $\int_M f\mathrm{vol}_g = 0$, contradicting $f > 0$. Therefore $\lambda = 0$, hence f is harmonic:

$$\Delta_g f = 0 \Rightarrow f = C > 0.$$

And

$$0 = \gamma_g^*(C) = -\,C\mathrm{Ric}(g)$$

$$\Rightarrow \mathrm{Ric}(g) = 0.]$$

[Note: There may be no g at which L_φ is stationary.]

<u>LEMMA</u> View Ein as a map

$$\mathrm{Ein}:\underline{M} \rightarrow S_2(M).$$

Then

$$(D_g\mathrm{Ein})(h) = \frac{d}{d\varepsilon}\,\mathrm{Ein}(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\,[\Delta_L\bar{h} + \Gamma_{\mathrm{div}_g\,\bar{h}} + (\delta_g\mathrm{div}_g\,\bar{h})g]$$

$$+ \frac{1}{2}\,[g[{}^0_2](\mathrm{Ric}(g),\bar{h})g - S(g)\bar{h}].$$

[Note: Here

$$\bar{h} = h - \frac{1}{2}\,\mathrm{tr}_g(h)g,$$

thus locally,

$$\bar{h}_{ij} = h_{ij} - \frac{1}{2}\,h^a{}_a g_{ij}.]$$

<u>FACT</u> Take $h = L_X g$ $(X \in \mathcal{D}^1_c(M))$ -- then

$$(D_g\mathrm{Ein})(L_X g) = L_X(\mathrm{Ein}(g)).$$

It is sometimes necessary to consider second order issues, the downside being that the computations can be involved.

Example: Put

$$\begin{cases} \mathrm{div}'_{g,h} = \dfrac{d}{d\varepsilon}\,\mathrm{div}_{g+\varepsilon h}\Big|_{\varepsilon=0} \\[2ex] \mathrm{div}''_{g,h} = \dfrac{d^2}{d\varepsilon^2}\,\mathrm{div}_{g+\varepsilon h}\Big|_{\varepsilon=0} . \end{cases}$$

- Differentiate the identity

$$\mathrm{div}_{g+\varepsilon h}\mathrm{Ein}(g+\varepsilon h) = 0$$

once w.r.t. ε and then set $\varepsilon = 0$ to get

$$\mathrm{div}'_{g,h}\mathrm{Ein}(g) + \mathrm{div}_g(D_g\mathrm{Ein})(h) = 0.$$

Therefore

$$\mathrm{div}_g(D_g\mathrm{Ein})(h) = 0$$

if $\mathrm{Ein}(g) = 0$.

- Differentiate the identity

$$\mathrm{div}_{g+\varepsilon h}\mathrm{Ein}(g+\varepsilon h) = 0$$

twice w.r.t. ε and then set $\varepsilon = 0$ to get

$$\mathrm{div}''_{g,h}\mathrm{Ein}(g) + 2\mathrm{div}'_{g,h}(D_g\mathrm{Ein})(h) + \mathrm{div}_g(D_g^2\mathrm{Ein})(h,h) = 0.$$

Therefore

$$\mathrm{div}_g(D_g^2\mathrm{Ein})(h,h) = 0$$

if

$$\mathrm{Ein}(g) = 0 \ \& \ (D_g\mathrm{Ein})(h) = 0.$$

[Note: Strictly speaking, $\text{div}''_{g,h}$ should be denoted by $\text{div}''_{g,(h,h)}$.]

Observation: Let X be an infinitesimal isometry per g and suppose that $s \in S_2(M)$ is divergence free (i.e., $\text{div}_g\, s = 0$). Define $X \cdot s$ by

$$(X\cdot s)_i = X^j s_{ij}.$$

Then

$$\delta_g X\cdot s = 0.$$

[In fact,

$$\begin{aligned}\delta_g X\cdot s &= -\nabla^i (X\cdot s)_i \\ &= -\nabla^i (X^j s_{ij}) \\ &= -(\nabla^i X^j) s_{ij} - X^j \nabla^i s_{ij} \\ &= -(\nabla^i X^j) s_{ij}.\end{aligned}$$

But

$$\nabla_i X_j + \nabla_j X_i = 0$$

$$\Rightarrow$$

$$\nabla^i X^j + \nabla^j X^i = 0$$

$$\Rightarrow$$

$$-(\nabla^i X^j) s_{ij} = (\nabla^j X^i) s_{ij} = (\nabla^i X^j) s_{ij}.]$$

Application: Suppose that

$$\text{Ein}(g) = 0 \;\&\; (D_g \text{Ein})(h) = 0.$$

Then for any infinitesimal isometry X per g,

$$\delta_g X\cdot (D_g^2 \text{Ein})(h,h) = 0.$$

<u>LEMMA</u> Suppose that $\text{Ric}(g) = 0$ -- then $\forall\ h \in S_{2,c}(M)$,

$$\int_M (D_g^2 S)(h,h)\text{vol}_g$$

$$= -\frac{1}{2}\int_M g[{}^0_2](h,\Delta_L h)\text{vol}_g$$

$$-\frac{1}{2}\int_M g[{}^0_1](d\text{tr}_g(h),d\text{tr}_g(h))\text{vol}_g$$

$$+\int_M g[{}^0_1](\text{div}_g\, h,\text{div}_g\, h)\text{vol}_g.$$

[We have

$$\int_M (D_g^2 S)(h,h)\text{vol}_g$$

$$= \int_M \frac{d}{d\varepsilon}(D_{g + \varepsilon h}S)(h)\Big|_{\varepsilon=0}\text{vol}_g.$$

But

$$\frac{d}{d\varepsilon}[(D_{g + \varepsilon h}S)(h)\text{vol}_{g + \varepsilon h}]\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}(D_{g + \varepsilon h}S)(h)\Big|_{\varepsilon=0}\text{vol}_g + (D_g S)(h)\frac{d}{d\varepsilon}\text{vol}_{g + \varepsilon h}\Big|_{\varepsilon=0}.$$

Therefore

$$\int_M (D_g^2 S)(h,h)\text{vol}_g$$

$$= D_g[\int_M (D_g S)(h)\text{vol}_g](h) - \int_M (D_g S)(h)(D_g\text{vol})(h)$$

$$= D_g[\int_M (-\Delta_g\text{tr}_g(h) - \delta_g\text{div}_g\, h - g[{}^0_2](\text{Ric}(g),h))\text{vol}_g](h)$$

$$- \int_M (D_g S)(h)(D_g\text{vol})(h)$$

$$= - D_g[\int_M g[{}^0_2](\mathrm{Ric}(g),h)\mathrm{vol}_g](h)$$

$$- \int_M (D_g S)(h)(D_g \mathrm{vol})(h)$$

$$= - \int_M g[{}^0_2](h,(D_g\mathrm{Ric})(h))\mathrm{vol}_g$$

$$- \int_M (-\Delta_g \mathrm{tr}_g(h) - \delta_g \mathrm{div}_g\, h)\, \frac{1}{2}\, \mathrm{tr}_g(h)\mathrm{vol}_g$$

$$= - \frac{1}{2} \int_M g[{}^0_2](h, \Delta_L h + \Gamma_{\mathrm{div}_g\, h} - H_{\mathrm{tr}_g(h)})\mathrm{vol}_g$$

$$+ \frac{1}{2} \int_M (\Delta_g \mathrm{tr}_g(h) + \delta_g \mathrm{div}_g\, h)\mathrm{tr}_g(h)\mathrm{vol}_g.$$

The term

$$- \frac{1}{2} \int_M g[{}^0_2](h,\Delta_L h)\mathrm{vol}_g$$

requires no further attention, hence can be set aside. Next

$$g[{}^0_2](h,\Gamma_{\mathrm{div}_g\, h})$$

$$= h^{ij}(\Gamma_{\mathrm{div}_g\, h})_{ij}$$

$$= h^{ij}((\mathrm{div}_g\, h)_{i;j} + (\mathrm{div}_g\, h)_{j;i})$$

$\Rightarrow$

$$- \frac{1}{2} \int_M g[{}^0_2](h,\Gamma_{\mathrm{div}_g\, h})\mathrm{vol}_g$$

$$= - \frac{1}{2}\,(-2) \int_M (\mathrm{div}_g\, h)_i \nabla_j h^{ij}\mathrm{vol}_g$$

$$= \int_M (\text{div}_g\ h)_i (\text{div}_g\ h)^i \text{vol}_g$$

$$= \int_M g[{}^0_1](\text{div}_g\ h, \text{div}_g\ h)\text{vol}_g.$$

And

$$g[{}^0_2](h, H_{\text{tr}_g(h)})$$

$$= h^{ij}(H_{\text{tr}_g(h)})_{ij}$$

$$= h^{ij} g^{ab} h_{ab;i;j}$$

$$= h^{ij}(\nabla_i \text{tr}_g(h))_{;j}$$

$\Rightarrow$

$$\frac{1}{2}\int_M g[{}^0_2](h, H_{\text{tr}_g(h)})\text{vol}_g$$

$$= \frac{1}{2}\int_M h^{ij}(\nabla_i \text{tr}_g(h))_{;j}$$

$$= -\frac{1}{2}\int_M (\nabla_j h^{ij})\nabla_i \text{tr}_g(h)\text{vol}_g + \frac{1}{2}\int_M \nabla_j(h^{ij}\nabla_i \text{tr}_g(h))\text{vol}_g$$

$$= -\frac{1}{2}\int_M (\nabla_j h^{ij})\nabla_i \text{tr}_g(h)\text{vol}_g$$

$$= -\frac{1}{2}\int_M g[{}^0_1](\text{div}_g\ h, d\text{tr}_g(h))\text{vol}_g.$$

On the other hand,

$$\frac{1}{2}\int_M (\delta_g \text{div}_g\ h)\text{tr}_g(h)\text{vol}_g$$

$$= \frac{1}{2} \int_M g[{}^0_1](\mathrm{div}_g\, h, d\mathrm{tr}_g(h))\mathrm{vol}_g.$$

Thus these terms cancel out, leaving

$$\frac{1}{2} \int_M (\Delta_g \mathrm{tr}_g(h))\mathrm{tr}_g(h)\mathrm{vol}_g$$

or still,

$$- \frac{1}{2} \int_M g[{}^1_0](\mathrm{grad}_g\, \mathrm{tr}_g(h), \mathrm{grad}_g\, \mathrm{tr}_g(h))\mathrm{vol}_g$$

or still,

$$- \frac{1}{2} \int_M g[{}^0_1](d\mathrm{tr}_g(h), d\mathrm{tr}_g(h))\mathrm{vol}_g,$$

as desired.]

Example: Take M compact and $n > 2$. Put

$$L(g) = \int_M S(g)\mathrm{vol}_g.$$

Then

$$(D_g^2 L)(h,h) = \int_M (D_g^2 S)(h,h)\mathrm{vol}_g$$

$$+ 2 \int_M (D_g S)(h)(D_g \mathrm{vol})(h)$$

$$+ \int_M S(g)(D_g^2 \mathrm{vol})(h,h).$$

Suppose now that g is a critical point: $\mathrm{Ein}(g) = 0 \Rightarrow \mathrm{Ric}(g) = 0$ & $S(g) = 0$, thus

$$(D_g^2 L)(h,h) = - \frac{1}{2} \int_M g[{}^0_2](h, \Delta_L h)\mathrm{vol}_g$$

$$- \frac{1}{2} \int_M g[{}^0_1](d\mathrm{tr}_g(h), d\mathrm{tr}_g(h))\mathrm{vol}_g + \int_M g[{}^0_1](\mathrm{div}_g\, h, \mathrm{div}_g\, h)\mathrm{vol}_g$$

$$+ 2 \int_M (-\Delta_g \mathrm{tr}_g(h) - \delta_g \mathrm{div}_g\, h) \frac{1}{2} \mathrm{tr}_g(h) \mathrm{vol}_g$$

$$= -\frac{1}{2} \int_M g[{}^0_2](h, \Delta_L h) \mathrm{vol}_g$$

$$+ \frac{1}{2} \int_M g[{}^0_1](d\mathrm{tr}_g(h), d\mathrm{tr}_g(h)) \mathrm{vol}_g + \int_M g[{}^0_1](\mathrm{div}_g\, h, \mathrm{div}_g\, h) \mathrm{vol}_g$$

$$- \int_M g[{}^0_1](\mathrm{div}_g\, h, d\mathrm{tr}_g(h)) \mathrm{vol}_g.$$

<u>LEMMA</u> We have

$$(D_g^2 S)(h,h) = -\frac{1}{2} g[{}^0_3](\nabla h, \nabla h) + 2g[{}^0_2](\mathrm{Ric}(g), h*h)$$

$$- \frac{1}{2} g[{}^0_1](d\mathrm{tr}_g(h), d\mathrm{tr}_g(h)) + \nabla h * \nabla h$$

$$+ 2g[{}^0_2](h, H_{\mathrm{tr}_g(h)}) + 2g[{}^0_1](\mathrm{div}_g\, h, d\mathrm{tr}_g(h))$$

$$+ \Delta_g\, g[{}^0_2](h,h) + 2\delta_g \mathrm{div}_g(h*h).$$

[Note: Here $\nabla h * \nabla h$ stands for the combination

$$\nabla_k h^{ij} \nabla_i h_j{}^k.]$$

<u>Reality Check</u> This amounts to calculating the integral

$$\int_M (D_g^2 S)(h,h) \mathrm{vol}_g$$

directly from the expression for $(D_g^2 S)(h,h)$ provided by the lemma and comparing the result with the formula obtained earlier (which was derived under the assumption that $\mathrm{Ric}(g) = 0$).

[Note: If Ric(g) = 0, then

$$\Delta_L h = -\Delta_{con} h - 2R(s).$$

Locally,

$$(\Delta_L h)_{ij} = -g^{ab} h_{ij;a;b} - 2R^{a\ b}_{\ i\ j} h_{ab}.]$$

● By definition,

$$g[^0_3](\nabla h, \nabla h) = (\nabla h)^{ijk} (\nabla h)_{ijk}$$

$$= \nabla^k h^{ij} \nabla_k h_{ij}.$$

Therefore

$$-\frac{1}{2} \int_M g[^0_3](\nabla h, \nabla h) \mathrm{vol}_g$$

$$= -\frac{1}{2} \int_M \nabla^k h^{ij} \nabla_k h_{ij} \mathrm{vol}_g$$

$$= -\frac{1}{2} \int_M [\nabla^k (h^{ij} \nabla_k h_{ij}) - h^{ij} \nabla^k \nabla_k h_{ij}] \mathrm{vol}_g$$

$$= \frac{1}{2} \int_M h^{ij} \nabla^k \nabla_k h_{ij} \mathrm{vol}_g$$

$$= \frac{1}{2} \int_M g[^0_2](h, \Delta_{con} h) \mathrm{vol}_g.$$

● Write

$$\nabla_k (h^{ij} \nabla_i h_j^{\ k}) = \nabla_k h^{ij} \nabla_i h_j^{\ k} + h^{ij} \nabla_k \nabla_i h_j^{\ k}$$

$\Rightarrow$

$$\int_M \nabla h * \nabla h \mathrm{vol}_g = -\int_M h^{ij} \nabla_k \nabla_i h_j^{\ k} \mathrm{vol}_g$$

$$= - \int_M h^{ij} g^k{}_a \nabla_k \nabla_i h_{ja} \mathrm{vol}_g$$

$$= - \int_M h^{ij} g^k{}_a [\nabla_i \nabla_k h_{ja} + h_{\ell a} R^{\ell}{}_{jik} + h_{j\ell} R^{\ell}{}_{aik}] \mathrm{vol}_g$$

$$= - \int_M h^{ij} \nabla_i \nabla_k h_j{}^k \mathrm{vol}_g$$

$$- \int_M h^{ij} R^{\ell}{}_{jik} h_{\ell}{}^k \mathrm{vol}_g - \int_M h^{ij} h_{j\ell} R^{\ell k}{}_{ik} \mathrm{vol}_g$$

$$= - \int_M h^{ij} \nabla_i \nabla_k h_j{}^k \mathrm{vol}_g$$

$$- \int_M h^{ij} R_j{}^{\ell}{}_{ki} h_{\ell}{}^k \mathrm{vol}_g - \int_M h^{ij} h_{j\ell} R^{k\ell}{}_{ki} \mathrm{vol}_g .$$

Both of these integrals will contribute.

$$\longrightarrow \; - \int_M h^{ij} \nabla_i \nabla_k h_j{}^k \mathrm{vol}_g$$

$$= - \int_M h^{ij} \nabla_i (\mathrm{div}_g\ h)_j \mathrm{vol}_g$$

$$= - \int_M [\nabla_i (h^{ij} (\mathrm{div}_g\ h)_j) - \nabla_i h^{ij} (\mathrm{div}_g\ h)_j] \mathrm{vol}_g$$

$$= \int_M \nabla_i h^{ij} (\mathrm{div}_g\ h)_j \mathrm{vol}_g$$

$$= \int_M (\mathrm{div}_g\ h)^j (\mathrm{div}_g\ h)_j \mathrm{vol}_g$$

$$= \int_M g[{}^0_1] (\mathrm{div}_g\ h, \mathrm{div}_g\ h) \mathrm{vol}_g .$$

$$\longrightarrow \; - \int_M h^{ij} R_j{}^{\ell}{}_{ki} h_{\ell}{}^{k} \mathrm{vol}_g$$

$$= - \int_M h^{ij} - R^{\ell}{}_j{}^{k}{}_i h_{\ell k} \mathrm{vol}_g$$

$$= \int_M h^{ij} R(h)_{ij} \mathrm{vol}_g$$

$$= \int_M g[{}^0_2](h, R(h)) \mathrm{vol}_g .$$

$$\longrightarrow \; - \int_M h^{ij} h_{j\ell} R^{k\ell}{}_{ki} \mathrm{vol}_g$$

$$= - \int_M R^{\ell}{}_i h^{ij} h_{j\ell} \mathrm{vol}_g$$

$$= - \int_M R^{\ell i} h_{j\ell} h_i{}^j \mathrm{vol}_g$$

$$= - \int_M R^{\ell i} h_{\ell j} h^j{}_i \mathrm{vol}_g$$

$$= - \int_M R^{\ell i} (h * h)_{\ell i} \mathrm{vol}_g$$

$$= - \int_M g[{}^0_2](\mathrm{Ric}(g), h * h) \mathrm{vol}_g .$$

• As has been already established,

$$2 \int_M g[{}^0_2](h, H_{\mathrm{tr}_g(h)}) \mathrm{vol}_g$$

$$= - 2 \int_M g[{}^0_1](\mathrm{div}_g\, h, d\mathrm{tr}_g(h)) \mathrm{vol}_g ,$$

thereby cancelling the contribution coming from

$$2 g[{}^0_1](\mathrm{div}_g\, h, d\mathrm{tr}_g(h)) .$$

• Both

$$\Delta_g g[{}^0_2](h,h)$$

and

$$2\delta_g \mathrm{div}_g(h*h)$$

integrate to zero.

Summary: We have

$$\int_M (D_g^2 S)(h,h)\,\mathrm{vol}_g$$

$$= \frac{1}{2}\int_M g[{}^0_2](h,\Delta_{\mathrm{con}} h)\,\mathrm{vol}_g$$

$$+ \int_M g[{}^0_2](h,R(h))\,\mathrm{vol}_g$$

$$+ \int_M g[{}^0_2](\mathrm{Ric}(g),h*h)\,\mathrm{vol}_g$$

$$- \frac{1}{2}\int_M g[{}^0_1](d\mathrm{tr}_g(h),d\mathrm{tr}_g(h))\,\mathrm{vol}_g$$

$$+ \int_M g[{}^0_1](\mathrm{div}_g\, h,\mathrm{div}_g\, h)\,\mathrm{vol}_g,$$

which reduces to the formula established previously when $\mathrm{Ric}(g) = 0$.

<u>Section 30</u>: <u>Splittings</u> Let M be a connected C^∞ manifold of dimension n. Assume: M is compact and orientable and $n > 1$.

Equip $\mathcal{D}_q^p(M)$ with the C^∞ topology -- then $\mathcal{D}_q^p(M)$ is a Fréchet space. In particular: $\mathcal{D}_2^0(M)$ is a Fréchet space, as is $S_2(M)$ (being a closed subspace of $\mathcal{D}_2^0(M)$).

Abbreviate $\underline{M}_{0,n}$ to $\underline{M}_0$ -- then $\underline{M}_0$ is open in $S_2(M)$, hence is a Fréchet manifold modeled on $S_2(M)$.

Given $g \in \underline{M}_0$, define

$$\alpha_g : \mathcal{D}^1(M) \to S_2(M)$$

by

$$\alpha_g(X) = L_X g$$

and define

$$\alpha_g^* : S_2(M) \to \mathcal{D}^1(M)$$

by

$$\alpha_g^*(s) = - 2g^\# \text{div}_g\, s.$$

Then

$$< \alpha_g(X), s > = \int_M g[{}^0_2](L_X g, s)\text{vol}_g$$

$$= - 2 \int_M g[{}^0_1](g^\flat X, \text{div}_g\, s)\text{vol}_g$$

$$= - 2 \int_M g[{}^1_0](X, g^\# \text{div}_g\, s)\text{vol}_g$$

$$= < X, \alpha_g^*(s) >.$$

<u>LEMMA</u> $\forall\, x \in M$ & $\forall\, \xi \in T_x^* M - \{0\}$, the symbol

$$\sigma_\xi(\alpha_g;x) : T_x M \to \mathrm{Sym}^2\, T_x^* M$$

of α_g is injective.

[Given $V \in T_x M$, we have

$$\sigma_\xi(\alpha_g;x)(V) = \xi \otimes g_x^\flat V + g_x^\flat V \otimes \xi.$$

So, if $\sigma_\xi(\alpha_g;x)(V) = 0$, then $\forall\, i$ & $\forall\, j$,

$$\xi_i V_j + V_i \xi_j = 0$$

$\Rightarrow$

$$\xi^i V_i = g_x^{ij} \xi_j V_i$$

$$= - g_x^{ij} \xi_i V_j$$

$$= - g_x^{ij} \xi_j V_i$$

$\Rightarrow$

$$\xi^i V_i = 0$$

$\Rightarrow$

$$\xi^i \xi_i V_j + \xi^i V_i \xi_j = \xi^i \xi_i V_j = 0$$

$\Rightarrow$

$$V_j = 0.$$

I.e.: $V = 0$, hence $\sigma_\xi(\alpha_g;x)$ is injective.]

By elliptic theory, it then follows that there is an orthogonal decomposition

$$S_2(M) = \text{Ran } \alpha_g \oplus \text{Ker } \alpha_g^*,$$

where both Ran α_g and Ker α_g^* are closed subspaces of $S_2(M)$.

Consequently, every $s \in S_2(M)$ can be split into two pieces:

$$s = s_0 + L_X g.$$

Here $\text{div}_g\, s_0 = 0$ and $L_X g$ is unique in X up to infinitesimal isometries.

Notation: Given $X \in \mathcal{D}^1(M)$, put

$$(X \cdot \text{Ric}(g))_i = X_j R^j{}_i.$$

<u>SUBLEMMA</u> Let $X \in \mathcal{D}^1(M)$ — then

$$(\text{div}_g L_X g)_i = (\Delta_g g^\flat X)_i - (d\delta_g g^\flat X)_i + 2(X \cdot \text{Ric}(g))_i.$$

[By definition,

$$(\text{div}_g L_X g)_i = \nabla^j (X_{i;j} + X_{j;i})$$

$$= \nabla^j \nabla_j X_i + \nabla^j \nabla_i X_j.$$

And (Weitzenboeck)

$$\nabla^j \nabla_j X_i = (\Delta_g g^\flat X)_i + X_j R^j{}_i.$$

Turning to $\nabla^j \nabla_i X_j$, write

$$\nabla^j\nabla_i X_j = g^{jk}\nabla_k\nabla_i X_j$$

$$= g^{jk}X_{j;i;k}$$

$$= g^{jk}(X_{j;k;i} + X_\ell R^\ell{}_{jik})$$

$$= g^{jk}\nabla_i\nabla_k X_j + X_\ell g^{jk}R^\ell{}_{jik}$$

$$= \nabla_i\nabla^j X_j + X_\ell R^{\ell k}{}_{ik}$$

$$= \nabla_i(-\delta_g g^\flat X) + X_\ell R^{k\ell}{}_{ki}$$

$$= -(d\delta_g g^\flat X)_i + X_\ell R^\ell{}_i,$$

from which the result.]

[Note: This computation does not use the assumption that M is compact and is valid for any $g \in \underline{M}$.]

Application: Suppose that $\text{Ric}(g) = 0$ -- then

$$\Delta_g \text{tr}_g L_X g + \delta_g \text{div}_g L_X g = 0.$$

[Consider $\Delta_g \text{tr}_g L_X g$:

$$\text{tr}_g L_X g = g^{ij}(X_{i;j} + X_{j;i})$$

$$= 2\nabla^i X_i$$

$$= -2\delta_g g^\flat X$$

$\Rightarrow$

$$\Delta_g \mathrm{tr}_g L_X g = - 2\Delta_g \delta_g g^\flat X$$

$$= - 2\delta_g \Delta_g g^\flat X$$

$$= 2\nabla^i (\Delta_g g^\flat X)_i .$$

Consider $\delta_g \mathrm{div}_g L_X g$:

$$\delta_g \mathrm{div}_g L_X g$$

$$= - \nabla^i (\mathrm{div}_g L_X g)_i$$

$$= - \nabla^i (\Delta_g g^\flat X)_i + \nabla^i (d\delta_g g^\flat X)_i .$$

But

$$\nabla^i (d\delta_g g^\flat X)_i = \nabla^i \nabla_i \delta_g g^\flat X$$

$$= \Delta_{con} \delta_g g^\flat X$$

$$= \Delta_g \delta_g g^\flat X$$

$$= \delta_g \Delta_g g^\flat X$$

$$= - \nabla^i (\Delta_g g^\flat X)_i .]$$

Rappel: Suppose that $\mathrm{Ric}(g) = 0$ — then $\forall\ h \in S_2(M)$,

$$\int_M (D_g^2 S)(h,h)\mathrm{vol}_g$$

$$= - \int_M g[{}^0_2](h, (D_g \mathrm{Ric})(h))\mathrm{vol}_g$$

$$+ \frac{1}{2} \int_M (\Delta_g \mathrm{tr}_g(h) + \delta_g \mathrm{div}_g\, h) \mathrm{tr}_g(h) \mathrm{vol}_g.$$

Example: Suppose that $\mathrm{Ric}(g) = 0$. Let $h \in S_2(M) : h = h_0 + L_X g$ $(\mathrm{div}_g\, h_0 = 0)$ — then

$$\begin{aligned}
(D_g \mathrm{Ric})(h) &= (D_g \mathrm{Ric})(h_0 + L_X g) \\
&= (D_g \mathrm{Ric})(h_0) + (D_g \mathrm{Ric})(L_X g) \\
&= (D_g \mathrm{Ric})(h_0) + L_X(\mathrm{Ric}(g)) \\
&= (D_g \mathrm{Ric})(h_0) \\
&= \frac{1}{2} [\Delta_L h_0 + \Gamma_{\mathrm{div}_g\, h_0} - H_{\mathrm{tr}_g(h_0)}] \\
&= \frac{1}{2} [\Delta_L h_0 - H_{\mathrm{tr}_g(h_0)}].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_M (D_g^2 S)(h,h) \mathrm{vol}_g \\
&= -\frac{1}{2} \int_M g[{}^0_2](h, \Delta_L h_0 - H_{\mathrm{tr}_g(h_0)}) \mathrm{vol}_g \\
&\quad + \frac{1}{2} \int_M (\Delta_g \mathrm{tr}_g(h_0) + \Delta_g \mathrm{tr}_g L_X g \\
&\qquad + \delta_g \mathrm{div}_g\, h_0 + \delta_g \mathrm{div}_g L_X g) \mathrm{tr}_g(h) \mathrm{vol}_g \\
&= -\frac{1}{2} \int_M g[{}^0_2](h, \Delta_L h_0 - H_{\mathrm{tr}_g(h_0)}) \mathrm{vol}_g
\end{aligned}$$

$$+ \frac{1}{2} \int_M (\Delta_g \mathrm{tr}_g(h_0))\mathrm{tr}_g(h)\mathrm{vol}_g$$

$$= - \frac{1}{2} \int_M g[{}^0_2](h, \Delta_L h_0)\mathrm{vol}_g$$

$$- \frac{1}{2} \int_M g[{}^0_1](\mathrm{div}_g\, h, d\mathrm{tr}_g(h_0))\mathrm{vol}_g$$

$$- \frac{1}{2} \int_M g[{}^0_1](d\mathrm{tr}_g(h), d\mathrm{tr}_g(h_0))\mathrm{vol}_g.$$

There are some additional simplifications that can be made. First, since $\mathrm{Ric}(g) = 0$,

$$\mathrm{div}_g \circ \Delta_L = - \Delta_{con} \circ \mathrm{div}_g$$

$$\Rightarrow$$

$$\Delta_L h_0 \in \mathrm{Ker}\ \alpha_g^*$$

$$\Rightarrow$$

$$- \frac{1}{2} \int_M g[{}^0_2](h, \Delta_L h_0)\mathrm{vol}_g$$

$$= - \frac{1}{2} \int_M g[{}^0_2](h_0, \Delta_L h_0).$$

Next

$$\mathrm{div}_g\, h = \mathrm{div}_g L_X g$$

$$= \Delta_g g^\flat X - d\delta_g g^\flat X$$

$$= - (d \circ \delta_g + \delta_g \circ d) g^\flat X - d\delta_g g^\flat X$$

$$= - (2d\delta_g g^\flat X + \delta_g d g^\flat X)$$

$\Rightarrow$

$$\operatorname{div}_g h + d\operatorname{tr}_g(h)$$

$$= -2d\delta_g g^\flat X - \delta_g dg^\flat X + d\operatorname{tr}_g(h_0) + d\operatorname{tr}_g(L_X g)$$

$$= -4d\delta_g g^\flat X - \delta_g dg^\flat X + d\operatorname{tr}_g(h_0).$$

Therefore

$$-\frac{1}{2}\int_M g[{}^0_1](\operatorname{div}_g h + d\operatorname{tr}_g(h), d\operatorname{tr}_g(h_0))\operatorname{vol}_g$$

$$= -\frac{1}{2}\int_M g[{}^0_1](d\operatorname{tr}_g(h_0), d\operatorname{tr}_g(h_0))\operatorname{vol}_g$$

$$+ 2\int_M g[{}^0_1](d\delta_g g^\flat X, d\operatorname{tr}_g(h_0))\operatorname{vol}_g$$

$$+ \frac{1}{2}\int_M g[{}^0_1](\delta_g dg^\flat X, d\operatorname{tr}_g(h_0))\operatorname{vol}_g.$$

Finally

$$\int_M g[{}^0_1](\delta_g dg^\flat X, d\operatorname{tr}_g(h_0))\operatorname{vol}_g$$

$$= \int_M g[{}^0_2](dg^\flat X, d^2\operatorname{tr}_g(h_0))\operatorname{vol}_g$$

$$= 0.$$

So, in conclusion,

$$\int_M (D_g^2 S)(h,h)\operatorname{vol}_g$$

$$= -\frac{1}{2}\int_M g[{}^0_2](h_0, \Delta_L h_0)\,\mathrm{vol}_g$$

$$-\frac{1}{2}\int_M g[{}^0_1](d\,\mathrm{tr}_g(h_0), d\,\mathrm{tr}_g(h_0))\,\mathrm{vol}_g$$

$$+\ 2\int_M g[{}^0_1](d\delta_g g^\flat X, d\,\mathrm{tr}_g(h_0))\,\mathrm{vol}_g.$$

Example: If $g \in \underline{M}_0$ is a critical point for

$$L(g) = \int_M S(g)\,\mathrm{vol}_g \quad (n > 2),$$

then

$$(D_g^2 L)(h,h) = -\frac{1}{2}\int_M g[{}^0_2](h_0, \Delta_L h_0)\,\mathrm{vol}_g$$

$$+\ \frac{1}{2}\int_M g[{}^0_1](d\,\mathrm{tr}_g(h_0), d\,\mathrm{tr}_g(h_0))\,\mathrm{vol}_g.$$

[Note that

$$(D_g S)(h) = -\Delta_g \mathrm{tr}_g(h) - \delta_g \mathrm{div}_g\, h$$

$$= -\Delta_g \mathrm{tr}_g(h_0) - \Delta_g \mathrm{tr}_g L_X g - \delta_g \mathrm{div}_g L_X g$$

$$= -\Delta_g \mathrm{tr}_g(h_0).]$$

<u>FACT</u> We have

$$(D_g^2 L)(h,h) = (D_g^2 L)(h_0,h_0).$$

Rappel:

$$\left[\begin{array}{l} \gamma_g : S_2(M) \to C^\infty(M) \\ \\ \gamma_g(h) = -\Delta_g \mathrm{tr}_g(h) - \delta_g \mathrm{div}_g h - g[{}^0_2](\mathrm{Ric}(g), h) \end{array}\right.$$

and

$$\left[\begin{array}{l} \gamma_g^* : C^\infty(M) \to S_2(M) \\ \\ \gamma_g^*(f) = -(\Delta_g f)g + H_f - f\mathrm{Ric}(g). \end{array}\right.$$

<u>LEMMA</u> $\forall\, x \in M$ & $\forall\, \xi \in T_x^*M - \{0\}$, the symbol

$$\sigma_\xi(\gamma_g^*;x) : \underline{R} \to \mathrm{Sym}^2\, T_x^*M$$

of γ_g^* is injective.

[Given $r \in \underline{R}$, we have

$$\sigma_\xi(\gamma_g^*;x)(r) = (-g_x[{}^0_1](\xi,\xi)g_x + \xi \otimes \xi)r.$$

But the trace of the RHS is

$$(1-n)g_x[{}^0_1](\xi,\xi)r.$$

Therefore

$$\sigma_\xi(\gamma_g^*;x)(r) = 0 \Rightarrow r = 0 \quad (n > 1).]$$

By elliptic theory, it then follows that there is an orthogonal decomposition

$$S_2(M) = \mathrm{Ker}\, \gamma_g \oplus \mathrm{Ran}\, \gamma_g^*,$$

where both Ker γ_g and Ran γ_g^* are closed subspaces of $S_2(M)$.

Consequently, every $h \in S_2(M)$ can be written in the form

$$h = \tilde{h} + (-(\Delta_g f)g + H_f - f\mathrm{Ric}(g)),$$

where

$$\Delta_g \mathrm{tr}_g(\tilde{h}) + \delta_g \mathrm{div}_g\, h + g[{}^0_2](\mathrm{Ric}(g), h) = 0.$$

Assume now that $\mathrm{Ric}(g) = \lambda g$, thus M (or rather the pair (M,g)) is an Einstein manifold (and $\lambda = S(g)/n$ $(n > 1)$).

Rappel: $\forall\, h \in S_2(M)$,

$$\gamma_g(h) = (D_g S)(h).$$

Therefore

$$\begin{aligned}\gamma_g(L_X g) &= (D_g S)(L_X g)\\ &= L_X(S(g))\\ &= 0\end{aligned}$$

$\Rightarrow$

$$\mathrm{Ran}\ \alpha_g \subset \mathrm{Ker}\ \gamma_g$$

$\Rightarrow$

$$S_2(M) = (\mathrm{Ker}\ \gamma_g \cap \mathrm{Ker}\ \alpha_g^*) \oplus \mathrm{Ran}\ \alpha_g \oplus \mathrm{Ran}\ \gamma_g^*.$$

So, if $h \in S_2(M)$, then

$$h = \tilde{h}_0 + L_X g + (-(\Delta_g f)g + H_f - \lambda f g),$$

where

$$\operatorname{div}_g \tilde{h}_0 = 0 \ \& \ \Delta_g \operatorname{tr}_g(\tilde{h}_0) + g[{}^0_2](\operatorname{Ric}(g), \tilde{h}_0) = 0.$$

<u>LEMMA</u> We have

$$\begin{cases} \lambda \neq 0 \Rightarrow \operatorname{tr}_g(\tilde{h}_0) = 0 \\ \lambda = 0 \Rightarrow \operatorname{tr}_g(\tilde{h}_0) = C_0. \end{cases}$$

[Consider the relation

$$\begin{aligned} \Delta_g \operatorname{tr}_g(\tilde{h}_0) &= - g[{}^0_2](\operatorname{Ric}(g), \tilde{h}_0) \\ &= (-\lambda) g[{}^0_2](g, \tilde{h}_0) \\ &= - \lambda \operatorname{tr}_g(\tilde{h}_0). \end{aligned}$$

If $\lambda = 0$, then $\operatorname{tr}_g(\tilde{h}_0)$ is harmonic, hence equals a constant C_0. If $\lambda < 0$, then $\operatorname{tr}_g(\tilde{h}_0) = 0$ (since the eigenvalues of Δ_g are ≤ 0). If $\lambda > 0$ and if $\lambda_1 < 0$ is the first strictly negative eigenvalue of Δ_g, then the Lichnerowicz inequality says that

$$\lambda_1 \leq \frac{n}{n-1} (-\lambda) \quad \text{(see below)}.$$

But

$$\frac{n}{n-1} (-\lambda) < - \lambda,$$

thus $\operatorname{tr}_g(\tilde{h}_0) = 0$.]

[Note: To explicate C_0 when $\text{tr}_g(\tilde{h}_0)$ is harmonic, observe that

$$\int_M [\text{tr}_g(\tilde{h}_0) - \frac{1}{\text{vol}_g(M)} \int_M \text{tr}_g(\tilde{h}_0)\text{vol}_g]\text{vol}_g = 0.$$

Therefore the difference

$$\text{tr}_g(\tilde{h}_0) - \frac{1}{\text{vol}_g(M)} \int_M \text{tr}_g(\tilde{h}_0)\text{vol}_g$$

is orthogonal to the constants, in particular is orthogonal to itself. I.e.:

$$C_0 = \frac{1}{\text{vol}_g(M)} \int_M \text{tr}_g(\tilde{h}_0)\text{vol}_g.]$$

Scholium: Suppose that M is Einstein $(n > 1)$ -- then $\forall\, h \in S_2(M)$,

$$\begin{cases} \lambda \neq 0: h = h^{TT} + L_X g + (-(\Delta_g f)g + H_f - \lambda f g) \\ \lambda = 0: h = (h^{TT} + (C_0/n)g) + L_X g + (-(\Delta_g f)g + H_f). \end{cases}$$

[Note: Here

$$h^{TT} = \begin{cases} \tilde{h}_0 & (\lambda \neq 0) \\ \tilde{h}_0 - (C_0/n)g & (\lambda = 0) \end{cases}$$

has zero divergence and zero trace, a circumstance which in the literature is referred to as being <u>transverse traceless</u> (cf. infra).]

<u>FACT</u> (The Lichnerowicz inequality) Suppose that $\text{Ric}(g) = \lambda g$ $(\lambda > 0)$. Let $\lambda_1 < 0$ be the first strictly negative eigenvalue of Δ_g -- then

$$\lambda_1 \le \frac{n}{n-1}\,(-\lambda).$$

[Fix $f \neq 0 : \Delta_g f = \lambda_1 f$ and integrate the equality

$$\frac{1}{2}\,\Delta_g(g(\mathrm{grad}_g\, f, \mathrm{grad}_g\, f))$$

$$= g[{}^0_2](H_f, H_f) + g(\mathrm{grad}_g\, f, \mathrm{grad}_g\, \Delta_g f) + \mathrm{Ric}(\mathrm{grad}_g\, f, \mathrm{grad}_g\, f)$$

over M to get

$$0 = \int_M g[{}^0_2](H_f, H_f)\mathrm{vol}_g$$

$$+ \lambda_1 \int_M g[{}^0_1](df, df)\mathrm{vol}_g + \lambda \int_M g[{}^0_1](df, df)\mathrm{vol}_g$$

or still,

$$0 = \|H_f\|^2 - \lambda_1 < \Delta_g f, f > - \lambda < \Delta_g f, f >$$

or still,

$$0 = \|H_f\|^2 - \|\Delta_g f\|^2 - \frac{\lambda}{\lambda_1}\,\|\Delta_g f\|^2.$$

But

$$\|\Delta_g f\|^2 \le n\,\|H_f\|^2.$$

Therefore

$$0 \ge \|H_f\|^2 - n\,\|H_f\|^2 - \frac{\lambda}{\lambda_1}\,n\,\|H_f\|^2$$

$\Rightarrow$

$$0 \ge 1 - n - \frac{\lambda}{\lambda_1}\,n$$

$\Rightarrow$

$$\lambda_1(1-n) - \lambda n \ge 0 \quad (\lambda_1 < 0)$$

$\Rightarrow$

$$\lambda_1 \leq \frac{n}{n-1} (-\lambda).]$$

Observation: Let $X \in \mathcal{D}^1(M)$, $s \in S_2(M)$ -- then

$$< -\frac{2}{n} (\mathrm{div}_g X)g,s > = \int_M g[{}^0_2](-\frac{2}{n} (\mathrm{div}_g X)g,s)\mathrm{vol}_g$$

$$= -\frac{2}{n} \int_M (\mathrm{div}_g X)g[{}^0_2](g,s)\mathrm{vol}_g$$

$$= -\frac{2}{n} \int_M (\mathrm{div}_g X)\mathrm{tr}_g(s)\mathrm{vol}_g$$

$$= \frac{2}{n} \int_M X\mathrm{tr}_g(s)\mathrm{vol}_g$$

$$= \frac{2}{n} \int_M g[{}^1_0](X,g^{\#}d\mathrm{tr}_g(s))\mathrm{vol}_g$$

$$= \frac{2}{n} \int_M g[{}^1_0](X,\mathrm{grad}_g\ \mathrm{tr}_g(s))\mathrm{vol}_g$$

$$= < X, \frac{2}{n} \mathrm{grad}_g\ \mathrm{tr}_g(s) >.$$

Given $g \in \underline{M}_0$, define

$$\tau_g : \mathcal{D}^1(M) \rightarrow S_2(M)$$

by

$$\tau_g(X) = L_X g + \frac{2}{n} (-\mathrm{div}_g X)g$$

and define

$$\tau_g^* : S_2(M) \rightarrow \mathcal{D}^1(M)$$

by

$$\tau_g^*(s) = -2g^{\#}\mathrm{div}_g\, s + \frac{2}{n}\,\mathrm{grad}_g\, \mathrm{tr}_g(s).$$

Then

$$< \tau_g(X), s > = < X, \tau_g^*(s) >.$$

<u>LEMMA</u> $\forall\ x \in M$ & $\forall\ \xi \in T_x^*M - \{0\}$, the symbol

$$\sigma_\xi(\tau_g;x) : T_xM \to \mathrm{Sym}^2\ T_x^*M$$

of τ_g is injective provided $n > 1$.

[Given $V \in T_xM$, we have

$$\sigma_\xi(\tau_g;x)(V) = \xi \otimes g_x^\flat V + g_x^\flat V \otimes \xi - \frac{2}{n}\,(V^a\xi_a)g_x.$$

So, if $\sigma_\xi(\tau_g;x)(V) = 0$, then $\forall$ i & $\forall$ j,

$$\xi_i V_j + V_i \xi_j - \frac{2}{n}\,(V^a\xi_a)(g_x)_{ij} = 0$$

$\Rightarrow$

$$\xi^j V^i \xi_i V_j + \xi^j V^i V_i \xi_j - \frac{2}{n}\,(V^a\xi_a)\xi^j V^i (g_x)_{ij} = 0$$

$\Rightarrow$

$$(V^i\xi_i)(V^j\xi_j) + (\xi^j\xi_j)(V^iV_i) - \frac{2}{n}\,(V^a\xi_a)(V^i\xi_i) = 0$$

$\Rightarrow$

$$(1 - \frac{2}{n})(V^a\xi_a)^2 + g_x[{}^0_1](\xi,\xi)\,g_x(g^\flat V, g^\flat V) = 0$$

$\Rightarrow$

$$g_x(g^\flat V, g^\flat V) = 0$$

$\Rightarrow$ $g^\flat V = 0 \Rightarrow V = 0 \quad (n > 1).]$

By elliptic theory, it then follows that there is an orthogonal decomposition

$$S_2(M) = \text{Ran } \tau_g \oplus \text{Ker } \tau_g^*,$$

where both Ran τ_g and Ker τ_g^* are closed subspaces of $S_2(M)$.

Consequently, every $s \in S_2(M)$ can be split into three parts:

$$s = s^0 + L_X g + \frac{2}{n}\,(-\text{div}_g\, X)g.$$

Here

$$-\,2g^\#\text{div}_g\, s^0 + \frac{2}{n}\,\text{grad}_g\, \text{tr}_g(s^0) = 0$$

or still,

$$-\,g^\#\text{div}_g\, s^0 + \frac{1}{n}\, g^\# d\text{tr}_g(s^0) = 0$$

or still,

$$-\,\text{div}_g\, s^0 + \frac{1}{n}\, d\text{tr}_g(s^0) = 0$$

or still,

$$-\,\text{div}_g\, s^0 + \frac{1}{n}\;\text{div}_g(\text{tr}_g(s^0)g) = 0$$

or still,

$$\text{div}_g(s^0 - \frac{1}{n}\,\text{tr}_g(s^0)g) = 0.$$

Remark: A vector field X is said to be <u>conformal</u> if

$$L_X g = \frac{2}{n} (\mathrm{div}_g X) g.$$

Every infinitesimal isometry is conformal, the converse being false in general.

[Note: According to Yano's formula,

$$\int_M [\mathrm{Ric}(X,X) - (\mathrm{div}_g X)^2 + \frac{1}{2} g[{}^0_2](L_X g, L_X g) - g[{}^1_1](\nabla X, \nabla X)] \mathrm{vol}_g = 0.$$

So, if X is conformal, then

$$\int_M [\mathrm{Ric}(X,X) - \frac{n-2}{n} (\mathrm{div}_g X)^2 - g[{}^1_1](\nabla X, \nabla X)] \mathrm{vol}_g = 0,$$

a relation which places an a priori restriction on the existence of X. E.g.: There are no nonzero conformal vector fields if the Ricci curvature is negative definite.]

Put

$$s^{TT} = s^0 - \frac{1}{n} \mathrm{tr}_g(s^0) g.$$

Then

$$s = s^{TT} + L_X g + \frac{2}{n} (-\mathrm{div}_g X) g + \frac{1}{n} \mathrm{tr}_g(s) g.$$

[Note: We have used the fact that

$$\mathrm{tr}_g(s) = \mathrm{tr}_g(s^0).$$

Proof:

$$\begin{aligned} \mathrm{tr}_g(s) &= \mathrm{tr}_g(s^0) + \mathrm{tr}_g L_X g + \frac{2}{n} (-\mathrm{div}_g X) \mathrm{tr}_g(g) \\ &= \mathrm{tr}_g(s^0) - 2\delta_g g^\flat X - 2(\mathrm{div}_g X) \\ &= \mathrm{tr}_g(s^0) - 2\delta_g g^\flat X - 2(-\delta_g g^\flat X) \\ &= \mathrm{tr}_g(s^0).] \end{aligned}$$

Notation: $S_2(M)^{TT}$ stands for the subspace of $S_2(M)$ consisting of those s such that

$$\mathrm{div}_g\, s = 0 \ \&\ \mathrm{tr}_g(s) = 0.$$

[Note: In other words, $S_2(M)^{TT}$ is the kernel of the map

$$S_2(M) \to \mathcal{D}_1(M) \times C^\infty(M)$$

that sends s to $(\mathrm{div}_g\, s, \mathrm{tr}_g(s))$.

The preceding considerations then imply that

$$S_2(M) = S_2(M)^{TT} \oplus \mathrm{Ran}\ \tau_g \oplus C^\infty(M)g.$$

Remark: It can be shown that $S_2(M)^{TT}$ is infinite dimensional provided $n > 2$.

Here is some terminology that can serve as a recapitulation.

Nomenclature:

(1) The splitting

$$s = s_0 + L_X g$$

is called the <u>canonical decomposition</u> of $s \in S_2(M)$.

(2) The splitting

$$h = \tilde{h} + (\ -(\Delta_g f)g + H_f - f\mathrm{Ric}(g))$$

is called the <u>BDBE decomposition</u> of $h \in S_2(M)$.

(3) The splitting

$$s = s^{TT} + L_X g + \frac{2}{n}\,(\ -\mathrm{div}_g\, X)g + \frac{1}{n}\,\mathrm{tr}_g(s)g$$

is called the <u>York decomposition</u> of $s \in S_2(M)$.

<u>Section 31</u>: <u>Metrics on Metrics</u> Let M be a connected C^∞ manifold of dimension n. Assume: M is compact and orientable and $n > 1$.

Rappel: $\underline{M}_0$ (the set of riemannian structures on M) is open in $S_2(M)$, hence is a Fréchet manifold modeled on $S_2(M)$.

Put

$$\begin{cases} T\underline{M}_0 = \underline{M}_0 \times S_2(M) \\ T^*\underline{M}_0 = \underline{M}_0 \times S_d^2(M). \end{cases}$$

Then $\forall\ g \in \underline{M}_0$,

$$\begin{cases} T_g\underline{M}_0 = S_2(M) \\ T_g^*\underline{M}_0 = S_d^2(M), \end{cases}$$

the pairing

$$< \ , \ > : T_g\underline{M}_0 \times T_g^*\underline{M}_0 \to \underline{R}$$

being

$$< u, v^\# \otimes |g|^{1/2} > = \int_M v^\#(u) \operatorname{vol}_g$$

$$= \int_M g[{}^0_2](u,v) \operatorname{vol}_g.$$

[Note: $T^*\underline{M}_0$ is the "L^2 cotangent bundle" of $\underline{M}_0$ (the fiber $T_g^*\underline{M}_0 = S_d^2(M)$ is a proper subspace of the topological dual of $T_g\underline{M}_0 = S_2(M)$).]

Given $\beta \in \underline{R}$, define

$$[\ ,\ ]_{\beta,g} : S_2(M) \times S_2(M) \to C^\infty(M)$$

by

$$[u,v]_{\beta,g} = [u - \frac{1}{n}\,tr_g(u)g, v - \frac{1}{n}\,tr_g(v)g]_g + \beta tr_g(u)tr_g(v)$$

and set

$$G_{\beta,g}(u,v) = \int_M [u,v]_{\beta,g}\mathrm{vol}_g.$$

Then

$$G_{\beta,g}: S_2(M) \times S_2(M) \to \underline{R}$$

is a smooth symmetric bilinear form.

[Note: Obviously,

$$[u,v]_{\beta,g} = [u,v]_g + (\beta - \frac{1}{n})tr_g(u)tr_g(v).$$

Therefore

$$[u,v]_{\frac{1}{n},g} = [u,v]_g$$

$$\Rightarrow$$

$$G_{\frac{1}{n},g}(u,v) = \int_M g[{}^0_2](u,v)\mathrm{vol}_g.]$$

Example: Take $\beta = \frac{1}{n} - 1$ -- then $G_{\frac{1}{n} - 1,g}$ $(\equiv G_g)$ is called the <u>DeWitt metric</u>, thus

$$G_g(u,v) = \int_M ([u,v]_g - tr_g(u)tr_g(v))\mathrm{vol}_g.$$

<u>LEMMA</u> $\forall\ \beta \neq 0$, $G_{\beta,g}$ is nondegenerate.

[Fix $u \in S_2(M)$ and suppose that $G_{\beta,g}(u,v) = 0\ \forall\ v \in S_2(M)$ -- then in particular

$$G_{\beta,g}(u,u - \frac{\beta n-1}{\beta n^2}\,tr_g(u)g) = 0.$$

We have

(1) $$[u,u - \frac{\beta n-1}{\beta n^2}\,tr_g(u)g]_g$$

$$= [u,u]_g - \frac{\beta n-1}{\beta n^2}\,tr_g(u)\,[u,g]_g$$

$$= [u,u]_g - \frac{\beta n-1}{\beta n^2}\,tr_g(u)^2.$$

(2) $$(\beta - \frac{1}{n})\,tr_g(u)\,tr_g(u - \frac{\beta n-1}{\beta n^2}\,tr_g(u)g)$$

$$= (\beta - \frac{1}{n})\,tr_g(u)\,[tr_g(u) - \frac{\beta n-1}{\beta n^2}\,tr_g(u)\,tr_g(g)]$$

$$= (\beta - \frac{1}{n})\,tr_g(u)^2[1 - \frac{\beta n-1}{\beta n}]$$

$$= \frac{\beta n-1}{\beta n^2}\,tr_g(u)^2.$$

Therefore

$$0 = \int_M ([u,u]_g - \frac{\beta n-1}{\beta n^2}\,tr_g(u)^2$$

$$+ \frac{\beta n-1}{\beta n^2}\,tr_g(u)^2)\,vol_g$$

$$= \int_M [u,u]_g vol_g$$

$\Rightarrow$

$u = 0.$]

Rappel: Denote by Diff^+M the normal subgroup of Diff M consisting of the orientation preserving diffeomorphisms — then there are two possibilities.

- $[\text{Diff } M:\text{Diff}^+M] = 1$, in which case M is <u>irreversible</u>.

- $[\text{Diff } M:\text{Diff}^+M] = 2$, in which case M is <u>reversible</u>.

[Note: There is then an orientation reversing diffeomorphism of M and a short exact sequence

$$1 \to \text{Diff}^+M \to \text{Diff } M \overset{\varepsilon_M}{\to} \underline{Z}_2 \to 1,$$

where $\varepsilon_M(\varphi) = +1$ if φ is orientation preserving and $\varepsilon_M(\varphi) = -1$ if φ is orientation reversing.]

Remark: When equipped with the C^∞ topology, Diff M is a topological group. The normal subgroup Diff^+M is both open and closed and contains Diff_0M, the identity component of Diff M.

The group Diff^+M operates to the right on $\underline{M}_0$ via pullback: $\forall\ \varphi\in\text{Diff}^+M$,

$$g\cdot\varphi = \varphi^*g \quad (g\in\underline{M}_0).$$

<u>FACT</u> View G_β as a semiriemannian structure on $\underline{M}_0$ $(\beta \neq 0)$ — then $\forall\ \varphi\in\text{Diff}^+M$,

$$(\varphi^*)^*G_\beta = G_\beta.$$

[Note: In other words, Diff^+M can be identified with a subgroup of the isometry group of $(\underline{M}_0, G_\beta)$.]

In what follows, it will always be assumed that $\beta \neq 0$.

<u>$G^\flat_{\beta,g}:S_2(M) \to S^2_d(M)$</u> Here

$$G^{\flat}_{\beta,g}(u)(v) = G_{\beta,g}(u,v)$$

$$= \int_M ([u,v]_g + (\beta - \tfrac{1}{n})\mathrm{tr}_g(u)\mathrm{tr}_g(v))\mathrm{vol}_g.$$

But

$$(u + (\beta - \tfrac{1}{n})\mathrm{tr}_g(u)g)^{\#}(v)$$

$$= (u^{ab} + (\beta - \tfrac{1}{n})\mathrm{tr}_g(u)g^{ab})v_{ab}$$

$$= [u,v]_g + (\beta - \tfrac{1}{n})\mathrm{tr}_g(u)\mathrm{tr}_g(v).$$

Therefore

$$G^{\flat}_{\beta,g}(u)(v) = \int_M (u + (\beta - \tfrac{1}{n})\mathrm{tr}_g(u)g)^{\#}(v)\mathrm{vol}_g$$

$$= < v, (u + (\beta - \tfrac{1}{n})\mathrm{tr}_g(u)g)^{\#} \otimes |g|^{1/2} >$$

$\Rightarrow$

$$G^{\flat}_{\beta,g}(u) = (u + (\beta - \tfrac{1}{n})\mathrm{tr}_g(u)g)^{\#} \otimes |g|^{1/2}.$$

[Note: By construction, $G^{\flat}_{\beta,g}$ is injective. More is true: $G^{\flat}_{\beta,g}$ is bijective with inverse

$$G^{\#}_{\beta,g}: S^2_d(M) \rightarrow S_2(M)$$

given by

$$G^{\#}_{\beta,g}(s^{\#} \otimes |g|^{1/2}) = s + \frac{1}{\beta n}(\frac{1}{n} - \beta)\mathrm{tr}_g(s)g.$$

In fact,

$$G^{\#}_{\beta,g}(G^{\flat}_{\beta,g}(s)) = G^{\#}_{\beta,g}((s + (\beta - \tfrac{1}{n})\mathrm{tr}_g(s)g)^{\#} \otimes |g|^{1/2})$$

$$= s + (\beta - \frac{1}{n})\,tr_g(s)g$$

$$+ \frac{1}{\beta n}\,(\frac{1}{n} - \beta)\,tr_g(s + (\beta - \frac{1}{n})\,tr_g(s)g)g$$

$$= s + (\beta - \frac{1}{n})\,tr_g(s)g$$

$$+ \frac{1}{\beta n}\,(\frac{1}{n} - \beta)\,[tr_g(s) + (\beta - \frac{1}{n})\,n\,tr_g(s)]g$$

$$= s + [(\beta - \frac{1}{n}) + \frac{1}{\beta n}\,(\frac{1}{n} - \beta)\,(1 + \beta n - 1)]\,tr_g(s)g$$

$$= s + (\beta - \frac{1}{n})\,[1 - \frac{\beta n}{\beta n}]\,tr_g(s)g$$

$$= s.]$$

From the definitions,

$$TT\underline{M}_0 = T\underline{M}_0 \times TS_2(M)$$

$$= (\underline{M}_0 \times S_2(M)) \times (S_2(M) \times S_2(M)).$$

Therefore a vector field X on $T\underline{M}_0$ can be thought of as a map

$$\begin{cases} \underline{M}_0 \times S_2(M) \to S_2(M) \times S_2(M) \\ (g,s) \longmapsto (u,v). \end{cases}$$

Observation: There is a commutative diagram

$$\begin{array}{ccc} (\underline{M}_0 \times S_2(M)) \times (S_2(M) \times S_2(M)) & \xrightarrow{T\pi} & \underline{M}_0 \times S_2(M) \\ \pi_T \downarrow & & \downarrow \pi \\ \underline{M}_0 \times S_2(M) & \xrightarrow[\pi]{} & \underline{M}_0 \end{array},$$

where

$$\begin{cases} \pi(g,s) = g \\ \pi_T((g,s),(u,v)) = (g,s) \\ T\pi((g,s),(u,v)) = (g,u). \end{cases}$$

Definition: A vector field X on $T\underline{M}_0$ is said to be <u>second order</u> if

$$T\pi \circ X = \mathrm{id}_{T\underline{M}_0}.$$

[Note: Of course, $\pi_T \circ X = \mathrm{id}_{T\underline{M}_0}$ is automatic.]

Let

$$X:\underline{M}_0 \times S_2(M) \to S_2(M) \times S_2(M)$$

be a vector field on $T\underline{M}_0$ -- then X has two components: $X = (X_1,X_2)$, where

$$\begin{cases} X_1:\underline{M}_0 \times S_2(M) \to S_2(M) \\ X_2:\underline{M}_0 \times S_2(M) \to S_2(M). \end{cases}$$

This said, it is then clear that X is second order iff

$$X(g,s) = (s,X_2(g,s)) \quad (X_1(g,s) = s).$$

Remark: If X is second order and if $\gamma(t) = (g(t),s(t)) \in \underline{M}_0 \times S_2(M)$ is an integral curve for X, then

$$\frac{d\gamma}{dt} = (\frac{dg}{dt}, \frac{ds}{dt})$$

$$= X(g(t),s(t))$$

$$= (s(t),X_2(g(t),s(t))),$$

so

$$\begin{cases} \frac{dg}{dt} = s(t) \\ \frac{d^2g}{dt^2} = \frac{ds}{dt} = X_2(g(t), \frac{dg}{dt}) \end{cases}$$

or, in brief,

$$\begin{cases} \dot{g} = s \\ \ddot{g} = X_2(g,\dot{g}). \end{cases}$$

[Note: The <u>geodesics</u> of X are, by definition, the projection to $\underline{M}_0$ of its integral curves.]

Definition: A <u>spray</u> is a second order vector field X on $T\underline{M}_0$ which satisfies the following condition: $\forall\ \lambda \in \underline{R}$,

$$X_2(g,\lambda s) = \lambda^2 X_2(g,s).$$

[Note: In other words, X_2 is homogeneous of degree 2 in the variable s, hence

$$X_2(g,s) = \frac{1}{2} D_2^2 X_2(g,0)(s,s).]$$

THEOREM Fix $\beta \neq 0$ — then there exists a unique spray X_β on $\underline{TM}_0$ whose second component Γ_β has the property that

$$G_{\beta,g}(\Gamma_\beta(g,s),h)$$

$$= \frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g} + \varepsilon h^{(s,s)}\Big|_{\varepsilon=0} - \frac{d}{d\varepsilon} G_{\beta,g} + \varepsilon s^{(s,h)}\Big|_{\varepsilon=0}.$$

[Note: The significance of this result will become apparent in the next section.]

The uniqueness of X_β is obvious. As for its existence, let

$$X_\beta(g,s) = (s,\Gamma_\beta(g,s)),$$

where

$$\Gamma_\beta(g,s) = s*s - \frac{1}{2}\,\mathrm{tr}_g(s)s + \frac{1}{4\beta n}\,[s,s]_g g + \frac{\beta n-1}{4\beta n^2}\,\mathrm{tr}_g(s)^2 g$$

or still,

$$\Gamma_\beta(s,s) = s*s - \frac{1}{2}\,\mathrm{tr}_g(s)s + \frac{1}{4\beta n}\,[s,s]_{\beta,g} g.$$

Then

$$\Gamma_\beta(g,\lambda s) = \lambda^2\Gamma_\beta(g,s).$$

[Note: Put

$$B_\beta(g;u,v) = \frac{1}{2}\,[\Gamma_\beta(g,u + v) - \Gamma_\beta(g,u) - \Gamma_\beta(g,v).]$$

Then B_β is bilinear and

$$B_\beta(g;u,u) = \frac{1}{2}\,[\Gamma_\beta(g,2u) - 2\Gamma_\beta(g,u)]$$

$$= \frac{1}{2}\,[4\Gamma_\beta(g,u) - 2\Gamma_\beta(g,u)]$$

$$= \Gamma_\beta(g,u).]$$

Example: Take $\beta = \frac{1}{n} - 1$ — then $\Gamma_{\frac{1}{n}-1}$ ($\equiv \Gamma$) is called the <u>DeWitt spray</u>, thus

$$\Gamma(g,s) = s*s - \frac{1}{2}\operatorname{tr}_g(s)s + \frac{1}{4(n-1)}(\operatorname{tr}_g(s)^2 - [s,s]_g)g.$$

To verify the equality stated in the theorem, start with the LHS:

$$G_{\beta,g}(\Gamma_\beta(g,s),h) = \int_M [\Gamma_\beta(g,s),h]_{\beta,g}\operatorname{vol}_g$$

$$= \int_M [\Gamma_\beta(g,s),h]_g\operatorname{vol}_g$$

$$+ (\beta - \frac{1}{n})\int_M \operatorname{tr}_g(\Gamma_\beta(g,s))\operatorname{tr}_g(h)\operatorname{vol}_g$$

$$= \int_M \{[s*s,h]_g - \frac{1}{2}[s,h]_g\operatorname{tr}_g(s)$$

$$+ \frac{1}{4\beta n}[s,s]_g\operatorname{tr}_g(h) + \frac{\beta n-1}{4\beta n^2}\operatorname{tr}_g(s)^2\operatorname{tr}_g(h)\}\operatorname{vol}_g$$

$$+ (\beta - \frac{1}{n})\int_M \{[s,s]_g\operatorname{tr}_g(h) - \frac{1}{2}\operatorname{tr}_g(s)^2\operatorname{tr}_g(h)$$

$$+ \frac{1}{4\beta}[s,s]_g\operatorname{tr}_g(h) + \frac{\beta n-1}{4\beta n}\operatorname{tr}_g(s)^2\operatorname{tr}_g(h)\}\operatorname{vol}_g$$

$$= \int_M \{[s*s,h]_g - \frac{1}{2}[s,h]_g\operatorname{tr}_g(s)\}\operatorname{vol}_g$$

$$+ (\beta - \frac{1}{n} + \frac{1}{4})\int_M [s,s]_g\operatorname{tr}_g(h)\operatorname{vol}_g$$

$$+ \frac{1-\beta n}{4n}\int_M \operatorname{tr}_g(s)^2\operatorname{tr}_g(h)\operatorname{vol}_g.$$

Consider now the RHS.

- $\frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g+\varepsilon h}(s,s)\Big|_{\varepsilon=0}$

$$= \frac{1}{2}\frac{d}{d\varepsilon}\int_M [s,s]_{g+\varepsilon h}\mathrm{vol}_{g+\varepsilon h}\Big|_{\varepsilon=0}$$

$$+ \frac{1}{2}(\beta-\frac{1}{n})\frac{d}{d\varepsilon}\int_M \mathrm{tr}_{g+\varepsilon h}(s)^2\mathrm{vol}_{g+\varepsilon h}\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\int_M -2[s*s,h]_g\mathrm{vol}_g + \frac{1}{2}\int_M [s,s]_g\frac{1}{2}\mathrm{tr}_g(h)\mathrm{vol}_g$$

$$+ \frac{1}{2}(\beta-\frac{1}{n})\int_M 2\mathrm{tr}_g(s)(-[s,h]_g)\mathrm{vol}_g$$

$$+ \frac{1}{2}(\beta-\frac{1}{n})\int_M \mathrm{tr}_g(s)^2\frac{1}{2}\mathrm{tr}_g(h)\mathrm{vol}_g$$

$$= -\int_M [s*s,h]_g\mathrm{vol}_g + \frac{1}{4}\int_M [s,s]_g\mathrm{tr}_g(h)\mathrm{vol}_g$$

$$- (\beta-\frac{1}{n})\int_M \mathrm{tr}_g(s)[s,h]_g\mathrm{vol}_g$$

$$+ \frac{1}{4}(\beta-\frac{1}{n})\int_M \mathrm{tr}_g(s)^2\mathrm{tr}_g(h)\mathrm{vol}_g.$$

- $-\frac{d}{d\varepsilon} G_{\beta,g+\varepsilon s}(s,h)\Big|_{\varepsilon=0}$

$$= -\frac{d}{d\varepsilon}\int_M [s,h]_{g+\varepsilon s}\mathrm{vol}_{g+\varepsilon s}\Big|_{\varepsilon=0}$$

$$- (\beta-\frac{1}{n})\frac{d}{d\varepsilon}\int_M \mathrm{tr}_{g+\varepsilon s}(s)\mathrm{tr}_{g+\varepsilon s}(h)\mathrm{vol}_{g+\varepsilon s}\Big|_{\varepsilon=0}$$

$$= - \int_M - 2[s*h,s]_g \mathrm{vol}_g - \int_M [s,h]_g \frac{1}{2} \mathrm{tr}_g(s)\mathrm{vol}_g$$

$$- (\beta - \frac{1}{n}) \int_M (-[s,s]_g) \mathrm{tr}_g(h)\mathrm{vol}_g$$

$$- (\beta - \frac{1}{n}) \int_M \mathrm{tr}_g(s) (-[s,h]_g)\mathrm{vol}_g$$

$$- (\beta - \frac{1}{n}) \int_M \mathrm{tr}_g(s)\mathrm{tr}_g(h) \frac{1}{2} \mathrm{tr}_g(s)\mathrm{vol}_g$$

$$= 2 \int_M [s*h,s]_g \mathrm{vol}_g - \frac{1}{2} \int_M [s,h]_g \mathrm{tr}_g(s)\mathrm{vol}_g$$

$$+ (\beta - \frac{1}{n}) \int_M [s,s]_g \mathrm{tr}_g(h)\mathrm{vol}_g$$

$$+ (\beta - \frac{1}{n}) \int_M \mathrm{tr}_g(s) [s,h]_g \mathrm{vol}_g$$

$$- \frac{1}{2} (\beta - \frac{1}{n}) \int_M \mathrm{tr}_g(s)^2 \mathrm{tr}_g(h)\mathrm{vol}_g.$$

<u>N.B.</u> We have

$$2[s*h,s]_g - [s*s,h]_g$$

$$= 2s^{ij}(s*h)_{ij} - h^{ij}(s*s)_{ij}$$

$$= 2s^{ij}s_{ik}h^k{}_j - h^{ij}s_{ik}s^k{}_j.$$

And

$$h^{ij}s_{ik}s^k{}_j$$

$$= g^{j\ell}h^i{}_\ell s_{ik}s^k{}_j$$

$$= h^{i}{}_{\ell} s_{ik} g^{j\ell} s^{k}{}_{j}$$

$$= h^{i}{}_{\ell} s_{ik} s^{k\ell}$$

$$= h^{k}{}_{\ell} s_{ik} s^{i\ell}$$

$$= h^{k}{}_{j} s_{ik} s^{ij}$$

$$= s^{ij} s_{ik} h^{k}{}_{j}.$$

Therefore

$$[s*h,s]_g = [s*s,h]_g$$

$$\Rightarrow$$

$$2[s*h,s]_g - [s*s,h]_g = [s*s,h]_g.$$

Combining terms then gives

$$\frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g+\varepsilon h}(s,s)\Big|_{\varepsilon=0} - \frac{d}{d\varepsilon} G_{\beta,g+\varepsilon s}(s,h)\Big|_{\varepsilon=0}$$

$$= \int_M \{[s*s,h]_g - \frac{1}{2}[s,h]_g \mathrm{tr}_g(s)\}\mathrm{vol}_g$$

$$+ (\beta - \frac{1}{n} + \frac{1}{4}) \int_M [s,s]_g \mathrm{tr}_g(h)\mathrm{vol}_g$$

$$+ \frac{1-\beta n}{4n} \int_M \mathrm{tr}_g(s)^2 \mathrm{tr}_g(h)\mathrm{vol}_g,$$

which is precisely the expression derived above for

$$G_{\beta,g}(\Gamma_\beta(g,s),h).$$

[Note: There is a cancellation

$$- (\beta - \frac{1}{n}) \int_M tr_g(s)[s,h]_g vol_g$$

$$+ (\beta - \frac{1}{n}) \int_M tr_g(s)[s,h]_g vol_g.]$$

The governing equation for the geodesics of X_β is

$$\ddot{g} = \Gamma_\beta(g,\dot{g})$$

or, written out,

$$\ddot{g} = \dot{g}*\dot{g} - \frac{1}{2} tr_g(\dot{g})\dot{g} + \frac{1}{4\beta n} [\dot{g},\dot{g}]_{\beta,g} g.$$

Remark: This equation is an ODE and the evolution of a solution $g(t)$ depends only on

$$\begin{cases} g(0) \\ \dot{g}(0). \end{cases}$$

To be precise: Given (g_0, s_0), there exists a unique integral curve $\gamma :]-\varepsilon, \varepsilon[\to \underline{M}_0 \times S_2(M)$ for X_β such that $\gamma(0) = (g_0, s_0)$, i.e.,

$$\begin{cases} g(0) = g_0 \\ \dot{g}(0) = s_0. \end{cases}$$

[Note: The geodesics can be found explicitly but the formulas are not particularly enlightening (they do show, however, that the geodesics exist for a short time only in that they eventually run out of $\underline{M}_0$ into $S_2(M)$).

<u>Section 32:</u> <u>The Symplectic Structure</u> Let M be a connected C^∞ manifold of dimension n. Assume: M is compact and orientable and n > 1.

Rappel: There is an arrow of evaluation

$$\begin{cases} S_2(M) \times S_d^2(M) \to C_d^\infty(M) \\ (s,\Lambda) \longrightarrow \Lambda(s) \end{cases}$$

and a nondegenerate bilinear functional

$$< , > : S_2(M) \times S_d^2(M) \to \underline{R},$$

viz.

$$< s,\Lambda > = \int_M \Lambda(s).$$

Consider $T^*\underline{M}_0 = \underline{M}_0 \times S_d^2(M)$ -- then

$$TT^*\underline{M}_0 = (\underline{M}_0 \times S_d^2(M)) \times (S_2(M) \times S_d^2(M))$$

$\Rightarrow$

$$T_{(g,\Lambda)}T^*\underline{M}_0 = S_2(M) \times S_d^2(M).$$

<u>The Canonical 1-Form Θ</u> This is the map

$$\Theta_{(g,\Lambda)} : T_{(g,\Lambda)}T^*\underline{M}_0 \to \underline{R}$$

defined by the prescription

$$\Theta_{(g,\Lambda)}(s',\Lambda') = < s',\Lambda >.$$

<u>The Canonical 2-Form Ω</u> This is the map

$$\Omega_{(g,\Lambda)} : T_{(g,\Lambda)}T^*\underline{M}_0 \times T_{(g,\Lambda)}T^*\underline{M}_0 \to \underline{R}$$

defined by the prescription

$$\Omega_{(g,\Lambda)}((s_1,\Lambda_1),(s_2,\Lambda_2)) = < s_1,\Lambda_2 > - < s_2,\Lambda_1 >$$

or, in determinant notation,

$$\Omega_{(g,\Lambda)}((s_1,\Lambda_1),(s_2,\Lambda_2)) = \begin{vmatrix} s_1 & \Lambda_1 \\ s_2 & \Lambda_2 \end{vmatrix} .$$

<u>LEMMA</u> We have

$$\Omega = - d\Theta.$$

[In fact,

$$d\Theta\Big|_{(g,\Lambda)}((s_1,\Lambda_1),(s_2,\Lambda_2))$$

$$= \frac{d}{d\varepsilon}\,\Theta_{(g + \varepsilon s_1,\Lambda + \varepsilon\Lambda_1)}(s_2,\Lambda_2)\Big|_{\varepsilon=0} - \frac{d}{d\varepsilon}\,\Theta_{(g + \varepsilon s_2,\Lambda + \varepsilon\Lambda_2)}(s_1,\Lambda_1)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} < s_2,\Lambda + \varepsilon\Lambda_1 > \Big|_{\varepsilon=0} - \frac{d}{d\varepsilon} < s_1,\Lambda + \varepsilon\Lambda_2 > \Big|_{\varepsilon=0}$$

$$= < s_2,\Lambda_1 > - < s_1,\Lambda_2 >$$

$$= - \Omega_{(g,\Lambda)}((s_1,\Lambda_1),(s_2,\Lambda_2)).]$$

Therefore Ω is exact and the pair $(T^*\underline{M}_0,\Omega)$ is a symplectic manifold.

Fix $\beta \neq 0$ and define

$$\phi_\beta : T\underline{M}_0 \rightarrow T^*\underline{M}_0$$

by

$$\phi_\beta(g,s) = (g,G^\flat_{\beta,g}(s)).$$

Then ϕ_β is an isomorphism of vector bundles, hence

$$\Omega_\beta = \phi_\beta^*\Omega$$

is nondegenerate. On the other hand,

$$\phi_\beta^*\Omega = \phi_\beta^*(-d\Theta)$$

$$= - d\phi_\beta^*\Theta,$$

which implies that Ω_β is exact.

Conclusion: The pair $(T\underline{M}_0,\Omega_\beta)$ is a symplectic manifold.

[Given $\beta_i \neq 0$ $(i = 1,2)$, the bijection

$$\phi_{\beta_2}^{-1} \circ \phi_{\beta_1}:T\underline{M}_0 \rightarrow T\underline{M}_0$$

is a canonical transformation:

$$(\phi_{\beta_2}^{-1} \circ \phi_{\beta_1})^*\Omega_{\beta_2} = \Omega_{\beta_1}.$$

For the LHS equals

$$\phi_{\beta_1}^* \circ (\phi_{\beta_2}^*)^{-1} \circ \phi_{\beta_2}^*\Omega = \phi_{\beta_1}^*\Omega = \Omega_{\beta_1}.]$$

SUBLEMMA The tangent map

$$T\phi_\beta:TT\underline{M}_0 \rightarrow TT^*\underline{M}_0$$

is given by

$$T\phi_\beta(g,s,u,v) = (g,G^\flat_{\beta,g}(s),u,DG^\flat_{\beta,g}(u)(s) + G^\flat_{\beta,g}(v)).$$

[Note: Since

$$G^\flat_\beta:\underline{M}_0 \to \text{Hom}(S_2(M),S^2_d(M)),$$

it follows that

$$DG^\flat_\beta:\underline{M}_0 \to \text{Hom}(S_2(M),\text{Hom}(S_2(M),S^2_d(M))),$$

where

$$DG^\flat_{\beta,g}(u) = \frac{d}{d\varepsilon} G^\flat_{\beta,g + \varepsilon u}\Big|_{\varepsilon=0}.$$

Explicated:

$$< w,DG^\flat_{\beta,g}(u)(v) >$$

$$= \frac{d}{d\varepsilon} G^\flat_{\beta,g + \varepsilon u}(v)(w)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} G_{\beta,g + \varepsilon u}(v,w)\Big|_{\varepsilon=0}.]$$

<u>LEMMA</u> We have

$$(\Omega_\beta)_{(g,s)}((u_1,v_1),(u_2,v_2))$$

$$= G_{\beta,g}(u_1,v_2) - G_{\beta,g}(u_2,v_1)$$

$$+ < u_1,DG^\flat_{\beta,g}(u_2)(s) > - < u_2,DG^\flat_{\beta,g}(u_1)(s) >.$$

[Thanks to the sublemma,

$$(\phi_\beta^*\Omega)_{(g,s)}((u_1,v_1),(u_2,v_2))$$

$$= \Omega_{(g,G^\flat_{\beta,g}(s))}((u_1,DG^\flat_{\beta,g}(u_1)(s) + G^\flat_{\beta,g}(v_1)),(u_2,DG^\flat_{\beta,g}(u_2)(s) + G^\flat_{\beta,g}(v_2)))$$

$$= < u_1,DG^\flat_{\beta,g}(u_2)(s) > + < u_1,G^\flat_{\beta,g}(v_2) >$$

$$- < u_2,DG^\flat_{\beta,g}(u_1)(s) > - < u_2,G^\flat_{\beta,g}(v_1) >$$

$$= G_{\beta,g}(u_1,v_2) - G_{\beta,g}(u_2,v_1)$$

$$+ < u_1,DG^\flat_{\beta,g}(u_2)(s) > - < u_2,DG^\flat_{\beta,g}(u_1)(s) >.]$$

Maintaining the assumption that $\beta \neq 0$, define $K_\beta:T\underline{M}_0 \to \underline{R}$ by

$$K_\beta(g,s) = \frac{1}{2} G_{\beta,g}(s,s).$$

N.B. Consider dK_β, thus

$$dK_\beta\Big|_{(g,s)}:T_{(g,s)}T\underline{M}_0 \to \underline{R}$$

with

$$dK_\beta\Big|_{(g,s)}(u,v) = \frac{d}{d\varepsilon} K_\beta(g + \varepsilon u,s + \varepsilon v)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} K_\beta(g + \varepsilon u,s)\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon} K_\beta(g,s + \varepsilon v)\Big|_{\varepsilon=0}.$$

And

- $\frac{d}{d\varepsilon} K_\beta(g + \varepsilon u,s)\Big|_{\varepsilon=0}$

$$= \frac{d}{d\varepsilon} \frac{1}{2} G_{\beta,g + \varepsilon u}(s,s)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} < s, DG^{\flat}_{\beta,g}(u)(s) >.$$

• $\frac{d}{d\varepsilon} K_\beta(g, s + \varepsilon v)\Big|_{\varepsilon=0}$

$$= \frac{d}{d\varepsilon} \frac{1}{2} G_{\beta,g}(s + \varepsilon v, s + \varepsilon v)\Big|_{\varepsilon=0}$$

$$= G_{\beta,g}(s,v).$$

<u>THEOREM</u> For all vector fields X on $T\underline{M}_0$,

$$\Omega_\beta(X_\beta, X) = dK_\beta(X).$$

[Suppose that $X(g,s) = (u,v)$ — then

$$(\Omega_\beta)_{(g,s)}(X_\beta(g,s), X(g,s))$$

$$= (\Omega_\beta)_{(g,s)}((s, \Gamma_\beta(g,s)), (u,v))$$

$$= G_{\beta,g}(s,v) - G_{\beta,g}(u, \Gamma_\beta(g,s))$$

$$+ < s, DG^{\flat}_{\beta,g}(u)(s) > - < u, DG^{\flat}_{\beta,g}(s)(s) >$$

$$= G_{\beta,g}(s,v)$$

$$- \frac{1}{2} \frac{d}{d\varepsilon} G_{\beta,g + \varepsilon u}(s,s)\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon} G_{\beta,g + \varepsilon s}(s,u)\Big|_{\varepsilon=0}$$

$$+ \frac{d}{d\varepsilon} G_{\beta,g + \varepsilon u}(s,s)\Big|_{\varepsilon=0} - \frac{d}{d\varepsilon} G_{\beta,g + \varepsilon s}(s,u)\Big|_{\varepsilon=0}$$

$$= G_{\beta,g}(s,v) + \frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g + \varepsilon u}(s,s)\Big|_{\varepsilon=0}$$

$$= G_{\beta,g}(s,v) + \frac{1}{2} < s, DG^{\flat}_{\beta,g}(u)(s) >$$

$$= dK_\beta\Big|_{(g,s)}(u,v).]$$

Interpretation: Per Ω_β, X_β is a hamiltonian vector field on $T\underline{M}_0$ with energy K_β.

<u>FACT</u> (Conservation of Energy) Let $\gamma(t)$ be an integral curve for X_β -- then the function $t \to K_\beta(\gamma(t))$ is constant in t.

[Simply note that

$$\frac{d}{dt} K_\beta(\gamma(t)) = dK_\beta\Big|_{\gamma(t)}(\dot{\gamma}(t))$$

$$= (\Omega_\beta)_{\gamma(t)}(X_\beta(\gamma(t)),\dot{\gamma}(t))$$
$$= (\Omega_\beta)_{\gamma(t)}(X_\beta(\gamma(t)),X_\beta(\gamma(t)))$$
$$= 0.]$$

<u>Construction</u> Let $X \in \mathcal{D}^1(M)$ -- then X induces a vector field $\bar{X}:\underline{M}_0 \to S_2(M)$ on $\underline{M}_0$ via the prescription

$$\bar{X}(g) = L_X g.$$

Put $\Phi_t = \phi_t^*$, where ϕ_t is the flow of X -- then there is a commutative diagram

$$\begin{array}{ccc} T\underline{M}_0 & \xrightarrow{T\Phi_t} & T\underline{M}_0 \\ \pi \downarrow & & \downarrow \pi \\ \underline{M}_0 & \xrightarrow[\Phi_t]{} & \underline{M}_0 \end{array}.$$

Here

$$T\Phi_t(g,s) = (\Phi_t(g), D\Phi_t\big|_g(s))$$

and

$$\frac{d}{dt}\,(D\Phi_t\big|_g(s)) = D\bar{X}\big|_{\Phi_t(g)}\,(D\Phi_t\big|_g(s)).$$

<u>SUBLEMMA</u> We have

$$K_\beta = K_\beta \circ T\Phi_t.$$

Application: At any point $(g,s)\in T\underline{M}_0$,

$$0 = \frac{d}{dt}\,K_\beta(\Phi_t(g), D\Phi_t\big|_g(s))\,\Big|_{t=0}$$

$$= \frac{1}{2}\,\frac{d}{d\varepsilon}\,G_{\beta,g} + \varepsilon L_X g(s,s)\,\Big|_{\varepsilon=0} + G_{\beta,g}(s, D\bar{X}\big|_g(s)).$$

Rappel: A <u>first integral</u> for a vector field on $T\underline{M}_0$ is a function $f:T\underline{M}_0 \to \underline{R}$ which is constant on integral curves.

So, e.g., K_β is a first integral for X_β.

<u>LEMMA</u> $\forall\ X\in\mathcal{D}^1(M)$, the function

$$(g,s) \to G_{\beta,g}(s, L_X g)$$

is a first integral for X_β.

[Let $\gamma(t) = (g(t),s(t))$ be an integral curve for X_β -- then $\dot{g} = s$ and

$$G_{\beta,g}(\ddot{g},h) = G_{\beta,g}(\Gamma_\beta(g,\dot{g}),h)$$

$$= \frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g} + \varepsilon h(\dot{g},\dot{g})\Big|_{\varepsilon=0} - \frac{d}{d\varepsilon} G_{\beta,g} + \varepsilon\dot{g}(\dot{g},h)\Big|_{\varepsilon=0}$$

$\Rightarrow$

$$G_{\beta,g}(\ddot{g},h) + \frac{d}{d\varepsilon} G_{\beta,g} + \varepsilon\dot{g}(\dot{g},h)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g} + \varepsilon h(\dot{g},\dot{g})\Big|_{\varepsilon=0}$$

or, restoring the dependence on t,

$$\frac{d}{dt} G_{\beta,g(t)}(s(t),h) = \frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g(t)} + \varepsilon h(s(t),s(t))\Big|_{\varepsilon=0}.$$

Now replace h by $L_X g(t)$ — then

$$\frac{d}{dt} G_{\beta,g(t)}(s(t),L_X g(t))$$

$$= \frac{1}{2}\frac{d}{d\varepsilon} G_{\beta,g(t)} + \varepsilon L_X g(t)(s(t),s(t))\Big|_{\varepsilon=0}$$

$$+ G_{\beta,g(t)}(s(t), \frac{d}{dt} L_X g(t)).$$

But

$$\frac{d}{dt} L_X g(t) = \frac{d}{dt}\bar{X}(g(t))$$

$$= D\bar{X}\Big|_{g(t)}(\dot{g}(t))$$

$$= D\bar{X}\Big|_{g(t)}(s(t)).$$

Therefore

$$\frac{d}{dt} G_{\beta,g(t)}(s(t), L_X g(t)) = 0.]$$

[Note: We have

- $g(t)g(t)^{-1} = I$

$\Rightarrow$

$$\frac{dg^{-1}}{dt} = - g^{-1}\dot{g}g^{-1}.$$

- $(g(t) + \varepsilon s(t))(g(t) + \varepsilon s(t))^{-1} = I$

$\Rightarrow$

$$\frac{d}{d\varepsilon}(g + \varepsilon s)^{-1}\Big|_{\varepsilon=0} = - (g + \varepsilon s)^{-1}\Big|_{\varepsilon=0} \frac{d}{d\varepsilon}(g + \varepsilon s)\Big|_{\varepsilon=0}(g + \varepsilon s)^{-1}\Big|_{\varepsilon=0}$$

$$= - g^{-1}\dot{g}g^{-1}.]$$

Write

$$G_{\beta,g}(s, L_X g) = G_{\beta,g}(L_X g, s)$$

$$= \int_M [L_X g, s]_{\beta,g} \text{vol}_g$$

$$= \int_M ([L_X g, s]_g + (\beta - \frac{1}{n}) \text{tr}_g(L_X g)\text{tr}_g(s))\text{vol}_g$$

$$= \int_M ([L_X g, s]_g + (\beta - \frac{1}{n}) [L_X g, g]_g \text{tr}_g(s))\text{vol}_g$$

$$= \int_M [L_X g, s + (\beta - \frac{1}{n}) \text{tr}_g(s) g]_g \text{vol}_g$$

$$= \langle L_X g, (s + (\beta - \tfrac{1}{n})\mathrm{tr}_g(s)g)^{\#} \otimes |g|^{1/2} \rangle$$

$$= -2 \langle X, \mathrm{div}_g(s + (\beta - \tfrac{1}{n})\mathrm{tr}_g(s)g) \otimes |g|^{1/2} \rangle$$

$$= -2 \int_M \mathrm{div}_g(s + (\beta - \tfrac{1}{n})\mathrm{tr}_g(s)g)(X)\mathrm{vol}_g.$$

Let

$$\pi_{\beta,g}(s) = s + (\beta - \tfrac{1}{n})\mathrm{tr}_g(s)g.$$

Then it follows that the function

$$(g,s) \to \int_M \mathrm{div}_g(\pi_{\beta,g}(s))(X)\mathrm{vol}_g$$

is a first integral for X_β.

Conservation Principle Suppose that $\gamma(t) = (g(t),s(t))$ is an integral curve for X_β. Abbreviating $\pi_{\beta,g(t)}(s(t))$ to $\pi_\beta(t)$, $\forall\, X \in \mathcal{D}^1(M)$,

$$\int_M \mathrm{div}_{g(t)}\pi_\beta(t)(X)\mathrm{vol}_{g(t)}$$

is a constant function of t, which implies that

$$\mathrm{div}_{g(t)}\pi_\beta(t) \otimes |g(t)|^{1/2} \in \Lambda_d^1(M)$$

is a constant function of t. Consequently, if

$$\mathrm{div}_{g(0)}\pi_\beta(0) = 0,$$

then $\forall$ t,

$$\mathrm{div}_{g(t)}\pi_\beta(t) = 0.$$

<u>Section 33:</u> <u>Motion in a Potential</u> Let M be a connected C^∞ manifold of dimension n. Assume: M is compact and orientable and $n > 1$.

Given $N \in C^\infty(M)$, put

$$V_N(g) = \int_M NS(g)\text{vol}_g \quad (g \in \underline{M}_0).$$

Then $V_N : \underline{M}_0 \to \underline{R}$ and

$$dV_N\Big|_g(h) = \frac{d}{d\varepsilon} V_N(g + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \int_M N \frac{d}{d\varepsilon} S(g + \varepsilon h)\Big|_{\varepsilon=0}\text{vol}_g + \int_M NS(g) \frac{d}{d\varepsilon} \text{vol}_{g + \varepsilon h}\Big|_{\varepsilon=0}$$

$$= \int_M N[-\Delta_g \text{tr}_g(h) - \delta_g \text{div}_g h - g[{}^0_2](\text{Ric}(g), h)]\text{vol}_g$$

$$+ \int_N NS(g) \frac{1}{2} \text{tr}_g(h)\text{vol}_g.$$

• $\int_M N(-\Delta_g \text{tr}_g(h))\text{vol}_g$

$$= \int_M (-\Delta_g N)\text{tr}_g(h)\text{vol}_g$$

$$= \int_M [(-\Delta_g N)g, h]_g \text{vol}_g.$$

• $\int_M N(-\delta_g \text{div}_g h)\text{vol}_g$

$$= -\int_M g[{}^0_1](dN, \text{div}_g h)\text{vol}_g$$

$$= \int_M g[{}^0_2](H_N, h)\text{vol}_g$$

$$= \int_M [H_N, h]_g \text{vol}_g.$$

- $\int_M [\ -Ng[{}^0_2](\mathrm{Ric}(g),h) + NS(g)\ \frac{1}{2}\ \mathrm{tr}_g(h)]\mathrm{vol}_g$

$$= \int_M - N([\mathrm{Ric}(g),h]_g - \frac{1}{2}\ S(g)\,[g,h]_g)\mathrm{vol}_g$$

$$= \int_N\ [\ -N(\mathrm{Ric}(g)\ - \frac{1}{2}\ S(g)g),h]_g\mathrm{vol}_g.$$

Therefore

$$dV_N\Big|_g(h)$$

$$= \int_M\ [(\ -\Delta_g N)g + H_N - N(\mathrm{Ric}(g)\ - \frac{1}{2}\ S(g)g),h]_g\mathrm{vol}_g$$

$$= \int_M\ ((\ -\Delta_g N)g + H_N - N(\mathrm{Ric}(g)\ - \frac{1}{2}\ S(g)g))^{\#}(h)\,\mathrm{vol}_g$$

$$= < h,((\ -\Delta_g N)g + H_N - N(\mathrm{Ric}(g)\ - \frac{1}{2}\ S(g)g))^{\#} \otimes |g|^{1/2} >$$

$\Rightarrow$

$$dV_N\Big|_g = ((\ -\Delta_g N)g + H_N - N(\mathrm{Ric}(g)\ - \frac{1}{2}\ S(g)g))^{\#} \otimes |g|^{1/2}.$$

Now fix $\beta \neq 0$ and let

$$\mathrm{grad}_{\beta,g}V_N = G^{\#}_{\beta,g}(dV_N\Big|_g).$$

1. We have

$$G^{\#}_{\beta,g}(((\ -\Delta_g N)g + H_N)^{\#} \otimes |g|^{1/2})$$

$$= (\ -\Delta_g N)g + H_N + \frac{1}{\beta n}\ (\frac{1}{n} - \beta)\,\mathrm{tr}_g((\ -\Delta_g N)g + H_N)g$$

$$= (\ -\Delta_g N)g + H_N$$

$$+ \frac{1}{\beta}\left(\frac{1}{n} - \beta\right)(-\Delta_g N)g + \frac{1}{\beta n}\left(\frac{1}{n} - \beta\right)(\Delta_g N)g$$

$$= H_N + \frac{1-n-\beta n}{\beta n^2}(\Delta_g N)g.$$

2. We have

$$G^{\#}_{\beta,g}(-N(\mathrm{Ric}(g) - \frac{1}{2}S(g)g)^{\#} \otimes |g|^{1/2})$$

$$= - N(\mathrm{Ric}(g) - \frac{1}{2}S(g)g) + \frac{1}{\beta n}\left(\frac{1}{n} - \beta\right)\mathrm{tr}_g(-N(\mathrm{Ric}(g) - \frac{1}{2}S(g)g))g$$

$$= - N(\mathrm{Ric}(g) - \frac{1}{2}S(g)g) + \frac{1}{\beta n}\left(\frac{1}{n} - \beta\right)(-NS(g) + \frac{1}{2}NnS(g))g$$

$$= - N\mathrm{Ric}(g) - N\left(\frac{2-n-2\beta n}{2\beta n^2}\right)S(g)g.$$

Combining 1 and 2 then gives

$$\mathrm{grad}_{\beta,g} V_N = H_N + \frac{1-n-\beta n}{\beta n^2}(\Delta_g N)g$$

$$- N\mathrm{Ric}(g) - N\left(\frac{2-n-2\beta n}{2\beta n^2}\right)S(g)g.$$

Example: Take $\beta = \frac{1}{n} - 1$ -- then

$$\begin{bmatrix} 1 - n - (\frac{1}{n} - 1)n = 0 \\ 2 - n - 2(\frac{1}{n} - 1)n = n \\ 2(\frac{1}{n} - 1)n^2 = 2n(1-n), \end{bmatrix}$$

thus in this case the gradient of V_N at g (denoted by $\mathrm{grad}_g V_N$) equals

$$H_N - N\mathrm{Ric}(g) + \frac{1}{2(n-1)} NS(g)g.$$

[Note: When N = 1, the hessian drops out and there remains

$$- \mathrm{Ric}(g) + \frac{1}{2(n-1)} S(g)g.]$$

Define a vector field

$$Y_{\beta,N}: \underline{M}_0 \times S_2(M) \to S_2(M) \times S_2(M)$$

on $T\underline{M}_0$ by

$$\begin{aligned} Y_{\beta,N}(g,s) &= (s, \Gamma_\beta(g,s)) + (0, -\mathrm{grad}_{\beta,g} V_N) \\ &= (s, \Gamma_\beta(g,s) - \mathrm{grad}_{\beta,g} V_N). \end{aligned}$$

Then $Y_{\beta,N}$ is second order and the equation determining its geodesics reads

$$\ddot{g} = Y_{\beta,N}(g,\dot{g}) = \Gamma_\beta(g,\dot{g}) - \mathrm{grad}_{\beta,g} V_N.$$

Example: Take $\beta = \frac{1}{n} - 1$ and $N = 1$ -- then

$$\begin{aligned} \ddot{g} = \dot{g}*\dot{g} - \frac{1}{2} \mathrm{tr}_g(\dot{g})\dot{g} + \frac{1}{4(n-1)} (\mathrm{tr}_g(\dot{g})^2 - [\dot{g},\dot{g}]_g)g \\ + \mathrm{Ric}(g) - \frac{1}{2(n-1)} S(g)g. \end{aligned}$$

<u>THEOREM</u> For all vector fields Y on $T\underline{M}_0$,

$$\Omega_\beta(Y_{\beta,N}, Y) = dE_{\beta,N}(Y),$$

where

$$E_{\beta,N} = K_\beta + V_N.$$

[Suppose that $Y(g,s) = (u,v)$ -- then

$$(\Omega_\beta)_{(g,s)}(Y_{\beta,N}(g,s),Y(g,s))$$

$$= (\Omega_\beta)_{(g,s)}((s,\Gamma_\beta(g,s)) + (0, -\mathrm{grad}_{\beta,g}V_N),(u,v))$$

$$= (\Omega_\beta)_{(g,s)}((s,\Gamma_\beta(g,s)),(u,v))$$

$$+ (\Omega_\beta)_{(g,s)}((0, -\mathrm{grad}_{\beta,g}V_N),(u,v))$$

$$= dK_\beta\Big|_{(g,s)}(u,v)$$

$$+ (\Omega_\beta)_{(g,s)}((0, -\mathrm{grad}_{\beta,g}V_N),(u,v))$$

$$= dK_\beta\Big|_{(g,s)}(u,v) - G_{\beta,g}(u, -\mathrm{grad}_{\beta,g}V_N).$$

And

$$G_{\beta,g}(u,\mathrm{grad}_{\beta,g}V_N)$$

$$= G_{\beta,g}(u,G^{\#}_{\beta,g}(dV_N\Big|_g))$$

$$= G_{\beta,g}(G^{\#}_{\beta,g}(dV_N\Big|_g),u)$$

$$= G^{\flat}_{\beta,g}(G^{\#}_{\beta,g}(dV_N\Big|_g))(u)$$

$$= dV_N\Big|_g(u).]$$

Bearing in mind that the pair $(T\underline{M}_0,\Omega_\beta)$ is a symplectic manifold, it follows that $Y_{\beta,N}$ is a hamiltonian vector field on $T\underline{M}_0$ with energy $E_{\beta,N}$.

[Note: As before, energy is conserved, i.e., on an integral curve $\gamma(t)$ for $Y_{\beta,N}$, the function $t \to E_{\beta,N}(\gamma(t))$ is constant in t.]

Take N = 1 and write V in place of V_1, hence

$$V(g) = \int_M S(g)\mathrm{vol}_g \quad (g \in \underline{M}_0)$$

and

$$V = V \circ \Phi_t$$

$$\Rightarrow$$

$$0 = \frac{d}{dt} V(\Phi_t(g))\Big|_{t=0}$$

$$= dV_g(L_X g).$$

<u>LEMMA</u> $\forall\, X \in \mathcal{D}^1(M)$, the function

$$(g,s) \to G_{\beta,g}(s, L_X g)$$

is a first integral for Y_β $(\equiv Y_{\beta,1})$.

[The only new point is that

$$G_{\beta,g(t)}(\mathrm{grad}_{\beta,g(t)} V, L_X g(t))$$

$$= dV\Big|_{g(t)}(L_X g(t))$$

$$= 0.]$$

Therefore the function

$$(g,s) \to \int_M \mathrm{div}_g\,(\pi_{\beta,g}(s))(X)\mathrm{vol}_g$$

is a first integral for Y_β. But $X \in \mathcal{D}^1(M)$ is arbitrary. So, along an integral curve $\gamma(t)$ for Y_β,

$$\mathrm{div}_{g(t)}\pi_\beta(t) \otimes |g(t)|^{1/2} \in \Lambda^1_d(M)$$

is necessarily a constant.

Notation: Let

$$\pi_g(s) = s - \mathrm{tr}_g(s)g.$$

Then

$$\pi_g = \pi_{\beta,g}$$

for the choice $\beta = \frac{1}{n} - 1$.

<u>LEMMA</u> We have

$$-\Delta_g \mathrm{tr}_g(s) - \delta_g \mathrm{div}_g\, s = -\delta_g \mathrm{div}_g\, \pi_g(s).$$

[In fact,

$$\begin{aligned} -\Delta_g \mathrm{tr}_g(s) &= -\mathrm{div}_g\, \mathrm{grad}_g\, \mathrm{tr}_g(s) \\ &= -\mathrm{div}_g\, g^{\#}(d\mathrm{tr}_g(s)) \\ &= \delta_g g^{\flat} g^{\#}(d\mathrm{tr}_g(s)) \\ &= \delta_g(d\mathrm{tr}_g(s)) \\ &= \delta_g \mathrm{div}_g\, (\mathrm{tr}_g(s)g). \end{aligned}$$

Therefore

$$\begin{aligned} &-\Delta_g \mathrm{tr}_g(s) - \delta_g \mathrm{div}_g\, s \\ &\qquad = \delta_g \mathrm{div}_g\, (\mathrm{tr}_g(s)g) - \delta_g \mathrm{div}_g\, s \end{aligned}$$

$$= \delta_g \mathrm{div}_g \ (\mathrm{tr}_g(s)g - s)$$

$$= - \delta_g \mathrm{div}_g \ \pi_g(s).]$$

Define a function $\Phi_\beta : T\underline{M}_0 \to C_d^\infty(M)$ by

$$\Phi_\beta(g,s) = (\tfrac{1}{2} [s,s]_{\beta,g} + S(g)) \otimes |g|^{1/2}.$$

Then Φ_β is the <u>energy density</u>:

$$E_\beta(g,s) = K_\beta(g,s) + V(g)$$

$$= \int_M \Phi_\beta(g,s).$$

<u>THEOREM</u> On the integral curves for Y_β,

$$\frac{d}{dt} \Phi_\beta(g,\dot{g}) + \delta_g \mathrm{div}_g \ \pi_g(\dot{g}) \otimes |g|^{1/2} = 0.$$

[First

$$\frac{d}{dt} \frac{1}{2} [\dot{g},\dot{g}]_{\beta,g} \otimes |g|^{1/2}$$

$$= ([\dot{g},\ddot{g}]_g + (\beta - \tfrac{1}{n}) \mathrm{tr}_g(\dot{g}) \mathrm{tr}_g(\ddot{g})) \otimes |g|^{1/2}$$

$$+ (-[\dot{g},\dot{g}*\dot{g}]_g - (\beta - \tfrac{1}{n}) \mathrm{tr}_g(\dot{g}) [\dot{g},\dot{g}]_g) \otimes |g|^{1/2}$$

$$+ (\tfrac{1}{2} [\dot{g},\dot{g}]_{\beta,g} \frac{\mathrm{tr}_g(\dot{g})}{2}) \otimes |g|^{1/2}.$$

Now insert the explicit expression for $\ddot{g}$ derived above.

- $[\dot{g},\ddot{g}]_g$ is the sum of five terms:

1. $[\dot{g},\dot{g}*\dot{g}]_g$.

2. $-\frac{1}{2}\,tr_g(\dot{g})\,[\dot{g},\dot{g}]_g$.

3. $\frac{1}{4\beta n}\,tr_g(\dot{g})\,[\dot{g},\dot{g}]_{\beta,g}$.

4. $[\dot{g},Ric(g)]_g$.

5. $\frac{2-n-2\beta n}{2\beta n^2}\,tr_g(\dot{g})S(g)$.

- $(\beta - \frac{1}{n})tr_g(\dot{g})tr_g(\ddot{g})$ is the sum of five terms:

6. $(\beta - \frac{1}{n})tr_g(\dot{g})\,[\dot{g},\dot{g}]_g$.

7. $-\frac{1}{2}\,(\beta - \frac{1}{n})tr_g(\dot{g})^3$.

8. $(\beta - \frac{1}{n})tr_g(\dot{g})\,\frac{1}{4\beta}\,[\dot{g},\dot{g}]_{\beta,g}$.

9. $(\beta - \frac{1}{n})tr_g(\dot{g})S(g)$.

10. $(\beta - \frac{1}{n})\,\frac{2-n-2\beta n}{2\beta n}\,tr_g(\dot{g})S(g)$.

There are two immediate cancellations, viz. term 1 cancels with $-[\dot{g},\dot{g}*\dot{g}]_g$ and term 6 cancels with $-(\beta - \frac{1}{n})tr_g(\dot{g})\,[\dot{g},\dot{g}]_g$. Consider next term 3 and term 8

$$+\frac{1}{4}\,tr_g(\dot{g})\,[\dot{g},\dot{g}]_{\beta,g}.$$

I.e.:

$$\left(\frac{1}{4\beta n} + \frac{1}{4\beta}\left(\beta - \frac{1}{n}\right) + \frac{1}{4}\right)\mathrm{tr}_g(\dot{g})\,[\dot{g},\dot{g}]_{\beta,g}$$

$$= \frac{1}{2}\,\mathrm{tr}_g(\dot{g})\,[\dot{g},\dot{g}]_{\beta,g}$$

or still,

$$\frac{1}{2}\,\mathrm{tr}_g(\dot{g})\left([\dot{g},\dot{g}]_g + \left(\beta - \frac{1}{n}\right)\mathrm{tr}_g(\dot{g})^2\right),$$

which cancels with term 2 + term 7. There remains

$$[\dot{g},\mathrm{Ric}(g)]_g$$

$$+ \left(\frac{2-n-2\beta n}{2\beta n^2} + \left(\beta - \frac{1}{n}\right) + \left(\beta - \frac{1}{n}\right)\frac{2-n-2\beta n}{2\beta n}\right)\mathrm{tr}_g(\dot{g})\,S(g).$$

But

$$\frac{2-n-2\beta n}{2\beta n^2} + \left(\beta - \frac{1}{n}\right)\left(1 + \frac{2-n-2\beta n}{2\beta n}\right)$$

$$= \frac{2-n-2\beta n}{2\beta n^2} + \left(\beta - \frac{1}{n}\right)\left(\frac{2\beta n+2-n-2\beta n}{2\beta n}\right)$$

$$= \frac{2-n-2\beta n}{2\beta n^2} + \frac{(n\beta-1)(2-n)}{2\beta n^2}$$

$$= \frac{2-n-2\beta n+2n\beta-2-n^2\beta+n}{2\beta n^2}$$

$$= -\frac{n^2\beta}{2\beta n^2} = -\frac{1}{2}.$$

Thus matters reduce to

$$[\dot{g},\mathrm{Ric}(g)]_g - \frac{1}{2}\,\mathrm{tr}_g(\dot{g})\,S(g).$$

However

$$\frac{d}{dt} S(g) \otimes |g|^{1/2}$$

$$= (-\Delta_g \mathrm{tr}_g(\dot{g}) - \delta_g \mathrm{div}_g \, \dot{g} - [\mathrm{Ric}(g),\dot{g}]_g) \otimes |g|^{1/2}$$

$$+ \frac{1}{2} \mathrm{tr}_g(\dot{g}) S(g) \otimes |g|^{1/2}.$$

Therefore

$$\frac{d}{dt} \Phi_\beta(g,\dot{g}) = \frac{d}{dt} \frac{1}{2} [\dot{g},\dot{g}]_{\beta,g} \otimes |g|^{1/2} + \frac{d}{dt} S(g) \otimes |g|^{1/2}$$

$$= (-\Delta_g \mathrm{tr}_g(\dot{g}) - \delta_g \mathrm{div}_g \, \dot{g}) \otimes |g|^{1/2}$$

$$= - \delta_g \mathrm{div}_g \, \pi_g(\dot{g}) \otimes |g|^{1/2},$$

which completes the proof.]

While this result is valid $\forall \beta \neq 0$, it is hybrid in character and points to the significance of the DeWitt metric: The choice $\beta = \frac{1}{n} - 1$ is the parameter value per π_g, hence along an integral curve $\gamma(t)$ for $Y_{\frac{1}{n} - 1}$,

$$\mathrm{div}_{g(t)} \pi(t) \otimes |g(t)|^{1/2}$$

is a constant. Accordingly, if at $t = 0$,

$$\mathrm{div}_{g(0)} \pi(0) = 0,$$

then $\forall$ t,

$$\mathrm{div}_{g(t)} \pi(t) = 0,$$

thus

$$\frac{d}{dt}\Phi_{\frac{1}{n}-1} = 0$$

and so $\Phi_{\frac{1}{n}-1}$ is pointwise constant in time.

[Note: Here $\pi(t)$ stands for $\pi_{g(t)}(s(t))$, where $\gamma(t) = (g(t),s(t))$.]

Remark: Let C be a nonzero constant. Replace V by CV (a.k.a. V_C) and define a function $\Phi_{\beta,C}:T\underline{M}_0 \rightarrow C_d^\infty(M)$ by

$$\Phi_{\beta,C}(g,s) = (\frac{1}{2}[s,s]_{\beta,g} + CS(g)) \otimes |g|^{1/2}.$$

Then, on the integral curves of $Y_{\beta,C}$,

$$\frac{d}{dt}\Phi_{\beta,C}(g,\dot{g}) + C\delta_g \mathrm{div}_g\, \pi_g(\dot{g}) \otimes |g|^{1/2} = 0.$$

Let

$$H_\beta = \Phi_\beta \circ \phi_\beta^{-1}.$$

Then

$$H_\beta:T^*\underline{M}_0 \rightarrow C_d^\infty(M).$$

<u>LEMMA</u> We have

$$H_\beta(g,\Lambda) = (\frac{1}{2}[s,s]_g - \frac{1}{2\beta n}(\beta - \frac{1}{n})\mathrm{tr}_g(s)^2 + S(g)) \otimes |g|^{1/2}$$

if $\Lambda = s^\# \otimes |g|^{1/2}$.

[Since

$$\phi_\beta(g,s) = (g, G^\flat_{\beta,g}(s)),$$

it follows that

$$\phi_\beta^{-1}(g,\Lambda) = (g, G^\#_{\beta,g}(\Lambda)).$$

Therefore

$$H_\beta(g, s^\# \otimes |g|^{1/2})$$

$$= \Phi_\beta(s, G^\#_{\beta,g}(s^\# \otimes |g|^{1/2}))$$

$$= \Phi_\beta(g, s + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)g)$$

$$= (\frac{1}{2}[s + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)g, s + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)g]_{\beta,g} + S(g)) \otimes |g|^{1/2}.$$

• $\frac{1}{2}[s + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)g, s + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)g]_g$

$$= \frac{1}{2}[s,s]_g + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)[s,s]_g$$

$$+ \frac{1}{2}\frac{1}{\beta^2 n^2}(\frac{1}{n} - \beta)^2 \operatorname{tr}_g(s)^2 [g,g]_g$$

$$= \frac{1}{2}[s,s]_g + (\frac{1}{\beta n}(\frac{1}{n} - \beta) + \frac{1}{2}\frac{1}{\beta^2 n}(\frac{1}{n} - \beta)^2)\operatorname{tr}_g(s)^2.$$

• $\frac{1}{2}(\beta - \frac{1}{n})(\operatorname{tr}_g(s + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)g))^2$

$$= \frac{1}{2}(\beta - \frac{1}{n})(\operatorname{tr}_g(s) + \frac{1}{\beta n}(\frac{1}{n} - \beta)\operatorname{tr}_g(s)\operatorname{tr}_g(g))^2$$

$$= \frac{1}{2}(\beta - \frac{1}{n})(\operatorname{tr}_g(s) + \frac{1}{\beta}(\frac{1}{n} - \beta)\operatorname{tr}_g(s))^2$$

$$= \frac{1}{2}\,(\beta - \frac{1}{n})\,(1 + \frac{1}{\beta}\,(\frac{1}{n} - \beta))^2 \mathrm{tr}_g(s)^2$$

$$= \frac{1}{2}\,\frac{1}{\beta^2 n^2}\,(\beta - \frac{1}{n})\mathrm{tr}_g(s)^2.$$

Thus the coefficient of $\mathrm{tr}_g(s)^2$ is

$$\frac{1}{\beta n}\,(\frac{1}{n} - \beta)\; + \frac{1}{2}\,\frac{1}{\beta^2 n}\,(\frac{1}{n} - \beta)^2 + \frac{1}{2}\,\frac{1}{\beta^2 n^2}\,(\beta - \frac{1}{n})$$

or still,

$$\frac{1}{\beta n}\,(\frac{1}{n} - \beta)\,(1 + \frac{1}{2\beta}\,(\frac{1}{n} - \beta)\; - \frac{1}{2\beta n})$$

$$= - \frac{1}{2\beta n}\,(\beta - \frac{1}{n}).]$$

Example: Take $\beta = \frac{1}{n} - 1$ -- then

$$- \frac{1}{2\beta n}\,(\beta - \frac{1}{n}) = - \frac{1}{2(n-1)}\,,$$

so

$$H_{\frac{1}{n} - 1}(g,\Lambda) = (\frac{1}{2}[s,s]_g - \frac{1}{2(n-1)}\,\mathrm{tr}_g(s)^2 + S(g))\otimes |g|^{1/2}$$

if $\Lambda = s^\# \otimes |g|^{1/2}$, which implies that

$$H_{\frac{1}{n} - 1}(g,G_g^\flat(s))$$

$$= H_{\frac{1}{n} - 1}(g,(s - \mathrm{tr}_g(s)g)^\# \otimes |g|^{1/2})$$

$$= (\frac{1}{2}[s,s]_g - \frac{1}{2}\,\mathrm{tr}_g(s)^2 + S(g))\otimes |g|^{1/2}.$$

Define a function $\mathcal{H}_\beta : T^*\underline{M}_0 \to \underline{R}$ by

$$\mathcal{H}_\beta(g,\Lambda) = \int_M H_\beta(g,\Lambda).$$

Then

$$d\mathcal{H}_\beta\Big|_{(g,\Lambda)} : T_{(g,\Lambda)} T^*\underline{M}_0 \to \underline{R},$$

where

$$d\mathcal{H}_\beta\Big|_{(g,\Lambda)}(s,\Lambda') = \frac{d}{d\varepsilon}\,\mathcal{H}_\beta(g + \varepsilon s, \Lambda + \varepsilon\Lambda')\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\,\mathcal{H}_\beta(g + \varepsilon s, \Lambda)\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon}\,\mathcal{H}_\beta(g, \Lambda + \varepsilon\Lambda')\Big|_{\varepsilon=0}$$

$$= < s, \frac{\delta\mathcal{H}_\beta}{\delta g} > + < \frac{\delta\mathcal{H}_\beta}{\delta\Lambda}, \Lambda' >.$$

Definition: The <u>hamiltonian vector field</u>

$$Z_\beta : \underline{M}_0 \times \mathcal{S}^2_d(M) \to \mathcal{S}_2(M) \times \mathcal{S}^2_d(M)$$

on $T^*\underline{M}_0$ corresponding to $\mathcal{H}_\beta$ is given by the prescription

$$Z_\beta(g,\Lambda) = \left(\frac{\delta\mathcal{H}_\beta}{\delta\Lambda}, -\frac{\delta\mathcal{H}_\beta}{\delta g} \right).$$

To justify the terminology, let Z be any vector field on $T^*\underline{M}_0$. Suppose that $Z(g,\Lambda) = (s,\Lambda')$ -- then

$$\Omega_{(g,\Lambda)}(Z_\beta(g,\Lambda), Z(g,\Lambda))$$

$$= \Omega_{(g,\Lambda)}\left(\left(\frac{\delta\mathcal{H}_\beta}{\delta\Lambda}, -\frac{\delta\mathcal{H}_\beta}{\delta g} \right), (s,\Lambda')\right)$$

$$= \ < \frac{\delta H_\beta}{\delta \Lambda}, \Lambda' > - < s, -\frac{\delta H_\beta}{\delta g} >$$

$$= < s, \frac{\delta H_\beta}{\delta g} > + < \frac{\delta H_\beta}{\delta \Lambda}, \Lambda' >$$

$$= dH_\beta \Big|_{(g,\Lambda)} (s,\Lambda').$$

Observation: The diagram

$$\begin{array}{ccc} T T\underline{M}_0 & \xrightarrow{T\phi_\beta} & T T^*\underline{M}_0 \\ Y_\beta \uparrow & & \uparrow Z_\beta \\ T\underline{M}_0 & \xrightarrow{\phi_\beta} & T^*\underline{M}_0 \\ E_\beta \downarrow & & \downarrow H_\beta \\ \underline{R} & & \underline{R} \end{array}$$

commutes.

Therefore

$$(\phi_\beta)_* Y_\beta = Z_\beta.$$

Moreover, if $\gamma(t)$ is an integral curve for Y_β and $c(t)$ is an integral curve for Z_β and if $\phi_\beta\gamma(0) = c(0)$, then $\phi_\beta\gamma(t) = c(t)$, hence the projections of $\gamma(t)$ and $c(t)$ onto $\underline{M}_0$ coincide.

Remark: Hamilton's equations are, by definition, the system of differential equations defined by Z_β:

$$\frac{dc}{dt} = Z_\beta \Big|_{c(t)}.$$

Section 34: Constant Lapse, Zero Shift Let M be a connected C^∞ manifold of dimension $n > 2$. Fix ε $(0 < \varepsilon \leq \infty)$ and assume that

$$M =]-\varepsilon,\varepsilon[\times \Sigma,$$

where Σ is compact and orientable (hence $\dim \Sigma = n - 1$).

[Note: Σ is going to play the role of the M from the previous section, so when quoting results from there, one must replace n by n - 1.]

Notation: Q is the set of riemannian structures on Σ, thus now

$$\begin{cases} TQ = Q \times S_2(\Sigma) \\ T^*Q = Q \times S_d^2(\Sigma). \end{cases}$$

Fix a nonzero constant N (the lapse). Suppose that $t \to q(t)$ $(= q_t)$ $(t \in]-\varepsilon,\varepsilon[)$ is a path in Q -- then the prescription

$$g_{(t,x)}((r,X),(s,Y))$$

$$= -rsN^2 + q_x(t)(X,Y) \quad (r,s \in \underline{R} \ \& \ X,Y \in T_x\Sigma)$$

defines an element of $\underline{M}_{1,n-1}$ $(g_{00} = g(\partial_0,\partial_0) = -N^2)$.

Notation: Indices a,b,c run from 1 to n - 1.

SUBLEMMA In adapted coordinates, the connection coefficients of g are given by

$$\begin{cases} \Gamma^c{}_{ab}(t,x) = (\Gamma_t)^c{}_{ab}(x) \\ \Gamma^0{}_{ab}(t,x) = \dfrac{1}{2N^2}(\dot{q}_t)_{ab}(x) \\ \Gamma^c{}_{0b}(t,x) = \dfrac{1}{2}(\dot{q}_t)^c{}_b(x) \end{cases}$$

and

$$\Gamma^0{}_{00}(t,x) = \Gamma^0{}_{0b}(t,x) = \Gamma^c{}_{00}(t,x) = 0.$$

<u>LEMMA</u> In adapted coordinates, the components of Ric(g) are given by

- $R_{a0}(t,x) = -\frac{1}{2}\,[dtr_{q_t}(\dot{q}_t)_a(x) - (div_{q_t}\dot{q}_t)_a(x)]$

- $R_{00}(t,x) = -\frac{1}{2}\,tr_{q_t}(\ddot{q}_t)(x) + \frac{1}{4}\,[\dot{q}_t,\dot{q}_t]_{q_t}(x)$

- $R_{ab}(t,x) = \frac{1}{2N^2}\,(\ddot{q}_t)_{ab}(x)$

$$-\frac{1}{2N^2}\,(\dot{q}_t * \dot{q}_t)_{ab}(x) + \frac{1}{4N^2}\,tr_{q_t}(\dot{q}_t)(x)\,(\dot{q}_t)_{ab}(x)$$

$$+\,Ric(q_t)_{ab}(x).$$

<u>THEOREM</u> Ric(g) = 0 iff q_t satisfies the differential equation

$$\ddot{q}_t = \Gamma(q_t,\dot{q}_t) + 2N^2\,grad_{q_t}V$$

and the constraints

$$\begin{cases} div_{q_t}(\dot{q}_t - tr_{q_t}(\dot{q}_t)q_t) = 0 \\ \frac{1}{2}\,([\dot{q}_t,\dot{q}_t]_{q_t} - tr_{q_t}(\dot{q}_t)^2) - 2N^2\,S(q_t) = 0. \end{cases}$$

We shall start with the assumption that Ric(g) = 0.

Rappel:

$$\Gamma(q_t,\dot{q}_t) = \dot{q}_t * \dot{q}_t - \frac{1}{2}\,\mathrm{tr}_{q_t}(\dot{q}_t)\dot{q}_t$$

$$+ \frac{1}{4(n-2)}\,(\mathrm{tr}_{q_t}(\dot{q}_t)^2 - [\dot{q}_t,\dot{q}_t]_{q_t})q_t$$

and

$$\mathrm{grad}_{q_t} V = -\,\mathrm{Ric}(q_t) + \frac{1}{2(n-2)}\,S(q_t)q_t.$$

[Note: Recall that $\mathrm{grad}_{q_t} V$ stands for the gradient of V in the DeWitt metric (which here amounts to choosing $\beta = \frac{1}{n-1} - 1$).]

• $R_{ab} = 0$

$\Rightarrow$

$$-\frac{1}{2N^2}\,(\ddot{q}_t)_{ab}$$

$$= -\frac{1}{2N^2}\,(\dot{q}_t * \dot{q}_t)_{ab} + \frac{1}{4N^2}\,\mathrm{tr}_{q_t}(\dot{q}_t)(\dot{q}_t)_{ab}$$

$$+ \mathrm{Ric}(q_t)_{ab}$$

$\Rightarrow$

$$\ddot{q}_t = \dot{q}_t * \dot{q}_t - \frac{1}{2}\,\mathrm{tr}_{q_t}(\dot{q}_t)\dot{q}_t - 2N^2\,\mathrm{Ric}(q_t).$$

• $R_{00} = 0$

$\Rightarrow$

$$\mathrm{tr}_{q_t}(\ddot{q}_t) = \frac{1}{2}\,[\dot{q}_t,\dot{q}_t]_{q_t}$$

$\Rightarrow$

$$\mathrm{tr}_{q_t}(\dot{q}_t * \dot{q}_t) - \frac{1}{2}\,\mathrm{tr}_{q_t}(\dot{q}_t)^2 - 2N^2\,\mathrm{tr}_{q_t}\,\mathrm{Ric}(q_t) = \frac{1}{2}\,[\dot{q}_t,\dot{q}_t]_{q_t}$$

$\Rightarrow$

$$[\dot{q}_t,\dot{q}_t]_{q_t} - \frac{1}{2}\,\mathrm{tr}_{q_t}(\dot{q}_t)^2 - 2N^2\,S(q_t) = \frac{1}{2}\,[\dot{q}_t,\dot{q}_t]_{q_t}$$

$\Rightarrow$

$$\frac{1}{2}\,([\dot{q}_t,\dot{q}_t]_{q_t} - \mathrm{tr}_{q_t}(\dot{q}_t)^2) - 2N^2\,S(q_t) = 0.$$

Therefore

$$\frac{1}{4(n-2)}\,(\mathrm{tr}_{q_t}(\dot{q}_t)^2 - [\dot{q}_t,\dot{q}_t]_{q_t})q_t + \frac{1}{n-2}\,N^2\,S(q_t)q_t$$

$$= \frac{1}{n-2}\,(\frac{1}{4}\,(\mathrm{tr}_{q_t}(\dot{q}_t)^2 - [\dot{q}_t,\dot{q}_t]_{q_t}) + N^2\,S(q_t))q_t$$

$$= 0.$$

But then

$$\ddot{q}_t = \Gamma(q_t,\dot{q}_t) + 2N^2\,\mathrm{grad}_{q_t}V,$$

as claimed.

[Note: Since

$$2N^2\,\mathrm{grad}_{q_t}V = -\,(\,-2N^2\,\mathrm{grad}_{q_t}V)$$

$$= -\,\mathrm{grad}_{q_t}V_{-2N^2},$$

it follows that the curve $t \to q(t)$ $(t\in\,]-\varepsilon,\varepsilon[)$ is a geodesic per

$$Y_{\frac{1}{n-1} - 1, \; - 2N^2}.]$$

Finally

$$R_{a0} = 0$$

$\Rightarrow$

$$(\mathrm{div}_{q_t} \dot{q}_t)_a = \mathrm{dtr}_{q_t}(\dot{q}_t)_a$$

$\Rightarrow$

$$(\mathrm{div}_{q_t} \dot{q}_t)_a = (\mathrm{div}_{q_t}(\mathrm{tr}_{q_t}(\dot{q}_t)q_t))_a$$

$\Rightarrow$

$$\mathrm{div}_{q_t}(\dot{q}_t - \mathrm{tr}_{q_t}(\dot{q}_t)q_t) = 0.$$

Thus, in summary, the stated conditions on q_t are necessary. That they are also sufficient can be established by running the argument in reverse.

Remark: By definition,

$$\pi(t) = \dot{q}_t - \mathrm{tr}_{q_t}(\dot{q}_t)q_t.$$

Therefore

$$\mathrm{div}_{q_t}(\dot{q}_t - \mathrm{tr}_{q_t}(\dot{q}_t)q_t) = \mathrm{div}_{q_t}\pi(t).$$

On the other hand,

$$E_{\frac{1}{n-1} - 1, \; -2N^2}(q_t, \dot{q}_t) = K_{\frac{1}{n-1} - 1}(q_t, \dot{q}_t) + V_{-2N^2}(q_t)$$

$$= \int_\Sigma \Phi_{\frac{1}{n-1} - 1,\ -2N^2}(q_t, \dot{q}_t),$$

where

$$\Phi_{\frac{1}{n-1} - 1,\ -2N^2}(q_t, \dot{q}_t)$$

$$= (\tfrac{1}{2}\ ([\dot{q}_t, \dot{q}_t]_{q_t} - \mathrm{tr}_{q_t}(\dot{q}_t)^2) - 2N^2\ S(q_t)) \otimes |q_t|^{1/2}.$$

<u>FACT</u> If $\mathrm{Ric}(g) = 0$ and if

$$\ddot{q}_t = \Gamma_\beta(q_t, \dot{q}_t) + 2N^2\ \mathrm{grad}_{\beta, q_t} V$$

subject to

$$\frac{1}{2}\ [\dot{q}_t, \dot{q}_t]_{\beta, q_t} - 2N^2\ S(q_t) = 0$$

for some $\beta \neq \frac{1}{n-1} - 1$, then $q_t = q_0$ for all t and $\mathrm{Ric}(q_0) = 0$.

We shall now transfer the theory from TQ to T^*Q. For this purpose, it will be simplest to first change the initial data, which is the path

$$t \rightarrow (q_t, \dot{q}_t)$$

in TQ.

Let $\underline{n}_t = \frac{1}{N}\,\partial_t$ — then

$$g(\underline{n}_t, \underline{n}_t) = \frac{1}{N^2}\, g(\partial_t, \partial_t)$$

$$= -\frac{N^2}{N^2} = -1.$$

Given $t \in\ ]-\varepsilon, \varepsilon[$, put $\Sigma_t = \{t\} \times \Sigma$ and let $i_t : \Sigma \approx \Sigma_t \rightarrow M$ be the embedding.

Working with the metric connection of g, let $\varkappa_t \in S_2(\Sigma)$ be the extrinsic curvature, thus

$$\varkappa_t(V,W) = q_t(-i_t^*\nabla_V \underline{n}_t, W)\, g(\underline{n}_t, \underline{n}_t)$$

$$= q_t(i_t^*\nabla_V \underline{n}_t, W).$$

<u>LEMMA</u> We have

$$(\varkappa_t)_{ab} = \frac{1}{2N}(\dot{q}_t)_{ab}.$$

[In fact,

$$\begin{cases} [\partial_t, \partial_a] = 0 \\ [\partial_t, \partial_b] = 0 \end{cases} \Rightarrow \begin{cases} \nabla_{\partial_t}\partial_a = \nabla_{\partial_a}\partial_t \\ \nabla_{\partial_t}\partial_b = \nabla_{\partial_b}\partial_t \end{cases}$$

$\Rightarrow$

$$\partial_t(q_t)_{ab} = q_t(i_t^*\nabla_{\partial_a}(N\underline{n}_t), \partial_b) + q_t(\partial_a, i_t^*\nabla_{\partial_b}(N\underline{n}_t))$$

$$= N((\varkappa_t)_{ab} + (\varkappa_t)_{ba})$$

$$= 2N(\varkappa_t)_{ab}$$

$\Rightarrow$

$$(\varkappa_t)_{ab} = \frac{1}{2N}(\dot{q}_t)_{ab}.]$$

So, instead of the path

$$t \to (q_t, \dot{q}_t),$$

we can just as well work with the path

$$t \to (q_t, \varkappa_t).$$

Put $K_t = \operatorname{tr}_{q_t}(\varkappa_t)$ — then

$$\operatorname{tr}_{q_t}(\dot{q}_t) = 2NK_t.$$

Definition: The <u>momentum</u> of the theory is the path $t \to p_t$ in $S_d^2(\Sigma)$ defined by the prescription

$$p_t = \pi_t \otimes |q_t|^{1/2},$$

where

$$\pi_t = (\varkappa_t - K_t q_t)^{\#}.$$

<u>LEMMA</u> We have

$$\varkappa_t = \pi_t^{\flat} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t^{\flat}) q_t.$$

[Simply observe that

$$\pi_t^{\flat} = \varkappa_t - K_t q_t$$

$\Rightarrow$

$$\varkappa_t = \pi_t^{\flat} + K_t q_t$$

$\Rightarrow$

$$K_t = \operatorname{tr}_{q_t}(\pi_t^{\flat}) + K_t \operatorname{tr}_{q_t}(q_t)$$

$$= \mathrm{tr}_{q_t}(\pi_t^{\flat}) + (n-1)K_t$$

$\Rightarrow$

$$\mathrm{tr}_{q_t}(\pi_t^{\flat}) = (2-n)K_t$$

$\Rightarrow$

$$\varkappa_t = \pi_t^{\flat} - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t^{\flat})q_t.]$$

Therefore

$$\dot{q}_t = 2N\varkappa_t$$

$$= 2N(\pi_t^{\flat} - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t^{\flat})q_t).$$

Consider the relations figuring in the theorem, beginning with the constraints.

• $\mathrm{div}_{q_t}(\dot{q}_t - \mathrm{tr}_{q_t}(\dot{q}_t)q_t) = 0.$

In terms of $\varkappa_t$, this reads

$$\mathrm{div}_{q_t}(2N\varkappa_t - 2NK_tq_t) = 0$$

or still,

$$\mathrm{div}_{q_t}(\varkappa_t - K_tq_t) = 0.$$

But $\mathrm{div}_{q_t}p_t$ is, by definition,

$$\mathrm{div}_{q_t}(\varkappa_t - K_tq_t) \otimes |q_t|^{1/2},$$

thus our constraint becomes

$$\mathrm{div}_{q_t} p_t = 0.$$

- $\frac{1}{2}([\dot{q}_t, \dot{q}_t]_{q_t} - \mathrm{tr}_{q_t}(\dot{q}_t)^2) - 2N^2 S(q_t) = 0.$

In terms of κ_t, this reads

$$\frac{1}{2}((2N)^2[\kappa_t, \kappa_t]_{q_t} - (2N)^2 K_t^2) - 2N^2 S(q_t) = 0$$

or still,

$$([\kappa_t, \kappa_t]_{q_t} - K_t^2) - S(q_t) = 0$$

or still,

$$[\pi_t^\flat - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t^\flat) q_t,\ \pi_t^\flat - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t^\flat) q_t]_{q_t}$$

$$-\ (\mathrm{tr}_{q_t}(\pi_t^\flat) - \frac{n-1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t^\flat))^2 - S(q_t) = 0$$

or still,

$$[\pi_t^\flat, \pi_t^\flat]_{q_t} + (-\frac{2}{n-2} + \frac{n-1}{(n-2)^2} - \frac{1}{(n-2)^2})\mathrm{tr}_{q_t}(\pi_t^\flat)^2 - S(q_t) = 0$$

or still,

$$[\pi_t^\flat, \pi_t^\flat]_{q_t} - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t^\flat)^2 - S(q_t) = 0$$

or still,

$$([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t)^2 - S(q_t)) \otimes |q_t|^{1/2} = 0,$$

where we have set

$$\begin{cases} [\pi_t, \pi_t]_{q_t} = q_t[{}^2_0](\pi_t, \pi_t) \quad (= q_t[{}^0_2](\pi_t^\flat, \pi_t^\flat)) \\ \mathrm{tr}_{q_t}(\pi_t) = q_t[{}^2_0](\pi_t, q_t^\#) \quad (= q_t[{}^0_2](\pi_t^\flat, q_t)). \end{cases}$$

It remains to reformulate the differential equation

$$\ddot{q}_t = \Gamma(q_t,\dot{q}_t) + 2N^2 \operatorname{grad}_{q_t} V$$

$$= \dot{q}_t * \dot{q}_t - \frac{1}{2} \operatorname{tr}_{q_t}(\dot{q}_t)\dot{q}_t$$

$$+ \frac{1}{4(n-2)} (\operatorname{tr}_{q_t}(\dot{q}_t)^2 - [\dot{q}_t,\dot{q}_t]_{q_t})q_t$$

$$+ 2N^2 (- \operatorname{Ric}(q_t) + \frac{1}{2(n-2)} S(q_t)q_t)$$

in terms of p_t.

We have

$$\dot{p}_t = \frac{d}{dt} \pi_t \otimes |q_t|^{1/2} + \pi_t \otimes \frac{d}{dt} |q_t|^{1/2},$$

where

$$\frac{d}{dt} \pi_t = \frac{d}{dt} (\varkappa_t - K_t q_t)^{\#}$$

$$= \frac{d}{dt} \varkappa_t^{\#} - \frac{d}{dt} (K_t q_t^{\#})$$

$$= \frac{d}{dt} \varkappa_t^{\#} - (\frac{d}{dt} K_t) q_t^{\#} - K_t (\frac{d}{dt} q_t^{\#}).$$

Formulas

- $\frac{d}{dt} \varkappa_t^{\#} = (\dot{\varkappa}_t)^{\#} - 4N(\varkappa_t * \varkappa_t)^{\#}.$

- $\frac{d}{dt} K_t = - 2N[\varkappa_t,\varkappa_t]_{q_t} + \operatorname{tr}_{q_t}(\dot{\varkappa}_t).$

- $\frac{d}{dt} q_t^{\#} = - 2N\varkappa_t^{\#}$.

- $\frac{d}{dt} |q_t|^{1/2} = NK_t|q_t|^{1/2}$.

To isolate $\dot{\varkappa}_t$, one need only divide $\ddot{q}_t$ by 2N.

1. $\frac{1}{2N} \dot{q}_t * \dot{q}_t$

$$= \frac{1}{2N} (2N)^2 \varkappa_t * \varkappa_t$$

$$= 2N(\varkappa_t * \varkappa_t).$$

2. $\frac{1}{2N} (- \frac{1}{2} \operatorname{tr}_{q_t} (\dot{q}_t)\dot{q}_t)$

$$= \frac{1}{2N} (- \frac{1}{2} \operatorname{tr}_{q_t} (2N\varkappa_t) 2N\varkappa_t)$$

$$= - NK_t \varkappa_t.$$

3. $\frac{1}{2N} \frac{1}{4(n-2)} (\operatorname{tr}_{q_t} (\dot{q}_t)^2 - [\dot{q}_t, \dot{q}_t]_{q_t}) q_t$

$$= \frac{1}{2N} \frac{1}{4(n-2)} (\operatorname{tr}_{q_t} (2N\varkappa_t)^2 - [2N\varkappa_t, 2N\varkappa_t]_{q_t}) q_t$$

$$= \frac{N}{2(n-2)} (K_t^2 - [\varkappa_t, \varkappa_t]_{q_t}) q_t.$$

4. $\frac{1}{2N} (2N^2(- \operatorname{Ric}(q_t) + \frac{1}{2(n-2)} S(q_t) q_t))$

$$= - N\mathrm{Ric}(q_t) + \frac{N}{2(n-2)} S(q_t)q_t.$$

Therefore

$$\dot{\varkappa}_t = 1 + 2 + 3 + 4.$$

And then

$$\mathrm{tr}_{q_t}(\dot{\varkappa}_t) = \mathrm{tr}_{q_t}(1) + \mathrm{tr}_{q_t}(2) + \mathrm{tr}_{q_t}(3) + \mathrm{tr}_{q_t}(4)$$

$$= 2N[\varkappa_t,\varkappa_t]_{q_t} - NK_t^2$$

$$+ \frac{N}{2}\frac{(n-1)}{(n-2)}(K_t^2 - [\varkappa_t,\varkappa_t]_{q_t})$$

$$- NS(q_t) + \frac{N}{2}\frac{(n-1)}{(n-2)} S(q_t).$$

From the above,

$$\dot{p}_t = (\dot{\varkappa}_t)^\# \otimes |q_t|^{1/2} - 4N(\varkappa_t * \varkappa_t)^\# \otimes |q_t|^{1/2}$$

$$+ 2N[\varkappa_t,\varkappa_t]_{q_t} q_t^\# \otimes |q_t|^{1/2} - \mathrm{tr}_{q_t}(\dot{\varkappa}_t)q_t^\# \otimes |q_t|^{1/2}$$

$$+ 2NK_t\varkappa_t^\# \otimes |q_t|^{1/2} + NK_t\pi_t \otimes |q_t|^{1/2}.$$

To assemble the terms involving $\mathrm{Ric}(q_t)$ and $S(q_t)$, note that

$$4^\# = - N\mathrm{Ric}(q_t)^\# + \frac{N}{2(n-2)} S(q_t)q_t^\#.$$

However, there is also a contribution from $- \mathrm{tr}_{q_t}(\dot{\varkappa}_t)q_t^\#$, viz.

$$(NS(q_t) - \frac{N}{2}\frac{(n-1)}{(n-2)} S(q_t))q_t^\#.$$

But

$$\frac{N}{2(n-2)} + N - \frac{N}{2}\,\frac{(n-1)}{(n-2)}$$

$$= N\left(\frac{1}{2(n-2)}\,(1 - n + 1) + 1\right)$$

$$= N\left(\frac{2-n}{2(n-2)} + 1\right) = \frac{N}{2}.$$

Thus we are left with

$$- N\left(\mathrm{Ric}(q_t) - \frac{1}{2}\,S(q_t)q_t\right)^{\#}$$

$$= - N\mathrm{Ein}(q_t)^{\#}.$$

Next

$$1^{\#} - 4N(\varkappa_t * \varkappa_t)^{\#} = - 2N(\varkappa_t * \varkappa_t)^{\#},$$

which leaves

$$- NK_t\varkappa_t^{\#} + \frac{N}{2(n-2)}\,(K_t^2 - [\varkappa_t,\varkappa_t]_{q_t})q_t^{\#}$$

$$+ 2N[\varkappa_t,\varkappa_t]_{q_t}q_t^{\#} - 2N[\varkappa_t,\varkappa_t]_{q_t}q_t^{\#} + NK_t^2 q_t^{\#}$$

$$- \frac{N}{2}\,\frac{(n-1)}{(n-2)}\,(K_t^2 - [\varkappa_t,\varkappa_t]_{q_t})q_t^{\#} + 2NK_t\varkappa_t^{\#} + NK_t(\varkappa_t^{\#} - K_t q_t^{\#}).$$

Now collate the data and collect terms.

- The coefficient of $K_t\varkappa_t^{\#}$ is

$$- N + 2N + N = 2N.$$

- The coefficent of $K_t^2 q_t^\#$ is

$$\frac{N}{2(n-2)} + N - \frac{N}{2}\frac{(n-1)}{(n-2)} - N$$

$$= \frac{N}{2(n-2)}\ (1 - n + 1) = -\frac{N}{2}.$$

- The coefficent of $[\varkappa_t, \varkappa_t]_{q_t} q_t^\#$ is

$$-\frac{N}{2(n-2)} + 2N - 2N + \frac{N}{2}\frac{(n-1)}{(n-2)}$$

$$= \frac{N}{2(n-2)}\ (-1 + n - 1) = \frac{N}{2}.$$

To recapitulate:

$$\dot{p}_t = -2N(\varkappa_t * \varkappa_t)^\# \otimes |q_t|^{1/2} + 2NK_t \varkappa_t^\# \otimes |q_t|^{1/2}$$

$$-\frac{N}{2} K_t^2 q_t^\# \otimes |q_t|^{1/2} + \frac{N}{2} [\varkappa_t, \varkappa_t]_{q_t} q_t^\# \otimes |q_t|^{1/2}$$

$$- N\mathrm{Ein}(q_t)^\# \otimes |q_t|^{1/2}.$$

But we are not done yet: It is best to replace $\varkappa_t$ by π_t.

Observation: Since

$$\varkappa_t = \pi_t^\flat - \frac{1}{n-2} \mathrm{tr}_{q_t}(\pi_t^\flat) q_t$$

$$= \pi_t^\flat - \frac{1}{n-2} (2-n) K_t q_t$$

$$= \pi_t^\flat + K_t q_t,$$

it follows that

$$(\varkappa_t * \varkappa_t)_{ab} = ((\pi_t^\flat + K_t q_t) * (\pi_t^\flat + K_t q_t))_{ab}$$

$$= (\pi_t^\flat + K_t q_t)_{ac} (\pi_t^\flat + K_t q_t)^c{}_b$$

$$= (\pi_t^\flat)_{ac} (\pi_t^\flat)^c{}_b$$

$$+ (K_t q_t)_{ac} (\pi_t^\flat)^c{}_b + (\pi_t^\flat)_{ac} (K_t q_t)^c{}_b$$

$$+ (K_t)^2 (q_t)_{ac} (q_t)^c{}_b$$

$$= (\pi_t^\flat * \pi_t^\flat)_{ab} + 2K_t (\pi_t^\flat)_{ab} + (K_t)^2 (q_t)_{ab}.$$

Accordingly,

$$- 2N(\varkappa_t * \varkappa_t)^\#$$

$$= - 2N(\pi_t * \pi_t + 2K_t \pi_t + (K_t)^2 q_t^\#),$$

where, by definition,

$$\pi_t * \pi_t = (\pi_t^\flat * \pi_t^\flat)^\#.$$

Therefore

$$- 2N(\varkappa_t * \varkappa_t)^\# + 2NK_t \varkappa_t^\#$$

$$= - 2N(\pi_t * \pi_t) - 4NK_t \pi_t - 2N(K_t)^2 q_t^\#$$

$$+ 2NK_t \pi_t + 2N(K_t)^2 q_t^\#$$

$$= -2N(\pi_t * \pi_t) - 2NK_t \pi_t$$

$$= -2N(\pi_t * \pi_t - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t)\pi_t).$$

The last item of detail is

$$-\frac{N}{2} K_t^2 q_t^\# + \frac{N}{2} [\varkappa_t, \varkappa_t]_{q_t} q_t^\#.$$

Write

$$[\varkappa_t, \varkappa_t]_{q_t} = [\pi_t^\flat + K_t q_t, \pi_t^\flat + K_t q_t]_{q_t}$$

$$= [\pi_t^\flat, \pi_t^\flat]_{q_t} + 2K_t [\pi_t^\flat, q_t]_{q_t} + (K_t)^2 [q_t, q_t]_{q_t}$$

$$= [\pi_t, \pi_t]_{q_t} + 2K_t \mathrm{tr}_{q_t}(\pi_t) + (n-1)(K_t)^2$$

$$= [\pi_t, \pi_t]_{q_t} + 2K_t (2-n) K_t + (n-1)(K_t)^2$$

$$= [\pi_t, \pi_t]_{q_t} + (3-n) K_t^2.$$

Then

$$-\frac{N}{2} K_t^2 q_t^\# + \frac{N}{2} [\varkappa_t, \varkappa_t]_{q_t} q_t^\#$$

$$= \frac{N}{2} ([\pi_t, \pi_t]_{q_t} + (3-n) K_t^2 - K_t^2) q_t^\#$$

$$= \frac{N}{2} ([\pi_t, \pi_t]_{q_t} + (2-n) K_t^2) q_t^\#$$

$$= \frac{N}{2} ([\pi_t, \pi_t]_{q_t} + \frac{2-n}{(2-n)^2}\,\mathrm{tr}_{q_t}(\pi_t)^2) q_t^\#$$

$$= \frac{N}{2} \left([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2 \right) q_t^{\#}.$$

Summary: We have

$$\dot{p}_t = -2N\left(\pi_t * \pi_t - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)\pi_t\right) \otimes |q_t|^{1/2}$$

$$+ \frac{N}{2} \left([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2 \right) q_t^{\#} \otimes |q_t|^{1/2}$$

$$- N\mathrm{Ein}(q_t)^{\#} \otimes |q_t|^{1/2}.$$

Section 35: Variable Lapse, Zero Shift Let M be a connected C^∞ manifold of dimension $n > 2$. Fix $\varepsilon\,(0 < \varepsilon \leq \infty)$ and assume that

$$M =]-\varepsilon,\varepsilon[\, \times \Sigma,$$

where Σ is compact and orientable (hence $\dim \Sigma = n - 1$).

Let $N \in C^\infty(M)$ be strictly positive (or strictly negative) (the lapse). Put

$$N_t(x) = N(t,x) \quad (x \in \Sigma).$$

Suppose that $t \to q(t)$ ($= q_t$) ($t \in\,]-\varepsilon,\varepsilon[$) is a path in Q -- then the prescription

$$g_{(t,x)}((r,X),(s,Y))$$

$$= -rsN_t^2(x) + q_x(t)(X,Y) \quad (r,s \in \underline{R}\ \&\ X,Y \in T_x\Sigma)$$

defines an element of $\underline{M}_{1,n-1}$ ($g_{00} = g(\partial_0,\partial_0) = -N^2$).

Let $\underline{n}_t = \frac{1}{N_t}\partial_t$ -- then

$$g(\underline{n}_t,\underline{n}_t) = \frac{1}{N_t^2} g(\partial_t,\partial_t)$$

$$= -\frac{N_t^2}{N_t^2} = -1.$$

Working with the metric connection of g, let $\kappa_t \in S_2(\Sigma)$ be the extrinsic curvature, thus

$$\kappa_t(V,W) = q_t(-i_t^*\nabla_V \underline{n}_t, W)\, g(\underline{n}_t,\underline{n}_t)$$

$$= q_t(i_t^*\nabla_V n_t, W).$$

And, as in the case of constant N,

$$\varkappa_t = \frac{1}{2N_t}\dot{q}_t.$$

Remark: The focus below will be on the computation of $\dot{\varkappa}_t$ rather than $\ddot{q}_t$.

At each t, submanifold theory is applicable to the pair (M,Σ) (per $\bar{g} = i_t^* g = q_t$). To help keep things straight, overbars are sometimes used to distinguish objects on Σ from the corresponding objects on M.

<u>LEMMA</u> In adapted coordinates (and abbreviated notation), the connection coefficients of g are given by

$$\left[\begin{array}{l} \Gamma^c{}_{ab} = \bar{\Gamma}^c{}_{ab} \\ \\ \Gamma^0{}_{ab} = \dfrac{1}{2N^2} q_{ab,0} \\ \\ \Gamma^c{}_{0b} = \dfrac{1}{2} q^{cd} q_{db,0} \end{array}\right. \qquad \text{and} \qquad \left[\begin{array}{l} \Gamma^0{}_{00} = \dfrac{N_{,0}}{N} \\ \\ \Gamma^0{}_{0b} = \dfrac{N_{,b}}{N} \\ \\ \Gamma^c{}_{00} = Nq^{cd}N_{,d}. \end{array}\right.$$

[The computation is carried out using

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{k\ell}(g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}).$$

- $\Gamma^c{}_{ab} = \frac{1}{2} g^{c\ell}(g_{\ell a,b} + g_{\ell b,a} - g_{ab,\ell})$

$$= \frac{1}{2} g^{c0}(g_{0a,b} + g_{0b,a} - g_{ab,0})$$

$$+ \frac{1}{2} g^{cd}(g_{da,b} + g_{db,a} - g_{ab,d})$$

$$= \frac{1}{2} q^{cd}(q_{da,b} + q_{db,a} - q_{ab,d})$$

$$= \bar{\Gamma}^{c}{}_{ab}.$$

- $\Gamma^{0}{}_{ab} = \frac{1}{2} g^{0\ell}(g_{\ell a,b} + g_{\ell b,a} - g_{ab,\ell})$

$$= \frac{1}{2} g^{00}(g_{0a,b} + g_{0b,a} - g_{ab,0})$$

$$= \frac{1}{2} g^{00}(- g_{ab,0})$$

$$= \frac{1}{2N^2} q_{ab,0}.$$

- $\Gamma^{c}{}_{0b} = \frac{1}{2} g^{c\ell}(g_{\ell 0,b} + g_{\ell b,0} - g_{0b,\ell})$

$$= \frac{1}{2} g^{c\ell}(g_{\ell 0,b} + g_{\ell b,0})$$

$$= \frac{1}{2} g^{cd}(g_{d0,b} + g_{db,0})$$

$$= \frac{1}{2} q^{cd} q_{db,0}.$$

- $\Gamma^{0}{}_{00} = \frac{1}{2} g^{0\ell}(g_{\ell 0,0} + g_{\ell 0,0} - g_{00,\ell})$

$$= \frac{1}{2} g^{00}(g_{00,0} + g_{00,0} - g_{00,0})$$

$$= \frac{1}{2} g^{00}(g_{00,0})$$

$$= \frac{1}{2}\left(-\frac{1}{N^2}\right)\partial_0(-N^2)$$

$$= \frac{N_{,0}}{N}.$$

$$\bullet\ \Gamma^0{}_{0b} = \frac{1}{2} g^{0\ell}(g_{\ell 0,b} + g_{\ell b,0} - g_{0b,\ell})$$

$$= \frac{1}{2} g^{00}(g_{00,b} + g_{0b,0} - g_{0b,0})$$

$$= \frac{1}{2} g^{00}(g_{00,b})$$

$$= \frac{1}{2}\left(-\frac{1}{N^2}\right)\partial_b(-N^2)$$

$$= \frac{N_{,b}}{N}.$$

$$\bullet\ \Gamma^c{}_{00} = \frac{1}{2} g^{c\ell}(g_{\ell 0,0} + g_{\ell 0,0} - g_{00,\ell})$$

$$= \frac{1}{2} g^{cd}(g_{d0,0} + g_{d0,0} - g_{00,d})$$

$$= \frac{1}{2} g^{cd}(-g_{00,d})$$

$$= \frac{1}{2} g^{cd}(-\partial_d(-N^2))$$

$$= N g^{cd} N_{,d}$$

$$= N q^{cd} N_{,d}.$$

Example: We have

$$\varkappa_{ab} = \nabla_b \underline{n}_a = \underline{n}_{a,b} - \Gamma^i{}_{ab}\underline{n}_i$$

$$= -\Gamma^{i}{}_{ab}\underline{n}_i = -\Gamma^{0}{}_{ab}(-N) = \frac{1}{2N}\, q_{ab,0}.$$

Recall now our indexing conventions for the curvature tensor:

$$R_{ijk\ell} = g(\partial_i, R(\partial_k,\partial_\ell)\partial_j).$$

Rappel: We have

$$g(W_1, R(V_1,V_2)W_2)$$

$$= \bar{g}(W_1, \bar{R}(V_1,V_2)W_2)$$

$$+ g(\Pi_\nabla(V_1,W_2), \Pi_\nabla(V_2,W_1)) - g(\Pi_\nabla(V_1,W_1), \Pi_\nabla(V_2,W_2)).$$

[Note: Here it is understood that ∇ is the metric connection of g. Moreover, the dependence on t is implicit:

$$\Pi_\nabla(V,W) = \kappa_\nabla(V,W)\underline{n}$$

$$= \kappa(V,W)\underline{n} \equiv \kappa_t(V,W)\underline{n}_t.]$$

Specialize and take

$$W_1 = \partial_a, V_1 = \partial_c, V_2 = \partial_d, W_2 = \partial_b.$$

Then

$$R_{abcd} = \bar{R}_{abcd}$$

$$+ g(\kappa(\partial_c,\partial_b)\underline{n}, \kappa(\partial_d,\partial_a)\underline{n}) - g(\kappa(\partial_c,\partial_a)\underline{n}, \kappa(\partial_d,\partial_b)\underline{n})$$

$$= \bar{R}_{abcd} + \kappa_{ac}\kappa_{bd} - \kappa_{ad}\kappa_{bc}.$$

Rappel: We have

$$g(\underline{n},R(V_1,V_2)W)$$

$$= g(\underline{n},(\nabla^{\perp}_{V_1}\Pi_{\nabla})(V_2,W)) - g(\underline{n},(\nabla^{\perp}_{V_2}\Pi_{\nabla})(V_1,W))$$

or still,

$$g(\underline{n},R(V_1,V_2)W)$$

$$= (\bar{\nabla}_{V_2}\kappa)(V_1,W) - (\bar{\nabla}_{V_1}\kappa)(V_2,W).$$

[Note: $\bar{\nabla}$ is the metric connection of $\bar{g}$, hence is torsion free. Therefore

$$\bar{g}(S_{\underline{n}}\bar{T}(V_1,V_2),W) = 0.]$$

<u>Details</u> It is a question of supplying the omitted steps in the preceding manipulation. To begin with,

$$(\bar{\nabla}_{V_1}\kappa)(V_2,W) = V_1(\kappa(V_2,W)) - \kappa(\bar{\nabla}_{V_1}V_2,W) - \kappa(V_2,\bar{\nabla}_{V_1}W).$$

On the other hand,

$$g(\underline{n},(\nabla^{\perp}_{V_1}\Pi_{\nabla})(V_2,W))$$

$$= g(\underline{n},\nabla^{\perp}_{V_1}\Pi_{\nabla}(V_2,W))$$

$$- g(\underline{n},\Pi_{\nabla}(\bar{\nabla}_{V_1}V_2,W)) - g(\underline{n},\Pi_{\nabla}(V_2,\bar{\nabla}_{V_1}W)).$$

- $g(\underline{n},\nabla^{\perp}_{V_1}\Pi_{\nabla}(V_2,W))$

$$= g(\underline{n}, \text{nor } i^*\nabla_{V_1}\Pi_\nabla(V_2,W))$$

$$= g(\underline{n}, \text{nor } i^*\nabla_{V_1}(\varkappa(V_2,W)\underline{n}))$$

$$= g(\underline{n}, \text{nor}(V_1(\varkappa(V_2,W))\underline{n} + \varkappa(V_2,W)i^*\nabla_{V_1}\underline{n}))$$

$$= V_1(\varkappa(V_2,W))g(\underline{n},\underline{n}) + \varkappa(V_2,W)g(\underline{n},i^*\nabla_{V_1}\underline{n})$$

$$= - V_1(\varkappa(V_2,W)) + \varkappa(V_2,W)g(\underline{n}, - S_{\underline{n}}V_1)$$

$$= - V_1(\varkappa(V_2,W)).$$

• $$\begin{bmatrix} g(\underline{n},\Pi_\nabla(\bar{\nabla}_{V_1}V_2,W)) = g(\underline{n},\varkappa(\bar{\nabla}_{V_1}V_2,W)\underline{n}) = -\varkappa(\bar{\nabla}_{V_1}V_2,W) \\ g(\underline{n},\Pi_\nabla(V_2,\bar{\nabla}_{V_1}W)) = g(\underline{n},\varkappa(V_2,\bar{\nabla}_{V_1}W)\underline{n}) = -\varkappa(V_2,\bar{\nabla}_{V_1}W). \end{bmatrix}$$

Therefore

$$g(\underline{n},(\nabla^{\perp}_{V_1}\Pi_\nabla)(V_2,W)) = - V_1(\varkappa(V_2,W)) - \varkappa(\bar{\nabla}_{V_1}V_2,W) - \varkappa(V_2,\bar{\nabla}_{V_1}W)$$

$$= - (\bar{\nabla}_{V_1}\varkappa)(V_2,W).$$

Specialize and take

$$V_1 = \partial_b, V_2 = \partial_c, W = \partial_a.$$

Then

$$R_{0abc} = N(\bar{\nabla}_c\varkappa_{ab} - \bar{\nabla}_b\varkappa_{ac}).$$

It will also be necessary to compute R_{0a0b} which, by definition, is

$$g(\partial_0, R(\partial_0, \partial_b)\partial_a).$$

But $\partial_0 = N\underline{n}$, thus

$$\frac{R_{0a0b}}{N^2} = g(\underline{n}, R(\underline{n}, \partial_b)\partial_a)$$

$$= g(R(\underline{n}, \partial_b)\partial_a, \underline{n})$$

$$= g(R(\partial_a, \underline{n})\underline{n}, \partial_b)$$

$$= - g(R(\underline{n}, \partial_a)\underline{n}, \partial_b).$$

Write

$$R(\underline{n}, \partial_a)\underline{n} = \nabla_{\underline{n}}\nabla_a\underline{n} - \nabla_a\nabla_{\underline{n}}\underline{n} - \nabla_{[\underline{n}, \partial_a]}\underline{n}.$$

Then the calculation divides into three parts, viz.

1. $g(\nabla_{\underline{n}}\nabla_a\underline{n}, \partial_b)$

2. $g(\nabla_a\nabla_{\underline{n}}\underline{n}, \partial_b)$

3. $g(\nabla_{[\underline{n}, \partial_a]}\underline{n}, \partial_b)$.

Ad 1: First,

$$(\nabla_a\underline{n})^b = \kappa_a{}^b.$$

Second,

$$(\nabla_a\underline{n})^0 = (\nabla_a(\frac{1}{N}\partial_0))^0$$

$$= (\frac{1}{N} \nabla_a \partial_0 + \partial_a (\frac{1}{N}) \partial_0)^0$$

$$= (\frac{1}{N} \Gamma^k{}_{a0} \partial_k - \frac{N_{,a}}{N^2} \partial_0)^0$$

$$= \frac{1}{N} \Gamma^0{}_{a0} - \frac{N_{,a}}{N^2}$$

$$= \frac{1}{N} (\frac{N_{,a}}{N}) - \frac{N_{,a}}{N^2}$$

$$= 0.$$

Third, $\forall\, X \in \mathcal{D}^1(M)$,

$$(\nabla_{\underline{n}} X)^c = \frac{1}{N} (\nabla_0 X)^c$$

$$= \frac{1}{N} [X^c{}_{,0} + \Gamma^c{}_{0j} X^j]$$

$$= \frac{1}{N} [X^c{}_{,0} + \Gamma^c{}_{00} X^0 + \Gamma^c{}_{0d} X^d]$$

$$= \frac{1}{N} [X^c{}_{,0} + N g^{cd} N_{,d} X^0 + \frac{1}{2} g^{cb} g_{bd,0} X^d]$$

$$= \frac{X^c{}_{,0}}{N} + g^{cd} N_{,d} X^0 + \frac{1}{2N} g^{cb} 2N \varkappa_{bd} X^d$$

$$= \frac{X^c{}_{,0}}{N} + g^{cd} N_{,d} X^0 + \varkappa^c{}_d X^d.$$

Therefore

$$g(\nabla_{\underline{n}} \nabla_a \underline{n}, \partial_b)$$

$$= (\nabla_{\underline{n}} \nabla_a \underline{n})^i (\partial_b)_i$$

$$= g_{ij}(\partial_b)^j(\nabla_{\underline{n}}\nabla_a\underline{n})^i$$

$$= g_{ib}(\nabla_{\underline{n}}\nabla_a\underline{n})^i$$

$$= g_{0b}(\nabla_{\underline{n}}\nabla_a\underline{n})^0 + g_{cb}(\nabla_{\underline{n}}\nabla_a\underline{n})^c$$

$$= g_{bc}(\nabla_{\underline{n}}\nabla_a\underline{n})^c$$

$$= g_{bc}[\frac{1}{N}(\nabla_a\underline{n})^c{}_{,0} + g^{cd}N_{,d}(\nabla_a\underline{n})^0 + \varkappa^c{}_d(\nabla_a\underline{n})^d]$$

$$= g_{bc}[\frac{1}{N}\varkappa_a{}^c{}_{,0} + \varkappa^c{}_d\varkappa_a{}^d]$$

$$= \frac{1}{N}[\frac{\partial}{\partial t}\varkappa_{ab} - \varkappa_a{}^c\frac{\partial}{\partial t}g_{bc}] + (\varkappa*\varkappa)_{ab}$$

$$= \frac{1}{N}\frac{\partial}{\partial t}\varkappa_{ab} - \varkappa_a{}^c\frac{1}{N}(2N\varkappa_{bc}) + (\varkappa*\varkappa)_{ab}$$

$$= \frac{1}{N}\frac{\partial}{\partial t}\varkappa_{ab} - (\varkappa*\varkappa)_{ab}.$$

<u>Ad 2</u>: Analogously,

$$g(\nabla_a\nabla_{\underline{n}}\underline{n},\partial_b) = \frac{N_{;b;a}}{N} - \frac{N_{,b}N_{,a}}{N^2}.$$

<u>Ad 3</u>: On the one hand,

$$[\underline{n},\partial_a]^c = 0,$$

while on the other,

$$[\underline{n},\partial_a]^0 = \underline{n}^i(\partial_a)^0{}_{,i} - (\partial_a)^i(\underline{n})^0{}_{,i}$$

$$= - (\partial_a)^i(\underline{n})^0{}_{,i}$$

$$= - (\underline{n})^0{}_{,a}$$

$$= - \partial_a \left(\frac{1}{N}\right) = \frac{N_{,a}}{N^2} .$$

So

$$\nabla_{[\underline{n},\partial_a]}\underline{n} = [\underline{n},\partial_a]^i\nabla_i\underline{n}$$

$$= \frac{N_{,a}}{N^2} \nabla_0\underline{n}$$

$$= \frac{N_{,a}}{N^2} \nabla_0\left(\frac{1}{N} \partial_0\right)$$

$$= \frac{N_{,a}}{N^2} \left[\frac{1}{N} \nabla_0\partial_0 + \partial_0\left(\frac{1}{N}\right)\partial_0\right]$$

$$= \frac{N_{,a}}{N^2} \left[\frac{1}{N} \Gamma^k{}_{00}\partial_k + \partial_0\left(\frac{1}{N}\right)\partial_0\right]$$

$\Rightarrow$

$$g(\nabla_{[\underline{n},\partial_a]}\underline{n},\partial_b)$$

$$= \frac{N_{,a}}{N^2} \left(\frac{1}{N} \Gamma^c{}_{00}g(\partial_c,\partial_b)\right)$$

$$= \frac{N_{,a}}{N^2} \frac{1}{N} (Ng^{cd}N_{,d})g_{cb}$$

$$= \frac{N_{,a}}{N^2} g^{cd} g_{cb} N_{,d}$$

$$= \frac{N_{,a}}{N^2} \delta^d{}_b N_{,d}$$

$$= \frac{N_{,a} N_{,b}}{N^2} .$$

Now combine terms:

$$\frac{R_{0a0b}}{N^2} = - g(R(\underline{n},\partial_a)\underline{n},\partial_b)$$

$$= - [\frac{1}{N} \frac{\partial}{\partial t} \varkappa_{ab} - (\varkappa * \varkappa)_{ab}$$

$$+ \frac{N_{,a} N_{,b}}{N^2} - \frac{N_{;b;a}}{N} - \frac{N_{,a} N_{,b}}{N^2}]$$

$$= - [\frac{1}{N} \frac{\partial}{\partial t} \varkappa_{ab} - (\varkappa * \varkappa)_{ab} - \frac{1}{N} (H_N)_{ab}] .$$

<u>THEOREM</u> Ric(g) = 0 iff $\varkappa_t$ satisfies the differential equation

$$\dot{\varkappa}_t = 2N_t(\varkappa_t * \varkappa_t) - N_t K_t \varkappa_t - N_t \mathrm{Ric}(q_t) + H_{N_t}$$

and the constraints

$$\left[\begin{array}{l} \mathrm{div}_{q_t}(\varkappa_t - K_t q_t) = 0 \\ \\ ([\varkappa_t, \varkappa_t]_{q_t} - K_t^2) - S(q_t) = 0. \end{array}\right.$$

It will be enough to establish the necessity of the stated conditions (sufficiency follows by retracement). So suppose that Ric(g) = 0.

• $R_{ab} = 0$

$$0 = R^{i}{}_{aib} = g^{ij}R_{jaib}$$

⇒

$$0 = g^{00}R_{0a0b} + g^{cd}R_{dacb}$$

⇒

$$= -\frac{1}{N^2}R_{0a0b} + g^{cd}R_{dacb}$$

$$= \frac{1}{N}\frac{\partial}{\partial t}\varkappa_{ab} - (\varkappa*\varkappa)_{ab} - \frac{1}{N}(H_N)_{ab}$$

$$+ g^{cd}(\bar{R}_{dacb} + \varkappa_{dc}\varkappa_{ab} - \varkappa_{db}\varkappa_{ac})$$

$$= \frac{1}{N}\frac{\partial}{\partial t}\varkappa_{ab} - (\varkappa*\varkappa)_{ab} - \frac{1}{N}(H_N)_{ab}$$

$$+ \bar{R}_{ab} + K\varkappa_{ab} - (\varkappa*\varkappa)_{ab}$$

⇒

$$\frac{1}{N}\frac{\partial}{\partial t}\varkappa_{ab} = 2(\varkappa*\varkappa)_{ab} - K\varkappa_{ab} - \bar{R}_{ab} + \frac{1}{N}(H_N)_{ab}$$

⇒

$$\frac{\partial}{\partial t}\varkappa_{ab} = 2N(\varkappa*\varkappa)_{ab} - NK\varkappa_{ab} - N\bar{R}_{ab} + (H_N)_{ab}.$$

I.e.:

$$\dot{\varkappa}_t = 2N_t(\varkappa_t * \varkappa_t) - N_t K_t \varkappa_t - N_t \mathrm{Ric}(q_t) + H_{N_t}.$$

• $R_{0a} = 0$

$\Rightarrow$

$$0 = R^i{}_{0ia} = g^{ij} R_{j0ia}$$

$$= g^{bc} R_{c0ba}$$

$$= - g^{bc} R_{0cba}$$

$\Rightarrow$

$$0 = g^{bc} N(\bar{\nabla}_a \varkappa_{cb} - \bar{\nabla}_b \varkappa_{ca})$$

$$= N(\bar{\nabla}_a \varkappa^b{}_b - \bar{\nabla}_b \varkappa^b{}_a)$$

$\Rightarrow$

$$\bar{\nabla}_b \varkappa^b{}_a - \bar{\nabla}_a \varkappa^b{}_b = 0.$$

But

$$\left[\begin{array}{l} (\mathrm{div}_q\, \varkappa)_a = \bar{\nabla}_b \varkappa^b{}_a \\ \\ (\mathrm{div}_q (Kq))_a = \bar{\nabla}_a \varkappa^b{}_b. \end{array}\right.$$

Therefore

$$\operatorname{div}_q(\varkappa - Kq) = 0.$$

I.e.:

$$\operatorname{div}_{q_t}(\varkappa_t - K_t q_t) = 0.$$

- $R_{00} = 0$

$\Rightarrow$

$$0 = R^i{}_{0i0} = g^{ij} R_{j0i0}$$

$$= g^{ab} R_{b0a0}$$

$$= - g^{ab} R_{0ba0}$$

$$= g^{ab} R_{0b0a}$$

$$= g^{ab} R_{0a0b}$$

$$= g^{ab}(-N^2)[\frac{1}{N}\frac{\partial}{\partial t}\varkappa_{ab} - (\varkappa * \varkappa)_{ab} - \frac{1}{N}(H_N)_{ab}]$$

$\Rightarrow$

$$0 = g^{ab}[\frac{1}{N}\frac{\partial}{\partial t}\varkappa_{ab} - (\varkappa * \varkappa)_{ab} - \frac{1}{N}(H_N)_{ab}]$$

$$= g^{ab}[2(\varkappa * \varkappa)_{ab} - K\varkappa_{ab} - \bar{R}_{ab} + \frac{1}{N}(H_N)_{ab}$$

$$- (\varkappa * \varkappa)_{ab} - \frac{1}{N} (H_N)_{ab}]$$

$$= g^{ab}[(\varkappa * \varkappa)_{ab} - K\varkappa_{ab} - \bar{R}_{ab}]$$

$$= ([\varkappa,\varkappa]_q - K^2) - S(q).$$

I.e.:

$$([\varkappa_t,\varkappa_t]_{q_t} - K_t^2) - S(q_t) = 0.$$

The necessity of the stated conditions is thereby established.

Observation:

$$[\varkappa_t,\varkappa_t]_{q_t} - K_t^2 = S(q_t)$$

$$\Rightarrow$$

$$\dot{\varkappa}_t = 2N_t(\varkappa_t * \varkappa_t) - N_t K_t \varkappa_t - N_t \mathrm{Ric}(q_t) + H_{N_t}$$

$$= 2N_t(\varkappa_t * \varkappa_t) - N_t K_t \varkappa_t + \frac{N_t}{2(n-2)} (K_t^2 - [\varkappa_t,\varkappa_t]_{q_t})q_t$$

$$- N_t \mathrm{Ric}(q_t) + H_{N_t} + \frac{N_t}{2(n-2)} S(q_t)q_t$$

$$= \frac{1}{2N_t} \Gamma(q_t, 2N_t\varkappa_t) + \mathrm{grad}_{q_t} V_{N_t}.$$

Definition: The <u>momentum</u> of the theory is the path $t \to p_t$ in $S_d^2(\Sigma)$ defined by the prescription

$$p_t = \pi_t \otimes |q_t|^{1/2},$$

where

$$\pi_t = (\varkappa_t - K_t q_t)^{\#}.$$

The discussion in the previous section can now be repeated virtually verbatim.

<u>Constraint Equations</u> These are the relations

$$\left[\begin{array}{l} ([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2 - S(q_t)) \otimes |q_t|^{1/2} = 0 \\ \operatorname{div}_{q_t} p_t = 0. \end{array}\right.$$

<u>Evolution Equations</u> These are the relations

$$\dot{q}_t = 2N_t(\pi_t^{\flat} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t^{\flat})q_t)$$

and

$$\begin{aligned} \dot{p}_t = & -2N_t(\pi_t * \pi_t - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)\pi_t) \otimes |q_t|^{1/2} \\ & + \frac{N_t}{2}([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2)q_t^{\#} \otimes |q_t|^{1/2} \\ & - N_t \operatorname{Ein}(q_t)^{\#} \otimes |q_t|^{1/2} \\ & + (H_{N_t} - (\Delta_{q_t} N_t)q_t)^{\#} \otimes |q_t|^{1/2}. \end{aligned}$$

[Note: The explanation for the appearance of the laplacian $\Delta_{q_t} N_t$ is the fact that $\operatorname{tr}_{q_t}(\dot{\varkappa}_t)$ figures in the formula for $\dot{p}_t$.]

Section 36: Incorporation of the Shift Let M be a connected C^∞ manifold of dimension $n > 2$. Fix ε $(0 < \varepsilon \leq \infty)$ and assume that

$$M =]-\varepsilon,\varepsilon[\times \Sigma,$$

where Σ is compact and orientable (hence $\dim \Sigma = n - 1$).

Definition: A shift is a time dependent vector vield $\vec{N}$ on Σ (thus $\vec{N}:]-\varepsilon,\varepsilon[\rightarrow T\Sigma$ has the property that $\vec{N}_t(x) = \vec{N}(t,x) \in T_x\Sigma \ \forall \ x \in \Sigma$).

Fix a lapse N and a shift $\vec{N}$. Suppose that $t \rightarrow q(t)$ $(= q_t)$ $(t \in]-\varepsilon,\varepsilon[)$ is a path in Q. Then the prescription

$$\begin{aligned} & g_{(t,x)}((r,X),(s,Y)) \\ & = - rs(N_t^2(x) - q_x(t)(\vec{N}_t|_x,\vec{N}_t|_x)) \\ & + sq_x(t)(X,\vec{N}_t|_x) + rq_x(t)(Y,\vec{N}_t|_x) \\ & + q_x(t)(X,Y) \quad (r,s \in \underline{R} \ \& \ X,Y \in T_x\Sigma) \end{aligned}$$

defines an element of $\underline{M}_{1,n-1}$.

[Note: In adapted coordinates (with $\vec{N} = N^a \partial_a$),

$$[g_{ij}] = \begin{bmatrix} -N^2 + N^a N_a & N_b \\ & \\ N_a & q_{ab} \end{bmatrix}$$

and

$$[g^{ij}] = \frac{1}{N^2}\begin{bmatrix} -1 & N^b \\ & \\ N^a & N^2q^{ab} - N^aN^b \end{bmatrix}.]$$

Remark: We can write

$$g = -(N^2 - q(\vec{N},\vec{N}))dt \otimes dt + \vec{N}^\flat \otimes dt + dt \otimes \vec{N}^\flat + q,$$

modulo, of course, the obvious agreements.

Let $\underline{n}_t = \frac{1}{N_t}(\frac{\partial}{\partial t} - \vec{N}_t)$ -- then

$$g(\underline{n}_t,\partial_a)$$

$$= \frac{1}{N_t} g(\frac{\partial}{\partial t} - \vec{N}_t,\partial_a)$$

$$= \frac{1}{N_t}(g(\partial_0,\partial_a) - N^b g(\partial_b,\partial_a))$$

$$= \frac{1}{N_t}(N_a - N^b q_{ab})$$

$$= \frac{1}{N_t}(N_a - N_a) = 0.$$

On the other hand,

$$g(\underline{n}_t,\underline{n}_t)$$

$$= \frac{1}{N_t^2} g(\frac{\partial}{\partial t} - \vec{N}_t, \frac{\partial}{\partial t} - \vec{N}_t)$$

$$= \frac{1}{N_t^2} (g(\partial_0, \partial_0) - 2g(\partial_0, \vec{N}_t) + g(\vec{N}_t, \vec{N}_t))$$

$$= \frac{1}{N_t^2} (g_{00} - 2N^a g_{0a} + N^a N^b q_{ab})$$

$$= \frac{1}{N_t^2} (- N_t^2 + N^a N_a - 2N^a N_a + N_a N_a)$$

$$= - \frac{N_t^2}{N_t^2} = - 1.$$

Remark: Obviously,

$$\underline{n}^0 = \frac{1}{N}, \ \underline{n}^a = - \frac{N^a}{N} .$$

In addition,

$$\underline{n}^\flat = - Ndt.$$

<u>FACT</u> We have

$$\begin{cases} \nabla_0 \partial_a = (\partial_a N + \varkappa_{ab} N^b)\underline{n} + (N\varkappa_a{}^b + \bar{\nabla}_a N^b)\partial_b \\ \nabla_a \underline{n} = \varkappa_a{}^b \partial_b \\ \nabla_0 \underline{n} = (\partial_b N + \varkappa_{bc} N^c) q^{ba} \partial_a. \end{cases}$$

<u>LEMMA</u> Let $\varkappa_t \in S_2(\Sigma)$ be the extrinsic curvature (per the metric connection

of g) -- then

$$\dot{q}_t = 2N_t \varkappa_t + L_{\vec{N}_t} q_t.$$

[For

$$\partial_t (q_t)_{ab} = q_t (i_t^* \nabla_{\partial_a} (N_t \underline{n}_t + \vec{N}_t), \partial_b)$$

$$+ q_t (\partial_a, i_t^* \nabla_{\partial_b} (N_t \underline{n}_t + \vec{N}_t))$$

$$= 2N_t (\varkappa_t)_{ab} + q_t (\bar{\nabla}_{\partial_a} \vec{N}_t, \partial_b) + q_t (\partial_a, \bar{\nabla}_{\partial_b} \vec{N}_t)$$

$$= 2N_t (\varkappa_t)_{ab} + (L_{\vec{N}_t} q_t)(\partial_a, \partial_b).]$$

Application:

$$\frac{d}{dt} |q_t|^{1/2} = (\mathrm{div}_{q_t} \vec{N}_t + N_t K_t) |q_t|^{1/2}.$$

The presence of the shift is not a problem: It adds one more term to the equation of motion.

<u>THEOREM</u> Ric(g) = 0 iff $\varkappa_t$ satisfies the differential equation

$$\dot{\varkappa}_t = 2N_t (\varkappa_t * \varkappa_t) - N_t K_t \varkappa_t - N_t \mathrm{Ric}(q_t) + H_{N_t} + L_{\vec{N}_t} \varkappa_t$$

and the constraints

$$\begin{cases} \mathrm{div}_{q_t} (\varkappa_t - K_t q_t) = 0 \\ ([\varkappa_t, \varkappa_t]_{q_t} - K_t^2) - S(q_t) = 0. \end{cases}$$

Apart from a few additional wrinkles, the argument runs along the by now familiar lines.

• $R^0{}_{a0b}$

$$= \frac{1}{N}\,[\dot{\varkappa}_{ab} - N\varkappa_a{}^c\varkappa_{cb} - \varkappa_{bc}\bar{\nabla}_a N^c - \bar{\nabla}_b(\bar{\nabla}_a N + \varkappa_{ac}N^c)].$$

[In fact,

$$R^0{}_{a0b} = dt(\nabla_0\nabla_b\partial_a - \nabla_b\nabla_0\partial_a)$$

$$= dt(\nabla_0(\varkappa_{ab}\underline{n} + \bar{\Gamma}^c{}_{ba}\partial_c) - \nabla_b\nabla_0\partial_a)$$

$$= \frac{1}{N}\,\dot{\varkappa}_{ab} + dt(\bar{\Gamma}^c{}_{ab}\nabla_0\partial_c - \nabla_b((\partial_a N + \varkappa_{ac}N^c)\underline{n} + (N\varkappa_a{}^c + \bar{\nabla}_a N^c)\partial_c))$$

$$= \frac{1}{N}\,[\dot{\varkappa}_{ab} + \bar{\Gamma}^c{}_{ba}(\partial_c N + \varkappa_{cd}N^d) - \partial_b(\partial_a N + \varkappa_{ac}N^c) - (N\varkappa_a{}^c + \bar{\nabla}_a N^c)\varkappa_{bc}]$$

$$= \frac{1}{N}\,[\dot{\varkappa}_{ab} - N\varkappa_a{}^c\varkappa_{cb} - \varkappa_{bc}\bar{\nabla}_a N^c - \bar{\nabla}_b(\bar{\nabla}_a N + \varkappa_{ac}N^c)].]$$

• $R^c{}_{acb}$

$$= \bar{R}_{ab} + K\varkappa_{ab} - \varkappa_{ac}\varkappa^c{}_b + \frac{N^c}{N}\,(\bar{\nabla}_b\varkappa_{ac} - \bar{\nabla}_c\varkappa_{ab}).$$

[In fact,

$$R^c{}_{acb} = g^{ci}R_{iacb}$$

$$= g^{cd}R_{dacb} + g^{c0}R_{0acb}$$

$$= q^{cd}(\bar{R}_{dacb} + \varkappa_{dc}\varkappa_{ab} - \varkappa_{db}\varkappa_{ac}) - \frac{N^c N^d}{N^2} R_{dacb}$$

$$+ \frac{N^c}{N^2} [g(N\underline{n}, R(\partial_c, \partial_b)\partial_a) + g(\vec{N}, R(\partial_c, \partial_b)\partial_a)]$$

$$= \bar{R}_{ab} + K\varkappa_{ab} - \varkappa_{ac}\varkappa^c{}_b - \frac{N^c N^d}{N^2} R_{dacb}$$

$$+ \frac{N^c}{N^2} [N(\bar{\nabla}_b \varkappa_{ac} - \bar{\nabla}_c \varkappa_{ab}) + N^d R_{dacb}]$$

$$= \bar{R}_{ab} + K\varkappa_{ab} - \varkappa_{ac}\varkappa^c{}_b + \frac{N^c}{N} (\bar{\nabla}_b \varkappa_{ac} - \bar{\nabla}_c \varkappa_{ab}).]$$

Therefore

$$R_{ab} = R^0{}_{a0b} + R^c{}_{acb}$$

$$= \bar{R}_{ab} + K\varkappa_{ab} - 2\varkappa_{ac}\varkappa^c{}_b$$

$$+ \frac{1}{N} [\dot{\varkappa}_{ab} - \bar{\nabla}_a \bar{\nabla}_b N - N^c \bar{\nabla}_c \varkappa_{ab} - \varkappa_{bc}\bar{\nabla}_a N^c - \varkappa_{ac}\bar{\nabla}_b N^c]$$

$$= \bar{R}_{ab} + K\varkappa_{ab} - 2(\varkappa * \varkappa)_{ab} + \frac{1}{N} [\dot{\varkappa}_{ab} - (H_N)_{ab} - L_{\vec{N}}\varkappa_{ab}].$$

But then $R_{ab} = 0$ iff

$$\dot{\varkappa}_{ab} = 2N(\varkappa * \varkappa)_{ab} - NK\varkappa_{ab} - N\bar{R}_{ab} + (H_N)_{ab} + L_{\vec{N}}\varkappa_{ab}.$$

N.B. Since there is no torsion,

$$L_{\vec{N}}\varkappa_{ab} = \bar{\nabla}_{\vec{N}}\varkappa_{ab} + \varkappa(\bar{\nabla}_a \vec{N}, \partial_b) + \varkappa(\partial_a, \bar{\nabla}_b \vec{N})$$

$$= \bar{\nabla}_{\vec{N}}\varkappa_{ab} + \varkappa([\partial_a, \vec{N}], \partial_b) + \varkappa(\partial_a, [\partial_b, \vec{N}])$$

$$= N^c \bar{\nabla}_c \varkappa_{ab} + \varkappa((\bar{\nabla}_a N^c)\partial_c, \partial_b) + \varkappa(\partial_a, (\bar{\nabla}_b N^c)\partial_c)$$

$$= N^c \bar{\nabla}_c \varkappa_{ab} + \varkappa_{bc} \bar{\nabla}_a N^c + \varkappa_{ac} \bar{\nabla}_b N^c.$$

<u>FACT</u> We have

$$\mathrm{tr}_q(L_{\vec{N}} \varkappa) = L_{\vec{N}} \mathrm{tr}_q(\varkappa) + [\varkappa, L_{\vec{N}} q]_q.$$

Remark: The evolution of K_t follows from the evolution of $\varkappa_t$. Indeed,

$$\dot{K}_t = - [\dot{q}_t, \varkappa_t]_{q_t} + \mathrm{tr}_{q_t}(\dot{\varkappa}_t)$$

$$= - [2N_t \varkappa_t + L_{\vec{N}_t} q_t, \varkappa_t]_{q_t} + \mathrm{tr}_{q_t}(\dot{\varkappa}_t)$$

$$= - [\varkappa_t, L_{\vec{N}_t} q_t]_{q_t} - 2N_t [\varkappa_t, \varkappa_t]_{q_t} + \mathrm{tr}_{q_t}(\dot{\varkappa}_t)$$

$$= L_{\vec{N}_t} \mathrm{tr}_{q_t}(\varkappa_t) - \mathrm{tr}_{q_t}(L_{\vec{N}_t} \varkappa_t) - 2N_t [\varkappa_t, \varkappa_t]_{q_t} + \mathrm{tr}_{q_t}(\dot{\varkappa}_t).$$

Notation: Let

$$h = g + \underline{n}^\flat \otimes \underline{n}^\flat.$$

Then

$$[h_{ij}] = \begin{bmatrix} N^a N_a & N_b \\ & \\ N_a & q_{ab} \end{bmatrix}$$

and

$$[h^{ij}] = \begin{bmatrix} 0 & 0 \\ & \\ 0 & q^{ab} \end{bmatrix}.$$

To discuss the first constraint, write

$$\begin{aligned}
\mathrm{Ric}(\underline{n}, \partial_a) &= \underline{n}^i R^j{}_{ija} \\
&= g^{ik} \underline{n}_k R^j{}_{ija} \\
&= g^{ki} \underline{n}_i R^j{}_{kja} \\
&= \underline{n}_i g^{ik} R^j{}_{kja} \\
&= \underline{n}_i R^{ji}{}_{ja} \\
&= - \underline{n}_i R^{ij}{}_{ja} \\
&= - \underline{n}_i g^{j\ell} R^i{}_{\ell ja} \\
&= - \underline{n}_i R^i{}_{\ell ja} g^{j\ell} \\
&= - \underline{n}_i R^i{}_{\ell ja} (h^{j\ell} - \underline{n}^j \underline{n}^\ell) \\
&= - \underline{n}_i R^i{}_{cba} q^{cb} \qquad \text{(see below)}
\end{aligned}$$

$$= - \underline{n}_i g^{ik} R_{kcba} q^{cb}$$

$$= - g^{ki} \underline{n}_i R_{kcba} q^{cb}$$

$$= - \underline{n}^k R_{kcba} q^{cb}$$

$$= - g(\underline{n}, R(\partial_b, \partial_a)\partial_c) q^{cb}$$

$$= - (\bar{\nabla}_a \varkappa_{bc} - \bar{\nabla}_b \varkappa_{ac}) q^{cb}$$

$$= \bar{\nabla}^c \varkappa_{ac} - \bar{\nabla}_a \varkappa^c{}_c$$

$$= \bar{\nabla}^b \varkappa_{ab} - \bar{\nabla}_a \varkappa^b{}_b$$

$$= \bar{\nabla}_b \varkappa^b{}_a - \bar{\nabla}_a \varkappa^b{}_b.$$

Accordingly, if Ric(g) = 0, then

$$\bar{\nabla}_b \varkappa^b{}_a - \bar{\nabla}_a \varkappa^b{}_b = 0.$$

Therefore

$$\mathrm{div}_q (\varkappa - Kq) = 0.$$

I.e.:

$$\mathrm{div}_{q_t} (\varkappa_t - K_t q_t) = 0.$$

Conversely, under the stated conditions,

$$\mathrm{Ric}(\underline{n}, \partial_a) = 0$$

$$\Rightarrow$$

$$\mathrm{Ric}(N\underline{n}, \partial_a) = 0$$

$$\Rightarrow$$

$$\mathrm{Ric}(\partial_0 - N^b \partial_b, \partial_a) = 0$$

$$\Rightarrow$$

$$R_{0a} - N^b R_{ba} = 0$$

$$\Rightarrow$$

$$R_{0a} = 0.$$

<u>Details</u> We claim that

$$\underline{n}_i R^i{}_{\ell ja} \underline{n}^j \underline{n}^\ell = 0,$$

a not completely obvious point. Thus

$$\underline{n}_i R^i{}_{\ell ja} \underline{n}^j \underline{n}^\ell = \underline{n}_0 R^0{}_{\ell ja} \underline{n}^j \underline{n}^\ell$$

$$= - N R^0{}_{\ell ja} \underline{n}^j \underline{n}^\ell.$$

And

$$R^0{}_{\ell ja} \underline{n}^j \underline{n}^\ell = g^{0k} R_{k\ell ja} \underline{n}^\ell \underline{n}^j$$

$$= g^{00} R_{0\ell ja} \underline{n}^\ell \underline{n}^j + g^{0b} R_{b\ell ja} \underline{n}^\ell \underline{n}^j$$

$$= g^{00}R_{000a}\underline{n}^0\underline{n}^0 + g^{00}R_{00da}\underline{n}^0\underline{n}^d$$

$$+ g^{00}R_{0c0a}\underline{n}^c\underline{n}^0 + g^{00}R_{0cda}\underline{n}^c\underline{n}^d$$

$$+ g^{0b}R_{b00a}\underline{n}^0\underline{n}^0 + g^{0b}R_{b0da}\underline{n}^0\underline{n}^d$$

$$+ g^{0b}R_{bc0a}\underline{n}^c\underline{n}^0 + g^{0b}R_{bcda}\underline{n}^c\underline{n}^d$$

- $R_{000a} = 0$ & $R_{00da} = 0$.
- $g^{00}R_{0c0a}\underline{n}^c\underline{n}^0$

$$= g^{00}\underline{n}^c\underline{n}^0 R_{0c0a}$$

$$= -\frac{1}{N^2} \cdot -\frac{N^c}{N} \cdot \frac{1}{N} R_{0c0a}$$

$$= \frac{N^c}{N^4} R_{0c0a}$$

$$= -\frac{N^b}{N^4} R_{b00a}.$$

But

$$g^{0b}R_{b00a}\underline{n}^0\underline{n}^0$$

$$= g^{0b}\underline{n}^0\underline{n}^0 R_{b00a}$$

$$= \frac{N^b}{N^2} \cdot \frac{1}{N} \cdot \frac{1}{N} R_{b00a}$$

$$= \frac{N^b}{N^4} R_{b00a}.$$

• $g^{00}R_{0cda}\underline{n}^c\underline{n}^d$

$$= g^{00}\underline{n}^c\underline{n}^d R_{0cda}$$

$$= -\frac{1}{N^2}\cdot -\frac{N^c}{N}\cdot -\frac{N^d}{N} R_{0cda}$$

$$= -\frac{N^cN^d}{N^4} R_{0cda}$$

$$= \frac{N^cN^d}{N^4} R_{c0da}$$

$$= \frac{N^bN^d}{N^4} R_{b0da}.$$

But

$g^{0b}R_{b0da}\underline{n}^0\underline{n}^d$

$$= g^{0b}\underline{n}^0\underline{n}^d R_{b0da}$$

$$= \frac{N^b}{N^2}\cdot\frac{1}{N}\cdot -\frac{N^d}{N} R_{b0da}$$

$$= -\frac{N^bN^d}{N^4} R_{b0da}.$$

This leaves

$$g^{0b}R_{bc0a}\underline{n}^c\underline{n}^0 + g^{0b}R_{bcda}\underline{n}^c\underline{n}^d.$$

• $R_{bc0a} = R_{0abc} = -R_{a0bc}.$

And

$$R_{a0bc} + R_{abc0} + R_{ac0b} = 0.$$

But

$$g^{0b}\underline{n}^c\underline{n}^0 R_{ac0b}$$

$$= - g^{0b}\underline{n}^c\underline{n}^0 R_{acb0}$$

$$= - g^{0c}\underline{n}^b\underline{n}^0 R_{abc0}$$

$$= \frac{N^c N^b}{N^4} R_{abc0}.$$

On the other hand,

$$g^{0b}\underline{n}^c\underline{n}^0 = - \frac{N^b N^c}{N^4} .$$

Consequently,

$$g^{0b}\underline{n}^c\underline{n}^0 R_{bc0a} = 0.$$

- $R_{bcda} = R_{dabc} = - R_{adbc}.$

And

$$R_{adbc} + R_{abcd} + R_{acdb} = 0.$$

But

$$g^{0b}\underline{n}^c\underline{n}^d R_{acdb}$$

$$= - g^{0b}\underline{n}^c\underline{n}^d R_{acbd}$$

$$= - g^{0c}\underline{n}^b\underline{n}^d R_{abcd}$$

$$= - \frac{N^c N^b N^d}{N^4} R_{abcd}.$$

On the other hand,

$$g^{0b} \underline{n}^c \underline{n}^d = \frac{N^b N^c N^d}{N^4} .$$

Consequently,

$$g^{0b} \underline{n}^c \underline{n}^d R_{bcda} = 0.$$

Turning now to the second constraint, write

$$S(g) = R_{ikj\ell} g^{ij} g^{k\ell}$$

$$= R_{ikj\ell} (h^{ij} - \underline{n}^i \underline{n}^j)(h^{k\ell} - \underline{n}^k \underline{n}^\ell)$$

$$= R_{ikj\ell} h^{ij} h^{k\ell} - 2R^k{}_{ikj} \underline{n}^i \underline{n}^j \qquad \text{(see below)}$$

$$= R_{acbd} q^{ab} q^{cd} - 2\mathrm{Ric}(\underline{n},\underline{n})$$

$$= (\bar{R}_{acbd} + \varkappa_{ab}\varkappa_{cd} - \varkappa_{ad}\varkappa_{cb}) q^{ab} q^{cd} - 2\mathrm{Ric}(\underline{n},\underline{n})$$

$$= S(q) + K^2 - [\varkappa,\varkappa]_q - 2\mathrm{Ric}(\underline{n},\underline{n}).$$

Therefore

$$\mathrm{Ein}(\underline{n},\underline{n}) = \mathrm{Ric}(\underline{n},\underline{n}) + \frac{1}{2} S(g)$$

$$= \mathrm{Ric}(\underline{n},\underline{n}) + \frac{1}{2} (S(q) + K^2 - [\varkappa,\varkappa]_q - 2\mathrm{Ric}(\underline{n},\underline{n}))$$

$$= \frac{1}{2} (S(q) + K^2 - [\varkappa,\varkappa]_q).$$

So, if Ric(g) = 0, then Ein(g) = 0, hence

$$S(q) + K^2 - [\varkappa,\varkappa]_q = 0$$

or still,

$$([\varkappa,\varkappa]_q - K^2) - S(q) = 0.$$

I.e.:

$$([\varkappa_t,\varkappa_t]_{q_t} - K_t^2) - S(q_t) = 0.$$

Conversely, under the stated conditions,

$$\mathrm{Ein}(\underline{n},\underline{n}) = 0$$

$$\Rightarrow$$

$$\mathrm{Ric}(\underline{n},\underline{n}) + \frac{1}{2} S(g) = 0$$

$$\Rightarrow$$

$$\frac{1}{N^2} \mathrm{Ric}(\partial_0 - N^a\partial_a, \partial_0 - N^b\partial_b) + \frac{1}{2} g^{ij}R_{ij} = 0$$

$$\Rightarrow$$

$$\frac{1}{N^2} R_{00} + \frac{1}{2} g^{00}R_{00} = 0$$

$$\Rightarrow$$

$$\frac{1}{N^2} R_{00} - \frac{1}{2}\frac{1}{N^2} R_{00} = 0$$

$$\Rightarrow$$

$$\frac{1}{2} R_{00} = 0 \Rightarrow R_{00} = 0.$$

<u>Details</u> The claim is that

$$- R_{ikj\ell}h^{ij}\underline{n}^k\underline{n}^\ell - R_{ikj\ell}h^{k\ell}\underline{n}^i\underline{n}^j - R_{ikj\ell}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell$$

equals

$$- 2R^k{}_{ikj}\underline{n}^i\underline{n}^j.$$

• $- R_{ikj\ell}h^{ij}\underline{n}^k\underline{n}^\ell$

$$= - R_{ki\ell j}h^{k\ell}\underline{n}^i\underline{n}^j$$

$$= - R_{\ell ikj}h^{k\ell}\underline{n}^i\underline{n}^j.$$

• $- R_{ikj\ell}h^{k\ell}\underline{n}^i\underline{n}^j$

$$= - R_{ki\ell j}h^{k\ell}\underline{n}^i\underline{n}^j$$

$$= - R_{\ell ikj}h^{k\ell}\underline{n}^i\underline{n}^j.$$

• $- R_{ikj\ell}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell$

$$= - R_{ki\ell j}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell$$

$$= - R_{\ell ikj}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell.$$

• $- 2R^k{}_{ikj}\underline{n}^i\underline{n}^j$

$$= - 2g^{k\ell}R_{\ell ikj}\underline{n}^i\underline{n}^j$$

$$= - 2(h^{k\ell} - \underline{n}^k\underline{n}^\ell)R_{\ell ikj}\underline{n}^i\underline{n}^j$$

$$= 2R_{\ell ikj}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell - 2R_{\ell ikj}h^{k\ell}\underline{n}^i\underline{n}^j.$$

Matters thus reduce to showing that

$$R_{\ell ikj}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell = 0.$$

To this end, we shall use the fact that

$$R_{\ell ikj} + R_{\ell kji} + R_{\ell jik} = 0.$$

- $R_{\ell kji}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell$

$$= - R_{\ell kij}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell$$

$$= - R_{\ell ikj}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell.$$

- $R_{\ell jik}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell$

$$= - R_{\ell jki}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell$$

$$= - R_{\ell kji}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell.$$

Therefore

$$R_{\ell ikj}\underline{n}^i\underline{n}^j\underline{n}^k\underline{n}^\ell = 0.$$

<u>LEMMA</u> We have

$$S(g) = S(q) + [\varkappa,\varkappa]_q - K^2$$

$$- 2\nabla_i(\underline{n}^j\nabla_j\underline{n}^i - \underline{n}^i\nabla_j\underline{n}^j).$$

[As was noted above,

$$S(g) = S(q) + K^2 - [\varkappa,\varkappa]_q - 2\text{Ric}(\underline{n},\underline{n}).$$

But

$$\text{Ric}(\underline{n},\underline{n}) = (\nabla_i\nabla_j\underline{n}^i - \nabla_j\nabla_i\underline{n}^i)\underline{n}^j$$

$\Rightarrow$

$$- 2\text{Ric}(\underline{n},\underline{n})$$

$$= 2[(\nabla_i\underline{n}^j)(\nabla_j\underline{n}^i) - (\nabla_j\underline{n}^j)(\nabla_i\underline{n}^i)]$$

$$- 2[\nabla_i(\underline{n}^j\nabla_j\underline{n}^i) - \nabla_j(\underline{n}^j\nabla_i\underline{n}^i)].$$

Therefore

$$S(g) = S(q) + K^2 - [\varkappa,\varkappa]_q + 2([\varkappa,\varkappa]_q - K^2)$$

$$- 2\nabla_i(\underline{n}^j\nabla_j\underline{n}^i - \underline{n}^i\nabla_j\underline{n}^j)$$

$$= S(q) + [\varkappa,\varkappa]_q - K^2$$

$$- 2\nabla_i(\underline{n}^j\nabla_j\underline{n}^i - \underline{n}^i\nabla_j\underline{n}^j).]$$

<u>Formulas</u>

• $\nabla_i(\underline{n}^i\nabla_j\underline{n}^j)$

$$= dx^i(\nabla_i(K\underline{n}))$$

$$= dt(\nabla_0(K\underline{n})) + dx^a(\nabla_a(K\underline{n}))$$

$$= \frac{1}{N} (q^{ab}\varkappa_{ab})^{\cdot} + K^2 - \frac{N^a}{N} \nabla_a K$$

$$= \frac{1}{N} (q^{ab})^{\cdot}\varkappa_{ab} + \frac{1}{N} q^{ab}\dot{\varkappa}_{ab} + K^2 - \frac{1}{N} L_{\vec{N}}(q^{cd}\varkappa_{cd})$$

$$= - \frac{1}{N} q^{ac}\dot{q}_{cd}q^{db}\varkappa_{ab} + \frac{1}{N} \mathrm{tr}_q(\dot{\varkappa}) + K^2$$

$$- \frac{1}{N} (L_{\vec{N}} q^{cd})\varkappa_{cd} - \frac{1}{N} q^{cd}L_{\vec{N}}\varkappa_{cd}$$

$$= - \frac{1}{N} q^{ac}(2N\varkappa_{cd} + L_{\vec{N}}q_{cd})q^{db})\varkappa_{ab} + \frac{1}{N} \mathrm{tr}_q(\dot{\varkappa}) + K^2$$

$$- \frac{1}{N} (L_{\vec{N}}q^{cd})\varkappa_{cd} - \frac{1}{N} \mathrm{tr}_q(L_{\vec{N}}\varkappa)$$

$$= - 2[\varkappa,\varkappa]_q + \frac{1}{N} (L_{\vec{N}}q^{ab})\varkappa_{ab} + \frac{1}{N} \mathrm{tr}_q(\dot{\varkappa}) + K^2$$

$$- \frac{1}{N} (L_{\vec{N}}q^{cd})\varkappa_{cd} - \frac{1}{N} \mathrm{tr}_q(L_{\vec{N}}\varkappa)$$

$$= - 2[\varkappa,\varkappa]_q + \frac{1}{N} \mathrm{tr}_q(\dot{\varkappa}) + K^2 - \frac{1}{N} \mathrm{tr}_q(L_{\vec{N}}\varkappa).$$

• $\nabla_i(\underline{n}^j\nabla_j\underline{n}^i)$

$$= dx^i(\nabla_i(\underline{n}^j\nabla_j\underline{n}))$$

$$= dx^i(\nabla_i[\frac{1}{N}(\partial_c N + \varkappa_{bc}N^b)q^{cd}\partial_d - \frac{N^b}{N} \varkappa_b{}^d\partial_d])$$

$$= dx^i(\nabla_i[\frac{\partial_c N}{N} q^{cd}\partial_d])$$

$$= dt(\frac{\partial_c N}{N} q^{cd}\nabla_0\partial_d) + dx^a(\nabla_a[\frac{\partial_c N}{N} q^{cd}\partial_d])$$

$$= \frac{\partial_c N}{N^2} q^{cd}(\partial_d N + N^b \kappa_{bd}) + \bar{\nabla}_a(\frac{1}{N}\bar{\nabla}^a N) - \frac{\partial_c N}{N^2}\kappa^c{}_a N^a$$

$$= \frac{1}{N}\bar{\nabla}^a\bar{\nabla}_a N$$

$$= \frac{1}{N}\Delta_q N.$$

Substituting these relations into the lemma then gives

$$S(g) = S(q) - 3[\kappa,\kappa]_q + K^2$$

$$+ \frac{2}{N}(\mathrm{tr}_q(\dot{\kappa}) - \mathrm{tr}_q(L_{\vec{N}}\kappa) - \Delta_q N).$$

Scholium: We have

$$G_{ab} = R_{ab} - \frac{1}{2}S(g)g_{ab}$$

$$= \bar{G}_{ab} + K\kappa_{ab} - 2(\kappa*\kappa)_{ab} + \frac{3}{2}[\kappa,\kappa]_q q_{ab} - \frac{1}{2}K^2 q_{ab}$$

$$+ \frac{1}{N}(\dot{\kappa}_{ab} - \mathrm{tr}_q(\dot{\kappa})q_{ab})$$

$$+ \frac{1}{N}[(\Delta_q N)q_{ab} - (H_N)_{ab} + \mathrm{tr}_q(L_{\vec{N}}\kappa)q_{ab} - L_{\vec{N}}\kappa_{ab}].$$

Therefore

$$G_{ab} = 0$$

iff

$$\dot{\varkappa}_{ab} - \operatorname{tr}_q(\dot{\varkappa})q_{ab} = L_{\vec{N}}\varkappa_{ab} - \operatorname{tr}_q(L_{\vec{N}}\varkappa)q_{ab}$$

$$+ 2N(\varkappa * \varkappa)_{ab} - NK\varkappa_{ab} - \frac{3}{2} N[\varkappa,\varkappa]_q q_{ab} + \frac{1}{2} NK^2 q_{ab}$$

$$- N\bar{G}_{ab} + (H_N)_{ab} - (\Delta_q N)q_{ab}.$$

Remark: Locally,

$$G = G(\underline{n},\underline{n})\underline{n}^\flat \otimes \underline{n}^\flat + G(\underline{n},\partial_a)(\underline{n}^\flat \otimes dx^a + dx^a \otimes \underline{n}^\flat) + G(\partial_a,\partial_b)dx^a \otimes dx^b.$$

The preceding result admits an interpretation in the language of lagrangian mechanics. For this purpose, we shall use the following notation.

- q will stand for an arbitrary element of Q and v will stand for an arbitrary element of $S_2(\Sigma)$.

- N will stand for an arbitrary element of $C^\infty_{>0}(\Sigma) \cup C^\infty_{<0}(\Sigma)$.

[Note: Earlier N was a time dependent element of $C^\infty_{>0}(\Sigma) \cup C^\infty_{<0}(\Sigma)$.]

- $\vec{N}$ will stand for an arbitrary element of $\mathcal{D}^1(\Sigma)$.

[Note: Earlier $\vec{N}$ was a time dependent element of $\mathcal{D}^1(\Sigma)$.]

Given $(q,v;N,\vec{N})$, put

$$\varkappa = \frac{v - L_{\vec{N}}q}{2N}.$$

[Note: It is clear that $\varkappa \in S_2(\Sigma)$, thus it makes sense to form $[\varkappa,\varkappa]_q$ and $K = \mathrm{tr}_q(\varkappa)$.]

Definition: The <u>lagrangian</u> of the theory is the function

$$L: TQ \to C_d^\infty(\Sigma)$$

defined by the rule

$$L(q,v;N,\vec{N}) = N(S(q) + [\varkappa,\varkappa]_q - K^2) \otimes |q|^{1/2}.$$

[Note: Accordingly, N and $\vec{N}$ are merely external variables.]

<u>Heuristics</u> Here is the motivation for this seemingly off the wall definition. Returning to the original setup, let

$$f = -2\nabla_i(\underline{n}^j\nabla_j\underline{n}^i - \underline{n}^i\nabla_j\underline{n}^j).$$

Then f is a divergence (on M). So, ignoring boundary terms and all issues of convergence,

$$\int_M S(g)\mathrm{vol}_g = \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma (S(g)\circ i_t) i_t^*(\iota_{\partial/\partial t}\mathrm{vol}_g)$$

$$= \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma (S(g)\circ i_t) g(\underline{n}_t,\underline{n}_t) g(\partial/\partial t,\underline{n}_t)\mathrm{vol}_{q_t}$$

$$= \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma (S(g_t)\circ i_t)(-1) g(N_t\underline{n}_t,\underline{n}_t)\mathrm{vol}_{q_t}$$

$$= \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma N_t(S(g)\circ i_t)\mathrm{vol}_{q_t}$$

$$= \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma N_t(S(q_t) + [\varkappa_t,\varkappa_t]_{q_t} - K_t^2 + f\circ i_t)\mathrm{vol}_{q_t}$$

$$= \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma N_t(S(q_t) + [\kappa_t,\kappa_t]_{q_t} - K_t^2)\mathrm{vol}_{q_t}$$

$$+ \int_M f\mathrm{vol}_g$$

$$= \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma N_t(S(q_t) + [\kappa_t,\kappa_t]_{q_t} - K_t^2)\mathrm{vol}_{q_t}$$

$$= \int_{-\varepsilon}^{\varepsilon} dt \int_\Sigma L(q_t,\dot{q}_t;N_t,\vec{N}_t).$$

[Note: Working in adapted coordinates,

$$\mathrm{vol}_g = |g|^{1/2} dt\wedge dx^1 \wedge \ldots \wedge dx^{n-1},$$

thus

$$dt\wedge\iota_{\partial/\partial t}\mathrm{vol}_g$$

$$= |g|^{1/2} dt\wedge\iota_{\partial/\partial t} dt\wedge(dx^1 \wedge \ldots \wedge dx^{n-1})$$

$$- |g|^{1/2} dt\wedge dt\wedge\iota_{\partial/\partial t}(dx^1 \wedge \ldots \wedge dx^{n-1})$$

$$= \mathrm{vol}_g.]$$

Let

$$L(q,v;N,\vec{N}) = \int_\Sigma L(q,v;N,\vec{N}).$$

Example: Consider the simplest case: $N = 1$, $\vec{N} = \vec{0}$ -- then

$$L(q,v;1,\vec{0}) = \int_\Sigma L(q,v;1,\vec{0})$$

$$= \int_\Sigma (S(q) + [\tfrac{v}{2},\tfrac{v}{2}]_q - \mathrm{tr}_q(\tfrac{v}{2})^2)\mathrm{vol}_q$$

$$= \frac{1}{4}\int_\Sigma ([v,v]_q - \mathrm{tr}_q(v)^2)\mathrm{vol}_q - \int_\Sigma - S(q)\mathrm{vol}_q$$

$$= \frac{1}{4} G_q(v,v) - V_{-1}(q)$$

$$= \frac{1}{2} K_{\frac{1}{n}-1}(q,v) - V_{-1}(q).$$

Notation: Write

$$\frac{\delta L}{\delta q_{ab}} = (\frac{\delta L}{\delta q})^{ab} \text{ and } \frac{\delta L}{\delta v_{ab}} = (\frac{\delta L}{\delta v})^{ab}.$$

[Note: On general grounds,

$$\frac{\delta L}{\delta q} \in S_d^2(\Sigma) \text{ and } \frac{\delta L}{\delta v} \in S_d^2(\Sigma).]$$

<u>SUBLEMMA</u> We have

$$\frac{\delta L}{\delta q_{ab}} = L_{\vec{N}}[(\kappa^{ab} - Kq^{ab}) \otimes |q|^{1/2}]$$

$$+ [- 2N((\kappa*\kappa)^{ab} - K\kappa^{ab}) + \frac{N}{2} ([\kappa,\kappa]_q - K^2)q^{ab}$$

$$- N\bar{G}^{ab} + (H_N)^{ab} - (\Delta_q N)q^{ab}] \otimes |q|^{1/2}.$$

<u>SUBLEMMA</u> We have

$$\frac{\delta L}{\delta v_{ab}} = (\kappa^{ab} - Kq^{ab}) \otimes |q|^{1/2}.$$

Consider now the original situation, viz. the triple $(q_t,N_t,\vec{N}_t)$ $(t\in]-\varepsilon,\varepsilon[)$ -- then insertion of this data into the formulas for the functional derivatives leads to two functions of t:

$$\frac{\delta L}{\delta q_{ab}}(t) \quad \& \quad \frac{\delta L}{\delta v_{ab}}(t).$$

THEOREM $G^{ab} = 0$ iff the equations of Lagrange are satisfied, i.e., iff

$$\frac{d}{dt}\frac{\delta L}{\delta v_{ab}}(t) = \frac{\delta L}{\delta q_{ab}}(t).$$

It is a question of first calculating

$$\frac{d}{dt}\frac{\delta L}{\delta v_{ab}}(t)$$

and then comparing terms.

Step 1: From the definitions,

$$\frac{d}{dt}\frac{\delta L}{\delta v_{ab}}(t) = \frac{d}{dt}((\varkappa^{ab} - Kq^{ab}) \otimes |q|^{1/2})$$

$$= \frac{d}{dt}(\varkappa_{cd}(q^{ac}q^{bd} - q^{ab}q^{cd}) \otimes |q|^{1/2}).$$

• $(\frac{d}{dt}\varkappa_{cd})q^{ac}q^{bd}$

$$= q^{ac}q^{bd}\dot{\varkappa}_{cd}$$

$$= \dot{\varkappa}^{ab}(\neq (\varkappa^{ab})^{\cdot}).$$

• $\varkappa_{cd}(\frac{d}{dt}q^{ac})q^{bd}$

$$= \varkappa_{cd}(-q^{au}\dot{q}_{uv}q^{vc})q^{bd}$$

$$= - \varkappa_{cd}\dot{q}^{ac}q^{bd}.$$

• $\varkappa_{cd}q^{ac}(\frac{d}{dt}\, q^{bd})$

$$= \varkappa_{cd}q^{ac}(- q^{bu}\dot{q}_{uv}q^{vd})$$

$$= - \varkappa_{cd}q^{ac}\dot{q}^{bd}.$$

• $- (\frac{d}{dt}\, \varkappa_{cd})q^{ab}q^{cd}$

$$= (- q^{cd}\dot{\varkappa}_{cd})q^{ab}$$

$$= - \mathrm{tr}_q(\dot{\varkappa})q^{ab}.$$

• $- \varkappa_{cd}(\frac{d}{dt}\, q^{ab})q^{cd}$

$$= - \varkappa_{cd}(- q^{au}\dot{q}_{uv}q^{vb})q^{cd}$$

$$= \varkappa_{cd}\dot{q}^{ab}q^{cd}.$$

• $- \varkappa_{cd}q^{ab}(\frac{d}{dt}\, q^{cd})$

$$= - \varkappa_{cd}q^{ab}(- q^{cu}\dot{q}_{uv}q^{vd})$$

$$= \varkappa_{cd}q^{ab}\dot{q}^{cd}.$$

• $(\varkappa^{ab} - Kq^{ab}) \otimes \frac{d}{dt}\, |q|^{1/2}$

$$= \frac{1}{2}\, (\varkappa^{ab} - Kq^{ab})q^{cd}\dot{q}_{cd} \otimes |q|^{1/2}.$$

Summary: We have

$$\frac{d}{dt}\left((\kappa^{ab} - Kq^{ab}) \otimes |q|^{1/2}\right)$$

$$= (\dot{\kappa}^{ab} - \mathrm{tr}_q(\dot{\kappa})q^{ab}) \otimes |q|^{1/2}$$

$$- \kappa_{cd}(\dot{q}^{ac}q^{bd} + q^{ac}\dot{q}^{bd} - \dot{q}^{ab}q^{cd} - q^{ab}\dot{q}^{cd}) \otimes |q|^{1/2}$$

$$+ \frac{1}{2}(\kappa^{ab} - Kq^{ab})q^{cd}\dot{q}_{cd} \otimes |q|^{1/2}.$$

<u>Step 2</u>: Write

$$- \kappa_{cd}(\dot{q}^{ac}q^{bd} + q^{ac}\dot{q}^{bd} - \dot{q}^{ab}q^{cd} - q^{ab}\dot{q}^{cd})$$

$$= - \kappa_{cd}(2N\kappa^{ac} + (L_{\vec{N}}q)^{ac})q^{bd}$$

$$- \kappa_{cd}(2N\kappa^{bd} + (L_{\vec{N}}q)^{bd})q^{ac}$$

$$+ \kappa_{cd}(2N\kappa^{ab} + (L_{\vec{N}}q)^{ab})q^{cd}$$

$$+ \kappa_{cd}(2N\kappa^{cd} + (L_{\vec{N}}q)^{cd})q^{ab}.$$

Then

$$\left[\begin{array}{l}(\mathcal{L}_{\vec{N}}q)^{ac} = q^{au}(\mathcal{L}_{\vec{N}}q_{uv})q^{vc} = -\mathcal{L}_{\vec{N}}q^{ac} \\ (\mathcal{L}_{\vec{N}}q)^{bd} = q^{bu}(\mathcal{L}_{\vec{N}}q_{uv})q^{vd} = -\mathcal{L}_{\vec{N}}q^{bd} \\ (\mathcal{L}_{\vec{N}}q)^{ab} = q^{au}(\mathcal{L}_{\vec{N}}q_{uv})q^{vb} = -\mathcal{L}_{\vec{N}}q^{ab} \\ (\mathcal{L}_{\vec{N}}q)^{cd} = q^{cu}(\mathcal{L}_{\vec{N}}q_{uv})q^{vd} = -\mathcal{L}_{\vec{N}}q^{cd}.\end{array}\right.$$

Therefore

$$- \kappa_{cd}(\dot{q}^{ac}q^{bd} + q^{ac}\dot{q}^{bd} - \dot{q}^{ab}q^{cd} - q^{ab}\dot{q}^{cd})$$

$$= - 2N\kappa_{cd}\kappa^{ac}q^{bd} - 2N\kappa_{cd}\kappa^{bd}q^{ac}$$

$$+ 2N\kappa_{cd}\kappa^{ab}q^{cd} + 2N\kappa_{cd}\kappa^{cd}q^{ab}$$

$$+ \kappa_{cd}((\mathcal{L}_{\vec{N}}q^{ac})q^{bd} + (\mathcal{L}_{\vec{N}}q^{bd})q^{ac} - (\mathcal{L}_{\vec{N}}q^{ab})q^{cd} - (\mathcal{L}_{\vec{N}}q^{cd})q^{ab})$$

$$= - 4N(\kappa * \kappa)^{ab} + 2NK\kappa^{ab} + 2N[\kappa,\kappa]_q q^{ab}$$

$$+ \kappa_{cd}\mathcal{L}_{\vec{N}}(q^{ac}q^{bd} - q^{ab}q^{cd}).$$

Next

$$\frac{1}{2}(\kappa^{ab} - Kq^{ab})q^{cd}\dot{q}_{cd}$$

$$= \frac{1}{2}(\kappa^{ab} - Kq^{ab})q^{cd}(2N\kappa_{cd} + \mathcal{L}_{\vec{N}}q_{cd})$$

$$= N(\varkappa^{ab} - Kq^{ab})q^{cd}\varkappa_{cd}$$

$$+ \frac{1}{2} (\varkappa^{ab} - Kq^{ab})q^{cd}(N_{c;d} + N_{d;c})$$

$$= N(\varkappa^{ab} - Kq^{ab})K + (\varkappa^{ab} - Kq^{ab})\bar{\nabla}_c N^c.$$

Summary: We have

$$\frac{d}{dt} ((\varkappa^{ab} - Kq^{ab}) \otimes |q|^{1/2})$$

$$= (\dot{\varkappa}^{ab} - \mathrm{tr}_q(\dot{\varkappa})q^{ab}) \otimes |q|^{1/2}$$

$$- (4N(\varkappa * \varkappa)^{ab} - 2NK\varkappa^{ab} - 2N[\varkappa,\varkappa]_q q^{ab}) \otimes |q|^{1/2}$$

$$+ \varkappa_{cd} L_{\vec{N}}(q^{ac}q^{bd} - q^{ab}q^{cd}) \otimes |q|^{1/2}$$

$$+ N(\varkappa^{ab} - Kq^{ab})K \otimes |q|^{1/2} + (\varkappa^{ab} - Kq^{ab})\bar{\nabla}_c N^c \otimes |q|^{1/2}$$

$$= (\dot{\varkappa}^{ab} - \mathrm{tr}_q(\dot{\varkappa})q^{ab}) \otimes |q|^{1/2}$$

$$- (4N(\varkappa * \varkappa)^{ab} - 3NK\varkappa^{ab} - 2N[\varkappa,\varkappa]_q q^{ab} + NK^2 q^{ab}) \otimes |q|^{1/2}$$

$$+ \varkappa_{cd} L_{\vec{N}}(q^{ac}q^{bd} - q^{ab}q^{cd}) \otimes |q|^{1/2}$$

$$+ (\varkappa^{ab} - Kq^{ab})\bar{\nabla}_c N^c \otimes |q|^{1/2}.$$

The final point is to note that

$$L_{\vec{N}}(\varkappa^{ab} - Kq^{ab})$$

$$= L_{\vec{N}}(\varkappa_{cd}(q^{ac}q^{bd} - q^{ab}q^{cd}))$$

$$= (L_{\vec{N}}\varkappa_{cd})(q^{ac}q^{bd} - q^{ab}q^{cd})$$

$$+ \varkappa_{cd}L_{\vec{N}}(q^{ac}q^{bd} - q^{ab}q^{cd}).$$

Since $\bar{\nabla}_c N^c = \mathrm{div}_q \vec{N}$, it follows that

$$\varkappa_{cd}L_{\vec{N}}(q^{ac}q^{bd} - q^{ab}q^{cd}) \otimes |q|^{1/2}$$

$$+ (\varkappa^{ab} - Kq^{ab})\bar{\nabla}_c N^c \otimes |q|^{1/2}$$

$$= L_{\vec{N}}[(\varkappa^{ab} - Kq^{ab}) \otimes |q|^{1/2}]$$

$$- (L_{\vec{N}}\varkappa_{cd})(q^{ac}q^{bd} - q^{ab}q^{cd}) \otimes |q|^{1/2}.$$

Observation:

$$\left[\begin{array}{l} q^{ac}q^{bd}(L_{\vec{N}}\varkappa_{cd}) = (L_{\vec{N}}\varkappa)^{ab} \\ \\ q^{cd}(L_{\vec{N}}\varkappa_{cd})q^{ab} = \mathrm{tr}_q(L_{\vec{N}}\varkappa)q^{ab}. \end{array}\right.$$

Taking the preceding considerations into account, we then find that

$$\frac{d}{dt}\frac{\delta L}{\delta v_{ab}}(t) = \frac{\delta L}{\delta q_{ab}}(t)$$

iff

$$\dot{\varkappa}^{ab} - \mathrm{tr}_q(\dot{\varkappa})q^{ab} = (L_{\vec{N}}\varkappa)^{ab} - \mathrm{tr}_q(L_{\vec{N}}\varkappa)q^{ab}$$

$$+ 2N(\varkappa * \varkappa)^{ab} - NK\varkappa^{ab} - \frac{3}{2} N[\varkappa,\varkappa]_q q^{ab} + \frac{1}{2} NK^2 q^{ab}$$

$$- N\bar{G}^{ab} + (H_N)^{ab} - (\Delta_q N) q^{ab},$$

which is equivalent to the assertion of the theorem.

One can also arrive at the constraint equations by demanding that $\forall$ t,

$$\begin{cases} \dfrac{\delta L}{\delta N} = 0 \\[2ex] \dfrac{\delta L}{\delta \vec{N}} = 0, \end{cases}$$

relationships which should be expected to hold on purely formal grounds (due to the absence of the corresponding velocities in the definition of L).

[Note: Here

$$\begin{cases} \dfrac{\delta L}{\delta N} \in C_d^{\infty}(\Sigma) \\[2ex] \dfrac{\delta L}{\delta \vec{N}} \in \Lambda_d^1(\Sigma).] \end{cases}$$

<u>Ad $\frac{\delta L}{\delta N}$</u>: We have

$$\frac{d}{d\varepsilon} L(q,v;N + \varepsilon N',\vec{N}) \Big|_{\varepsilon=0}$$

$$= \int_\Sigma \frac{d}{d\varepsilon} L(q,v;N + \varepsilon N',\vec{N}) \Big|_{\varepsilon=0}.$$

- $\frac{d}{d\varepsilon}\,(N + \varepsilon N')S(q)\Big|_{\varepsilon=0}$

$$= N'S(q).$$

- $\frac{d}{d\varepsilon}\,(N + \varepsilon N')\,[\varkappa,\varkappa]_q\Big|_{\varepsilon=0}$

$$= \frac{d}{d\varepsilon}\,(N + \varepsilon N')\,\Big[\frac{v - L_{\vec{N}}q}{2(N + \varepsilon N')}\,,\,\frac{v - L_{\vec{N}}q}{2(N + \varepsilon N')}\Big]_q\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\,\frac{1}{4(N + \varepsilon N')}\,[v - L_{\vec{N}}q,\ v - L_{\vec{N}}q]_q\Big|_{\varepsilon=0}$$

$$= -\frac{1}{4}\,\frac{N'}{N^2}\,[v - L_{\vec{N}}q,\ v - L_{\vec{N}}q]_q$$

$$= -N'[\varkappa,\varkappa]_q.$$

- $\frac{d}{d\varepsilon}\,(N + \varepsilon N')(-K^2)\Big|_{\varepsilon=0}$

$$= -\frac{d}{d\varepsilon}\,(N + \varepsilon N')\,\mathrm{tr}_q(\varkappa)^2\Big|_{\varepsilon=0}$$

$$= -\frac{d}{d\varepsilon}\,(N + \varepsilon N')\,\Big[q,\,\frac{v - L_{\vec{N}}q}{2(N + \varepsilon N')}\Big]_q^2\Big|_{\varepsilon=0}$$

$$= -\frac{d}{d\varepsilon}\,\frac{1}{4(N + \varepsilon N')}\,[q, v - L_{\vec{N}}q]_q^2\Big|_{\varepsilon=0}$$

$$= \frac{1}{4}\frac{N'}{N^2}[q, v - L_{\vec{N}} q]_q^2$$

$$= N' \mathrm{tr}_q(\varkappa)^2$$

$$= N'K^2.$$

Thus

$$\frac{\delta L}{\delta N} = (S(q) - [\varkappa,\varkappa]_q + K^2) \otimes |q|^{1/2}$$

and so

$$\frac{\delta L}{\delta N}(t) = 0 \Leftrightarrow ([\varkappa_t,\varkappa_t]_{q_t} - K_t^2) - S(q_t) = 0.$$

<u>Ad $\frac{\delta L}{\delta \vec{N}}$</u>: We have

$$\frac{d}{d\varepsilon} L(q,v;N,\vec{N} + \varepsilon\vec{N}')\Big|_{\varepsilon=0}$$

$$= \int_\Sigma \frac{d}{d\varepsilon} L(q,v;N,\vec{N} + \varepsilon\vec{N}')\Big|_{\varepsilon=0}.$$

$$\bullet\, N\frac{d}{d\varepsilon}\left[\frac{v - L_{\vec{N}+\varepsilon\vec{N}'}q}{2N}, \frac{v - L_{\vec{N}+\varepsilon\vec{N}'}q}{2N}\right]_q\Big|_{\varepsilon=0}$$

$$= N\frac{d}{d\varepsilon}\frac{1}{(2N)^2}\Big(-[v, L_{\varepsilon\vec{N}'}q]_q + [L_{\vec{N}}q, L_{\varepsilon\vec{N}'}q]_q$$

$$- [L_{\varepsilon\vec{N}'}q, v]_q + [L_{\varepsilon\vec{N}'}q, L_{\vec{N}}q]_q\Big)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} \frac{1}{4N} \left(-2[v, L_{\varepsilon\vec{N}'} q]_q + 2[L_{\vec{N}} q, L_{\varepsilon\vec{N}'} q]_q \right) \Big|_{\varepsilon=0}$$

$$= -\frac{d}{d\varepsilon} [\varkappa, L_{\varepsilon\vec{N}'} q]_q \Big|_{\varepsilon=0}$$

$$= - [\varkappa, L_{\vec{N}'} q]_q$$

$\Rightarrow$

$$\int_\Sigma - [\varkappa, L_{\vec{N}'} q]_q \otimes |q|^{1/2}$$

$$= - < L_{\vec{N}'} q, \varkappa^{\#} \otimes |q|^{1/2} >$$

$$= 2 < \vec{N}', \mathrm{div}_q\, \varkappa \otimes |q|^{1/2} >.$$

$$\bullet\ - N \frac{d}{d\varepsilon} [q, \frac{v - L_{\vec{N} + \varepsilon\vec{N}'} q}{2N}]_q^2 \Big|_{\varepsilon=0}$$

$$= - 2N[q,\varkappa]_q \frac{d}{d\varepsilon} [q, - \frac{L_{\varepsilon\vec{N}'} q}{2N}]_q \Big|_{\varepsilon=0}$$

$$= [q,\varkappa]_q \, [q, L_{\vec{N}'} q]_q$$

$$= \mathrm{tr}_q(\varkappa) \, [q, L_{\vec{N}'} q]_q$$

$$= K[q, L_{\vec{N}'} q]_q$$

$\Rightarrow$

$$\int_\Sigma K[q, L_{\vec{N}'} q]_q \otimes |q|^{1/2}$$

$$= \int_\Sigma Kq^{ij}(\bar{\nabla}_j(\vec{N}')_i + \bar{\nabla}_i(\vec{N}')_j)\mathrm{vol}_q$$

$$= \int_\Sigma K(\bar{\nabla}_j(\vec{N}')^j + \bar{\nabla}_i(\vec{N}')^i)\mathrm{vol}_q$$

$$= 2\int_\Sigma K\mathrm{div}_q \vec{N}'\mathrm{vol}_q$$

$$= -2\int_\Sigma \vec{N}'K\mathrm{vol}_q$$

$$= -2\int_\Sigma dK(\vec{N}')\mathrm{vol}_q$$

$$= -2\int_\Sigma (\mathrm{div}_q Kq)(\vec{N}')\mathrm{vol}_q$$

$$= -2 < \vec{N}', \mathrm{div}_q Kq \otimes |q|^{1/2} >.$$

Thus

$$\frac{\delta L}{\delta \vec{N}} = 2(\mathrm{div}_q \varkappa - \mathrm{div}_q Kq) \otimes |q|^{1/2}$$

and so

$$\frac{\delta L}{\delta \vec{N}}(t) = 0 \Leftrightarrow \mathrm{div}_{q_t}(\varkappa_t - K_t q_t) = 0.$$

Section 37: Dynamics Let M be a connected C^∞ manifold of dimension $n > 2$. Fix ε $(0 < \varepsilon \leq \infty)$ and assume that

$$M =]-\varepsilon,\varepsilon[\times \Sigma,$$

where Σ is compact and orientable (hence $\dim \Sigma = n - 1$).

Suppose given a triple $(q_t,N_t,\vec{N}_t)$ $(t \in]-\varepsilon,\varepsilon[)$ (subject to the customary stipulations).

Definition: The momentum of the theory is the path $t \rightarrow p_t$ in $S_d^2(\Sigma)$ defined by the prescription

$$p_t = \pi_t \otimes |q_t|^{1/2},$$

where

$$\pi_t = (\kappa_t - K_t q_t)^{\#}.$$

[Note: The motivation lying behind the definition of π_t is the fact that

$$\frac{\delta L}{\delta v} = p.$$

Here p stands for $\pi^{\#} \otimes |q|^{1/2}$ with $\pi = \kappa - Kq$.]

One can then reformulate the results from the last section along the following lines.

Constraint Equations These are the relations

$$\begin{cases} ([\pi_t,\pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2 - S(q_t)) \otimes |q_t|^{1/2} = 0 \\ \operatorname{div}_{q_t} p_t = 0. \end{cases}$$

Evolution Equations These are the relations

$$\dot{q}_t = 2N_t(\pi_t^\flat - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t^\flat)q_t) + L_{\vec{N}_t} q_t$$

and

$$\dot{p}_t = -\,2N_t(\pi_t * \pi_t - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t)\pi_t) \otimes |q_t|^{1/2}$$

$$+ \frac{N_t}{2}([\pi_t,\pi_t]_{q_t} - \frac{1}{n-2}\,\mathrm{tr}_{q_t}(\pi_t)^2)q_t^\# \otimes |q_t|^{1/2}$$

$$- N_t \mathrm{Ein}(q_t)^\# \otimes |q_t|^{1/2}$$

$$+ (H_{N_t} - (\Delta_{q_t}N_t)q_t)^\# \otimes |q_t|^{1/2} + L_{\vec{N}_t} p_t.$$

<u>THEOREM</u> $\mathrm{Ein}(g) = 0$ iff the constraint equations and the evolution equations are satisfied by the pair (q_t, p_t).

<u>Derivatives</u> Given a function $F: T^*Q \to C_d^\infty(\Sigma)$, define

$$D_{(q,\Lambda)}F: S_2(\Sigma) \times S_d^2(\Sigma) \to C_d^\infty(\Sigma)$$

by

$$(D_{(q,\Lambda)}F)(v,\Lambda') = \frac{d}{d\varepsilon} F(q + \varepsilon v, \Lambda + \varepsilon\Lambda')\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} F(q + \varepsilon v, \Lambda)\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon} F(q, \Lambda + \varepsilon\Lambda')\Big|_{\varepsilon=0}.$$

Write

$$\begin{cases} D_q F(q,\Lambda)v = \dfrac{d}{d\varepsilon} F(q + \varepsilon v, \Lambda)\Big|_{\varepsilon=0} \\ D_\Lambda F(q,\Lambda)\Lambda' = \dfrac{d}{d\varepsilon} F(q, \Lambda + \varepsilon\Lambda')\Big|_{\varepsilon=0} \end{cases}$$

and put

$$\mathcal{F}(q,\Lambda) = \int_\Sigma F(q,\Lambda).$$

Then

$$\begin{cases} \frac{d}{d\varepsilon} F(q + \varepsilon v, \Lambda)\Big|_{\varepsilon=0} = \int_\Sigma D_q F(q,\Lambda)v = < v, \frac{\delta F}{\delta q} > \\ \\ \frac{d}{d\varepsilon} F(q,\Lambda + \varepsilon\Lambda')\Big|_{\varepsilon=0} = \int_\Sigma D_\Lambda F(q,\Lambda)\Lambda' = < \frac{\delta F}{\delta\Lambda}, \Lambda' > \end{cases}$$

provided, of course, that the relevant functional derivatives exist.

[Note: Analogous conventions are used in other situations as well, e.g., if instead $F:T^*Q \to \Lambda^1_d(\Sigma)$.]

Define a function $H:T^*Q \to C^\infty_d(\Sigma)$ by

$$H(q,\Lambda) = ([s,s]_q - \frac{1}{n-2} \operatorname{tr}_q(s)^2 - S(q)) \otimes |q|^{1/2}$$

if $\Lambda = s^\# \otimes |q|^{1/2}$ and for any $f \in C^\infty(\Sigma)$, put

$$H_f(q,\Lambda) = \int_\Sigma fH(q,\Lambda).$$

<u>LEMMA</u> We have

$$\begin{aligned} D_q H(q,\Lambda)v = {} & 2([v,s*s]_q - \frac{1}{n-2} \operatorname{tr}_q(s)[v,s]_q) \otimes |q|^{1/2} \\ & - \frac{1}{2}([s,s]_q - \frac{1}{n-2} \operatorname{tr}_q(s)^2)\operatorname{tr}_q(v) \otimes |q|^{1/2} \\ & + q[^0_2](v,\operatorname{Ein}(q)) \otimes |q|^{1/2} \\ & + (\Delta_q \operatorname{tr}_q(v) + \delta_q \operatorname{div}_q v) \otimes |q|^{1/2} \end{aligned}$$

and

$$D_\Lambda H(q,\Lambda)\Lambda' = 2(\operatorname{ev}(s,\Lambda') - \frac{1}{n-2} \operatorname{tr}_q(s)\operatorname{ev}(q,\Lambda')).$$

So, as a corollary,

$$\frac{\delta H_f}{\delta q} = 2f(s*s - \frac{1}{n-2}\,\mathrm{tr}_q(s)s)^{\#} \otimes |q|^{1/2}$$

$$- \frac{f}{2}\,([s,s]_q - \frac{1}{n-2}\,\mathrm{tr}_q(s)^2)q^{\#} \otimes |q|^{1/2}$$

$$+ f\mathrm{Ein}(q)^{\#} \otimes |q|^{1/2}$$

$$- (H_f - (\Delta_q f)q)^{\#} \otimes |q|^{1/2}$$

and

$$\frac{\delta H_f}{\delta \Lambda} = 2f(s - \frac{1}{n-2}\,\mathrm{tr}_q(s)q).$$

Define a function $I:T^*Q \to \Lambda_d^1(\Sigma)$ by

$$I(q,\Lambda) = -\,2\mathrm{div}_q\,\Lambda.$$

Each $X \in \mathcal{D}^1(\Sigma)$ thus gives rise to a map $I_X:T^*Q \to C_d^\infty(\Sigma)$, viz.

$$I_X(q,\Lambda) = -\,2\mathrm{div}_q\,\Lambda(X) \quad (= \mathrm{ev}(X,I(q,\Lambda))).$$

Let

$$\mathcal{I}_X(q,\Lambda) = \int_\Sigma I_X(q,\Lambda).$$

Then

$$\left[\begin{array}{l} \dfrac{\delta \mathcal{I}_X}{\delta q} = -\,\mathcal{L}_X\Lambda \\[2ex] \dfrac{\delta \mathcal{I}_X}{\delta \Lambda} = \mathcal{L}_X q. \end{array}\right.$$

[Note: Recall that

$$L_X \Lambda = L_X s^{\#} \otimes |q|^{1/2} + s^{\#} \otimes (\mathrm{div}_q X)|q|^{1/2}.]$$

Heuristics To see the origin of the preceding definitions, consider the fiber derivative of L:

$$\begin{bmatrix} FL : TQ \to T^*Q \\ FL(q,v) = (q, \frac{\delta L}{\delta v}). \end{bmatrix}$$

Then

$$< v, \frac{\delta L}{\delta v} > - L(q,v;N,\vec{N})$$

$$= < 2N\kappa, \frac{\delta L}{\delta v} > + < L_{\vec{N}} q, \frac{\delta L}{\delta v} > - L(q,v;N,\vec{N}).$$

But

$$< 2N\kappa, \frac{\delta L}{\delta v} > = \int_\Sigma [2N\kappa, \kappa - Kq]_q \mathrm{vol}_q$$

$$= 2 \int_\Sigma N([\kappa,\kappa]_q - K^2)\mathrm{vol}_q$$

$$\Rightarrow$$

$$< 2N\kappa, \frac{\delta L}{\delta v} > - L(q,v;N,\vec{N})$$

$$= 2 \int_\Sigma N([\kappa,\kappa]_q - K^2)\mathrm{vol}_q$$

$$- \int_\Sigma N(S(q) + [\kappa,\kappa]_q - K^2)\mathrm{vol}_q$$

$$= \int_\Sigma N([\kappa,\kappa]_q - K^2 - S(q))\mathrm{vol}_q$$

$$= \int_{\Sigma} N([\pi,\pi]_q + (3-n)K^2 - K^2 - S(q))\mathrm{vol}_q$$

$$= \int_{\Sigma} N([\pi,\pi]_q + (2-n)K^2 - S(q))\mathrm{vol}_q$$

$$= \int_{\Sigma} N([\pi,\pi]_q - \frac{1}{n-2}\,\mathrm{tr}_q(\pi)^2 - S(q))\mathrm{vol}_q$$

$$= \int_{\Sigma} N([\pi^\flat,\pi^\flat]_q - \frac{1}{n-2}\,\mathrm{tr}_q(\pi^\flat)^2 - S(q))\mathrm{vol}_q$$

$$= \mathcal{H}_N(q,p).$$

As for the term involving the Lie derivative,

$$< L_{\vec{N}}q,\ \frac{\delta L}{\delta v} >$$

$$= < L_{\vec{N}}q, (\kappa - Kq)^{\#} \otimes |q|^{1/2} >$$

$$= -\ 2 < \vec{N}, \mathrm{div}_q(\kappa - Kq) \otimes |q|^{1/2} >$$

$$= < \vec{N}, -\ 2\mathrm{div}_q\ p >$$

$$= \int_{\Sigma} -\ 2\mathrm{div}_q\ p(\vec{N})$$

$$= \int_{\Sigma} I_{\vec{N}}(q,p)$$

$$= \mathcal{I}_{\vec{N}}(q,p).$$

<u>FACT</u> $\forall\ X \in \mathcal{D}^1(\Sigma)$,

$$\begin{cases} (D_{(q,\Lambda)}H)(L_X q, L_X \Lambda) = L_X(H(q,\Lambda)) \\ (D_{(q,\Lambda)}I)(L_X q, L_X \Lambda) = L_X(I(q,\Lambda)). \end{cases}$$

[Note: Keep in mind that

$$\begin{cases} D_{(q,\Lambda)}H: S_2(\Sigma) \times S_d^2(\Sigma) \to C_d^\infty(\Sigma) \\ D_{(q,\Lambda)}I: S_2(\Sigma) \times S_d^2(\Sigma) \to \Lambda_d^1(\Sigma). \end{cases}$$]

Let

$$H_{f,X}(q,\Lambda) = H_f(q,\Lambda) + I_X(q,\Lambda).$$

Then the hamiltonian vector field

$$Z_{f,X}: Q \times S_d^2(\Sigma) \to S_2(\Sigma) \times S_d^2(\Sigma)$$

on T^*Q corresponding to $H_{f,X}$ is characterized by the condition

$$\Omega(Z_{f,X}, \text{—}) = dH_{f,X}$$

and can be represented in terms of functional derivatives:

$$Z_{f,X}(q,\Lambda) = \left(\frac{\delta H_{f,X}}{\delta \Lambda}, -\frac{\delta H_{f,X}}{\delta q}\right).$$

Now specialize and take $f = N_t$, $X = \vec{N}_t$. Replacing (q,Λ) by (q_t,p_t), the evolution equations state that

$$(\dot{q}_t, \dot{p}_t) = Z_{N_t,\vec{N}_t}(q_t,p_t).$$

Otherwise said: The curve

$$t \to (q_t, p_t) \in T^*Q$$

is an integral curve for $Z_{N_t, \vec{N}_t}$.

Definition: Let $t \to f(t)$ be a path in $C^\infty_{>0}(\Sigma)$ (or $C^\infty_{<0}(\Sigma)$) and let $t \to X(t)$ be a path in $\mathcal{D}^1(\Sigma)$ -- then a curve $t \to (q(t), \Lambda(t))$ in T^*Q is said to satisfy the <u>evolution equations</u> if

$$\begin{cases} \dot{q} = \dfrac{\delta H_{f,X}}{\delta \Lambda} \\ \\ \dot{\Lambda} = - \dfrac{\delta H_{f,X}}{\delta q}. \end{cases}$$

[Note: Here t lies in some open interval centered at the origin.]

Example: If $\text{Ein}(g) = 0$, then the curve $t \to (q_t, p_t)$ satisfies the evolution equations, where

$$\begin{cases} f(t) = N_t \\ \\ X(t) = \vec{N}_t. \end{cases}$$

<u>LEMMA</u> Under the conditions of the preceding definition, along $(q(t), \Lambda(t))$, we have

$$\begin{cases} \dfrac{dH}{dt} = - \dfrac{1}{f(t)} \delta_{q(t)} (f(t)^2 I(q(t), \Lambda(t))) + L_{X(t)} (H(q(t), \Lambda(t))) \\ \\ \dfrac{dI}{dt} = (df(t)) \otimes H(q(t), \Lambda(t)) + L_{X(t)} I(q(t), \Lambda(t)) \end{cases}$$

or, in brief,

$$\begin{cases} \dfrac{dH}{dt} = -\dfrac{1}{f}\,\delta_q(f^2 I) + L_X H \\[2ex] \dfrac{dI}{dt} = (df)H + L_X I. \end{cases}$$

We shall first consider $\frac{dH}{dt}$:

$$\frac{dH}{dt} = \frac{d}{dt}\,H(q(t),\Lambda(t))$$

$$= (D_{(q,\Lambda)}H)\,(\dot{q},\dot{\Lambda})$$

$$= (D_{(q,\Lambda)}H)\,\left(\frac{\delta H_f}{\delta\Lambda}, -\frac{\delta H_f}{\delta q}\right)$$

$$+ (D_{(q,\Lambda)}H)\,\left(\frac{\delta I_X}{\delta\Lambda}, -\frac{\delta I_X}{\delta q}\right).$$

But

$$\begin{cases} \dfrac{\delta I_X}{\delta\Lambda} = L_X q \\[2ex] \dfrac{\delta I_X}{\delta q} = -\,L_X\Lambda. \end{cases}$$

Therefore

$$(D_{(q,\Lambda)}H)\,\left(\frac{\delta I_X}{\delta\Lambda}, -\frac{\delta I_X}{\delta q}\right)$$

$$= (D_{(q,\Lambda)}H)\,(L_X q, L_X\Lambda)$$

$$= L_X(H(q,\Lambda)).$$

It remains to deal with

$$(D_{(q,\Lambda)}H)\left(\frac{\delta H_f}{\delta \Lambda}, -\frac{\delta H_f}{\delta q}\right)$$

or still,

$$D_q H(q,\Lambda)\left(\frac{\delta H_f}{\delta \Lambda}\right) - D_\Lambda H(q,\Lambda)\left(\frac{\delta H_f}{\delta q}\right)$$

or still,

$$\left(\Delta_q \mathrm{tr}_q\left(\frac{\delta H_f}{\delta \Lambda}\right) + \delta_q \mathrm{div}_q\left(\frac{\delta H_f}{\delta \Lambda}\right)\right) \otimes |q|^{1/2}$$

$$+ D_\Lambda H(q,\Lambda)((H_f - (\Delta_q f)q)^{\#} \otimes |q|^{1/2})$$

or still,

$$\Delta_q \mathrm{tr}_q(2f(\mathrm{tr}_q(s) - \frac{1}{n-2}\mathrm{tr}_q(s)\mathrm{tr}_q(q))) \otimes |q|^{1/2}$$

$$+ \delta_q \mathrm{div}_q(2f(s - \frac{1}{n-2}\mathrm{tr}_q(s)q)) \otimes |q|^{1/2}$$

$$+ 2\mathrm{ev}(s, (H_f - (\Delta_q f)q)^{\#} \otimes |q|^{1/2})$$

$$- \frac{2}{n-2}\mathrm{tr}_q(s)\mathrm{ev}(q, (H_f - (\Delta_q f)q)^{\#} \otimes |q|^{1/2})$$

$$= -\frac{2}{n-2}\Delta_q(f\mathrm{tr}_q(s)) \otimes |q|^{1/2}$$

$$+ \delta_q \mathrm{div}_q(2fs) \otimes |q|^{1/2}$$

$$- \frac{2}{n-2}\delta_q \mathrm{div}_q(f\mathrm{tr}_q(s)q) \otimes |q|^{1/2}$$

$$+ 2[s, H_f]_q \otimes |q|^{1/2} - 2(\Delta_q f)\mathrm{tr}_q(s) \otimes |q|^{1/2}$$

$$- \frac{2}{n-2}\, tr_q(s)\, tr_q(H_f) \otimes |q|^{1/2} + \frac{2n-2}{n-2}\, (\Delta_q f)\, tr_q(s) \otimes |q|^{1/2}.$$

Since

$$\delta_q div_q(f tr_q(s) q)$$

$$= \delta_q d(f tr_q(s))$$

$$= - \Delta_q(f tr_q(s))$$

and

$$tr_q(H_f) = \Delta_q f,$$

matters reduce to

$$- \frac{2}{n-2}\, \Delta_q(f tr_q(s)) \otimes |q|^{1/2}$$

$$+ \delta_q div_q(2fs) \otimes |q|^{1/2} + 2[s,H_f]_q \otimes |q|^{1/2}$$

$$+ \frac{2}{n-2}\, \Delta_q(f tr_q(s)) \otimes |q|^{1/2}$$

$$+ (\Delta_q f)\, tr_q(s) [- 2 - \frac{2}{n-2} + \frac{2n-2}{n-2}] \otimes |q|^{1/2}$$

or still,

$$\delta_q div_q(2fs) \otimes |q|^{1/2} + 2[s,H_f]_q \otimes |q|^{1/2}.$$

<u>SUBLEMMA</u> We have

$$\delta_q div_q(fs) = - [s,H_f]_q + \frac{1}{f}\, \delta_q(f^2 div_q\, s).$$

[Start by writing

$$- \delta_q \mathrm{div}_q(fs) = \bar{\nabla}^a (\mathrm{div}_q(fs))_a$$

$$= \bar{\nabla}^a \bar{\nabla}^b (fs)_{ab}$$

$$= (fs^{ab})_{;a;b}$$

$$= (f_{;a} s^{ab} + fs^{ab}_{;a})_{;b}$$

$$= f_{;a;b} s^{ab} + f_{;a} s^{ab}_{;b} + (fs^{ab}_{;a})_{;b}$$

$$= [s, H_f]_q + f_{;a} s^{ab}_{;b} + (fs^{ab}_{;a})_{;b}.$$

But

$$f_{;a} s^{ab}_{;b} + (fs^{ab}_{;a})_{;b}$$

$$= f_{;a} s^{ab}_{;b} + f_{;b} s^{ab}_{;a} + fs^{ab}_{;a;b}$$

$$= f_{;a} s^{ab}_{;b} + f_{;a} s^{ab}_{;b} + fs^{ab}_{;a;b}$$

$$= 2f_{;a} s^{ab}_{;b} + fs^{ab}_{;a;b}.$$

On the other hand,

$$- \delta_q (f^2 \mathrm{div}_q\, s) = \bar{\nabla}^a (f^2 \mathrm{div}_q\, s)_a$$

$$= \bar{\nabla}_a (f^2 \mathrm{div}_q\, s)^a$$

$$= (f^2 \mathrm{div}_q\ s)^a_{;a}$$

$$= (f^2)_{;a}(\mathrm{div}_q\ s)^a + f^2(\mathrm{div}_q\ s)^a_{;a}$$

$$= (2f)f_{;a}s^{ab}_{;b} + f^2 s^{ab}_{;a;b}.$$

Therefore

$$\delta_q \mathrm{div}_q(fs) = -\ [s,H_f]_q + \frac{1}{f}\ \delta_q(f^2 \mathrm{div}_q\ s).]$$

Accordingly,

$$(\delta_q \mathrm{div}_q(2fs) + 2[s,H_f]_q) \otimes |q|^{1/2}$$

$$= \frac{1}{f}\ \delta_q(2f^2 \mathrm{div}_q\ s) \otimes |q|^{1/2}$$

$$= \frac{1}{f}\ \delta_q(f^2 2\mathrm{div}_q\ s \otimes |q|^{1/2})$$

$$= -\ \frac{1}{f}\ \delta_q(f^2(-\ 2\mathrm{div}_q\ \Lambda))$$

$$= -\ \frac{1}{f}\ \delta_q(f^2 I(q,\Lambda)).$$

This establishes the formula for $\frac{dH}{dt}$. Turning to $\frac{dI}{dt}$, fix $Y \in \mathcal{D}^1(\Sigma)$ -- then

$$< Y, \frac{dI}{dt} > = < Y, \frac{d}{dt}\ I(q(t),\Lambda(t)) >$$

$$= < Y, (D_{(q,\Lambda)}I)(\dot{q},\dot{\Lambda}) >$$

$$= -\ < f, (D_{(q,\Lambda)}H)(L_Y q, L_Y \Lambda) >$$

$$- < X, (D_{(q,\Lambda)} I)(L_Y q, L_Y \Lambda) >$$

$$= - < f, L_Y(H(q,\Lambda)) > - < X, L_Y(I(q,\Lambda)) >$$

$$= - \int_\Sigma [f L_Y(H(q,\Lambda)) + ev(X, L_Y(I(q,\Lambda)))]$$

$$= \int_\Sigma [(L_Y f) H(q,\Lambda) + ev(L_Y X, I(q,\Lambda))]$$

$$= \int_\Sigma [Y(df) H(q,\Lambda) - ev(L_X Y, I(q,\Lambda))]$$

$$= \int_\Sigma [ev(Y, (df) H(q,\Lambda)) + ev(Y, L_X I(q,\Lambda))]$$

$$= \int_\Sigma ev(Y, (df) H(q,\Lambda) + L_X I(q,\Lambda))$$

$$= < Y, (df) H(q,\Lambda) + L_X I(q,\Lambda) >.$$

The formula for $\frac{dI}{dt}$ thus follows, Y being arbitrary.

[Note: Integration by parts has been used several times and can be justified in the usual way.]

<u>Poisson Brackets</u> Given functions $F_1, F_2 : T^*Q \to C_d^\infty(\Sigma)$, put

$$\begin{cases} \mathcal{F}_1(q,\Lambda) = \int_\Sigma F_1(q,\Lambda) \\ \mathcal{F}_2(q,\Lambda) = \int_\Sigma F_2(q,\Lambda) \end{cases}$$

and let

$$\begin{cases} Z_1 \\ Z_2 \end{cases}$$

be the corresponding hamiltonian vector fields:

$$\begin{cases} Z_1(q,\Lambda) = (\dfrac{\delta F_1}{\delta\Lambda}, -\dfrac{\delta F_1}{\delta q}) \\[2ex] Z_2(q,\Lambda) = (\dfrac{\delta F_2}{\delta\Lambda}, -\dfrac{\delta F_2}{\delta q}). \end{cases}$$

Then the <u>Poisson bracket</u> of F_1,F_2 is the function

$$\{F_1,F_2\}:T^*Q \to \underline{R}$$

defined by the rule

$$\{F_1,F_2\}(q,\Lambda) = \Omega(Z_1(q,\Lambda),Z_2(q,\Lambda)).$$

Therefore

$$\begin{aligned}\{F_1,F_2\} &= <\frac{\delta F_1}{\delta\Lambda}, -\frac{\delta F_2}{\delta q}> - <\frac{\delta F_2}{\delta\Lambda}, -\frac{\delta F_1}{\delta q}> \\ &= <\frac{\delta F_2}{\delta\Lambda}, \frac{\delta F_1}{\delta q}> - <\frac{\delta F_1}{\delta\Lambda}, \frac{\delta F_2}{\delta q}>.\end{aligned}$$

[Note: Tacitly, it is assumed that the functional derivatives exist.]

<u>LEMMA</u> Let $F:T^*Q \to C_d^\infty(\Sigma)$ -- then, in the presence of the evolution equations, along $(q(t),\Lambda(t))$, we have

$$\frac{dF}{dt} = \{F,H_{f,X}\}.$$

[In fact,

$$\frac{dF}{dt} = \int_\Sigma \frac{d}{dt} F(q(t),\Lambda(t))$$

$$= \int_\Sigma (D_{(q,\Lambda)}F)(\dot{q},\dot{\Lambda})$$

$$= \int_\Sigma D_q F(q,\Lambda)\dot{q} + \int_\Sigma D_\Lambda F(q,\Lambda)\dot{\Lambda}$$

$$= < \dot{q}, \frac{\delta F}{\delta q} > + < \frac{\delta F}{\delta \Lambda}, \dot{\Lambda} >$$

$$= < \frac{\delta H_{f,X}}{\delta \Lambda}, \frac{\delta F}{\delta q} > + < \frac{\delta F}{\delta \Lambda}, - \frac{\delta H_{f,X}}{\delta q} >$$

$$= < \frac{\delta H_{f,X}}{\delta \Lambda}, \frac{\delta F}{\delta q} > - < \frac{\delta F}{\delta \Lambda}, \frac{\delta H_{f,X}}{\delta q} >$$

$$= \{F, H_{f,X}\}.]$$

<u>THEOREM</u> Suppose that $f_1, f_2 \in C^\infty(\Sigma)$ and $X_1, X_2 \in \mathcal{D}^1(\Sigma)$ -- then

$$\{H_{f_1,X_1}, H_{f_2,X_2}\} = \int_\Sigma (L_{X_1} f_2 - L_{X_2} f_1) H$$

$$+ \int_\Sigma \text{ev}(f_1(\text{grad } f_2) - f_2(\text{grad } f_1), I)$$

$$+ \int_\Sigma \text{ev}([X_1, X_2], I).$$

[Fix a point (q_0, Λ_0) and take f_2 in $C^\infty_{>0}(\Sigma)$. Choose paths $t \rightarrow f_2(t)$ in $C^\infty_{>0}(\Sigma)$ and $t \rightarrow X_2(t)$ in $\mathcal{D}^1(\Sigma)$ such that $f_2(0) = f_2$, $X_2(0) = X_2$. Let $t \rightarrow (q(t), \Lambda(t))$ be the curve in T^*Q satisfying the evolution equations subject to $(q(0), \Lambda(0)) = (q_0, \Lambda_0)$ -- then along $(q(t), \Lambda(t))$, we have

$$\{H_{f_1,X_1}, H_{f_2,X_2}\}$$

$$= \frac{d}{dt} \mathcal{H}_{f_1,X_1}$$

$$= \frac{d}{dt} \int_\Sigma (f_1 H + I_{X_1})$$

$$= \int_\Sigma (f_1 \frac{dH}{dt} + \frac{d}{dt} I_{X_1})$$

$$= \int_\Sigma f_1 (- \frac{1}{f_2} \delta_q (f_2^2 I) + \mathcal{L}_{X_2} H)$$

$$+ \int_\Sigma ev(X_1, (df_2) H + \mathcal{L}_{X_2} I)$$

$$= \int_\Sigma (\mathcal{L}_{X_1} f_2 - \mathcal{L}_{X_2} f_1) H$$

$$- \int_\Sigma \frac{f_1}{f_2} \delta_q (f_2^2 I)$$

$$+ \int_\Sigma ev([X_1, X_2], I).$$

And (cf. infra)

$$- \int_\Sigma \frac{f_1}{f_2} \delta_q (f_2^2 I)$$

$$= \int_\Sigma ev(f_1 (grad_q f_2) - f_2 (grad_q f_1), I).$$

Setting $t = 0$ completes the proof when f_2 is strictly positive. Assume now that f_2 is arbitrary. Fix $C > 0 : f_2 + C \in C^\infty_{>0}(\Sigma)$ -- then

$$\{\mathcal{H}_{f_1,X_1}, \mathcal{H}_{f_2+C,X_2}\}$$

$$= \{H_{f_1,X_1}, H_{f_2,X_2}\} + \{H_{f_1,X_1}, H_{C,0}\}$$

or still,

$$\{H_{f_1,X_1}, H_{f_2,X_2}\}$$

$$= \{H_{f_1,X_1}, H_{f_2+C,X_2}\} - \{H_{f_1,X_1}, H_{C,0}\}$$

$$= \int_\Sigma (L_{X_1}(f_2+C) - L_{X_2} f_1)H$$

$$+ \int_\Sigma ev(f_1 grad(f_2+C) - (f_2+C) grad\ f_1, I)$$

$$+ \int_\Sigma ev([X_1,X_2], I)$$

$$- \int_\Sigma ev(-C grad\ f_1, I)$$

$$= \int_\Sigma (L_{X_1} f_2 - L_{X_2} f_1)H$$

$$+ \int_\Sigma ev(f_1(grad\ f_2) - f_2(grad\ f_1), I)$$

$$+ \int_\Sigma ev([X_1,X_2], I).]$$

[Note: There are results in PDE theory that guarantee existence (and uniqueness) of solutions to the evolution equations, a fact which was taken for granted in the above. This accounts for the initial restriction on f_2.]

<u>Details</u> At a point (q,Λ),

$$- \int_\Sigma \frac{f_1}{f_2}\, \delta_q(f_2^2 I(q,\Lambda))$$

$$= \int_\Sigma \frac{f_1}{f_2}\, \delta_q(2f_2^2 \mathrm{div}_q\ s)\mathrm{vol}_q$$

$$= 2 \int_\Sigma q[{}^0_1](d(\frac{f_1}{f_2}),\ f_2^2 \mathrm{div}_q\ s)\mathrm{vol}_q$$

$$= 2 \int_\Sigma q[{}^0_1](\frac{f_2(df_1) - f_1(df_2)}{f_2^2},\ f_2^2 \mathrm{div}_q\ s)\mathrm{vol}_q$$

$$= 2 \int_\Sigma q[{}^0_1](f_2(df_1) - f_1(df_2),\ \mathrm{div}_q\ s)\mathrm{vol}_q$$

$$= 2 \int_\Sigma (\mathrm{div}_q\ s)(f_2(\mathrm{grad}_q f_1))\mathrm{vol}_q$$

$$- 2 \int_\Sigma (\mathrm{div}_q\ s)(f_1(\mathrm{grad}_q f_2))\mathrm{vol}_q$$

$$= 2 \int_\Sigma \mathrm{div}_q\ \Lambda(f_2(\mathrm{grad}_q f_1))$$

$$- 2 \int_\Sigma \mathrm{div}_q\ \Lambda(f_1(\mathrm{grad}_q f_2))\mathrm{vol}_q$$

$$= \int_\Sigma \mathrm{ev}(f_1(\mathrm{grad}_q f_2) - f_2(\mathrm{grad}_q f_1),\ I(q,\Lambda)).$$

Scholium: The following formulas are special cases of the theorem:

$$\{\int_\Sigma fH,\ \int_\Sigma I_X\} = - \int_\Sigma (L_X f)H$$

$$\{\int_\Sigma f_1 H,\ \int_\Sigma f_2 H\} = \int_\Sigma \mathrm{ev}(f_1(\mathrm{grad}\ f_2) - f_2(\mathrm{grad}\ f_1),\ I)$$

$$\{\int_\Sigma I_{X_1}, \int_\Sigma I_{X_2}\} = \int_\Sigma I_{[X_1,X_2]}.$$

These relations can also be derived directly, i.e., without an appeal to the evolution equations.

The first formula is easy to establish:

$$\{\int_\Sigma fH, \int_\Sigma I_X\}(q,\Lambda)$$

$$= < \frac{\delta I_X}{\delta\Lambda}, \frac{\delta H_f}{\delta q} > - < \frac{\delta H_f}{\delta\Lambda}, \frac{\delta I_X}{\delta q} >$$

$$= < L_X q, \frac{\delta H_f}{\delta q} > + < \frac{\delta H_f}{\delta\Lambda}, L_X\Lambda >$$

$$= \int_\Sigma fD_q H(q,\Lambda)(L_X q) + \int_\Sigma fD_\Lambda H(q,\Lambda)(L_X\Lambda)$$

$$= \int_\Sigma f(D_{(q,\Lambda)}H)(L_X q, L_X\Lambda)$$

$$= \int_\Sigma fL_X(H(q,\Lambda))$$

$$= - \int_\Sigma (L_X f)H(q,\Lambda).$$

To discuss the second, let

$$\begin{cases} H_1 = H_{f_1} \\ H_2 = H_{f_2} \end{cases}$$

and write

$$\begin{cases} \dfrac{\delta H_1}{\delta q} = f_1 A - (H_{f_1} - (\Delta_q f_1) q)^{\#} \otimes |q|^{1/2} \\[2ex] \dfrac{\delta H_2}{\delta q} = f_2 A - (H_{f_2} - (\Delta_q f_2) q)^{\#} \otimes |q|^{1/2}. \end{cases}$$

Then

$$\{\textstyle\int_\Sigma f_1 H, \int_\Sigma f_2 H\}(q,\Lambda)$$

$$= \left< \frac{\delta H_2}{\delta \Lambda}, \frac{\delta H_1}{\delta q} \right> - \left< \frac{\delta H_1}{\delta \Lambda}, \frac{\delta H_2}{\delta q} \right>$$

$$= \left< 2f_2\left(s - \frac{1}{n-2}\,tr_q(s)q\right), f_1 A - (H_{f_1} - (\Delta_q f_1)q)^{\#} \otimes |q|^{1/2} \right>$$

$$- \left< 2f_1\left(s - \frac{1}{n-2}\,tr_q(s)q\right), f_2 A - (H_{f_2} - (\Delta_q f_2)q)^{\#} \otimes |q|^{1/2} \right>$$

$$= \left< 2f_1 f_2\left(s - \frac{1}{n-2}\,tr_q(s)q\right), A \right> - \left< 2f_1 f_2\left(s - \frac{1}{n-2}\,tr_q(s)q\right), A \right>$$

$$- 2\int_\Sigma f_2 (H_{f_1} - (\Delta_q f_1)q)^{\#}\left(s - \frac{1}{n-2}\,tr_q(s)q\right) vol_q$$

$$+ 2\int_\Sigma f_1 (H_{f_2} - (\Delta_q f_2)q)^{\#}\left(s - \frac{1}{n-2}\,tr_q(s)q\right) vol_q$$

$$= -2\int_\Sigma f_2 (\nabla^a \nabla^b f_1 - q^{ab}\nabla^c \nabla_c f_1)\left(s_{ab} - \frac{1}{n-2}\,tr_q(s) q_{ab}\right) vol_q$$

$$+ 2\int_\Sigma f_1 (\nabla^a \nabla^b f_2 - q^{ab}\nabla^c \nabla_c f_2)\left(s_{ab} - \frac{1}{n-2}\,tr_q(s) q_{ab}\right) vol_q$$

$$= 2\int_\Sigma [f_1 \nabla^a \nabla^b f_2 - f_2 \nabla^a \nabla^b f_1] s_{ab}\, vol_q$$

$$+ 2 \int_\Sigma f_2 \nabla^c \nabla_c f_1 (q^{ab} s_{ab} - \frac{1}{n-2} \operatorname{tr}_q(s) q^{ab} q_{ab}) \mathrm{vol}_q$$

$$- 2 \int_\Sigma f_1 \nabla^c \nabla_c f_2 (q^{ab} s_{ab} - \frac{1}{n-2} \operatorname{tr}_q(s) q^{ab} q_{ab}) \mathrm{vol}_q$$

$$+ \frac{2}{n-2} \int_\Sigma (f_2 q_{ab} \nabla^a \nabla^b f_1) \operatorname{tr}_q(s) \mathrm{vol}_q$$

$$- \frac{2}{n-2} \int_\Sigma (f_1 q_{ab} \nabla^a \nabla^b f_1) \operatorname{tr}_q(s) \mathrm{vol}_q$$

$$= 2 \int_\Sigma [f_1 \nabla^a \nabla^b f_2 - f_2 \nabla^a \nabla^b f_1] s_{ab} \mathrm{vol}_q$$

$$- \frac{2}{n-2} \int_\Sigma (f_2 \nabla^c \nabla_c f_1) \operatorname{tr}_q(s) \mathrm{vol}_q$$

$$+ \frac{2}{n-2} \int_\Sigma (f_1 \nabla^c \nabla_c f_2) \operatorname{tr}_q(s) \mathrm{vol}_q$$

$$+ \frac{2}{n-2} \int_\Sigma (f_2 \nabla^a \nabla_a f_1) \operatorname{tr}_q(s) \mathrm{vol}_q$$

$$- \frac{2}{n-2} \int_\Sigma (f_1 \nabla^a \nabla_a f_2) \operatorname{tr}_q(s) \mathrm{vol}_q$$

$$= 2 \int_\Sigma [f_1 \nabla^a \nabla^b f_2 - f_2 \nabla^a \nabla^b f_1] s_{ab} \mathrm{vol}_q$$

$$= \int_\Sigma [\nabla_a (f_1 \nabla_b f_2 - f_2 \nabla_b f_1) + \nabla_b (f_1 \nabla_a f_2 - f_2 \nabla_a f_1)] s^{ab} \mathrm{vol}_q$$

$$= \int_\Sigma s^{ab} L_{(f_1 (\mathrm{grad}_q f_2) - f_2 (\mathrm{grad}_q f_1))} q_{ab} \mathrm{vol}_q$$

$$= < L_{(f_1(\mathrm{grad}_q f_2) - f_2(\mathrm{grad}_q f_1))} q, \ s^{\#} \otimes |q|^{1/2} >$$

$$= - 2 < f_1(\mathrm{grad}_q f_2) - f_2(\mathrm{grad}_q f_1), \ \mathrm{div}_q \ s \otimes |q|^{1/2} >$$

$$= \int_\Sigma \mathrm{ev}(f_1(\mathrm{grad}_q f_2) - f_2(\mathrm{grad}_q f_1), \ I(q,\Lambda)).$$

As for the third formula, we have

$$\{\int_\Sigma I_{X_1}, \int_\Sigma I_{X_2}\}(q,\Lambda)$$

$$= < \frac{\delta I_{X_2}}{\delta \Lambda}, \frac{\delta I_{X_1}}{\delta q} > - < \frac{\delta I_{X_1}}{\delta \Lambda}, \frac{\delta I_{X_2}}{\delta q} >$$

$$= < L_{X_2} q, - L_{X_1} \Lambda > - < L_{X_1} q, - L_{X_2} \Lambda >$$

$$= - \int_\Sigma (L_{X_1} \Lambda)(L_{X_2} q) + \int_\Sigma (L_{X_2} \Lambda)(L_{X_1} q)$$

$$= \int_\Sigma \Lambda(L_{X_1} L_{X_2} q) - \int_\Sigma \Lambda(L_{X_2} L_{X_1} q)$$

$$= \int_\Sigma \Lambda((L_{X_1} L_{X_2} - L_{X_2} L_1) q)$$

$$= \int_\Sigma \Lambda(L_{[X_1,X_2]} q)$$

$$= - 2 \int_\Sigma \mathrm{div}_q \ \Lambda([X_1,X_2])$$

$$= \int_\Sigma I_{[X_1,X_2]}(q,\Lambda).$$

Remark: The set whose elements are the $H_{f,X}$ is a vector space over $\underline{R}$ but it is not closed under the Poisson bracket operation since

$$\{H_{f_1}, H_{f_2}\}(q,\Lambda) = \int_\Sigma ev(f_1(grad_q f_2) - f_2(grad_q f_1), I(q,\Lambda))$$

and the vector field

$$f_1(grad_q f_2) - f_2(grad_q f_1)$$

depends on q. On the other hand, the set whose elements are the I_X is a vector space over $\underline{R}$ which is closed under the Poisson bracket operation:

$$\{I_{X_1}, I_{X_2}\} = I_{[X_1,X_2]}.$$

So, in view of the Jacobi identity

$$\{I_{X_1},\{I_{X_2},I_{X_3}\}\} + \{I_{X_2},\{I_{X_3},I_{X_1}\}\} + \{I_{X_3},\{I_{X_1},I_{X_2}\}\} = 0,$$

it is a Lie algebra over $\underline{R}$. The arrow $X \to I_X$ is thus a homomorphism of Lie algebras.

Let

$$\begin{cases} Con_H = \{(q,\Lambda) \in T^*Q : H(q,\Lambda) = 0\} \\ Con_D = \{(q,\Lambda) \in T^*Q : I(q,\Lambda) = 0\}. \end{cases}$$

Then

$$Con_Q = Con_H \cap Con_D \subset T^*Q$$

is called the physical phase space of the theory.

[Note: The constraint equations imply that $\forall\, t$, $(q_t, p_t) \in \mathrm{Con}_Q$.]

Remark: Con_Q is not a submanifold of T^*Q.

A function

$$\mathcal{F} = \int_\Sigma F \quad (F: T^*Q \to C_d^\infty(\Sigma))$$

is said to be a constraint if

$$\mathcal{F} | \mathrm{Con}_Q = 0.$$

In particular: The

$$\left[\begin{array}{l} \mathcal{H}_f \\ \\ \mathcal{I}_X \end{array}\right.$$

are constraints, these being termed primary.

Observation: The Poisson bracket of two primary constraints is a constraint.

[Note: In traditional terminology, this says that GR is a first class system.]

Section 38: Causality In this section we shall provide a proofless summary of the relevant facts.

Let M be a connected C^∞ manifold of dimension $n > 2$.

Rappel: If M is noncompact or if M is compact and has zero Euler characteristic, then $\underline{M}_{1,n-1}$ is not empty.

Assume henceforth that M is noncompact. Fix $g \in \underline{M}_{1,n-1}$ -- then the pair (M,g) is said to be a spacetime if M is orientable and time orientable (i.e., admits a timelike vector field).

Remark: The tangent space T_xM at a given $x \in M$ is $\underline{R}^{1,n-1}$. Therefore a vector $X \in T_xM$ is timelike if $g_x(X,X) < 0$, lightlike if $g_x(X,X) = 0$, and spacelike if $g_x(X,X) > 0$. The complement in T_xM of the closure of the spacelike points has two components ("timecones") and there is no intrinsic way to distinguish them. If one of these cones is singled out and called the future cone $V_+(x)$, then T_xM is said to be time oriented. A timelike or lightlike vector in or on $V_+(x)$ is said to be future directed. The other cone is denoted by $V_-(x)$. A timelike or lightlike vector in or on $V_-(x)$ is said to be past directed.

[Note: If T is a timelike vector field, then T_xM can be time oriented by specifying the time cone containing T_x.]

Assume henceforth that (M,g) is a spacetime.

FACT Let $g_1, g_2 \in \underline{M}_{1,n-1}$. Suppose that $\forall\ x \in M$ & $\forall\ X \in T_xM$,

$$(g_1)_x(X,X) = 0 \text{ iff } (g_2)_x(X,X) = 0.$$

Then

$$g_1 = \varphi g_2,$$

where $\varphi \in C^\infty_{>0}(M)$.

A curve in M is timelike, lightlike, or spacelike if its tangent vectors are timelike, lightlike, or spacelike.

A curve in M is <u>causal</u> if its tangent vectors are timelike or lightlike. A causal curve is <u>future directed</u> (<u>past directed</u>) if its tangent vectors have this property.

A future directed causal curve $\gamma: I \to M$ is said to have a <u>future endpoint</u> (<u>past endpoint</u>) if $\gamma(t)$ converges to some point in M as $t \uparrow \sup I$ $(t \downarrow \inf I)$.

A past directed causal curve $\gamma: I \to M$ is said to have a <u>past endpoint</u> <u>(future endpoint)</u> if $\gamma(t)$ converges to some point in M as $t \uparrow \sup I$ $(t \downarrow \inf I)$.

A future (past) directed causal curve γ is said to <u>start at a point $p \in M$</u> provided that p is the past (future) endpoint of γ.

A future (past) directed causal curve γ is said to be <u>future (past) inextendible</u> if it possesses no future (past) endpoint.

Notation: $\forall$ p,q in M,

- $p \ll q$: $\exists$ a future directed timelike curve from p to q.
- $p < q$: $\exists$ a future directed causal curve from p to q.

[Note: It may or may not be the case that $p \ll p$ but it's always true that $p < p$ (conventionally, a constant curve is lightlike and both future and past directed).]

Definition: The <u>chronological future of p</u> is

$$I^+(p) = \{q : p \ll q\}$$

and the <u>causal future of p</u> is

$$J^+(p) = \{q : p < q\}.$$

The <u>chronological past of p</u> is

$$I^-(p) = \{q : q \ll p\}$$

and the <u>causal past of p</u> is

$$J^-(p) = \{q : q < p\}.$$

[Note: For a nonempty subset $S \subset M$, the sets $I^{\pm}(S)$, $J^{\pm}(S)$ are defined analogously. E.g.: $I^+(S) = \{q : p \ll q \ (\exists\, p \in S)\}$ and $J^+(S) = \{q : p < q \ (\exists\, p \in S)\}$. Obviously, $I^+(S) = \bigcup_{p \in S} I^+(p)$ and $J^+(S) = \bigcup_{p \in S} J^+(p)$. Furthermore, $J^+(S) \supset S \cup I^+(S)$.]

<u>LEMMA</u> If $x \ll y$ and $y < z$ or if $x < y$ and $y \ll z$, then $x \ll z$.

Application: We have

$$I^+(S) = I^+(I^+S) = I^+(J^+S)$$

$$= J^+(I^+S) \subset J^+(J^+S) = J^+(S).$$

<u>LEMMA</u> If $p \ll q$, then $\exists$ neighborhoods N_p of p and N_q of q such that

$$\left.\begin{array}{l} p' \in N_p \\ q' \in N_q \end{array}\right] \Rightarrow p' \ll q'.$$

Application: $\forall\, p \in M$, $I^+(p)$ is open.

<u>Topological Properties</u> $\forall\, p \in M$,

1. $\operatorname{int} \overline{I^+(p)} = I^+(p)$;

2. $\overline{I^+(p)} = \{x : I^+(x) \subset I^+(p)\}$;

3. $\text{fr } I^+(p) = \{x : x \notin I^+(p) \ \& \ I^+(x) \subset I^+(p)\}$;

4. $\text{int } J^+(p) = I^+(p)$;

5. $J^+(p) \subset \overline{I^+(p)}$.

Remark: In general, $J^+(p)$ is not closed, hence may very well be a proper subset of $\overline{I^+(p)}$.

Let (M,g) be a spacetime — then (M,g) is

- <u>future distinguishing</u> if $x \neq y \Rightarrow I^+(x) \neq I^+(y)$
- <u>past distinguishing</u> if $x \neq y \Rightarrow I^-(x) \neq I^-(y)$.

[Note: Call (M,g) <u>distinguishing</u> if it is both future and past distinguishing.]

Let (M,g), (M',g') be spacetimes. Suppose that $f:M \rightarrow M'$ is a diffeomorphism -- then f is said to be a <u>chronal isomorphism</u> provided

$$x \ll y \Leftrightarrow f(x) \ll f(y).$$

<u>THEOREM</u> If (M,g) and (M',g') are distinguishing and if $f:M \rightarrow M'$ is a chronal isomorphism, then f is a conformal isometry.

[Note: Spelled out, $\exists\ \varphi \in C^\infty_{>0}(M) : \forall\ x \in M$,

$$g'_{f(x)}(f_*X, f_*Y) = \varphi(x) g_x(X,Y) \quad (X,Y \in T_xM),$$

thus

$$f^*g' = \varphi g.]$$

Given $p,q \in M$, put

$$[p,q] = \{x : p < x < q\}.$$

I.e.:

$$[p,q] = J^+(p) \cap J^-(q).$$

Let S be a nonempty subset of M — then S is <u>causally convex</u> if $\forall\ p,q \in S$, $[p,q] \subset S$.

Definition: A spacetime (M,g) is said to be <u>strongly causal</u> if each $x \in M$ has a basis of open neighborhoods consisting of causally convex sets.

[Note: A strongly causal spacetime is necessarily distinguishing.]

<u>FACT</u> Suppose that (M,g) is strongly causal -- then the $I^+(p) \cap I^-(q)$ $(p,q \in M)$ are a basis for the topology on M.

A <u>time function</u> is a surjective C^∞ function $\tau : M \to \underline{R}$ whose gradient grad τ is timelike.

Definition: A spacetime (M,g) is said to be <u>stably causal</u> if it admits a time function $\tau : M \to \underline{R}$.

<u>FACT</u> Every stably causal spacetime is strongly causal.

Definition: A spacetime (M,g) is said to be <u>globally hyperbolic</u> if it is strongly causal and $\forall\ p,q \in M$, $[p,q]$ is compact.

<u>LEMMA</u> If (M,g) is globally hyperbolic, then $\forall\ p$, $J^+(p)$ is closed.

[Note: More generally, K compact $\Rightarrow$ $J(K)$ closed.]

Example: $\underline{R}^{1,n-1}$ is globally hyperbolic but $\underline{R}^{1,n-1} - \{0\}$ is not.

<u>FACT</u> Let (M,g), (M',g') be distinguishing chronally isomorphic spacetimes --

then (M,g) is globally hyperbolic iff (M',g') is globally hyperbolic.

Remark: If (M,g) is globally hyperbolic, then so is $(M,\varphi g)$ $(\varphi \in C^{\infty}_{>0}(M))$. On the other hand, if (M,g) and (M,g') are globally hyperbolic and if the identity map is a chronal isomorphism, then $g = \varphi g'$ for some $\varphi \in C^{\infty}_{>0}(M)$.

Let (M,g) be a spacetime. Suppose that S is a nonempty subset of M -- then the future domain of dependence $D^+(S)$ of S is the set of all points $p \in M$ such that every past inextendible causal curve starting at p meets S.

[Note: The definition of $D^-(S)$ is dual. The union $D(S) = D^+(S) \cup D^-(S)$ is the domain of dependence of S.]

LEMMA If S is a closed achronal subset of M, then int D(S), if nonempty, is globally hyperbolic.

[Note: S is achronal provided $S \cap I^{\pm}(S) = \emptyset$.]

Definition: Let (M,g) be a spacetime -- then a Cauchy hypersurface is a closed achronal hypersurface $\Sigma \subset M$ with the property that $D(\Sigma) = M$, hence is met exactly once by every inextendible timelike curve in M.

[Note: A hypersurface per se is an embedded connected submanifold of dimension n - 1.]

Example: In $\underline{R}^{1,n-1}$, the hyperplanes x_0 = constant are Cauchy hypersurfaces.

FACT If Σ_1 and Σ_2 are Cauchy hypersurfaces in M, then Σ_1 and Σ_2 are diffeomorphic.

In view of the preceding lemma, if (M,g) admits a Cauchy hypersurface, then (M,g) is globally hyperbolic. The converse is also true: Every globally hyperbolic spacetime admits a Cauchy hypersurface but one can say considerably more than this.

LEMMA If (M,g) is globally hyperbolic, then (M,g) admits a spacelike Cauchy hypersurface.

FACT A spacelike Cauchy hypersurface Σ is acausal, i.e., $\Sigma \cap J^{\pm}(\Sigma) = \emptyset$.

STRUCTURE THEOREM Suppose that (M,g) is globally hyperbolic -- then there exists a connected (n-1)-dimensional manifold Σ and a diffeomorphism $\Psi:\underline{R} \times \Sigma \to M$ such that $\forall$ t, $\Sigma_t = \Psi(\{t\} \times \Sigma)$ is a spacelike Cauchy hypersurface in M, hence

$$M = \coprod_t \Sigma_t.$$

Addenda

1. The spacelike leaves Σ_t of the foliation figuring in the theorem are the level hypersurfaces of a time function τ, i.e., $\forall$ t, $\Sigma_t = \tau^{-1}(t)$.

2. The vector field grad τ is past directed but possibly incomplete. To remedy this technicality, let

$$X_\tau = \frac{\text{grad } \tau}{||\text{grad } \tau||}.$$

Here the norm is taken relative to some complete riemannian metric, thus X_τ is a complete vector field. Put $\Sigma = \tau^{-1}(0)$ and define a diffeomorphism $\Phi:M \to \underline{R} \times \Sigma$ by

$$\Phi(p) = (\tau(p),\rho(p)),$$

where $\rho(p)$ is the unique point of Σ crossed by the maximal integral curve of X_τ through p. Let $\Psi = \Phi^{-1}$ -- then $\forall$ t,

$$\Psi(\{t\} \times \Sigma) = \tau^{-1}(t).$$

3. Put

$$\frac{\partial}{\partial \tau} = \Psi_*(\frac{\partial}{\partial t}).$$

Given $x \in \Sigma$, let

$$\gamma_x(t) = \Psi(t,x).$$

Then $\gamma_x : \underline{R} \to M$ is an integral curve for $\frac{\partial}{\partial \tau}$. It is timelike and

$$t < t' \Rightarrow \gamma_x(t) \ll \gamma_x(t').$$

Furthermore, $\frac{\partial}{\partial \tau}$ is parallel to grad τ: $\forall$ t,

$$\begin{aligned} (t,x) &= \Phi \circ \Psi(t,x) \\ &= \Phi(\gamma_x(t)) \\ &= (\tau(\gamma_x(t)), \rho(\gamma_x(t))) \end{aligned}$$

$$\Rightarrow$$

$$\rho(\gamma_x(t)) = x.$$

So $\forall$ t, $\gamma_x(t)$ lies on the trajectory of X_τ containing x.

4. If $\Sigma_0 \subset M$ is a Cauchy hypersurface, then $\gamma_x(t)$ intersects Σ_0 exactly once at the parameter value $t_{\Sigma_0}(x)$. The function $t_{\Sigma_0} : \Sigma \to \underline{R}$ is C^∞ and

$\Sigma_0 = \{\Psi(t_{\Sigma_0}(x),x):x\in\Sigma\}$. In addition, if Σ_1,Σ_2 are Cauchy hypersurfaces, then the map $\Sigma_1 \to \Sigma_2$ which sends $\Psi(t_{\Sigma_1}(x),x)$ to $\Psi(t_{\Sigma_2}(x),x)$ is a diffeomorphism.

5. Since $\tau = t \circ \Phi$, it follows that

$$d\tau(\frac{\partial}{\partial\tau}) = \frac{\partial}{\partial\tau}(\tau)$$

$$= \Psi_*(\frac{\partial}{\partial t})(t \circ \Phi)$$

$$= \frac{d}{dt}(t \circ \Phi \circ \Psi)$$

$$= 1.$$

Therefore

$$g(\frac{\partial}{\partial\tau}, \text{grad } \tau) = d\tau(\frac{\partial}{\partial\tau}) = 1$$

$$\Rightarrow$$

$$\frac{\partial}{\partial\tau} = \frac{1}{g(\text{grad } \tau, \text{grad } \tau)} \text{grad } \tau$$

$$\Rightarrow$$

$$\Psi^*g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \frac{1}{g(\text{grad } \tau, \text{grad } \tau)}.$$

6. Let $q(t)$ be the riemannian structure on Σ determined by pulling back g via the arrow

$$\Sigma \approx \{t\} \times \Sigma \xrightarrow{\Psi_t} \Sigma_t \xrightarrow{i_t} M.$$

Put

$$N_t(x) = \frac{1}{|g_{\Psi(t,x)}(\text{grad } \tau, \text{grad } \tau)|^{1/2}} \quad (x\in\Sigma).$$

Define $g_\tau \in \underline{M}_{1,n-1}$ (per $\underline{R} \times \Sigma$) by the prescription

$$(g_\tau)_{(t,x)}((r,X),(s,Y))$$

$$= - rsN_t^2(x) + q_x(t)(X,Y) \quad (r,s \in \underline{R} \ \& \ X,Y \in T_x\Sigma).$$

Then

$$g_\tau = \Psi^* g.$$

But this implies that

$$\mathrm{Ein}(g_\tau) = \Psi^*\mathrm{Ein}(g),$$

thus the vanishing of $\mathrm{Ein}(g_\tau)$ is equivalent to the vanishing of $\mathrm{Ein}(g)$.

<u>Terminology</u> Let (M,g) be a globally hyperbolic spacetime.

- M is <u>spatially compact</u> if its Cauchy hypersurfaces are compact.
- M is <u>spatially noncompact</u> if its Cauchy hypersurfaces are noncompact.

<u>FACT</u> Suppose that (M,g) is globally hyperbolic. Let $\Sigma \subset M$ be a closed achronal hypersurface. Assume: Σ is compact -- then Σ is a Cauchy hypersurface.

<u>LEMMA</u> The Cauchy hypersurfaces in a globally hyperbolic spacetime are orientable.

Let

$$M =]-\varepsilon,\varepsilon[\times \Sigma \quad (0 < \varepsilon \leq \infty),$$

where Σ is orientable (hence $\dim \Sigma = n - 1$). Suppose given a triple $(q_t, N_t, \vec{N}_t)$ satisfying the usual conditions and let g be the element of $\underline{M}_{1,n-1}$ determined thereby (for this, it is not necessary to assume that Σ is compact) -- then, in general, the pair (M,g) is not globally hyperbolic.

[Note: The spacetime (M,g) is, however, stably causal. Thus take for τ the projection $(t,x) \rightarrow t$ -- then

$$\begin{aligned} \text{grad } t &= (dt)^{\#} \\ &= \left(-\frac{n^{\flat}}{N}\right)^{\#} \\ &= -\frac{n}{N} \end{aligned}$$

$\Rightarrow$

$$g(\text{grad } t, \text{grad } t) = -\frac{1}{N^2} < 0.$$

Therefore grad t is timelike.]

• Assume that $\exists$ a complete $q \in Q$ and positive constants $A > 0$, $B > 0$: $\forall$ t and $\forall$ $X \in \mathcal{D}^1(\Sigma)$,

$$Aq(X,X) \leq q_t(X,X) \leq Bq(X,X).$$

• Assume that $\exists$ positive constants $C > 0$, $D > 0$: $\forall$ t & $\forall$ $x \in \Sigma$:

$$0 < C \leq N_t(x) \leq D.$$

• Assume that $\exists$ a positive constant $K > 0$: $\forall$ t,

$$q_t(\vec{N}_t, \vec{N}_t) \leq K.$$

<u>FACT</u> Under these conditions, (M,g) is globally hyperbolic and the slices

$\{t\} \times \Sigma$ are spacelike Cauchy hypersurfaces.

[Note: There is also a converse: Make the same assumptions on the data except for the completeness of q, form (M,g), and suppose that it is globally hyperbolic -- then q is necessarily complete.]

Example: When $\vec{N} = 0$ and q and N are independent of t, g is said to be static. So, in this situation, (M,g) is globally hyperbolic if (Σ,q) is complete and N is bounded above and below on Σ (matters being automatic if Σ is compact).

Section 39: The Standard Setup The point here is to initiate the transition from a theory based on metrics to a theory based on forms.

LEMMA Every connected orientable 3-manifold Σ is parallelizable.

[For the proof, it will be convenient to admit manifolds with boundary. Thus let

$$\begin{cases} w_1(\Sigma) = 1^{\text{st}} \text{ Stiefel-Whitney class} \\ w_2(\Sigma) = 2^{\text{nd}} \text{ Stiefel-Whitney class.} \end{cases}$$

Then Σ is parallelizable provided $w_1(\Sigma) = 0 = w_2(\Sigma)$. But $w_1(\Sigma) = 0$ is automatic (Σ being orientable).

Case 1: Σ compact and $\partial\Sigma = \emptyset$. Proof: $w_1(\Sigma) = 0 \Rightarrow w_2(\Sigma) = w_1^2(\Sigma) = 0$ (Wu relations).

Case 2: Σ compact and $\partial\Sigma \neq \emptyset$. Proof: Consider the double of Σ and apply Case 1.

Case 3: Σ noncompact and $\partial\Sigma = \emptyset$. Proof: Let $\alpha \in H_2(\Sigma;\underline{Z}/2\underline{Z})$ be arbitrary -- then α is represented by a compact surface $S \to \Sigma$ (Thom), hence $< w_2(\Sigma), \alpha > = 0$ (pass to a tubular neighborhood of S).

Case 4: Σ noncompact and $\partial\Sigma \neq \emptyset$. Proof: Consider $\Sigma - \partial\Sigma$ and apply Case 3.]

Take $n > 3$ and let Σ be a connected compact $(n-1)$-dimensional orientable C^∞ manifold.

Assumption Σ is parallelizable.

Put

$$M = \underline{R} \times \Sigma.$$

Then M is also parallelizable.

Notation: Indices a,b,c run from 1 to n - 1.

Let $E_1,\ldots,E_{n-1}$ be time dependent vector fields on Σ such that $\forall$ t,

$$\{E_1(t),\ldots,E_{n-1}(t)\}$$

is a basis for $\mathcal{D}^1(\Sigma)$. Complete this to a basis

$$\left[\begin{array}{l} E_0 \\ \\ E_a \end{array}\right.$$

for $\mathcal{D}^1(M)$.

<u>Construction</u> Let q(t) be the element of Q determined by stipulating that the $E_a(t)$ are to be an orthonormal frame -- then the prescription

$$g_{(t,x)}(rE_0\big|_{(t,x)} + X,\ sE_0\big|_{(t,x)} + Y)$$

$$= - rs + q_x(t)(X,Y) \quad (r,s\in\underline{R}\ \&\ X,Y\in T_x\Sigma)$$

defines an element of $\underline{M}_{1,n-1}$.

Remark: This procedure gives rise to a certain class of spacetimes (M,g) (E_0 is a timelike vector field). In general, however, if $g\in\underline{M}_{1,n-1}$ is arbitrary, then one has no guarantee that $g|\Sigma$ is nondegenerate, let alone spacelike. On the other hand, there is a gauge-theoretic ambiguity: Distinct E may lead to the same g.

[Note: While not necessarily globally hyperbolic, the spacetime (M,g) is at least stably causal (the projection $(t,x) \rightarrow t$ is a time function).]

In view of the definitions, $\exists\ C^\infty$ functions N and N^a on M such that

$$\frac{\partial}{\partial t} = NE_0 + N^a E_a.$$

[Note: N has constant sign, i.e., N is strictly positive (or strictly negative).]

Terminology: N is called the <u>lapse</u> and $\vec{N} = N^a E_a$ is called the <u>shift</u>.

<u>Reality Check</u> Suppose given a triple $(q_t, N_t, \vec{N}_t)$ satisfying the usual conditions. Fix time dependent vector fields $E_1, \ldots, E_{n-1}$ on Σ which at each t constitute an orthonormal frame for q(t). Take

$$E_0 = \underline{n} = \frac{1}{N}\left(\frac{\partial}{\partial t} - \vec{N}\right).$$

Then

$$g_{(t,x)}((r,X),(s,Y))$$

$$= g_{(t,x)}\left(r\frac{\partial}{\partial t} + X,\ s\frac{\partial}{\partial t} + Y\right)$$

$$= g_{(t,x)}\left(rN_t(x)E_0\Big|_{(t,x)} + r\vec{N}_t\Big|_x + X, sN_t(x)E_0\Big|_{(t,x)} + s\vec{N}_t\Big|_x + Y\right)$$

$$= -\, rsN_t^2(x) + q_x(t)\left(r\vec{N}_t\Big|_x + X, s\vec{N}_t\Big|_x + Y\right)$$

$$= -\, rsN_t^2(x) + sq_x(t)\left(X,\vec{N}_t\Big|_x\right) + rq_x(t)\left(Y,\vec{N}_t\Big|_x\right)$$

$$+ rsq_x(t)\left(\vec{N}_t\Big|_x,\vec{N}_t\Big|_x\right) + q_x(t)(X,Y)$$

$$= -\, rs\left(N_t^2(x) - q_x(t)\left(\vec{N}_t\Big|_x,\vec{N}_t\Big|_x\right)\right)$$

$$+ sq_x(t)\left(X,\vec{N}_t\Big|_x\right) + rq_x(t)\left(Y,\vec{N}_t\Big|_x\right) + q_x(t)(X,Y),$$

which is in agreement with the earlier considerations.

Let $i_t:\Sigma \approx \Sigma_t \to M$ be the embedding ($\Sigma_t = \{t\} \times \Sigma$).

Notation: Given $T \in \mathcal{D}^0_q(M)$, put

$$\dot{\bar{T}} = \frac{d}{dt} i_t^* T \ \left(= \frac{d}{dt} \bar{T}\right).$$

<u>LEMMA</u> We have

$$\dot{\bar{T}} = i_t^* L_{\partial/\partial t} T.$$

[In fact,

$$i_{t+s} = \phi_s \circ i_t,$$

where ϕ_s is the flow attached to $\frac{\partial}{\partial t}$. Therefore

$$\dot{\bar{T}} = \frac{d}{ds}\Big|_{s=t} (i_s^* T)$$

$$= \lim_{s \to 0} \frac{i_{t+s}^* T - i_t^* T}{s}$$

$$= \lim_{s \to 0} \frac{i_t^* \phi_s^* T - i_t^* T}{s}$$

$$= i_t^* \lim_{s \to 0} \frac{\phi_s^* T - T}{s}$$

$$= i_t^* L_{\partial/\partial t} T.]$$

Example: By construction,

$$i_t^* g = \bar{g} = q(t) \ \ (= q_t).$$

So

$$\dot{q}_t = \frac{d}{dt} q(t) = \frac{d}{dt} i_t^* g = \dot{\bar{g}} = i_t^* L_{\partial/\partial t} g.$$

Let ∇ be the metric connection associated with g (thus $\bar{\nabla}$ is the metric connection associated with $\bar{g}$) -- then the $\bar{\omega}^a{}_b$ are the connection 1-forms of $\bar{\nabla}$.

Consider now the coframe $\{\omega^0, \omega^1, \ldots, \omega^{n-1}\}$ per the frame $\{E_0, E_1, \ldots, E_{n-1}\}$ -- then

$$g = - \omega^0 \otimes \omega^0 + \omega^a \otimes \omega^a$$

and

$$\begin{cases} \omega^0 = N dt \\ \omega^a = N^a dt + \bar{\omega}^a. \end{cases}$$

[Note: On Σ,

$$\bar{g} = \bar{\omega}^a \otimes \bar{\omega}^a.]$$

• $$\begin{aligned} d\omega^0 &= d(Ndt) \\ &= dN \wedge dt \\ &= (\bar{d}N + dt \wedge \partial_t N) \wedge dt \\ &= \bar{d}N \wedge dt \\ &= (E_a N) \bar{\omega}^a \wedge dt \\ &= (E_a N)(\omega^a - N^a dt) \wedge dt \end{aligned}$$

$$= (E_a N)\,\omega^a \wedge dt$$

$$= \frac{E_a N}{N}\,\omega^a \wedge \omega^0.$$

• $d\omega^a = d(N^a dt + \bar{\omega}^a)$

$$= dN^a \wedge dt + d\bar{\omega}^a$$

$$= (\bar{d}N^a + dt \wedge \partial_t N) \wedge dt$$

$$+ \bar{d}\bar{\omega}^a + dt \wedge \partial_t \bar{\omega}^a$$

$$= \bar{d}N^a \wedge dt + dt \wedge \partial_t \bar{\omega}^a - \bar{\omega}^a{}_b \wedge \bar{\omega}^b.$$

Let $\varkappa_t$ be the extrinsic curvature:

$$\varkappa_t = \varkappa_{ab}\bar{\omega}^a \otimes \bar{\omega}^b \quad (\varkappa_{ab} = (\varkappa_t)_{ab}).$$

Rappel: We have

$$\bar{\omega}^0{}_b = \varkappa_{ab}\bar{\omega}^a$$

or still,

$$\omega^0{}_a(E_b) = \varkappa_{ab}.$$

<u>$\omega^0{}_a(E_0)$:</u>

$$d\omega^0 = -\,\omega^0{}_i \wedge \omega^i$$

$$= -\,\omega^0{}_a \wedge \omega^a$$

$\Rightarrow$

$$\iota_{E_0} d\omega^0 = - \iota_{E_0} (\omega^0{}_a \wedge \omega^a)$$

$$= - \omega^0{}_a (E_0) \omega^a.$$

But

$$d\omega^0 = \frac{E_a N}{N} \omega^a \wedge \omega^0$$

$\Rightarrow$

$$\iota_{E_0} d\omega^0 = \iota_{E_0} (\frac{E_a N}{N} \omega^a \wedge \omega^0)$$

$$= - \frac{E_a N}{N} \omega^a.$$

Therefore

$$\omega^0{}_a (E_0) = \frac{E_a N}{N} .$$

Corollary:

$$\omega^0{}_a = \frac{E_a N}{N} \omega^0 + \varkappa_{ab} \omega^b.$$

[Note:

$$\omega^0{}_a = - \varepsilon_0 \varepsilon_a \omega^a{}_0 \quad \text{(no sum)}$$

$$= - (-1)(+1) \omega^a{}_0$$

$$= \omega^a{}_0 .]$$

Rappel: We have

$$\omega^a{}_b(E_c) = \bar{\omega}^a{}_b(E_c).$$

<u>$\omega^a{}_b(E_0)$:</u>

$$d\omega^a = -\,\omega^a{}_i\wedge\omega^i$$

$$= -\,\omega^a{}_0\wedge\omega^0 - \omega^a{}_b\wedge\omega^b$$

or still,

$$d\omega^a = -\,\left(\frac{E_a N}{N}\,\omega^0 + \varkappa^a{}_b\omega^b\right)\wedge\omega^0 - \omega^a{}_b\wedge\omega^b$$

$$= -\,\varkappa^a{}_b\omega^b\wedge\omega^0 - \omega^a{}_b\wedge\omega^b$$

$\Rightarrow$

$$\iota_{E_0} d\omega^a = (\varkappa^a{}_b - \omega^a{}_b(E_0))\,\omega^b$$

$\Rightarrow$

$$\iota_{E_b}\iota_{E_0} d\omega^a = \varkappa^a{}_b - \omega^a{}_b(E_0).$$

But

$$d\omega^a = \bar{d}N^a\wedge dt + dt\wedge\partial_t\bar{\omega}^a - \bar{\omega}^a{}_c\wedge\bar{\omega}^c$$

$\Rightarrow$

$$\iota_{E_0} d\omega^a = \iota_{E_0}(\bar{d}N^a\wedge dt + dt\wedge\partial_t\bar{\omega}^a - \bar{\omega}^a{}_c\wedge\bar{\omega}^c)$$

$$= -\,\frac{1}{N}\,\bar{d}N^a + \frac{1}{N}\,\partial_t\bar{\omega}^a + \frac{1}{N}\,\iota_{\vec{N}}(\bar{\omega}^a{}_c\wedge\bar{\omega}^c)$$

$\Rightarrow$

$$\iota_{E_b}\iota_{E_0} d\omega^a = \frac{1}{N}\,[\,-\,\bar{d}N^a(E_b) + \partial_t\bar{\omega}^a(E_b) + \bar{\omega}^a{}_c\wedge\bar{\omega}^c(\vec{N},E_b)\,]$$

$$= \frac{1}{N}\,\partial_t\bar{\omega}^a(E_b) + \frac{1}{N}\,[\,-\,E_bN^a + \bar{\omega}^a{}_c(\vec{N})\,\bar{\omega}^c(E_b) - \bar{\omega}^c(\vec{N})\,\bar{\omega}^a{}_c(E_b)\,]$$

$$= \frac{1}{N}\,\partial_t\bar{\omega}^a(E_b) + \frac{1}{N}\,[\bar{\omega}^a{}_b(\vec{N}) - E_bN^a - N^c\bar{\omega}^a{}_c(E_b)]$$

$$= \frac{1}{N}\,\partial_t\bar{\omega}^a(E_b) + \frac{1}{N}\,[\bar{\omega}^a{}_b(\vec{N}) - (\bar{\nabla}\vec{N})\,(\bar{\omega}^a,E_b)]$$

$$= \frac{1}{N}\,\partial_t\bar{\omega}^a(E_b) + \frac{1}{N}\,[\bar{\omega}^a{}_b(\vec{N}) - \bar{\nabla}_bN^a].$$

Therefore

$$\varkappa_{ab} - \omega_{ab}(E_0)$$

$$= \frac{1}{N}\,\dot{\bar{\omega}}_a(E_b) + \frac{1}{N}\,[\bar{\omega}_{ab}(\vec{N}) - \bar{\nabla}_bN_a]$$

$$= \frac{1}{2N}\,[\dot{\bar{\omega}}_a(E_b) + \dot{\bar{\omega}}_b(E_a)] - \frac{1}{2N}\,[\bar{\nabla}_bN_a + \bar{\nabla}_aN_b] + \frac{1}{2N}\,[\dot{\bar{\omega}}_a(E_b) - \dot{\bar{\omega}}_b(E_a)] - \frac{1}{2N}\,[\bar{\nabla}_bN_a - \bar{\nabla}_aN_b] + \frac{1}{N}\,\bar{\omega}_{ab}(\vec{N})$$

$\Rightarrow$

$$\varkappa_{ab} = \frac{1}{2N}\,[\dot{\bar{\omega}}_a(E_b) + \dot{\bar{\omega}}_b(E_a)] - \frac{1}{2N}\,[\bar{\nabla}_bN_a + \bar{\nabla}_aN_b]$$

and

$$\omega_{ab}(E_0) = -\frac{1}{2N}[\dot{\bar{\omega}}_a(E_b) - \dot{\bar{\omega}}_b(E_a)] + \frac{1}{2N}[\bar{\nabla}_b N_a - \bar{\nabla}_a N_b]$$

$$-\frac{1}{N}\bar{\omega}_{ab}(\vec{N}).$$

[Note: $\varkappa_{ab}$ is symmetric while ω_{ab} is antisymmetric.]

Remark: Since $\bar{g} = \bar{\omega}^a \otimes \bar{\omega}^a$, it follows that

$$\dot{\bar{g}}_{ab} = \dot{\bar{\omega}}_a(E_b) + \dot{\bar{\omega}}_b(E_a).$$

Therefore

$$\varkappa_{ab} = \frac{1}{2N}\dot{\bar{g}}_{ab} - \frac{1}{2N}(L_{\vec{N}}\bar{g})_{ab}.$$

I.e.:

$$\dot{q}_t = 2N_t\varkappa_t + L_{\vec{N}_t} q_t.$$

Definition: The <u>rotational parameter</u> of the theory is the function

$$\bar{Q}^a{}_b = -N_t i_t^* \omega^a{}_b(E_0).$$

<u>LEMMA</u> We have

$$\dot{\bar{\omega}}^a = N_t \bar{\omega}^a{}_0 + \bar{Q}^a{}_b \bar{\omega}^b + L_{\vec{N}_t}\bar{\omega}^a.$$

[It is a question of explicating the relation

$$\dot{\bar{\omega}}^a = i_t^* L_{\partial/\partial t}\omega^a.$$

Write

$$L_{\partial/\partial t} = L_{NE_0} + L_{\vec{N}}$$

$$= \iota_{NE_0} \circ d + d \circ \iota_{NE_0} + L_{\vec{N}}.$$

Then

$$L_{\partial/\partial t}\omega^a = \iota_{NE_0} d\omega^a + d\omega^a(NE_0) + L_{\vec{N}}\omega^a$$

$$= \iota_{NE_0} d\omega^a + L_{\vec{N}}\omega^a.$$

But

$$\iota_{NE_0} d\omega^a = N\iota_{E_0} d\omega^a$$

$$= - N\iota_{E_0}(\omega^a{}_i \wedge \omega^i)$$

$$= - N(\iota_{E_0}\omega^a{}_i \wedge \omega^i - \omega^a{}_i \wedge \iota_{E_0}\omega^i)$$

$$= - N(\omega^a{}_i(E_0)\omega^i - \omega^i(E_0)\omega^a{}_i)$$

$$= - N(\omega^a{}_i(E_0)\omega^i - \omega^a{}_0).$$

Therefore

$$\dot{\bar{\omega}}^a = N_t\bar{\omega}^a{}_0 - N_t i_t^* \omega^a{}_b(E_0)\bar{\omega}^b + L_{\vec{N}_t}\bar{\omega}^a$$

$$= N_t\bar{\omega}^a{}_0 + \bar{Q}^a{}_b\bar{\omega}^b + L_{\vec{N}_t}\bar{\omega}^a.]$$

[Note: In terms of the extrinsic curvature,

$$\dot{\bar{\omega}}^a = N_t\kappa^a{}_b\bar{\omega}^b + \bar{Q}^a{}_b\bar{\omega}^b + L_{\vec{N}_t}\bar{\omega}^a.]$$

Notation: Put

$$\begin{cases} \dot{\bar{\omega}}^a{}_S = \frac{1}{2}\,(\dot{\bar{\omega}}^a + \bar{g}(\dot{\bar{\omega}}^c,\bar{\omega}^a)\bar{\omega}_c) \\ \dot{\bar{\omega}}^a{}_A = \frac{1}{2}\,(\dot{\bar{\omega}}^a - \bar{g}(\dot{\bar{\omega}}^c,\bar{\omega}^a)\bar{\omega}_c). \end{cases}$$

Then

$$\dot{\bar{\omega}}^a = \dot{\bar{\omega}}^a{}_S + \dot{\bar{\omega}}^a{}_A.$$

Notation: Put

$$\begin{cases} \mathcal{L}_{\vec{N}_t}\bar{\omega}^a{}_S = \frac{1}{2}\,(\mathcal{L}_{\vec{N}_t}\bar{\omega}^a + \bar{g}(\mathcal{L}_{\vec{N}_t}\bar{\omega}^c,\bar{\omega}^a)\bar{\omega}_c) \\ \mathcal{L}_{\vec{N}_t}\bar{\omega}^a{}_A = \frac{1}{2}\,(\mathcal{L}_{\vec{N}_t}\bar{\omega}^a - \bar{g}(\mathcal{L}_{\vec{N}_t}\bar{\omega}^c,\bar{\omega}^a)\bar{\omega}_c). \end{cases}$$

Then

$$\mathcal{L}_{\vec{N}_t}\bar{\omega}^a = \mathcal{L}_{\vec{N}_t}\bar{\omega}^a{}_S + \mathcal{L}_{\vec{N}_t}\bar{\omega}^a{}_A.$$

<u>LEMMA</u> We have

$$\begin{cases} \dot{\bar{\omega}}^a{}_S = \mathcal{L}_{\vec{N}_t}\bar{\omega}^a{}_S + N_t\bar{\omega}^a{}_0 \\ \dot{\bar{\omega}}^a{}_A = \mathcal{L}_{\vec{N}_t}\bar{\omega}^a{}_A + \bar{Q}^a{}_b\bar{\omega}^b. \end{cases}$$

[Consider the first relation. Thus

$$\dot{\bar{\omega}}^a{}_S = \frac{1}{2}\,(\dot{\bar{\omega}}^a + \bar{g}(\dot{\bar{\omega}}^c,\bar{\omega}^a)\bar{\omega}_c)$$

$$= \frac{1}{2}\,(N_t\bar{\omega}^a{}_0 + \bar{Q}^a{}_b\bar{\omega}^b + L_{\vec{N}_t}\bar{\omega}^a)$$

$$+ \frac{1}{2}\,\bar{g}(N_t\bar{\omega}^c{}_0 + \bar{Q}^c{}_d\bar{\omega}^d + L_{\vec{N}_t}\bar{\omega}^c,\bar{\omega}^a)\bar{\omega}_c$$

$$= \frac{1}{2}\,(L_{\vec{N}_t}\bar{\omega}^a + \bar{g}(L_{\vec{N}_t}\bar{\omega}^c,\bar{\omega}^a)\bar{\omega}_c)$$

$$+ \frac{1}{2}\,(N_t\bar{\omega}^a{}_0 + \bar{g}(N_t\bar{\omega}^c{}_0,\bar{\omega}^a)\bar{\omega}_c$$

$$+ \frac{1}{2}\,(\bar{Q}^a{}_b\bar{\omega}^b + \bar{Q}^c{}_a\bar{\omega}_c)$$

$$= \frac{1}{2}\,(L_{\vec{N}_t}\bar{\omega}^a + \bar{g}(L_{\vec{N}_t}\bar{\omega}^c,\bar{\omega}^a)\bar{\omega}_c)$$

$$+ \frac{1}{2}\,(N_t\kappa^a{}_d\bar{\omega}^d + N_t\kappa^c{}_a\bar{\omega}_c)$$

$$+ \frac{1}{2}\,(\bar{Q}^a{}_b\bar{\omega}^b - \bar{Q}^a{}_c\bar{\omega}^c)$$

$$= L_{\vec{N}_t}\bar{\omega}^a{}_S + N_t\bar{\omega}^a{}_0.]$$

[Note:

$$\left[\begin{array}{l} \bar{g}(\dot{\bar{\omega}}^a{}_S,\bar{\omega}^b) = \bar{g}(\dot{\bar{\omega}}^b{}_S,\bar{\omega}^a) \\ \\ \bar{g}(\bar{\omega}^a{}_0,\bar{\omega}^b) = \bar{g}(\bar{\omega}^b{}_0,\bar{\omega}^a) \end{array}\right.$$

$\Rightarrow$

$$\bar{g}(L_{\vec{N}_t} \bar{\omega}^a{}_S, \bar{\omega}^b) = \bar{g}(L_{\vec{N}_t} \bar{\omega}^b{}_S, \bar{\omega}^a)$$

and

$$\left[\begin{array}{l} \bar{g}(\dot{\bar{\omega}}^a{}_A, \bar{\omega}^b) = -\,\bar{g}(\dot{\bar{\omega}}^b{}_A, \bar{\omega}^a) \\ \\ \bar{g}(\bar{Q}^a{}_c \bar{\omega}^c, \bar{\omega}^b) = -\,\bar{g}(\bar{Q}^b{}_d \bar{\omega}^d, \bar{\omega}^a) \end{array}\right.$$

$\Rightarrow$

$$\bar{g}(L_{\vec{N}_t} \bar{\omega}^a{}_A, \bar{\omega}^b) = -\,\bar{g}(L_{\vec{N}_t} \bar{\omega}^b{}_A, \bar{\omega}^a).]$$

Let μ, ν be indices that run between 1 and n-1. Working locally, write

$$\frac{\partial}{\partial x^\mu} = e^a{}_\mu E_a.$$

Then

$$\bar{g}_{\mu\nu} = \bar{g}\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$$

$$= \bar{g}(e^a{}_\mu E_a, e^b{}_\nu E_b)$$

$$= \eta_{ab} e^a{}_\mu e^b{}_\nu.$$

<u>LEMMA</u> We have

$$\dot{\bar{g}}_{\mu\nu} = (\bar{g}(\dot{\bar{\omega}}_a, \bar{\omega}_b) + \bar{g}(\dot{\bar{\omega}}_b, \bar{\omega}_a)) e^a{}_\mu e^b{}_\nu.$$

To simplify this, write

$$\bar{g}(\dot{\bar{\omega}}_a,\bar{\omega}_b) + \bar{g}(\dot{\bar{\omega}}_b,\bar{\omega}_a)$$

$$= \bar{g}(\dot{\bar{\omega}}_{a,S} + \dot{\bar{\omega}}_{a,A},\bar{\omega}_b) + \bar{g}(\dot{\bar{\omega}}_{b,S} + \dot{\bar{\omega}}_{b,A},\bar{\omega}_a)$$

$$= \bar{g}(\dot{\bar{\omega}}_{a,S},\bar{\omega}_b) + \bar{g}(\dot{\bar{\omega}}_{b,S},\bar{\omega}_a)$$

$$+ \bar{g}(\dot{\bar{\omega}}_{a,A},\bar{\omega}_b) + \bar{g}(\dot{\bar{\omega}}_{b,A},\bar{\omega}_a)$$

$$= \bar{g}(\dot{\bar{\omega}}_{a,S},\bar{\omega}_b) + \bar{g}(\dot{\bar{\omega}}_{a,S},\bar{\omega}_b)$$

$$+ \bar{g}(\dot{\bar{\omega}}_{a,A},\bar{\omega}_b) - \bar{g}(\dot{\bar{\omega}}_{a,A},\bar{\omega}_b)$$

$$= 2\bar{g}(\dot{\bar{\omega}}_{a,S},\bar{\omega}_b).$$

<u>Reality Check</u> The claim is that

$$2N_t\varkappa_{\mu\nu} + (\mathcal{L}_{\vec{N}_t}\bar{g})_{\mu\nu}$$

equals

$$2\bar{g}(\dot{\bar{\omega}}_{a,S},\bar{\omega}_b)e^a{}_\mu e^b{}_\nu$$

or still,

$$2\bar{g}(N_t\bar{\omega}_{a0},\bar{\omega}_b)e^a{}_\mu e^b{}_\nu$$

$$+ 2\bar{g}(\mathcal{L}_{\vec{N}_t}\bar{\omega}_{a,S},\bar{\omega}_b)e^a{}_\mu e^b{}_\nu.$$

• $2N_t \varkappa_{\mu\nu}$

$$= 2N_t \varkappa_t (\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu})$$

$$= 2N_t \varkappa_t (e^a{}_\mu E_a, e^b{}_\nu E_b)$$

$$= 2N_t e^a{}_\mu e^b{}_\nu \varkappa_t (E_a, E_b)$$

$$= 2N_t e^a{}_\mu e^b{}_\nu \varkappa_{ab}.$$

On the other hand,

$$2\bar{g}(N_t \bar{\omega}_{a0}, \bar{\omega}_b) e^a{}_\mu e^b{}_\nu$$

$$= 2N_t e^a{}_\mu e^b{}_\nu \bar{g}(\varkappa_{ac}\bar{\omega}^c, \bar{\omega}_b)$$

$$= 2N_t e^a{}_\mu e^b{}_\nu \varkappa_{ab}.$$

• $(L_{\vec{N}_t} \bar{g})_{\mu\nu}$

$$= (L_{\vec{N}_t} \bar{g}) (\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu})$$

$$= \overrightarrow{\bar{\nabla} g} \vec{N}_t (\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) + \overrightarrow{\bar{\nabla} g} \vec{N}_t (\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\mu})$$

$$= \overrightarrow{\bar{\nabla} g} \vec{N}_t (E_a, E_b) e^a{}_\mu e^b{}_\nu$$

$$+ \overrightarrow{\bar{\nabla} g} \vec{N}_t (E_b, E_a) e^a{}_\mu e^b{}_\nu$$

$$= (\bar{\nabla}_b N_a) e^a{}_\mu e^b{}_\nu + (\bar{\nabla}_a N_b) e^a{}_\mu e^b{}_\nu .$$

But

$$2\bar{g}(L_{\vec{N}_t} \bar{\omega}_{a,S}, \bar{\omega}_b) e^a{}_\mu e^b{}_\nu$$

$$= \bar{g}(L_{\vec{N}_t} \bar{\omega}_a + \bar{g}(L_{\vec{N}_t} \bar{\omega}^c, \bar{\omega}_a)\bar{\omega}_c, \bar{\omega}_b) e^a{}_\mu e^b{}_\nu$$

$$= \bar{g}(L_{\vec{N}_t} \bar{\omega}_a, \bar{\omega}_b) e^a{}_\mu e^b{}_\nu + \bar{g}(L_{\vec{N}_t} \bar{\omega}_b, \bar{\omega}_a) e^a{}_\mu e^b{}_\nu .$$

Now use the following relations

$$\left[\begin{array}{l} (\bar{\nabla}_c N_a)\bar{\omega}_c = L_{\vec{N}_t} \bar{\omega}_a + \bar{\omega}_{ac}(\vec{N}_t)\bar{\omega}_c \\ \\ (\bar{\nabla}_c N_b)\bar{\omega}_c = L_{\vec{N}_t} \bar{\omega}_b + \bar{\omega}_{bc}(\vec{N}_t)\bar{\omega}_c \end{array}\right.$$

to get

1. $\bar{g}(L_{\vec{N}_t} \bar{\omega}_a, \bar{\omega}_b) e^a{}_\mu e^b{}_\nu$

$$= \bar{g}((\bar{\nabla}_c N_a)\bar{\omega}_c - \bar{\omega}_{ac}(\vec{N}_t)\bar{\omega}_c, \bar{\omega}_b) e^a{}_\mu e^b{}_\nu$$

$$= (\bar{\nabla}_b N_a) e^a{}_\mu e^b{}_\nu - \bar{\omega}_{ab}(\vec{N}_t) e^a{}_\mu e^b{}_\nu .$$

2. $\bar{g}(L_{\vec{N}_t} \bar{\omega}_b, \bar{\omega}_a) e^a{}_\mu e^b{}_\nu$

$$= \bar{g}((\bar{\nabla}_c N_b)\bar{\omega}_c - \bar{\omega}_{bc}(\vec{N}_t)\bar{\omega}_c, \bar{\omega}_a) e^a{}_\mu e^b{}_\nu$$

$$= (\bar{\nabla}_a N_b) e^a{}_\mu e^b{}_\nu - \bar{\omega}_{ba}(\vec{N}_t) e^a{}_\mu e^b{}_\nu .$$

Therefore

$$1 + 2 = (\bar{\nabla}_b N_a) e^a{}_\mu e^b{}_\nu + (\bar{\nabla}_a N_b) e^a{}_\mu e^b{}_\nu$$

$$- \bar{\omega}_{ab}(\vec{N}_t) e^a{}_\mu e^b{}_\nu - \bar{\omega}_{ba}(\vec{N}_t) e^a{}_\mu e^b{}_\nu$$

$$= (\bar{\nabla}_b N_a) e^a{}_\mu e^b{}_\nu + (\bar{\nabla}_a N_b) e^a{}_\mu e^b{}_\nu$$

$$- \bar{\omega}_{ab}(\vec{N}_t) e^a{}_\mu e^b{}_\nu + \bar{\omega}_{ab}(\vec{N}_t) e^a{}_\mu e^b{}_\nu$$

$$= (\bar{\nabla}_b N_a) e^a{}_\mu e^b{}_\nu + (\bar{\nabla}_a N_b) e^a{}_\mu e^b{}_\nu .$$

<u>N.B.</u> $\forall\ X \in \mathcal{D}^1(\Sigma)$,

$$(\bar{\nabla}_b X^a)\bar{\omega}^b = L_X \bar{\omega}^a + \bar{\omega}^a{}_b(X)\bar{\omega}^b .$$

[Note: The verification is an exercise in the definitions and will be detailed later on.]

<u>Section 40</u>: <u>Isolating the Lagrangian</u> The assumptions and notation are those of the standard setup.

Rappel:

$$\theta^{ij} = *(\omega^i \wedge \omega^j) \quad (i,j = 0,1,\ldots,n-1).$$

[Note: θ^{ij} is an (n-2)-form and the Hodge star is taken per

$$g = -\omega^0 \otimes \omega^0 + \omega^a \otimes \omega^a.]$$

Consider

$$\theta^{ij} \wedge \Omega_{ij} \ (= S(g)\mathrm{vol}_g).$$

Write

$$\theta^{ij} \wedge \Omega_{ij} = \theta^{0j} \wedge \Omega_{0j} + \theta^{i0} \wedge \Omega_{i0}$$

$$+ \theta^{bc} \wedge \Omega_{bc}$$

$$= 2\theta^{0a} \wedge \Omega_{0a} + \theta^{bc} \wedge \Omega_{bc}.$$

[Note: Obviously, $\theta^{ij} = -\theta^{ji}$. In addition,

$$\Omega^i{}_j = -\varepsilon_i \varepsilon_j \Omega^j{}_i \quad \text{(no sum)}$$

$$\Rightarrow$$

$$\Omega_{ij} = \varepsilon_i(-\varepsilon_i \varepsilon_j \Omega^j{}_i)$$

$$= -\varepsilon_j \Omega^j{}_i$$

$$= -\Omega_{ji}.]$$

Since

$$\Omega_{0a} = d\omega_{0a} + \omega_{0b}\wedge\omega^{b}{}_{a},$$

it follows that

$$\theta^{ij}\wedge\Omega_{ij} = 2\theta^{0a}\wedge d\omega_{0a} + 2\theta^{0a}\wedge\omega_{0b}\wedge\omega^{b}{}_{a}$$

$$+ \theta^{bc}\wedge(\Omega_{bc} - \omega_{b0}\wedge\omega^{0}{}_{c}) + \theta^{bc}\wedge\omega_{b0}\wedge\omega^{0}{}_{c}.$$

Rappel: We have

$$d\theta^{ij} = -\omega^{i}{}_{k}\wedge\theta^{kj} - \omega^{j}{}_{k}\wedge\theta^{ik}.$$

Consequently,

$$d(\theta^{0a}\wedge\omega_{0a}) = d\theta^{0a}\wedge\omega_{0a} + (-1)^{n-2}\theta^{0a}\wedge d\omega_{0a}$$

$$= (-\omega^{0}{}_{b}\wedge\theta^{ba} - \omega^{a}{}_{b}\wedge\theta^{0b})\wedge\omega_{0a} + (-1)^{n-2}\theta^{0a}\wedge d\omega_{0a}$$

$\Rightarrow$

$$\theta^{0a}\wedge d\omega_{0a} = (-1)^{n-2}d(\theta^{0a}\wedge\omega_{0a})$$

$$+ (-1)^{n-2}(\omega^{0}{}_{b}\wedge\theta^{ba}\wedge\omega_{0a} + \omega^{a}{}_{b}\wedge\theta^{0b}\wedge\omega_{0a})$$

$$= d(\omega_{0a}\wedge\theta^{0a}) + \theta^{ba}\wedge\omega^{0}{}_{b}\wedge\omega_{0a} + \theta^{0b}\wedge\omega^{a}{}_{b}\wedge\omega_{0a}$$

$$= d(\omega_{0a}\wedge\theta^{0a}) + \theta^{ab}\wedge\omega^{0}{}_{a}\wedge\omega_{0b} + \theta^{0a}\wedge\omega^{b}{}_{a}\wedge\omega_{0b}$$

$$= d(\omega_{0a}\wedge\theta^{0a}) + \theta^{ab}\wedge\omega^{0}{}_{a}\wedge\omega_{0b} - \theta^{0a}\wedge\omega_{0b}\wedge\omega^{b}{}_{a}.$$

Therefore

$$\theta^{ij}\wedge\Omega_{ij} = 2d(\omega_{0a}\wedge\theta^{0a}) + 2\theta^{ab}\wedge\omega^0{}_a\wedge\omega_{0b}$$

$$+ \theta^{bc}\wedge(\Omega_{bc} - \omega_{b0}\wedge\omega^0{}_c) + \theta^{bc}\wedge\omega_{b0}\wedge\omega^0{}_c.$$

But

$$\theta^{ab}\wedge\omega^0{}_a\wedge\omega_{0b} = \theta^{ab}\wedge\omega_{0a}\wedge\omega^0{}_b$$

and

$$\theta^{bc}\wedge\omega_{b0}\wedge\omega^0{}_c = \theta^{ab}\wedge\omega_{a0}\wedge\omega^0{}_b$$

$$= - \theta^{ab}\wedge\omega_{0a}\wedge\omega^0{}_b$$

$\Rightarrow$

$$\theta^{ij}\wedge\Omega_{ij} = 2d(\omega_{0a}\wedge\theta^{0a}) + \theta^{ab}\wedge\omega_{0a}\wedge\omega^0{}_b$$

$$+ \theta^{bc}\wedge(\Omega_{bc} - \omega_{b0}\wedge\omega^0{}_c)$$

$$= 2d(\omega_{0a}\wedge\theta^{0a}) - \theta^{ab}\wedge\omega_{0a}\wedge\omega_{0b}$$

$$+ \theta^{ab}\wedge(\Omega_{ab} - \omega_{a0}\wedge\omega^0{}_b).$$

Remark: The explanation for singling out the term

$$\Omega_{ab} - \omega_{a0}\wedge\omega^0{}_b$$

is the fact that

$$\bar{\Omega}_{ab} - \bar{\omega}_{a0}\wedge\bar{\omega}^0{}_b = {}^{(n-1)}\Omega_{ab},$$

the overbar standing for pullback by i_t^*.

Notation: Put

$$\bar{\theta}^{ab} = *(\bar{\omega}^a\wedge\bar{\omega}^b).$$

[Note: The Hodge star is taken per

$$i_t^*g = \bar{g} = q(t) \ \ (= q_t)$$

but there is a caveat: $\bar{\theta}^{ab}$ is not equal to $i_t^*\theta^{ab}$ (which, in fact, is identically zero (cf. infra)).]

Proceeding formally, set aside the differential

$$2d(\omega_{0a}\wedge\theta^{0a})$$

and ignore all issues of convergence -- then

$$\int_M \theta^{ij}\wedge\Omega_{ij}$$

$$= \int_{\underline{R}} dt \int_\Sigma i_t^*\iota_{\partial/\partial t}[\theta^{ab}\wedge((\Omega_{ab} - \omega_{a0}\wedge\omega^0{}_b) - \omega_{0a}\wedge\omega_{0b})]$$

$$= \int_{\underline{R}} dt \int_\Sigma N_t\bar{\theta}^{ab}\wedge({}^{(n-1)}\Omega_{ab} - \bar{\omega}_{0a}\wedge\bar{\omega}_{0b}).$$

<u>Details</u> To see the passage from

$$i_t^*\iota_{\partial/\partial t}[\theta^{ab}\wedge((\Omega_{ab} - \omega_{a0}\wedge\omega^0{}_b) - \omega_{0a}\wedge\omega_{0b})]$$

to

$$N_t\bar{\theta}^{ab}\wedge({}^{(n-1)}\Omega_{ab} - \bar{\omega}_{0a}\wedge\bar{\omega}_{0b}),$$

recall first that $\omega^0 = Ndt$ ($\Rightarrow \iota_{\partial/\partial t}\omega^0 = N\iota_{\partial/\partial t}dt = N$), hence $i_t^*\omega^0 = N_t i_t^* dt = 0$. This said, write

$$\theta^{ab} = \frac{1}{(n-2)!}\,\varepsilon_{abj_3\ldots j_n}\omega^{j_3}\wedge\ldots\wedge\omega^{j_n}$$

$$= \frac{1}{(n-2)!}\,\varepsilon_{ab0j_4\ldots j_n}\omega^0\wedge\omega^{j_4}\wedge\ldots\wedge\omega^{j_n}$$

$$+ \ldots + \frac{1}{(n-2)!}\,\varepsilon_{abj_3\ldots j_{n-1}0}\omega^{j_3}\wedge\ldots\wedge\omega^{j_{n-1}}\wedge\omega^0$$

$$+ \frac{1}{(n-2)!}\,\varepsilon_{abc_3\ldots c_n}\omega^{c_3}\wedge\ldots\wedge\omega^{c_n}$$

$$= \frac{1}{(n-2)!}\,\varepsilon_{ab0j_4\ldots j_n}\omega^0\wedge\omega^{j_4}\wedge\ldots\wedge\omega^{j_n}$$

$$+ \ldots + \frac{1}{(n-2)!}\,\varepsilon_{abj_3\ldots j_{n-1}0}\omega^{j_3}\wedge\ldots\wedge\omega^{j_{n-1}}\wedge\omega^0$$

$$= \frac{(n-2)}{(n-2)!}\,\varepsilon_{ab0c_4\ldots c_n}\omega^0\wedge\omega^{c_4}\wedge\ldots\wedge\omega^{c_n}$$

$$= \omega^0\wedge\frac{1}{(n-3)!}\,\varepsilon_{ab0c_4\ldots c_n}\omega^{c_4}\wedge\ldots\wedge\omega^{c_n}$$

$$= \omega^0\wedge\frac{1}{(n-3)!}\,\varepsilon_{0abc_4\ldots c_n}\omega^{c_4}\wedge\ldots\wedge\omega^{c_n}.$$

Put

$$T_{ab} = (\Omega_{ab} - \omega_{a0}\wedge\omega^0{}_b) - \omega_{0a}\wedge\omega_{0b}.$$

Then

$$i_t^* \iota_{\partial/\partial t}[\theta^{ab} \wedge T_{ab}]$$

$$= i_t^* \iota_{\partial/\partial t}[\omega^0 \wedge \frac{1}{(n-3)!} \varepsilon_{0abc_4 \ldots c_n} \omega^{c_4} \wedge \ldots \wedge \omega^{c_n} \wedge T_{ab}]$$

$$= i_t^*[(\iota_{\partial/\partial t}\omega^0) \wedge \frac{1}{(n-3)!} \varepsilon_{0abc_4 \ldots c_n} \omega^{c_4} \wedge \ldots \wedge \omega^{c_n} \wedge T_{ab}$$

$$- \omega^0 \wedge \iota_{\partial/\partial t}(\frac{1}{(n-3)!} \varepsilon_{0abc_4 \ldots c_n} \omega^{c_4} \wedge \ldots \wedge \omega^{c_n} \wedge T_{ab})]$$

$$= N_t \frac{1}{(n-3)!} \varepsilon_{0abc_4 \ldots c_n} \bar{\omega}^{c_4} \wedge \ldots \wedge \bar{\omega}^{c_n} \wedge \bar{T}_{ab}$$

$$= N_t \frac{1}{(n-3)!} \varepsilon_{0abc_4 \ldots c_n} \bar{\omega}^{c_4} \wedge \ldots \wedge \bar{\omega}^{c_n} \wedge ({}^{(n-1)}\Omega_{ab} - \bar{\omega}_{0a} \wedge \bar{\omega}_{0b}).$$

And

$$\bar{\theta}^{ab} = *(\bar{\omega}^a \wedge \bar{\omega}^b)$$

$$= \frac{1}{(n-3)!} \varepsilon_{abc_3 \ldots c_{n-1}} \bar{\omega}^{c_3} \wedge \ldots \wedge \bar{\omega}^{c_{n-1}}$$

$$= \frac{1}{(n-3)!} \varepsilon_{0abc_4 \ldots c_n} \bar{\omega}^{c_4} \wedge \ldots \wedge \bar{\omega}^{c_n}.$$

[Note: To discuss the effect of omitting

$$2d(\omega_{0a} \wedge \theta^{0a})$$

from these considerations, observe that

$$\int_M d(\omega_{0a}\wedge\theta^{0a}) = \int_{\underline{R}} dt \int_\Sigma i_t^* \iota_{\partial/\partial t} d(\omega_{0a}\wedge\theta^{0a})$$

$$= \int_{\underline{R}} dt \int_\Sigma i_t^* L_{\partial/\partial t}(\omega_{0a}\wedge\theta^{0a}) - \int_{\underline{R}} dt \int_\Sigma i_t^* d\iota_{\partial/\partial t}(\omega_{0a}\wedge\theta^{0a})$$

$$= \int_{\underline{R}} dt \int_\Sigma \frac{d}{dt} i_t^*(\omega_{0a}\wedge\theta^{0a}) - \int_{\underline{R}} dt \int_\Sigma d i_t^* \iota_{\partial/\partial t}(\omega_{0a}\wedge\theta^{0a})$$

$$= \int_{\underline{R}} \frac{d}{dt} [\int_\Sigma i_t^*(\omega_{0a}\wedge\theta^{0a})]dt.]$$

It remains to examine the integrand:

$$\bar{\theta}^{ab}\wedge({}^{(n-1)}\Omega_{ab} - \bar{\omega}_{0a}\wedge\bar{\omega}_{0b})$$

$$= \bar{\theta}^{ab}\wedge{}^{(n-1)}\Omega_{ab} - *(\bar{\omega}^a\wedge\bar{\omega}^b)\wedge(\bar{\omega}_{0a}\wedge\bar{\omega}_{0b})$$

$$= S(\bar{g})\mathrm{vol}_{\bar{g}} - (-1)^{2(n-3)}(\bar{\omega}_{0a}\wedge\bar{\omega}_{0b})\wedge*(\bar{\omega}^a\wedge\bar{\omega}^b)$$

$$= S(\bar{g})\mathrm{vol}_{\bar{g}} - \bar{g}(\bar{\omega}_{0a}\wedge\bar{\omega}_{0b},\bar{\omega}^a\wedge\bar{\omega}^b)\mathrm{vol}_{\bar{g}}.$$

And

$$- \bar{g}(\bar{\omega}_{0a}\wedge\bar{\omega}_{0b},\bar{\omega}^a\wedge\bar{\omega}^b)$$

$$= - \det\begin{bmatrix} \bar{g}(\bar{\omega}_{0a},\bar{\omega}^a) & \bar{g}(\bar{\omega}_{0a},\bar{\omega}^b) \\ \bar{g}(\bar{\omega}_{0b},\bar{\omega}^a) & \bar{g}(\bar{\omega}_{0b},\bar{\omega}^b) \end{bmatrix}.$$

But

$$\left[\begin{array}{l} \bar{g}(\bar{\omega}_{0a},\bar{\omega}^{a}) = \bar{g}(-\varkappa_{ac}\bar{\omega}^{c},\bar{\omega}^{a}) = -\varkappa_{aa} \\ \\ \bar{g}(\bar{\omega}_{0a},\bar{\omega}^{b}) = \bar{g}(-\varkappa_{ac}\bar{\omega}^{c},\bar{\omega}^{b}) = -\varkappa_{ab}. \end{array}\right.$$

Therefore

$$-\bar{g}(\bar{\omega}_{0a}\wedge\bar{\omega}_{0b},\bar{\omega}^{a}\wedge\bar{\omega}^{b})$$

$$= -\det\begin{bmatrix} -\varkappa_{aa} & -\varkappa_{ab} \\ \\ -\varkappa_{ab} & -\varkappa_{bb} \end{bmatrix}$$

$$= -(\varkappa_{aa}\varkappa_{bb} - (\varkappa_{ab})^{2})$$

$$= [\varkappa,\varkappa]_{\bar{g}} - \operatorname{tr}_{\bar{g}}(\varkappa)^{2}.$$

Accordingly, at each instant of time,

$$\int_{\Sigma} N_{t}\bar{\theta}^{ab}\wedge({}^{(n-1)}\Omega_{ab} - \bar{\omega}_{0a}\wedge\bar{\omega}_{0b})$$

$$= \int_{\Sigma} N_{t}(S(q_{t}) + [\varkappa_{t},\varkappa_{t}]_{q_{t}} - K_{t}^{2})\mathrm{vol}_{q_{t}},$$

which is in complete agreement with what has been established earlier.

<u>Section 41</u>: <u>The Momentum Form</u> The assumptions and notation are those of the standard setup.

Recall that the momentum of the theory is the path $t \rightarrow p_t$ in $S_d^2(\Sigma)$ defined by the prescription

$$p_t = \pi_t \otimes |q_t|^{1/2},$$

where

$$\pi_t = (\kappa_t - K_t q_t)^{\#}.$$

In this section, we shall show that p_t is closely related to a certain element $\vec{p}_t \in \Lambda^{n-2}(\Sigma; T_1^0(\Sigma))$.

Notation: Let

$$p_a = \bar{\omega}_{0b} \wedge * (\bar{\omega}^a \wedge \bar{\omega}^b).$$

Definition: The <u>momentum form</u> of the theory is the path $t \rightarrow \vec{p}_t$ in $\Lambda^{n-2}(\Sigma; T_1^0(\Sigma))$ defined by the prescription

$$\vec{p}_t(X_1, \ldots, X_{n-2}) = p_a(X_1, \ldots, X_{n-2}) \bar{\omega}^a.$$

<u>LEMMA</u> We have

$$p_a = q_t(\bar{\omega}_{0b}, \bar{\omega}^b) * \bar{\omega}^a - q_t(\bar{\omega}_{0b}, \bar{\omega}^a) * \bar{\omega}^b.$$

[To begin with,

$$\iota_{\bar{\omega}^b}(\bar{\omega}_{0b} \wedge * \bar{\omega}^a) = \iota_{\bar{\omega}^b} \bar{\omega}_{0b} \wedge * \bar{\omega}^a - \bar{\omega}_{0b} \wedge \iota_{\bar{\omega}^b} * \bar{\omega}^a.$$

But

$$\iota_{\bar{\omega}^b} * \bar{\omega}^a = * (\bar{\omega}^a \wedge \bar{\omega}^b).$$

Therefore

$$p_a = \bar{\omega}_{0b} \wedge * (\bar{\omega}^a \wedge \bar{\omega}^b)$$

$$= \iota_{\bar{\omega}^b} \bar{\omega}_{0b} \wedge * \bar{\omega}^a - \iota_{\bar{\omega}^b} (\bar{\omega}_{0b} \wedge * \bar{\omega}^a)$$

$$= q_t(\bar{\omega}_{0b}, \bar{\omega}^b) * \bar{\omega}^a - \iota_{\bar{\omega}^b} (q_t(\bar{\omega}_{0b}, \bar{\omega}^a) \mathrm{vol}_{q_t})$$

$$= q_t(\bar{\omega}_{0b}, \bar{\omega}^b) * \bar{\omega}^a - q_t(\bar{\omega}_{0b}, \bar{\omega}^a) \iota_{\bar{\omega}^b} \mathrm{vol}_{q_t}$$

$$= q_t(\bar{\omega}_{0b}, \bar{\omega}^b) * \bar{\omega}^a - q_t(\bar{\omega}_{0b}, \bar{\omega}^a) * \bar{\omega}^b .]$$

Consider now

$$\frac{1}{2} (\bar{\omega}^a \wedge p_b + \bar{\omega}^b \wedge p_a).$$

Write

$$\begin{cases} p_a = q_t(\bar{\omega}_{0c}, \bar{\omega}^c) * \bar{\omega}^a - q_t(\bar{\omega}_{0c}, \bar{\omega}^a) * \bar{\omega}^c \\ p_b = q_t(\bar{\omega}_{0d}, \bar{\omega}^d) * \bar{\omega}^b - q_t(\bar{\omega}_{0d}, \bar{\omega}^b) * \bar{\omega}^d . \end{cases}$$

Then

$$\bar{\omega}^a \wedge p_b = q_t(\bar{\omega}_{0d}, \bar{\omega}^d) \bar{\omega}^a \wedge * \bar{\omega}^b - q_t(\bar{\omega}_{0d}, \bar{\omega}^b) \bar{\omega}^a \wedge * \bar{\omega}^d$$

$$= q_t(\bar{\omega}_{0d}, \bar{\omega}^d) q_t(\bar{\omega}^a, \bar{\omega}^b) \mathrm{vol}_{q_t} - q_t(\bar{\omega}_{0d}, \bar{\omega}^b) q_t(\bar{\omega}^a, \bar{\omega}^d) \mathrm{vol}_{q_t}$$

and

$$\bar{\omega}^b \wedge p_a = q_t(\bar{\omega}_{0c}, \bar{\omega}^c)\bar{\omega}^b \wedge * \bar{\omega}^a - q_t(\bar{\omega}_{0c}, \bar{\omega}^a)\bar{\omega}^b \wedge * \bar{\omega}^c$$

$$= q_t(\bar{\omega}_{0c}, \bar{\omega}^c) q_t(\bar{\omega}^b, \bar{\omega}^a) \mathrm{vol}_{q_t} - q_t(\bar{\omega}_{0c}, \bar{\omega}^a) q_t(\bar{\omega}^b, \bar{\omega}^c) \mathrm{vol}_{q_t}$$

$\Rightarrow$

$$\frac{1}{2}(\bar{\omega}^a \wedge p_b + \bar{\omega}^b \wedge p_a) = C_{ab} \mathrm{vol}_{q_t}.$$

<u>a≠b</u>: In this case,

$$2C_{ab} = - q_t(\bar{\omega}_{0c}, \bar{\omega}^a) q_t(\bar{\omega}^b, \bar{\omega}^c) - q_t(\bar{\omega}_{0d}, \bar{\omega}^b) q_t(\bar{\omega}^a, \bar{\omega}^d)$$

$$= - q_t(\bar{\omega}_{0b}, \bar{\omega}^a) - q_t(\bar{\omega}_{0a}, \bar{\omega}^b)$$

$$= - q_t(\bar{\omega}_{0b}, E_a^\flat) - q_t(\bar{\omega}_{0a}, E_b^\flat)$$

$$= - \bar{\omega}_{0b}(E_a) - \bar{\omega}_{0a}(E_b)$$

$$= \varkappa_{ab} + \varkappa_{ba}$$

$$= 2\varkappa_{ab}.$$

<u>a=b</u>: In this case,

$$2C_{aa} = q_t(\bar{\omega}_{0c}, \bar{\omega}^c) - q_t(\bar{\omega}_{0c}, \bar{\omega}^a) q_t(\bar{\omega}^a, \bar{\omega}^c)$$

$$+ q_t(\bar{\omega}_{0d}, \bar{\omega}^d) - q_t(\bar{\omega}_{0d}, \bar{\omega}^a) q_t(\bar{\omega}^a, \bar{\omega}^d)$$

$$= 2q_t(\bar{\omega}_{0c}, \bar{\omega}^c) - 2q_t(\bar{\omega}_{0a}, \bar{\omega}^a)$$

$$= -2q_t(\bar{\omega}_{0a},\bar{\omega}^a) + 2q_t(\bar{\omega}_{0c},\bar{\omega}^c)$$

$$= 2\varkappa_{aa} - 2K_t.$$

Since

$$(\varkappa_t - K_t q_t)_{ab} = \begin{cases} \varkappa_{ab} & (a \neq b) \\ \varkappa_{aa} - K_t & (a=b), \end{cases}$$

it follows that

$$\frac{1}{2}(\bar{\omega}^a \wedge p_b + \bar{\omega}^b \wedge p_a) = (\varkappa_t - K_t q_t)_{ab} \mathrm{vol}_{q_t}.$$

By definition,

$$p_t = \pi_t \otimes |q_t|^{1/2},$$

where

$$\pi_t = (\varkappa_t - K_t q_t)^{\#}.$$

And, as elements of $\mathcal{D}^2_{n-1}(\Sigma)$,

$$E_a \otimes E_b \otimes \frac{1}{2}(\bar{\omega}^a \wedge p_b + \bar{\omega}^b \wedge p_a)$$

$$= (\varkappa_t - K_t q_t)_{ab}(E_a \otimes E_b) \otimes \mathrm{vol}_{q_t}.$$

But

$$(\varkappa_t - K_t q_t)_{ab}(E_a \otimes E_b) = (\varkappa_t - K_t q_t)^{\#}.$$

Indeed,

$$(\varkappa_t - K_t q_t)^{\#}(\bar{\omega}^a,\bar{\omega}^b)$$

$$= (\varkappa_t - K_t q_t)(E_a, E_b)$$

$$= (\varkappa_t - K_t q_t)_{ab}.$$

Let

$$P_t = q_t(p_a, *\bar{\omega}^a).$$

Then

$$P_t = q_t(q_t(\bar{\omega}_{0b}, \bar{\omega}^b) *\bar{\omega}^a - q_t(\bar{\omega}_{0b}, \bar{\omega}^a) *\bar{\omega}^b, *\bar{\omega}^a)$$

$$= \quad q_t(\bar{\omega}_{0b}, \bar{\omega}^b) q_t(*\bar{\omega}^a, *\bar{\omega}^a)$$

$$- q_t(\bar{\omega}_{0b}, \bar{\omega}^a) q_t(*\bar{\omega}^b, *\bar{\omega}^a)$$

$$= (n-1) q_t(\bar{\omega}_{0b}, \bar{\omega}^b) - q_t(\bar{\omega}_{0a}, \bar{\omega}^a)$$

$$= (n-1) q_t(\bar{\omega}_{0a}, \bar{\omega}^a) - q_t(\bar{\omega}_{0a}, \bar{\omega}^a)$$

$$= (n-2) q_t(\bar{\omega}_{0a}, \bar{\omega}^a).$$

<u>LEMMA</u> We have

$$\bar{\omega}_{0a} = - q_t(p_b, *\bar{\omega}^a)\bar{\omega}^b + \frac{1}{n-2} P_t \delta^b{}_a \bar{\omega}^b.$$

[Write

$$\bar{\omega}_{0a} = q_t(\bar{\omega}_{0a}, \bar{\omega}^b)\bar{\omega}^b.$$

Then

$$p_b = \frac{1}{n-2} P_t * \bar{\omega}^b - q_t(\bar{\omega}_{0a}, \bar{\omega}^b) *\bar{\omega}^a$$

$\Rightarrow$

$$q_t(p_b, *\bar{\omega}^a) = \frac{1}{n-2} P_t \delta^b{}_a - q_t(\bar{\omega}_{0a}, \bar{\omega}^b)$$

$\Rightarrow$

$$q_t(\bar{\omega}_{0a}, \bar{\omega}^b) = - q_t(p_b, *\bar{\omega}^a) + \frac{1}{n-2} P_t \delta^b{}_a.]$$

Application:

$$\bar{\omega}^a \wedge p_b = \bar{\omega}^b \wedge p_a.$$

[In fact,

$$- \bar{\omega}_{0a} = \varkappa_{ba} \bar{\omega}^b$$

$\Rightarrow$

$$\varkappa_{ba} = q_t(p_b, *\bar{\omega}^a) - \frac{1}{n-2} P_t \delta^b{}_a$$

$\Leftrightarrow$

$$q_t(p_b, *\bar{\omega}^a) = q_t(p_a, *\bar{\omega}^b) \quad (\varkappa_{ba} = \varkappa_{ab})$$

$\Leftrightarrow$

$$p_b \wedge ** \bar{\omega}^a = p_a \wedge ** \bar{\omega}^b$$

$\Rightarrow$

$$p_b \wedge \bar{\omega}^a = p_a \wedge \bar{\omega}^b$$

$\Rightarrow$

$$\bar{\omega}^a \wedge p_b = \bar{\omega}^b \wedge p_a.]$$

<u>Section 42:</u> <u>Elimination of the Metric</u> The assumptions and notation are those of the standard setup.

Let $\vec{\omega}_t$ be the element of $\Lambda^1(\Sigma;T_0^1(\Sigma))$ given by

$$\vec{\omega}_t(X) = \bar{\omega}^a(X)E_a \quad (X\in\mathcal{D}^1(\Sigma)).$$

Then the dynamics can be formulated in terms of $(\vec{\omega}_t,\vec{p}_t)$ as opposed to (q_t,p_t), i.e., there are again constraint equations and evolution equations. While this approach does not lead to new results, the methods are instructive, thus are worth examining.

Let $\underset{=}{Q}$ be the set of ordered coframes on Σ -- then each $\vec{\omega}\in\underset{=}{Q}$ gives rise to a riemannian structure $q\in Q$, viz.

$$q = \omega^a \otimes \omega^a.$$

Conversely, each $q\in Q$ gives rise to a coframe $\vec{\omega}\in\underset{=}{Q}$ which, however, is only determined up to a local rotation.

[Note: At this point, M does not play a role, hence the absence of overbars in the notation.]

Put

$$\begin{cases} T\underset{=}{Q} = \underset{=}{Q} \times \Lambda^1(\Sigma;T_0^1(\Sigma)) \\ T^*\underset{=}{Q} = \underset{=}{Q} \times \Lambda^{n-2}(\Sigma;T_1^0(\Sigma)). \end{cases}$$

Observation: There is a canonical pairing $< , >$

$$\begin{cases} \Lambda^1(\Sigma;T_0^1(\Sigma)) \times \Lambda^{n-2}(\Sigma;T_1^0(\Sigma)) \xrightarrow{\wedge} \Lambda^{n-1}(\Sigma;T_1^1(\Sigma)) \xrightarrow{\int_\Sigma} \underset{=}{R} \\ (\alpha,\beta) \longrightarrow \alpha\wedge\beta \longrightarrow \int_\Sigma \alpha\wedge\beta. \end{cases}$$

[Note: Explicated, on general grounds,

$$\Lambda^1(\Sigma;T_0^1(\Sigma)) \times \Lambda^{n-2}(\Sigma;T_1^0(\Sigma)) \xrightarrow{\wedge} \Lambda^{n-1}(\Sigma;T_0^1(\Sigma) \otimes T_1^0(\Sigma)).$$

But

$$\Lambda^{n-1}(\Sigma;T_1^1(\Sigma)) = \Lambda^{n-1}(\Sigma;T_0^1(\Sigma) \otimes T_1^0(\Sigma))$$

$$= \Lambda^0(\Sigma;T_0^1(\Sigma) \otimes T_1^0(\Sigma)) \otimes_{C^\infty(\Sigma)} \Lambda^{n-1}\Sigma$$

$$= (\Lambda^0(\Sigma;T_0^1(\Sigma)) \otimes_{C^\infty(\Sigma)} \Lambda^0(\Sigma;T_1^0(\Sigma))) \otimes_{C^\infty(\Sigma)} \Lambda^{n-1}\Sigma$$

$$= (\mathcal{D}^1(\Sigma) \otimes_{C^\infty(\Sigma)} \mathcal{D}_1(\Sigma)) \otimes_{C^\infty(\Sigma)} \Lambda^{n-1}\Sigma.$$

One then puts

$$\int_\Sigma X \otimes \omega \otimes \gamma = \langle X,\omega \rangle \cdot \int_\Sigma \gamma.]$$

Consider $T^*\underline{Q} = \underline{Q} \times \Lambda^{n-2}(\Sigma;T_1^0(\Sigma))$ -- then

$$TT^*\underline{Q} = (\underline{Q} \times \Lambda^{n-2}(\Sigma;T_1^0(\Sigma))) \times (\Lambda^1(\Sigma;T_0^1(\Sigma)) \times \Lambda^{n-2}(\Sigma;T_1^0(\Sigma)))$$

$\Rightarrow$

$$T_{(\vec{\omega},\vec{p})}T^*\underline{Q} = \Lambda^1(\Sigma;T_0^1(\Sigma)) \times \Lambda^{n-2}(\Sigma;T_1^0(\Sigma)).$$

<u>The Canonical 1-Form Θ</u> This is the map

$$\Theta_{(\vec{\omega},\vec{p})}:T_{(\vec{\omega},\vec{p})}T^*\underline{Q} \to \underline{R}$$

defined by the prescription

$$\Theta_{(\vec{\omega},\vec{p})}(\alpha,\beta) = \int_{\Sigma} \alpha\wedge\vec{p}.$$

<u>The Canonical 2-Form Ω</u> This is the map

$$\Omega_{(\vec{\omega},\vec{p})} : T_{(\vec{\omega},\vec{p})}T^*\underline{Q} \times T_{(\vec{\omega},\vec{p})}T^*\underline{Q} \rightarrow \underline{R}$$

defined by the prescription

$$\Omega_{(\vec{\omega},\vec{p})}((\alpha,\beta),(\alpha',\beta')) = \int_{\Sigma} (\alpha\wedge\beta' - \alpha'\wedge\beta).$$

<u>LEMMA</u> We have

$$\Omega = - d\Theta.$$

[In fact,

$$d\Theta\Big|_{(\vec{\omega},\vec{p})}((\alpha,\beta),(\alpha',\beta'))$$

$$= \frac{d}{d\varepsilon}\,\Theta_{(\vec{\omega}+\varepsilon\alpha,\vec{p}+\varepsilon\beta)}(\alpha',\beta')\Big|_{\varepsilon=0} - \frac{d}{d\varepsilon}\,\Theta_{(\vec{\omega}+\varepsilon\alpha',\vec{p}+\varepsilon\beta')}(\alpha,\beta)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\,[\int_{\Sigma} \alpha'\wedge(\vec{p}+\varepsilon\beta)]\Big|_{\varepsilon=0} - \frac{d}{d\varepsilon}\,[\int_{\Sigma} \alpha\wedge(\vec{p}+\varepsilon\beta')]\Big|_{\varepsilon=0}$$

$$= \int_{\Sigma} (\alpha'\wedge\beta - \alpha\wedge\beta')$$

$$= - \Omega_{(\vec{\omega},\vec{p})}((\alpha,\beta),(\alpha',\beta')).]$$

Therefore Ω is exact and the pair $(T^*\underline{Q},\Omega)$ is a symplectic manifold.

Suppose given a function $f:T^*\underset{\sim}{Q} \rightarrow \underset{\sim}{R}$.

$\underset{\sim}{\Lambda}^{n-2}$: Write

$$\frac{\delta f}{\delta \vec{\omega}}$$

for that element of $\Lambda^{n-2}(\Sigma;T_1^0(\Sigma))$ characterized by the relation

$$\frac{d}{d\varepsilon} f(\vec{\omega} + \varepsilon\vec{\omega}',\vec{p})\Big|_{\varepsilon=0} = \int_\Sigma \vec{\omega}' \wedge \frac{\delta f}{\delta \vec{\omega}}.$$

$\underset{\sim}{\Lambda}^{1}$: Write

$$\frac{\delta f}{\delta \vec{p}}$$

for that element of $\Lambda^{1}(\Sigma;T_0^1(\Sigma))$ characterized by the relation

$$\frac{d}{d\varepsilon} f(\vec{\omega},\vec{p} + \varepsilon\vec{p}')\Big|_{\varepsilon=0} = \int_\Sigma \frac{\delta f}{\delta \vec{p}} \wedge \vec{p}'.$$

[Note: Both $\frac{\delta f}{\delta \vec{\omega}}$ and $\frac{\delta f}{\delta \vec{p}}$ depend on $(\vec{\omega},\vec{p})$, thus

$$\begin{cases} \dfrac{\delta f}{\delta \vec{\omega}} : T^*\underset{\sim}{Q} \rightarrow \Lambda^{n-2}(\Sigma;T_1^0(\Sigma)) \\[2ex] \dfrac{\delta f}{\delta \vec{p}} : T^*\underset{\sim}{Q} \rightarrow \Lambda^{1}(\Sigma;T_0^1(\Sigma)).] \end{cases}$$

Definition: The <u>hamiltonian vector field</u>

$$X_f:T^*\underset{\sim}{Q} \rightarrow TT^*\underset{\sim}{Q}$$

attached to f is defined by

$$X_f = (\frac{\delta f}{\delta \vec{p}}, - \frac{\delta f}{\delta \vec{\omega}}).$$

To justify the terminology, let X be any vector field on $T^*\underset{\sim}{Q}$. Suppose that $X(\vec{\omega},\vec{p}) = (\vec{\omega}',\vec{p}')$ -- then

$$\Omega_{(\vec{\omega},\vec{p})}(X_f(\vec{\omega},\vec{p}),X(\vec{\omega},\vec{p}))$$

$$= \Omega_{(\vec{\omega},\vec{p})}((\frac{\delta f}{\delta \vec{p}}, - \frac{\delta f}{\delta \vec{\omega}}),(\vec{\omega}',\vec{p}'))$$

$$= \int_\Sigma (\frac{\delta f}{\delta \vec{p}} \wedge \vec{p}' - \vec{\omega}' \wedge - \frac{\delta f}{\delta \vec{\omega}})$$

$$= \int_\Sigma (\frac{\delta f}{\delta \underset{\sim}{\vec{p}}} \wedge \vec{p}' + \vec{\omega}' \wedge \frac{\delta f}{\delta \vec{\omega}})$$

$$= df \Big|_{(\vec{\omega},\vec{p})} (\vec{\omega}',\vec{p}').$$

Example: Each coframe $\vec{\omega}$ determines by duality a frame $\vec{E}$, thus

$$\vec{\omega}(X) = \omega^a(X)E_a \quad (X \in \mathcal{D}^1(\Sigma)).$$

Moreover, $\forall\ \vec{p} \in \Lambda^{n-2}(\Sigma;T_1^0(\Sigma))$,

$$\vec{p}(X_1,\ldots,X_{n-2}) = p_a(X_1,\ldots,X_{n-2})\omega^a.$$

This said, let

$$f(\vec{\omega},\vec{p}) = \int_\Sigma \omega^a \wedge p_a.$$

Then it is clear that

$$\left[\begin{array}{l} \frac{\delta f}{\delta \omega^a} \ (= (\frac{\delta f}{\delta \vec{\omega}})_a) = p_a \\ \\ \frac{\delta f}{\delta p_a} \ (= (\frac{\delta f}{\delta \vec{p}})^a) = \omega^a. \end{array}\right.$$

Definition: The <u>configuration space</u> of the theory is $\underline{Q}$, the <u>velocity phase space</u> of the theory is $T\underline{Q}$, and the <u>momentum phase space</u> of the theory is $T^*\underline{Q}$.

Elements of $\underline{Q}$ are denoted by $\vec{\omega}$, elements of $T\underline{Q}$ are denoted by $(\vec{\omega},\vec{v})$, and elements of $T^*\underline{Q}$ are denoted by $(\vec{\omega},\vec{p})$.

The theory carries three external variables, namely

$$\left[\begin{array}{l} N \in C^\infty_{>0}(\Sigma) \cup C^\infty_{<0}(\Sigma) \\ \\ \vec{N} \in \mathcal{D}^1(\Sigma) \end{array}\right.$$

and

$$W = [W^a{}_b],$$

where $W^a{}_b \in C^\infty(\Sigma)$ and $W^a{}_b = - W^b{}_a$.

Given $(\vec{\omega},\vec{v};N,\vec{N},W)$, put

$$N\omega^a{}_0 = v^a - W^a{}_b\omega^b - \mathcal{L}_{\vec{N}}\,\omega^a.$$

Definition: The <u>lagrangian</u> of the theory is the function

$$L: T\underline{Q} \rightarrow \Lambda^{n-1}\Sigma$$

defined by the rule

$$L(\vec{\omega},\vec{v};N,\vec{N},W)$$

$$= N*(\omega^a \wedge \omega^b) \wedge ({}^{(n-1)}\Omega_{ab} - \omega_{0a} \wedge \omega_{0b}).$$

[Note: As usual, the ${}^{(n-1)}\Omega_{ab}$ are the curvature forms of the metric connection ∇^q associated with q and, of course, the Hodge star is taken per q.]

Let

$$L(\vec{\omega},\vec{v};N,\vec{N},W) = \frac{1}{2} \int_\Sigma L(\vec{\omega},\vec{v};N,\vec{N},W).$$

Then, in order to transfer the theory from $T\underline{Q}$ to $T^*\underline{Q}$, it will be necessary to calculate the functional derivative

$$\frac{\delta L}{\delta \vec{v}}$$

which, a priori, is an element of $\Lambda^{n-2}(\Sigma;T_1^0(\Sigma))$:

$$\frac{d}{d\varepsilon} L(\vec{\omega},\vec{v} + \varepsilon\vec{v}';N,\vec{N},W)\Big|_{\varepsilon=0} = \int_\Sigma \vec{v}' \wedge \frac{\delta L}{\delta \vec{v}}.$$

Notation: Let

$$p_a = \omega_{0b} \wedge *(\omega^a \wedge \omega^b).$$

[Note: Therefore

$$p_a = q(\omega_{0b},\omega^b)*\omega^a - q(\omega_{0b},\omega^a)*\omega^b.]$$

<u>LEMMA</u> We have

$$\frac{\delta L}{\delta v^a} = p_a.$$

[To facilitate the computation, "variational notation" will be employed, i.e., we shall replace the symbol $v^{a'}$ by δv^a and abbreviate D_{v^a} to δ_a -- then

$$\delta_a \frac{1}{2} [N * (\omega^c \wedge \omega^d) \wedge ({}^{(n-1)}\Omega_{cd} - \omega_{0c} \wedge \omega_{0d})]$$

$$= \frac{1}{2} * (\omega^c \wedge \omega^d) \delta_a (- N\omega_{0c}) \wedge \omega_{0d}$$

$$+ \frac{1}{2} * (\omega^c \wedge \omega^d) \omega_{0c} \wedge \delta_a (- N\omega_{0d})$$

$$= \frac{1}{2} * (\omega^a \wedge \omega^d) \wedge \delta v^a \wedge \omega_{0d}$$

$$+ \frac{1}{2} * (\omega^c \wedge \omega^a) \wedge \omega_{0c} \wedge \delta v^a$$

$$= \frac{1}{2} \delta v^a \wedge \omega_{0d} \wedge * (\omega^a \wedge \omega^d)$$

$$+ \frac{1}{2} \omega_{0c} \wedge \delta v^a \wedge * (\omega^c \wedge \omega^a)$$

$$= \frac{1}{2} \delta v^a \wedge \omega_{0b} \wedge * (\omega^a \wedge \omega^b)$$

$$+ \frac{1}{2} - (\delta v^a \wedge \omega_{0b}) \wedge - * (\omega^a \wedge \omega^b)$$

$$= \delta v^a \wedge p_a .]$$

[Note: This result is the reason for the "1/2" prefacing the integral $\int_\Sigma L$.]

Remark: The method employed above for the calculation of $\frac{\delta L}{\delta v^a}$ is widely applicable and will be used without comment whenever it is convenient to do so.

[Note: The interior derivative is not a participant, hence the possibility of misinterpretation is minimal.]

Consider now the fiber derivative of L:

$$\begin{cases} FL: T\underline{Q} \rightarrow T^*\underline{Q} \\ FL(\vec{\omega},\vec{v}) = (\vec{\omega}, \dfrac{\delta L}{\delta \vec{v}}). \end{cases}$$

Then

$$< \vec{v}, \frac{\delta L}{\delta \vec{v}} > - L(\vec{\omega},\vec{v};N,\vec{N},W)$$

$$= \int_\Sigma v^a \wedge p_a - \frac{1}{2} \int_\Sigma N*(\omega^a \wedge \omega^b) \wedge ({}^{(n-1)}\Omega_{ab} - \omega_{0a} \wedge \omega_{0b}).$$

To simplify this, write

$$\int_\Sigma v^a \wedge p_a$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge p_a + \int_\Sigma W^a{}_b \omega^b \wedge p_a - \int_\Sigma N\omega_{0a} \wedge p_a.$$

Let

$$P = q(p_a, *\omega^a) \quad (= (n-2) q(\omega_{0a}, \omega^a)).$$

Then

$$\omega_{0a} = - q(p_b, *\omega^a)\omega^b + \frac{P}{n-2}\,\omega^a$$

$$\Rightarrow$$

$$*\omega_{0a} = - q(p_b, *\omega^a) *\omega^b + \frac{P}{n-2} *\omega^a.$$

Therefore

$$\omega_{0a}\wedge p_a = (-1)^{n-2} p_a\wedge\omega_{0a}$$

$$= (-1)^n p_a\wedge\omega_{0a}$$

$$= (-1)^n p_a\wedge(-1)^{(n-1-1)} **\omega_{0a}$$

$$= p_a\wedge **\omega_{0a}$$

$$= q(p_a, *\omega_{0a})\,\mathrm{vol}_q$$

$$= \left(-\,q(p_a, *\omega^b)q(p_b, *\omega^a) + \frac{P}{n-2}\,q(p_a, *\omega^a)\right)\mathrm{vol}_q$$

$$= \left(-\,q(p_a, *\omega^b)q(p_b, *\omega^a) + \frac{P^2}{n-2}\right)\mathrm{vol}_q$$

$\Rightarrow$

$$-\int_\Sigma N\omega_{0a}\wedge p_a$$

$$= \int_\Sigma N\left(q(p_a, *\omega^b)q(p_b, *\omega^a) - \frac{P^2}{n-2}\right)\mathrm{vol}_q.$$

On the other hand,

$$-\frac{1}{2}\int_\Sigma N*(\omega^a\wedge\omega^b)\wedge({}^{(n-1)}\Omega_{ab} - \omega_{0a}\wedge\omega_{0b})$$

$$= -\frac{1}{2}\int_\Sigma N*(\omega^a\wedge\omega^b)\wedge{}^{(n-1)}\Omega_{ab}$$

$$+\frac{1}{2}\int_\Sigma N*(\omega^a\wedge\omega^b)\wedge(\omega_{0a}\wedge\omega_{0b})$$

$$= -\frac{1}{2}\int_\Sigma NS(q)\,\mathrm{vol}_q$$

$$+ \frac{1}{2} \int_\Sigma N(\omega_{0a} \wedge \omega_{0b}) \wedge * (\omega^a \wedge \omega^b)$$

$$= - \frac{1}{2} \int_\Sigma NS(q) \mathrm{vol}_q$$

$$+ \frac{1}{2} \int_\Sigma N\omega_{0a} \wedge p_a.$$

Consequently,

$$- \int_\Sigma N\omega_{0a} \wedge p_a$$

$$- \frac{1}{2} \int_\Sigma N * (\omega^a \wedge \omega^b) \wedge ({}^{(n-1)}\Omega_{ab} - \omega_{0a} \wedge \omega_{0b})$$

$$= \int_\Sigma \frac{N}{2} [q(p_a, *\omega^b) q(p_b, *\omega^a) - \frac{P^2}{n-2} - S(q)] \mathrm{vol}_q.$$

Motivated by these considerations, let

$$H : T^*\underset{=}{Q} \to \underset{-}{R}$$

be the function defined by the prescription

$$H(\vec{\omega}, \vec{p}; N, \vec{N}, W)$$

$$= \int_\Sigma L_{\vec{N}} \omega^a \wedge p_a + \int_\Sigma W^a{}_b \omega^b \wedge p_a$$

$$+ \int_\Sigma \frac{N}{2} [q(p_a, *\omega^b) q(p_b, *\omega^a) - \frac{P^2}{n-2} - S(q)] \mathrm{vol}_q.$$

[Note: Here the external variable N is unrestricted, i.e., N can be any element of $C^\infty(\Sigma)$.]

Definition: The <u>physical phase space</u> of the theory (a.k.a. the <u>constraint surface</u> of the theory) is the subset $\mathrm{Con}_{\underset{=}{Q}}$ of $T^*\underset{=}{Q}$ whose elements are the points

$(\vec{\omega},\vec{p})$ such that simultaneously

$$\frac{\delta H}{\delta N} = 0,\ \frac{\delta H}{\delta \vec{N}} = 0,\ \frac{\delta H}{\delta W} = 0.$$

The calculation of $\frac{\delta H}{\delta N}$ is trivial. Thus define

$$E: T^*\underline{Q} \to \Lambda^{n-1}\Sigma$$

by

$$E(\vec{\omega},\vec{p}) = \frac{1}{2}\,[q(p_a,*\omega^b)q(p_b,*\omega^a) - \frac{P^2}{n-2} - S(q)]\mathrm{vol}_q.$$

Then

$$\frac{\delta H}{\delta N} = E.$$

Turning to $\frac{\delta H}{\delta W^a{}_b}$, observe that

$$\delta^a{}_b(W^c{}_d\omega^d\wedge p_c)$$

$$= \delta W^a{}_b\omega^b\wedge p_a - \delta W^a{}_b\omega^a\wedge p_b.$$

Therefore

$$\frac{\delta H}{\delta W^a{}_b} = \omega^b\wedge p_a - \omega^a\wedge p_b.$$

There remains the determination of $\frac{\delta H}{\delta \vec{N}}$. To this end, fix a -- then

$$\delta_a[\,L_{\vec{N}}\omega^b\wedge p_b\,]$$

$$= L_{(\delta N^a)E_a}\omega^b\wedge p_b.$$

Write

$$L_{(\delta N^a)E_a}\omega^b \wedge p_b$$

$$= (\iota_{(\delta N^a)E_a} \circ d + d \circ \iota_{(\delta N^a)E_a})\omega^b \wedge p_b$$

$$= \delta N^a(\iota_{E_a} d\omega^b \wedge p_b) + d(\delta N^a \iota_{E_a}\omega^b)\wedge p_b.$$

But

$$d(\delta N^a \iota_{E_a}\omega^b)\wedge p_b$$

$$= d(\delta N^a \omega^b(E_a))\wedge p_b$$

$$= d\delta N^a \wedge p_a.$$

And

$$d(\delta N^a \wedge p_a) = d\delta N^a \wedge p_a + \delta N^a \wedge dp_a$$

$\Rightarrow$

$$d\delta N^a \wedge p_a = d(\delta N^a \wedge p_a) - \delta N^a \wedge dp_a.$$

Since

$$\int_\Sigma d(\delta N^a \wedge p_a) = 0,$$

it follows that

$$\frac{\delta H}{\delta N^a} = - dp_a + \iota_{E_a} d\omega^b \wedge p_b.$$

[Note: The integral

$$\int_\Sigma L_{\vec{N}}\omega^a \wedge p_a$$

can be rewritten as

$$\int_\Sigma N^a I_a.$$

Here

$$I_a : T^*\underline{Q} \to \Lambda^{n-1}\Sigma$$

is defined by

$$I_a(\vec{\omega},\vec{p}) = -\,dp_a + \iota_{E_a} d\omega^b \wedge p_b.]$$

Scholium: $\mathrm{Con}_{\underline{Q}}$ is the subset of $T^*\underline{Q}$ consisting of those pairs $(\vec{\omega},\vec{p})$ such that

$$E(\vec{\omega},\vec{p}) = 0$$

subject to

$$\begin{cases} \omega^a \wedge p_b = \omega^b \wedge p_a \\ \\ -\,dp_a + \iota_{E_a} d\omega^b \wedge p_b = 0. \end{cases}$$

Definition: The <u>ADM sector</u> of $T^*\underline{Q}$ consists of the pairs $(\vec{\omega},\vec{p})$ for which

$$\omega^a \wedge p_b = \omega^b \wedge p_a.$$

In the ADM sector of $T^*\underline{Q}$, the functional derivative $\frac{\delta H}{\delta N_a}$ can be expressed in terms of the $\underline{R}$-linear operator

$$d^{\nabla^q} : \Lambda^{n-2}(\Sigma; T^0_1(\Sigma)) \to \Lambda^{n-1}(\Sigma; T^0_1(\Sigma)).$$

To see this, recall that

$$d^{\nabla^q} p_a = dp_a - \omega^b{}_a \wedge p_b.$$

And

$$d\omega^b = - \omega^b{}_c \wedge \omega^c$$

$\Rightarrow$

$$\iota_{E_a} d\omega^b = - \iota_{E_a} (\omega^b{}_c \wedge \omega^c)$$

$$= - [\iota_{E_a} \omega^b{}_c \wedge \omega^c - \omega^b{}_c \wedge \iota_{E_a} \omega^c]$$

$$= - [\omega^b{}_c (E_a) \wedge \omega^c - \omega^b{}_c \wedge \omega^c (E_a)]$$

$$= - \omega^b{}_c (E_a) \wedge \omega^c + \omega^b{}_a .$$

Therefore

$$- dp_a + \iota_{E_a} d\omega^b \wedge p_b$$

$$= - d^{\nabla^q} p_a - \omega^b{}_a \wedge p_b$$

$$- \omega^b{}_c (E_a) \omega^c \wedge p_b + \omega^b{}_a \wedge p_b$$

$$= - d^{\nabla^q} p_a - \omega^b{}_c (E_a) \omega^c \wedge p_b .$$

But

$$\omega^c \wedge p_b = \omega^b \wedge p_c$$

$\Rightarrow$

$$- \omega^b{}_c (E_a) \omega^c \wedge p_b$$

$$= - \omega^b{}_c (E_a) \omega^b \wedge p_c$$

$$= \omega^c{}_b(E_a)\omega^b \wedge p_c$$

$$= \omega^b{}_c(E_a)\omega^c \wedge p_b$$

$\Rightarrow$

$$\frac{\delta H}{\delta N^a} = - d^{\nabla^q} p_a.$$

Since $n = \dim M > 2$, the vanishing of $\mathrm{Ein}(g)$ is equivalent to the vanishing of $\mathrm{Ric}(g)$ and for the latter, conditions have been given in terms of the path $t \to (q_t, \varkappa_t)$ in TQ or the path $t \to (q_t, p_t)$ in T*Q. However, one can also work instead with the path $t \to (\vec{\omega}_t, \vec{p}_t)$ in $T^*\underline{Q}$, there being, as always, two aspects to the analysis: Constraints (i.e., the $G_{0i} = 0$ equations) and evolution (i.e., the $G_{ab} = 0$ equations). In the next section, we shall treat the constraints and, in the section after that, evolution.

Rappel: The symmetry of the extrinsic curvature implies that the components p_a of the momentum form $\vec{p}_t$ satisfy the constraint

$$\bar{\omega}^a \wedge p_b = \bar{\omega}^b \wedge p_a,$$

i.e., the path $t \to (\vec{\omega}_t, \vec{p}_t)$ lies in the ADM sector of $T^*\underline{Q}$.

Section 43: Constraints in the Coframe Picture The assumptions and notation are those of the standard setup.

Rappel: $\forall$ t,

$$\bar{G}_{00} = \frac{1}{2} S(q_t) + \frac{1}{2} (K_t^2 - [\varkappa_t, \varkappa_t]_{q_t}).$$

LEMMA $\forall$ t,

$$\frac{\delta H}{\delta N}\Big|_{(\vec{\omega}_t, \vec{p}_t)} = - \bar{G}_{00} \text{vol}_{q_t}.$$

[Since

$$\varkappa_{ab} = q_t(p_a, *\bar{\omega}^b) - \frac{1}{n-2} P_t \delta_{ab},$$

we have

1. $$- [\varkappa_t, \varkappa_t]_{q_t} = - \varkappa^{ab}\varkappa_{ab} = - \varkappa_{ab}\varkappa_{ab} = - \varkappa_{ab}\varkappa_{ba}$$

$$= \frac{2}{n-2} P_t \delta_{ab} q_t(p_a, *\bar{\omega}^b) - q_t(p_a, *\bar{\omega}^b) q_t(p_b, *\bar{\omega}^a) - \frac{1}{(n-2)^2} P_t^2 \delta_{ab}\delta_{ab}$$

$$= \frac{2}{n-2} P_t q_t(p_a, *\bar{\omega}^a) - q_t(p_a, *\bar{\omega}^b) q_t(p_b, *\bar{\omega}^a) - \frac{(n-1)}{(n-2)^2} P_t^2$$

$$= \frac{2}{n-2} P_t^2 - q_t(p_a, *\bar{\omega}^b) q_t(p_b, *\bar{\omega}^a) - \frac{(n-1)}{(n-2)^2} P_t^2$$

$$= [\frac{2}{n-2} - \frac{(n-1)}{(n-2)^2}] P_t^2 - q_t(p_a, *\bar{\omega}^b) q_t(p_b, *\bar{\omega}^a)$$

$$= \frac{n-3}{(n-2)^2} P_t^2 - q_t(p_a, *\bar{\omega}^b) q_t(p_b, *\bar{\omega}^a).$$

2. $$K_t^2 = \varkappa_{aa}\varkappa_{bb}$$

$$= \sum_a (q_t(p_a, *\bar{\omega}^a) - \frac{1}{n-2} P_t) \cdot \sum_b (q_t(p_b, *\bar{\omega}^b) - \frac{1}{n-2} P_t)$$

$$= (P_t - \frac{n-1}{n-2} P_t) \cdot (P_t - \frac{n-1}{n-2} P_t)$$

$$= \frac{1}{(n-2)^2} P_t^2.$$

Therefore

$$\frac{1}{2} [q_t(p_a, *\bar{\omega}^b) q_t(p_b, *\bar{\omega}^a) - \frac{1}{n-2} P_t^2 - S(q_t)]$$

$$= \frac{1}{2}[[\varkappa_t, \varkappa_t]_{q_t} + \frac{n-3}{(n-2)^2} P_t^2] - \frac{1}{2(n-2)} P_t^2 - \frac{1}{2} S(q_t)$$

$$= -\frac{1}{2} S(q_t) + \frac{1}{2} [\varkappa_t, \varkappa_t]_{q_t} + [\frac{n-3}{2(n-2)^2} - \frac{1}{2(n-2)}] P_t^2$$

$$= -\frac{1}{2} S(q_t) - \frac{1}{2(n-2)^2} P_t^2 + \frac{1}{2} [\varkappa_t, \varkappa_t]_{q_t}$$

$$= -\frac{1}{2} S(q_t) - \frac{1}{2} K_t^2 + \frac{1}{2} [\varkappa_t, \varkappa_t]_{q_t}$$

$$= -\frac{1}{2} S(q_t) - \frac{1}{2} (K_t^2 - [\varkappa_t, \varkappa_t]_{q_t})$$

$$= -\bar{G}_{00}$$

$$\Rightarrow$$

$$\left.\frac{\delta H}{\delta N}\right|_{(\vec{\omega}_t, \vec{P}_t)} = -\bar{G}_{00} \mathrm{vol}_{q_t}.]$$

Rappel: $\forall$ t,

$$\bar{G}_{0a} = \bar{\nabla}_b \varkappa_{ab} - \bar{\nabla}_a K_t.$$

[Note: $\bar{\nabla}$ stands for ∇^{q_t}.]

<u>LEMMA</u> $\forall$ t,

$$\frac{\delta H}{\delta N^a}\bigg|_{(\vec{\omega}_t, \vec{p}_t)} = - \bar{G}_{0a} \mathrm{vol}_{q_t}.$$

It suffices to deal with

$$- d^{\bar{\nabla}} p_a$$

as opposed to

$$- dp_a + \iota_{E_a} d\bar{\omega}^b \wedge p_b.$$

[Note: Bear in mind that

$$\bar{\omega}^a \wedge p_b = \bar{\omega}^b \wedge p_a.]$$

This said,

$$p_a = (\varkappa_t - K_t q_t)_{ab} * \bar{\omega}_b$$

$$\Rightarrow$$

$$d^{\bar{\nabla}} p_a = d^{\bar{\nabla}} (\varkappa_t - K_t q_t)_{ab} \wedge * \bar{\omega}_b$$

$$+ (\varkappa_t - K_t q_t)_{ab} d^{\bar{\nabla}} * \bar{\omega}_b.$$

Using the definitions, one finds that

$$d^{\bar{\nabla}}(\varkappa_t - K_t q_t)_{ab} \wedge * \bar{\omega}_b$$

$$= d^{\bar{\nabla}}(\varkappa_t - K_t q_t)_{ab} \wedge \iota_{E_b} \mathrm{vol}_{q_t}$$

$$= \bar{\nabla}_b(\varkappa_t - K_t q_t)_{ab} \mathrm{vol}_{q_t} \qquad \text{(see below)}$$

$$= (\bar{\nabla}_b \varkappa_{ab} - \delta_{ab} \bar{\nabla}_b K_t) \mathrm{vol}_{q_t}$$

$$= (\bar{\nabla}_b \varkappa_{ab} - \bar{\nabla}_a K_t) \mathrm{vol}_{q_t}$$

$$= \bar{G}_{0a} \mathrm{vol}_{q_t}.$$

On the other hand,

$$d^{\bar{\nabla}} * \bar{\omega}^b = d * \bar{\omega}^b + \bar{\omega}^b{}_c \wedge * \bar{\omega}^c$$

$$= - \bar{\omega}^b{}_c \wedge * \bar{\omega}^c + \bar{\omega}^b{}_c \wedge * \bar{\omega}^c$$

$$= 0.$$

Therefore

$$\left.\frac{\delta H}{\delta N^a}\right|_{(\vec{\omega}_t, \vec{p}_t)} = - \bar{G}_{0a} \mathrm{vol}_{q_t}.$$

<u>Details</u> The claim is that

$$d^{\bar{\nabla}}(\varkappa_t - K_t q_t)_{ab} \wedge \iota_{E_b} \mathrm{vol}_{q_t}$$

$$= \bar{\nabla}_b(\varkappa_t - K_t q_t)_{ab} \mathrm{vol}_{q_t},$$

a relation which is a special case of the following generalities. Thus let

$$T = T_{ab}\omega^a \otimes \omega^b \in \mathcal{D}_2^0(\Sigma).$$

Fix a $\nabla \in \text{con}\, T\Sigma$ — then $\forall\, X \in \mathcal{D}^1(\Sigma)$,

$$\nabla_X T = \nabla_X (T_{ab}\omega^a \otimes \omega^b)$$

$$= (XT_{ab})(\omega^a \otimes \omega^b) + T_{ab}(\nabla_X \omega^a) \otimes \omega^b + T_{ab}\omega^a \otimes \nabla_X \omega^b$$

$$= (XT_{ab})(\omega^a \otimes \omega^b) + T_{ab}(-\omega^a{}_c(X)\omega^c) \otimes \omega^b + T_{ab}\omega^a \otimes (-\omega^b{}_d(X)\omega^d)$$

$$= dT_{ab}(X)(\omega^a \otimes \omega^b) - \omega^c{}_a(X)T_{cb}(\omega^a \otimes \omega^b) - \omega^d{}_b(X)T_{ad}(\omega^a \otimes \omega^b)$$

$$= (dT_{ab}(X) - \omega^c{}_a(X)T_{cb} - \omega^d{}_b(X)T_{ad})(\omega^a \otimes \omega^b)$$

$$= \langle X, dT_{ab} - \omega^c{}_a \wedge T_{cb} - \omega^d{}_b \wedge T_{ad} \rangle (\omega^a \otimes \omega^b)$$

$$= \langle X, d^\nabla T_{ab} \rangle (\omega^a \otimes \omega^b).$$

But

1. $\nabla T = \nabla T(E_r, E_s, E_c)\omega^r \otimes \omega^s \otimes \omega^c$

$$= (\nabla_{E_c} T)(E_r, E_s)\omega^r \otimes \omega^s \otimes \omega^c$$

$$= \langle E_c, d^\nabla T_{rs} \rangle \omega^r \otimes \omega^s \otimes \omega^c.$$

2. $\omega^a \otimes \omega^b \otimes d^\nabla T_{ab}(E_r, E_s, E_c)$

$$= \omega^a(E_r)\omega^b(E_s)d^\nabla T_{ab}(E_c)$$

$$= d^\nabla T_{rs}(E_c)$$

$$= < E_c, d^\nabla T_{rs} >.$$

Therefore

$$\nabla T = \omega^a \otimes \omega^b \otimes d^\nabla T_{ab}$$

$$\Rightarrow$$

$$\nabla_b T_{ab} = T_{ab;b}$$

$$= (\nabla T)_{abb}$$

$$= \nabla T(E_a, E_b, E_b)$$

$$= d^\nabla T_{ab}(E_b)$$

$$= \iota_{E_b} d^\nabla T_{ab}.$$

And, $\forall\ q \in Q$,

$$0 = \iota_{E_b}(d^\nabla T_{ab} \wedge \text{vol}_q)$$

$$= \iota_{E_b} d^\nabla T_{ab} \wedge \text{vol}_q - d^\nabla T_{ab} \wedge \iota_{E_b} \text{vol}_q$$

$$\Rightarrow$$

$$d^\nabla T_{ab} \wedge \iota_{E_b} \text{vol}_q = \iota_{E_b} d^\nabla T_{ab} \wedge \text{vol}_q$$

$$= \nabla_b T_{ab} \mathrm{vol}_q.$$

[Note: These considerations apply in particular to the choices $q = q_t$, $\nabla = \bar{\nabla}$ ($= \nabla^{q_t}$), and $T = \varkappa_t - K_t q_t$.]

<u>Section 44:</u> <u>Evolution in the Coframe Picture</u> The assumptions and notation are those of the standard setup.

Rappel:

$$H(\vec{\omega},\vec{p};N,\vec{N},W)$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge p_a + \int_\Sigma W^a{}_b\omega^b \wedge p_a + \int_\Sigma NE,$$

where

$$E(\vec{\omega},\vec{p}) = \frac{1}{2}\,[q(p_a,*\omega^b)q(p_b,*\omega^a) - \frac{P^2}{n-2} - S(q)]\mathrm{vol}_q.$$

There are now two central objectives:

1. Compute $\dfrac{\delta H}{\delta p_a}$;

2. Compute $\dfrac{\delta H}{\delta \omega^a}$.

We shall start with $\dfrac{\delta H}{\delta p_a}$, which turns out to be the easier of the two.

Obviously

$$\frac{\delta H}{\delta p_a} = L_{\vec{N}}\omega^a + W^a{}_b\omega^b + \frac{\delta}{\delta p_a}\,[\int_\Sigma NE].$$

And:

I. $\delta_a \frac{1}{2}\,(q(p_b,*\omega^c)q(p_c,*\omega^b)\mathrm{vol}_q) = q(p_b,*\omega^a)\omega^b\wedge\delta p_a.$

II. $\delta_a\,(-\frac{P^2}{2(n-2)}\,\mathrm{vol}_q) = -\frac{P}{n-2}\,\omega^a\wedge\delta p_a.$

Granted I and II, it follows that

$$\frac{\delta}{\delta p_a}\,[\int_\Sigma NE] = N(q(p_b,*\omega^a)\omega^b - \frac{P}{n-2}\,\omega^a).$$

Ad I: Consider

$$q(p_b,*\omega^c)q(p_c,*\omega^b)\mathrm{vol}_q.$$

Then

$$\left[\begin{array}{l} q(p_b,*\omega^c)\mathrm{vol}_q = p_b\wedge{*}{*}\omega^c = \omega^c\wedge p_b \\ \\ q(p_c,*\omega^b) = *(\omega^b\wedge p_c) \end{array}\right.$$

$\Rightarrow$

$$\delta_a(q(p_b,*\omega^c)q(p_c,*\omega^b)\mathrm{vol}_q)$$

$$= \delta_a((\omega^c\wedge p_b)\wedge*(\omega^b\wedge p_c))$$

$$= \omega^c\wedge\delta_a p_b\wedge*(\omega^b\wedge p_c) + (\omega^c\wedge p_b)\wedge*(\omega^b\wedge\delta_a p_c).$$

But

- $\omega^c\wedge\delta_a p_b\wedge*(\omega^b\wedge p_c)$

$$= \omega^c\wedge\delta p_a\wedge*(\omega^a\wedge p_c)$$

$$= \omega^b\wedge\delta p_a\wedge*(\omega^a\wedge p_b)$$

$$= \omega^b\wedge\delta p_a\wedge q(p_b,*\omega^a)$$

$$= q(p_b,*\omega^a)\,\omega^b\wedge\delta p_a.$$

- $(\omega^c\wedge p_b)\wedge*(\omega^b\wedge\delta_a p_c)$

$$= (\omega^a\wedge p_b)\wedge*(\omega^b\wedge\delta p_a)$$

$$= q(p_b,*\omega^a)\mathrm{vol}_q \wedge*(\omega^b\wedge\delta p_a)$$

$$= q(p_b, *\omega^a) * (\omega^b \wedge \delta p_a) \mathrm{vol}_q$$

$$= q(p_b, *\omega^a) \omega^b \wedge \delta p_a .$$

So

$$\delta_a (q(p_b, *\omega^c) q(p_c, *\omega^b) \mathrm{vol}_q)$$

$$= 2q(p_b, *\omega^a) \omega^b \wedge \delta p_a ,$$

thereby establishing I.

Ad II:

$$\delta_a (- \frac{P^2}{2(n-2)} \mathrm{vol}_q)$$

$$= - \frac{P}{n-2} (\delta_a P) \mathrm{vol}_q$$

$$= - \frac{P}{n-2} q(\delta p_a, *\omega^a) \mathrm{vol}_q$$

$$= - \frac{P}{n-2} \omega^a \wedge \delta p_a .$$

Summary: We have

$$\frac{\delta H}{\delta p_a} = L_{\vec{N}} \omega^a + W^a{}_b \omega^b + N(q(p_b, *\omega^a) \omega^b - \frac{P}{n-2} \omega^a) .$$

The calculation of $\frac{\delta H}{\delta \omega^a}$ is more difficult. However, it is at least clear that

$$\frac{\delta H}{\delta \omega^a} = - L_{\vec{N}} p_a + W^b{}_a p_b + \frac{\delta}{\delta \omega^a} [\int_\Sigma NE] .$$

[Note: Pinned down,

$$\int_\Sigma L_{\vec{N}}(\omega^a \wedge p_a)$$

$$= \int_\Sigma (\iota_{\vec{N}} \circ d + d \circ \iota_{\vec{N}})(\omega^a \wedge p_a)$$

$$= \int_\Sigma d(\iota_{\vec{N}}(\omega^a \wedge p_a))$$

$$= 0$$

$\Rightarrow$

$$\int_\Sigma L_{\vec{N}}\omega^a \wedge p_a = - \int_\Sigma \omega^a \wedge L_{\vec{N}} p_a .]$$

<u>LEMMA</u> We have

$$\delta_a \mathrm{vol}_q = \delta\omega^a \wedge *\omega^a .$$

[There are two points:

1. $\mathrm{vol}_q = \frac{1}{(n-1)!} \varepsilon_{b_1 \ldots b_{n-1}} \omega^{b_1} \wedge \ldots \wedge \omega^{b_{n-1}}$.

2. $*\omega^a = \frac{1}{(n-2)!} \varepsilon_{ac_2 \ldots c_{n-1}} \omega^{c_2} \wedge \ldots \wedge \omega^{c_{n-1}}$.

Accordingly,

$$\delta_a \mathrm{vol}_q = \frac{1}{(n-1)!} \varepsilon_{b_1 \ldots b_{n-1}} \delta_a \omega^{b_1} \wedge \ldots \wedge \omega^{b_{n-1}}$$

$$+ \cdots + \frac{1}{(n-1)!} \varepsilon_{b_1 \ldots b_{n-1}} \omega^{b_1} \wedge \ldots \wedge \delta_a \omega^{b_{n-1}}$$

$$= \frac{1}{(n-1)!}\,\varepsilon_{ab_2\ldots b_{n-1}}\,\delta\omega^a\wedge\omega^{b_2}\wedge\ldots\wedge\omega^{b_{n-1}}$$

$$+ \cdots + \frac{1}{(n-1)!}\,\varepsilon_{b_1\ldots b_{n-2}a}\,\omega^{b_1}\wedge\ldots\wedge\omega^{b_{n-2}}\wedge\delta\omega^a$$

$$= \frac{(n-1)}{(n-1)!}\,\varepsilon_{ac_2\ldots c_{n-1}}\,\delta\omega^a\wedge\omega^{c_2}\wedge\ldots\wedge\omega^{c_{n-1}}$$

$$= \delta\omega^a\wedge\frac{1}{(n-2)!}\,\varepsilon_{ac_2\ldots c_{n-1}}\,\omega^{c_2}\wedge\ldots\wedge\omega^{c_{n-1}}$$

$$= \delta\omega^a\wedge *\omega^a.]$$

Claim:

I. $$\delta_a\,\frac{1}{2}\,(q(p_b,*\omega^c)\,q(p_c,*\omega^b)\,\mathrm{vol}_q)$$

$$= \delta\omega^a\wedge(q(p_a,*\omega^b)\,p_b - \frac{1}{2}\,q(p_b,*\omega^c)\,q(p_c,*\omega^b)\,*\omega^a).$$

II. $$\delta_a(-\frac{p^2}{2(n-2)}\,\mathrm{vol}_q) = -\,\delta\omega^a\wedge(\frac{p}{n-2}\,p_a - \frac{p^2}{2(n-2)}\,*\omega^a).$$

Ad I: Proceeding as above, write

$$q(p_b,*\omega^c)\,q(p_c,*\omega^b)\,\mathrm{vol}_q$$

$$= (\omega^c\wedge p_b)\wedge*(\omega^b\wedge p_c).$$

Then

$$\delta_a((\omega^c\wedge p_b)\wedge*(\omega^b\wedge p_c))$$

$$= \delta_a(\omega^c \wedge p_b) \wedge *(\omega^b \wedge p_c) + (\omega^c \wedge p_b) \wedge \delta_a *(\omega^b \wedge p_c)$$

$$= \delta\omega^a \wedge p_b \wedge *(\omega^b \wedge p_a) + (\omega^c \wedge p_b) \wedge \delta_a *(\omega^b \wedge p_c)$$

$$= \delta\omega^a \wedge q(p_a, *\omega^b) p_b + (\omega^c \wedge p_b) \wedge \delta_a *(\omega^b \wedge p_c).$$

Next

$$\delta_a(*(\omega^b \wedge p_c) \wedge \mathrm{vol}_q)$$

$$= \delta_a *(\omega^b \wedge p_c) \wedge \mathrm{vol}_q + *(\omega^b \wedge p_c) \wedge \delta_a \mathrm{vol}_q.$$

Therefore

$$(\omega^c \wedge p_b) \wedge \delta_a *(\omega^b \wedge p_c)$$

$$= q(p_b, *\omega^c)\, \mathrm{vol}_q \wedge \delta_a *(\omega^b \wedge p_c)$$

$$= q(p_b, *\omega^c)\, \delta_a *(\omega^b \wedge p_c) \wedge \mathrm{vol}_q$$

$$= q(p_b, *\omega^c)\, (\delta_a(*(\omega^b \wedge p_c) \wedge \mathrm{vol}_q)$$

$$- *(\omega^b \wedge p_c) \wedge \delta_a \mathrm{vol}_q).$$

But

$$\delta_a(*(\omega^b \wedge p_c) \wedge \mathrm{vol}_q)$$

$$= \delta_a(\mathrm{vol}_q \wedge *(\omega^b \wedge p_c))$$

$$= \delta_a((\omega^b \wedge p_c) \wedge *\mathrm{vol}_q)$$

$$= \delta_a(\omega^b \wedge p_c)$$

$$= \delta_a \omega^b \wedge p_c.$$

So, in view of the lemma, it follows that

$$(\omega^c \wedge p_b) \wedge \delta_a * (\omega^b \wedge p_c)$$

$$= q(p_b, *\omega^c)(\delta_a \omega^b \wedge p_c - *(\omega^b \wedge p_c)\delta\omega^a \wedge *\omega^a)$$

$$= q(p_b, *\omega^c)\delta_a \omega^b \wedge p_c$$

$$- q(p_b, *\omega^c) * (\omega^b \wedge p_c)\delta\omega^a \wedge *\omega^a$$

$$= q(p_a, *\omega^b)\delta\omega^a \wedge p_b$$

$$- q(p_b, *\omega^c) q(p_c, *\omega^b)\delta\omega^a \wedge *\omega^a$$

$$= \delta\omega^a \wedge (q(p_a, *\omega^b) p_b - q(p_b, *\omega^c) q(p_c, *\omega^b) *\omega^a).$$

<u>Ad II</u>:

$$\delta_a\left(-\frac{P^2}{2(n-2)} \mathrm{vol}_q\right)$$

$$= -\frac{P}{n-2}(\delta_a P)\mathrm{vol}_q - \frac{P^2}{2(n-2)}\delta_a \mathrm{vol}_q$$

$$= -\frac{P}{n-2}(\delta_a(P\mathrm{vol}_q) - P\delta_a \mathrm{vol}_q)$$

$$- \frac{P^2}{2(n-2)} \delta_a \mathrm{vol}_q$$

$$= - \frac{P}{n-2} (\delta_a(\omega^b \wedge p_b) - P\delta\omega^a \wedge *\omega^a)$$

$$- \frac{P^2}{2(n-2)} \delta\omega^a \wedge *\omega^a$$

$$= \delta\omega^a \wedge (- \frac{P}{n-2} p_a)$$

$$+ \delta\omega^a \wedge (\frac{1}{n-2} - \frac{1}{2(n-2)}) P^2 *\omega^a$$

$$= - \delta\omega^a \wedge (\frac{P}{n-2} p_a - \frac{P^2}{2(n-2)} *\omega^a).$$

Now add I and II to get:

$$\delta_a \frac{1}{2} (q(p_b, *\omega^c) q(p_c, *\omega^b) \mathrm{vol}_q)$$

$$+ \delta_a (- \frac{P^2}{2(n-2)} \mathrm{vol}_q)$$

$$= \delta\omega^a \wedge (q(p_a, *\omega^b) p_b - \frac{P}{n-2} p_a)$$

$$+ \delta\omega^a \wedge \frac{1}{2} (\frac{P^2}{n-2} - q(p_b, *\omega^c) q(p_c, *\omega^b)) *\omega^a.$$

It remains to evaluate

$$\delta_a (- \frac{1}{2} S(q) \mathrm{vol}_q)$$

or still,

$$\delta_a (- \frac{1}{2} \Omega_{bc} \wedge *(\omega^b \wedge \omega^c))$$

or still,

$$-\frac{1}{2}\,[\delta_a\Omega_{bc}\wedge *(\omega^b\wedge\omega^c) + \Omega_{bc}\wedge\delta_a *(\omega^b\wedge\omega^c)].$$

<u>LEMMA</u> We have

$$\delta_a *(\omega^b\wedge\omega^c) = \delta\omega^a\wedge *(\omega^a\wedge\omega^b\wedge\omega^c).$$

[In fact,

$$\delta_a *(\omega^b\wedge\omega^c)$$

$$= \delta_a(\frac{1}{(n-3)!}\,\varepsilon_{bcd_3\ldots d_{n-1}}\omega^{d_3}\wedge\ldots\wedge\omega^{d_{n-1}})$$

$$= \frac{1}{(n-3)!}\,\varepsilon_{bcd_3\ldots d_{n-1}}\delta_a\omega^{d_3}\wedge\ldots\wedge\omega^{d_{n-1}}$$

$$+ \cdots + \frac{1}{(n-3)!}\,\varepsilon_{bcd_3\ldots d_{n-1}}\omega^{d_3}\wedge\ldots\wedge\delta_a\omega^{d_{n-1}}$$

$$= \frac{1}{(n-3)!}\,\varepsilon_{bcad_4\ldots d_{n-1}}\delta\omega^a\wedge\omega^{d_4}\wedge\ldots\wedge\omega^{d_{n-1}}$$

$$+ \cdots + \frac{1}{(n-3)!}\,\varepsilon_{bcd_3\ldots d_{n-2}a}\omega^{d_3}\wedge\ldots\wedge\omega^{d_{n-2}}\wedge\delta\omega_a$$

$$= \frac{(n-3)}{(n-3)!}\,\varepsilon_{bcad_4\ldots d_{n-1}}\delta\omega^a\wedge\omega^{d_4}\wedge\ldots\wedge\omega^{d_{n-1}}$$

$$= \delta\omega^a\wedge\frac{1}{(n-4)!}\,\varepsilon_{abcd_4\ldots d_{n-1}}\omega^{d_4}\wedge\ldots\wedge\omega^{d_{n-1}}$$

$$= \delta\omega^a\wedge *(\omega^a\wedge\omega^b\wedge\omega^c).]$$

Application:

$$\Omega_{bc}\wedge\delta_a*(\omega^b\wedge\omega^c) = \delta\omega^a\wedge\Omega_{bc}\wedge*(\omega^a\wedge\omega^b\wedge\omega^c).$$

Rappel: By definition,

$$\iota_{E_a}\Omega^a{}_b \quad (= \iota_{E_a}\Omega_{ab})$$

is the Ricci 1-form Ric_b, hence

$$\begin{aligned} Ric_b(E_b) &= \Omega_{ab}(E_a,E_b) \\ &= R_{abab} \\ &= R^a{}_{bab} \\ &= S(q). \end{aligned}$$

[Note: There is an expansion

$$Ric_b = R_{bc}\omega^c,$$

where

$$R_{bc} = R_{cb}$$

$\Rightarrow$

$$q(Ric_b,\omega^c) = q(Ric_c,\omega^b).]$$

<u>LEMMA</u> Let $\alpha\in\Lambda^2\Sigma$ -- then

$$\alpha\wedge*(\omega^a\wedge\omega^b\wedge\omega^c)$$

$$= q(\alpha,\omega^a\wedge\omega^b)*\omega^c + q(\alpha,\omega^b\wedge\omega^c)*\omega^a + q(\alpha,\omega^c\wedge\omega^a)*\omega^b.$$

[For any index d between 1 and n-1,

$$q(\alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c), *\omega^d)\mathrm{vol}_q$$

$$= \alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c)\wedge **\omega^d$$

$$= (-1)^n\alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c)\wedge\omega^d$$

$$= (-1)^n(-1)^{n-4}\alpha\wedge\omega^d\wedge *(\omega^a\wedge\omega^b\wedge\omega^c)$$

$$= \omega^d\wedge\alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c)$$

$$= q(\omega^d\wedge\alpha, \omega^a\wedge\omega^b\wedge\omega^c)\mathrm{vol}_q$$

$$= \iota_{\omega^d\wedge\alpha}(\omega^a\wedge\omega^b\wedge\omega^c)\mathrm{vol}_q$$

$$= \iota_\alpha\iota_{\omega^d}(\omega^a\wedge\omega^b\wedge\omega^c)\mathrm{vol}_q,$$

an expression which surely vanishes if $d \neq a,b,c$. Therefore

$$\alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c) = C_a *\omega^a + C_b *\omega^b + C_c *\omega^c.$$

Here

$$\begin{cases} C_a = q(\alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c), *\omega^a) \\ C_b = q(\alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c), *\omega^b) \\ C_c = q(\alpha\wedge *(\omega^a\wedge\omega^b\wedge\omega^c), *\omega^c). \end{cases}$$

But

$$\bullet\, C_a \mathrm{vol}_q = q(\alpha\wedge*(\omega^a\wedge\omega^b\wedge\omega^c), *\omega^a)\mathrm{vol}_q$$

$$= \iota_\alpha \iota_{\omega^a}(\omega^a\wedge\omega^b\wedge\omega^c)\mathrm{vol}_q$$

$$= \iota_\alpha(\omega^b\wedge\omega^c)\mathrm{vol}_q$$

$$= q(\alpha,\omega^b\wedge\omega^c)\mathrm{vol}_q.$$

$$\bullet\, C_b \mathrm{vol}_q = q(\alpha\wedge*(\omega^a\wedge\omega^b\wedge\omega^c), *\omega^b)\mathrm{vol}_q$$

$$= \iota_\alpha \iota_{\omega^b}(\omega^a\wedge\omega^b\wedge\omega^c)\mathrm{vol}_q$$

$$= - \iota_\alpha(\omega^a\wedge\omega^c)\mathrm{vol}_q$$

$$= \iota_\alpha(\omega^c\wedge\omega^a)\mathrm{vol}_q$$

$$= q(\alpha,\omega^c\wedge\omega^a)\mathrm{vol}_q.$$

$$\bullet\, C_c \mathrm{vol}_q = q(\alpha\wedge*(\omega^a\wedge\omega^b\wedge\omega^c), *\omega^c)\mathrm{vol}_q$$

$$= \iota_\alpha \iota_{\omega^c}(\omega^a\wedge\omega^b\wedge\omega^c)\mathrm{vol}_q$$

$$= \iota_\alpha(\omega^a\wedge\omega^b)\mathrm{vol}_q$$

$$= q(\alpha,\omega^a\wedge\omega^b)\mathrm{vol}_q.]$$

Application:

$$\Omega_{bc}\wedge*(\omega^a\wedge\omega^b\wedge\omega^c)$$

$$= q(\Omega_{bc},\omega^a\wedge\omega^b)*\omega^c + q(\Omega_{bc},\omega^b\wedge\omega^c)*\omega^a + q(\Omega_{bc},\omega^c\wedge\omega^a)*\omega^b.$$

• $q(\Omega_{bc},\omega^a\wedge\omega^b)*\omega^c$

$$= - q(\Omega_{bc},\omega^b\wedge\omega^a)*\omega^c$$

$$= - q(\iota_{\omega^b}\Omega_{bc},\omega^a)*\omega^c$$

$$= - q(\iota_{E_b}\Omega_{bc},\omega^a)*\omega^c$$

$$= - q(\mathrm{Ric}_c,\omega^a)*\omega^c$$

$$= - q(\mathrm{Ric}_a,\omega^c)*\omega^c$$

$$= - *(\mathrm{Ric}_a).$$

• $q(\Omega_{bc},\omega^b\wedge\omega^c)*\omega^a$

$$= \iota_{\omega^b\wedge\omega^c}\Omega_{bc}*\omega^a$$

$$= \iota_{\omega^c}\iota_{\omega^b}\Omega_{bc}*\omega^a$$

$$= (\iota_{\omega^b}\Omega_{bc})(E_c)*\omega^a$$

$$= \Omega_{bc}(E_b,E_c)*\omega^a$$

$$= S(q)*\omega^a.$$

$$\bullet\ q(\Omega_{bc}, \omega^c \wedge \omega^a) * \omega^b$$

$$= -\ q(\Omega_{cb}, \omega^c \wedge \omega^a) * \omega^b$$

$$= -\ q(\iota_{\omega^c} \Omega_{cb}, \omega^a) * \omega^b$$

$$= -\ q(\iota_{E_c} \Omega_{cb}, \omega^a) * \omega^b$$

$$= -\ q(Ric_b, \omega^a) * \omega^b$$

$$= -\ q(Ric_a, \omega^b) * \omega^b$$

$$= -\ *(Ric_a).$$

Therefore

$$-\frac{1}{2}\, \Omega_{bc} \wedge \delta_a * (\omega^b \wedge \omega^c)$$

$$= \delta\omega^a \wedge \left(-\frac{1}{2}\left(-2*(Ric_a) + S(q) * \omega^a \right) \right)$$

$$= \delta\omega^a \wedge * \left(Ric_a - \frac{1}{2} S(q)\, \omega^a\right).$$

The final point is the analysis of

$$\delta_a \Omega_{bc} \wedge * (\omega^b \wedge \omega^c)$$

or, as is preferable, of

$$N\delta_a \Omega_{bc} \wedge * (\omega^b \wedge \omega^c).$$

Put

$$\theta^{bc} = * (\omega^b \wedge \omega^c).$$

Then the θ^{bc} are the components of an element

$$\theta \in \Lambda^{n-3}(\Sigma; T_0^2(\Sigma)),$$

thus

$$d^{\nabla^q}\theta \in \Lambda^{n-2}(\Sigma; T_0^2(\Sigma))$$

and

$$(d^{\nabla^q}\theta)^{bc} = d\theta^{bc} + \omega^b{}_d \wedge \theta^{dc} + \omega^c{}_d \wedge \theta^{bd}.$$

Rappel: We have

$$d^{\nabla^q}\theta = 0.$$

But then

$$\delta_a \Omega_{bc} \wedge *(\omega^b \wedge \omega^c)$$

$$= \delta_a \Omega_{bc} \wedge \theta^{bc}$$

$$= \delta_a (d\omega_{bc} + \omega_{bd} \wedge \omega^d{}_c) \wedge \theta^{bc}$$

$$= d\delta_a \omega_{bc} \wedge \theta^{bc} + \delta_a \omega_{bd} \wedge \omega^d{}_c \wedge \theta^{bc} + \omega_{bd} \wedge \delta_a \omega^d{}_c \wedge \theta^{bc}$$

$$= d(\delta_a \omega_{bc} \wedge \theta^{bc})$$

$$+ \delta_a \omega_{bc} \wedge d\theta^{bc} + \delta_a \omega_{bd} \wedge \omega^d{}_c \wedge \theta^{bc} + \omega_{bd} \wedge \delta_a \omega^d{}_c \wedge \theta^{bc}$$

$$= d(\delta_a \omega_{bc} \wedge \theta^{bc})$$

$$+ \delta_a \omega_{bc} \wedge d\theta^{bc} + \delta_a \omega_{bc} \wedge \omega^c{}_d \wedge \theta^{bd} + \omega_{db} \wedge \delta_a \omega_{bc} \wedge \theta^{dc}$$

$$= d(\delta_a\omega_{bc}\wedge\theta^{bc})$$

$$+ \delta_a\omega_{bc}\wedge d\theta^{bc} + \delta_a\omega_{bc}\wedge\omega^c{}_d\wedge\theta^{bd} + \delta_a\omega_{bc}\wedge\omega^b{}_d\wedge\theta^{dc}$$

$$= d(\delta_a\omega_{bc}\wedge\theta^{bc}) + \delta_a\omega_{bc}\wedge(d^{\nabla^q}\theta)^{bc}$$

$$= d(\delta_a\omega_{bc}\wedge\theta^{bc})$$

$\Rightarrow$

$$N\delta_a\Omega_{bc}\wedge*(\omega^b\wedge\omega^c)$$

$$= Nd(\delta_a\omega_{bc}\wedge*(\omega^b\wedge\omega^c))$$

$$= d(N\delta_a\omega_{bc}\wedge*(\omega^b\wedge\omega^c)) - dN\wedge\delta_a\omega_{bc}\wedge*(\omega^b\wedge\omega^c).$$

The differential

$$d(N\delta_a\omega_{bc}\wedge*(\omega^b\wedge\omega^c))$$

integrates to zero, hence can be set aside. Write

$$dN\wedge\delta_a\omega_{bc}\wedge*(\omega^b\wedge\omega^c)$$

$$= \omega^b\wedge\omega^c\wedge*(dN\wedge\delta_a\omega_{bc})$$

$$= \omega^b\wedge\omega^c\wedge\iota_{\delta_a\omega_{bc}}*dN.$$

Then

$$0 = \iota_{\delta_a\omega_{bc}}(\omega^b\wedge\omega^c\wedge*dN)$$

$$= \iota_{\delta_a \omega_{bc}} \omega^b \wedge \omega^c \wedge {*dN} - \omega^b \wedge \iota_{\delta_a \omega_{bc}} \omega^c \wedge {*dN}$$

$$+ \omega^b \wedge \omega^c \wedge \iota_{\delta_a \omega_{bc}} {*dN}$$

$$= q(\delta_a \omega_{bc}, \omega^b) \omega^c \wedge {*dN} - q(\delta_a \omega_{bc}, \omega^c) \omega^b \wedge {*dN}$$

$$+ \omega^b \wedge \omega^c \wedge \iota_{\delta_a \omega_{bc}} {*dN}$$

$$= q(\delta_a \omega_{bc}, \omega^b) \omega^c \wedge {*dN} - q(\delta_a \omega_{cb}, \omega^b) \omega^c \wedge {*dN}$$

$$+ \omega^b \wedge \omega^c \wedge \iota_{\delta_a \omega_{bc}} {*dN}$$

$$= 2q(\delta_a \omega_{bc}, \omega^b) \omega^c \wedge {*dN}$$

$$+ \omega^b \wedge \omega^c \wedge \iota_{\delta_a \omega_{bc}} {*dN}$$

$\Rightarrow$

$$- dN \wedge \delta_a \omega_{bc} \wedge * (\omega^b \wedge \omega^c)$$

$$= 2q(\delta_a \omega_{bc}, \omega^b) \omega^c \wedge {*dN}.$$

But

$$d\omega^b = - \omega^b{}_c \wedge \omega^c$$

$\Rightarrow$

$$\delta_a d\omega^b = - \delta_a \omega^b{}_c \wedge \omega^c - \omega^b{}_c \wedge \delta_a \omega^c$$

$\Rightarrow$

$$\iota_{\omega^b}\delta_a d\omega^b = -\iota_{\omega^b}(\delta_a\omega^b{}_c\wedge\omega^c) - \iota_{\omega^b}(\omega^b{}_c\wedge\delta_a\omega^c)$$

$$= -[\iota_{\omega^b}\delta_a\omega^b{}_c\wedge\omega^c - \delta_a\omega^b{}_c\wedge\iota_{\omega^b}\omega^c] - \iota_{\omega^b}(\omega^b{}_c\wedge\delta_a\omega^c)$$

$$= -[q(\delta_a\omega^b{}_c,\omega^b)\omega^c - \delta_a\omega^c{}_c] - \iota_{\omega^b}(\omega^b{}_c\wedge\delta_a\omega^c)$$

$$= -q(\delta_a\omega_{bc},\omega^b)\omega^c - \iota_{\omega^b}(\omega^b{}_c\wedge\delta_a\omega^c).$$

Therefore

$$-\frac{1}{2}dN\wedge\delta_a\omega_{bc}\wedge*(\omega^b\wedge\omega^c)$$

$$= q(\delta_a\omega_{bc},\omega^b)\omega^c\wedge*dN$$

$$= -\iota_{\omega^b}(d\delta_a\omega^b + \omega^b{}_c\wedge\delta_a\omega^c)\wedge*dN$$

$$= -\iota_{\omega^b}((d\delta_a\omega^b + \omega^b{}_c\wedge\delta_a\omega^c)\wedge*dN)$$

$$+ (d\delta_a\omega^b + \omega^b{}_c\wedge\delta_a\omega^c)\wedge\iota_{\omega^b}*dN$$

$$= 0 + (d\delta_a\omega^b + \omega^b{}_c\wedge\delta_a\omega^c)\wedge\iota_{\omega^b}*dN$$

$$= (d\delta_a\omega^b + \omega^b{}_c\wedge\delta_a\omega^c)\wedge*(dN\wedge\omega_b)$$

$$= d\delta_a\omega^b\wedge*(dN\wedge\omega_b)$$

$$+ \omega^b{}_c\wedge\delta_a\omega^c\wedge*(dN\wedge\omega_b)$$

$$= d(\delta_a \omega^b \wedge *(dN \wedge \omega_b)) + \delta_a \omega^b \wedge d*(dN \wedge \omega_b)$$

$$- \delta_a \omega^c \wedge \omega^b{}_c \wedge *(dN \wedge \omega_b).$$

Omit the differential

$$d(\delta_a \omega^b \wedge *(dN \wedge \omega_b))$$

which, of course, will not contribute -- then

$$\delta_a \omega^b \wedge d*(dN \wedge \omega_b) - \delta_a \omega^c \wedge \omega^b{}_c \wedge *(dN \wedge \omega_b)$$

$$= \delta\omega^a \wedge (d*(dN \wedge \omega_a) - \omega^b{}_a \wedge *(dN \wedge \omega_b))$$

$$= \delta\omega^a \wedge d^{\nabla^q} *(dN \wedge \omega_a).$$

<u>LEMMA</u> We have

$$d^{\nabla^q} *(dN \wedge \omega_a) = *(\nabla_a dN - (\Delta_q N)\omega^a).$$

[Write

$$dN = N_c \omega^c \quad (N_c = q(dN, \omega^c)).$$

Then

$$d^{\nabla^q} *(dN \wedge \omega_a)$$

$$= d*(dN \wedge \omega_a) - \omega^b{}_a \wedge *(dN \wedge \omega_b)$$

$$= d*(N_c \omega^c \wedge \omega^a) + \omega^a{}_b \wedge *(N_c \omega^c \wedge \omega^b).$$

But

$$d{*}(N_c\omega^c\wedge\omega^a) = d(N_c\wedge{*}(\omega^c\wedge\omega^a))$$

$$= dN_c\wedge{*}(\omega^c\wedge\omega^a) + N_c\wedge d{*}(\omega^c\wedge\omega^a)$$

$$= dN_c\wedge{*}(\omega^c\wedge\omega^a)$$

$$+ N_c\wedge(- \omega^c{}_b\wedge{*}(\omega^b\wedge\omega^a) - \omega^a{}_b\wedge{*}(\omega^c\wedge\omega^b))$$

$$= dN_c\wedge{*}(\omega^c\wedge\omega^a) + N_c\wedge(- \omega^c{}_b\wedge{*}(\omega^b\wedge\omega^a))$$

$$- \omega^a{}_b\wedge{*}(N_c\omega^c\wedge\omega^b).$$

Make the obvious cancellation -- then

$$d^{\nabla^q}{*}(dN\wedge\omega_a)$$

$$= dN_c\wedge{*}(\omega^c\wedge\omega^a) + N_c\wedge(- \omega^c{}_b\wedge{*}(\omega^b\wedge\omega^a)).$$

$$\bullet\, dN_c\wedge{*}(\omega^c\wedge\omega^a)$$

$$= q(dN_c,\omega^b)\,\omega^b\wedge{*}(\omega^c\wedge\omega^a)$$

$$= q(dN_c,\omega^b)\,(-1)^{n-3}{*}(\omega^c\wedge\omega^a)\wedge\omega^b$$

$$= (-1)^{n-3}q(dN_c,\omega^b)\,(-1)^{n-2}{*}(\iota_{\omega^b}(\omega^c\wedge\omega^a))$$

$$= -\ q(dN_c,\omega^b)*(q(\omega^b,\omega^c)\omega^a - q(\omega^b,\omega^a)\omega^c)$$

$$= *(\ -\ q(dN_c,\omega^c)\omega^a + q(dN_c,\omega^a)\omega^c).$$

• $N_c\wedge(\ -\ \omega^c{}_b\wedge*(\omega^b\wedge\omega^a))$

$$= N_c\wedge(-1)(-1)^{n-3}*(\omega^b\wedge\omega^a)\wedge\omega^c{}_b$$

$$= (-1)^n N_c\wedge*(\omega^b\wedge\omega^a)\wedge\omega^c{}_b$$

$$= (-1)^n N_c\wedge(-1)^{n-2}*(\iota_{\omega^c{}_b}(\omega^b\wedge\omega^a))$$

$$= N_c*(q(\omega^c{}_b,\omega^b)\omega^a - q(\omega^c{}_b,\omega^a)\omega^b)$$

$$= *(N_c q(\omega^c{}_b,\omega^b)\omega^a - N_c q(\omega^c{}_b,\omega^a)\omega^b).$$

However, by definition,

$$H_N = \nabla dN = \omega^b \otimes (dN_b - \omega^c{}_b N_c)$$

$\Rightarrow$

$$\nabla_a dN = < E_a, dN_b - \omega^c{}_b N_c > \omega^b$$

$$= < E_a, dN_b > \omega^b - N_c < E_a, \omega^c{}_b > \omega^b$$

$$= q(dN_c,\omega^a)\omega^c - N_c q(\omega^c{}_b,\omega^a)\omega^b.$$

In addition,

$$\Delta_q N = H_N(E_d, E_d)$$

$$= \omega^b \otimes (dN_b - \omega^c{}_b N_c)(E_d, E_d)$$

$$= \omega^b(E_d) < E_d, dN_b - \omega^c{}_b N_c >$$

$$= < E_b, dN_b - \omega^c{}_b N_c >$$

$$= q(dN_c, \omega^c) - N_c q(\omega^c{}_b, \omega^b).$$

Consequently,

$$*(q(dN_c, \omega^a)\omega^c - N_c q(\omega^c{}_b, \omega^a)\omega^b)$$

$$+ *(-q(dN_c, \omega^c) + N_c q(\omega^c{}_b, \omega^b))\omega^a$$

$$= *(\nabla_a dN - (\Delta_q N)\omega^a),$$

which completes the proof of the lemma.]

Putting everything together then leads to the conclusion that, modulo an exact form,

$$-\frac{N}{2}\delta_a \Omega_{bc} \wedge *(\omega^b \wedge \omega^c) = -\delta\omega^a \wedge *(\nabla_a dN - (\Delta_q N)\omega^a).$$

Summary: We have

$$\frac{\delta H}{\delta \omega^a} = -L_{\vec{N}} p_a + W^b{}_a p_b$$

$$+ N(q(p_a,*\omega^b)p_b - \frac{P}{n-2} p_a)$$

$$- \frac{N}{2} (q(p_b,*\omega^c)q(p_c,*\omega^b) - \frac{P^2}{n-2})*\omega^a$$

$$+ N*(Ric_a - \frac{1}{2} S(q)\omega^a) - *(\nabla_a dN - (\Delta_q N)\omega^a).$$

Constraint Equations These are the relations

$$\begin{cases} \frac{1}{2} [q_t(p_a,*\bar{\omega}^b)q_t(p_b,*\bar{\omega}^a) - \frac{1}{n-2} P_t^2 - S(q_t)]vol_{q_t} = 0 \\ - dp_a + \iota_{E_a} d\bar{\omega}^b p_b = - d^{\bar{\nabla}} p_a = 0. \end{cases}$$

Evolution Equations These are the relations

$$\dot{\bar{\omega}}^a = N_t \bar{\omega}^a{}_0 + \bar{Q}^a{}_b \bar{\omega}^b + L_{\vec{N}_t} \bar{\omega}^a$$

and

$$\dot{p}_a = - N_t(q_t(p_a,*\bar{\omega}^b)p_b - \frac{1}{n-2} P_t p_a)$$

$$+ \frac{N_t}{2} (q_t(p_b,*\bar{\omega}^c)q_t(p_c,*\bar{\omega}^b) - \frac{1}{n-2} P_t^2)*\bar{\omega}^a$$

$$- N_t*(Ric_a - \frac{1}{2} S(q_t)\bar{\omega}^a)$$

$$+ *(\bar{\nabla}_a dN_t - (\Delta_{q_t} N_t)\bar{\omega}^a) + L_{\vec{N}_t} p_a - \bar{Q}^b{}_a p_b.$$

In the last section, we saw that

$$\frac{\delta H}{\delta N}\bigg|_{(\vec{\omega}_t,\vec{p}_t)} = -\bar{G}_{00}\mathrm{vol}_{q_t},$$

i.e.,

$$\frac{1}{2}\,[q_t(p_a,*\bar{\omega}^b)q_t(p_b,*\bar{\omega}^a) - \frac{1}{n-2}\,P_t^2 - S(q_t)]\mathrm{vol}_{q_t} = -\bar{G}_{00}\mathrm{vol}_{q_t},$$

and

$$\frac{\delta H}{\delta N^a}\bigg|_{(\vec{\omega}_t,\vec{p}_t)} = -\bar{G}_{0a}\mathrm{vol}_{q_t},$$

i.e.,

$$-dp_a + \iota_{E_a}d\bar{\omega}^b\wedge p_b = -d^{\bar{\nabla}}p_a = -\bar{G}_{0a}\mathrm{vol}_{q_t}.$$

Therefore the constraint equations are equivalent to

$$\begin{cases} G_{00} = 0 \\ G_{0a} = 0. \end{cases}$$

Turning to the evolution equations, note that

$$\dot{\bar{\omega}}^a = N_t(q_t(p_b,*\bar{\omega}^a)\bar{\omega}^b - \frac{1}{n-2}\,P_t\bar{\omega}^a) + \bar{Q}^a{}_b\bar{\omega}^b + L_{\vec{N}_t}\bar{\omega}^a,$$

which is precisely the functional derivative $\frac{\delta H}{\delta p_a}$ evaluated at $(\vec{\omega}_t,\vec{p}_t;N_t,\vec{N}_t,\bar{Q}^a{}_b)$. In view of this, the evolution equations thus say that

$$\begin{cases} \dot{\bar{\omega}}^a = \dfrac{\delta H}{\delta p_a} \\ \dot{p}_a = -\dfrac{\delta H}{\delta \bar{\omega}^a}\,. \end{cases}$$

In other words: The curve

$$t \to (\vec{\omega}_t, \vec{p}_t) \in T^*\underline{Q}$$

is an integral curve for the hamiltonian vector field

$$X_H = \left(\frac{\delta H}{\delta \vec{p}}, - \frac{\delta H}{\delta \vec{\omega}} \right)$$

attached to H (all data taken at t).

<u>MAIN THEOREM</u> Suppose that the constraint equations and the evolution equations are satisfied by the pair $(\vec{\omega}_t, \vec{p}_t)$ -- then $\mathrm{Ein}(g) = 0$.

[Note: It is this result which justifies the passage to the coframe picture.]

To prove the theorem, it suffices to show that if $\forall$ b,

$$\dot{p}_b = - N_t (q_t(p_b, *\bar{\omega}^c) p_c - \frac{1}{n-2} P_t p_b)$$

$$+ \frac{N_t}{2} (q_t(p_c, *\bar{\omega}^d) q_t(p_d, *\bar{\omega}^c) - \frac{1}{n-2} P_t^2) *\bar{\omega}^b$$

$$- N_t * (\mathrm{Ric}_b - \frac{1}{2} S(q_t) \bar{\omega}^b)$$

$$+ *(\bar{\nabla}_b dN_t - (\Delta_{q_t} N_t) \bar{\omega}^b) + L_{\vec{N}_t} p_b - \bar{Q}^c{}_b p_c,$$

then

$$\dot{p}_t = - 2N_t (\pi_t * \pi_t - \frac{1}{n-2} \mathrm{tr}_{q_t}(\pi_t) \pi_t) \otimes |q_t|^{1/2}$$

$$+ \frac{N_t}{2} ([\pi_t,\pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2) q_t^{\#} \otimes |q_t|^{1/2}$$

$$- N_t \operatorname{Ein}(q_t)^{\#} \otimes |q_t|^{1/2}$$

$$+ (H_{N_t} - (\Delta_{q_t} N_t) q_t)^{\#} \otimes |q_t|^{1/2} + L_{\vec{N}_t} p_t.$$

And for this, one can work locally.

Let μ,ν be indices that run between 1 and n-1.

<u>Local Formulas</u>

1. $\frac{\partial}{\partial x^\mu} = e^a{}_\mu E_a$ & $E_a = e^\mu{}_a \frac{\partial}{\partial x^\mu}$.

2. $dx^\mu = e^\mu{}_a \bar{\omega}^a$ & $\bar{\omega}^a = e^a{}_\mu dx^\mu$.

3. $e^\mu{}_a e^a{}_\nu = \delta^\mu{}_\nu$ & $e^a{}_\mu e^\mu{}_b = \delta^a{}_b$.

4. $\bar{g}_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$ & $\bar{g}^{\mu\nu} = \eta^{ab} e^\mu{}_a e^\nu{}_b$.

5. $e^a{}_\mu = \eta^{ab} \bar{g}_{\mu\nu} e^\nu{}_b$ & $e^\mu{}_a = \eta_{ab} \bar{g}^{\mu\nu} e^b{}_\nu$.

[Note: Bear in mind that $\bar{g}$ and q_t are one and the same.]

<u>LEMMA</u> We have

$$(\bar{\omega}^a \wedge p_b) e^\mu{}_a e^\nu{}_b = (x_t^{\mu\nu} - K_t q_t^{\mu\nu}) \operatorname{vol}_{q_t},$$

i.e.,

$$(dx^{\mu}\wedge p_b)e^{\nu}{}_b = \pi_t^{\mu\nu}\mathrm{vol}_{q_t}.$$

Strictly speaking, $\pi_t\mathrm{vol}_{q_t}$ and $\pi_t \otimes |q_t|^{1/2}$ are different entities but for the purposes at hand, it is more convenient to use $\pi_t\mathrm{vol}_{q_t}$. Agreeing to denote it also by p_t, the evolution equation for $\dot{p}_t^{\mu\nu}$ is as above, the only change being that $|q_t|^{1/2}$ is replaced throughout by vol_{q_t}.

<u>LEMMA</u> We have

$$\dot{p}_t^{\mu\nu} = (dx^{\mu}\wedge\dot{p}_b)e^{\nu}{}_b - (dx^{\mu}\wedge p_b)\dot{\bar{\omega}}^c(E_b)e^{\nu}{}_c.$$

[In fact,

$$\dot{p}_t^{\mu\nu} = [(dx^{\mu}\wedge p_b)e^{\nu}{}_b]^{\bullet}$$

$$= (dx^{\mu}\wedge\dot{p}_b)e^{\nu}{}_b + (dx^{\mu}\wedge p_b)\dot{e}^{\nu}{}_b$$

$$= (dx^{\mu}\wedge\dot{p}_b)e^{\nu}{}_b - (dx^{\mu}\wedge p_b)e^{\nu}{}_c\dot{e}^c{}_{\nu'}e^{\nu'}{}_b$$

$$= (dx^{\mu}\wedge\dot{p}_b)e^{\nu}{}_b - (dx^{\mu}\wedge p_b)e^{\nu}{}_c\dot{\bar{\omega}}^c\left(\frac{\partial}{\partial x^{\nu'}}\right)e^{\nu'}{}_b$$

$$= (dx^{\mu}\wedge\dot{p}_b)e^{\nu}{}_b - (dx^{\mu}\wedge p_b)e^{\nu}{}_c\dot{\bar{\omega}}^c\left(e^{\nu'}{}_b\frac{\partial}{\partial x^{\nu'}}\right)$$

$$= (dx^{\mu} \wedge \dot{p}_b) e^{\nu}{}_b - (dx^{\mu} \wedge p_b) \dot{\bar{\omega}}^c (E_b) e^{\nu}{}_c.]$$

The point now is to apply the lemma and replace $\dot{p}_b$ by its evolution equation, the claim being that the result is the evolution equation for $\dot{p}_t^{\mu\nu}$.

The first item on the agenda is to check that there is no net contribution from the rotational terms.

<u>Rot_1</u>: The rotational contribution from $\dot{p}_b$ is

$$- (dx^{\mu} \wedge \bar{\Omega}^c{}_b p_c) e^{\nu}{}_b.$$

<u>Rot_2</u>: The rotational contribution from $\dot{\bar{\omega}}^c (E_b)$ is

$$- (dx^{\mu} \wedge p_b) \bar{\Omega}^c{}_d \bar{\omega}^d (E_b) e^{\nu}{}_c$$

or still,

$$- (dx^{\mu} \wedge p_b) \bar{\Omega}^c{}_b e^{\nu}{}_c.$$

But

$$- (dx^{\mu} \wedge p_b) \bar{\Omega}^c{}_b e^{\nu}{}_c$$

$$= - (dx^{\mu} \wedge \bar{\Omega}^c{}_b p_b) e^{\nu}{}_c$$

$$= - (dx^{\mu} \wedge \bar{\Omega}^c{}_d p_d) e^{\nu}{}_c$$

$$= - (dx^{\mu} \wedge \bar{\Omega}^b{}_d p_d) e^{\nu}{}_b$$

$$= - (dx^{\mu} \wedge - \bar{\Omega}^d{}_b p_d) e^{\nu}{}_b$$

$$= (dx^\mu \wedge \bar{Q}^c{}_b p_c) e^\nu{}_b.$$

So the two rotational terms do indeed cancel out.

<u>Item</u>:

$$dx^\mu \wedge (- N_t (q_t(p_b, *\bar{\omega}^c) p_c - \frac{1}{n-2} P_t p_b) e^\nu{}_b$$

$$- (dx^\mu \wedge p_b) N_t \bar{\omega}^c{}_0 (E_b) e^\nu{}_c$$

equals

$$- 2N_t ((\pi_t * \pi_t)^{\mu\nu} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t) \pi_t^{\mu\nu}) \operatorname{vol}_{q_t}.$$

Consider first

$$dx^\mu \wedge (N_t (\frac{1}{n-2}) P_t p_b) e^\nu{}_b.$$

Since

$$P_t = \operatorname{tr}_{q_t}(\pi_t),$$

we have

$$dx^\mu \wedge (N_t (\frac{1}{n-2}) P_t p_b) e^\nu{}_b$$

$$= N_t (\frac{1}{n-2}) \operatorname{tr}_{q_t}(\pi_t) (dx^\mu \wedge p_b) e^\nu{}_b$$

$$= N_t (\frac{1}{n-2}) \operatorname{tr}_{q_t}(\pi_t) \pi_t^{\mu\nu} \operatorname{vol}_{q_t}.$$

But there is more, viz.

$$- (dx^\mu \wedge p_b) N_t \bar{\omega}^c{}_0 (E_b) e^\nu{}_c$$

$$= - (dx^{\mu}\wedge p_b)N_t\varkappa^c{}_d\bar{\omega}^d(E_b)e^{\nu}{}_c$$

$$= - (dx^{\mu}\wedge p_b)N_t\varkappa^c{}_b e^{\nu}{}_c$$

$$= - (dx^{\mu}\wedge p_b)N_t(q_t(p_c,*\bar{\omega}^b) - \frac{1}{n-2} P_t\delta_{cb})e^{\nu}{}_c$$

$$= - (dx^{\mu}\wedge p_b)N_t q_t(p_c,*\bar{\omega}^b)e^{\nu}{}_c$$

$$+ (dx^{\mu}\wedge p_b)N_t(\frac{1}{n-2})P_t\delta_{cb}e^{\nu}{}_c.$$

And

$$(dx^{\mu}\wedge p_b)N_t(\frac{1}{n-2})P_t\delta_{cb}e^{\nu}{}_c$$

$$= (dx^{\mu}\wedge p_b)N_t(\frac{1}{n-2})P_t e^{\nu}{}_b$$

$$= N_t(\frac{1}{n-2})\operatorname{tr}_{q_t}(\pi_t)(dx^{\mu}\wedge p_b)e^{\nu}{}_b$$

$$= N_t(\frac{1}{n-2})\operatorname{tr}_{q_t}(\pi_t)\pi_t^{\mu\nu}\mathrm{vol}_{q_t}.$$

Thus

$$2N_t(\frac{1}{n-2})\operatorname{tr}_{q_t}(\pi_t)\pi_t^{\mu\nu}\mathrm{vol}_{q_t}$$

is accounted for. There remains

$$dx^{\mu}\wedge(- N_t q_t(p_b,*\bar{\omega}^c)p_c)e^{\nu}{}_b$$

$$- (dx^{\mu}\wedge p_b)N_t q_t(p_c,*\bar{\omega}^b)e^{\nu}{}_c$$

or still,

$$- 2N_t(dx^\mu \wedge p_c)q_t(p_b, *\bar{\omega}^c)e^\nu{}_b$$

or still,

$$- 2N_t(\bar{\omega}^a e^\mu{}_a \wedge p_c)q_t(p_b, *\bar{\omega}^c)e^\nu{}_b$$

or still,

$$- 2N_t q_t(p_c, *\bar{\omega}^a)q_t(p_b, *\bar{\omega}^c)e^\mu{}_a e^\nu{}_b \mathrm{vol}_{q_t}$$

or still,

$$- 2N_t q_t(p_a, *\bar{\omega}^c)q_t(p_b, *\bar{\omega}^c)e^\mu{}_a e^\nu{}_b \mathrm{vol}_{q_t}$$

or still,

$$- 2N_t q_t(p_a, p_b)e^\mu{}_a e^\nu{}_b \mathrm{vol}_{q_t}$$

or still,

$$- 2N_t(\pi_t * \pi_t)^{\mu\nu}\mathrm{vol}_{q_t}.$$

<u>Item</u>:

$$dx^\mu \wedge (\frac{N_t}{2}(q_t(p_c, *\bar{\omega}^d)q_t(p_d, *\bar{\omega}^c) - \frac{1}{n-2} P_t^2) * \bar{\omega}^b)e^\nu{}_b$$

equals

$$\frac{N_t}{2}([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2} \mathrm{tr}_{q_t}(\pi_t)^2)q_t^{\mu\nu}\mathrm{vol}_{q_t}.$$

To begin with,

$$- dx^\mu \wedge \frac{N_t}{2}(\frac{1}{n-2})P_t^2 * \bar{\omega}^b e^\nu{}_b$$

$$= -\frac{N_t}{2}\left(\frac{1}{n-2}\right)\mathrm{tr}_{q_t}(\pi_t)^2 dx^\mu \wedge *\bar{\omega}^b e^\nu{}_b$$

$$= -\frac{N_t}{2}\left(\frac{1}{n-2}\right)\mathrm{tr}_{q_t}(\pi_t)^2 q_t(dx^\mu, \bar{\omega}^b) e^\nu{}_b \mathrm{vol}_{q_t}$$

$$= -\frac{N_t}{2}\left(\frac{1}{n-2}\right)\mathrm{tr}_{q_t}(\pi_t)^2 e^\mu{}_b e^\nu{}_b \mathrm{vol}_{q_t}$$

$$= -\frac{N_t}{2}\left(\frac{1}{n-2}\right)\mathrm{tr}_{q_t}(\pi_t)^2 q_t^{\mu\nu} \mathrm{vol}_{q_t}.$$

This leaves

$$dx^\mu \wedge \frac{N_t}{2} q_t(p_c, *\bar{\omega}^d) q_t(p_d, *\bar{\omega}^c) *\bar{\omega}^b e^\nu{}_b$$

or still,

$$\frac{N_t}{2} q_t(p_c, *\bar{\omega}^d) q_t(p_d, *\bar{\omega}^c) dx^\mu \wedge *\bar{\omega}^b e^\nu{}_b$$

or still,

$$\frac{N_t}{2} q_t(p_c, p_c) q_t^{\mu\nu} \mathrm{vol}_{q_t}$$

or still,

$$\frac{N_t}{2} [\pi_t, \pi_t]_{q_t} q_t^{\mu\nu} \mathrm{vol}_{q_t}.$$

<u>Item</u>:

$$dx^\mu \wedge \left(-N_t *(\mathrm{Ric}_b - \frac{1}{2} S(q_t)\bar{\omega}^b)\right) e^\nu{}_b$$

equals

$$-N_t \mathrm{Ein}(q_t)^{\mu\nu} \mathrm{vol}_{q_t}.$$

Here

$$- N_t \mathrm{Ein}(q_t)^{\mu\nu} \mathrm{vol}_{q_t}$$

$$= - N_t (R^{\mu\nu} - \frac{1}{2} S(q_t) q_t^{\mu\nu}) \mathrm{vol}_{q_t}$$

$$= - N_t R^{\mu\nu} \mathrm{vol}_{q_t} + \frac{N_t}{2} S(q_t) q_t^{\mu\nu} \mathrm{vol}_{q_t}.$$

But

$$dx^{\mu} \wedge (- N_t * (\mathrm{Ric}_b - \frac{1}{2} S(q_t) \bar{\omega}^b)) e^{\nu}{}_b$$

$$= - N_t dx^{\mu} \wedge * (\mathrm{Ric}_b) e^{\nu}{}_b$$

$$+ \frac{N_t}{2} S(q_t) dx^{\mu} \wedge * \bar{\omega}^b e^{\nu}{}_b$$

$$= - N_t dx^{\mu} \wedge * (\mathrm{Ric}_b) e^{\nu}{}_b$$

$$+ \frac{N_t}{2} S(q_t) q_t^{\mu\nu} \mathrm{vol}_{q_t}.$$

And

$$- N_t dx^{\mu} \wedge * (\mathrm{Ric}_b) e^{\nu}{}_b$$

$$= - N_t dx^{\mu} \wedge * (R_{bc} \bar{\omega}^c) e^{\nu}{}_b$$

$$= - N_t dx^{\mu} \wedge (R_{bc} * \bar{\omega}^c) e^{\nu}{}_b$$

$$= - N_t R_{bc} dx^{\mu} \wedge * \bar{\omega}^{c} e^{\nu}{}_{b}$$

$$= - N_t R_{bc} e^{\mu}{}_{c} e^{\nu}{}_{b} \mathrm{vol}_{q_t}$$

$$= - N_t \mathrm{Ric}(E_b, E_c) e^{\mu}{}_{c} e^{\nu}{}_{b} \mathrm{vol}_{q_t}$$

$$= - N_t \mathrm{Ric}\left(e^{\nu'}{}_{b} \frac{\partial}{\partial x^{\nu'}}, e^{\mu'}{}_{c} \frac{\partial}{\partial x^{\mu'}}\right) e^{\mu}{}_{c} e^{\nu}{}_{b} \mathrm{vol}_{q_t}$$

$$= - N_t e^{\nu'}{}_{b} e^{\mu'}{}_{c} R_{\nu'\mu'} e^{\mu}{}_{c} e^{\nu}{}_{b} \mathrm{vol}_{q_t}$$

$$= - N_t e^{\mu}{}_{c} e^{\mu'}{}_{c} e^{\nu}{}_{b} e^{\nu'}{}_{b} R_{\mu'\nu'} \mathrm{vol}_{q_t}$$

$$= - N_t q_t^{\mu\mu'} q_t^{\nu\nu'} R_{\mu'\nu'} \mathrm{vol}_{q_t}$$

$$= - N_t R^{\mu\nu} \mathrm{vol}_{q_t}.$$

<u>Item</u>:

$$dx^{\mu} \wedge * (\bar{\nabla}_b dN_t - (\Delta_{q_t} N_t) \bar{\omega}^{b}) e^{\nu}{}_{b}$$

equals

$$(H_{N_t}^{\mu\nu} - (\Delta_{q_t} N_t) q_t^{\mu\nu}) \mathrm{vol}_{q_t}.$$

For it is clear that

$$- dx^{\mu} \wedge (\Delta_{q_t} N_t) * \bar{\omega}^{b} e^{\nu}{}_{b}$$

$$= - (\Delta_{q_t} N_t) q_t^{\mu\nu} \mathrm{vol}_{q_t}.$$

On the other hand,

$$dx^{\mu}\wedge*(\bar{\nabla}_b dN_t)e^{\nu}{}_b$$

$$= dx^{\mu}\wedge q_t(\bar{\nabla}_b dN_t,\bar{\omega}^c)*\bar{\omega}^c e^{\nu}{}_b$$

$$= dx^{\mu}\wedge H_{N_t}(E_c,E_b)*\bar{\omega}^c e^{\nu}{}_b$$

$$= H_{N_t}(E_b,E_c)dx^{\mu}\wedge*\bar{\omega}^c e^{\nu}{}_b$$

$$= H_{N_t}(E_b,E_c)e^{\mu}{}_c e^{\nu}{}_b \mathrm{vol}_{q_t},$$

which, in complete analogy with the discussion of Ric, reduces to

$$H_{N_t}^{\mu\nu}\mathrm{vol}_{q_t}.$$

<u>Item:</u>

$$(dx^{\mu}\wedge L_{\vec{N}_t}p_b)e^{\nu}{}_b - (dx^{\mu}\wedge p_b)(L_{\vec{N}_t}\bar{\omega}^c)(E_b)e^{\nu}{}_c$$

equals

$$L_{\vec{N}_t}p_t^{\mu\nu},$$

i.e., equals

$$(L_{\vec{N}_t}dx^{\mu}\wedge p_b)e^{\nu}{}_b + (dx^{\mu}\wedge L_{\vec{N}_t}p_b)e^{\nu}{}_b$$

$$+ (dx^{\mu}\wedge p_b)L_{\vec{N}_t}e^{\nu}{}_b.$$

Therefore the issue is the equality of

$$-\ (dx^{\mu}\wedge p_b)\,(L_{\vec{N}_t}\bar{\omega}^c)\,(E_b)\,e^{\nu}{}_c$$

and

$$(L_{\vec{N}_t} dx^{\mu}\wedge p_b)\,e^{\nu}{}_b + (dx^{\mu}\wedge p_b)\,L_{\vec{N}_t} e^{\nu}{}_b.$$

Write

$$L_{\vec{N}_t}(dx^{\nu}) = L_{\vec{N}_t}(e^{\nu}{}_c\bar{\omega}^c)$$

$$= (L_{\vec{N}_t} e^{\nu}{}_c)\bar{\omega}^c + e^{\nu}{}_c(L_{\vec{N}_t}\bar{\omega}^c)$$

to get

$$(L_{\vec{N}_t}\bar{\omega}^c)\,(E_b)\,e^{\nu}{}_c = L_{\vec{N}_t}(dx^{\nu})\,(E_b) - L_{\vec{N}_t} e^{\nu}{}_b.$$

Then

$$-\ (dx^{\mu}\wedge p_b)\,(L_{\vec{N}_t}\bar{\omega}^c)\,(E_b)\,e^{\nu}{}_c$$

$$= (dx^{\mu}\wedge p_b)\,L_{\vec{N}_t} e^{\nu}{}_b - (dx^{\mu}\wedge p_b)\,L_{\vec{N}_t}(dx^{\nu})\,(E_b),$$

so what's left is the equality of

$$(L_{\vec{N}_t} dx^{\mu}\wedge p_b)\,e^{\nu}{}_b$$

and

$$-\ (dx^{\mu}\wedge p_b)\,L_{\vec{N}_t}(dx^{\nu})\,(E_b)$$

or, equivalently, that

$$(L_{\vec{N}_t} dx^\mu \wedge p_b) dx^\nu (E_b) + (dx^\mu \wedge p_b) L_{\vec{N}_t} (dx^\nu)(E_b) = 0.$$

To see this, take $\mu = \nu$ (the general case is similar (because $p^{\mu\nu} = p^{\nu\mu}$)) -- then

$$L_{\vec{N}_t} dx^\mu \wedge p_b \wedge dx^\mu + dx^\mu \wedge p_b \wedge L_{\vec{N}_t} dx^\mu = 0.$$

Indeed, each term is an n-form while dim $\Sigma = n-1$. Apply ι_{E_b}:

1. $\iota_{E_b} L_{\vec{N}_t} dx^\mu \wedge p_b \wedge dx^\mu.$

2. $L_{\vec{N}_t} dx^\mu \wedge \iota_{E_b} p_b \wedge dx^\mu.$

3. $(-1)^{n-2} L_{\vec{N}_t} dx^\mu \wedge p_b \wedge \iota_{E_b} dx^\mu.$

4. $\iota_{E_b} dx^\mu \wedge p_b \wedge L_{\vec{N}_t} dx^\mu.$

5. $dx^\mu \wedge \iota_{E_b} p_b \wedge L_{\vec{N}_t} dx^\mu.$

6. $(-1)^{n-2} dx^\mu \wedge p_b \wedge \iota_{E_b} L_{\vec{N}_t} dx^\mu.$

The sum $1 + \cdots + 6$ is zero.

• 2 + 5 equals $(-1)^{n-3}$ times

$$L_{\vec{N}_t} dx^\mu \wedge dx^\mu \wedge \iota_{E_b} p_b + dx^\mu \wedge L_{\vec{N}_t} dx^\mu \wedge \iota_{E_b} p_b,$$

i.e., equals $(-1)^{n-3}$ times

$$(L_{\vec{N}_t} dx^\mu \wedge dx^\mu + dx^\mu \wedge L_{\vec{N}_t} dx^\mu) \wedge \iota_{E_b} p_b,$$

i.e., equals $(-1)^{n-3}$ times

$$L_{\vec{N}_t} (dx^\mu \wedge dx^\mu) \wedge \iota_{E_b} p_b,$$

which is zero.

- $\iota_{E_b} L_{\vec{N}_t} dx^\mu = L_{\vec{N}_t} (dx^\mu)(E_b)$

$\Rightarrow$

$$\iota_{E_b} L_{\vec{N}_t} dx^\mu \wedge p_b \wedge dx^\mu$$

$$= p_b \wedge dx^\mu \wedge \iota_{E_b} L_{\vec{N}_t} dx^\mu$$

$$= (-1)^{n-2} (dx^\mu \wedge p_b) L_{\vec{N}_t} (dx^\mu)(E_b)$$

$\Rightarrow$

$$1 + 6 = 2(-1)^{n-2} (dx^\mu \wedge p_b) L_{\vec{N}_t} (dx^\mu)(E_b).$$

- $\iota_{E_b} dx^\mu = dx^\mu (E_b)$

$\Rightarrow$

$$\iota_{E_b} dx^\mu \wedge p_b \wedge L_{\vec{N}_t} dx^\mu$$

$$= p_b \wedge L_{\vec{N}_t} dx^\mu \wedge \iota_{E_b} dx^\mu$$

$$= (-1)^{n-2} (L_{\vec{N}_t} dx^\mu \wedge p_b) dx^\mu (E_b)$$

$\Rightarrow$

$$3 + 4 = 2(-1)^{n-2} (L_{\vec{N}_t} dx^\mu \wedge p_b) dx^\mu (E_b).$$

Therefore

$$0 = (1+6) + (3+4)$$

$$= 2(-1)^{n-2} ((L_{\vec{N}_t} dx^\mu \wedge p_b) dx^\mu (E_b) + (dx^\mu \wedge p_b) L_{\vec{N}_t} (dx^\mu)(E_b))$$

$\Rightarrow$

$$(L_{\vec{N}_t} dx^\mu \wedge p_b) dx^\mu (E_b) + (dx^\mu \wedge p_b) L_{\vec{N}_t} (dx^\mu)(E_b) = 0.$$

Section 45: Computation of the Poisson Brackets The assumptions and notation are those of the standard setup.

Given functions $f_1, f_2 : T^*\underline{Q} \to \underline{R}$, let X_1, X_2 be the corresponding hamiltonian vector fields — then the Poisson bracket of f_1, f_2 is the function

$$\{f_1, f_2\} : T^*\underline{Q} \to \underline{R}$$

defined by the rule

$$\{f_1, f_2\}(\vec{\omega}, \vec{p}) = \Omega(X_1(\vec{\omega}, \vec{p}), X_2(\vec{\omega}, \vec{p})).$$

Therefore

$$\{f_1, f_2\} = \int_\Sigma \left[\frac{\delta f_2}{\delta \vec{p}} \wedge \frac{\delta f_1}{\delta \vec{\omega}} - \frac{\delta f_1}{\delta \vec{p}} \wedge \frac{\delta f_2}{\delta \vec{\omega}} \right].$$

[Note: Tacitly, it is assumed that the functional derivatives exist.]

Rappel:

$$\begin{aligned} &H(\vec{\omega}, \vec{p}; N, \vec{N}, W) \\ &\qquad = \int_\Sigma L_{\vec{N}} \omega^a \wedge p_a + \int_\Sigma W^a{}_b \omega^b \wedge p_a + \int_\Sigma NE, \end{aligned}$$

where

$$E(\vec{\omega}, \vec{p}) = \frac{1}{2} [q(p_a, *\omega^b) q(p_b, *\omega^a) - \frac{p^2}{n-2} - S(q)] \text{vol}_q.$$

Definition:

$$H_D(\vec{N}) = \int_\Sigma L_{\vec{N}} \omega^a \wedge p_a$$

is the integrated diffeomorphism constraint;

$$H_R(W) = \int_\Sigma W^a{}_b \omega^b \wedge p_a$$

is the integrated rotational constraint;

$$H_H(N) = \int_\Sigma NE$$

is the integrated hamiltonian constraint.

Therefore

$$H = H_D + H_R + H_H$$

and we have:

1. $\{H_D(\vec{N}_1), H_D(\vec{N}_2)\} = H_D([\vec{N}_1, \vec{N}_2]);$

2. $\{H_D(\vec{N}), H_R(W)\} = H_R(L_{\vec{N}} W);$

3. $\{H_D(\vec{N}), H_H(N)\} = H_H(L_{\vec{N}} N);$

4. $\{H_R(W_1), H_R(W_2)\} = H_R([W_1, W_2]);$

5. $\{H_R(W), H_H(N)\} = 0;$

6. $\{H_H(N_1), H_H(N_2)\}$

$$= H_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)$$

$$+ H_R(q(dN_1 \wedge dN_2, \omega^a \wedge \omega_b) + q(N_1 dN_2 - N_2 dN_1, \omega^a{}_b)).$$

Remark: A constraint is a function $f: T^*\underline{Q} \to \underline{R}$ such that $f|\text{Con}_{\underline{Q}} = 0$. Thus, by construction, $H_D(\vec{N})$, $H_R(W)$, and $H_H(N)$ are constraints, these being termed primary. The foregoing relations then imply that the Poisson bracket of two primary constraints is a constraint.

<u>LEMMA</u> $\forall\ X \in \mathcal{D}^1(\Sigma)$, $\forall\ \gamma \in \Lambda^{n-1}\Sigma$,

$$\int_\Sigma L_X \gamma = 0.$$

[Apply the formula

$$L_X = \iota_X \circ d + d \circ \iota_X.]$$

<u>Ad 1</u>: We have

$$\{H_D(\vec{N}_1), H_D(\vec{N}_2)\}$$

$$= \int_\Sigma \left[\frac{\delta H_D(\vec{N}_2)}{\delta \vec{p}} \wedge \frac{\delta H_D(\vec{N}_1)}{\delta \vec{\omega}} - \frac{\delta H_D(\vec{N}_1)}{\delta \vec{p}} \wedge \frac{\delta H_D(\vec{N}_2)}{\delta \vec{\omega}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_D(\vec{N}_2)}{\delta p_a} \wedge \frac{\delta H_D(\vec{N}_1)}{\delta \omega^a} - \frac{\delta H_D(\vec{N}_1)}{\delta p_a} \wedge \frac{\delta H_D(\vec{N}_2)}{\delta \omega^a} \right]$$

$$= \int_\Sigma [L_{\vec{N}_2} \omega^a \wedge - L_{\vec{N}_1} p_a - L_{\vec{N}_1} \omega^a \wedge - L_{\vec{N}_2} p_a]$$

$$= \int_\Sigma [- L_{\vec{N}_2} \omega^a \wedge L_{\vec{N}_1} p_a + L_{\vec{N}_1} \omega^a \wedge L_{\vec{N}_2} p_a].$$

But

$$\begin{cases} L_{\vec{N}_1}(L_{\vec{N}_2} \omega^a \wedge p_a) = L_{\vec{N}_1} L_{\vec{N}_2} \omega^a \wedge p_a + L_{\vec{N}_2} \omega^a \wedge L_{\vec{N}_1} p_a \\ L_{\vec{N}_2}(L_{\vec{N}_1} \omega^a \wedge p_a) = L_{\vec{N}_2} L_{\vec{N}_1} \omega^a \wedge p_a + L_{\vec{N}_1} \omega^a \wedge L_{\vec{N}_2} p_a \end{cases}$$

$\Rightarrow$

$$\left[\begin{array}{l} \int_\Sigma - L_{\vec{N}_2}\omega^a \wedge L_{\vec{N}_1} p_a = \int_\Sigma L_{\vec{N}_1} L_{\vec{N}_2}\omega^a \wedge p_a \\ \\ \int_\Sigma L_{\vec{N}_1}\omega^a \wedge L_{\vec{N}_2} p_a = -\int_\Sigma L_{\vec{N}_2} L_{\vec{N}_1}\omega^a \wedge p_a . \end{array}\right.$$

Therefore

$$\{H_D(\vec{N}_1), H_D(\vec{N}_2)\}$$

$$= \int_\Sigma (L_{\vec{N}_1} L_{\vec{N}_2} - L_{\vec{N}_2} L_{\vec{N}_1})\omega^a \wedge p_a$$

$$= \int_\Sigma L_{[\vec{N}_1,\vec{N}_2]}\omega^a \wedge p_a$$

$$= H_D([\vec{N}_1,\vec{N}_2]).$$

Remark: The canonical left action of Diff Σ on $T^*\underline{Q}$ is symplectic (i.e., $\forall\ \varphi \in \text{Diff}\ \Sigma,\ \varphi \cdot \Omega = \Omega$) and admits a momentum map

$$J: T^*\underline{Q} \to \text{Hom}(\mathcal{D}^1(\Sigma), \underline{R}),$$

namely

$$J(\vec{\omega},\vec{p})(\vec{N}) = \int_\Sigma L_{\vec{N}}\omega^a \wedge p_a$$

$$= H_D(\vec{N})(\vec{\omega},\vec{p}),$$

which provides an interpretation of H_D.

Ad 2: We have

$$\{H_D(\vec{N}), H_R(W)\}$$

$$= \int_\Sigma \left[\frac{\delta H_R(W)}{\delta \vec{p}} \wedge \frac{\delta H_D(\vec{N})}{\delta \vec{\omega}} - \frac{\delta H_D(\vec{N})}{\delta \vec{p}} \wedge \frac{\delta H_R(W)}{\delta \vec{\omega}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_R(W)}{\delta p_a} \wedge \frac{\delta H_D(\vec{N})}{\delta \omega^a} - \frac{\delta H_D(\vec{N})}{\delta p_a} \wedge \frac{\delta H_R(W)}{\delta \omega^a} \right]$$

$$= \int_\Sigma [W^a{}_b \omega^b \wedge - L_{\vec{N}} p_a - L_{\vec{N}} \omega^a \wedge W^b{}_a p_b]$$

$$= - \int_\Sigma [W^a{}_b \omega^b \wedge L_{\vec{N}} p_a + L_{\vec{N}} \omega^a \wedge W^b{}_a p_b].$$

But

$$L_{\vec{N}} (W^a{}_b \omega^b \wedge p_a)$$

$$= L_{\vec{N}} (W^a{}_b \omega^b) \wedge p_a + W^a{}_b \omega^b \wedge L_{\vec{N}} p_a$$

$$= (L_{\vec{N}} W^a{}_b) \omega^b \wedge p_a$$

$$+ W^a{}_b (L_{\vec{N}} \omega^b) \wedge p_a + W^a{}_b \omega^b \wedge L_{\vec{N}} p_a$$

$$= (L_{\vec{N}} W^a{}_b) \omega^b \wedge p_a$$

$$+ W^a{}_b \omega^b \wedge L_{\vec{N}} p_a + L_{\vec{N}} \omega^a \wedge W^b{}_a p_b$$

$\Rightarrow$

$$- \int_\Sigma [W^a{}_b \omega^b \wedge L_{\vec{N}} p_a + L_{\vec{N}} \omega^a \wedge W^b{}_a p_b]$$

$$= \int_\Sigma (L_{\vec{N}} W^a{}_b) \omega^b \wedge p_a.$$

Therefore

$$\{H_D(\vec{N}), H_R(W)\} = H_R(L_{\vec{N}} W).$$

<u>Ad 3</u>: We have

$$\{H_D(\vec{N}), H_H(N)\}$$

$$= \int_\Sigma \left[\frac{\delta H_H(N)}{\delta \vec{p}} \wedge \frac{\delta H_D(\vec{N})}{\delta \vec{\omega}} - \frac{\delta H_D(\vec{N})}{\delta \vec{p}} \wedge \frac{\delta H_H(N)}{\delta \vec{\omega}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_H(N)}{\delta p_a} \wedge \frac{\delta H_D(\vec{N})}{\delta \omega^a} - \frac{\delta H_D(\vec{N})}{\delta p_a} \wedge \frac{\delta H_H(N)}{\delta \omega^a} \right].$$

Let

$$\begin{cases} E_{kin}(\vec{\omega},\vec{p}) = \frac{1}{2} [q(p_b, *\omega^c) q(p_c, *\omega^b) - \frac{p^2}{n-2}] \mathrm{vol}_q \\ E_{pot}(\vec{\omega},\vec{p}) = -\frac{1}{2} S(q) \mathrm{vol}_q. \end{cases}$$

Then

$$E = E_{kin} + E_{pot},$$

thus

$$\mathcal{H}_H(N) = \int_\Sigma NE$$

$$= \int_\Sigma NE_{kin} + \int_\Sigma NE_{pot}$$

$$= \mathcal{H}_{H_{kin}}(N) + \mathcal{H}_{H_{pot}}(N)$$

and so

$$\{\mathcal{H}_D(\vec{N}),\mathcal{H}_H(N)\} = \{\mathcal{H}_D(\vec{N}),\mathcal{H}_{H_{kin}}(N)\} + \{\mathcal{H}_D(\vec{N}),\mathcal{H}_{H_{pot}}(N)\}.$$

• <u>kin</u>: We have

$$\{\mathcal{H}_D(\vec{N}),\mathcal{H}_{H_{kin}}(N)\}$$

$$= \int_\Sigma \left[\frac{\partial}{\partial p_a} (NE_{kin}) \wedge - L_{\vec{N}} p_a - L_{\vec{N}} \omega^a \wedge \frac{\partial}{\partial \omega^a} (NE_{kin})\right]$$

$$= - \int_\Sigma N\left[\frac{\partial}{\partial p_a} (E_{kin}) \wedge L_{\vec{N}} p_a + L_{\vec{N}} \omega^a \wedge \frac{\partial}{\partial \omega^a} (E_{kin})\right].$$

But

$$L_{\vec{N}}(E_{kin}) = \frac{\partial}{\partial p_a} E_{kin} \wedge L_{\vec{N}} p_a + L_{\vec{N}} \omega^a \wedge \frac{\partial}{\partial \omega^a} E_{kin}$$

$\Rightarrow$

$$- \int_\Sigma N\left[\frac{\partial}{\partial p_a} (E_{kin}) \wedge L_{\vec{N}} p_a + L_{\vec{N}} \omega^a \wedge \frac{\partial}{\partial \omega^a} (E_{kin})\right]$$

$$= - \int_\Sigma N L_{\vec{N}}(E_{kin})$$

$$= \int_\Sigma (L_{\vec{N}} N) E_{kin}.$$

Therefore

$$\{H_D(\vec{N}), H_{H_{kin}}(N)\} = H_{H_{kin}}(L_{\vec{N}}N).$$

• pot: We have

$$\{H_D(\vec{N}), H_{H_{pot}}(N)\}$$

$$= - \int_\Sigma L_{\vec{N}}\omega^a \wedge \frac{\delta H_{H_{pot}}(N)}{\delta \omega^a},$$

it being clear that

$$\frac{\delta H_{H_{pot}}(N)}{\delta p_a} = 0.$$

Write

$$\frac{\delta H_{H_{pot}}(N)}{\delta \omega^a} = - \frac{N}{2} (\Omega_{bc} \wedge *(\omega^b \wedge \omega^c \wedge \omega_a))$$

$$- *(\nabla_a dN - (\Delta_q N)\omega^a)$$

and hold the second term in abeyance for the moment -- then

$$\frac{1}{2} \int_\Sigma L_{\vec{N}}\omega^a \wedge N(\Omega_{bc} \wedge *(\omega^b \wedge \omega^c \wedge \omega_a))$$

$$= \frac{1}{2} \int_\Sigma L_{\vec{N}} *(\omega^b \wedge \omega^c) \wedge N\Omega_{bc}.$$

But

$$L_{\vec{N}}(N \wedge *(\omega^b \wedge \omega^c) \wedge \Omega_{bc})$$

$$= (L_{\vec{N}}N) \wedge *(\omega^b \wedge \omega^c) \wedge \Omega_{bc}$$

$$+ N \wedge L_{\vec{N}} *(\omega^b \wedge \omega^c) \wedge \Omega_{bc} + N*(\omega^b \wedge \omega^c) \wedge L_{\vec{N}} \Omega_{bc}$$

$\Rightarrow$

$$\frac{1}{2} \int_\Sigma L_{\vec{N}} *(\omega^b \wedge \omega^c) \wedge N\Omega_{bc}$$

$$= -\frac{1}{2} \int_\Sigma (L_{\vec{N}}N) \wedge *(\omega^b \wedge \omega^c) \wedge \Omega_{bc}$$

$$- \frac{1}{2} \int_\Sigma N*(\omega^b \wedge \omega^c) \wedge L_{\vec{N}} \Omega_{bc}$$

$$= -\frac{1}{2} \int_\Sigma (L_{\vec{N}}N) S(q) \text{vol}_q$$

$$- \frac{1}{2} \int_\Sigma N*(\omega^b \wedge \omega^c) \wedge L_{\vec{N}} \Omega_{bc}$$

$$= H_{H_{pot}}(L_{\vec{N}}N) - \frac{1}{2} \int_\Sigma N*(\omega^b \wedge \omega^c) \wedge L_{\vec{N}} \Omega_{bc}.$$

It remains to consider the contribution

$$\int_\Sigma L_{\vec{N}} \omega^a \wedge *(\nabla_a dN - (\Delta_q N)\omega^a).$$

Bearing in mind that this is a sum over the index a, replace δ_a in the earlier analysis by $L_{\vec{N}}$ — then

$$L_{\vec{N}} \omega^a \wedge *(\nabla_a dN - (\Delta_q N)\omega^a)$$

$$= \frac{N}{2} * (\omega^b \wedge \omega^c) \wedge L_{\vec{N}} \Omega_{bc}.$$

Therefore

$$\{H_D(\vec{N}), H_{H_{pot}}(N)\} = H_{H_{pot}}(L_{\vec{N}} N).$$

<u>Ad 4</u>: We have

$$\{H_R(W_1), H_R(W_2)\}$$

$$= \int_\Sigma \left[\frac{\delta H_R(W_2)}{\delta \vec{p}} \wedge \frac{\delta H_R(W_1)}{\delta \vec{\omega}} - \frac{\delta H_R(W_1)}{\delta \vec{p}} \wedge \frac{\delta H_R(W_2)}{\delta \vec{\omega}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_R(W_2)}{\delta p_c} \wedge \frac{\delta H_R(W_1)}{\delta \omega^c} - \frac{\delta H_R(W_1)}{\delta p_c} \wedge \frac{\delta H_R(W_2)}{\delta \omega^c} \right.$$

$$= \int_\Sigma \left[(W_2)^c{}_b \omega^b \wedge (W_1)^a{}_c p_a - (W_1)^c{}_b \omega^b \wedge (W_2)^a{}_c p_a \right]$$

$$= \int_\Sigma \left[(W_1)^a{}_c (W_2)^c{}_b - (W_2)^a{}_c (W_1)^c{}_b \right] \omega^b \wedge p_a$$

$$= \int_\Sigma [W_1, W_2]^a{}_b \omega^b \wedge p_a$$

$$= H_R([W_1, W_2]).$$

Remark: The elements figuring in the integrated rotational constraint are smooth functions $W:\Sigma \to \underline{so}(n-1)$. Agreeing to view $C^\infty(\Sigma;\underline{so}(n-1))$ as a Lie algebra, it follows that the arrow $W \to H_R(W)$ is a homomorphism.

[Note: On the basis of Items 2, 4, and 5, the integrated rotational constraints are an ideal in the full constraint algebra.]

<u>Ad 5</u>: We have

$$\{H_R(W), H_H(N)\}$$

$$= \int_\Sigma \left[\frac{\delta H_H(N)}{\delta \vec{p}} \wedge \frac{\delta H_R(W)}{\delta \vec{\omega}} - \frac{\delta H_R(W)}{\delta \vec{p}} \wedge \frac{\delta H_H(N)}{\delta \vec{\omega}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_H(N)}{\delta p_a} \wedge \frac{\delta H_R(W)}{\delta \omega^a} - \frac{\delta H_R(W)}{\delta p_a} \wedge \frac{\delta H_H(N)}{\delta \omega^a} \right]$$

$$= \int_\Sigma [N(q(p_c, *\omega^a)\omega^c - \frac{P}{n-2}\,\omega^a) \wedge W^b{}_a p_b$$

$$- W^a{}_b \omega^b \wedge N(q(p_a, *\omega^c)p_c - \frac{P}{n-2}\,p_a)$$

$$- W^a{}_b \omega^b \wedge -\frac{N}{2}\,(q(p_c, *\omega^d)q(p_d, *\omega^c) - \frac{P^2}{n-2}) *\omega^a$$

$$- W^a{}_b \omega^b \wedge N*(\mathrm{Ric}_a - \frac{1}{2}\,S(q)\omega^a)$$

$$- W^a{}_b \omega^b \wedge - *(\nabla_a dN - (\Delta_q N)\omega^a)].$$

Obviously,

- $N(-\frac{P}{n-2})\omega^a \wedge W^b{}_a p_b$

$$+ W^a{}_b \omega^b \wedge N(\frac{P}{n-2})p_a$$

$$= 0.$$

- $Nq(p_c, *\omega^a)\omega^c \wedge W^b{}_a p_b$

$$- W^a{}_b \omega^b \wedge N q(p_a, *\omega^c) p_c$$

$$= N W^b{}_a q(p_c, *\omega^a) q(p_b, *\omega^c) \mathrm{vol}_q$$

$$- N W^a{}_b q(p_c, *\omega^b) q(p_a, *\omega^c) \mathrm{vol}_q$$

$$= N W^b{}_a q(p_c, *\omega^a) q(p_b, *\omega^c) \mathrm{vol}_q$$

$$- N W^b{}_a q(p_c, *\omega^a) q(p_b, *\omega^c) \mathrm{vol}_q$$

$$= 0.$$

Proceeding, note that

$$W^a{}_b \omega^b \wedge *\omega^a = W^a{}_b q(\omega^b, \omega^a) \mathrm{vol}_q$$

$$= W^a{}_a \mathrm{vol}_q = 0.$$

So now, all that's left is

$$- W^a{}_b \omega^b \wedge N * \mathrm{Ric}_a$$

$$+ W^a{}_b \omega^b \wedge * \nabla_a dN.$$

Write

$$\mathrm{Ric}_a = \mathrm{Ric}_{ac} \omega^c.$$

Then

$$N W^a{}_b \omega^b \wedge * \mathrm{Ric}_a$$

$$= N W^a{}_b \mathrm{Ric}_{ac} \omega^b \wedge * \omega^c$$

$$= NW^a{}_b \mathrm{Ric}_{ac} q(\omega^b, \omega^c) \mathrm{vol}_q$$

$$= NW^a{}_b \mathrm{Ric}_{ab} \mathrm{vol}_q .$$

But Ric is symmetric and W is antisymmetric, hence

$$\int_\Sigma NW^a{}_b \mathrm{Ric}_{ab} \mathrm{vol}_q = 0.$$

Finally

$$W^a{}_b \omega^b \wedge * \nabla_a dN$$

$$= W^a{}_b \omega^b \wedge H_N(E_c, E_a) * \omega^c$$

$$= W^a{}_b H_N(E_c, E_a) \omega^b \wedge * \omega^c$$

$$= W^a{}_b H_N(E_c, E_a) q(\omega^b, \omega^c) \mathrm{vol}_q$$

$$= W^a{}_b H_N(E_b, E_a) \mathrm{vol}_q .$$

And, since H_N is symmetric,

$$\int_\Sigma W^a{}_b H_N(E_b, E_a) \mathrm{vol}_q = 0.$$

Ad 6: We have

$$\{\mathcal{H}_H(N_1), \mathcal{H}_H(N_2)\}$$

$$= \int_\Sigma \left[\frac{\delta \mathcal{H}_H(N_2)}{\delta \vec{p}} \wedge \frac{\delta \mathcal{H}_H(N_1)}{\delta \vec{\omega}} - \frac{\delta \mathcal{H}_H(N_1)}{\delta \vec{p}} \wedge \frac{\delta \mathcal{H}_H(N_2)}{\delta \vec{\omega}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_H(N_2)}{\delta p_a} \wedge \frac{\delta H_H(N_1)}{\delta \omega^a} - \frac{\delta H_H(N_1)}{\delta p_a} \wedge \frac{\delta H_H(N_2)}{\delta \omega^a} \right].$$

Insert the explicit formulas for

$$\left[\begin{array}{l} \dfrac{\delta H_H(N_i)}{\delta p_a} \\ \\ \dfrac{\delta H_H(N_i)}{\delta \omega^a} \end{array}\right. \qquad (i = 1,2).$$

Then, after cancellation, matters reduce to

$$\int_\Sigma [N_2(q(p_b, *\omega^a)\omega^b - \frac{P}{n-2}\,\omega^a) \wedge - *(\nabla_a dN_1 - (\Delta_q N_1)\omega^a)$$

$$+ N_1(q(p_b, *\omega^a)\omega^b - \frac{P}{n-2}\,\omega^a) \wedge *(\nabla_a dN_2 - (\Delta_q N_2)\omega^a)],$$

which we claim is the same as

$$\int_\Sigma (N_1 \nabla_a \nabla_b N_2 - N_2 \nabla_a \nabla_b N_1)\,\omega^a \wedge p_b.$$

To see this, recall that

$$q(p_b, *\omega^a)\omega^b - \frac{P}{n-2}\,\omega^a = -\,\omega_{0a}.$$

But

$$p_b = q(\omega_{0c}, \omega^c) *\omega^b - q(\omega_{0c}, \omega^b) *\omega^c$$

$\Rightarrow$

$$\omega^a \wedge p_b = (q(\omega_{0c}, \omega^c)\,\delta_{ab} - q(\omega_{0a}, \omega^b))\,\mathrm{vol}_q.$$

Therefore

$$N_1(q(p_b, *\omega^a)\,\omega^b - \frac{P}{n-2}\,\omega^a) \wedge (*\nabla_a dN_2)$$

$$= N_1(-\omega_{0a})\wedge\nabla_a\nabla_b N_2 *\omega^b$$

$$= N_1\nabla_a\nabla_b N_2(-q(\omega_{0a},\omega^b)\,\mathrm{vol}_q)$$

$$= N_1\nabla_a\nabla_b N_2(\omega^a\wedge p_b - q(\omega_{0c},\omega^c)\,\delta_{ab}\mathrm{vol}_q)$$

$$= (N_1\nabla_a\nabla_b N_2)\,\omega^a\wedge p_b - (N_1\nabla_a\nabla_a N_2)\,q(\omega_{0c},\omega^c)\,\mathrm{vol}_q$$

$$= (N_1\nabla_a\nabla_b N_2)\,\omega^a\wedge p_b - (N_1\Delta_q N_2)\,q(\omega_{0a},\omega^a)\,\mathrm{vol}_q.$$

On the other hand,

$$N_1(q(p_b,*\omega^a)\,\omega^b - \frac{p}{n-2}\,\omega^a)\wedge - (\Delta_q N_2)*\omega^a$$

$$= N_1(-\omega_{0a})\wedge - (\Delta_q N_2)*\omega^a$$

$$= (N_1\Delta_q N_2)\,\omega_{0a}\wedge *\omega^a$$

$$= (N_1\Delta_q N_2)\,q(\omega_{0a},\omega^a)\,\mathrm{vol}_q.$$

Reversing the roles of N_1 and N_2 then completes the verification. Moving on, write

$$\int_\Sigma (N_1\nabla_a\nabla_b N_2 - N_2\nabla_a\nabla_b N_1)\,\omega^a\wedge p_b$$

$$= \int_\Sigma [\nabla_a(N_1\nabla_b N_2 - N_2\nabla_b N_1)$$

$$- (\nabla_a N_1\nabla_b N_2 - \nabla_a N_2\nabla_b N_1)]\,\omega^a\wedge p_b.$$

Now use the identity

$$(\nabla_a X^b)\omega^a = L_X\omega^b + \omega^b{}_a(X)\omega^a$$

valid for any $X\in\mathcal{D}^1(\Sigma)$ (cf infra). Thus let

$$X = N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1.$$

Then

$$\nabla_a(N_1\nabla_b N_2 - N_2\nabla_b N_1)\omega^a\wedge p_b$$

$$= L_{(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)}\omega^b\wedge p_b$$

$$+ \omega^b{}_a(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)\omega^a\wedge p_b$$

=>

$$\int_\Sigma \nabla_a(N_1\nabla_b N_2 - N_2\nabla_b N_1)\omega^a\wedge p_b$$

$$= H_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)$$

$$+ H_R(q(N_1 dN_2 - N_2 dN_1, \omega^a{}_b)).$$

As for what remains, viz.

$$\int_\Sigma - (\nabla_a N_1\nabla_b N_2 - \nabla_a N_2\nabla_b N_1)\omega^a\wedge p_b,$$

observe that

$$q(dN_1 \wedge dN_2, \omega^b \wedge \omega^a)$$

$$= \det \begin{bmatrix} q(dN_1,\omega^b) & q(dN_1,\omega^a) \\ q(dN_2,\omega^b) & q(dN_2,\omega^a) \end{bmatrix}$$

$$= \det \begin{bmatrix} \nabla_b N_1 & \nabla_a N_1 \\ \nabla_b N_2 & \nabla_a N_2 \end{bmatrix}$$

$$= - (\nabla_a N_1 \nabla_b N_2 - \nabla_a N_2 \nabla_b N_1)$$

$\Rightarrow$

$$\int_\Sigma - (\nabla_a N_1 \nabla_b N_2 - \nabla_a N_2 \nabla_b N_1) \omega^a \wedge p_b$$

$$= \int_\Sigma q(dN_1 \wedge dN_2, \omega^b \wedge \omega^a) \omega^a \wedge p_b$$

$$= \int_\Sigma q(dN_1 \wedge dN_2, \omega^a \wedge \omega^b) \omega^b \wedge p_a$$

$$= H_R(q(dN_1 \wedge dN_2, \omega^a \wedge \omega_b)).$$

[Note: In the ADM sector of $T^*\underline{Q}$, the Poisson bracket

$$\{H_H(N_1), H_H(N_2)\}$$

equals

$$H_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1).]$$

<u>Details</u> Here is the proof that $\forall\ X \in \mathcal{D}^1(\Sigma)$,

$$(\nabla_b X^a)\omega^b = L_X\omega^a + \omega^a{}_b(X)\omega^b.$$

I.e.:

$$(\nabla_b X^a)\omega^b = L_X\omega^a - \nabla_X\omega^a.$$

Start with the RHS — then

$$\begin{cases} (L_X\omega^a)(Y) = X\omega^a(Y) - \omega^a([X,Y]) \\ (\nabla_X\omega^a)(Y) = X\omega^a(Y) - \omega^a(\nabla_X Y) \end{cases}$$

=>

$$(L_X\omega^a)(Y) - (\nabla_X\omega^a)(Y)$$

$$= \omega^a(\nabla_X Y - [X,Y])$$

$$= \omega^a(\nabla_Y X).$$

Write $X = X^c E_c$ and take $Y = E_b$:

$$\omega^a(\nabla_{E_b} X) = \omega^a(\nabla_{E_b}(X^c E_c))$$

$$= \omega^a((\nabla_{E_b} X^c)E_c + X^c\nabla_{E_b} E_c)$$

$$= \nabla_{E_b} X^a + X^c\omega^a(\omega^d{}_c(E_b)E_d)$$

$$= \nabla_{E_b} X^a + X^c\omega^a{}_c(E_b)$$

$$= E_b X^a + X^c \omega^a{}_c(E_b)$$

$$= dX^a(E_b) + X^c \omega^a{}_c(E_b).$$

Turning to the LHS,

$$\nabla X = E_a \otimes (dX^a + \omega^a{}_c X_c)$$

=>

$$\nabla_b X^a = \nabla X(\omega^a, E_b)$$

$$= dX^a(E_b) + \omega^a{}_c(E_b) X^c.$$

Remark: The relation

$$\omega_{0a} = - q(p_b, *\omega^a)\omega^b + \frac{P}{n-2}\omega^a$$

is really a definition, though, for consistency, one should check that

$$p_a = \omega_{0b} \wedge * (\omega^a \wedge \omega^b)$$

or still,

$$p_a = q(\omega_{0b}, \omega^b) * \omega^a - q(\omega_{0b}, \omega^a) * \omega^b.$$

1. $q(\omega_{0b}, \omega^b) * \omega^a$

$$= q(- q(p_c, *\omega^b)\omega^c + \frac{P}{n-2}\omega^b, \omega^b) * \omega^a$$

$$= - q(p_c, *\omega^b)\delta_{cb} * \omega^a + \frac{n-1}{n-2} P * \omega^a$$

$$= - q(p_b, *\omega^b) *\omega^a + \frac{n-1}{n-2} P*\omega^a$$

$$= - P*\omega^a + \frac{n-1}{n-2} P*\omega^a$$

$$= \frac{P}{n-2} *\omega^a.$$

2. $- q(\omega_{0b}, \omega^a) *\omega^b$

$$= - q(- q(p_c, *\omega^b)\omega^c + \frac{P}{n-2} \omega^b, \omega^a) *\omega^b$$

$$= q(p_c, *\omega^b)\delta_{ca} *\omega^b - \frac{P}{n-2} \delta_{ba} *\omega^b$$

$$= q(p_a, *\omega^b) *\omega^b - \frac{P}{n-2} *\omega^a.$$

So

$$1 + 2 = q(p_a, *\omega^b) *\omega^b = p_a.$$

Section 46: Field Equations Let M be a connected C^∞ manifold of dimension n.

Assume: M is parallelizable.

Notation: cof_M is the set of ordered coframes on M.

[Note: Each $\omega = \{\omega^1,\ldots,\omega^n\}$ in cof_M gives rise to an element $g \in \underline{M}_{k,n-k}$, viz.

$$g = -\omega^1 \otimes \omega^1 - \cdots - \omega^k \otimes \omega^k + \omega^{k+1} \otimes \omega^{k+1} + \cdots + \omega^n \otimes \omega^n.]$$

Definition: Let $\omega = \{\omega^1,\ldots,\omega^n\}$ be an element of cof_M -- then a variation of ω is a curve

$$\varepsilon \to \omega(\varepsilon) = (\omega^1(\varepsilon),\ldots,\omega^n(\varepsilon)),$$

where

$$\omega^i(\varepsilon) = \omega^i + \varepsilon\delta\omega^i$$

and the $\delta\omega^i \in \Lambda^1 M$ have compact support.

[Note: This usage of the symbol δ conflicts with that used for the interior derivative which, to eliminate any possibility of confusion, will be denoted in this section by d^*.]

Let $F:\text{cof}_M \to V$, where V is a vector space over $\underline{R}$ -- then by definition,

$$D_\omega F(\delta\omega) = \frac{d}{d\varepsilon} F(\omega(\varepsilon))\Big|_{\varepsilon=0}.$$

[Note: It is customary to write δF instead of $D_\omega F(\delta\omega)$ and F instead of $F(\omega)$. This shorthand is computationally convenient and normally should not lead to misunderstandings.]

In what follows, we shall use the abbreviation $\omega + \varepsilon\delta\omega$ to designate a variation of ω.

Rules

- Suppose that $\alpha:\text{cof}_M \to \Lambda^p M$ — then

$$\delta d\alpha = d\delta\alpha.$$

[Note: Spelled out,

$$\frac{d}{d\varepsilon} d(\alpha(\omega + \varepsilon\delta\omega))\Big|_{\varepsilon=0} = d \frac{d}{d\varepsilon} \alpha(\omega + \varepsilon\delta\omega)\Big|_{\varepsilon=0}.]$$

• Suppose that $\alpha:\text{cof}_M \to \Lambda^p M$ and $\beta:\text{cof}_M \to \Lambda^q M$ — then

$$\delta(\alpha\wedge\beta) = \delta\alpha\wedge\beta + \alpha\wedge\delta\beta.$$

[Note: Spelled out,

$$\frac{d}{d\varepsilon} \alpha(\omega + \varepsilon\delta\omega)\wedge\beta(\omega + \varepsilon\delta\omega)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} \alpha(\omega + \varepsilon\delta\omega)\Big|_{\varepsilon=0}\wedge\beta(\omega) + \alpha(\omega)\wedge \frac{d}{d\varepsilon} \beta(\omega + \varepsilon\delta\omega)\Big|_{\varepsilon=0}.]$$

<u>N.B.</u>

1. In general, δ does not commute with the Hodge star:

$$\delta \circ \star \neq \star \circ \delta.$$

2. In general, δ does not commute with the interior derivative:

$$\delta \circ d^* \neq d^* \circ \delta.$$

Rappel:

$$\omega_j = \varepsilon_j\omega^j$$

=>

$$\iota_{\omega_j}\omega^i = g(\omega^i,\omega_j) = \delta^i{}_j.$$

<u>LEMMA</u> We have

$$\delta(\omega^{i_1} \wedge \dots \wedge \omega^{i_p}) = \delta\omega^j \wedge \iota_{\omega_j}(\omega^{i_1} \wedge \dots \wedge \omega^{i_p}).$$

[For

$$\delta(\omega^{i_1} \wedge \dots \wedge \omega^{i_p})$$

$$= \delta\omega^{i_1} \wedge \omega^{i_2} \wedge \dots \wedge \omega^{i_p} + \cdots + \omega^{i_1} \wedge \dots \wedge \omega^{i_{p-1}} \wedge \delta\omega^{i_p}.$$

On the other hand,

$$\iota_{\omega_j}(\omega^{i_1} \wedge \dots \wedge \omega^{i_p})$$

$$= \iota_{\omega_j}(\omega^{i_1}) \wedge \omega^{i_2} \wedge \dots \wedge \omega^{i_p} - \omega^{i_1} \wedge \iota_{\omega_j}(\omega^{i_2}) \wedge \dots \wedge \omega^{i_p} + \cdots$$

$$= \delta^{i_1}{}_j \wedge \omega^{i_2} \wedge \dots \wedge \omega^{i_p} - \omega^{i_1} \wedge \delta^{i_2}{}_j \wedge \dots \wedge \omega^{i_p} + \cdots$$

$\Rightarrow$

$$\delta\omega^j \wedge \iota_{\omega_j}(\omega^{i_1} \wedge \dots \wedge \omega^{i_p})$$

$$= \delta\omega^{i_1} \wedge \omega^{i_2} \wedge \dots \wedge \omega^{i_p} - \delta\omega^{i_2} \wedge \omega^{i_1} \wedge \dots \wedge \omega^{i_p} + \cdots$$

$$= \delta(\omega^{i_1} \wedge \dots \wedge \omega^{i_p}).]$$

Rappel:

$$\theta^{i_1 \dots i_p} = *(\omega^{i_1} \wedge \dots \wedge \omega^{i_p})$$

$$= \frac{1}{(n-p)!} \varepsilon_{i_1} \cdots \varepsilon_{i_p} \varepsilon_{i_1 \ldots i_p j_{p+1} \ldots j_n} \omega^{j_{p+1}} \wedge \ldots \wedge \omega^{j_n}.$$

<u>LEMMA</u> We have

$$\delta\theta^{i_1 \ldots i_p} = \delta\omega^j \wedge \iota_{\omega_j} \theta^{i_1 \ldots i_p}.$$

[In fact,

$$\delta\theta^{i_1 \ldots i_p}$$

$$= \frac{n-p}{(n-p)!} \varepsilon_{i_1} \cdots \varepsilon_{i_p} \varepsilon_{i_1 \ldots i_p j_{p+1} j_{p+2} \ldots j_n} (\delta\omega^{j_{p+1}}) \wedge (\omega^{j_{p+2}} \wedge \ldots \wedge \omega^{j_n})$$

$$= \frac{1}{(n-p-1)!} \varepsilon_{i_1} \cdots \varepsilon_{i_p} \varepsilon_{j_{p+1}} \varepsilon_{j_{p+1}} \varepsilon_{i_1 \ldots i_p j_{p+1} j_{p+2} \ldots j_n} (\delta\omega^{j_{p+1}}) \wedge (\omega^{j_{p+2}} \wedge \ldots \wedge \omega^{j_n})$$

$$= \delta\omega^{j_{p+1}} \wedge \varepsilon_{j_{p+1}} \theta^{i_1 \ldots i_p j_{p+1}}$$

$$= \delta\omega^{j_{p+1}} \wedge \theta^{i_1 \ldots i_p}{}_{j_{p+1}}$$

$$= \delta\omega^{j_{p+1}} \wedge \iota_{\omega_{j_{p+1}}} \theta^{i_1 \ldots i_p}$$

$$= \delta\omega^j \wedge \iota_{\omega_j} \theta^{i_1 \ldots i_p}.]$$

Example:

- $\delta \mathrm{vol}_g = \delta * 1$

$$= \delta\omega^j \wedge \iota_{\omega_j} *1$$

$$= \delta\omega^j \wedge *(1 \wedge \omega_j)$$

$$= \delta\omega^j \wedge *\omega_j.$$

• $$\delta * \mathrm{vol}_g = \delta * (\omega^1 \wedge \ldots \wedge \omega^n)$$

$$= \delta\theta^{1\ldots n}$$

$$= \delta\omega^j \wedge \iota_{\omega_j} \theta^{1\ldots n}$$

$$= 0.$$

For another example, define

$$L : \mathrm{cof}_M \to \Lambda^n M$$

by

$$L(\omega) = \frac{1}{2}\, \Omega_{ij} \wedge \theta^{ij}.$$

Then

$$\delta L = \frac{1}{2}\, (\delta\Omega_{ij} \wedge \theta^{ij} + \Omega_{ij} \wedge \delta\theta^{ij}).$$

The computation of $\Omega_{ij} \wedge \delta\theta^{ij}$ is immediate:

$$\Omega_{ij} \wedge \delta\theta^{ij} = \Omega_{ij} \wedge \delta\omega^k \wedge \theta^{ij}{}_k$$

$$= \delta\omega^k \wedge \Omega_{ij} \wedge \theta^{ij}{}_k.$$

Turning to the computation of $\delta\Omega_{ij}\wedge\theta^{ij}$, note first that

$$\delta\Omega_{ij} = \delta(d\omega_{ij} + \omega_{ik}\wedge\omega^k{}_j)$$

$$= d\delta\omega_{ij} + \delta\omega_{ik}\wedge\omega^k{}_j + \omega_{ik}\wedge\delta\omega^k{}_j,$$

so

$$\delta\Omega_{ij}\wedge\theta^{ij} = \delta\Omega_{ij}\wedge*(\omega^i\wedge\omega^j)$$

$$= d\delta\omega_{ij}\wedge*(\omega^i\wedge\omega^j)$$

$$+ \delta\omega_{ik}\wedge\omega^k{}_j\wedge*(\omega^i\wedge\omega^j) + \omega_{ik}\wedge\delta\omega^k{}_j\wedge*(\omega^i\wedge\omega^j).$$

On the other hand,

$$d(\delta\omega_{ij}\wedge*(\omega^i\wedge\omega^j))$$

$$= d\delta\omega_{ij}\wedge*(\omega^i\wedge\omega^j) - \delta\omega_{ij}\wedge d*(\omega^i\wedge\omega^j).$$

And

$$d*(\omega^i\wedge\omega^j)$$

$$= -\omega^i{}_a\wedge*(\omega^a\wedge\omega^j) - \omega^j{}_a\wedge*(\omega^i\wedge\omega^a).$$

But

1. $\delta\omega_{ij}\wedge\omega^i{}_a\wedge*(\omega^a\wedge\omega^j)$

$$= \delta\omega_{aj}\wedge\omega^a{}_i\wedge*(\omega^i\wedge\omega^j)$$

$$= \delta\omega_{ai}\wedge\omega^{a}{}_{j}\wedge*(\omega^{j}\wedge\omega^{i})$$

$$= \delta\omega_{ki}\wedge\omega^{k}{}_{j}\wedge*(\omega^{j}\wedge\omega^{i})$$

$$= \delta\omega_{ik}\wedge\omega^{k}{}_{j}\wedge*(\omega^{i}\wedge\omega^{j}).$$

2. $\delta\omega_{ij}\wedge\omega^{j}{}_{a}\wedge*(\omega^{i}\wedge\omega^{a})$

$$= \delta\omega_{ia}\wedge\omega^{a}{}_{j}\wedge*(\omega^{i}\wedge\omega^{j})$$

$$= -\ \omega^{a}{}_{j}\wedge\delta\omega_{ia}\wedge*(\omega^{i}\wedge\omega^{j})$$

$$= -\ \omega_{aj}\wedge\delta\omega_{i}{}^{a}\wedge*(\omega^{i}\wedge\omega^{j})$$

$$= \omega_{aj}\wedge\delta\omega^{a}{}_{i}\wedge*(\omega^{i}\wedge\omega^{j})$$

$$= \omega_{kj}\wedge\delta\omega^{k}{}_{i}\wedge*(\omega^{i}\wedge\omega^{j})$$

$$= \omega_{ki}\wedge\delta\omega^{k}{}_{j}\wedge*(\omega^{j}\wedge\omega^{i})$$

$$= \omega_{ik}\wedge\delta\omega^{k}{}_{j}\wedge*(\omega^{i}\wedge\omega^{j}).$$

Therefore

$$\delta\Omega_{ij}\wedge\theta^{ij} = d(\delta\omega_{ij}\wedge*(\omega^{i}\wedge\omega^{j})) = d(\delta\omega_{ij}\wedge\theta^{ij}).$$

Modulo the usual provisos, put

$$\mathcal{L}(\omega) = \int_{M} L(\omega).$$

Since the exact term $\delta\Omega_{ij}\wedge\theta^{ij}$ is dynamically irrelevant, the formalism dictates that

$$\frac{\delta L}{\delta\omega^k} = \frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij}{}_k.$$

To see the significance of this, write

$$\frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij}{}_k$$

$$= \frac{1}{2}\,[g(\Omega_{ij},\omega^i\wedge\omega^j)*\omega_k + g(\Omega_{ij},\omega^j\wedge\omega_k)*\omega^i + g(\Omega_{ij},\omega_k\wedge\omega^i)*\omega^j]$$

$$= -*(\mathrm{Ric}_k - \frac{1}{2}\,S(g)\omega_k),$$

Ric_k the Ricci 1-form. Accordingly, if we define the Einstein 1-form by

$$\mathrm{Ein}_k = \mathrm{Ric}_k - \frac{1}{2}\,S(g)g_k,$$

then the vanishing of the $\frac{\delta L}{\delta\omega^k}$ $(k = 1,\ldots,n)$ is equivalent to the vanishing of $\mathrm{Ein}(g)$.

[Note:

$$\mathrm{Ein} = \mathrm{Ric} - \frac{1}{2}\,S(g)g$$

$\Rightarrow$

$$\mathrm{Ein}_k = \mathrm{Ric}_k - \frac{1}{2}\,S(g)g_k,$$

where

$$g_k = g_{k\ell}\omega^\ell$$

$$= g_{kk}\omega^k = \varepsilon_k\omega^k = \omega_k.$$

One can also incorporate a cosmological constant λ: Take

$$L_\lambda(\omega) = \frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij} - \lambda \mathrm{vol}_g$$

and let

$$\mathcal{L}_\lambda(\omega) = \int_M L_\lambda(\omega).$$

Since $\delta\mathrm{vol}_g = \delta\omega^k\wedge *\omega_k$, the foregoing analysis implies that

$$\frac{\delta \mathcal{L}_\lambda}{\delta\omega^k} = -*(\mathrm{Ric}_k - \frac{1}{2}\,S(g)\omega_k + \lambda\omega_k).]$$

Exercise: Compute $\frac{\delta \mathcal{L}}{\delta\omega^k}$ if L =

$$\left[\begin{array}{l} \frac{1}{2}\,\Omega^{ij}\wedge *\Omega_{ij} \\ \\ \frac{1}{2}\,\Omega^{ij}\wedge\omega_j\wedge *(\Omega_{ik}\wedge\omega^k) \\ \\ \frac{1}{2}\,\Omega^{ij}\wedge\omega^k\wedge *(\Omega_{ik}\wedge\omega_j) \\ \\ \frac{1}{2}\,\Omega^{ij}\wedge(\omega_i\wedge\omega_j)\wedge *(\Omega^{k\ell}\wedge(\omega_k\wedge\omega_\ell)) \\ \\ \frac{1}{2}\,\Omega^{ij}\wedge(\omega_j\wedge\omega_k)\wedge *(\Omega^{k\ell}\wedge(\omega_\ell\wedge\omega_i)) \\ \\ \frac{1}{2}\,\Omega^{ij}\wedge(\omega^k\wedge\omega^\ell)\wedge *(\Omega_{k\ell}\wedge(\omega_i\wedge\omega_j)). \end{array}\right.$$

Given $\alpha:\mathrm{cof}_M \rightarrow \Lambda^p M$, write

$$\alpha = \frac{1}{p!}\,\alpha_{i_1\ldots i_p}\omega^{i_1}\wedge\ldots\wedge\omega^{i_p}.$$

<u>$\delta{*}\alpha$</u>:

$$*\alpha = \frac{1}{p!}\,\alpha_{i_1\ldots i_p}\theta^{i_1\ldots i_p}$$

=>

$$\delta{*}\alpha = \frac{1}{p!}\,\delta\alpha_{i_1\ldots i_p}\theta^{i_1\ldots i_p}$$

$$+\frac{1}{p!}\,\alpha_{i_1\ldots i_p}\delta\theta^{i_1\ldots i_p}$$

$$= \frac{1}{p!}\,\delta\alpha_{i_1\ldots i_p}\theta^{i_1\ldots i_p}$$

$$+\frac{1}{p!}\,\alpha_{i_1\ldots i_p}\delta\omega^j\wedge\iota_{\omega_j}\theta^{i_1\ldots i_p}$$

$$= \frac{1}{p!}\,\delta\alpha_{i_1\ldots i_p}\theta^{i_1\ldots i_p}$$

$$+\ \delta\omega^j\wedge\iota_{\omega_j}{*}\alpha.$$

<u>$*\delta\alpha$</u>:

$$\delta\alpha = \frac{1}{p!}\,\delta\alpha_{i_1\ldots i_p}\omega^{i_1}\wedge\ldots\wedge\omega^{i_p}$$

$$+\frac{1}{p!}\,\alpha_{i_1\ldots i_p}\delta(\omega^{i_1}\wedge\ldots\wedge\omega^{i_p})$$

$$= \frac{1}{p!}\, \delta\alpha_{i_1 \ldots i_p} \omega^{i_1} \wedge \ldots \wedge \omega^{i_p}$$

$$+ \frac{1}{p!}\, \alpha_{i_1 \ldots i_p} \delta\omega^j \wedge \iota_{\omega_j} (\omega^{i_1} \wedge \ldots \wedge \omega^{i_p})$$

$$= \frac{1}{p!}\, \delta\alpha_{i_1 \ldots i_p} \omega^{i_1} \wedge \ldots \wedge \omega^{i_p}$$

$$+ \delta\omega^j \wedge \iota_{\omega_j} \alpha$$

$\Rightarrow$

$$\star\delta\alpha = \frac{1}{p!}\, \delta\alpha_{i_1 \ldots i_p} \theta^{i_1 \ldots i_p} + \star(\delta\omega^j \wedge \iota_{\omega_j} \alpha).$$

Therefore

$$\delta\star\alpha - \star\delta\alpha$$

$$= \delta\omega^j \wedge \iota_{\omega_j} \star\alpha - \star(\delta\omega^j \wedge \iota_{\omega_j} \alpha).$$

Remark:

$$\star(\iota_{\delta\omega^j} (\omega_j \wedge \alpha))$$

$$= (-1)^{n-1} \star(\omega_j \wedge \alpha) \wedge \delta\omega^j$$

$$= (-1)^{n-1} (-1)^{n-p-1} \delta\omega^j \wedge \star(\omega_j \wedge \alpha)$$

$$= (-1)^p \delta\omega^j \wedge * (\omega_j \wedge \alpha)$$

$$= (-1)^p (-1)^p \delta\omega^j \wedge * (\alpha \wedge \omega_j)$$

$$= \delta\omega^j \wedge \iota_{\omega_j} * \alpha.$$

Thus

$$\delta * \alpha - * \delta\alpha$$

$$= * (\iota_{\delta\omega^j} (\omega_j \wedge \alpha) - \delta\omega^j \wedge \iota_{\omega_j} \alpha)$$

or still,

$$\delta * \alpha - * \delta\alpha$$

$$= * (\iota_{\delta\omega_j} (\omega^j \wedge \alpha) - \delta\omega_j \wedge \iota_{\omega^j} \alpha).$$

<u>THEOREM</u> Suppose that $\alpha, \beta : \mathrm{cof}_M \to \Lambda^p M$ -- then

$$\delta(\alpha \wedge * \beta) = \delta\alpha \wedge * \beta + \alpha \wedge * \delta\beta - \delta\omega_\ell \wedge J^\ell,$$

where

$$J^\ell = \iota_{\omega^\ell} \beta \wedge * \alpha - (-1)^p \alpha \wedge \iota_{\omega^\ell} * \beta.$$

[We have

$$\delta(\alpha \wedge * \beta) = \delta\alpha \wedge * \beta + \alpha \wedge \delta * \beta$$

$$= \delta\alpha\wedge *\beta + \alpha\wedge *\delta\beta + \alpha\wedge(\delta\omega_\ell\wedge\iota_{\omega^\ell}*\beta - *(\delta\omega_\ell\wedge\iota_{\omega^\ell}\beta))$$

$$= \delta\alpha\wedge *\beta + \alpha\wedge *\delta\beta + (-1)^p\delta\omega_\ell\wedge\alpha\wedge\iota_{\omega^\ell}*\beta - \delta\omega_\ell\wedge\iota_{\omega^\ell}\beta\wedge *\alpha$$

$$= \delta\alpha\wedge *\beta + \alpha\wedge *\delta\beta + \delta\omega_\ell\wedge((-1)^p\alpha\wedge\iota_{\omega^\ell}*\beta - \iota_{\omega^\ell}\beta\wedge *\alpha)$$

$$= \delta\alpha\wedge *\beta + \alpha\wedge *\delta\beta - \delta\omega_\ell\wedge J^\ell.]$$

The J^ℓ are (n-1)-forms and the collection $\{J^1,\ldots,J^n\}$ is called the <u>current</u> attached to the pair (α,β).

<u>Construction</u> Let $J^\ell\in\Lambda^{n-1}M$ $(\ell = 1,\ldots,n)$. Write

$$J^\ell = J^{\ell k}*\omega_k.$$

Then

$$*(\omega^k\wedge J^\ell) = *(\omega^k\wedge J^{\ell m}*\omega_m)$$

$$= J^{\ell m}*(\omega^k\wedge *\omega_m)$$

$$= J^{\ell m}*(g(\omega^k,\omega_m)\mathrm{vol}_g)$$

$$= (-1)^\iota J^{\ell k}.$$

Therefore $J^{\ell k} = J^{k\ell}$ iff $\omega^k\wedge J^\ell = \omega^\ell\wedge J^k$.

- $\omega_\ell \wedge J^\ell = \omega_\ell \wedge J^{\ell k} * \omega_k$

$$= J^{\ell k} \omega_\ell \wedge * \omega_k$$

$$= J^{\ell k} g(\omega_\ell, \omega_k) \mathrm{vol}_g$$

$$= J^{\ell k} \epsilon_k g(\omega_\ell, \omega^k) \mathrm{vol}_g$$

$$= J^\ell{}_k \delta^k{}_\ell \mathrm{vol}_g$$

$$= J^\ell{}_\ell * 1.$$

- $\iota_{\omega_\ell} J^\ell = \iota_{\omega_\ell} J^{\ell k} * \omega_k$

$$= J^{\ell k} \iota_{\omega_\ell} * \omega_k$$

$$= J^{\ell k} * (\omega_k \wedge \omega_\ell)$$

$$= \frac{J^{\ell k}}{2} * (\omega_k \wedge \omega_\ell) + \frac{J^{\ell k}}{2} * (\omega_k \wedge \omega_\ell)$$

$$= \frac{J^{\ell k}}{2} * (\omega_k \wedge \omega_\ell) - \frac{J^{\ell k}}{2} * (\omega_\ell \wedge \omega_k)$$

$$= \frac{J^{\ell k}}{2} * (\omega_k \wedge \omega_\ell) - \frac{J^{k \ell}}{2} * (\omega_k \wedge \omega_\ell)$$

$$= \frac{1}{2}\,(J^{\ell k} - J^{k\ell}) * (\omega_k \wedge \omega_\ell)$$

$$= J^{[\ell k]} * (\omega_k \wedge \omega_\ell).$$

Consider the trace of the current attached to the pair (α,β):

$$(-1)^{\iota} J^{\ell}{}_{\ell} = *(\omega_\ell \wedge J^\ell)$$

$$= *(\omega_\ell \wedge (\iota_{\omega^\ell}\beta\wedge *\alpha - (-1)^p \alpha\wedge\iota_{\omega^\ell} *\beta))$$

$$= *(\omega_\ell\wedge\iota_{\omega^\ell}\beta\wedge *\alpha - \alpha\wedge\omega_\ell\wedge\iota_{\omega^\ell} *\beta)$$

$$= *(p\beta\wedge *\alpha - (n-p)\,\alpha\wedge *\beta)$$

$$= *(\alpha\wedge *p\beta - (n-p)\,\alpha\wedge *\beta)$$

$$= -\ (n-2p) * (\alpha\wedge *\beta).$$

Therefore $J^{\ell}{}_{\ell} = 0$ iff $n = 2p$.

Now take $\alpha = \beta$ — then

$$J^\ell = \iota_{\omega^\ell}\alpha\wedge *\alpha - (-1)^p \alpha\wedge\iota_{\omega^\ell} *\alpha.$$

Observation:

$$\iota_{\omega^\ell}(\alpha\wedge *\alpha) = \iota_{\omega^\ell}\alpha\wedge *\alpha + (-1)^p\alpha\wedge\iota_{\omega^\ell} *\alpha$$

$\Rightarrow$

$$-\frac{1}{2}\,\iota_{\omega^\ell}(\alpha\wedge *\alpha) = -\frac{1}{2}\,\iota_{\omega^\ell}\alpha\wedge *\alpha - \frac{1}{2}\,(-1)^p\alpha\wedge\iota_{\omega^\ell} *\alpha$$

=>

$$\iota_{\omega^\ell}\alpha\wedge*\alpha - \frac{1}{2}\,\iota_{\omega^\ell}(\alpha\wedge*\alpha)$$

$$= \frac{1}{2}\,\iota_{\omega^\ell}\alpha\wedge*\alpha - \frac{1}{2}\,(-1)^p\alpha\wedge\iota_{\omega^\ell}*\alpha$$

$$= \frac{1}{2}\,J^\ell.$$

Rappel: $\forall\ \alpha\in\Lambda^pM$,

$$\iota_{E^\ell}\alpha\wedge\iota_{E_\ell}*\alpha = 0.$$

I.e.:

$$\iota_{\omega^\ell}\alpha\wedge\iota_{\omega_\ell}*\alpha = 0.$$

Thus

$$\iota_{\omega_\ell}J^\ell = 2\iota_{\omega_\ell}(\iota_{\omega^\ell}\alpha\wedge*\alpha - \frac{1}{2}\,\iota_{\omega^\ell}(\alpha\wedge*\alpha))$$

$$= 2(\iota_{\omega_\ell}\iota_{\omega^\ell}\alpha\wedge*\alpha + (-1)^{p-1}\iota_{\omega^\ell}\alpha\wedge\iota_{\omega_\ell}*\alpha - \frac{1}{2}\,\iota_{\omega_\ell}\iota_{\omega^\ell}(\alpha\wedge*\alpha))$$

$$= 2(-1)^{p-1}\iota_{\omega^\ell}\alpha\wedge\iota_{\omega_\ell}*\alpha$$

$$= 0.$$

But then

$$J^{[\ell k]} = 0,$$

so in this case,

$$J^{\ell k} = J^{k\ell}.$$

Let $L:\mathrm{cof}_M \to \Lambda^n M$, where L depends on ω and $d\omega$:

$$L = L(\omega^1,\ldots,\omega^n,d\omega^1,\ldots,d\omega^n).$$

Then

$$\begin{aligned}\delta L &= D_\omega L(\omega)\\ &= \frac{d}{d\varepsilon} L(\omega + \varepsilon\delta\omega)\Big|_{\varepsilon=0}\\ &= \delta\omega^i \wedge \frac{\partial L}{\partial \omega^i} + \delta d\omega^i \wedge \frac{\partial L}{\partial d\omega^i}\ .\end{aligned}$$

Here

$$\begin{cases} \dfrac{\partial L}{\partial \omega^i} \in \Lambda^{n-1}M \\[2ex] \dfrac{\partial L}{\partial d\omega^i} \in \Lambda^{n-2}M. \end{cases}$$

Now rewrite δL as

$$\delta\omega^i \wedge \left[\frac{\partial L}{\partial \omega^i} + d\,\frac{\partial L}{\partial d\omega^i} \right] + d\left(\delta\omega^i \wedge \frac{\partial L}{\partial d\omega^i}\right).$$

Definition: ω satisfies the <u>field equations</u> per L provided $\forall$ i,

$$\frac{\partial L}{\partial \omega^i} + d\,\frac{\partial L}{\partial d\omega^i} = 0.$$

[Note: Formally, if $\mathcal{L} = \int_M L$, then

$$\frac{\delta L}{\delta\omega^i} = \frac{\partial L}{\partial\omega^i} + d\,\frac{\partial L}{\partial d\omega^i}\ .]$$

Example: Take n = 4 and put

$$L(\omega) = d\omega_i \wedge d\omega^i.$$

Then

$$\delta L = \delta d\omega_i \wedge d\omega^i + d\omega_i \wedge \delta d\omega^i$$

$$= \delta d\omega^i \wedge d\omega_i + \delta d\omega^i \wedge d\omega_i$$

$$= 2(\delta d\omega^i \wedge d\omega_i)$$

$$= 2(\delta\omega^i \wedge dd\omega_i + d(\delta\omega^i \wedge d\omega_i))$$

=>

$$\frac{\partial L}{\partial\omega^i} = 2dd\omega_i = 0.$$

Definition: The lagrangian of <u>teleparallel gravity</u> is the combination

$$L(\ = L(\rho_0,\rho_1,\rho_2,\rho_3)) = \frac{1}{2}\,(\rho_0 L^0 + \rho_1 L^1 + \rho_2 L^2 + \rho_3 L^3),$$

where the ρ_i are real and

$$\begin{bmatrix} L^0 = \frac{1}{n}\,(\omega_i \wedge *\omega^i) = \mathrm{vol}_g \\ L^1 = d\omega_i \wedge * d\omega^i \\ L^2 = (d\omega_i \wedge \omega^i) \wedge * (d\omega_j \wedge \omega^j) \\ L^3 = (d\omega_i \wedge \omega^j) \wedge * (d\omega_j \wedge \omega^i). \end{bmatrix}$$

Rappel: We have

$$\frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij} = -\,d(\omega_i\wedge*d\omega^i)$$

$$+\,\frac{1}{4}\,(d\omega_i\wedge\omega^i)\wedge*(d\omega_j\wedge\omega^j)\ -\ \frac{1}{2}\,(d\omega_i\wedge\omega^j)\wedge*(d\omega_j\wedge\omega^i).$$

Because of this, the choice $\rho_0 = 0$, $\rho_1 = 0$, $\rho_2 = \frac{1}{2}$, $\rho_3 = -1$ is called the teleparallel equivalent of GR (sometimes denoted $GR_{||}$).

[Note: If desired, a cosmological constant λ can be introduced by setting $\frac{\rho_0}{2} = -\lambda$.]

Rappel:

$$d\omega^i = \frac{1}{2}\,C^i{}_{jk}\,\omega^j\wedge\omega^k \quad (C^i{}_{jk} = -\,C^i{}_{kj}).$$

[Note: In terms of the interior product,

$$C^i{}_{jk} = \iota_{E_k}\iota_{E_j}d\omega^i.$$

Thus

$$\iota_{E_k}\iota_{E_j}d\omega^i$$

$$= \frac{1}{2}\,C^i{}_{uv}\,\iota_{E_k}\iota_{E_j}\,\omega^u\wedge\omega^v$$

$$= \frac{1}{2}\,C^i{}_{uv}\,\iota_{E_k}(\delta^u{}_j\omega^v - \omega^u\delta^v{}_j)$$

$$= \frac{1}{2}\,C^i{}_{jv}\,\iota_{E_k}\omega^v - \frac{1}{2}\,C^i{}_{uj}\,\iota_{E_k}\omega^u$$

$$= \frac{1}{2} C^i{}_{jk} - \frac{1}{2} C^i{}_{kj}$$

$$= C^i{}_{jk}.]$$

Example (Anti Yang-Mills): Consider

$$\frac{\rho}{2} d^* \omega_i \wedge * d^* \omega^i.$$

• $\iota_{\omega_j} d\omega^j = C^j{}_{jk} \omega^k$

=>

$$\iota_{\omega_i} \iota_{\omega_j} d\omega^j = C^j{}_{ji} = - C^j{}_{ij}$$

=>

$$\iota_{\omega^i} \iota_{\omega_j} d\omega^j = \varepsilon_i \iota_{\omega_i} \iota_{\omega_j} d\omega^j$$

$$= - \varepsilon_i C^j{}_{ij}$$

$$= d^* \omega^i.$$

• $\iota_{\omega_j} d\omega^j$

$$= (-1)^{\iota} (-1)^{2(n-2)} \iota_{\omega_j} {*}{*} d\omega^j$$

$$= (-1)^{\iota} * ({*} d\omega^j \wedge \omega_j)$$

$$= (-1)^{\iota} (-1)^{n-2} * (\omega_j \wedge * d\omega^j)$$

$$= (-1)^{\iota}(-1)^{n} * (\omega_j \wedge * d\omega^j)$$

=>

$$d^* \omega^i = \iota_{\omega^i} \iota_{\omega_j} d\omega^j$$

$$= (-1)^{\iota}(-1)^{n} \iota_{\omega^i} * (\omega_j \wedge * d\omega^j)$$

$$= (-1)^{\iota}(-1)^{n} * (\omega_j \wedge * d\omega^j \wedge \omega^i)$$

$$= (-1)^{\iota}(-1)^{n}(-1)^{n-2} * (\omega_j \wedge \omega^i \wedge * d\omega^j)$$

$$= (-1)^{\iota} * (\omega_j \wedge \omega^i \wedge * d\omega^j)$$

$$= (-1)^{\iota} * (\omega^j \wedge \omega^i \wedge * d\omega_j)$$

$$= (-1)^{\iota+1} * (\omega^i \wedge \omega^j \wedge * d\omega_j) .$$

Therefore

$$\frac{\rho}{2} d^* \omega_i \wedge * d^* \omega^i$$

$$= \frac{\rho}{2} (\iota_{\omega_i} \iota_{\omega_k} d\omega^k) \wedge (-1)^{\iota+1} ** (\omega^i \wedge \omega^j \wedge * d\omega_j)$$

$$= \frac{\rho}{2} (\iota_{\omega_i} \iota_{\omega_k} d\omega^k) \wedge (-1)^{\iota+1} (-1)^{\iota} \omega^i \wedge \omega^j \wedge * d\omega_j$$

$$= - \frac{\rho}{2} (\iota_{\omega_i} \iota_{\omega_k} d\omega^k) \wedge \omega^i \wedge \omega^j \wedge * d\omega_j$$

$$= -\frac{\rho}{2}\,\omega^i \wedge \iota_{\omega_i}(\iota_{\omega_k} d\omega^k)\wedge\omega^j\wedge * d\omega_j$$

$$= -\frac{\rho}{2}\,(\iota_{\omega_k} d\omega^k \wedge \omega^j)\wedge * d\omega_j$$

$$= -\frac{\rho}{2}\,(\iota_{\omega_k}(d\omega^k\wedge\omega^j) - d\omega^k\wedge\iota_{\omega_k}\omega^j)\wedge * d\omega_j$$

$$= -\frac{\rho}{2}\,(\iota_{\omega_k}(d\omega^k\wedge\omega^j) - d\omega^j)\wedge * d\omega_j$$

$$= \frac{\rho}{2}\,d\omega_j\wedge * d\omega^j - \frac{\rho}{2}\,\iota_{\omega_k}(d\omega^k\wedge\omega^j)\wedge * d\omega_j .$$

Write

$$\iota_{\omega_k}(d\omega^k\wedge\omega^j)$$

$$= (-1)^{\iota}(-1)^{(n-3)(3+1)} * (\omega_k \wedge * (d\omega^k\wedge\omega^j))$$

$$= (-1)^{\iota} * (\omega_k \wedge * (d\omega^k\wedge\omega^j)) .$$

Then

$$-\frac{\rho}{2}\,\iota_{\omega_k}(d\omega^k\wedge\omega^j)\wedge * d\omega_j$$

$$= (-1)^{\iota+1}\,\frac{\rho}{2} * (\omega_k\wedge * (d\omega^k\wedge\omega^j))\wedge * d\omega_j$$

$$= (-1)^{\iota+1}\,\frac{\rho}{2}\,d\omega_j\wedge ** (\omega_k\wedge * (d\omega^k\wedge\omega^j))$$

$$= (-1)^{\iota+1}\,\frac{\rho}{2}\,d\omega_j\wedge(-1)^{\iota}(-1)^{(n-2)(n-(n-2))}\,\omega_k\wedge * (d\omega^k\wedge\omega^j)$$

$$= -\frac{\rho}{2}\,(d\omega_j\wedge\omega_k)\wedge *(d\omega^k\wedge\omega^j)$$

$$= -\frac{\rho}{2}\,(d\omega_j\wedge\omega^k)\wedge *(d\omega_k\wedge\omega^j)\,.$$

Therefore

$$\frac{\rho}{2}\,d^*\omega_i\wedge *d^*\omega^i$$

$$= \frac{1}{2}\,(\rho L^1 - \rho L^3) \quad (= L(0,\rho,0,-\rho))\,.$$

Using the theorem, one can calculate δL^1, δL^2, and δL^3. The field equations obtained thereby are, however, rather unwieldly. To illustrate, consider δL^2.

$\underline{\delta L^2}$: We have

$$\delta((d\omega_i\wedge\omega^i)\wedge *(d\omega_j\wedge\omega^j))$$

$$= \delta(d\omega_i\wedge\omega^i)\wedge *(d\omega_j\wedge\omega^j)$$

$$+ (d\omega_i\wedge\omega^i)\wedge *\delta(d\omega_j\wedge\omega^j)$$

$$- \delta\omega_\ell\wedge J^{2,\ell}$$

$$= \delta(d\omega_i\wedge\omega^i)\wedge *(d\omega_j\wedge\omega^j)$$

$$+ \delta(d\omega_j\wedge\omega^j)\wedge *(d\omega_i\wedge\omega^i)$$

$$- \delta\omega_\ell\wedge J^{2,\ell}$$

$$= 2\delta(d\omega_i\wedge\omega^i)\wedge *(d\omega_j\wedge\omega^j) - \delta\omega_\ell\wedge J^{2,\ell}$$

$$= 2d\delta\omega_i \wedge \omega^i \wedge *(d\omega_j \wedge \omega^j)$$

$$+ 2d\omega_i \wedge \delta\omega^i \wedge *(d\omega_j \wedge \omega^j)$$

$$- \delta\omega_\ell \wedge J^{2,\ell}$$

$$= 2d(\delta\omega_i \wedge \omega^i \wedge *(d\omega_j \wedge \omega^j))$$

$$+ 2\delta\omega_i \wedge d(\omega^i \wedge *(d\omega_j \wedge \omega^j))$$

$$+ 2\delta\omega^i \wedge d\omega_i \wedge *(d\omega_j \wedge \omega^j)$$

$$- \delta\omega_\ell \wedge J^{2,\ell}$$

$$= 2d(\delta\omega_i \wedge \omega^i \wedge *(d\omega_j \wedge \omega^j))$$

$$- 2\delta\omega_i \wedge \omega^i \wedge d*(d\omega_j \wedge \omega^j)$$

$$+ 2\delta\omega_i \wedge d\omega^i \wedge *(d\omega_j \wedge \omega^j)$$

$$+ 2\delta\omega^i \wedge d\omega_i \wedge *(d\omega_j \wedge \omega^j)$$

$$- \delta\omega_\ell \wedge J^{2,\ell}$$

$$= 2d(\delta\omega_i \wedge \omega^i \wedge *(d\omega_j \wedge \omega^j))$$

$$- 2\delta\omega^i \wedge \omega_i \wedge d*(d\omega_j \wedge \omega^j)$$

$$+ 4\delta\omega^i \wedge d\omega_i \wedge *(d\omega_j \wedge \omega^j)$$

$$- \delta\omega_\ell \wedge J^{2,\ell},$$

where

$$J^{2,\ell} = \iota_{\omega^\ell}(d\omega_j \wedge \omega^j) \wedge *(d\omega_i \wedge \omega^i)$$

$$- (-1)^3 (d\omega_i \wedge \omega^i) \wedge \iota_{\omega^\ell} *(d\omega_j \wedge \omega^j)$$

$$= - \iota_{\omega^\ell}((d\omega_i \wedge \omega^i) \wedge *(d\omega_j \wedge \omega^j))$$

$$+ 2\iota_{\omega^\ell}(d\omega_i \wedge \omega^i) \wedge *(d\omega_j \wedge \omega^j).$$

But

$$- \delta\omega_\ell \wedge J^{2,\ell}$$

$$= - \delta\omega^\ell \wedge J^2{}_{,\ell}$$

$$= - \delta\omega^i \wedge J^2{}_{,i}$$

$$= - \delta\omega^i \wedge [- \iota_{\omega_i}((d\omega_j \wedge \omega^j) \wedge *(d\omega_k \wedge \omega^k))$$

$$+ 2\iota_{\omega_i}(d\omega_j \wedge \omega^j) \wedge *(d\omega_k \wedge \omega^k)]$$

$$= \delta\omega^i \wedge \iota_{\omega_i}((d\omega_j \wedge \omega^j) \wedge *(d\omega_k \wedge \omega^k))$$

$$- \delta\omega^i \wedge 2\iota_{\omega_i} d\omega_j \wedge \omega^j \wedge *(d\omega_k \wedge \omega^k)$$

$$- \delta\omega^i \wedge 2d\omega_j \wedge \iota_{\omega_i} \omega^j \wedge *(d\omega_k \wedge \omega^k)$$

$$= \delta\omega^i \wedge \iota_{\omega_i} ((d\omega_j \wedge \omega^j) \wedge *(d\omega_k \wedge \omega^k))$$

$$- \delta\omega^i \wedge 2\iota_{\omega_i} d\omega_j \wedge \omega^j \wedge *(d\omega_k \wedge \omega^k)$$

$$- 2\delta\omega^i \wedge d\omega_i \wedge *(d\omega_j \wedge \omega^j).$$

Consequently, the field equations for ω per L^2 are

$$- 2\omega_i \wedge d*(d\omega_j \wedge \omega^j) + 2d\omega_i \wedge *(d\omega_j \wedge \omega^j)$$

$$+ \iota_{\omega_i} ((d\omega_j \wedge \omega^j) \wedge *(d\omega_k \wedge \omega^k))$$

$$- 2\iota_{\omega_i} d\omega_j \wedge \omega^j \wedge *(d\omega_k \wedge \omega^k)$$

$$= 0.$$

Take $\rho_0 = 0$ — then there is another approach to the field equations for ω per

$$L = \frac{1}{2} (\rho_1 L^1 + \rho_2 L^2 + \rho_3 L^3)$$

which is more economical in its execution.

We have

$$\left[\begin{array}{l} L^1 = \frac{1}{2} C_{ijk}C^{ijk}{*1} \\ \\ L^2 = \frac{1}{2} C_{ijk}(C^{ijk} + C^{jki} + C^{kij}){*1} \\ \\ L^3 = \frac{1}{2} (C_{ijk}C^{ijk} - 2C^{i}{}_{ik}C_j{}^{jk}){*1}. \end{array}\right.$$

Put

$$\gamma^{ijkrst} = (\rho_1 + \rho_2 + \rho_3)\eta^{ir}\eta^{js}\eta^{kt}$$

$$+ \rho_2(\eta^{is}\eta^{jt}\eta^{kr} + \eta^{it}\eta^{jr}\eta^{ks})$$

$$- 2\rho_3\eta^{ik}\eta^{rt}\eta^{js}.$$

Then

$$L = \frac{1}{4} C_{ijk}C_{rst}\gamma^{ijkrst}{*1}$$

or still,

$$L = \frac{1}{4} C_{ijk}F^{ijk}{*1},$$

where

$$F^{ijk} = \gamma^{ijkrst}C_{rst}.$$

Notation:

- $C^i = \frac{1}{2} C^{ijk}\omega_j \wedge \omega_k \quad (= d\omega^i)$

- $F^i = \frac{1}{2} F^{ijk}\omega_j \wedge \omega_k.$

<u>FACT</u>

$$F^i = (\rho_1 + \rho_3)C^i + \rho_2 \iota_{\omega^i}(\omega_j \wedge C^j) - \rho_3 \omega^i \wedge \iota_{\omega_j} C^j.$$

<u>LEMMA</u> We have

$$L = \frac{1}{2} C_i \wedge * F^i.$$

[For

$$C_i \wedge * F^i$$

$$= \frac{1}{2} C_{ijk} \omega^j \wedge \omega^k \wedge \frac{1}{2} F^{iuv} * (\omega_u \wedge \omega_v)$$

$$= \frac{1}{4} C_{ijk} F^{iuv} g(\omega^j \wedge \omega^k, \omega_u \wedge \omega_v) * 1$$

$$= \frac{1}{4} C_{ijk} F^{iuv} \det \begin{bmatrix} g(\omega^j, \omega_u) & g(\omega^j, \omega_v) \\ g(\omega^k, \omega_u) & g(\omega^k, \omega_v) \end{bmatrix} * 1$$

$$= \frac{1}{4} C_{ijk} F^{iuv} (\delta^j{}_u \delta^k{}_v - \delta^j{}_v \delta^k{}_u) * 1$$

$$= \frac{1}{4} C_{ijk} F^{ijk} * 1 - \frac{1}{4} C_{ijk} F^{ikj} * 1$$

$$= \frac{1}{4} C_{ijk} F^{ijk} * 1 - \frac{1}{4} C_{ikj} F^{ijk} * 1$$

$$= \frac{1}{4} C_{ijk} F^{ijk} {*1} + \frac{1}{4} C_{ijk} F^{ijk} {*1}$$

$$= \frac{1}{2} C_{ijk} F^{ijk} {*1}$$

$$= 2L.]$$

We shall now turn to the calculation of δL.

First

$$\delta(C_{ijk} F^{ijk} {*1})$$

$$= (\delta C_{ijk}) F^{ijk} {*1} + C_{ijk} (\delta F^{ijk}) {*1} + C_{ijk} F^{ijk} \delta {*1}.$$

But

$$\delta F^{ijk} = \delta(\gamma^{ijkrst} C_{rst})$$

$$= \gamma^{ijkrst} \delta C_{rst}$$

$$= \gamma^{rstijk} \delta C_{rst}$$

$$\Rightarrow$$

$$C_{ijk} (\delta F^{ijk})$$

$$= C_{ijk} \gamma^{rstijk} \delta C_{rst}$$

$$= \gamma^{ijkrst} C_{rst} \delta C_{ijk}$$

$$= (\delta C_{ijk}) F^{ijk}$$

=>

$$\delta L = \frac{1}{4}\ (2(\delta C_{ijk}) F^{ijk} *1 + C_{ijk} F^{ijk} \delta *1)$$

$$= \frac{1}{2}\ (\delta C_{ijk}) F^{ijk} *1 + \frac{1}{4}\, C_{ijk} F^{ijk} \delta *1$$

$$= \frac{1}{2}\ (\delta C_{ijk}) F^{ijk} *1 + \frac{1}{4}\, C_{ijk} F^{ijk} \delta\omega^{\ell} \wedge * \omega_{\ell}.$$

Observation:

$$\iota_{\omega_\ell} L = \iota_{\omega_\ell} (\frac{1}{4}\, C_{ijk} F^{ijk} \mathrm{vol}_g)$$

$$= \frac{1}{4}\, C_{ijk} F^{ijk} \iota_{\omega_\ell} \mathrm{vol}_g$$

$$= \frac{1}{4}\, C_{ijk} F^{ijk} * \omega_{\ell}.$$

So

$$\delta L = \frac{1}{2}\ (\delta C_{ijk}) F^{ijk} *1 + \delta\omega^{\ell} \wedge \iota_{\omega_\ell} L.$$

<u>LEMMA</u> We have

$$\delta C_{ijk} *1$$

$$= \delta d\omega_i \wedge * (\omega_j \wedge \omega_k) + \delta\omega^{\ell} \wedge (C_{i\ell j} * \omega_k - C_{i\ell k} * \omega_j).$$

[From the definitions,

$$\delta d\omega_i = \frac{1}{2}\ \delta C_{ijk} \omega^j \wedge \omega^k + C_{ijk} \delta\omega^j \wedge \omega^k,$$

hence

$$\delta d\omega_i \wedge *(\omega_u \wedge \omega_v)$$

$$= \frac{1}{2}\, \delta C_{ijk}\omega^j \wedge \omega^k \wedge *(\omega_u \wedge \omega_v)$$

$$+ C_{ijk}\delta\omega^j \wedge \omega^k \wedge *(\omega_u \wedge \omega_v) .$$

Write

$$*(\iota_{\omega^k}(\omega_u \wedge \omega_v))$$

$$= (-1)^{n-1} *(\omega_u \wedge \omega_v) \wedge \omega^k$$

$$= (-1)^{n-1}(-1)^{n-2}\omega^k \wedge *(\omega_u \wedge \omega_v)$$

$$= -\omega^k \wedge *(\omega_u \wedge \omega_v)$$

to get

$$\delta d\omega_i \wedge *(\omega_u \wedge \omega_v)$$

$$= -\frac{1}{2}\, \delta C_{ijk}\omega^j \wedge *(\iota_{\omega^k}(\omega_u \wedge \omega_v))$$

$$- C_{ijk}\delta\omega^j \wedge *(\iota_{\omega^k}(\omega_u \wedge \omega_v))$$

$$= -\frac{1}{2}\, \delta C_{ijk}\omega^j \wedge *(\delta^k{}_u\omega_v - \omega_u\delta^k{}_v)$$

$$- C_{ijk}\delta\omega^j \wedge *(\delta^k{}_u\omega_v - \omega_u\delta^k{}_v)$$

$$= -\frac{1}{2}\,\delta C_{ijk}\omega^j\wedge\delta^k{}_u{*}\omega_v + \frac{1}{2}\,\delta C_{ijk}\omega^j\wedge\delta^k{}_v{*}\omega_u$$

$$- C_{ijk}\delta\omega^j\wedge\delta^k{}_u{*}\omega_v + C_{ijk}\delta\omega^j\wedge\delta^k{}_v{*}\omega_u$$

$$= -\frac{1}{2}\,\delta C_{iju}\omega^j\wedge{*}\omega_v + \frac{1}{2}\,\delta C_{ijv}\omega^j\wedge{*}\omega_u$$

$$- C_{iju}\delta\omega^j\wedge{*}\omega_v + C_{ijv}\delta\omega^j\wedge{*}\omega_u$$

$$= -\frac{1}{2}\,\delta C_{iju}\delta^j{}_v{*}1 + \frac{1}{2}\,\delta C_{ijv}\delta^j{}_u{*}1$$

$$- \delta\omega^j\wedge(C_{iju}{*}\omega_v - C_{ijv}{*}\omega_u)$$

$$= -\frac{1}{2}\,\delta C_{ivu}{*}1 + \frac{1}{2}\,\delta C_{iuv}{*}1$$

$$- \delta\omega^\ell\wedge(C_{i\ell u}{*}\omega_v - C_{i\ell v}{*}\omega_u)$$

$$= \delta C_{iuv}{*}1 - \delta\omega^\ell\wedge(C_{i\ell u}{*}\omega_v - C_{i\ell v}{*}\omega_u).$$

The replacements $u \to j$, $v \to k$ then serve to complete the proof.]

So

$$\delta L = \frac{1}{2}\,(\delta d\omega_i\wedge{*}(\omega_j\wedge\omega_k)$$

$$+ \delta\omega^\ell\wedge(C_{i\ell j}{*}\omega_k - C_{i\ell k}{*}\omega_j))F^{ijk}$$

$$+ \delta\omega^\ell\wedge\iota_{\omega_\ell}L.$$

• $d(\delta\omega_i \wedge F^{ijk} \wedge *(\omega_j \wedge \omega_k))$

$= \delta d\omega_i \wedge F^{ijk} \wedge *(\omega_j \wedge \omega_k)$

$- \delta\omega_i \wedge (dF^{ijk} \wedge *(\omega_j \wedge \omega_k) + F^{ijk} \wedge d*(\omega_j \wedge \omega_k)).$

• $d(F^{\ell jk} \wedge *(\omega_j \wedge \omega_k))$

$= dF^{\ell jk} \wedge *(\omega_j \wedge \omega_k) + F^{\ell jk} \wedge d*(\omega_j \wedge \omega_k).$

Thus

$$\delta L = \frac{1}{2}\, \delta\omega_\ell \wedge (d(F^{\ell jk} \wedge *(\omega_j \wedge \omega_k))$$

$$+ F^{ijk}(C_i{}^\ell{}_j *\omega_k - C_i{}^\ell{}_k *\omega_j) + 2\iota_{\omega^\ell} L)$$

$$+ \frac{1}{2}\, d(\delta\omega_i \wedge F^{ijk} \wedge *(\omega_j \wedge \omega_k))$$

$$= \delta\omega_\ell \wedge (d*F^\ell$$

$$+ \frac{1}{2}\, F^{ijk}(C_i{}^\ell{}_j *\omega_k - C_i{}^\ell{}_k *\omega_j) + \iota_{\omega^\ell} L)$$

$$+ d(\delta\omega_i \wedge *F^i).$$

• $\frac{1}{2}\, F^{ijk}(C_i{}^\ell{}_j *\omega_k - C_i{}^\ell{}_k *\omega_j)$

$$= \frac{1}{2}\, (F^{ijk} - F^{ikj}) C_i{}^\ell{}_j *\omega_k.$$

$\bullet \, *(\iota_{\omega^j} F^i)$

$$= *(\iota_{\omega^j} \tfrac{1}{2} F^{iuv} \omega_u \wedge \omega_v)$$

$$= *(\tfrac{1}{2} F^{iuv} (\delta^j{}_u \omega_v - \omega_u \delta^j{}_v)$$

$$= \tfrac{1}{2} F^{ijv} *\omega_v - \tfrac{1}{2} F^{iuj} *\omega_u$$

$$= \tfrac{1}{2} F^{ijk} *\omega_k - \tfrac{1}{2} F^{ikj} *\omega_k$$

$$= \tfrac{1}{2} (F^{ijk} - F^{ikj}) *\omega_k.$$

Thus

$$\delta L = \delta\omega_\ell \wedge (d*F^\ell + C_i{}^\ell{}_j *(\iota_{\omega^j} F^i) + \iota_{\omega^\ell} L)$$

$$+ d(\delta\omega_i \wedge *F^i)$$

$$= \delta\omega^\ell \wedge (d*F_\ell + C_{i\ell j} *(\iota_{\omega^j} F^i) + \iota_{\omega_\ell} L)$$

$$+ d(\delta\omega_i \wedge *F^i).$$

$\bullet \, *(\iota_{\omega^j} F^i)$

$$= (-1)^{n-1} *F^i \wedge \omega^j$$

$$= (-1)^{n-1} (-1)^{n-2} \omega^j \wedge *F^i$$

$$= - \omega^j \wedge *F^i.$$

• $\iota_{\omega_\ell} C_i$

$$= \iota_{\omega_\ell} d\omega_i$$

$$= \frac{1}{2} C_{iuv} \iota_{\omega_\ell} (\omega^u \wedge \omega^v)$$

$$= \frac{1}{2} C_{iuv} (\delta^u{}_\ell \omega^v - \omega^u \delta^v{}_\ell)$$

$$= \frac{1}{2} C_{i\ell v} \omega^v - \frac{1}{2} C_{iu\ell} \omega^u$$

$$= C_{i\ell j} \omega^j .$$

Thus

$$\delta L = \delta\omega^\ell \wedge (d{*}F_\ell - \iota_{\omega_\ell} C_i \wedge {*}F^i + \iota_{\omega_\ell} L)$$

$$+ d(\delta\omega_i \wedge {*}F^i) .$$

Notation:

$$J_\ell = \iota_{\omega_\ell} C_i \wedge {*}F^i - \iota_{\omega_\ell} L .$$

Scholium: We have

$$\delta L = \delta\omega^\ell \wedge (d{*}F_\ell - J_\ell) + d(\delta\omega^i \wedge {*}F_i) .$$

Definition: ω satisfies the field equations per L provided $\forall\ \ell$,

$$d{*}F_\ell = J_\ell .$$

[Note: Matters are consistent in that

$$\frac{\partial L}{\partial \omega^\ell} = - J_\ell \text{ and } \frac{\partial L}{\partial d\omega^\ell} = {*}F_\ell .]$$

<u>Reality Check</u> Take $\rho_1 = 0$, $\rho_3 = 0$ -- then the claim is that the field equations per L_2 derived earlier agree with those obtained above. For, in this situation,

$$F_i = \rho_2 \iota_{\omega_i}(\omega_j \wedge C^j)$$

$$= \rho_2 \iota_{\omega_i}(d\omega_j \wedge \omega^j)$$

$$= \rho_2 (-1)^{\iota}(-1)^{3(n-3)} \iota_{\omega_i} {**}(d\omega_j \wedge \omega^j)$$

$$= \rho_2 (-1)^{\iota}(-1)^{3(n-3)} * (*(d\omega_j \wedge \omega^j) \wedge \omega_i)$$

$$= \rho_2 (-1)^{\iota}(-1)^{3(n-3)} (-1)^{n-3} * (\omega_i \wedge *(d\omega_j \wedge \omega^j))$$

$$= \rho_2 (-1)^{\iota}(-1)^{(n-3)(3+1)} * (\omega_i \wedge *(d\omega_j \wedge \omega^j))$$

$$= \rho_2 (-1)^{\iota} * (\omega_i \wedge *(d\omega_j \wedge \omega^j))$$

=>

$$*F_i = \rho_2 (-1)^{\iota} {**}(\omega_i \wedge *(d\omega_j \wedge \omega^j))$$

$$= \rho_2 (-1)^{\iota}(-1)^{\iota}(-1)^{(n-2)(n-(n-2))} \omega_i \wedge *(d\omega_j \wedge \omega^j)$$

$$= \rho_2 (\omega_i \wedge *(d\omega_j \wedge \omega^j))$$

=>

$$d{*}F_i = \rho_2(d\omega_i\wedge{*}(d\omega_j\wedge\omega^j) - \omega_i\wedge d{*}(d\omega_j\wedge\omega^j)).$$

Therefore

$$\rho_2[-2\omega_i\wedge d{*}(d\omega_j\wedge\omega^j) + 2d\omega_i\wedge{*}(d\omega_j\wedge\omega^j)$$

$$+ \iota_{\omega_i}((d\omega_j\wedge\omega^j)\wedge{*}(d\omega_k\wedge\omega^k))$$

$$- 2\iota_{\omega_i}d\omega_j\wedge\omega^j\wedge{*}(d\omega_k\wedge\omega^k)]$$

$$= 2d{*}F_i$$

$$+ \rho_2[\iota_{\omega_i}((d\omega_j\wedge\omega^j)\wedge{*}(d\omega_k\wedge\omega^k))$$

$$- 2\iota_{\omega_i}d\omega_j\wedge\omega^j\wedge{*}(d\omega_k\wedge\omega^k)]$$

$$= 2d{*}F_i$$

$$+ \rho_2[\iota_{\omega_i}((d\omega^j\wedge\omega_j)\wedge{*}(d\omega_k\wedge\omega^k))$$

$$- 2\iota_{\omega_i}d\omega^j\wedge\omega_j\wedge{*}(d\omega_k\wedge\omega^k)]$$

$$= 2d{*}F_i$$

$$+ \rho_2(\iota_{\omega_i}d\omega^j\wedge\omega_j + d\omega^j\wedge\iota_{\omega_i}\omega_j)\wedge{*}(d\omega_k\wedge\omega^k)$$

$$- \rho_2(d\omega^j\wedge\omega_j)\wedge\iota_{\omega_i}{*}(d\omega_k\wedge\omega^k)$$

$$- 2\rho_2\iota_{\omega_i}d\omega^j\wedge\omega_j\wedge{*}(d\omega_k\wedge\omega^k)$$

$$= 2d{*}F_i$$

$$+ \iota_{\omega_i} C^j \wedge {*}F_j + \rho_2 d\omega_i \wedge {*}(d\omega_k \wedge \omega^k)$$

$$- \rho_2 (d\omega_j \wedge \omega^j) \wedge \iota_{\omega_i} {*}(d\omega_k \wedge \omega^k)$$

$$- 2\iota_{\omega_i} C^j \wedge {*}F_j$$

$$= 2d{*}F_i - \iota_{\omega_i} C^j \wedge {*}F_j$$

$$+ \rho_2 d\omega_i \wedge {*}(d\omega_k \wedge \omega^k)$$

$$- \rho_2 (d\omega_j \wedge \omega^j) \wedge \iota_{\omega_i} {*}(d\omega_k \wedge \omega^k) .$$

But

$$C^j \wedge \iota_{\omega_i} {*}F_j$$

$$= C^j \wedge \rho_2 \iota_{\omega_i} (\omega_j \wedge {*}(d\omega_k \wedge \omega^k))$$

$$= \rho_2 C^j \iota_{\omega_i} \omega_j \wedge {*}(d\omega_k \wedge \omega^k)$$

$$- \rho_2 C^j \wedge \omega_j \wedge \iota_{\omega_i} {*}(d\omega_k \wedge \omega^k)$$

$$= \rho_2 C_i \wedge {*}(d\omega_k \wedge \omega^k)$$

$$- \rho_2 C^j \wedge \omega_j \wedge \iota_{\omega_i} {*}(d\omega_k \wedge \omega^k)$$

$$= \rho_2 d\omega_i \wedge *(d\omega_k \wedge \omega^k)$$

$$- \rho_2 (d\omega_j \wedge \omega^j) \wedge \iota_{\omega_i} *(d\omega_k \wedge \omega^k).$$

Inserting this then leads to

$$2d*F_i - \iota_{\omega_i} C^j \wedge *F_j + C^j \wedge \iota_{\omega_i} *F_j$$

or still,

$$2d*F_i - \iota_{\omega_i} C^j \wedge *F_j$$

$$+ \iota_{\omega_i} (C^j \wedge *F_j) - \iota_{\omega_i} C^j \wedge *F_i$$

$$= 2(d*F_i - \iota_{\omega_i} C_j \wedge *F^j + \frac{1}{2} \iota_{\omega_i} (C_j \wedge *F^j))$$

$$= 2(d*F_i - \iota_{\omega_i} C_j \wedge *F^j + \iota_{\omega_i} L)$$

$$= 2(d*F_i - J_i),$$

from which the claim.

Remark: In $GR_{||}$, the field equations

$$d*F_\ell = J_\ell \quad (\ell = 1,\ldots,n)$$

are equivalent to the vanishing of Ein(g).

The J^ℓ are (n-1)-forms and the collection $\{J^1,\ldots,J^n\}$ is called the <u>energy-momentum current</u> attached to ω.

<u>LEMMA</u> We have

$$J^{\ell}{}_{\ell} = (2 - \frac{n}{2}) C_i \wedge * F^i.$$

[In fact,

$$\begin{aligned} J^{\ell}{}_{\ell} * 1 &= \omega^{\ell} \wedge J_{\ell} \\ &= \omega^{\ell} \wedge (\iota_{\omega_{\ell}} C_i \wedge * F^i - \iota_{\omega_{\ell}} L) \\ &= (\omega^{\ell} \wedge \iota_{\omega_{\ell}} C_i) \wedge * F^i - \omega^{\ell} \wedge \iota_{\omega_{\ell}} L \\ &= 2 C_i \wedge * F^i - nL \\ &= 2 C_i \wedge * F^i - \frac{n}{2} C_i \wedge * F^i \\ &= (2 - \frac{n}{2}) C_i \wedge * F^i.] \end{aligned}$$

Application: If $n = 4$, then $J^{\ell}{}_{\ell} = 0$.

Let

$$E_{\ell} = d * F_{\ell} - J_{\ell}.$$

Then

$$\iota_{\omega_{\ell}} E^{\ell} = E^{[\ell k]} * (\omega_k \wedge \omega_{\ell}).$$

<u>FACT</u> We have

$$E^{[\ell k]} = - 2(\rho_1 - 2\rho_2 - \rho_3) A_{[\ell k]} + (2\rho_2 + \rho_3) B_{[\ell k]}$$

for certain entities A and B.

So, if $\rho_1 = 0$ and $2\rho_2 + \rho_3 = 0$, then $E^{[\ell k]} = 0$.

[Note: This applies to $GR_{||}$.]

Section 47: Lovelock Gravity Let M be a connected C^∞ manifold of dimension n.

Assume: M is parallelizable.

Definition: The p^{th} Lovelock lagrangian is the function

$$L_p : \text{cof}_M \to \Lambda^n M$$

given by

$$L_p(\omega) = \frac{1}{2} \Omega_{i_1 j_1} \wedge \ldots \wedge \Omega_{i_p j_p} \wedge \theta^{i_1 j_1 \cdots i_p j_p} \quad (2p \leq n).$$

[Note: Conventionally, $L_0(\omega) = \frac{1}{2} \text{vol}_g$, where, as before,

$$g = -\omega^1 \otimes \omega^1 - \cdots - \omega^k \otimes \omega^k + \omega^{k+1} \otimes \omega^{k+1} + \cdots + \omega^n \otimes \omega^n.]$$

Rappel: The

$$\Xi(p)_k = \Omega_{i_1 j_1} \wedge \ldots \wedge \Omega_{i_p j_p} \wedge \theta^{i_1 j_1 \cdots i_p j_p}{}_k \quad (k = 1, \ldots, n)$$

are the Lovelock (n-1)-forms.

[Note: Recall that

$$\Xi(p)_k = -2(G_p)_{k\ell} {*\omega^\ell}.]$$

LEMMA Fix $p \geq 1$ $(n \geq 2p)$ -- then

$$\delta L_p = \frac{1}{2} \delta\omega^k \wedge \Xi(p)_k$$

$$+ \frac{p}{2} d(\delta\omega_{i_1 j_1} \wedge \Omega_{i_2 j_2} \wedge \ldots \wedge \Omega_{i_p j_p} \wedge \theta^{i_1 j_1 \cdots i_p j_p}).$$

The case p = 1 was treated in the last section. There we saw that

$$\delta L_1 = \frac{1}{2}\,\delta\omega^k \wedge \Omega_{ij} \wedge \theta^{ij}{}_k + \frac{1}{2}\, d(\delta\omega_{ij} \wedge \theta^{ij}).$$

And

$$\begin{aligned}
\frac{1}{2}\,\Omega_{ij} \wedge \theta^{ij}{}_k &= -\,*(\mathrm{Ric}_k - \frac{1}{2}\, S(g) g_k) \\
&= -\,*(R_{k\ell}\omega^\ell - \frac{1}{2}\, S(g) g_{k\ell}\omega^\ell) \\
&= -\,(G_1)_{k\ell} * \omega^\ell \\
&= \frac{1}{2}\,\Xi(1)_k.
\end{aligned}$$

Proceeding by iteration, take $p = 2$ -- then

$$\begin{aligned}
\delta L_2 &= \frac{1}{2}\,\delta(\Omega_{i_1 j_1} \wedge \Omega_{i_2 j_2} \wedge \theta^{i_1 j_1 i_2 j_2}) \\
&= \frac{1}{2}\,\delta(\Omega_{i_1 j_1} \wedge \Omega_{i_2 j_2}) \wedge \theta^{i_1 j_1 i_2 j_2} \\
&\quad + \frac{1}{2}\,\Omega_{i_1 j_1} \wedge \Omega_{i_2 j_2} \wedge \delta(\theta^{i_1 j_1 i_2 j_2}).
\end{aligned}$$

But

$$\begin{aligned}
&\Omega_{i_1 j_1} \wedge \Omega_{i_2 j_2} \wedge \delta(\theta^{i_1 j_1 i_2 j_2}) \\
&\quad = \Omega_{i_1 j_1} \wedge \Omega_{i_2 j_2} \wedge \delta\omega^k \wedge \theta^{i_1 j_1 i_2 j_2}{}_k \\
&\quad = \delta\omega^k \wedge \Omega_{i_1 j_1} \wedge \Omega_{i_2 j_2} \wedge \theta^{i_1 j_1 i_2 j_2}{}_k
\end{aligned}$$

$$= \delta\omega^k \wedge \Xi(2)_k.$$

As for what remains, observe first that

$$\delta(\Omega_{i_1j_1} \wedge \Omega_{i_2j_2}) \wedge \theta^{i_1j_1i_2j_2}$$

$$= (\delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} + \Omega_{i_1j_1} \wedge \delta\Omega_{i_2j_2}) \wedge \theta^{i_1j_1i_2j_2}$$

$$= \delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \theta^{i_1j_1i_2j_2}$$

$$+ \Omega_{i_2j_2} \wedge \delta\Omega_{i_1j_1} \wedge \theta^{i_2j_2i_1j_1}$$

$$= \delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \theta^{i_1j_1i_2j_2}$$

$$+ \delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \theta^{i_2j_2i_1j_1}$$

$$= \delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \theta^{i_1j_1i_2j_2}$$

$$+ \delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \theta^{i_1j_1i_2j_2}$$

$$= 2(\delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \theta^{i_1j_1i_2j_2}).$$

And

$$\delta\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \theta^{i_1j_1i_2j_2}$$

$$= \delta(d\omega_{i_1j_1} + \omega_{i_1k}\wedge\omega^k{}_{j_1})\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$= (d\delta\omega_{i_1j_1} + \delta\omega_{i_1k}\wedge\omega^k{}_{j_1} + \omega_{i_1k}\wedge\delta\omega^k{}_{j_1})$$

$$\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$= d\delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$+ \delta\omega_{i_1k}\wedge\omega^k{}_{j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2} + \omega_{i_1k}\wedge\delta\omega^k{}_{j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}.$$

Now write

$$d(\delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2})$$

$$= d\delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$- \delta\omega_{i_1j_1}\wedge d(\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2})$$

$$= d\delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$- \delta\omega_{i_1j_1}\wedge d\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2} - \delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}\wedge d\theta^{i_1j_1i_2j_2}.$$

Then this already accounts for

$$d\delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}.$$

To see how the other terms are taken care of, express

$$d\theta^{i_1 j_1 i_2 j_2}$$

as

$$-\omega^{i_1}{}_k \wedge *(\omega^k \wedge \omega^{j_1} \wedge \omega^{i_2} \wedge \omega^{j_2})$$

$$-\omega^{j_1}{}_k \wedge *(\omega^{i_1} \wedge \omega^k \wedge \omega^{i_2} \wedge \omega^{j_2})$$

$$-\omega^{i_2}{}_k \wedge *(\omega^{i_1} \wedge \omega^{j_1} \wedge \omega^k \wedge \omega^{j_2})$$

$$-\omega^{j_2}{}_k \wedge *(\omega^{i_1} \wedge \omega^{j_1} \wedge \omega^{i_2} \wedge \omega^k).$$

1.

$$\delta\omega_{i_1 j_1} \wedge \Omega_{i_2 j_2} \wedge \omega^{i_1}{}_k \wedge *(\omega^k \wedge \omega^{j_1} \wedge \omega^{i_2} \wedge \omega^{j_2})$$

$$= \delta\omega_{k j_1} \wedge \Omega_{i_2 j_2} \wedge \omega^k{}_{i_1} \wedge *(\omega^{i_1} \wedge \omega^{j_1} \wedge \omega^{i_2} \wedge \omega^{j_2})$$

$$= \delta\omega_{k i_1} \wedge \Omega_{i_2 j_2} \wedge \omega^k{}_{j_1} \wedge *(\omega^{j_1} \wedge \omega^{i_1} \wedge \omega^{i_2} \wedge \omega^{j_2})$$

$$= \delta\omega_{i_1 k} \wedge \Omega_{i_2 j_2} \wedge \omega^k{}_{j_1} \wedge *(\omega^{i_1} \wedge \omega^{j_1} \wedge \omega^{i_2} \wedge \omega^{j_2})$$

$$= \delta\omega_{i_1 k} \wedge \omega^k{}_{j_1} \wedge \Omega_{i_2 j_2} \wedge \theta^{i_1 j_1 i_2 j_2}.$$

2.

$$\delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}\wedge\omega^{j_1}{}_k\wedge*(\omega^{i_1}\wedge\omega^k\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= \delta\omega_{i_1k}\wedge\Omega_{i_2j_2}\wedge\omega^k{}_{j_1}\wedge*(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= -\,\omega^k{}_{j_1}\wedge\delta\omega_{i_1k}\wedge\Omega_{i_2j_2}\wedge*(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= -\,\omega_{kj_1}\wedge\delta\omega_{i_1}{}^k\wedge\Omega_{i_2j_2}\wedge*(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= \omega_{kj_1}\wedge\delta\omega^k{}_{i_1}\wedge\Omega_{i_2j_2}\wedge*(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= \omega_{ki_1}\wedge\delta\omega^k{}_{j_1}\wedge\Omega_{i_2j_2}\wedge*(\omega^{j_1}\wedge\omega^{i_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= \omega_{i_1k}\wedge\delta\omega^k{}_{j_1}\wedge\Omega_{i_2j_2}\wedge*(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= \omega_{i_1k}\wedge\delta\omega^k{}_{j_1}\wedge\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}.$$

So, to finish the verification, we must show that

$$-\,\delta\omega_{i_1j_1}\wedge d\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$+\,\delta\omega_{i_1j_1}\wedge\Omega_{i_2j_2}$$

$$\wedge(\omega^{i_2}{}_k\wedge *(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^k\wedge\omega^{j_2}) + \omega^{j_2}{}_k\wedge *(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^k))$$

$$= 0,$$

the key being that

$$- d\Omega_{i_2j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$= - (\Omega_{i_2k}\wedge\omega^k{}_{j_2} - \omega_{i_2k}\wedge\Omega^k{}_{j_2})\wedge\theta^{i_1j_1i_2j_2}$$

$$= \Omega^k{}_{j_2}\wedge\omega_{i_2k}\wedge\theta^{i_1j_1i_2j_2} - \Omega_{i_2k}\wedge\omega^k{}_{j_2}\wedge\theta^{i_1j_1i_2j_2}.$$

3.

$$\Omega^k{}_{j_2}\wedge\omega_{i_2k}\wedge\theta^{i_1j_1i_2j_2}$$

$$= \Omega^k{}_{j_2}\wedge\omega_{i_2k}\wedge *(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= \Omega_{kj_2}\wedge\omega_{i_2}{}^k\wedge *(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^{i_2}\wedge\omega^{j_2})$$

$$= \Omega_{i_2j_2}\wedge\omega_k{}^{i_2}\wedge *(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^k\wedge\omega^{j_2})$$

$$= - \Omega_{i_2j_2}\wedge\omega^{i_2}{}_k\wedge *(\omega^{i_1}\wedge\omega^{j_1}\wedge\omega^k\wedge\omega^{j_2}).$$

4.

$$- \Omega_{i_2k}\wedge\omega^k{}_{j_2}\wedge\theta^{i_1j_1i_2j_2}$$

$$= - \Omega_{i_2 k} \wedge \omega^k{}_{j_2} \wedge * (\omega^{i_1 j_1} \wedge \omega^{i_2 j_2})$$

$$= - \Omega_{i_2 j_2} \wedge \omega^{j_2}{}_k \wedge * (\omega^{i_1 j_1} \wedge \omega^{i_2 k}).$$

Thus the terms in question do in fact cancel one another.

Remark: The condition on $p \geq 1$ is that $2p \leq n$. If $n = 2p$, then $\Xi(\frac{n}{2}) = 0$, hence $\delta L_{\frac{n}{2}}$ is exact.

[Note: This also follows from an earlier observation, viz. that $L_{\frac{n}{2}}$ itself is exact:

$$L_p = (4\pi)^p p! d\Pi_p$$

$$\Rightarrow \qquad (n = 2p)$$

$$\delta L_p = (4\pi)^p p! d\delta\Pi_p.]$$

Notation: Let

$$\sigma(p)_i = - \omega^{jk} \wedge \theta_{ijki_2j_2 \ldots i_pj_p} \wedge \Omega^{i_2j_2} \wedge \ldots \wedge \Omega^{i_pj_p}$$

and

$$\tau(p)_i = (\omega_i{}^j \wedge \omega^{k\ell} \wedge \theta_{jk\ell i_2j_2 \ldots i_pj_p} - \omega^j{}_\ell \wedge \omega^{\ell k}\, \theta_{ijki_2j_2 \ldots i_pj_p})$$

$$\wedge \Omega^{i_2j_2} \wedge \ldots \wedge \Omega^{i_pj_p}.$$

[Note: Therefore

$$\left[\begin{array}{l} \sigma(p)_i \in \Lambda^{n-2}M \\ \\ \tau(p)_i \in \Lambda^{n-1}M.] \end{array}\right.$$

<u>LEMMA</u> We have

$$\Xi(p)_k = \tau(p)_k - d\sigma(p)_k.$$

Definition: ω satisfies the <u>field equations</u> per L_p provided $\forall\ k$,

$$\Xi(p)_k = 0.$$

[Note: In view of the lemma, this amounts to requiring that

$$d\sigma(p)_k = \tau(p)_k \quad (k = 1,\ldots,n).$$

<u>Reality Check</u> Take $p = 1$ -- then

$$d\sigma(1)_k = \tau(1)_k$$

<=>

$$\Xi(1)_k = 0$$

<=>

$$(G_1)_{k\ell} = 0$$

<=>

$$R_{k\ell} - \frac{1}{2} S(g) g_{k\ell} = 0$$

<=>

$$\text{Ein}_k = 0.$$

Therefore ω satisfies the field equations per L_1 iff Ein(g) = 0.

Remark: Suppose that the standard setup is in force -- then it would be of interest to transcribe the problem of the vanishing of $\Xi(p)$ $(1 \leq p)(n > 2p)$ to a time dependent issue on Σ. Thus, if p = 1, the vanishing of $\Xi(1)$ is equivalent to the vanishing of Ein(g) and for this, one has the constraint equations and the evolution equations in T^*Q or $T^*\underline{\underline{Q}}$. Nothing this precise is known for p > 1. If p = 2, one can isolate the lagrangian as was done when p = 1, but even in this situation, the passage to T^*Q or $T^*\underline{\underline{Q}}$ along the lines that I would like to see has never been carried out.

<u>Section 48</u>: <u>The Palatini Formalism</u> Let M be a connected C^∞ manifold of dimension $n > 2$.

Assume: M is parallelizable.

Rappel: con TM is an affine space with translation group $\mathcal{D}_2^1(M)$.

Let con_0TM stand for the set of torsion free connections on TM.

Denote by $S_2^1(M)$ the subspace of $\mathcal{D}_2^1(M)$ consisting of those Ψ such that

$$\Psi(\Lambda,X,Y) = \Psi(\Lambda,Y,X).$$

• Let $\nabla',\nabla''\in\text{con}_0\text{TM}$ — then the assignment

$$\begin{cases} \mathcal{D}_1(M) \times \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to C^\infty(M) \\ (\Lambda,X,Y) \to \Lambda(\nabla'_X Y - \nabla''_X Y) \end{cases}$$

defines an element of $S_2^1(M)$.

[In fact,

$$\Lambda(\nabla'_X Y - \nabla''_X Y)$$

$$= \Lambda(\nabla'_Y X + [X,Y] - (\nabla''_Y X + [X,Y]))$$

$$= \Lambda(\nabla'_Y X - \nabla''_Y X).]$$

• Let $\nabla\in\text{con}_0\text{TM}$ — then $\forall\ \Psi\in S_2^1(M)$, the assignment

$$\begin{cases} \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M) \\ (X,Y) \to \nabla_X Y + \Psi(X,Y) \end{cases}$$

is a torsion free connection.

[In fact,

$$\nabla_X Y + \Psi(X,Y) - \nabla_Y X - \Psi(Y,X)$$

$$= \nabla_X Y - \nabla_Y X + \Psi(X,Y) - \Psi(Y,X)$$

$$= [X,Y].]$$

Scholium: $\text{con}_0 TM$ is an affine space with translation group $S_2^1(M)$.

Definition: Let $\nabla \in \text{con}_0 TM$ -- then a <u>variation of ∇</u> is a curve

$$\varepsilon \to \nabla + \varepsilon\Psi,$$

where $\Psi \in S_2^1(M)$ has compact support.

Fix $\omega \in \text{cof}_M$ and define

$$L_\omega : \text{con}_0 TM \to \Lambda^n M$$

by

$$L_\omega(\nabla) = \frac{1}{2}\,\Omega_{ij}(\nabla)\wedge\theta^{ij}.$$

Here

$$\Omega_{ij}(\nabla)$$

is computed per ∇ while

$$\theta^{ij} = *(\omega^i \wedge \omega^j)$$

is computed per $g \in \underline{M}_{k,n-k}$ (conventions as in the previous section).

Remark: Actually, in the considerations that follow, it will be simplest to use the local representation of L_ω, i.e.,

$$L_\omega(\nabla) = \frac{1}{2} g^{ij} \mathrm{Ric}(\nabla)_{ij} \mathrm{vol}_g.$$

Of course, in this context, the indices refer to a chart $(U,\{x^1,\ldots,x^n\})$.

[Note: As a map,

$$\mathrm{Ric}:\mathrm{con}_0 TM \to \mathcal{D}^0_2(M)$$

but $\mathrm{Ric}(\nabla)$ need not be symmetric.]

Let $R^i{}_{jk\ell}(\nabla + \varepsilon\Psi)$ be the curvature components of $\nabla + \varepsilon\Psi$.

<u>LEMMA</u> We have

$$\frac{d}{d\varepsilon} R^i{}_{jk\ell}(\nabla + \varepsilon\Psi)\Big|_{\varepsilon=0}$$

$$= \partial_k \Psi^i{}_{\ell j} - \partial_\ell \Psi^i{}_{kj}$$

$$+ \Psi^a{}_{\ell j}\Gamma^i{}_{ka} + \Gamma^a{}_{\ell j}\Psi^i{}_{ka} - \Psi^a{}_{kj}\Gamma^i{}_{\ell a} - \Gamma^a{}_{kj}\Psi^i{}_{\ell a}.$$

[In fact,

$$R^i{}_{jk\ell}(\nabla + \varepsilon\Psi)$$

$$= \partial_k \Gamma^i{}_{\ell j}(\nabla + \varepsilon\Psi) - \partial_\ell \Gamma^i{}_{kj}(\nabla + \varepsilon\Psi)$$

$$+ \Gamma^a{}_{\ell j}(\nabla + \varepsilon\Psi)\Gamma^i{}_{ka}(\nabla + \varepsilon\Psi)$$

$$- \Gamma^a{}_{kj}(\nabla + \varepsilon\Psi)\Gamma^i{}_{\ell a}(\nabla + \varepsilon\Psi).$$

But, from the definitions,

$$\Gamma^a{}_{bc}(\nabla + \varepsilon\Psi) = \Gamma^a{}_{bc}(\nabla) + \varepsilon\Psi^a{}_{bc}.$$

Therefore

$$\frac{d}{d\varepsilon} R^i{}_{jk\ell}(\nabla + \varepsilon\Psi)\Big|_{\varepsilon=0}$$

$$= \partial_k\Psi^i{}_{\ell j} - \partial_\ell\Psi^i{}_{kj}$$

$$+ \Psi^a{}_{\ell j}\Gamma^i{}_{ka} + \Gamma^a{}_{\ell j}\Psi^i{}_{ka} - \Psi^a{}_{kj}\Gamma^i{}_{\ell a} - \Gamma^a{}_{kj}\Psi^i{}_{\ell a}$$

as contended.]

Application: We have

$$\frac{d}{d\varepsilon} \mathrm{Ric}(\nabla + \varepsilon\Psi)_{j\ell}\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} R^i{}_{ji\ell}(\nabla + \varepsilon\Psi)\Big|_{\varepsilon=0}$$

$$= \partial_k\Psi^k{}_{\ell j} - \partial_\ell\Psi^k{}_{kj}$$

$$+ \Psi^a{}_{\ell j}\Gamma^k{}_{ka} + \Gamma^a{}_{\ell j}\Psi^k{}_{ka} - \Psi^a{}_{kj}\Gamma^k{}_{\ell a} - \Gamma^a{}_{kj}\Psi^k{}_{\ell a}$$

$$= \partial_k\Psi^k{}_{j\ell} - \partial_\ell\Psi^k{}_{jk}$$

$$+ \Gamma^k{}_{ka}\Psi^a{}_{j\ell} + \Gamma^a{}_{\ell j}\Psi^k{}_{ak} - \Gamma^k{}_{\ell a}\Psi^a{}_{jk} - \Gamma^a{}_{kj}\Psi^k{}_{a\ell}$$

$$+ \Gamma^a{}_{\ell k}\Psi^k{}_{ja} - \Gamma^a{}_{k\ell}\Psi^k{}_{ja}.$$

[Note: Since ∇ is torsion free, Γ is symmetric in its covariant indices (by construction, the same holds for Ψ).]

Rappel: $\forall\ T \in \mathcal{D}^1_2(M)$,

$$\nabla_d T^c{}_{ab} = \partial_d T^c{}_{ab}$$

$$+ \Gamma^c{}_{de}T^e{}_{ab} - \Gamma^e{}_{da}T^c{}_{eb} - \Gamma^e{}_{db}T^c{}_{ae}.$$

$$\bullet\ \nabla_k\Psi^k{}_{j\ell} = \partial_k\Psi^k{}_{j\ell}$$

$$+ \Gamma^k{}_{ka}\Psi^a{}_{j\ell} - \Gamma^a{}_{kj}\Psi^k{}_{a\ell} - \Gamma^a{}_{k\ell}\Psi^k{}_{ja}.$$

$$\bullet\ - \nabla_\ell\Psi^k{}_{jk} = - \partial_\ell\Psi^k{}_{jk}$$

$$- \Gamma^k{}_{\ell a}\Psi^a{}_{jk} + \Gamma^a{}_{\ell j}\Psi^k{}_{ak} + \Gamma^a{}_{\ell k}\Psi^k{}_{ja}.$$

Therefore

$$\frac{d}{d\varepsilon}\,\mathrm{Ric}(\nabla + \varepsilon\Psi)_{j\ell}\Big|_{\varepsilon=0} = \nabla_k\Psi^k{}_{j\ell} - \nabla_\ell\Psi^k{}_{jk}.$$

Let ∇^g be the metric connection -- then $\nabla^g \in \mathrm{con}_0 TM$, hence the difference D defined by

$$\nabla = \nabla^g + D$$

is in $S^1_2(M)$.

Observation:

$$\nabla_d \Psi^c{}_{ab} = \nabla^g_d \Psi^c{}_{ab} + D^c{}_{de}\Psi^e{}_{ab} - D^e{}_{db}\Psi^c{}_{ae} - D^e{}_{da}\Psi^c{}_{be}.$$

Consequently,

$$g^{j\ell}\frac{d}{d\varepsilon}\operatorname{Ric}(\nabla + \varepsilon\Psi)_{j\ell}\Big|_{\varepsilon=0}$$

$$= g^{j\ell}(\nabla_k\Psi^k{}_{j\ell} - \nabla_\ell\Psi^k{}_{jk})$$

$$= g^{j\ell}(\nabla^g_k\Psi^k{}_{j\ell} + D^k{}_{ka}\Psi^a{}_{j\ell} - D^a{}_{k\ell}\Psi^k{}_{ja} - D^a{}_{kj}\Psi^k{}_{\ell a}$$

$$- \nabla^g_\ell\Psi^k{}_{jk} - D^k{}_{\ell a}\Psi^a{}_{jk} + D^a{}_{\ell k}\Psi^k{}_{ja} + D^a{}_{\ell j}\Psi^k{}_{ka})$$

$$= g^{j\ell}(\nabla^g_k\Psi^k{}_{j\ell} - \nabla^g_\ell\Psi^k{}_{jk})$$

$$+ g^{j\ell}(D^k{}_{ka}\Psi^a{}_{j\ell} - D^a{}_{kj}\Psi^k{}_{\ell a} - D^k{}_{\ell a}\Psi^a{}_{jk} + D^a{}_{\ell j}\Psi^k{}_{ka}).$$

[Note: The term

$$g^{j\ell}(\nabla^g_k\Psi^k{}_{j\ell} - \nabla^g_\ell\Psi^k{}_{jk})$$

is the divergence of a compactly supported vector field X_Ψ, hence integrates to zero.]

$$\bullet - g^{j\ell}D^{a}{}_{kj}\psi^{k}{}_{\ell a}$$

$$= - g^{j\ell}D^{k}{}_{aj}\psi^{a}{}_{\ell k}$$

$$= - g^{k\ell}D^{j}{}_{ak}\psi^{a}{}_{\ell j}$$

$$= - g^{k\ell}D^{j}{}_{ak}\psi^{a}{}_{j\ell}.$$

$$\bullet - g^{j\ell}D^{k}{}_{\ell a}\psi^{a}{}_{jk}$$

$$= - g^{jk}D^{\ell}{}_{ka}\psi^{a}{}_{j\ell}$$

$$= - g^{\ell k}D^{j}{}_{ka}\psi^{a}{}_{\ell j}$$

$$= - g^{k\ell}D^{j}{}_{ak}\psi^{a}{}_{j\ell}.$$

$$\bullet\, g^{j\ell}D^{a}{}_{\ell j}\psi^{k}{}_{ka}$$

$$= g^{j\ell}D^{a}{}_{\ell j}g^{kb}\psi_{bka}$$

$$= g^{j\ell}D^{k}{}_{\ell j}g^{ab}\psi_{bak}$$

$$= g^{b\ell}D^{k}{}_{\ell b}g^{aj}\psi_{jak}$$

$$= g^{bk}D^{\ell}{}_{kb}g^{aj}\psi_{ja\ell}$$

$$= g^{bk} D^{\ell}{}_{bk} \psi^{a}{}_{a\ell}$$

$$= g^{bk} D^{\ell}{}_{bk} \delta^{j}{}_{a} \psi^{a}{}_{j\ell}.$$

Therefore

$$g^{j\ell}(D^{k}{}_{ka}\psi^{a}{}_{j\ell} - D^{a}{}_{kj}\psi^{k}{}_{\ell a} - D^{k}{}_{\ell a}\psi^{a}{}_{jk} + D^{a}{}_{\ell j}\psi^{k}{}_{ka})$$

$$= (g^{j\ell}D^{k}{}_{ka} - 2g^{k\ell}D^{j}{}_{ak} + g^{bk}D^{\ell}{}_{bk}\delta^{j}{}_{a})\psi^{a}{}_{j\ell}$$

$$= (g^{j\ell}D^{k}{}_{ka} - 2D^{j}{}_{a}{}^{\ell} + \delta^{j}{}_{a}D^{\ell k}{}_{k})\psi^{a}{}_{j\ell}.$$

Let $T(\nabla)$ be the element of $\mathcal{D}^{1}_{2}(M)$ given locally by

$$T(\nabla)^{j\ell}{}_{a} = g^{j\ell}D^{k}{}_{ka} - 2D^{j}{}_{a}{}^{\ell} + \delta^{j}{}_{a}D^{\ell k}{}_{k}.$$

Then the conclusion is that

$$g^{j\ell}\frac{d}{d\varepsilon}\operatorname{Ric}(\nabla + \varepsilon\psi)_{j\ell}\Big|_{\varepsilon=0}$$

$$= \operatorname{div}_g X_\psi + \operatorname{tr}_g(T(\nabla),\psi).$$

[Note: Here tr_g stands for the pairing

$$\begin{cases} \mathcal{D}^{2}_{1}(M) \times \mathcal{D}^{1}_{2}(M) \to C^{\infty}(M) \\ \quad (T,S) \longrightarrow T^{ij}{}_{k}S^{k}{}_{ij}. \end{cases}$$]

Definition: An element $\nabla \in \mathrm{con}_0 TM$ is said to be critical if

$$T(\nabla) = 0.$$

[Note: To motivate this, adopt the usual shorthand and let

$$L_\omega(\nabla) = \int_M L_\omega(\nabla).$$

Then

$$\frac{d}{d\varepsilon} L_\omega(\nabla + \varepsilon \Psi)\Big|_{\varepsilon=0}$$

$$= \int_M \frac{d}{d\varepsilon} L_\omega(\nabla + \varepsilon \Psi)\Big|_{\varepsilon=0}$$

$$= \frac{1}{2} \int_M g^{j\ell} \frac{d}{d\varepsilon} \mathrm{Ric}(\nabla + \varepsilon \Psi)_{j\ell}\Big|_{\varepsilon=0} \mathrm{vol}_g$$

$$= \frac{1}{2} \int_M \mathrm{div}_g X_\Psi \mathrm{vol}_g + \frac{1}{2} \int_M \mathrm{tr}_g(T(\nabla), \Psi) \mathrm{vol}_g$$

$$= \frac{1}{2} \int_M \mathrm{tr}_g(T(\nabla), \Psi) \mathrm{vol}_g.$$

So ∇ is critical iff $\forall\ \Psi$,

$$\frac{d}{d\varepsilon} L_\omega(\nabla + \varepsilon \Psi)\Big|_{\varepsilon=0} = 0$$

or still, ∇ is critical iff

$$\frac{\delta L_\omega}{\delta \nabla} = 0.]$$

<u>THEOREM</u> Suppose that $n > 2$ — then $\nabla \in \text{con}_0 TM$ is critical iff $\nabla = \nabla^g$.

It is clear that ∇^g is critical $\forall$ n (since in this case $D = 0$).

To go the other way, the assumption that $T(\nabla) = 0$ implies that

$$\begin{cases} T(\nabla)^{ij}{}_k = g^{ij}D^{\ell}{}_{\ell k} - 2D^{i}{}_{k}{}^{j} + \delta^{i}{}_{k}D^{j\ell}{}_{\ell} = 0 \\ T(\nabla)^{ji}{}_k = g^{ji}D^{\ell}{}_{\ell k} - 2D^{j}{}_{k}{}^{i} + \delta^{j}{}_{k}D^{i\ell}{}_{\ell} = 0. \end{cases}$$

Thus

$$2g^{ij}D^{\ell}{}_{\ell k} - 2D^{i}{}_{k}{}^{j} - 2D^{j}{}_{k}{}^{i} + \delta^{i}{}_{k}D^{j\ell}{}_{\ell} + \delta^{j}{}_{k}D^{i\ell}{}_{\ell} = 0$$

=>

$$2g^{ij}D^{\ell}{}_{\ell j} - 2D^{i}{}_{j}{}^{j} - 2D^{j}{}_{j}{}^{i} + \delta^{i}{}_{j}D^{j\ell}{}_{\ell} + \delta^{j}{}_{j}D^{i\ell}{}_{\ell} = 0$$

=>

$$2g^{ij}D^{\ell}{}_{\ell j} - 2D^{i}{}_{\ell}{}^{\ell} - 2D^{\ell}{}_{\ell}{}^{i} + D^{i\ell}{}_{\ell} + \delta^{j}{}_{j}D^{i\ell}{}_{\ell} = 0$$

=>

$$2D^{\ell}{}_{\ell}{}^{i} - 2D^{i}{}_{\ell}{}^{\ell} - 2D^{\ell}{}_{\ell}{}^{i} + D^{i\ell}{}_{\ell} + nD^{i\ell}{}_{\ell} = 0$$

=>

$$(n+1)D^{i\ell}{}_{\ell} - 2D^{i}{}_{\ell}{}^{\ell} = 0.$$

But

$$\begin{cases} D^{i\ell}{}_{\ell} = g^{\ell k}D^{i}{}_{k\ell} \\ D^{i}{}_{\ell}{}^{\ell} = g^{\ell k}D^{i}{}_{\ell k} \end{cases} \quad \& \ D^{i}{}_{k\ell} = D^{i}{}_{\ell k}.$$

Therefore

$$(n-1)D^{i\ell}{}_{\ell} = 0$$

and, by the symmetry in i & j,

$$(n-1)D^{j\ell}{}_{\ell} = 0.$$

Similarly

$$2(n-2)D^{\ell}{}_{\ell k} = 0.$$

But then

$$0 = 2g^{ij}D^{\ell}{}_{\ell k} - 2D^{i}{}_{k}{}^{j} - 2D^{j}{}_{k}{}^{i} + \delta^{i}{}_{k}D^{j\ell}{}_{\ell} + \delta^{j}{}_{k}D^{i\ell}{}_{\ell}$$

$$= - 2D^{i}{}_{k}{}^{j} - 2D^{j}{}_{k}{}^{i}$$

=>

$$D^{i}{}_{k}{}^{j} = - D^{j}{}_{k}{}^{i}$$

=>

$$D_{ikj} = - D_{jki}$$

=>

$$D_{ijk} = - D_{jik}$$

=>

$$D_{ijk} + D_{jki} + D_{kij} + D_{ikj} + D_{jik} + D_{kji} = 0.$$

Add to this the relation

$$D_{ijk} + D_{jki} + D_{kij} - (D_{ikj} + D_{jik} + D_{kji}) = 0$$

to get

$$D_{ijk} + D_{jki} + D_{kij} = 0$$

or still,

$$D_{ijk} + D_{jik} + D_{kij} = 0$$

or still,

$$D_{kij} = 0.$$

I.e.:

$$D = 0$$

$$\Rightarrow$$

$$\nabla = \nabla^g.$$

Fix $\omega \in \text{cof}_M$ -- then instead of working with $\text{con}_0 TM$, one can work with $\text{con}_g TM$ which, as will be recalled, is an affine space with translation group $\mathcal{D}_2^1(M)_g$ (the subspace of $\mathcal{D}_2^1(M)$ consisting of those Ψ such that $\forall\ X,Y,Z \in \mathcal{D}^1(M)$,

$$g(\Psi(X,Y),Z) + g(Y,\Psi(X,Z)) = 0).$$

Definition: Let $\nabla \in \text{con}_g TM$ -- then a <u>variation of ∇</u> is a curve

$$\varepsilon \to \nabla + \varepsilon\Psi,$$

where $\Psi \in \mathcal{D}_2^1(M)_g$ has compact support.

[Note: Write

$$\Psi(E_k,E_j) = \Psi^i{}_{kj} E_i.$$

Then

$$g(\Psi(E_k,E_j),E_i) + g(E_j,\Psi(E_k,E_i)) = 0$$

=>

$$\Psi^i{}_{kj} = -\ \epsilon_i\epsilon_j\Psi^j{}_{ki} \quad \text{(no sum).]}$$

As before, define

$$L_\omega : \text{con}_g TM \to \Lambda^n M$$

by

$$L_\omega(\nabla) = \frac{1}{2}\,\Omega_{ij}(\nabla)\wedge\theta^{ij}.$$

Here

$$\Omega_{ij}(\nabla)$$

is computed per ∇ while

$$\theta^{ij} = \star(\omega^i\wedge\omega^j)$$

is computed per g.

Given $\nabla\in\text{con}_g TM$, consider

$$\frac{d}{d\epsilon}\,\Omega_{ij}(\nabla + \epsilon\Psi)\Big|_{\epsilon=0}\wedge\theta^{ij}$$

or, in brief,

$$\frac{d}{d\epsilon}\,\Omega_{ij}(\epsilon)\Big|_{\epsilon=0}\wedge\theta^{ij}.$$

[Note: Since

$$(\nabla + \epsilon\Psi)_X E_j = \nabla_X E_j + \epsilon\Psi(X,E_j)$$

$$= \nabla_X E_j + \varepsilon X^k \Psi(E_k, E_j)$$

$$= (\omega^i{}_j(X) + \varepsilon X^k \Psi^i{}_{kj}) E_i,$$

it follows that the connection 1-forms of $\nabla + \varepsilon\Psi$ are the

$$\omega^i{}_j(\varepsilon) = \omega^i{}_j + \varepsilon \Psi^i{}_{kj}\omega^k.]$$

Put

$$D = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}.$$

Then

$$\frac{d}{d\varepsilon}\,\Omega_{ij}(\varepsilon)\Big|_{\varepsilon=0} \wedge\theta^{ij}$$

$$= D(d\omega_{ij}(\varepsilon) + \omega_{ik}(\varepsilon)\wedge\omega^k{}_j(\varepsilon))\wedge\theta^{ij}$$

$$= (dD\omega_{ij}(\varepsilon) + D\omega_{ik}(\varepsilon)\wedge\omega^k{}_j + \omega_{ik}\wedge D\omega^k{}_j(\varepsilon))\wedge\theta^{ij}$$

$$= d(D\omega_{ij}(\varepsilon)\wedge *(\omega^i\wedge\omega^j))$$

$$+ D\omega_{ij}(\varepsilon)\wedge\Theta^a(\nabla)\wedge *(\omega^i\wedge\omega^j\wedge\omega_a)$$

$$= d(\Psi_{ikj}\omega^k\wedge *(\omega^i\wedge\omega^j))$$

$$+ \Psi_{ikj}\omega^k\wedge\Theta^a(\nabla)\wedge *(\omega^i\wedge\omega^j\wedge\omega_a).$$

<u>N.B.</u>

$$d*(\omega^i\wedge\omega^j)$$

$$= \Theta^a(\nabla)\wedge *(\omega^i\wedge\omega^j\wedge\omega_a) - \omega^i{}_a\wedge *(\omega^a\wedge\omega^j) - \omega^j{}_a\wedge *(\omega^i\wedge\omega^a).$$

Definition: An element $\nabla \in \text{con}_g TM$ is said to be critical if $\forall$ i,j:

$$\Theta^a(\nabla) \wedge * (\omega^i \wedge \omega^j \wedge \omega_a) = 0.$$

[Note: Set

$$L_\omega(\nabla) = \int_M \mathrm{L}_\omega(\nabla).$$

Then ∇ is critical iff $\forall$ Ψ,

$$\frac{d}{d\varepsilon} L_\omega(\nabla + \varepsilon\Psi)\Big|_{\varepsilon=0} = 0$$

or still, ∇ is critical iff

$$\frac{\delta L_\omega}{\delta \nabla} = 0.]$$

Remark: Our assumption is that $n > 2$. If n were 2, then

$$\omega^i \wedge \omega^j \wedge \omega_a = 0,$$

so every $\nabla \in \text{con}_g TM$ would be critical and the methods used below are not applicable.

<u>THEOREM</u> Suppose that $n > 2$ -- then $\nabla \in \text{con}_g TM$ is critical iff $\nabla = \nabla^g$.

It is clear that ∇^g is critical $\forall$ n (the metric connection is torsion free). As for the converse, it suffices to prove that

$$\nabla \text{ critical} \Rightarrow \nabla \text{ torsion free.}$$

I.e.:

$$\nabla \text{ critical} \Rightarrow \Theta^a(\nabla) = 0 \quad (a = 1,\ldots,n).$$

To see how the argument runs, take $a = 1$ -- then the claim is that

$$g(\Theta^1(\nabla),\omega^k\wedge\omega^\ell) = 0$$

for all $k \neq \ell$, there being two possibilities:

1. $k > 1$, $\ell > 1$

2. $k > 1$, $\ell = 1$.

Write

$$0 = \sum_a \Theta^a(\nabla)\wedge*(\omega^i\wedge\omega^j\wedge\omega_a)$$

$$= \sum_{a\neq i,j} \Theta^a(\nabla)\wedge*(\omega^i\wedge\omega^j\wedge\omega_a)$$

$$= \sum_{a\neq i,j} g(\Theta^a(\nabla),\omega^i\wedge\omega^j)*\omega_a$$

$$+ *\omega^i \sum_{a\neq i,j} g(\Theta^a(\nabla),\omega^j\wedge\omega_a) + *\omega^j \sum_{a\neq i,j} g(\Theta^a(\nabla),\omega_a\wedge\omega^i).$$

Then

$$g(\Theta^a(\nabla),\omega^i\wedge\omega^j) = 0 \quad (a \neq i,j)$$

=>

$$g(\Theta^1(\nabla),\omega^k\wedge\omega^\ell) = 0 \quad (k > 1,\ \ell > 1).$$

In addition,

$$\sum_{a\neq i,j} g(\Theta^a(\nabla),\omega^j\wedge\omega_a) = 0.$$

Now put

$$X^a = g(\Theta^a(\nabla),\omega^j\wedge\omega_a)$$

and

$$A = [A^i{}_a] \quad (A^i{}_a = 1 - \delta^i{}_a)$$

to get

$$A^i{}_a X^a$$

$$= (1 - \delta^i{}_a) g(\Theta^a(\nabla), \omega^j \wedge \omega_a)$$

$$= \sum_{a \neq j} g(\Theta^a(\nabla), \omega^j \wedge \omega_a) - g(\Theta^i(\nabla), \omega^j \wedge \omega_i)$$

$$= \sum_{a \neq i,j} g(\Theta^a(\nabla), \omega^j \wedge \omega_a)$$

$$= 0$$

$\Rightarrow$

$$AX = 0.$$

But A is nonsingular:

$$\det A = (-1)^{n+1}(n-1).$$

Therefore

$$X^a = g(\Theta^a(\nabla), \omega^j \wedge \omega_a) = 0.$$

In particular:

$$g(\Theta^1(\nabla), \omega^k \wedge \omega_1) = 0 \quad (k > 1).$$

Remark: It is not difficult to extend the Lovelock theory so as to incorporate $\mathrm{con}_g TM$: Simply define

$$L_{\omega,p}: \mathrm{con}_g TM \to \Lambda^n M$$

by

$$L_{\omega,p}(\nabla) = \frac{1}{2}\,\Omega_{i_1 j_1}(\nabla) \wedge \ldots \wedge \Omega_{i_p j_p}(\nabla) \wedge \theta^{i_1 j_1 \ldots i_p j_p} \quad (2p \le n).$$

The condition for criticality at level p then becomes the requirement that $\forall\ i_1, j_1$:

$$\Theta^a(\nabla) \wedge \Omega_{i_2 j_2}(\nabla) \wedge \ldots \wedge \Omega_{i_p j_p}(\nabla) \wedge \theta^{i_1 j_1 \ldots i_p j_p}{}_a = 0$$

[Note: If $p > 1$, then these equations do not necessarily imply that ∇ is torsion free.]

Section 49: Torsion Let M be a connected C^∞ manifold of dimension $n > 2$.

Assume: M is parallelizable.

Fix $\omega \in \mathrm{cof}_M$ and let ∇ be a g-connection -- then, per the previous section,

$$L_\omega(\nabla) = \frac{1}{2}\,\Omega_{ij}(\nabla)\wedge\theta^{ij}$$

and, as was shown there, ∇ is critical, i.e.,

$$\Theta^a(\nabla)\wedge *(\omega^i\wedge\omega^j\wedge\omega_a) = 0$$

$\forall$ i,j iff $\nabla = \nabla^g$.

Rappel: Suppose that $\nabla = \nabla^g$ (in which case we write Ω_{ij} in place of $\Omega_{ij}(\nabla^g)$) -- then

$$\frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij} = -\,d(\omega_i\wedge *d\omega^i)$$

$$+\,\frac{1}{4}\,(d\omega_i\wedge\omega^i)\wedge *(d\omega_j\wedge\omega^j) - \frac{1}{2}\,(d\omega_i\wedge\omega^j)\wedge *(d\omega_j\wedge\omega^i).$$

LEMMA We have

$$\frac{1}{2}\,\Omega_{ij}(\nabla)\wedge\theta^{ij} = \frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij} + d(\omega_i\wedge *\Theta^i(\nabla))$$

$$-\,\frac{1}{4}\,(\omega_i\wedge\Theta^i(\nabla))\wedge *(\omega_j\wedge\Theta^j(\nabla)) + \frac{1}{2}\,(\omega_i\wedge\Theta^j(\nabla))\wedge *(\omega_j\wedge\Theta^i(\nabla)).$$

Assume now that the standard setup is in force -- then

$$\nabla\in\mathrm{con}_g TM \Rightarrow \bar{\nabla}\in\mathrm{con}_{\bar{g}} T\Sigma.$$

[Note: By definition, $\bar{\nabla}$ is the connection on $T\Sigma$ which is obtained from the induced connection $i_t^*\nabla$ on i_t^*TM via the prescription

$$\bar{\nabla}_X Y = \tan i_t^*\nabla_X Y \quad (X,Y \in \mathcal{D}^1(\Sigma)).]$$

This said, let us consider the significance of the following conditions.

<u>Equation 1</u>: $i_t^*\Theta^0(\nabla) = 0.$

[We have

$$\begin{aligned}
i_t^*\Theta^0(\nabla) &= i_t^*(d\omega^0 + \omega^0{}_i \wedge \omega^i) \\
&= d i_t^*\omega^0 + i_t^*\omega^0{}_i \wedge i_t^*\omega^i \\
&= d i_t^*(N dt) + \bar{\omega}^0{}_i \wedge \bar{\omega}^i \\
&= dN_t\, i_t^* dt + \bar{\omega}^0{}_i \wedge \bar{\omega}^i \\
&= \bar{\omega}^0{}_a \wedge \bar{\omega}^a \\
&= (\bar{\omega}^0{}_a(E_b)\bar{\omega}^b) \wedge \bar{\omega}^a \\
&= \bar{\omega}^0{}_a(E_b)(\bar{\omega}^b \wedge \bar{\omega}^a).
\end{aligned}$$

Therefore

$$i_t^*\Theta^0(\nabla) = 0$$

$$\Longleftrightarrow$$

$$\bar{\omega}^0{}_a(E_b) = \bar{\omega}^0{}_b(E_a).$$

But

$$\left[\begin{array}{l} \bar{\omega}^0{}_a(E_b) = \kappa_{ba} = \kappa_t(E_b,E_a) \\ \\ \bar{\omega}^0{}_b(E_a) = \kappa_{ab} = \kappa_t(E_a,E_b). \end{array}\right. \qquad (\kappa = \mathcal{H})$$

Accordingly, $i_t^*\Theta^0(\nabla) = 0$ iff the extinsic curvature κ_t is symmetric.]

<u>Equation 2</u>: $i_t^*\Theta^a(\nabla) = 0 \quad (a = 1,\ldots,n-1)$.

[We have

$$i_t^*\Theta^a(\nabla) = i_t^*(d\omega^a + \omega^a{}_i\wedge\omega^i)$$

$$= d\bar{\omega}^a + \bar{\omega}^a{}_b\wedge\bar{\omega}^b.$$

So, if $i_t^*\Theta^a(\nabla) = 0 \ \forall\ a$, then $\bar{\nabla}$ is torsion free (and conversely).]

<u>Equation 3</u>: $i_t^*\iota_{E_0}\Theta^0(\nabla) = 0$.

[We have

$$i_t^*\iota_{E_0}\Theta^0(\nabla) = i_t^*\iota_{E_0}(d\omega^0 + \omega^0{}_i\wedge\omega^i)$$

$$= i_t^*\iota_{E_0}d\omega^0 + i_t^*\iota_{E_0}(\omega^0{}_i\wedge\omega^i).$$

1. $i_t^*\iota_{E_0}d\omega^0 = i_t^*\iota_{E_0}d(Ndt)$

$$= i_t^*\iota_{E_0}(dN\wedge dt)$$

$$= i_t^*(\iota_{E_0}dN\wedge dt - dN\wedge\iota_{E_0}dt)$$

$$= i_t^*(dN(E_0)\wedge dt - dN\wedge dt(E_0))$$

$$= dN(E_0)\, i_t^* dt - i_t^*(dN\wedge dt(E_0))$$

$$= -\ i_t^*(dN\wedge dt(E_0))$$

$$= -\ i_t^*\ (\frac{\omega^0(E_0)}{N}\, dN)$$

$$= -\ (\frac{1}{N_t}) dN_t.$$

2. $i_t^* \iota_{E_0}(\omega^0{}_i\wedge\omega^i)$

$$= i_t^*(\iota_{E_0}\omega^0{}_i\wedge\omega^i - \omega^0{}_i\wedge\iota_{E_0}\omega^i)$$

$$= i_t^*(\omega^0{}_i(E_0)\omega^i - \omega^i(E_0)\omega^0{}_i)$$

$$= i_t^*(\omega^0{}_a(E_0)\omega^a)$$

$$= -\ (i_t^*\omega_{0a}(E_0))\overline{\omega}^a.$$

Thus

$$i_t^*\iota_{E_0}\Theta^0(\nabla) = 0$$

$\Leftrightarrow$

$$(\frac{1}{N_t}) dN_t + (i_t^*\omega_{0a}(E_0))\overline{\omega}^a = 0$$

or still,

$$dN_t + (N_t i_t^*\omega_{0a}(E_0))\overline{\omega}^a = 0.]$$

<u>Equation 4</u>: $i_t^* \iota_{E_0} \Theta^a(\nabla) = 0 \quad (a = 1,\ldots,n-1).$

[We have

$$i_t^* \iota_{E_0} \Theta^a(\nabla) = i_t^* \iota_{E_0} (d\omega^a + \omega^a{}_i \wedge \omega^i)$$

$$= i_t^* \iota_{E_0} d\omega^a + i_t^* \iota_{E_0} (\omega^a{}_i \wedge \omega^i).$$

1. $i_t^* \iota_{E_0} d\omega^a$

$$= i_t^* (L_{E_0} - d \circ \iota_{E_0}) \omega^a$$

$$= i_t^* L_{E_0} \omega^a.$$

2. $i_t^* \iota_{E_0} (\omega^a{}_i \wedge \omega^i)$

$$= i_t^* (\iota_{E_0} \omega^a{}_i \wedge \omega^i - \omega^a{}_i \wedge \iota_{E_0} \omega^i)$$

$$= i_t^* (\omega^a{}_i (E_0) \omega^i - \omega^a{}_0)$$

$$= (i_t^* \omega^a{}_b (E_0)) \bar{\omega}^b - \bar{\omega}^a{}_0.$$

Consequently,

$$i_t^* \iota_{E_0} \Theta^a(\nabla) = 0$$

<=>

$$i_t^* L_{E_0} \omega^a = \bar{\omega}^a{}_0 - (i_t^* \omega^a{}_b (E_0)) \bar{\omega}^b$$

or still,

$$N_t i_t^* L_{E_0} \omega^a = N_t \bar{\omega}^a{}_0 - (N_t i_t^* \omega^a{}_b(E_0)) \bar{\omega}^b.$$

But

$$i_t^* L_{NE_0} \omega^a = i_t^*(N L_{E_0} \omega^a + dN \wedge \iota_{E_0} \omega^a)$$

$$= i_t^*(N L_{E_0} \omega^a)$$

$$= N_t i_t^* L_{E_0} \omega^a.$$

On the other hand,

$$i_t^* L_{NE_0} \omega^a = i_t^*(L_{\partial/\partial t} \omega^a - L_{\vec{N}} \omega^a)$$

$$= \dot{\bar{\omega}}^a - L_{\vec{N}_t} \bar{\omega}^a.$$

Therefore

$$i_t^* \iota_{E_0} \Theta^a(\nabla) = 0$$

<=>

$$\dot{\bar{\omega}}^a = N_t \bar{\omega}^a{}_0 - (N_t i_t^* \omega^a{}_b(E_0)) \bar{\omega}^b + L_{\vec{N}_t} \bar{\omega}^a.]$$

Notation: Put

$$\begin{bmatrix} \bar{P}_a = N_t i_t^* \omega_{0a}(E_0) & \text{(cf. Equation 3)} \\ \bar{Q}^a{}_b = - N_t i_t^* \omega^a{}_b(E_0) & \text{(cf. Equation 4).} \end{bmatrix}$$

LEMMA Suppose that Equations 1 - 4 are satisfied for all t -- then ∇ is torsion free, i.e., $\Theta(\nabla) = 0$.

[It is a question of showing that $\Theta^0(\nabla) = 0$ and $\Theta^a(\nabla) = 0$ $(a = 1,\ldots,n-1)$. Write

$$\Theta^0(\nabla) = C^0{}_{0a}\omega^0\wedge\omega^a$$

$$+ \frac{1}{2} C^0{}_{ab}\omega^a\wedge\omega^b \quad (C^0{}_{ab} = - C^0{}_{ba}).$$

Then

$$i_t^*\Theta^0(\nabla) = 0 \ \forall \ t$$

=>

$$\frac{1}{2} \bar{C}^0{}_{ab}\bar{\omega}^a\wedge\bar{\omega}^b = 0 \ \forall \ t$$

=>

$$\bar{C}^0{}_{ab} = 0 \ \forall \ t$$

=>

$$C^0{}_{ab} = 0$$

and

$$i_t^*\iota_{E_0}\Theta^0(\nabla) = 0 \ \forall \ t$$

=>

$$\bar{C}^0{}_{0a}\bar{\omega}^a = 0 \ \forall \ t$$

=>

$$\bar{C}^0{}_{0a} = 0 \ \forall \ t$$

=>

$$C^{0}{}_{0a} = 0.$$

So $\Theta^0(\nabla) = 0$. The proof that $\Theta^a(\nabla) = 0$ $(a = 1,\ldots,n-1)$ is analogous.]

<u>Section 50</u>: <u>Extending the Theory</u> The assumptions and notation are those of the standard setup.

Throughout this section, ∇ stands for an arbitrary element of $\text{con}_g TM$.

[Note: Here, of course,

$$g = -\omega^0 \otimes \omega^0 + \omega^a \wedge \omega^a.]$$

Rappel: If $\nabla = \nabla^g$, then

$$\Omega_{ij} \wedge \theta^{ij} = \theta^{ij} \wedge \Omega_{ij}$$

$$= 2d(\omega_{0a} \wedge \theta^{0a}) - \theta^{ab} \wedge \omega_{0a} \wedge \omega_{0b}$$

$$+ \theta^{ab} \wedge (\Omega_{ab} - \omega_{a0} \wedge \omega^0{}_b).$$

This relation was the initial step in isolating the lagrangian and our first objective is to generalize it in order to cover the case when $\nabla \neq \nabla^g$.

Since ∇ is a g-connection, it is still true that $\Omega_{ij}(\nabla) = -\Omega_{ji}(\nabla)$, hence

$$\theta^{ij} \wedge \Omega_{ij}(\nabla) = 2\theta^{0a} \wedge \Omega_{0a}(\nabla) + \theta^{bc} \wedge \Omega_{bc}(\nabla),$$

thus as before

$$\theta^{ij} \wedge \Omega_{ij}(\nabla) = 2\theta^{0a} \wedge d\omega_{0a} + 2\theta^{0a} \wedge \omega_{0b} \wedge \omega^b{}_a$$

$$+ \theta^{bc} \wedge (\Omega_{bc}(\nabla) - \omega_{b0} \wedge \omega^0{}_c) + \theta^{bc} \wedge \omega_{b0} \wedge \omega^0{}_c.$$

Write

$$d(\theta^{0a} \wedge \omega_{0a}) = d\theta^{0a} \wedge \omega_{0a} + (-1)^{n-2}\theta^{0a} \wedge d\omega_{0a}$$

to get

$$2\theta^{0a}\wedge d\omega_{0a}$$

$$= 2(-1)^{n-2}d(\theta^{0a}\wedge\omega_{0a}) - 2(-1)^{n-2}d\theta^{0a}\wedge\omega_{0a}$$

$$= 2(-1)^{n-2}(-1)^{n-2}d(\omega_{0a}\wedge\theta^{0a}) - 2(-1)^{n-2}(-1)^{n-1}\omega_{0a}\wedge d\theta^{0a}$$

$$= 2d(\omega_{0a}\wedge\theta^{0a}) + 2\omega_{0a}\wedge d\theta^{0a}.$$

Then

$$\theta^{ij}\wedge\Omega_{ij}(\nabla) = 2d(\omega_{0a}\wedge\theta^{0a}) + 2\omega_{0a}\wedge d\theta^{0a} + 2\theta^{0a}\wedge\omega_{0b}\wedge\omega^{b}{}_{a}$$

$$+ \theta^{ab}\wedge(\Omega_{ab}(\nabla) - \omega_{a0}\wedge\omega^{0}{}_{b}) + \theta^{ab}\wedge\omega_{a0}\wedge\omega^{0}{}_{b}.$$

But on general grounds,

$$d\theta^{0a} = d*(\omega^{0}\wedge\omega^{a})$$

$$= d\omega_{b}\wedge*(\omega^{0}\wedge\omega^{a}\wedge\omega^{b}),$$

so modulo the differential $2d(\omega_{0a}\wedge\theta^{0a})$,

$$\theta^{ij}\wedge\Omega_{ij}(\nabla) = 2\omega_{0a}\wedge d\omega_{b}\wedge*(\omega^{0}\wedge\omega^{a}\wedge\omega^{b}) + 2\theta^{0a}\wedge\omega_{0b}\wedge\omega^{b}{}_{a}$$

$$+ \theta^{ab}\wedge(\Omega_{ab}(\nabla) - \omega_{a0}\wedge\omega^{0}{}_{b}) + \theta^{ab}\wedge\omega_{a0}\wedge\omega^{0}{}_{b}.$$

•We have

$$0 = \iota_{E_0}(\omega_{0a}\wedge d\omega_b\wedge\theta^{ab})$$

$$= \iota_{E_0}\omega_{0a}\wedge d\omega_b\wedge\theta^{ab} - \omega_{0a}\wedge\iota_{E_0}(d\omega_b\wedge\theta^{ab})$$

$$= \iota_{E_0}\omega_{0a}\wedge d\omega_b\wedge\theta^{ab} - \omega_{0a}\wedge\iota_{E_0}d\omega_b\wedge\theta^{ab}$$

$$- \omega_{0a}\wedge d\omega_b\wedge\iota_{E_0}\theta^{ab}$$

=>

$$\omega_{0a}\wedge d\omega_b\wedge\iota_{E_0}\theta^{ab}$$

$$= \omega_{0a}(E_0)\,d\omega_b\wedge\theta^{ab} - \omega_{0a}\wedge\iota_{E_0}d\omega_b\wedge\theta^{ab}.$$

And

$$\iota_{E_0}\theta^{ab} = \iota_{E_0}*(\omega^a\wedge\omega^b)$$

$$= *(\omega^a\wedge\omega^b\wedge g^\flat E_0)$$

$$= *(\omega^a\wedge\omega^b\wedge\omega_0)$$

$$= *(\omega_0\wedge\omega^a\wedge\omega^b)$$

$$= - *(\omega^0\wedge\omega^a\wedge\omega^b).$$

• We have

$$0 = \iota_{E_0}(*\omega^a\wedge\omega_{0b}\wedge\omega^b{}_a)$$

$$= \iota_{E_0}*\omega^a\wedge\omega_{0b}\wedge\omega^b{}_a + (-1)^{n-1}*\omega^a\wedge\iota_{E_0}(\omega_{0b}\wedge\omega^b{}_a)$$

$$= \iota_{E_0}*\omega^a\wedge\omega_{0b}\wedge\omega^b{}_a + (-1)^{n-1}*\omega^a\wedge\iota_{E_0}\omega_{0b}\wedge\omega^b{}_a$$

$$+ (-1)^n*\omega^a\wedge\omega_{0b}\wedge\iota_{E_0}\omega^b{}_a$$

=>

$$\iota_{E_0}*\omega^a\wedge\omega_{0b}\wedge\omega^b{}_a$$

$$= (-1)^n\omega_{0b}(E_0)*\omega^a\wedge\omega^b{}_a + (-1)^{n-1}\omega^b{}_a(E_0)*\omega^a\wedge\omega_{0b}$$

$$= (-1)^n(-1)^{n-1}\omega_{0b}(E_0)\omega^b{}_a\wedge*\omega^a + (-1)^{n-1}(-1)^{n-1}\omega^b{}_a(E_0)\omega_{0b}\wedge*\omega^a$$

$$= -\omega_{0b}(E_0)\omega^b{}_a\wedge*\omega^a + \omega^b{}_a(E_0)\omega_{0b}\wedge*\omega^a.$$

And

$$\iota_{E_0}*\omega^a = *(\omega^a\wedge g^\flat E_0)$$

$$= *(\omega^a\wedge\omega_0)$$

$$= -*(\omega_0\wedge\omega^a)$$

$$= *(\omega^0\wedge\omega^a) = \theta^{0a}.$$

Therefore, up to the differential $2d(\omega_{0a}\wedge\theta^{0a})$,

$$\theta^{ij}\wedge\Omega_{ij}(\nabla) = 2\omega_{0a}\wedge\iota_{E_0}d\omega_b\wedge\theta^{ab} - 2\omega_{0a}(E_0)d\omega_b\wedge\theta^{ab}$$

$$- 2\omega_{0b}(E_0)\omega^b{}_a\wedge*\omega^a + 2\omega^b{}_a(E_0)\omega_{0b}\wedge*\omega^a$$

$$+ \theta^{ab}\wedge(\Omega_{ab}(\nabla) - \omega_{a0}\wedge\omega^0{}_b) + \theta^{ab}\wedge\omega_{a0}\wedge\omega^0{}_b.$$

N.B.

$$- 2\omega_{0b}(E_0)\omega^b{}_a\wedge*\omega^a + 2\omega^b{}_a(E_0)\omega_{0b}\wedge*\omega^a$$

$$= - 2\omega_{0a}(E_0)\omega^a{}_b\wedge*\omega^b + 2\omega^a{}_b(E_0)\omega_{0a}\wedge*\omega^b.$$

Rappel:

$$\begin{cases} \bar{P}_a = N_t i_t^*\omega_{0a}(E_0) \\ \bar{Q}^a{}_b = - N_t i_t^*\omega^a{}_b(E_0). \end{cases}$$

Using now the same methods that were employed in the study of $\theta^{ij}\wedge\Omega_{ij}$, we then find that

1. $i_t^*\iota_{\partial/\partial t}(\omega_{0a}\wedge\iota_{E_0}d\omega_b\wedge\theta^{ab})$

$$= \bar{\omega}_{0a}\wedge N_t i_t^*\iota_{E_0}d\omega_b\wedge*(\bar{\omega}^a\wedge\bar{\omega}^b).$$

2. $- i_t^*\iota_{\partial/\partial t}(\omega_{0a}(E_0)d\omega_b\wedge\theta^{ab})$

$$= - \bar{P}_a d\bar{\omega}_b\wedge*(\bar{\omega}^a\wedge\bar{\omega}^b).$$

3. $- i_t^* \iota_{\partial/\partial t}(\omega_{0a}(E_0)\omega^a{}_b \wedge *\omega^b)$

$$= - \bar{P}_a \bar{\omega}^a{}_b \wedge *\bar{\omega}^b.$$

4. $i_t^* \iota_{\partial/\partial t}(\omega^a{}_b(E_0)\omega_{0a} \wedge *\omega^b)$

$$= - \bar{Q}^a{}_b \bar{\omega}_{0a} \wedge *\bar{\omega}^b.$$

5. $i_t^* \iota_{\partial/\partial t}(\theta^{ab} \wedge (\Omega_{ab}(\nabla) - \omega_{a0} \wedge \omega^0{}_b))$

$$= N_t * (\bar{\omega}^a \wedge \bar{\omega}^b) \wedge {}^{(n-1)}\Omega_{ab}(\bar{\nabla}).$$

6. $i_t^* \iota_{\partial/\partial t}(\theta^{ab} \wedge \omega_{a0} \wedge \omega^0{}_b)$

$$= N_t * (\bar{\omega}^a \wedge \bar{\omega}^b) \bar{\omega}_{0a} \wedge \bar{\omega}_{0b}.$$

<u>Details</u> Items 1, 2, 5, and 6 are handled as before but one has to be careful with items 3 and 4 and make sure that the signs are correct. Thus write

$$*\omega^b = \frac{1}{(n-1)!} \epsilon_{bj_2 \ldots j_n} \omega^{j_2} \wedge \ldots \wedge \omega^{j_n}$$

$$= \frac{1}{(n-1)!} \epsilon_{b0j_3 \ldots j_n} \omega^0 \wedge \omega^{j_3} \wedge \ldots \wedge \omega^{j_n}$$

$$+ \cdots + \frac{1}{(n-1)!} \epsilon_{bj_2 \ldots j_{n-1}0} \omega^{j_2} \wedge \ldots \wedge \omega^{j_{n-1}} \wedge \omega^0$$

$$+ \frac{1}{(n-1)!} \epsilon_{bc_2 \ldots c_n} \omega^{c_2} \wedge \ldots \wedge \omega^{c_n}$$

$$= \frac{1}{(n-1)!}\,\varepsilon_{b0j_3\ldots j_n}\,\omega^0\wedge\omega^{j_3}\wedge\ldots\wedge\omega^{j_n}$$

$$+ \cdots + \frac{1}{(n-1)!}\,\varepsilon_{bj_2\ldots j_{n-1}0}\,\omega^{j_2}\wedge\ldots\wedge\omega^{j_{n-1}}\wedge\omega^0$$

$$= \frac{(n-1)}{(n-1)!}\,\varepsilon_{b0c_3\ldots c_n}\,\omega^0\wedge\omega^{c_3}\wedge\ldots\wedge\omega^{c_n}$$

$$= \omega^0\wedge\frac{1}{(n-2)!}\,\varepsilon_{b0c_3\ldots c_n}\,\omega^{c_3}\wedge\ldots\wedge\omega^{c_n}$$

$$= -\,\omega^0\wedge\frac{1}{(n-2)!}\,\varepsilon_{0bc_3\ldots c_n}\,\omega^{c_3}\wedge\ldots\wedge\omega^{c_n}.$$

Then

$$-\,i_t^*\iota_{\partial/\partial t}(\omega_{0a}(E_0)\,\omega^a{}_b\wedge *\omega^b)$$

$$= +\,i_t^*\iota_{\partial/\partial t}[\omega_{0a}(E_0)\,\omega^a{}_b\wedge\omega^0\wedge\frac{1}{(n-2)!}\,\varepsilon_{0bc_3\ldots c_n}\,\omega^{c_3}\wedge\ldots\wedge\omega^{c_n}]$$

$$= -\,i_t^*\iota_{\partial/\partial t}[\omega^0\wedge\omega_{0a}(E_0)\,\omega^a{}_b\wedge\frac{1}{(n-2)!}\,\varepsilon_{0bc_3\ldots c_n}\,\omega^{c_3}\wedge\ldots\wedge\omega^{c_n}]$$

$$= -\,i_t^*[(\iota_{\partial/\partial t}\omega^0)\wedge\omega_{0a}(E_0)\,\omega^a{}_b\wedge\frac{1}{(n-2)!}\,\varepsilon_{0bc_3\ldots c_n}\,\omega^{c_3}\wedge\ldots\wedge\omega^{c_n}$$

$$-\,\omega^0\wedge\iota_{\partial/\partial t}(\omega_{0a}(E_0)\,\omega^a{}_b\wedge\frac{1}{(n-2)!}\,\varepsilon_{0bc_3\ldots c_n}\,\omega^{c_3}\wedge\ldots\wedge\omega^{c_n})]$$

$$= -\,N_t\,i_t^*\omega_{0a}(E_0)\,\bar{\omega}^a{}_b\wedge\frac{1}{(n-2)!}\,\varepsilon_{0bc_3\ldots c_n}\,\bar{\omega}^{c_3}\wedge\ldots\wedge\bar{\omega}^{c_n}$$

$$= -\,\bar{P}_a\,\bar{\omega}^a{}_b\wedge *\bar{\omega}^b.$$

The fact that

$$i_t^* \iota_{\partial/\partial t}(\omega^a{}_b(E_0)\omega_{0a}\wedge *\omega^b)$$

$$= -\bar{Q}^a{}_b \bar{\omega}_{0a}\wedge *\bar{\omega}^b$$

is proved in exactly the same way.

Summary:

$$1 + 2 + 3 + 4 + 5 + 6$$

$$= 2[\bar{\omega}_{0a}\wedge N_t i_t^* \iota_{E_0} d\omega_b - \bar{P}_a d\bar{\omega}_b$$

$$+ \frac{1}{2} N_t({}^{(n-1)}\Omega_{ab}(\bar{\nabla}) + \bar{\omega}_{0a}\wedge\bar{\omega}_{0b})]\wedge *(\bar{\omega}^a\wedge\bar{\omega}^b)$$

$$- 2(\bar{Q}^a{}_b\bar{\omega}_{0a} + \bar{P}_a\bar{\omega}^a{}_b)\wedge *\bar{\omega}^b.$$

Claim:

$$\bar{Q}^a{}_b\bar{\omega}_{0a}\wedge *\bar{\omega}^b = \bar{Q}^c{}_b\bar{\omega}_c\wedge\bar{\omega}_{0a}\wedge *(\bar{\omega}^a\wedge\bar{\omega}^b).$$

[The issue is the equality of

$$\bar{Q}^a{}_b\bar{g}(\bar{\omega}_{0a},\bar{\omega}^b)$$

and

$$\bar{Q}^c{}_b\bar{g}(\bar{\omega}_c\wedge\bar{\omega}_{0a},\bar{\omega}^a\wedge\bar{\omega}^b).$$

But

$$\bar{g}(\bar{\omega}_c\wedge\bar{\omega}_{0a},\bar{\omega}^a\wedge\bar{\omega}^b)$$

$$= \bar{g}(\iota_{\bar{\omega}^a}(\bar{\omega}_c\wedge\bar{\omega}_{0a}),\bar{\omega}^b)$$

$$= \bar{g}(\iota_{\bar{\omega}^a}\bar{\omega}_c \wedge \bar{\omega}_{0a} - \bar{\omega}_c \wedge \iota_{\bar{\omega}^a}\bar{\omega}_{0a}, \bar{\omega}^b)$$

$$= \bar{g}(\bar{\omega}^a, \bar{\omega}_c)\bar{g}(\bar{\omega}_{0a}, \bar{\omega}^b) - \bar{g}(\bar{\omega}^a, \bar{\omega}_{0a})\bar{g}(\bar{\omega}_c, \bar{\omega}^b)$$

=>

$$\bar{Q}^c{}_b \bar{g}(\bar{\omega}_c \wedge \bar{\omega}_{0a}, \bar{\omega}^a \wedge \bar{\omega}^b)$$

$$= \bar{Q}^c{}_b \bar{g}(\bar{\omega}^a, \bar{\omega}_c)\bar{g}(\bar{\omega}_{0a}, \bar{\omega}^b) - \bar{Q}^c{}_b \bar{g}(\bar{\omega}^a, \bar{\omega}_{0a})\bar{g}(\bar{\omega}_c, \bar{\omega}^b)$$

$$= \bar{Q}^a{}_b \bar{g}(\bar{\omega}_{0a}, \bar{\omega}^b) - \bar{Q}^b{}_b \bar{g}(\bar{\omega}^a, \bar{\omega}_{0a})$$

$$= \bar{Q}^a{}_b \bar{g}(\bar{\omega}_{0a}, \bar{\omega}^b).]$$

Claim:

$$\bar{P}_a \bar{\omega}^a{}_b \wedge * \bar{\omega}^b = \bar{P}_a \bar{\omega}_{bc} \wedge \bar{\omega}^c \wedge * (\bar{\omega}^a \wedge \bar{\omega}^b).$$

[The issue is the equality of

$$\bar{P}_a \bar{g}(\bar{\omega}^a{}_b, \bar{\omega}^b)$$

and

$$\bar{P}_a \bar{g}(\bar{\omega}_{bc} \wedge \bar{\omega}^c, \bar{\omega}^a \wedge \bar{\omega}^b).$$

But

$$\bar{g}(\bar{\omega}_{bc} \wedge \bar{\omega}^c, \bar{\omega}^a \wedge \bar{\omega}^b)$$

$$= \bar{g}(\iota_{\bar{\omega}^a}(\bar{\omega}_{bc} \wedge \bar{\omega}^c), \bar{\omega}^b)$$

$$= \bar{g}(\iota_{\bar{\omega}^a}\bar{\omega}_{bc} \wedge \bar{\omega}^c - \bar{\omega}_{bc} \wedge \iota_{\bar{\omega}^a}\bar{\omega}^c, \bar{\omega}^b)$$

$$= \bar{g}(\bar{\omega}^a, \bar{\omega}_{bc})\bar{g}(\bar{\omega}^c, \bar{\omega}^b) - \bar{g}(\bar{\omega}^a, \bar{\omega}^c)\bar{g}(\bar{\omega}_{bc}, \bar{\omega}^b)$$

=>

$$\bar{P}_a \bar{g}(\bar{\omega}_{bc} \wedge \bar{\omega}^c, \bar{\omega}^a \wedge \bar{\omega}^b)$$

$$= \bar{P}_a \bar{g}(\bar{\omega}^a, \bar{\omega}_{bc})\bar{g}(\bar{\omega}^c, \bar{\omega}^b) - \bar{P}_a \bar{g}(\bar{\omega}^a, \bar{\omega}^c)\bar{g}(\bar{\omega}_{bc}, \bar{\omega}^b)$$

$$= \bar{P}_a \bar{g}(\bar{\omega}^a, \bar{\omega}_{bb}) - \bar{P}_a \bar{g}(\bar{\omega}_{ba}, \bar{\omega}^b)$$

$$= - \bar{P}_a \bar{g}(\bar{\omega}_{ba}, \bar{\omega}^b)$$

$$= \bar{P}_a \bar{g}(\bar{\omega}_{ab}, \bar{\omega}^b)$$

$$= \bar{P}_a \bar{g}(\bar{\omega}^a{}_b, \bar{\omega}^b).]$$

From these considerations, it follows that

$$1 + 2 + 3 + 4 + 5 + 6$$

$$= 2[\tfrac{1}{2} N_t({}^{(n-1)}\Omega_{ab}(\bar{\nabla}) + \bar{\omega}_{0a} \wedge \bar{\omega}_{0b}) - \bar{P}_a(d\bar{\omega}_b + \bar{\omega}_{bc} \wedge \bar{\omega}^c)$$

$$- \bar{Q}^c{}_b \bar{\omega}_c \wedge \bar{\omega}_{0a} + \bar{\omega}_{0a} \wedge N_t i_t^* \iota_{E_0} d\omega_b] \wedge *(\bar{\omega}^a \wedge \bar{\omega}^b).$$

Consequently, if we set aside the differential

$$2d(\omega_{0a} \wedge \theta^{0a}),$$

then formally

$$\int_M \theta^{ij} \wedge \Omega_{ij}(\nabla)$$

$$= \int_{\underline{R}} dt \int_\Sigma i_t^* \iota_{\partial/\partial t}[2\omega_{0a}\wedge\iota_{E_0} d\omega_b\wedge\theta^{ab} - 2\omega_{0a}(E_0)d\omega_b\wedge\theta^{ab}$$

$$- 2\omega_{0a}(E_0)\omega^a{}_b\wedge *\omega^b + 2\omega^a{}_b(E_0)\omega_{0a}\wedge *\omega^b$$

$$+ \theta^{ab}\wedge(\Omega_{ab}(\nabla) - \omega_{a0}\wedge\omega^0{}_b) + \theta^{ab}\wedge\omega_{a0}\wedge\omega^0{}_b]$$

$$= \int_{\underline{R}} dt \int_\Sigma (1 + 2 + 3 + 4 + 5 + 6)$$

$$= \int_{\underline{R}} dt \int_\Sigma 2[\frac{1}{2} N_t({}^{(n-1)}\Omega_{ab}(\bar\nabla) + \bar\omega_{0a}\wedge\bar\omega_{0b}) - \bar P_a(d\bar\omega_b + \bar\omega_{bc}\wedge\bar\omega^c)$$

$$- \bar Q^c{}_b\bar\omega_c\wedge\bar\omega_{0a} + \bar\omega_{0a}\wedge N_t i_t^*\iota_{E_0} d\omega_b]\wedge *(\bar\omega^a\wedge\bar\omega^b).$$

Remark: As far as I can tell, an analysis of the

$$\theta^{i_1 j_1 \cdots i_p j_p}\wedge\Omega_{i_1 j_1}(\nabla)\wedge\ldots\wedge\Omega_{i_p j_p}(\nabla) \quad (p > 1)$$

along the foregoing lines has never been carried out.

<u>LEMMA</u> We have

$$N_t i_t^*\iota_{E_0} d\omega_b = \dot{\bar\omega}^b - L_{\vec N_t}\bar\omega^b.$$

[In fact,

$$\dot{\bar\omega}^b = i_t^* L_{\partial/\partial t}\omega^b$$

$$= i_t^*(L_{NE_0} + L_{\vec N})\omega^b$$

$$= i_t^* \mathcal{L}_{NE_0}\omega^b + i_t^* \mathcal{L}_{\vec{N}}\omega^b$$

$$= i_t^*(N\mathcal{L}_{E_0}\omega^b + dN\wedge\iota_{E_0}\omega^b) + \mathcal{L}_{\vec{N}_t}\bar{\omega}^b$$

$$= i_t^*(N\mathcal{L}_{E_0}\omega^b) + \mathcal{L}_{\vec{N}_t}\bar{\omega}^b$$

$$= N_t i_t^* \mathcal{L}_{E_0}\omega^b + \mathcal{L}_{\vec{N}_t}\bar{\omega}^b$$

$$= N_t i_t^*(\iota_{E_0} \circ d + d \circ \iota_{E_0})\omega^b + \mathcal{L}_{\vec{N}_t}\bar{\omega}^b$$

$$= N_t i_t^* \iota_{E_0} d\omega^b + \mathcal{L}_{\vec{N}_t}\bar{\omega}^b.]$$

Because of this, one can replace

$$\bar{\omega}_{0a}\wedge N_t i_t^* \iota_{E_0} d\omega_b$$

by

$$\bar{\omega}_{0a}\wedge(\dot{\bar{\omega}}^b - \mathcal{L}_{\vec{N}_t}\bar{\omega}^b).$$

Summary:

$$\int_{\underline{R}} dt \int_{\Sigma} (1 + 2 + 3 + 4 + 5 + 6)$$

$$= \int_{\underline{R}} dt \int_{\Sigma} 2[\tfrac{1}{2} N_t({}^{(n-1)}\Omega_{ab}(\bar{\nabla}) + \bar{\omega}_{0a}\wedge\bar{\omega}_{0b}) - \bar{P}_a(d\bar{\omega}_b + \bar{\omega}_{bc}\wedge\bar{\omega}^c)$$

$$+ \bar{Q}^c{}_a\bar{\omega}_c\wedge\bar{\omega}_{0b} + \dot{\bar{\omega}}^a\wedge\bar{\omega}_{0b} - \mathcal{L}_{\vec{N}_t}\bar{\omega}^a\wedge\bar{\omega}_{0b}]\wedge*(\bar{\omega}^a\wedge\bar{\omega}^b).$$

<u>Reality Check</u> Specialize and take $\nabla = \nabla^g$ -- then

$$1 + 2 + 3 + 4 + 5 + 6$$

reduces to

$$N_t({}^{(n-1)}\Omega_{ab} - \bar{\omega}_{0a}\wedge\bar{\omega}_{0b})\wedge *(\bar{\omega}^a\wedge\bar{\omega}^b)$$

as it should. First, since ∇^g is torsion free,

$$0 = \bar{\Theta}_b = d\bar{\omega}_b + \bar{\omega}_{bc}\wedge\bar{\omega}^c,$$

hence

$$\bar{P}_a(d\bar{\omega}_b + \bar{\omega}_{bc}\wedge\bar{\omega}^c) = 0.$$

It remains to consider

$$N_t(\bar{\omega}_{0a}\wedge\bar{\omega}_{0b}) + 2[\bar{Q}^c{}_a\bar{\omega}_c\wedge\bar{\omega}_{0b} + \dot{\bar{\omega}}^a\wedge\bar{\omega}_{0b} - L_{\vec{N}_t}\bar{\omega}^a\wedge\bar{\omega}_{0b}]$$

or still,

$$[N_t\bar{\omega}_{0a} + 2(\bar{Q}^c{}_a\bar{\omega}_c + \dot{\bar{\omega}}^a - L_{\vec{N}_t}\bar{\omega}^a)]\wedge\bar{\omega}_{0b}$$

or still,

$$[N_t\bar{\omega}_{0a} + 2(-\bar{Q}^a{}_c\bar{\omega}^c + \dot{\bar{\omega}}^a - L_{\vec{N}_t}\bar{\omega}^a)]\wedge\bar{\omega}_{0b}$$

or still,

$$[N_t\bar{\omega}_{0a} + 2(-\dot{\bar{\omega}}^a + N_t\bar{\omega}^a{}_0 + L_{\vec{N}_t}\bar{\omega}^a)$$

$$+ 2(\dot{\bar{\omega}}^a - L_{\vec{N}_t}\bar{\omega}^a)]\wedge\bar{\omega}_{0b}$$

or still,

$$[N_t\bar{\omega}_{0a} + 2N_t\bar{\omega}^a{}_0]\wedge\bar{\omega}_{0b}$$

or still,

$$[N_t\bar{\omega}_{0a} + 2N_t\bar{\omega}_{a0}]\wedge\bar{\omega}_{0b}$$

or still,

$$[N_t\bar{\omega}_{0a} - 2N_t\bar{\omega}_{0a}]\wedge\bar{\omega}_{0b}$$

which equals

$$- N_t\bar{\omega}_{0a}\wedge\bar{\omega}_{0b}.$$

Before extrapolating the foregoing, let us recall the notation: Elements of $\underline{Q}$ are denoted by $\vec{\omega}$, elements of $T\underline{Q}$ are denoted by $(\vec{\omega},\vec{v})$, and elements of $T^*\underline{Q}$ are denoted by $(\vec{\omega},\vec{p})$.

<u>External Variables</u> These are N, $\vec{N}$, and W plus three others, viz.:

1. $\underline{\omega} = [\omega^a{}_b]\in\Lambda^1(\Sigma;\underline{so}(n-1))$.

2. $\underline{\omega}_0 = [\omega_{0a}]\in\Lambda^1(\Sigma;\underline{R}^{n-1})$.

3. $\vec{B} = [B_a]\in C^\infty(\Sigma;\underline{R}^{n-1})$.

Definition: The <u>lagrangian</u> of the theory is the function

$$L:T\underline{Q} \to \Lambda^{n-1}\Sigma$$

defined by the rule

$$L(\vec{\omega},\vec{v};N,\vec{N},W;\underline{\omega},\underline{\omega}_0,\vec{B})$$

$$= 2[\frac{N}{2}(\Omega_{ab}(\underline{\omega}) + \omega_{0a}\wedge\omega_{0b}) - B_a(d\omega_b + \omega_{bc}\wedge\omega^c)$$

$$+ W^c{}_a\omega_c\wedge\omega_{0b} + v^a\wedge\omega_{0b} - L_{\vec{N}}\omega^a\wedge\omega_{0b}]\wedge*(\omega^a\wedge\omega^b).$$

[Note: The precise meaning of the symbol $\Omega_{ab}(\underline{\omega})$ is this. Let $\vec{E}$ be the frame associated with $\vec{\omega}$ by duality — then the prescription

$$\nabla_X Y = < X, dY^a + \omega^a{}_b Y^b > E_a$$

defines a q-connection $\nabla(\underline{\omega})$ (since $\omega_{ab} + \omega_{ba} = 0$) and the

$$\Omega_{ab}(\underline{\omega}) = d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b$$

are the associated curvature forms.]

<u>Reality Check</u> Let $\underline{\omega} = [\omega^a{}_b]$ be the connection 1-forms per the metric connection ∇^q associated with q and, as in the earlier theory, put

$$N\omega^a{}_0 = v^a - W^a{}_b\omega^b - L_{\vec{N}}\omega^a.$$

Since ∇^q is torsion free, with these specializations,

$$L(\vec{\omega},\vec{v};N,\vec{N},W;\underline{\omega},\underline{\omega}_0,\vec{B})$$

$$= N({}^{(n-1)}\Omega_{ab} + \omega_{0a}\wedge\omega_{0b})\wedge*(\omega^a\wedge\omega^b)$$

$$+ 2(W^c{}_a\omega_c + v^a - L_{\vec{N}}\omega^a)\wedge\omega_{0b}\wedge*(\omega^a\wedge\omega^b)$$

$$= N({}^{(n-1)}\Omega_{ab} + \omega_{0a}\wedge\omega_{0b})\wedge*(\omega^a\wedge\omega^b)$$

$$+ 2(W^c{}_a\omega_c + N\omega^a{}_0 + W^a{}_b\omega^b + L_{\vec{N}}\omega^a - L_{\vec{N}}\omega^a)\wedge\omega_{0b}\wedge*(\omega^a\wedge\omega^b)$$

$$= N({}^{(n-1)}\Omega_{ab} + \omega_{0a}\wedge\omega_{0b})\wedge*(\omega^a\wedge\omega^b)$$

$$+ 2(W^b{}_a\omega_b + N\omega_{a0} + W^a{}_b\omega^b)\wedge\omega_{0b}\wedge *(\omega^a\wedge\omega^b)$$

$$= N({}^{(n-1)}\Omega_{ab} + \omega_{0a}\wedge\omega_{0b})\wedge *(\omega^a\wedge\omega^b)$$

$$+ 2(- W^a{}_b\omega^b - N\omega_{0a} + W^a{}_b\omega^b)\wedge\omega_{0b}\wedge *(\omega^a\wedge\omega^b)$$

$$= N({}^{(n-1)}\Omega_{ab} - \omega_{0a}\wedge\omega_{0b})\wedge *(\omega^a\wedge\omega^b)$$

$$= N*(\omega^a\wedge\omega^b)\wedge({}^{(n-1)}\Omega_{ab} - \omega_{0a}\wedge\omega_{0b})$$

$$= L(\vec{\omega},\vec{v};N,\vec{N},W).$$

Let

$$L(\vec{\omega},\vec{v};N,\vec{N},W;\underline{\omega},\underline{\omega}_0,\vec{B})$$

$$= \frac{1}{2}\int_\Sigma L(\vec{\omega},\vec{v};N,\vec{N},W;\underline{\omega},\underline{\omega}_0,\vec{B}).$$

<u>LEMMA</u> We have

$$\frac{\delta L}{\delta v^a} = p_a = \omega_{0b}\wedge *(\omega^a\wedge\omega^b).$$

Let

$$FL: T\underline{Q} \to T^*\underline{Q}$$

be the fiber derivative of L:

$$FL(\vec{\omega},\vec{v}) = (\vec{\omega}, \frac{\delta L}{\delta\vec{v}}).$$

Then

$$< \vec{v}, \frac{\delta L}{\delta \vec{v}} > - L(\vec{\omega},\vec{v};N,\vec{N},W;\underline{\omega},\underline{\omega}_0,\vec{B})$$

$$= \int_\Sigma v^a \wedge p_a - \int_\Sigma (W^c{}_a \omega_c + v^a - L_{\vec{N}} \omega^a) \wedge p_a$$

$$+ \int_\Sigma B_a (d\omega_b + \omega_{bc} \wedge \omega^c) \wedge * (\omega^a \wedge \omega^b)$$

$$- \int_\Sigma \frac{N}{2} (\Omega_{ab}(\underline{\omega}) + \omega_{0a} \wedge \omega_{0b}) \wedge * (\omega^a \wedge \omega^b)$$

$$= \int_\Sigma L_{\vec{N}} \omega^a \wedge p_a + \int_\Sigma W^a{}_b \omega^b \wedge p_a$$

$$+ \int_\Sigma B_a (d\omega_b + \omega_{bc} \wedge \omega^c) \wedge * (\omega^a \wedge \omega^b)$$

$$- \int_\Sigma \frac{N}{2} (\Omega_{ab}(\underline{\omega}) + \omega_{0a} \wedge \omega_{0b}) \wedge * (\omega^a \wedge \omega^b) .$$

But

- $\Omega_{ab}(\underline{\omega}) \wedge * (\omega^a \wedge \omega^b)$

$$= q(\omega^a \wedge \omega^b, \Omega_{ab}(\underline{\omega})) \mathrm{vol}_q$$

$$= \iota_{\omega^a \wedge \omega^b} \Omega_{ab}(\underline{\omega}) \mathrm{vol}_q$$

$$= \iota_{\omega^b} \iota_{\omega^a} \Omega_{ab}(\underline{\omega}) \mathrm{vol}_q$$

$$= S(\underline{\omega}) \mathrm{vol}_q .$$

$$\bullet\ \omega_{0a}\wedge\omega_{0b}\wedge *(\omega^a\wedge\omega^b) = \omega_{0a}\wedge p_a$$

$$= (-q(p_a, *\omega^b)q(p_b, *\omega^a) + \frac{p^2}{n-2})\text{vol}_q.$$

Therefore

$$-\int_\Sigma \frac{N}{2}(\Omega_{ab}(\underline{\omega}) + \omega_{0a}\wedge\omega_{0b})\wedge *(\omega^a\wedge\omega^b)$$

$$= \int_\Sigma \frac{N}{2}[q(p_a, *\omega^b)q(p_b, *\omega^a) - \frac{p^2}{n-2} - S(\underline{\omega})]\text{vol}_q.$$

We now shift the theory from $T\underline{Q}$ to $T^*\underline{Q}$ and let

$$H: T^*\underline{Q} \to \underline{R}$$

be the function defined by the prescription

$$H(\vec{\omega}, \vec{p}; N, \vec{N}, W; \underline{\omega}, \vec{B})$$

$$= \int_\Sigma L_{\vec{N}}\omega^a\wedge p_a + \int_\Sigma W^a{}_b\omega^b\wedge p_a$$

$$+ \int_\Sigma \frac{N}{2}[q(p_a, *\omega^b)q(p_b, *\omega^a) - \frac{p^2}{n-2} - S(\underline{\omega})]\text{vol}_q$$

$$+ \int_\Sigma B_a(d\omega_b + \omega_{bc}\wedge\omega^c)\wedge *(\omega^a\wedge\omega^b).$$

[Note: Here the external variable N is unrestricted, i.e., N can be any element of $C^\infty(\Sigma)$.]

Remark: Let $\underline{\omega} = [\omega^a{}_b]$ be the connection 1-forms per the metric connection ∇^q associated with q -- then it is clear that

$$H(\vec{\omega}, \vec{p}; N, \vec{N}, W; \underline{\omega}, \vec{B})$$

$$= H(\vec{\omega},\vec{p};N,\vec{N},W).$$

The theory has five constraints, characterized by the conditions

$$\frac{\delta H}{\delta N} = 0, \quad \frac{\delta H}{\delta \vec{N}} = 0, \quad \frac{\delta H}{\delta W} = 0, \quad \frac{\delta H}{\delta \underline{\omega}} = 0, \quad \frac{\delta H}{\delta \vec{B}} = 0.$$

Of these, the first three are familiar while the last two are new.

We have

$$\left[\begin{array}{l} \dfrac{\delta H}{\delta N} = \dfrac{1}{2}[q(p_a,*\omega^b)q(p_b,*\omega^a) - \dfrac{P^2}{n-2} - S(\underline{\omega})]\mathrm{vol}_q \\[2ex] \dfrac{\delta H}{\delta N^a} = -\, dp_a + \iota_{E_a} d\omega^b \wedge p_b \\[2ex] \dfrac{\delta H}{\delta W^a{}_b} = \omega^b \wedge p_a - \omega^a \wedge p_b. \end{array}\right.$$

Let $\Theta^a(\underline{\omega})$ be the torsion forms associated with $\nabla(\underline{\omega})$ -- then

$$\frac{\delta H}{\delta B_a} = \Theta_b(\underline{\omega}) \wedge *(\omega^a \wedge \omega^b).$$

• Define

$$I_a : T^*\underline{Q} \to \Lambda^{n-1}\Sigma$$

by

$$I_a(\vec{\omega},\vec{p}) = -\, dp_a + \iota_{E_a} d\omega^b \wedge p_b.$$

• Define

$$Av^b{}_a : T^*\underline{Q} \to \Lambda^{n-1}\Sigma$$

by

$$Av^b{}_a(\vec{\omega},\vec{p}) = \frac{1}{2}(\omega^b \wedge p_a - \omega^a \wedge p_b).$$

• Define

$$E: T^*\underline{Q} \to \Lambda^{n-1}\Sigma$$

by

$$E(\vec{\omega},\vec{p};\underline{\omega}) = \frac{1}{2}[q(p_a,*\omega^b)q(p_b,*\omega^a) - \frac{p^2}{n-2} - S(\underline{\omega})]\mathrm{vol}_q.$$

• Define

$$T^a: T^*\underline{Q} \to \Lambda^{n-1}\Sigma$$

by

$$T^a(\vec{\omega},\vec{p};\underline{\omega}) = (d\omega_b + \omega_{bc}\wedge\omega^c)\wedge*(\omega^a\wedge\omega^b).$$

Then

$$H(\vec{\omega},\vec{p};N,\vec{N},W;\underline{\omega},\vec{B})$$

$$= \int_\Sigma N^a I_a(\vec{\omega},\vec{p}) + \int_\Sigma W^a{}_b Av^b{}_a(\vec{\omega},\vec{p})$$

$$+ \int_\Sigma NE(\vec{\omega},\vec{p};\underline{\omega}) + \int_\Sigma B_a T^a(\vec{\omega},\vec{p};\underline{\omega}).$$

[Note: Accordingly, in contrast to the earlier theory, one of the constraints is not part of H.]

<u>LEMMA</u> We have

$$\frac{\delta H}{\delta\omega_{ab}} = -(dN + B_c\omega^c)\wedge*(\omega^a\wedge\omega^b)$$

$$- N\Theta^c(\underline{\omega})\wedge*(\omega^a\wedge\omega^b\wedge\omega_c).$$

[There are two contributions to the variation w.r.t. ω_{ab}. The first is

$$B_c(\omega^b\wedge*(\omega^c\wedge\omega^a) - \omega^a\wedge*(\omega^c\wedge\omega^b))$$

or still,

$$B_c(\omega^b\wedge\iota_{\omega^a}*\omega^c - \omega^a\wedge\iota_{\omega^b}*\omega^c)$$

or still,

$$B_c(\iota_{\omega^a}\omega^b\wedge *\omega^c - \iota_{\omega^a}(\omega^b\wedge *\omega^c)$$

$$+ \iota_{\omega^b}(\omega^a\wedge *\omega^c) - \iota_{\omega^b}\omega^a\wedge *\omega^c)$$

or still,

$$B_c(- \iota_{\omega^a}(\omega^b\wedge *\omega^c) + \iota_{\omega^b}(\omega^a\wedge *\omega^c))$$

or still,

$$B_c(- \iota_{\omega^a}q(\omega^b,\omega^c)\mathrm{vol}_q + \iota_{\omega^b}q(\omega^a,\omega^c)\mathrm{vol}_q)$$

or still,

$$B_c(q(\omega^a,\omega^c)*\omega^b - q(\omega^b,\omega^c)*\omega^a).$$

But

$$\begin{aligned}(-1)^{2+1}\omega^c\wedge *(\omega^a\wedge\omega^b) &= *\iota_{\omega^c}(\omega^a\wedge\omega^b)\\ &= *(\iota_{\omega^c}\omega^a\wedge\omega^b - \omega^a\wedge\iota_{\omega^c}\omega^b)\\ &= *(q(\omega^a,\omega^c)\omega^b - q(\omega^b,\omega^c)\omega^a)\\ &= q(\omega^a,\omega^c)*\omega^b - q(\omega^b,\omega^c)*\omega^a.\end{aligned}$$

The term

$$- B_c\omega^c\wedge *(\omega^a\wedge\omega^b)$$

is thus accounted for. What's left comes from consideration of

$$- \frac{N}{2}\,S(\underline{\omega})\mathrm{vol}_q = - \frac{N}{2}\,\Omega_{cd}(\underline{\omega})\wedge *(\omega^c\wedge\omega^d).$$

However, on the basis of what was said during our discussion of the Palatini formalism,

$$\delta_{ab}(\Omega_{cd}(\underline{\omega})\wedge*(\omega^c\wedge\omega^d))$$

$$= d(\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b)) + \delta\omega_{ab}\wedge\Theta^c(\underline{\omega})\wedge*(\omega^a\wedge\omega^b\wedge\omega_c)$$

$$- d(\delta\omega_{ab}\wedge*(\omega^b\wedge\omega^a)) - \delta\omega_{ab}\wedge\Theta^c(\underline{\omega})\wedge*(\omega^b\wedge\omega^a\wedge\omega_c)$$

$$= 2d(\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b)) + 2\delta\omega_{ab}\wedge\Theta^c(\underline{\omega})\wedge*(\omega^a\wedge\omega^b\wedge\omega_c).$$

This explains the occurrence of

$$-N\Theta^c(\underline{\omega})\wedge*(\omega^a\wedge\omega^b\wedge\omega_c).$$

Finally

$$d(N\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b))$$

$$= dN\wedge\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b) + Nd(\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b))$$

$\Rightarrow$

$$Nd(\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b))$$

$$= d(N\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b)) - dN\wedge\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b)$$

$$= d(N\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b)) + \delta\omega_{ab}\wedge dN\wedge*(\omega^a\wedge\omega^b).$$

Since

$$\int_\Sigma d(N\delta\omega_{ab}\wedge*(\omega^a\wedge\omega^b)) = 0,$$

incorporation of the minus sign leads to

$$-dN\wedge*(\omega^a\wedge\omega^b).]$$

If $\nabla(\underline{\omega})$ is torsion free, then $\frac{\delta H}{\delta B_a} = 0$ and if further $dN + B_c\omega^c = 0$, then, in view of the lemma, $\frac{\delta H}{\delta \omega_{ab}} = 0$.

There is also a partial converse. Thus assume that $\forall$ a,

$$\Theta^c(\underline{\omega})\wedge *(\omega^a\wedge\omega_c) = 0$$

and $\forall$ a & $\forall$ b,

$$-(dN + B_c\omega^c)\wedge *(\omega^a\wedge\omega^b) - N\Theta^c(\underline{\omega})\wedge *(\omega^a\wedge\omega^b\wedge\omega_c) = 0.$$

- $*(\omega^a\wedge\omega^b)\wedge\omega^b$

$$= (-1)^{(n-1)-1} * \iota_{\omega^b}(\omega^a\wedge\omega^b)$$

$$= (-1)^n * (\iota_{\omega^b}\omega^a\wedge\omega^b - \omega^a\wedge\iota_{\omega^b}\omega^b)$$

$$= (-1)^n * (q(\omega^a,\omega^b)\omega^b)$$

$$= (-1)^n * \omega^a.$$

- $*(\omega^a\wedge\omega^b\wedge\omega^c)\wedge\omega^b$

$$= (-1)^{(n-1)-1} * \iota_{\omega^b}(\omega^a\wedge\omega^b\wedge\omega^c)$$

$$= (-1)^n * (\iota_{\omega^b}\omega^a\wedge\omega^b\wedge\omega^c - \omega^a\wedge\iota_{\omega^b}\omega^b\wedge\omega^c + \omega^a\wedge\omega^b\wedge\iota_{\omega^b}\omega^c)$$

$$= (-1)^n * (\omega^a\wedge\omega^c + \omega^a\wedge\omega^c)$$

$$= 2(-1)^n * (\omega^a\wedge\omega^c).$$

Then

$$0 = [\,-\,(dN + B_c\omega^c)\wedge *(\omega^a\wedge\omega^b)\;-\;N\Theta^c(\underline{\omega})\wedge *(\omega^a\wedge\omega^b\wedge\omega_c)\,]\wedge\omega^b$$

$$= (-1)^{n+1}(dN + B_c\omega^c)\wedge *\omega^a + 2(-1)^{n+1}N\Theta^c(\underline{\omega})\wedge *(\omega^a\wedge\omega_c)$$

=>

$$(dN + B_c\omega^c)\wedge *\omega^a = 0 \;\forall\; a$$

=>

$$q(dN + B_c\omega^c,\omega^a) = 0 \;\forall\; a$$

=>

$$dN + B_c\omega^c = 0.$$

So, under the supposition that

$$N\in C^\infty_{>0}(\Sigma)\cup C^\infty_{<0}(\Sigma),$$

it follows that

$$\Theta^c(\underline{\omega})\wedge *(\omega^a\wedge\omega^b\wedge\omega_c) = 0.$$

But this means that

$$\nabla(\underline{\omega})\in \mathrm{con}_q T\Sigma$$

is critical, hence $\nabla(\underline{\omega})$ is torsion free.

[Note: Bear in mind that $\dim \Sigma > 2$.]

Definition: The relations

$$\begin{bmatrix} d\omega_b + \omega_{bc}\wedge\omega^c = 0 \\ dN + B_c\omega^c = 0 \end{bmatrix}$$

are called <u>auxiliary constraints</u>.

[Note: They are simpler to use and nothing of substance is lost in so doing.]

The central theorem in the coframe picture is that Ein(g) = 0 provided the constraint equations and the evolution equations are satisfied by the pair $(\vec{\omega}_t, \vec{p}_t)$. Is there a similar detection principle at work which will imply that $\nabla = \nabla^g$? It turns out that the answer is "yes" but no time development of the induced connection is involved: The situation is basically controlled by the imposition of certain constraints.

Let ∇ be a g-connection — then, as we know $\Theta(\nabla) = 0$ if Equations 1 - 4 are satisfied $\forall$ t:

1. $\forall$ a & $\forall$ b:

$$\bar{\omega}^0{}_a(E_b) = \bar{\omega}^0{}_b(E_a).$$

2. $\forall$ a:

$$d\bar{\omega}^a + \bar{\omega}^a{}_b \wedge \bar{\omega}^b = 0.$$

3. $dN_t + \bar{P}_c \bar{\omega}^c = 0.$

4. $\forall$ a:

$$\dot{\bar{\omega}}^a = N_t \bar{\omega}^a{}_0 + \bar{Q}^a{}_b \bar{\omega}^b + L_{\vec{N}_t} \bar{\omega}^a.$$

Consider the one parameter family

$$t \to (\vec{\omega}_t, \vec{p}_t; N_t, \vec{N}_t, [\bar{Q}^a{}_b]; [\bar{\omega}^a{}_b], [\bar{P}_c])$$

associated with the pair (g,∇).

Assume: $\forall$ t, the pair $(\vec{\omega}_t,\vec{p}_t)$ lies in the ADM sector of T*$\underline{Q}$, i.e.,

$$\bar{\omega}^a \wedge p_b = \bar{\omega}^b \wedge p_a$$

for all a,b. The claim is that Equation 1 is satisfied. This is obvious if a = b, so suppose that a ≠ b — then

$$\begin{cases} \bar{\omega}^a \wedge p_b = - q_t(\bar{\omega}_{0a},\bar{\omega}^b)\mathrm{vol}_{q_t} \\ \bar{\omega}^b \wedge p_a = - q_t(\bar{\omega}_{0b},\bar{\omega}^a)\mathrm{vol}_{q_t} \end{cases}$$

=>

$$q_t(\bar{\omega}_{0a},\bar{\omega}^b) = q_t(\bar{\omega}_{0b},\bar{\omega}^a)$$

=>

$$\bar{\omega}^0{}_a(E_b) = \bar{\omega}^0{}_b(E_a).$$

Stipulate next that the auxiliary constraints are in force $\forall$ t:

$$\begin{cases} d\bar{\omega}_b + \bar{\omega}_{bc}\wedge\bar{\omega}^c = 0 \\ dN_t + \bar{P}_c\bar{\omega}^c = 0, \end{cases}$$

thus taking care of Equations 2 - 3. As for Equation 4, we shall simply assume that it holds at each t (but see the next section on evolution).

Conclusion: Under the stated conditions, $\Theta(\nabla) = 0 \Rightarrow \nabla = \nabla^g$.

<u>Section 51:</u> <u>Evolution in the Palatini Picture</u> The assumptions and notation are those of the standard setup.

Rappel:

$$H(\vec{\omega},\vec{p};N,\vec{N},W;\underline{\omega},\vec{B})$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge p_a + \int_\Sigma W^a{}_b\omega^b \wedge p_a$$

$$+ \int_\Sigma NE + \int_\Sigma B_a(d\omega_b + \omega_{bc}\wedge\omega^c)\wedge *(\omega^a\wedge\omega^b),$$

where

$$E(\vec{\omega},\vec{p};\underline{\omega}) = \frac{1}{2}\,[q(p_a,*\omega^b)q(p_b,*\omega^a) - \frac{P^2}{n-2} - S(\underline{\omega})]\mathrm{vol}_q.$$

There are then two points:

1. Compute $\frac{\delta H}{\delta p_a}$;

2. Compute $\frac{\delta H}{\delta \omega^a}$.

The discussion of $\frac{\delta H}{\delta p_a}$ is verbatim the same as in the coframe picture, the result being that

$$\frac{\delta H}{\delta p_a} = L_{\vec{N}}\omega^a + W^a{}_b\omega^b + N(q(p_b,*\omega^a)\omega^b - \frac{P}{n-2}\,\omega^a).$$

Turning to $\frac{\delta H}{\delta \omega^a}$, we have

$$\frac{\delta H}{\delta \omega^a} = - L_{\vec{N}}p_a + W^b{}_a p_b$$

$$+ \frac{\delta}{\delta \omega^a}\,[\int_\Sigma NE]$$

$$+ \frac{\delta}{\delta\omega^a} [\int_\Sigma B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge *(\omega^b\wedge\omega^c)].$$

Repeating the earlier analysis word-for-word then leads to

$$\frac{\delta}{\delta\omega^a} [\int_\Sigma NE]$$

$$= \frac{\delta}{\delta\omega^a} [\int_\Sigma \frac{N}{2} q(p_b, *\omega^c) q(p_c, *\omega^b) \mathrm{vol}_q]$$

$$+ \frac{\delta}{\delta\omega^a} [\int_\Sigma N(- \frac{p^2}{2(n-2)}) \mathrm{vol}_q]$$

$$+ \frac{\delta}{\delta\omega^a} [\int_\Sigma N(- \frac{1}{2} S(\underline{\omega})) \mathrm{vol}_q]$$

$$= N(q(p_a, *\omega^b)p_b - \frac{p}{n-2} p_a)$$

$$- \frac{N}{2} (q(p_b, *\omega^c) q(p_c, *\omega^b) - \frac{p^2}{n-2}) *\omega^a$$

$$+ \frac{\delta}{\delta\omega^a} [\int_\Sigma N(- \frac{1}{2} S(\underline{\omega})) \mathrm{vol}_q].$$

But

$$S(\underline{\omega}) \mathrm{vol}_q = \Omega_{bc}(\underline{\omega}) \wedge *(\omega^b\wedge\omega^c)$$

$\Rightarrow$

$$\delta_a(- \frac{1}{2} S(\underline{\omega}) \mathrm{vol}_q)$$

$$= \delta_a(- \frac{1}{2} \Omega_{bc}(\underline{\omega}) \wedge *(\omega^b\wedge\omega^c))$$

$$= - \frac{1}{2} \Omega_{bc}(\underline{\omega}) \wedge \delta_a *(\omega^b\wedge\omega^c)$$

$$= \delta\omega^a \wedge *(\mathrm{Ric}_a(\underset{\sim}{\omega}) - \frac{1}{2} S(\underset{\sim}{\omega})\omega^a)$$

=>

$$\frac{\delta}{\delta\omega^a} [\int_\Sigma N(- \frac{1}{2} S(\underset{\sim}{\omega}))\mathrm{vol}_q] = N*(\mathrm{Ric}_a(\underset{\sim}{\omega}) - \frac{1}{2} S(\underset{\sim}{\omega})\omega^a).$$

[Note: There is no need to deal with $\delta_a\Omega_{bc}(\underset{\sim}{\omega})\wedge*(\omega^b\wedge\omega^c)$, $\underset{\sim}{\omega}$ being independent of $\vec{\omega}$.]

There remains the calculation of

$$\frac{\delta}{\delta\omega^a} [\int_\Sigma B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge*(\omega^b\wedge\omega^c)].$$

To this end, write

$$\delta_a[B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge*(\omega^b\wedge\omega^c)]$$

$$= \delta_a(B_b(d\omega_c + \omega_{cd}\wedge\omega^d))\wedge*(\omega^b\wedge\omega^c)$$

$$+ B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge\delta_a*(\omega^b\wedge\omega^c)$$

$$= B_b d\delta\omega_a\wedge*(\omega^b\wedge\omega^a) + B_b\omega_{ca}\wedge\delta\omega^a\wedge*(\omega^b\wedge\omega^c)$$

$$+ B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge\delta\omega^a\wedge*(\omega^a\wedge\omega^b\wedge\omega^c)$$

$$= d(\delta\omega_a\wedge B_b\wedge*(\omega^b\wedge\omega^a)) + \delta\omega_a\wedge d(B_b\wedge*(\omega^b\wedge\omega^a))$$

$$+ \delta\omega^a\wedge B_b\omega_{ac}\wedge*(\omega^b\wedge\omega^c)$$

$$+ \delta\omega^a\wedge B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge*(\omega^a\wedge\omega^b\wedge\omega^c).$$

Thus

$$\frac{\delta}{\delta\omega^a}\,[\textstyle\int_\Sigma B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge *(\omega^b\wedge\omega^c)]$$

$$= d(B_b\wedge *(\omega^b\wedge\omega^a)) + B_b\omega_{ac}\wedge *(\omega^b\wedge\omega^c)$$

$$+ B_b(d\omega_c + \omega_{cd}\wedge\omega^d)\wedge *(\omega^a\wedge\omega^b\wedge\omega^c).$$

Impose now the auxiliary constraints:

$$\begin{cases} d\omega_b + \omega_{bc}\wedge\omega^c = 0 \\ dN + B_c\omega^c = 0. \end{cases}$$

Then the term prefacing $*(\omega^a\wedge\omega^b\wedge\omega^c)$ disappears and the claim is that

$$d(B_b\wedge *(\omega^b\wedge\omega^a)) + B_b\omega_{ac}\wedge *(\omega^b\wedge\omega^c)$$

$$= - *(\nabla_a dN - (\Delta_q N)\omega^a).$$

• $- d*(dN\wedge\omega^a)$

$$= - d*(- B_b\omega^b\wedge\omega^a)$$

$$= d*(B_b\omega^b\wedge\omega^a)$$

$$= d(B_b\wedge *(\omega^b\wedge\omega^a)).$$

• $\omega_{ca}\wedge *(dN\wedge\omega^c)$

$$= \omega_{ac}\wedge *(-dN\wedge\omega^c)$$

$$= \omega_{ac}\wedge *(B_b\omega^b\wedge\omega^c)$$

$$= B_b\omega_{ac}\wedge *(\omega^b\wedge\omega^c).$$

And

$$- d*(dN\wedge\omega^a) + \omega_{ca}\wedge *(dN\wedge\omega^c)$$

$$= - d*(dN\wedge\omega_a) + \omega^c{}_a\wedge *(dN\wedge\omega_c)$$

$$= - d^{\nabla^g} *(dN\wedge\omega_a)$$

$$= - *(\nabla_a dN - (\Delta_g N)\omega^a).$$

Consider the one parameter family

$$t \rightarrow (\vec{\omega}_t,\vec{p}_t;N_t,\vec{N}_t,[\bar{Q}^a{}_b];\ [\bar{\omega}^a{}_b],\ [\bar{P}_c])$$

associated with the pair (g,∇) $(\nabla\in \mathrm{con}_g TM)$.

Assume: The evolution equations

$$\begin{cases} \dot{\bar{\omega}}^a = \dfrac{\delta H}{\delta p_a} \\[2ex] \dot{p}_a = - \dfrac{\delta H}{\delta \bar{\omega}^a} \end{cases}$$

are satisfied by the pair $(\vec{\omega}_t,\vec{p}_t)$.

If further the data is subject to the auxiliary constraints

$$\begin{cases} d\bar{\omega}_b + \bar{\omega}_{bc} \wedge \bar{\omega}^c = 0 \\ dN_t + \bar{P}_c \bar{\omega}^c = 0, \end{cases}$$

then the evolution equations reduce to those of the coframe picture. This said, suppose finally that $\forall$ t, the pair $(\vec{\omega}_t, \vec{p}_t)$ lies in the ADM sector of $T^*\underset{\sim}{Q}$. Equations 1 - 4 are therefore satisfied, hence $\nabla = \nabla^g$. Consequently, if the constraint equations of the coframe picture also hold, then Ein(g) = 0.

Section 52: Expansion of the Phase Space The assumptions and notation are those of the standard setup.

Rappel:

$$H(\vec{\omega},\vec{p};N,\vec{N},W;\underline{\omega},\vec{B})$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge p_a + \int_\Sigma W^a{}_b\omega^b\wedge p_a$$

$$+ \int_\Sigma NE + \int_\Sigma B_a(d\omega_b + \omega_{bc}\wedge\omega^c)\wedge *(\omega^a\wedge\omega^b),$$

where

$$E(\vec{\omega},\vec{p};\underline{\omega}) = \frac{1}{2}[q(p_a,*\omega^b)q(p_b,*\omega^a) - \frac{p^2}{n-2} - S(\underline{\omega})]\mathrm{vol}_q.$$

Definition:

$$H_D(\vec{N}) = \int_\Sigma L_{\vec{N}}\omega^a\wedge p_a$$

is the integrated diffeomorphism constraint;

$$H_R(W) = \int_\Sigma W^a{}_b\omega^b\wedge p_a$$

is the integrated rotational constraint;

$$H_H(N) = \int_\Sigma NE$$

is the integrated hamiltonian constraint.

Therefore

$$H = H_D + H_R + H_H$$

$$+ \int_\Sigma B_a\Theta_b(\underline{\omega})\wedge *(\omega^a\wedge\omega^b).$$

In the coframe picture, six relations were obtained for the Poisson brackets of the H_D, H_R, and H_H. Some of these computations carry over to the present setting and we have

$$\begin{cases} \{H_D(\vec{N}_1), H_D(\vec{N}_2)\} = H_D([\vec{N}_1, \vec{N}_2]) \\ \{H_D(\vec{N}), H_R(W)\} = H_R(L_{\vec{N}} W) \\ \{H_R(W_1), H_R(W_2)\} = H_R([W_1, W_2]). \end{cases}$$

But there are differences: This time $\frac{\delta H_H(N)}{\delta \omega^a}$ is linear in N (as is, of course, $\frac{\delta H_H(N)}{\delta p_a}$), hence

$$\{H_H(N_1), H_H(N_2)\} = 0.$$

There are also problems with

$$\{H_D(\vec{N}), H_H(N)\}$$

and

$$\{H_R(W), H_H(N)\}.$$

E.g.:

$$\{H_R(W), H_H(N)\}$$

$$= \int_\Sigma - W^a{}_b \omega^b \wedge N * (\mathrm{Ric}_a(\underline{\omega}) - \frac{1}{2} S(\underline{\omega}) \omega^a)$$

$$= - \int_\Sigma N W^a{}_b \omega^b \wedge * \mathrm{Ric}_a(\underline{\omega})$$

$$= - \int_\Sigma N W^a{}_b \mathrm{Ric}_{ab}(\underline{\omega}) \mathrm{vol}_q.$$

But, in general, $\mathrm{Ric}_{ab}(\underline{\omega}) \neq \mathrm{Ric}_{ba}(\underline{\omega})$, so there is no guarantee that the integral vanishes.

To resolve these issues (and others), it will be convenient to enlarge our horizons and promote two of the external variables to configuration status.

- $\underline{\omega} = [\omega^a{}_b] \in \Lambda^1(\Sigma;\underline{so}(n-1))$

is an (n-1)-by-(n-1) matrix of 1-forms with $\omega_{ab} + \omega_{ba} = 0$. Generically, $p_{\underline{\omega}} = [p_{\omega^a{}_b}] \in \Lambda^{n-2}(\Sigma;\underline{so}(n-1))$ is an (n-1)-by-(n-1) matrix of (n-2)-forms with $p_{\omega_{ab}} + p_{\omega_{ba}} = 0$. The prescription

$$\Omega((\underline{\omega},p_{\underline{\omega}}),(\underline{\omega}',p_{\underline{\omega}'}))$$

$$= \int_\Sigma (\omega_{ab} \wedge p_{\omega'_{ab}} - \omega'_{ab} \wedge p_{\omega_{ab}})$$

defines a symplectic structure on

$$\Lambda^1(\Sigma;\underline{so}(n-1)) \times \Lambda^{n-2}(\Sigma;\underline{so}(n-1)).$$

- $\vec{B} = [B_a] \in C^\infty(\Sigma;\underline{R}^{n-1})$

is a 1-by-(n-1) matrix of C^∞ functions on Σ. Generically, $p_{\vec{B}} = [p_{B_a}] \in \Lambda^{n-1}(\Sigma;\underline{R}^{n-1})$ is a 1-by-(n-1) matrix of (n-1)-forms on Σ. The prescription

$$\Omega((\vec{B},p_{\vec{B}}),\ (\vec{B}',p_{\vec{B}'}))$$

$$= \int_\Sigma (B_a \wedge p_{B'_a} - B'_a \wedge p_{B_a})$$

defines a symplectic structure on

$$C^{\infty}(\Sigma;\underline{R}^{n-1}) \times \Lambda^{n-1}(\Sigma;\underline{R}^{n-1}).$$

Definition: The <u>expanded configuration space</u> is

$$\mathcal{C} = \underline{Q} \times \Lambda^{1}(\Sigma;\underline{so}(n-1)) \times C^{\infty}(\Sigma;\underline{R}^{n-1}).$$

We shall then operate in

$$\begin{aligned} T^*\mathcal{C} = T^*\underline{Q} \\ &\times \Lambda^{1}(\Sigma;\underline{so}(n-1)) \times \Lambda^{n-2}(\Sigma;\underline{so}(n-1)) \\ &\times C^{\infty}(\Sigma;\underline{R}^{n-1}) \times \Lambda^{n-1}(\Sigma;\underline{R}^{n-1}) \end{aligned}$$

equipped with the obvious symplectic structure.

[Note: A typical point in $T^*\mathcal{C}$ is the pair of triples

$$(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}}).]$$

<u>N.B.</u> Functions on $T^*\underline{Q}$ lift to functions on $T^*\mathcal{C}$.

In particular: $H_D(\vec{N})$ and $H_R(W)$ are functions on $T^*\mathcal{C}$ which are independent of $(\underline{\omega},p_{\underline{\omega}};\vec{B},p_{\vec{B}})$. By contrast, $H_H(N)$ is a function on $T^*\mathcal{C}$ which definitely depends on $\underline{\omega}$ (but not on $p_{\underline{\omega}}$).

• Given $\vec{\alpha}\in\Lambda^{n-3}(\Sigma;\underline{R}^{n-1})$, define

$$H_T(\vec{\alpha}):T^*\mathcal{C} \to \underline{R}$$

by

$$H_T(\vec{\alpha})(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}})$$

$$= \int_\Sigma \alpha_a \wedge (d\omega^a + \omega^a{}_b \wedge \omega^b).$$

• Given $f \in C^\infty(\Sigma)$ and $\beta \in \Lambda^{n-2}\Sigma$, define

$$H_f(\beta) : T^*C \to \underline{R}$$

by

$$H_f(\beta)(\vec{\omega}, \underline{\omega}, \vec{B}; \vec{p}, p_{\underline{\omega}}, p_{\vec{B}})$$

$$= \int_\Sigma (df + B_a \omega^a) \wedge \beta.$$

• Given $\underline{\rho} \in \Lambda^1(\Sigma; \underline{so}(n-1))$,

define

$$H_1(\underline{\rho}) : T^*C \to \underline{R}$$

by

$$H_1(\rho)(\vec{\omega}, \underline{\omega}, \vec{B}; \vec{p}, p_{\underline{\omega}}, p_{\vec{B}})$$

$$= \frac{1}{2} \int_\Sigma \rho_{ab} \wedge p_{\omega_{ab}}.$$

• Given $\vec{R} \in C^\infty(\Sigma; \underline{R}^{n-1})$, define

$$H_2(\vec{R}) : T^*C \to \underline{R}$$

by

$$H_2(\vec{R})(\vec{\omega}, \underline{\omega}, \vec{B}; \vec{p}, p_{\underline{\omega}}, p_{\vec{B}})$$

$$= \int_\Sigma R_a p_{B_a}.$$

There are four constraint surfaces associated with these functions.

Con_T: This is the subset of T^*C whose elements are the points

$$(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}})$$

such that

$$d\omega^a + \omega^a{}_b \wedge \omega^b = 0 \quad (a = 1,\ldots,n-1).$$

Con_f: This is the subset of T^*C whose elements are the points

$$(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}})$$

such that

$$df + B_a\omega^a = 0.$$

Con_1: This is the subset of T^*C whose elements are the points

$$(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}})$$

such that

$$p_{\omega_{ab}} = 0 \quad (a,b = 1,\ldots,n-1).$$

Con_2: This is the subset of T^*C whose elements are the points

$$(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}})$$

such that

$$p_{B_a} = 0 \quad (a = 1,\ldots,n-1).$$

Example:

$$\{H_R(W), H_H(N)\}|Con_T$$

$$= - \int_\Sigma NW^a{}_b \mathrm{Ric}_{ab}(\underset{\sim}{\omega}) \mathrm{vol}_q$$

$$= 0.$$

Indeed, $\nabla(\underset{\sim}{\omega}) = \nabla^q$, hence $\mathrm{Ric}_{ab}(\underset{\sim}{\omega}) = \mathrm{Ric}_{ba}(\underset{\sim}{\omega})$.

Rappel: Let $f_1, f_2 : T^*C \to \underset{\sim}{R}$ -- then their Poisson bracket $\{f_1, f_2\}$ is the function

$$\{f_1, f_2\} : T^*C \to \underset{\sim}{R}$$

defined by the rule

$$\{f_1, f_2\} = \int_\Sigma \left[\frac{\delta f_2}{\delta \vec{p}} \wedge \frac{\delta f_1}{\delta \vec{\omega}} - \frac{\delta f_1}{\delta \vec{p}} \wedge \frac{\delta f_2}{\delta \vec{\omega}} \right]$$

$$+ \int_\Sigma \left[\frac{\delta f_2}{\delta p_{\underset{\sim}{\omega}}} \wedge \frac{\delta f_1}{\delta \underset{\sim}{\omega}} - \frac{\delta f_2}{\delta p_{\underset{\sim}{\omega}}} \wedge \frac{\delta f_2}{\delta \underset{\sim}{\omega}} \right]$$

$$+ \int_\Sigma \left[\frac{\delta f_2}{\delta p_{\vec{B}}} \wedge \frac{\delta f_1}{\delta \vec{B}} - \frac{\delta f_1}{\delta p_{\vec{B}}} \wedge \frac{\delta f_2}{\delta \vec{B}} \right].$$

Example: We have

$$\left[\begin{array}{l} \{H_T(\vec{\alpha}), H_T(\vec{\alpha}')\} = 0, \ \{H_f(\beta), H_f(\beta')\} = 0 \\ \{H_1(\underset{\sim}{\rho}), H_1(\underset{\sim}{\rho}')\} = 0, \ \{H_2(\vec{R}), H_2(\vec{R}')\} = 0. \end{array}\right.$$

Example: We have

$$\left[\begin{array}{l} \{H_T(\vec{\alpha}), H_f(\beta)\} = 0, \ \{H_T(\vec{\alpha}), H_2(\vec{R})\} = 0 \\ \{H_f(\beta), H_1(\underset{\sim}{\rho})\} = 0, \ \{H_1(\underset{\sim}{\rho}), H_2(\vec{R})\} = 0. \end{array}\right.$$

$$\bullet\ \delta_{ab}(\alpha_c\wedge(d\omega_c + \omega_{cd}\wedge\omega^d))$$

$$= \alpha_c\wedge\delta_{ab}(\omega_{cd}\wedge\omega^d)$$

$$= \alpha_a\wedge\delta\omega_{ab}\wedge\omega^b - \alpha_b\wedge\delta\omega_{ab}\wedge\omega^a$$

$$= (-1)^{n-3}\delta\omega_{ab}\wedge(\alpha_a\wedge\omega^b - \alpha_b\wedge\omega^a)$$

$$= \delta\omega_{ab}\wedge(\omega^b\wedge\alpha_a - \omega^a\wedge\alpha_b)$$

=>

$$\frac{\delta H_T(\vec{\alpha})}{\delta\omega_{ab}} = \omega^b\wedge\alpha_a - \omega^a\wedge\alpha_b.$$

$$\bullet\ \delta_{ab}(\tfrac{1}{2}\ \rho_{cd}\wedge p_{\omega_{cd}})$$

$$= \frac{1}{2}\ (\rho_{ab}\wedge\delta p_{\omega_{ab}} - \rho_{ba}\wedge\delta p_{\omega_{ab}})$$

$$= \rho_{ab}\wedge\delta p_{\omega_{ab}}$$

=>

$$\frac{\delta H_1(\underline{\rho})}{\delta p_{\omega_{ab}}} = \rho_{ab}.$$

Therefore

$$\{H_T(\vec{\alpha}), H_1(\underline{\rho})\}$$

$$= \int_\Sigma \left[\frac{\delta H_1(\underline{\rho})}{\delta p_{\omega_{ab}}} \wedge \frac{\delta H_T(\vec{\alpha})}{\delta \omega_{ab}} - \frac{\delta H_T(\vec{\alpha})}{\delta p_{\omega_{ab}}} \wedge \frac{\delta H_1(\underline{\rho})}{\delta \omega_{ab}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_1(\underline{\rho})}{\delta p_{\omega_{ab}}} \wedge \frac{\delta H_T(\vec{\alpha})}{\delta \omega_{ab}} \right]$$

$$= \int_\Sigma [\rho_{ab} \wedge (\omega^b \wedge \alpha_a - \omega^a \wedge \alpha_b)]$$

$$= \int_\Sigma \rho_{ab} \wedge \omega^b \wedge \alpha_a - \int_\Sigma \rho_{ab} \wedge \omega^a \wedge \alpha_b$$

$$= \int_\Sigma \rho_{ab} \wedge \omega^b \wedge \alpha_a + \int_\Sigma \rho_{ba} \wedge \omega^a \wedge \alpha_b$$

$$= 2 \int_\Sigma \rho_{ab} \wedge \omega^b \wedge \alpha_a .$$

- $\delta_a((df + B_b \omega^b) \wedge \beta)$

$$= \delta_a(B_b \omega^b) \wedge \beta$$

$$= \delta B_a \wedge \omega^a \wedge \beta$$

=>

$$\frac{\delta H_f(\beta)}{\delta B_a} = \omega^a \wedge \beta .$$

- $\delta_a(R_b p_{B_b})$

$$= R_a \delta p_{B_a}$$

=>

$$\frac{\delta H_2(\vec{R})}{\delta p_{B_a}} = R_a.$$

Therefore

$$\{H_f(\beta), H_2(\vec{R})\}$$

$$= \int_\Sigma \left[\frac{\delta H_2(\vec{R})}{\delta p_{B_a}} \wedge \frac{\delta H_f(\beta)}{\delta B_a} - \frac{\delta H_f(\beta)}{\delta p_{B_a}} \wedge \frac{\delta H_2(\vec{R})}{\delta B_a} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_2(\vec{R})}{\delta p_{B_a}} \wedge \frac{\delta H_f(\beta)}{\delta B_a} \right]$$

$$= \int_\Sigma R_a \omega^a \wedge \beta.$$

• $\delta_a(\alpha_c \wedge (d\omega^c + \omega^c{}_d \wedge \omega^d))$

$$= \delta_a(\alpha_c \wedge d\omega^c) + \delta_a(\alpha_c \wedge \omega^c{}_d \wedge \omega^d)$$

$$= \alpha_a \wedge d\delta\omega^a + \alpha_c \wedge \omega^c{}_a \wedge \delta\omega^a$$

$$= d\delta\omega^a \wedge \alpha_a + (-1)^{n-2} \delta\omega^a \wedge \alpha_c \wedge \omega^c{}_a$$

$$= d\delta\omega^a \wedge \alpha_a + (-1)^n (-1)^{n-3} \delta\omega^a \wedge \omega^c{}_a \wedge \alpha_c$$

$$= d\delta\omega^a \wedge \alpha_a - \delta\omega^a \wedge \omega^c{}_a \wedge \alpha_c$$

$$= d(\delta\omega^a \wedge \alpha_a) + \delta\omega^a \wedge d\alpha_a - \delta\omega^a \wedge \omega^c{}_a \wedge \alpha_c$$

=>

$$\frac{\delta H_T(\vec{\alpha})}{\delta\omega^a} = d\alpha_a - \omega^c{}_a \wedge \alpha_c.$$

Application:

$$\{H_D(\vec{N}), H_T(\vec{\alpha})\} = -\int_\Sigma L_{\vec{N}}\omega^a \wedge (d\alpha_a - \omega^c{}_a \wedge \alpha_c)$$

$$\{H_R(W), H_T(\vec{\alpha})\} = -\int_\Sigma W^a{}_b \omega^b \wedge (d\alpha_a - \omega^c{}_a \wedge \alpha_c)$$

$$\{H_H(N), H_T(\vec{\alpha})\} = -\int_\Sigma N(q(p_b, *\omega^a)\omega^b - \frac{P}{n-2}\omega^a) \wedge (d\alpha_a - \omega^c{}_a \wedge \alpha_c).$$

• $\delta_a((df + B_b\omega^b) \wedge \beta)$

$$= \delta_a(B_b\omega^b \wedge \beta)$$

$$= B_a\delta\omega^a \wedge \beta$$

$$= \delta\omega^a \wedge B_a\beta$$

=>

$$\frac{\delta H_f(\beta)}{\delta\omega^a} = B_a\beta.$$

Application:

$$\{H_D(\vec{N}), H_f(\beta)\} = -\int_\Sigma L_{\vec{N}}\omega^a \wedge B_a\beta$$

$$\{H_R(W), H_f(\beta)\} = -\int_\Sigma W^a{}_b \omega^b \wedge B_a\beta$$

$$\{H_H(N), H_f(\beta)\} = -\int_\Sigma N(q(p_b, *\omega^a)\omega^b - \frac{P}{n-2}\omega^a) \wedge B_a\beta.$$

Inspecting the definitions, we see at once that

$$\{H_D(\vec{N}),H_1(\underline{\rho})\} = 0, \ \{H_R(W),H_1(\underline{\rho})\} = 0$$

and

$$\{H_D(\vec{N}),H_2(\vec{R})\} = 0, \ \{H_R(W),H_2(\vec{R})\} = 0, \ \{H_H(N),H_2(\vec{R})\} = 0.$$

As regards $\{H_H(N),H_1(\underline{\rho})\}$, the situation is not so simple.

<u>LEMMA</u> We have

$$\frac{\delta H_H(N)}{\delta\omega_{ab}} = - d(N*(\omega^a\wedge\omega^b)) - N\omega_{ac}\wedge*(\omega^c\wedge\omega^b) - N\omega_{bc}\wedge*(\omega^a\wedge\omega^c).$$

[It is a question of explicating

$$-\frac{N}{2}\,\delta_{ab}(S(\underline{\omega})\mathrm{vol}_q)$$

or still,

$$-\frac{N}{2}\,\delta_{ab}(\Omega_{cd}(\underline{\omega})\wedge*(\omega^c\wedge\omega^d))$$

or still,

$$-\frac{N}{2}\,\delta_{ab}(d\omega_{cd} + \omega_{cr}\wedge\omega_{rd})\wedge*(\omega^c\wedge\omega^d)$$

or still,

$$-\frac{N}{2}\,(\delta_{ab}d\omega_{cd} + \delta_{ab}\omega_{cr}\wedge\omega_{rd} + \omega_{cr}\wedge\delta_{ab}\omega_{rd})\wedge*(\omega^c\wedge\omega^d).$$

But

$$-\frac{N}{2}\,\delta_{ab}d\omega_{cd}\wedge*(\omega^c\wedge\omega^d)$$

$$= -\frac{N}{2}\, d\delta\omega_{ab}\wedge *(\omega^a\wedge\omega^b) + \frac{N}{2}\, d\delta\omega_{ab}\wedge *(\omega^b\wedge\omega^a)$$

$$= -Nd\delta\omega_{ab}\wedge *(\omega^a\wedge\omega^b)$$

$$= -d\delta\omega_{ab}\wedge N*(\omega^a\wedge\omega^b)$$

$$= -d(\delta\omega_{ab}\wedge N*(\omega^a\wedge\omega^b)) - \delta\omega_{ab}\wedge d(N*(\omega^a\wedge\omega^b)).$$

And

$$\bullet -\frac{N}{2}\,\delta_{ab}\omega_{cr}\wedge\omega_{rd}\wedge *(\omega^c\wedge\omega^d)$$

$$= -\frac{N}{2}\,\delta\omega_{ab}\wedge\omega_{bd}\wedge *(\omega^a\wedge\omega^d) + \frac{N}{2}\,\delta\omega_{ab}\wedge\omega_{ad}\wedge *(\omega^b\wedge\omega^d)$$

$$= \frac{N}{2}\,\delta\omega_{ab}\wedge(\omega_{ad}\wedge *(\omega^b\wedge\omega^d) - \omega_{bd}\wedge *(\omega^a\wedge\omega^d)).$$

$$\bullet -\frac{N}{2}\,\omega_{cr}\wedge\delta_{ab}\omega_{rd}\wedge *(\omega^c\wedge\omega^d)$$

$$= -\frac{N}{2}\,\omega_{ca}\wedge\delta\omega_{ab}\wedge *(\omega^c\wedge\omega^b) + \frac{N}{2}\,\omega_{cb}\wedge\delta\omega_{ab}\wedge *(\omega^c\wedge\omega^a)$$

$$= \frac{N}{2}\,\delta\omega_{ab}\wedge(\omega_{ca}\wedge *(\omega^c\wedge\omega^b) - \omega_{cb}\wedge *(\omega^c\wedge\omega^a)).$$

To combine these terms, write

$$\omega_{ad}\wedge *(\omega^b\wedge\omega^d)$$

$$= \omega_{ac}\wedge *(\omega^b\wedge\omega^c)$$

$$= \omega_{ca}\wedge *(\omega^c\wedge\omega^b)$$

and

$$\omega_{bd}\wedge*(\omega^a\wedge\omega^d)$$

$$= \omega_{bc}\wedge*(\omega^a\wedge\omega^c)$$

$$= \omega_{cb}\wedge*(\omega^c\wedge\omega^a).$$

Then

$$-\frac{N}{2}(\delta_{ab}\omega_{cr}\wedge\omega_{rd} + \omega_{cr}\wedge\delta_{ab}\omega_{rd})\wedge*(\omega^c\wedge\omega^d)$$

$$= \delta\omega_{ab}\wedge(N\omega_{ca}\wedge*(\omega^c\wedge\omega^b) - N\omega_{cb}\wedge*(\omega^c\wedge\omega^a))$$

$$= \delta\omega_{ab}\wedge(-N\omega_{ac}\wedge*(\omega^c\wedge\omega^b) - N\omega_{bc}\wedge*(\omega^a\wedge\omega^c)).]$$

Application:

$$\{H_H(N), H_1(\underline{\rho})\}$$

$$= \int_\Sigma \left[\frac{\delta H_1(\underline{\rho})}{\delta p_{\omega_{ab}}}\wedge\frac{\delta H_H(N)}{\delta\omega_{ab}} - \frac{\delta H_H(N)}{\delta p_{\omega_{ab}}}\wedge\frac{\delta H_1(\underline{\rho})}{\delta\omega_{ab}}\right]$$

$$= \int_\Sigma \left[\frac{\delta H_1(\underline{\rho})}{\delta p_{\omega_{ab}}}\wedge\frac{\delta H_H(N)}{\delta\omega_{ab}}\right]$$

$$= \int_\Sigma \rho_{ab}\,(-d(N*(\omega^a\wedge\omega^b)) - N\omega^a{}_c\wedge*(\omega^c\wedge\omega^b) - N\omega^b{}_c\wedge*(\omega^a\wedge\omega^c)).$$

[Note: On Con_T,

$$d*(\omega^a\wedge\omega^b) = -\omega^a{}_c\wedge*(\omega^c\wedge\omega^b) - \omega^b{}_c\wedge*(\omega^a\wedge\omega^c).$$

Therefore

$$-d(N*(\omega^a\wedge\omega^b)) - N\omega^a{}_c\wedge*(\omega^c\wedge\omega^b) - N\omega^b{}_c\wedge*(\omega^a\wedge\omega^c)$$

$$= -dN\wedge*(\omega^a\wedge\omega^b) - Nd*(\omega^a\wedge\omega^b) + Nd*(\omega^a\wedge\omega^b)$$

$$= -dN\wedge*(\omega^a\wedge\omega^b)$$

=>

$$\{H_H(N), H_\perp(\underline{\rho})\}|\mathrm{Con}_T$$

$$= -\int_\Sigma \rho_{ab}\wedge dN\wedge*(\omega^a\wedge\omega^b).]$$

Given N, let

$$H_T(*(dN\wedge\omega_a))$$

stand for the function $T^*C \to \underline{R}$ that sends

$$(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}})$$

to

$$\int_\Sigma *(dN\wedge\omega_a)\wedge(d\omega^a + \omega^a{}_b\wedge\omega^b).$$

[Note: Strictly speaking, this is not consistent with the earlier agreements in that here $\alpha_a = *(dN\wedge\omega_a)$ depends on $\vec{\omega}$. However, no difficulties will arise therefrom. So, e.g.,

$$\int_\Sigma B_a(d\omega_b + \omega_{bc}\wedge\omega^c)\wedge*(\omega^a\wedge\omega^b)$$

$$= H_T(B_c*(\omega^c\wedge\omega^a)).]$$

Observation: To begin with,

$$(-1)^n[*(dN\wedge\omega_a)\wedge(d\omega^a + \omega^a{}_b\wedge\omega^b)]$$

$$= (-1)^n*(dN\wedge\omega_a)\wedge d\omega^a$$

$$+ (-1)^n*(dN\wedge\omega_a)\wedge\omega^a{}_b\wedge\omega^b$$

$$= (-1)^n*(dN\wedge\omega_a)\wedge d\omega^a$$

$$+ (-1)^n(-1)^{n-3}\omega^a{}_b\wedge*(dN\wedge\omega_a)\wedge\omega^b$$

$$= (-1)^n*(dN\wedge\omega_a)\wedge d\omega^a$$

$$- \omega^b{}_a\wedge*(dN\wedge\omega_b)\wedge\omega^a.$$

In addition,

$$- d[*(dN\wedge\omega_a)\wedge\omega^a]$$

$$= - [d*(dN\wedge\omega_a)\wedge\omega^a + (-1)^{n-3}*(dN\wedge\omega_a)\wedge d\omega^a]$$

$$= - d*(dN\wedge\omega_a)\wedge\omega^a + (-1)^n*(dN\wedge\omega_a)\wedge d\omega^a.$$

Therefore

$$(-1)^n \int_\Sigma *(dN\wedge\omega_a)\wedge(d\omega^a + \omega^a{}_b\wedge\omega^b)$$

$$= \int_\Sigma [d*(dN\wedge\omega_a) - \omega^b{}_a\wedge*(dN\wedge\omega_b)]\wedge\omega^a$$

$$- \int_\Sigma d[*(dN\wedge\omega_a)\wedge\omega^a]$$

$$= \int_\Sigma [d*(dN\wedge\omega_a) - \omega^b{}_a\wedge*(dN\wedge\omega_b)]\wedge\omega^a$$

$$= \int_\Sigma d^{\nabla(\underline{\omega})}*(dN\wedge\omega_a)\wedge\omega^a$$

$$= \int_\Sigma *(\nabla_a(\underline{\omega})dN - (\Delta_{con}(\underline{\omega})N)\omega^a)\wedge\omega^a,$$

where

$$\Delta_{con}(\underline{\omega})N = \nabla^a(\underline{\omega})\nabla_a(\underline{\omega})N$$

$$= \nabla(\underline{\omega})dN(E_a,E_a).$$

Write

$$\nabla_a(\underline{\omega})dN = \nabla(\underline{\omega})dN(E_c,E_a)\omega^c.$$

Then

$$\int_\Sigma *(\nabla_a(\underline{\omega})dN)\wedge\omega^a$$

$$= \int_\Sigma \nabla(\underline{\omega})dN(E_c,E_a)*\omega^c\wedge\omega^a$$

$$= (-1)^{n-2} \int_\Sigma \nabla(\underline{\omega})dN(E_c,E_a)\omega^a\wedge*\omega^c$$

$$= (-1)^n \int_\Sigma \nabla(\underline{\omega})dN(E_c,E_a)q(\omega^a,\omega^c)\text{vol}_q$$

$$= (-1)^n \int_\Sigma (\Delta_{con}(\underline{\omega})N)\mathrm{vol}_q.$$

On the other hand,

$$-\int_\Sigma (\Delta_{con}(\underline{\omega})N) * \omega^a \wedge \omega^a$$

$$= -(-1)^{n-2} \int_\Sigma (\Delta_{con}(\underline{\omega})N)\omega^a \wedge * \omega^a$$

$$= -(-1)^n (n-1) \int_\Sigma (\Delta_{con}(\underline{\omega})N)\mathrm{vol}_q.$$

Cancelling the $(-1)^n$, we thus conclude that

$$\mathcal{H}_T(*(dN\wedge\omega_a))$$

$$= \int_\Sigma (\Delta_{con}(\underline{\omega})N)\mathrm{vol}_q - (n-1) \int_\Sigma (\Delta_{con}(\underline{\omega})N)\mathrm{vol}_q$$

$$= (2-n) \int_\Sigma (\Delta_{con}(\underline{\omega})N)\mathrm{vol}_q,$$

which brings us to the point of the computation: In general, the integral

$$\int_\Sigma (\Delta_{con}(\underline{\omega})N)\mathrm{vol}_q$$

does not vanish, hence $\mathcal{H}_T$ is nontrivial.

[Note: If $\nabla(\underline{\omega}) = \nabla^q$, then

$$\Delta_{con}(\underline{\omega}) = \Delta_q$$

and, of course,

$$\int_\Sigma (\Delta_q N)\mathrm{vol}_q = 0.]$$

<u>LEMMA</u> We have

$$\rho_{ab} \wedge dN \wedge *(\omega^a \wedge \omega^b) + 2\rho_{ab} \wedge \omega^b \wedge *(dN \wedge \omega^a)$$

$$= 0.$$

[Write

$$dN = N_c \omega^c.$$

Then

• $\rho_{ab} \wedge dN \wedge *(\omega^a \wedge \omega^b)$

$$= \rho_{ab} \wedge N_c \omega^c \wedge *(\omega^a \wedge \omega^b)$$

$$= (-1)^{n-3} \rho_{ab} \wedge N_c *(\omega^a \wedge \omega^b) \wedge \omega^c$$

$$= (-1)^{n-3} \rho_{ab} \wedge N_c (-1)^{n-2} *(\iota_{\omega^c} (\omega^a \wedge \omega^b))$$

$$= - \rho_{ab} \wedge N_c *(\delta^c{}_a \omega^b - \omega^a \delta^c{}_b)$$

$$= - \rho_{ab} \wedge N_a * \omega^b + \rho_{ab} \wedge N_b * \omega^a$$

$$= - \rho_{ba} \wedge N_b * \omega^a + \rho_{ab} \wedge N_b * \omega^a$$

$$= \rho_{ab} \wedge N_b * \omega^a + \rho_{ab} \wedge N_b * \omega^a$$

$$= 2\rho_{ab} \wedge N_b * \omega^a.$$

$$\bullet\ 2\rho_{ab}\wedge\omega^b\wedge *(dN\wedge\omega^a)$$

$$= 2\rho_{ab}\wedge N_c\omega^b\wedge *(\omega^c\wedge\omega^a)$$

$$= 2(-1)^{n-3}\rho_{ab}\wedge N_c * (\omega^c\wedge\omega^a)\wedge\omega^b$$

$$= 2(-1)^{n-3}\rho_{ab}\wedge N_c(-1)^{n-2} * (\iota_{\omega^b}(\omega^c\wedge\omega^a))$$

$$= -\ 2\rho_{ab}\wedge N_c * (\delta^b{}_c\omega^a - \omega^c\delta^b{}_a)$$

$$= -\ 2\rho_{ab}\wedge N_b * \omega^a + 2\rho_{aa}\wedge N_c * \omega^c$$

$$= -\ 2\rho_{ab}\wedge N_b * \omega^a.]$$

Notation: Put

$$\tilde{H}_H(N) = H_H(N) - H_T(*(dN\wedge\omega_a)).$$

Accordingly,

$$\{\tilde{H}_H(N), H_1(\underset{\sim}{\rho})\}$$

$$= \{H_H(N), H_1(\underset{\sim}{\rho})\} - \{H_T(*(dN\wedge\omega_a)), H_1(\underset{\sim}{\rho})\},$$

which, upon restriction to Con_T, equals

$$-\int_\Sigma \rho_{ab}\wedge dN\wedge *(\omega^a\wedge\omega^b) - 2\int_\Sigma \rho_{ab}\wedge\omega^b\wedge *(dN\wedge\omega^a).$$

I.e.:

$$\{\tilde{H}_H(N), H_1(\rho)\}|Con_T = 0.$$

Remark: Since

$$\{H_T(*(dN\wedge\omega_a)), H_2(\vec{R})\} = 0,$$

it is still the case that

$$\{\tilde{H}_H(N), H_2(\vec{R})\} = 0.$$

<u>N.B.</u> The correction term

$$H_T(*(dN\wedge\omega_a))$$

is identically zero on Con_T.

In terms of $\tilde{H}_H$, we have:

I. $\{H_D(\vec{N}), \tilde{H}_H(N)\}|Con_T = \tilde{H}_H(L_{\vec{N}} N)|Con_T;$

II. $\{H_R(W), \tilde{H}_H(N)\}|Con_T = 0;$

III. $\{\tilde{H}_H(N_1), \tilde{H}_H(N_2)\}|Con_T$

$$= H_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)|Con_T$$

$$+ H_R(q(dN_1\wedge dN_2, \omega^a\wedge\omega_b) + q(N_1 dN_2 - N_2 dN_1, \omega^a{}_b))|Con_T.$$

<u>Ad I</u>: Proceeding as in the coframe picture, let

$$E = E_{kin} + E_{pot},$$

where

$$E_{kin}(\vec{\omega},\vec{p};\underline{\omega}) = \frac{1}{2}\,[q(p_b,*\omega^c)q(p_c,*\omega^b) - \frac{p^2}{n-2}\,]\mathrm{vol}_q$$

and

$$E_{pot}(\vec{\omega},\vec{p};\underline{\omega}) = -\frac{1}{2}\,S(\underline{\omega})\mathrm{vol}_q$$

$$= -\frac{1}{2}\,\Omega_{bc}(\underline{\omega})\wedge*(\omega^b\wedge\omega^c).$$

Note that E_{kin} does not depend on $\underline{\omega}$, while E_{pot} does not depend on $\vec{p}$. This said, in obvious notation,

$$H_H(N) = H_{H_{kin}}(N) + H_{H_{pot}}(N),$$

and, as before,

$$\{H_D(\vec{N}),H_{H_{kin}}(N)\} = H_{H_{kin}}(L_{\vec{N}}N).$$

However, $H_{H_{pot}}(N)$ has to be treated a little bit differently. Thus, in the present setting,

$$\frac{\delta H_{H_{pot}}(N)}{\delta\omega^a} = -\frac{N}{2}\,(\Omega_{bc}(\underline{\omega})\wedge*(\omega^b\wedge\omega^c\wedge\omega_a)),$$

so

$$\{H_D(\vec{N}),H_{H_{pot}}(N)\}$$

$$= \frac{1}{2}\int_\Sigma L_{\vec{N}}\omega^a\wedge N(\Omega_{bc}(\underline{\omega})\wedge*(\omega^b\wedge\omega^c\wedge\omega_a)).$$

=>

$$\{H_D(\vec{N}), H_{H_{pot}}(N)\}|Con_T$$

$$= \frac{1}{2} \int_\Sigma L_{\vec{N}}\omega^a \wedge N(\Omega_{bc} \wedge *(\omega^b \wedge \omega^c \wedge \omega_a)),$$

where Ω_{bc} is per ∇^q. This integral was encountered earlier: It computes to

$$H_{H_{pot}}(L_{\vec{N}}N)|Con_T$$

$$- \int_\Sigma L_{\vec{N}}\omega^a \wedge *(\nabla_a dN - (\Delta_q N)\omega^a).$$

But

$$- \int_\Sigma L_{\vec{N}}\omega^a \wedge *(\nabla_a dN - (\Delta_q N)\omega^a)$$

$$= - \int_\Sigma L_{\vec{N}}\omega^a \wedge d^{\nabla^q} *(dN \wedge \omega_a)$$

$$= - \int_\Sigma L_{\vec{N}}\omega^a \wedge (d*(dN \wedge \omega_a) - \omega^c{}_a \wedge *(dN \wedge \omega_c))$$

$$= \{H_D(\vec{N}), H_T(*(dN \wedge \omega_a))\}|Con_T.$$

Therefore

$$\{H_D(\vec{N}), \tilde{H}_H(N)\}|Con_T$$

$$= \{H_D(\vec{N}), H_H(N) - H_T(*(dN \wedge \omega_a))\}|Con_T$$

$$= \{H_D(\vec{N}), H_{H_{kin}}(N) + H_{H_{pot}}(N)\}|Con_T$$

$$- \{H_D(\vec{N}), H_T(*(dN\wedge\omega_a))\}|Con_T$$

$$= \{H_D(\vec{N}), H_{H_{kin}}(N)\}|Con_T$$

$$+ \{H_D(\vec{N}), H_{H_{pot}}(N)\}|Con_T$$

$$- \{H_D(\vec{N}), H_T(*(dN\wedge\omega_a))\}|Con_T$$

$$= H_{H_{kin}}(L_{\vec{N}}N)|Con_T$$

$$+ H_{H_{pot}}(L_{\vec{N}}N)|Con_T + \{H_D(\vec{N}), H_T(*(dN\wedge\omega_a))\}|Con_T$$

$$- \{H_D(\vec{N}), H_T(*(dN\wedge\omega_a))\}|Con_T$$

$$= H_{H_{kin}}(L_{\vec{N}}N)|Con_T + H_{H_{pot}}(L_{\vec{N}}N)|Con_T$$

$$= H_H(L_{\vec{N}}N)|Con_T$$

$$= H_H(L_{\vec{N}}N)|Con_T$$

$$- H_T(*(dL_{\vec{N}}N\wedge\omega_a))|Con_T \quad (\equiv 0!)$$

$$= \tilde{H}_H(L_{\vec{N}}N)|Con_T.$$

<u>Ad II</u>: In fact,

$$\{H_R(W), \tilde{H}_H(N)\}|Con_T$$

$$= \{\mathcal{H}_R(W), \mathcal{H}_H(N)\}|\mathrm{Con}_T - \{\mathcal{H}_R(W), \mathcal{H}_T(*(dN\wedge\omega_a))\}|\mathrm{Con}_T$$

$$= -\{\mathcal{H}_R(W), \mathcal{H}_T(*(dN\wedge\omega_a))\}|\mathrm{Con}_T$$

$$= \int_\Sigma W^a{}_b\omega^b\wedge(d*(dN\wedge\omega_a) - \omega^c{}_a\wedge*(dN\wedge\omega_c))|\mathrm{Con}_T$$

$$= \int_\Sigma W^a{}_b\omega^b\wedge*(\nabla_a dN - (\Delta_q N)\omega^a)$$

$$= \int_\Sigma W^a{}_b\omega^b\wedge*\nabla_a dN - \int_\Sigma (\Delta_q N)W^a{}_b\omega^b\wedge*\omega^a$$

$$= \int_\Sigma W^a{}_b\omega^b\wedge*\nabla_a dN - \int_\Sigma (\Delta_q N)W^a{}_a \mathrm{vol}_q$$

$$= \int_\Sigma W^a{}_b\omega^b\wedge*\nabla_a dN$$

$$= \int_\Sigma W^a{}_b H_N(E_b, E_a)\mathrm{vol}_q$$

$$= 0,$$

H_N being symmetric.

<u>Ad III:</u> It has been pointed out at the beginning of this section that here

$$\{\mathcal{H}_H(N_1), \mathcal{H}_H(N_2)\} = 0.$$

Therefore

$$\{\tilde{\mathcal{H}}_H(N_1), \tilde{\mathcal{H}}_H(N_2)\}$$

$$= \{\mathcal{H}_H(N_1) - \mathcal{H}_T(*(dN_1\wedge\omega_a)), \mathcal{H}_H(N_2) - \mathcal{H}_T(*(dN_2\wedge\omega_a))\}$$

$$= - \{H_T(*(dN_1 \wedge \omega_a)), H_H(N_2)\}$$

$$- \{H_H(N_1), H_T(*(dN_2 \wedge \omega_a))\}$$

$$= \{H_H(N_2), H_T(*(dN_1 \wedge \omega_a))\}$$

$$- \{H_H(N_1), H_T(*(dN_2 \wedge \omega_a))\}.$$

Using the explicit formulas for these Poisson brackets and then restricting to Con_T leads immediately to the claimed result.

Summary: On Con_T, the fundamental Poisson bracket relations are the same as those in the coframe picture provided one works with $\tilde{H}_H(N)$ rather than $H_H(N)$.

The next step is to find modifications

$$\begin{cases} H_D \rightarrow \bar{H}_D \\ H_R \rightarrow \bar{H}_R \\ \tilde{H}_H \rightarrow \bar{H}_H \end{cases}$$

such that on Con_T,

$$\begin{cases} \{\bar{H}_D(\vec{N}), H_T(\vec{\alpha})\} = 0 \\ \{\bar{H}_R(W), H_T(\vec{\alpha})\} = 0 \\ \{\bar{H}_H(N), H_T(\vec{\alpha})\} = 0, \end{cases} \qquad \begin{cases} \{\bar{H}_D(\vec{N}), H_f(\beta)\} = 0 \\ \{\bar{H}_R(W), H_f(\beta)\} = 0 \\ \{\bar{H}_H(N), H_f(\beta)\} = 0. \end{cases}$$

[Note: It will also be clear from the construction that on Con_T,

$$\begin{cases} \{\bar{H}_D(\vec{N}), H_1(\underline{\rho})\} = 0 \\ \{\bar{H}_R(W), H_1(\underline{\rho})\} = 0 \\ \{\bar{H}_H(N), H_1(\underline{\rho})\} = 0, \end{cases} \qquad \begin{cases} \{\bar{H}_D(\vec{N}), H_2(\vec{R})\} = 0 \\ \{\bar{H}_R(W), H_2(\vec{R})\} = 0 \\ \{\bar{H}_H(N), H_2(\vec{R})\} = 0.] \end{cases}$$

$\underline{\bar{H}_D(\vec{N})}$:

#1: We have

$$\{H_D(\vec{N}), H_T(\vec{\alpha})\} = - \int_\Sigma L_{\vec{N}}\omega^a \wedge (d\alpha_a - \omega^c{}_a \wedge \alpha_c).$$

$\bullet\, d(L_{\vec{N}}\omega^a \wedge \alpha_a)$

$$= d(L_{\vec{N}}\omega^a) \wedge \alpha_a - L_{\vec{N}}\omega^a \wedge d\alpha_a$$

=>

$$- L_{\vec{N}}\omega^a \wedge d\alpha_a = d(L_{\vec{N}}\omega^a \wedge \alpha_a) - d(L_{\vec{N}}\omega^a) \wedge \alpha_a.$$

$\bullet\, L_{\vec{N}}(\omega^a \wedge \omega^c{}_a \wedge \alpha_c)$

$$= L_{\vec{N}}\omega^a \wedge \omega^c{}_a \wedge \alpha_c + \omega^a \wedge L_{\vec{N}}\omega^c{}_a \wedge \alpha_c + \omega^a \wedge \omega^c{}_a \wedge L_{\vec{N}}\alpha_c$$

=>

$$L_{\vec{N}}\omega^a \wedge \omega^c{}_a \wedge \alpha_c = L_{\vec{N}}(\omega^a \wedge \omega^c{}_a \wedge \alpha_c)$$

$$- \omega^a \wedge L_{\vec{N}}\omega^c{}_a \wedge \alpha_c - \omega^a \wedge \omega^c{}_a \wedge L_{\vec{N}}\alpha_c.$$

Thus

$$\{H_D(\vec{N}), H_T(\vec{\alpha})\} = - \int_\Sigma d(L_{\vec{N}}\omega^a)\wedge\alpha_a$$

$$- \int_\Sigma (\omega^a\wedge L_{\vec{N}}\omega^b{}_a\wedge\alpha_b + \omega^a\wedge\omega^b{}_a\wedge L_{\vec{N}}\alpha_b)$$

$$= - \int_\Sigma d(L_{\vec{N}}\omega^a)\wedge\alpha_a - \int_\Sigma \omega^a\wedge\omega^b{}_a\wedge L_{\vec{N}}\alpha_b$$

$$+ \int_\Sigma L_{\vec{N}}\omega_{ba}\wedge\omega^a\wedge\alpha_b$$

$$= - \int_\Sigma d(L_{\vec{N}}\omega^a)\wedge\alpha_a - \int_\Sigma \omega^a\wedge\omega^b{}_a\wedge L_{\vec{N}}\alpha_b$$

$$+ \int_\Sigma L_{\vec{N}}\omega_{ab}\wedge\omega^b\wedge\alpha_a.$$

But

$$\frac{1}{2}\{H_1(L_{\vec{N}}\omega_{ab}), H_T(\vec{\alpha})\} = - \int_\Sigma L_{\vec{N}}\omega_{ab}\wedge\omega^b\wedge\alpha_a.$$

Therefore

$$\{H_D(\vec{N}), H_T(\vec{\alpha})\} + \frac{1}{2}\{H_1(L_{\vec{N}}\omega_{ab}), H_T(\vec{\alpha})\}$$

$$= - \int_\Sigma d(L_{\vec{N}}\omega^a)\wedge\alpha_a - \int_\Sigma \omega^a\wedge\omega^b{}_a\wedge L_{\vec{N}}\alpha_b.$$

Now restrict to Con_T -- then

$$d\omega^b = - \omega^b{}_a\wedge\omega^a$$

$\Rightarrow$

$$d(L_{\vec{N}}\omega^a)\wedge\alpha_a + \omega^a\wedge\omega^b{}_a\wedge L_{\vec{N}}\alpha_b$$

$$= L_{\vec{N}}d\omega^a\wedge\alpha_a - \omega^b{}_a\wedge\omega^a\wedge L_{\vec{N}}\alpha_b$$

$$= L_{\vec{N}}d\omega^a\wedge\alpha_a + d\omega^b\wedge L_{\vec{N}}\alpha_b$$

$$= L_{\vec{N}}d\omega^a\wedge\alpha_a + d\omega^a\wedge L_{\vec{N}}\alpha_a$$

$$= L_{\vec{N}}(d\omega^a\wedge\alpha_a)$$

=>

$$\int_\Sigma (d(L_{\vec{N}}\omega^a)\wedge\alpha_a + \omega^a\wedge\omega^b{}_a\wedge L_{\vec{N}}\alpha_b)$$

$$= 0.$$

I.e.:

$$\{H_D(\vec{N}) + \frac{1}{2} H_1(L_{\vec{N}}\omega_{ab}), H_T(\vec{\alpha})\}|\mathrm{Con}_T = 0.$$

#2: We have

$$\{H_D(\vec{N}), H_f(\beta)\} = - \int_\Sigma L_{\vec{N}}\omega^a\wedge B_a\beta.$$

On the other hand,

$$- \{H_2(B_b q(L_{\vec{N}}\omega^b, \omega_a)), H_f(\beta)\}$$

$$= \int_\Sigma B_b q(L_{\vec{N}}\omega^b, \omega_a)\omega^a\wedge\beta$$

$$= \int_\Sigma q(L_{\vec{N}}\omega^a,\omega_b)\omega^b \wedge B_a\beta$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge B_a\beta.$$

So, with no conditions,

$$\{H_D(\vec{N}) - H_2(B_b q(L_{\vec{N}}\omega^b,\omega_a)), H_f(\beta)\} = 0.$$

Notation: Put

$$\bar{H}_D(\vec{N}) = H_D(\vec{N}) + K_D(\vec{N}),$$

where

$$K_D(\vec{N}) = \frac{1}{2} H_1(L_{\vec{N}}\omega_{ab}) - H_2(B_b q(L_{\vec{N}}\omega^b,\omega_a)).$$

Then on Con_T,

$$\{\bar{H}_D(\vec{N}), H_T(\vec{\alpha})\} = 0, \ \{\bar{H}_D(\vec{N}), H_f(\beta)\} = 0.$$

<u>$\bar{H}_R(W)$</u>:

#1: We have

$$\{H_R(W), H_T(\vec{\alpha})\} = - \int_\Sigma W^a{}_b \omega^b \wedge (d\alpha_a - \omega^c{}_a \wedge \alpha_c).$$

• $d(W^a{}_b \wedge \omega^b \wedge \alpha_a)$

$$= dW^a{}_b \wedge \omega^b \wedge \alpha_a + W^a{}_b \wedge d(\omega^b \wedge \alpha_a)$$

$$= dW^a{}_b \wedge \omega^b \wedge \alpha_a + W^a{}_b \wedge d\omega^b \wedge \alpha_a - W^a{}_b \wedge \omega^b \wedge d\alpha_a$$

=>

$$- W^a{}_b \wedge \omega^b \wedge d\alpha_a$$

$$= d(W^a{}_b \wedge \omega^b \wedge \alpha_a)$$

$$- dW^a{}_b \wedge \omega^b \wedge \alpha_a - W^a{}_b \wedge d\omega^b \wedge \alpha_a.$$

• $- W^a{}_b \wedge d\omega^b \wedge \alpha_a$

$$= - W^a{}_b \wedge (\Theta^b(\underline{\omega}) - \omega^b{}_c \wedge \omega^c) \wedge \alpha_a$$

$$= - W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a + W_{ab} \wedge \omega^b{}_c \wedge \omega^c \wedge \alpha_a$$

$$= - W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a + \omega^b{}_c \wedge W_{ab} \wedge \omega^c \wedge \alpha_a$$

$$= - W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a - \omega^c{}_b \wedge W_{ab} \wedge \omega^c \wedge \alpha_a$$

$$= - W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a - \omega^b{}_c \wedge W_{ac} \wedge \omega^b \wedge \alpha_a$$

$$= - W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a + \omega^c{}_b \wedge W_{ac} \wedge \omega^b \wedge \alpha_a.$$

• $W^a{}_b \wedge \omega^b \wedge \omega^c{}_a \wedge \alpha_c$

$$= - W_{ab} \wedge \omega^c{}_a \wedge \omega^b \wedge \alpha_c$$

$$= \omega^a{}_c \wedge W_{ab} \wedge \omega^b \wedge \alpha_c$$

$$= \omega^c{}_a \wedge W_{cb} \wedge \omega^b \wedge \alpha_a.$$

$$\bullet - dW^a{}_b \wedge \omega^b \wedge \alpha_a$$

$$- W^a{}_b \wedge d\omega^b \wedge \alpha_a + W^a{}_b \wedge \omega^b \wedge \omega^c{}_a \wedge \alpha_c$$

$$= - W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a$$

$$+ (- dW_{ab} + \omega^c{}_a \wedge W_{cb} + \omega^c{}_b \wedge W_{ac}) \wedge \omega^b \wedge \alpha_a$$

$$= - W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a - d^{\nabla(\underline{\omega})} W_{ab} \wedge \omega^b \wedge \alpha_a .$$

Thus

$$\{H_R(W), H_T(\vec{\alpha})\}$$

$$= - \int_\Sigma W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a - \int_\Sigma d^{\nabla(\underline{\omega})} W_{ab} \wedge \omega^b \wedge \alpha_a .$$

But

$$- \frac{1}{2} \{H_1(d^{\nabla(\underline{\omega})} W_{ab}), H_T(\vec{\alpha})\} = \int_\Sigma d^{\nabla(\underline{\omega})} W_{ab} \wedge \omega^b \wedge \alpha_a .$$

Therefore

$$\{H_R(W), H_T(\vec{\alpha})\} - \frac{1}{2} \{H_1(d^{\nabla(\underline{\omega})} W_{ab}), H_T(\vec{\alpha})\}$$

$$= - \int_\Sigma W^a{}_b \wedge \Theta^b(\underline{\omega}) \wedge \alpha_a .$$

Now restrict to Con_T — then $\Theta^b(\underline{\omega}) = 0$, hence

$$\{H_R(W) - \frac{1}{2} H_1(d^{\nabla(\underline{\omega})} W_{ab}), H_T(\vec{\alpha})\} | \text{Con}_T = 0.$$

#2: We have

$$\{H_R(W), H_f(\beta)\} = - \int_\Sigma W^a{}_b \omega^b \wedge B_a \beta$$

$$= - \int_\Sigma W_{ab} B^a \omega^b \wedge \beta$$

$$= - \int_\Sigma W_{ba} B^b \omega^a \wedge \beta$$

$$= \int_\Sigma W_{ab} B^b \omega^a \wedge \beta.$$

On the other hand,

$$\{H_2(W_{ab}B^b), H_f(\beta)\} = - \int_\Sigma W_{ab} B^b \omega^a \wedge \beta.$$

So, with no conditions,

$$\{H_R(W) + H_2(W_{ab}B^b), H_f(\beta)\} = 0.$$

Notation: Put

$$\bar{H}_R(W) = H_R(W) + K_R(W),$$

where

$$K_R(W) = - \frac{1}{2} H_1(d^{\nabla(\underline{\omega})} W_{ab}) + H_2(W_{ab}B^b).$$

Then on Con_T,

$$\{\bar{H}_R(W), H_T(\vec{\alpha})\} = 0, \quad \{\bar{H}_R(W), H_f(\beta)\} = 0.$$

$\underline{\bar{H}_H(N)}$:

#1: We have

$$\{\tilde{H}_H(N), H_T(\vec{\alpha})\} = \{H_H(N), H_T(\vec{\alpha})\} - \{H_T(*(dN\wedge\omega_a)), H_T(\vec{\alpha})\}$$

$$= \{H_H(N), H_T(\vec{\alpha})\}$$

$$= - \int_\Sigma N(q(p_b, *\omega^a)\omega^b - \frac{p}{n-2}\,\omega^a)\wedge(d\alpha_a - \omega^c{}_a\wedge\alpha_c).$$

Let

$$\zeta^a = N(q(p_b, *\omega^a)\omega^b - \frac{p}{n-2}\,\omega^a).$$

- $d(\zeta^a\wedge\alpha_a) = d\zeta^a\wedge\alpha_a - \zeta^a\wedge d\alpha_a$

=>

$$- \zeta^a\wedge d\alpha_a = d(\zeta^a\wedge\alpha_a) - d\zeta^a\wedge\alpha_a.$$

- $\zeta^a\wedge\omega^c{}_a\wedge\alpha_c$

$$= - \omega^c{}_a\wedge\zeta^a\wedge\alpha_c$$

$$= - \omega^a{}_c\wedge\zeta^c\wedge\alpha_a$$

$$= \omega^c{}_a\wedge\zeta_c\wedge\alpha_a.$$

Thus

$$\{H_H(N), H_T(\vec{\alpha})\}$$

$$= \int_\Sigma (- d\zeta_a + \omega^c{}_a\wedge\zeta_c)\wedge\alpha_a$$

$$= - \int_\Sigma d^{\nabla(\underline{\omega})}\zeta_a\wedge\alpha_a.$$

The combination

$$Z_{ab}(\underline{\omega}) = \frac{1}{2}\,(\iota_{E_a} d^{\nabla(\underline{\omega})}\zeta_b - \iota_{E_b} d^{\nabla(\underline{\omega})}\zeta_a) + \frac{1}{2}\,\omega^c\iota_{E_b}\iota_{E_a} d^{\nabla(\underline{\omega})}\zeta_c$$

is antisymmetric and

$$-\frac{1}{2}\{H_1(Z_{ab}(\underline{\omega})),H_T(\vec{\alpha})\} = \int_\Sigma Z_{ab}(\underline{\omega})\wedge\omega^b\wedge\alpha_a.$$

But

$$\int_\Sigma Z_{ab}(\underline{\omega})\wedge\omega^b\wedge\alpha_a = \int_\Sigma d^{\nabla(\underline{\omega})}\zeta_a\wedge\alpha_a.$$

To see this, write

$$d^{\nabla(\underline{\omega})}\zeta_a = \frac{1}{2}C^a{}_{uv}\omega^u\wedge\omega^v \quad (C^a{}_{uv} = -C^a{}_{vu})$$

and recall that

$$C^a{}_{uv} = \iota_{E_v}\iota_{E_u}d^{\nabla(\underline{\omega})}\zeta_a.$$

- $\iota_{E_a}d^{\nabla(\underline{\omega})}\zeta_b$

$$= \frac{1}{2}C^b{}_{uv}\iota_{E_a}(\omega^u\wedge\omega^v)$$

$$= \frac{1}{2}C^b{}_{uv}(\delta^u{}_a\omega^v - \omega^u\delta^v{}_a)$$

$$= \frac{1}{2}C^b{}_{av}\omega^v - \frac{1}{2}C^b{}_{ua}\omega^u.$$

- $-\iota_{E_b}d^{\nabla(\underline{\omega})}\zeta_a$

$$= -\frac{1}{2}C^a{}_{uv}\iota_{E_b}(\omega^u\wedge\omega^v)$$

$$= -\frac{1}{2}C^a{}_{uv}(\delta^u{}_b\omega^v - \omega^u\delta^v{}_b)$$

$$= \frac{1}{2}C^a{}_{ub}\omega^u - \frac{1}{2}C^a{}_{bv}\omega^v$$

$\Rightarrow$

$$\frac{1}{2}\,(\iota_{E_a} d^{\nabla(\underline{\omega})}\zeta_b - \iota_{E_b} d^{\nabla(\underline{\omega})}\zeta_a)\wedge\omega^b$$

$$= \frac{1}{4}\,(C^b{}_{av}\omega^v\wedge\omega^b - C^b{}_{ua}\omega^u\wedge\omega^b$$

$$+ C^a{}_{ub}\omega^u\wedge\omega^b - C^a{}_{bv}\omega^v\wedge\omega^b)$$

$$= \frac{1}{4}\,(C^b{}_{au}\omega^u\wedge\omega^b + C^b{}_{au}\omega^u\wedge\omega^b$$

$$+ C^a{}_{vb}\omega^v\wedge\omega^b + C^a{}_{vb}\omega^v\wedge\omega^b)$$

$$= \frac{1}{2}\,C^b{}_{au}\omega^u\wedge\omega^b + \frac{1}{2}\,C^a{}_{vb}\omega^v\wedge\omega^b$$

$$= \frac{1}{2}\,C^u{}_{ab}\omega^b\wedge\omega^u + \frac{1}{2}\,C^a{}_{ub}\omega^u\wedge\omega^b$$

$$= \frac{1}{2}\,C^c{}_{ab}\omega^b\wedge\omega^c + \frac{1}{2}\,C^a{}_{uv}\omega^u\wedge\omega^v$$

$$= -\,\frac{1}{2}\,C^c{}_{ab}\omega^c\wedge\omega^b + d^{\nabla(\underline{\omega})}\zeta_a.$$

However

$$\frac{1}{2}\,\omega^c\iota_{E_b}\iota_{E_a} d^{\nabla(\underline{\omega})}\zeta_c\wedge\omega^b$$

$$= \frac{1}{2}\,\omega^c C^c{}_{ab}\wedge\omega^b$$

$$= \frac{1}{2}\,C^c{}_{ab}\omega^c\wedge\omega^b.$$

Therefore

$$Z_{ab}(\underline{\omega})\wedge\omega^b = -\,\frac{1}{2}\,C^c{}_{ab}\omega^c\wedge\omega^b + d^{\nabla(\underline{\omega})}\zeta_a$$

$$+ \frac{1}{2} C^{c}{}_{ab}\omega^{c}\wedge\omega^{b}$$

$$= d^{\nabla(\underline{\omega})}\zeta_a.$$

Consequently,

$$\{H_H(N) - \frac{1}{2} H_1(Z_{ab}(\underline{\omega})), H_T(\vec{\alpha})\} = 0$$

on the nose.

#2: We have

$$\{\tilde{H}_H(N), H_f(\beta)\} = \{H_H(N), H_f(\beta)\} - \{H_T(*(dN\wedge\omega_a)), H_f(\beta)\}$$

$$= \{H_H(N), H_f(\beta)\}$$

$$= - \int_\Sigma N(q(p_b, *\omega^a)\omega^b - \frac{p}{n-2}\,\omega^a)\wedge B_a\beta$$

$$= - \int_\Sigma B_a N(q(p_b, *\omega^a)\omega^b\wedge\beta - \frac{p}{n-2}\,\omega^a\wedge\beta).$$

On the other hand,

$$- \{H_2(B_b N(q(p_a, *\omega^b) - \frac{p}{n-2}\,\eta^b{}_a)), H_f(\beta)\}$$

$$= \int_\Sigma B_b N(q(p_a, *\omega^b) - \frac{p}{n-2}\,\eta^b{}_a)\omega^a\wedge\beta$$

$$= \int_\Sigma B_b N(q(p_a, *\omega^b)\omega^a\wedge\beta - \frac{p}{n-2}\,\eta^b{}_a\omega^a\wedge\beta)$$

$$= \int_\Sigma B_a N(q(p_b, *\omega^a)\omega^b\wedge\beta - \frac{p}{n-2}\,\eta^a{}_b\omega^b\wedge\beta)$$

$$= \int_\Sigma B_a N(q(p_b, *\omega^a)\,\omega^b \wedge \beta - \frac{p}{n-2}\,\omega^a \wedge \beta).$$

So, with no conditions,

$$\{\mathcal{H}_H(N) - \mathcal{H}_2(B_b N(q(p_a, *\omega^b) - \frac{p}{n-2}\,\eta^b{}_a)), \mathcal{H}_f(\beta)\} = 0.$$

Notation: Put

$$\bar{\mathcal{H}}_H(N) = \tilde{\mathcal{H}}_H(N) + \mathcal{K}_H(N),$$

where

$$\mathcal{K}_H(N) = -\frac{1}{2}\,\mathcal{H}_1(Z_{ab}(\underline{\omega})) - \mathcal{H}_2(B_b N(q(p_a, *\omega^b) - \frac{p}{n-2}\,\eta^b{}_a)).$$

Then on Con_T,

$$\{\bar{\mathcal{H}}_H(N), \mathcal{H}_T(\vec{\alpha})\} = 0, \quad \{\bar{\mathcal{H}}_H(N), \mathcal{H}_f(\beta)\} = 0.$$

Remark: Since

$$\{\bar{\mathcal{H}}_H(N), \mathcal{H}_1(\underline{\rho})\} = \{\tilde{\mathcal{H}}_H(N), \mathcal{H}_1(\underline{\rho})\} + \{\mathcal{K}_H(N), \mathcal{H}_1(\underline{\rho})\}$$

$$= \{\tilde{\mathcal{H}}_H(N), \mathcal{H}_1(\underline{\rho})\},$$

it follows that

$$\{\bar{\mathcal{H}}_H(N), \mathcal{H}_1(\underline{\rho})\}|\text{Con}_T = \{\tilde{\mathcal{H}}_H(N), \mathcal{H}_1(\underline{\rho})\}|\text{Con}_T$$

$$= 0.$$

<u>THEOREM</u> On Con_T, we have

1. $\{\bar{\mathcal{H}}_D(\vec{N}_1), \bar{\mathcal{H}}_D(\vec{N}_2)\} = \bar{\mathcal{H}}_D([\vec{N}_1, \vec{N}_2])$;

2. $\{\bar{H}_D(\vec{N}),\bar{H}_R(W)\} = \bar{H}_R(L_{\vec{N}}W);$

3. $\{\bar{H}_D(\vec{N}),\bar{H}_H(N)\} = \bar{H}_H(L_{\vec{N}}N);$

4. $\{\bar{H}_R(W_1),\bar{H}_R(W_2)\} = \bar{H}_R([W_1,W_2]);$

5. $\{\bar{H}_R(W),\bar{H}_H(N)\} = 0;$

6. $\{\bar{H}_H(N_1),\bar{H}_H(N_2)\}$

$$= \bar{H}_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)$$

$$+ \bar{H}_R(q(dN_1 \wedge dN_2, \omega^a \wedge \omega_b) + q(N_1 dN_2 - N_2 dN_1, \omega^a{}_b)).$$

Notation: $\text{Con}_{\underline{Q}}$ is the subset of T^*C whose elements are the points

$$(\vec{\omega},\underline{\omega},\vec{B};\vec{p},p_{\underline{\omega}},p_{\vec{B}})$$

such that simultaneously

$$\frac{\delta H}{\delta N} = 0,\ \frac{\delta H}{\delta \vec{N}} = 0,\ \frac{\delta H}{\delta W} = 0.$$

Definition: The <u>physical phase space</u> of the theory (a.k.a. the <u>constraint surface</u> of the theory) is the subset Con_{Pal} of T^*C defined by

$$\text{Con}_{\text{Pal}} = \text{Con}_{\underline{Q}} \cap \text{Con}_T \cap \text{Con}_1 \cap \text{Con}_2.$$

A <u>constraint</u> is a function $\phi: T^*C \to \underline{R}$ such that $\phi | \mathrm{Con}_{\mathrm{Pal}} = 0$.

Example: $H_T(\vec{\alpha}), H_1(\underline{\rho})$, and $H_2(\vec{R})$ are obviously constraints, thus, by construction, so are $\bar{H}_D(\vec{N})$, $\bar{H}_R(W)$, and $\bar{H}_H(N)$.

Definition: A function $\phi: T^*C \to \underline{R}$ is said to be <u>first class</u> if

$$\{\phi, \Phi\} | \mathrm{Con}_{\mathrm{Pal}} = 0,$$

where

$$\Phi = \bar{H}_D(\vec{N}), \bar{H}_R(W), \bar{H}_H(N)$$

or

$$\Phi = H_T(\vec{\alpha}), H_1(\underline{\rho}), H_2(\vec{R}).$$

[Note: Here the parameters

$$\vec{N}, W, N, \vec{\alpha}, \underline{\rho}, \vec{R}$$

are arbitrary.]

Example: $\bar{H}_D(\vec{N})$, $\bar{H}_R(W)$, and $\bar{H}_H(N)$ are first class.

A function that is not first class is called <u>second class</u>. E.g.: H_D, H_R, and H_H are second class, as is H.

The fact that H is second class can be partially remedied. To this end, let

$$\bar{H} = H + K_D(\vec{N}) + K_R(W) + K_H(N).$$

Then

$$\bar{H} = \bar{H}_D(\vec{N}) + \bar{H}_R(W) + H_T(B_C * (\omega^c \wedge \omega^a))$$

$$+ H_H(N) + K_H(N)$$

$$= \bar{H}_D(N) + \bar{H}_R(W) + H_T(B_c * (\omega^c \wedge \omega^a))$$

$$+ \tilde{H}_H(N) + H_T(*(dN) \wedge \omega_a) + K_H(N)$$

$$= \bar{H}_D(\vec{N}) + \bar{H}_R(W) + \bar{H}_H(N)$$

$$+ H_T(B_c * (\omega^c \wedge \omega^a)) + H_T(*(dN \wedge \omega_a)).$$

Thanks to this last representation of $\bar{H}$, on Con_{Pal}, we have:

$$\begin{cases} \{\bar{H}, \bar{H}_D\} = 0, \ \{\bar{H}, \bar{H}_R\} = 0, \ \{\bar{H}, \bar{H}_H\} = 0 \\ \{\bar{H}, H_T(\vec{\alpha})\} = 0, \ \{\bar{H}, H_2(\vec{R})\} = 0. \end{cases}$$

Still, this does not say that $\bar{H}$ is first class since $\{\bar{H}, H_1(\underline{\rho})\}$ has yet to be considered and therein lies the rub.

Notation: Let

$$Con_{Pal}(N) = Con_{Pal} \cap Con_N.$$

[Note: $Con_T \cap Con_N$ is the subset of T^*C consisting of those points

$$(\vec{\omega}, \underline{\omega}, \vec{B}; \vec{p}, p_{\underline{\omega}}, p_{\vec{B}})$$

such that the auxiliary constraints

$$\begin{cases} d\omega^a + \omega^a{}_b \wedge \omega^b = 0 \\ dN + B_c \omega^c = 0 \end{cases}$$

are in force.]

We then claim that on $\text{Con}_{\text{Pal}}(N)$,

$$\{\bar{H}, H_1(\underline{\rho})\} = 0.$$

In fact,

$$\{\bar{H}, H_1(\underline{\rho})\}|\text{Con}_{\text{Pal}}(N)$$

$$= \{H_T(B_c * (\omega^c \wedge \omega^a)) + H_T(*(dN \wedge \omega_a)), H_1(\underline{\rho})\}|\text{Con}_{\text{Pal}}(N)$$

$$= 2 \int_\Sigma \rho_{ab} \wedge \omega^b \wedge (B_c * (\omega^c \wedge \omega^a) + *(dN \wedge \omega_a))|\text{Con}_{\text{Pal}}(N)$$

$$= 2 \int_\Sigma \rho_{ab} \wedge \omega^b \wedge (B_c * (\omega^c \wedge \omega^a) + *(-B_c \omega^c \wedge \omega_a))$$

$$= 0.$$

So, while $\bar{H}$ is not, strictly speaking, first class, it is at least first class in a restricted sense.

[Note: For the record, observe too that

$$\{\bar{H}, H_f(\beta)\}|\text{Con}_{\text{Pal}} = 0.]$$

Remark: The correction terms $K_D(\vec{N})$, $K_R(W)$, and $K_H(N)$ involve H_1 and H_2, hence they vanish on $\text{Con}_1 \cap \text{Con}_2$. Nevertheless, working with $\bar{H}$ is not the same as working with H.

<u>Section 53:</u> <u>Extension of the Scalars</u> Let M be a connected C^∞ manifold of dimension n,

$$\mathcal{D}(M) = \bigoplus_{p,q=0}^{\infty} \mathcal{D}_q^p(M)$$

its tensor algebra.

Notation: Put

$$\mathcal{D}(M;\underline{C}) = \bigoplus_{p,q=0}^{\infty} \mathcal{D}_q^p(M;\underline{C}),$$

the complexified tensor algebra.

[Note: Here, $\mathcal{D}_0^0(M;\underline{C}) = C^\infty(M;\underline{C})$, $\mathcal{D}_0^1(M;\underline{C}) = \mathcal{D}^1(M;\underline{C})$, the derivations of $C^\infty(M;\underline{C})$ (a.k.a. the complex vector fields on M), and $\mathcal{D}_1^0(M;\underline{C}) = \mathcal{D}_1(M;\underline{C})$, the linear forms on $\mathcal{D}^1(M;\underline{C})$ (viewed as a module over $C^\infty(M;\underline{C})$).]

The operation of conjugation in $C^\infty(M;\underline{C})$ induces a similar operation in each $\mathcal{D}_q^p(M;\underline{C})$.

$\bar{X}$: Given $X \in \mathcal{D}^1(M;\underline{C})$, define $\bar{X} \in \mathcal{D}^1(M;\underline{C})$ by

$$\bar{X}f = \overline{(X\bar{f})}.$$

$\bar{\omega}$: Given $\omega \in \mathcal{D}_1(M;\underline{C})$, define $\bar{\omega} \in \mathcal{D}_1(M;\underline{C})$ by

$$\bar{\omega}(X) = \overline{\omega(\bar{X})}.$$

In general, the conjugation $T \to \bar{T}$ is defined by

$$\bar{T}(\omega_1,\ldots,\omega_p,X_1,\ldots,X_q)$$

$$= T(\bar{\omega}_1,\ldots,\bar{\omega}_p,\bar{X}_1,\ldots,\bar{X}_q)^{-}.$$

Remark: There is an arrow of inclusion

$$\mathcal{D}_q^p(M) \to \mathcal{D}_q^p(M;\underline{C}).$$

For example, each $X \in \mathcal{D}^1(M)$ can be regarded as a complex vector field via the prescription

$$Xf = X\left(\frac{1}{2}(f + \bar{f})\right) + \sqrt{-1}\, X\left(\frac{1}{2\sqrt{-1}}(f - \bar{f})\right).$$

A <u>complex metric</u> is an element of $\mathcal{D}_2^0(M;\underline{C})$ which is symmetric and nondegenerate.

Notation: $\underline{M}_{\underline{C}}$ is the set of complex metrics on M.

[Note: There is an arrow of inclusion $\underline{M} \to \underline{M}_{\underline{C}}$.]

Example: Suppose that M is parallelizable. Let $\{E_1,\ldots,E_n\}$ be a complex frame. Given $X,Y \in \mathcal{D}^1(M;\underline{C})$, put

$$g(X,Y) = \eta_{ij}X^iY^j \quad \left[\begin{array}{l} X = X^iE_i \\ \\ Y = Y^jE_j. \end{array}\right.$$

Then g is a complex metric on M.

[Note: In terms of the associated coframe $\{\omega^1,\ldots\omega^n\}$,

$$g = -\omega^1 \otimes \omega^1 - \cdots - \omega^k \otimes \omega^k + \omega^{k+1} \otimes \omega^{k+1} + \cdots + \omega^n \otimes \omega^n.]$$

Let $g \in \underline{M}_{\underline{C}}$ — then a connection ∇ on $TM \otimes \underline{C}$ is said to be a <u>g-connection</u> if $\nabla g = 0$. As in the real case, among all g-connections there is exactly one with zero torsion, the <u>metric connection</u>.

[Note: Likewise, other entities associated with g still make sense (e.g. Ein(g)), a point that will be taken for granted in the sequel.]

Section 54: Selfdual Algebra In this section we shall develop the machinery that will be needed for complex general relativity in dimension 4.

Rappel: Let V be a vector space over $\underline{R}$ — then a complex structure on V is an $\underline{R}$-linear map $J:V \to V$ such that $J^2 = - I$, where $I = \text{id}_V$ is the identity map.

LEMMA The arrow

$$J:\underline{so}(1,3) \to \underline{so}(1,3)$$

defined by

$$(JA)_{ij} = \frac{1}{2} \varepsilon_{ij}{}^{k\ell} A_{k\ell}$$

is a complex structure on $\underline{so}(1,3)$.

Before we give the proof, it is necessary to explain the index convention on the Levi-Civita symbol. Thus, as usual, $\varepsilon^{ijk\ell}$ is the upper Levi-Civita symbol ($\varepsilon^{0123} = 1$). Indices are then lowered by means of

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} \underline{\varepsilon}_{ijk\ell} &= \eta_{ir}\eta_{js}\eta_{ku}\eta_{\ell v}\varepsilon^{rsuv} \\ &= \varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_\ell\varepsilon^{ijk\ell} \end{aligned}$$

is not the lower Levi-Civita symbol ($\underline{\varepsilon}_{0123} = - 1$).

<u>FACT</u> We have

$$\varepsilon^{i_1 i_2 i_3 i_4} \varepsilon_{j_1 j_2 j_3 j_4} = - \delta^{i_1 i_2 i_3 i_4}{}_{j_1 j_2 j_3 j_4}.$$

A matrix $A = [A^i{}_j] \in \underline{so}(1,3)$ is characterized by the condition

$$A^i{}_j = - \varepsilon_i \varepsilon_j A^j{}_i \quad \text{(no sum)}.$$

Thus to check that $JA \in \underline{so}(1,3)$, one must compare

$$(JA)^i{}_j$$

with

$$- \varepsilon_i \varepsilon_j (JA)^j{}_i.$$

But

$$(JA)^i{}_j = \varepsilon_i \left(\frac{1}{2} \varepsilon_{ij}{}^{k\ell} A_{k\ell} \right),$$

while

$$- \varepsilon_i \varepsilon_j (JA)^j{}_i = - \varepsilon_i \varepsilon_j \varepsilon_j \left(\frac{1}{2} \varepsilon_{ji}{}^{k\ell} A_{k\ell} \right).$$

And

$$\begin{aligned} - \varepsilon_i \varepsilon_j \varepsilon_j \varepsilon_{ji}{}^{k\ell} &= - \varepsilon_i \varepsilon_{ji}{}^{k\ell} \\ &= - \varepsilon_i \varepsilon_j \varepsilon_i \varepsilon^{jik\ell} \\ &= \varepsilon_i \varepsilon_i \varepsilon_j \varepsilon^{ijk\ell} \\ &= \varepsilon_i \varepsilon_{ij}{}^{k\ell}. \end{aligned}$$

Moving on, write

$$J(JA)_{ij} = \frac{1}{2}\,\varepsilon_{ij}{}^{k\ell}(JA)_{k\ell}$$

$$= \frac{1}{2}\,\varepsilon_{ij}{}^{k\ell}\left(\frac{1}{2}\,\varepsilon_{k\ell}{}^{uv}A_{uv}\right)$$

$$= \frac{1}{4}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v\varepsilon^{ijk\ell}\varepsilon_{k\ell uv}A_{uv}$$

$$= -\frac{1}{4}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v\delta^{ijk\ell}{}_{k\ell uv}A_{uv}$$

$$= -\frac{1}{4}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v\delta^{ijk\ell}{}_{uvk\ell}A_{uv}$$

$$= -\frac{1}{4}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v\,\frac{(4-2)!}{(4-4)!}\,\delta^{ij}{}_{uv}A_{uv}$$

$$= -\frac{1}{2}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v\left(\begin{vmatrix}\delta^i{}_u & \delta^i{}_v\\ \delta^j{}_u & \delta^j{}_v\end{vmatrix}\right)A_{uv}$$

$$= -\frac{1}{2}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v(\delta^i{}_u\delta^j{}_v - \delta^i{}_v\delta^j{}_u)A_{uv}$$

$$= -\frac{1}{2}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v\delta^i{}_u\delta^j{}_vA_{uv}$$

$$+\frac{1}{2}\,\varepsilon_i\varepsilon_j\varepsilon_u\varepsilon_v\delta^i{}_v\delta^j{}_uA_{uv}$$

$$= -\frac{1}{2}\,\varepsilon_i\varepsilon_j\varepsilon_i\varepsilon_jA_{ij} + \frac{1}{2}\,\varepsilon_i\varepsilon_j\varepsilon_j\varepsilon_iA_{ji}$$

$$= -\frac{1}{2}A_{ij} + \frac{1}{2}A_{ji}$$

$$= -A_{ij}.$$

Therefore

$$J^2 = -I,$$

as contended.

N.B. One can, of course, view J as an endomorphism of $\underline{g\ell}(4,\underline{R})$ but then J is no longer a complex structure.

Pass now to the complexification $\underline{so}(1,3) \otimes \underline{C}$ ($\equiv \underline{so}(1,3)_{\underline{C}}$) and extend J by linearity -- then there is a direct sum decomposition

$$\underline{so}(1,3)_{\underline{C}} = \underline{so}(1,3)^{+}_{\underline{C}} \oplus \underline{so}(1,3)^{-}_{\underline{C}},$$

where

$$\underline{so}(1,3)^{\pm}_{\underline{C}} = \{A \in \underline{so}(1,3)_{\underline{C}} : JA = \pm\sqrt{-1}\, A\}.$$

[Note: The elements of $\underline{so}(1,3)^{+}_{\underline{C}}$ ($\underline{so}(1,3)^{-}_{\underline{C}}$) are said to be selfdual (antiselfdual).]

LEMMA $\forall\, A,B \in \underline{so}(1,3)_{\underline{C}}$,

$$[JA,B] = J[A,B] = [A,JB].$$

Application: $\forall\, A,B \in \underline{so}(1,3)_{\underline{C}}$,

$$[JA,JB] = -[A,B].$$

[In fact,

$$[JA,JB] = J[A,JB]$$

$$= JJ[A,B] = - [A,B].]$$

Let

$$P^{\pm}:\underline{so}(1,3)_{\underline{C}} \rightarrow \underline{so}(1,3)_{\underline{C}}^{\pm}$$

be the projections, so that

$$P^{\pm} = \frac{1}{2}(I \mp \sqrt{-1}\, J).$$

Then

$$P^{\pm}[A,B] = [P^{\pm}A,B] = [A,P^{\pm}B] = [P^{\pm}A,P^{\pm}B].$$

Therefore $\underline{so}(1,3)_{\underline{C}}^{\pm}$ are ideals in $\underline{so}(1,3)_{\underline{C}}$.

Remark: $\underline{SO}(1,3)_{\underline{C}}$ ($\approx \underline{SO}(4,\underline{C})$) is connected and there is a covering map

$$\underline{SL}(2,\underline{C}) \times \underline{SL}(2,\underline{C}) \rightarrow \underline{SO}(1,3)_{\underline{C}}$$

which is universal, the product

$$\underline{SL}(2,\underline{C}) \times \underline{SL}(2,\underline{C})$$

being simply connected.

[Note: It is not difficult to see that

$$\underline{so}(1,3)_{\underline{C}}^{\pm} \approx \underline{s\ell}(2,\underline{C}) \approx \underline{so}(3,\underline{C}).]$$

Let M be a connected C^{∞} manifold of dimension 4. Fix a semiriemannian structure $g \in \underline{M}_{1,3}$.

Assume: The orthonormal frame bundle LM(g) is trivial.

Suppose that $E = \{E_1, \ldots, E_n\}$ is an orthonormal frame. Let $\nabla \in \text{con}_g TM$ and put

$$\omega_\nabla = [\omega^i{}_j].$$

Then

$$\omega^i{}_j = -\varepsilon_i \varepsilon_j \omega^j{}_i \quad \text{(no sum)}$$

$\Rightarrow$

$$\omega_\nabla \in \Lambda^1(M; \underline{\underline{so}}(1,3)).$$

<u>LEMMA</u> We have

$$J\omega_\nabla \wedge J\omega_\nabla = -\omega_\nabla \wedge \omega_\nabla.$$

[Write

$$(J\omega_\nabla \wedge J\omega_\nabla)^i{}_j = (J\omega_\nabla)^i{}_k \wedge (J\omega_\nabla)^k{}_j$$

$$= \varepsilon_i \left(\frac{1}{2} \varepsilon_{ik}{}^{rs} \omega_{rs}\right) \wedge \varepsilon_k \left(\frac{1}{2} \varepsilon_{kj}{}^{uv} \omega_{uv}\right)$$

$$= \frac{1}{4} \varepsilon_i \varepsilon_k \varepsilon_{ik}{}^{rs} \varepsilon_{kj}{}^{uv} \omega_{rs} \wedge \omega_{uv}$$

$$= \frac{1}{4} \varepsilon_u \varepsilon_v \varepsilon^{ikrs} \underline{\varepsilon}_{kjuv} \omega_{rs} \wedge \omega_{uv}$$

$$= -\frac{1}{4} \varepsilon_u \varepsilon_v \delta^{ikrs}{}_{kjuv} \omega_{rs} \wedge \omega_{uv}$$

$$= \frac{1}{4} \varepsilon_u \varepsilon_v \delta^{irsk}{}_{juvk} \omega_{rs} \wedge \omega_{uv}$$

$$= \frac{1}{4}\,\varepsilon_u\varepsilon_v\,\frac{(4-3)!}{(4-4)!}\,\delta^{irs}{}_{juv}\omega_{rs}\wedge\omega_{uv}$$

$$= \frac{1}{4}\,\varepsilon_u\varepsilon_v\delta^{irs}{}_{juv}\omega_{rs}\wedge\omega_{uv}.$$

But

$$\delta^{irs}{}_{juv} = \begin{vmatrix} \delta^i{}_j & \delta^i{}_u & \delta^i{}_v \\ \delta^r{}_j & \delta^r{}_u & \delta^r{}_v \\ \delta^s{}_j & \delta^s{}_u & \delta^s{}_v \end{vmatrix}$$

$$= \delta^i{}_j\delta^r{}_u\delta^s{}_v - \delta^i{}_j\delta^r{}_v\delta^s{}_u - \delta^i{}_u\delta^r{}_j\delta^s{}_v$$

$$+ \delta^i{}_u\delta^r{}_v\delta^s{}_j + \delta^i{}_v\delta^r{}_j\delta^s{}_u - \delta^i{}_v\delta^r{}_u\delta^s{}_j.$$

And

1. $\frac{1}{4}\,\varepsilon_u\varepsilon_v\delta^i{}_j\delta^r{}_u\delta^s{}_v\omega_{rs}\wedge\omega_{uv}$

$$= \frac{1}{4}\,\varepsilon_u\varepsilon_v\delta^i{}_j\omega_{uv}\wedge\omega_{uv}$$

$$= 0.$$

2. $-\frac{1}{4}\,\varepsilon_u\varepsilon_v\delta^i{}_j\delta^r{}_v\delta^s{}_u\omega_{rs}\wedge\omega_{uv}$

$$= -\frac{1}{4}\,\varepsilon_u\varepsilon_v\delta^i{}_j\omega_{vu}\wedge\omega_{uv}$$

$$= \frac{1}{4}\,\varepsilon_u\varepsilon_v\delta^i{}_j\omega_{uv}\wedge\omega_{uv}$$

$= 0.$

3. $-\frac{1}{4}\varepsilon_u\varepsilon_v\delta^i{}_u\delta^r{}_j\delta^s{}_v\omega_{rs}\wedge\omega_{uv}$

$$= -\frac{1}{4}\varepsilon_i\varepsilon_v\omega_{jv}\wedge\omega_{iv}$$

$$= \frac{1}{4}\varepsilon_i\varepsilon_v\omega_{vj}\wedge\omega_{iv}.$$

4. $\frac{1}{4}\varepsilon_u\varepsilon_v\delta^i{}_u\delta^r{}_v\delta^s{}_j\omega_{rs}\wedge\omega_{uv}$

$$= \frac{1}{4}\varepsilon_i\varepsilon_v\omega_{vj}\wedge\omega_{iv}.$$

5. $\frac{1}{4}\varepsilon_u\varepsilon_v\delta^i{}_v\delta^r{}_j\delta^s{}_u\omega_{rs}\wedge\omega_{uv}$

$$= \frac{1}{4}\varepsilon_u\varepsilon_i\omega_{ju}\wedge\omega_{ui}.$$

6. $-\frac{1}{4}\varepsilon_u\varepsilon_v\delta^i{}_v\delta^r{}_u\delta^s{}_j\omega_{rs}\wedge\omega_{uv}$

$$= -\frac{1}{4}\varepsilon_u\varepsilon_i\omega_{uj}\wedge\omega_{ui}$$

$$= \frac{1}{4}\varepsilon_u\varepsilon_i\omega_{ju}\wedge\omega_{ui}.$$

So

$$3 + 4 = \frac{1}{2}\varepsilon_i\varepsilon_v\omega_{vj}\wedge\omega_{iv}$$

$$= -\frac{1}{2}\varepsilon_i\varepsilon_v\omega_{iv}\wedge\omega_{vj}$$

$$= -\frac{1}{2}\,\omega^{i}{}_{v}\wedge\omega^{v}{}_{j}$$

while

$$5 + 6 = \frac{1}{2}\,\varepsilon_u\varepsilon_i\omega_{ju}\wedge\omega_{ui}$$

$$= -\frac{1}{2}\,\varepsilon_i\varepsilon_u\omega_{ui}\wedge\omega_{ju}$$

$$= -\frac{1}{2}\,\varepsilon_i\varepsilon_u(-\omega_{iu})\wedge(-\omega_{uj})$$

$$= -\frac{1}{2}\,\omega^{i}{}_{u}\wedge\omega^{u}{}_{j}.$$

Therefore

$$(3 + 4) + (5 + 6) = -(\omega_\nabla\wedge\omega_\nabla)^{i}{}_{j}.]$$

<u>Variant</u>

$$J\omega_\nabla\wedge J\omega_\nabla = \frac{1}{2}\,[J\omega_\nabla, J\omega_\nabla]$$

$$= -\frac{1}{2}\,[\omega_\nabla, \omega_\nabla]$$

$$= -\omega_\nabla\wedge\omega_\nabla.$$

<u>LEMMA</u> We have

$$J(\omega_\nabla\wedge\omega_\nabla) = \frac{1}{2}\,(J\omega_\nabla\wedge\omega_\nabla + \omega_\nabla\wedge J\omega_\nabla).$$

[Write

$$J(\omega_\nabla\wedge\omega_\nabla)_{ij} = -(J(J\omega_\nabla\wedge J\omega_\nabla))_{ij}$$

$$= -\frac{1}{2}\,\varepsilon_{ij}{}^{k\ell}(J\omega_\nabla \wedge J\omega_\nabla)_{k\ell}$$

$$= -\frac{1}{2}\,\varepsilon_{ij}{}^{k\ell}(J\omega_\nabla)_{kr}\wedge(J\omega_\nabla)^{r}{}_{\ell}$$

$$= -\frac{1}{2}\,\varepsilon_{ij}{}^{k\ell}\left(\frac{1}{2}\,\varepsilon_{kr}{}^{st}\omega_{st}\right)\wedge\left(\frac{1}{2}\,\varepsilon^{r}{}_{\ell}{}^{uv}\omega_{uv}\right)$$

$$= -\frac{1}{8}\,\varepsilon_{ij}{}^{k\ell}\varepsilon^{r}{}_{\ell}{}^{uv}\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}$$

$$= -\frac{1}{8}\,\varepsilon_{i}\varepsilon_{j}\varepsilon_{r}\varepsilon_{u}\varepsilon_{v}\varepsilon^{ijk\ell}\varepsilon_{r\ell uv}\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}$$

$$= \frac{1}{8}\,\varepsilon_{i}\varepsilon_{j}\varepsilon_{r}\varepsilon_{u}\varepsilon_{v}\delta^{ijk\ell}{}_{r\ell uv}\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}$$

$$= \frac{1}{8}\,\varepsilon_{i}\varepsilon_{j}\varepsilon_{r}\varepsilon_{u}\varepsilon_{v}\delta^{ijk\ell}{}_{ruv\ell}\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}$$

$$= \frac{1}{8}\,\varepsilon_{i}\varepsilon_{j}\varepsilon_{r}\varepsilon_{u}\varepsilon_{v}\,\frac{(4-3)!}{(4-4)!}\,\delta^{ijk}{}_{ruv}\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}$$

$$= \frac{1}{8}\,\varepsilon_{i}\varepsilon_{j}\varepsilon_{r}\varepsilon_{u}\varepsilon_{v}\delta^{ijk}{}_{ruv}\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}.$$

But

$$\delta^{ijk}{}_{ruv} = \begin{vmatrix} \delta^{i}{}_{r} & \delta^{i}{}_{u} & \delta^{i}{}_{v} \\ \delta^{j}{}_{r} & \delta^{j}{}_{u} & \delta^{j}{}_{v} \\ \delta^{k}{}_{r} & \delta^{k}{}_{u} & \delta^{k}{}_{v} \end{vmatrix}$$

$$= \delta^{i}{}_{r}\delta^{j}{}_{u}\delta^{k}{}_{v} - \delta^{i}{}_{r}\delta^{j}{}_{v}\delta^{k}{}_{u} - \delta^{i}{}_{u}\delta^{j}{}_{r}\delta^{k}{}_{v}$$

$$+ \delta^i{}_u \delta^j{}_v \delta^k{}_r + \delta^i{}_v \delta^j{}_r \delta^k{}_u - \delta^i{}_v \delta^j{}_u \delta^k{}_r.$$

And

1. $\frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_r \varepsilon_u \varepsilon_v \delta^i{}_r \delta^j{}_u \delta^k{}_v \varepsilon_{kr}{}^{st} \omega_{st} \wedge \omega_{uv}$

$$= \frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_i \varepsilon_j \varepsilon_v \varepsilon_{vi}{}^{st} \omega_{st} \wedge \omega_{jv}$$

$$= \frac{1}{8} \varepsilon_v \varepsilon_{vi}{}^{st} \omega_{st} \wedge \omega_{jv}.$$

2. $- \frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_r \varepsilon_u \varepsilon_v \delta^i{}_r \delta^j{}_v \delta^k{}_u \varepsilon_{kr}{}^{st} \omega_{st} \wedge \omega_{uv}$

$$= - \frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_i \varepsilon_u \varepsilon_j \varepsilon_{ui}{}^{st} \omega_{st} \wedge \omega_{uj}$$

$$= - \frac{1}{8} \varepsilon_u \varepsilon_{ui}{}^{st} \omega_{st} \wedge \omega_{uj}.$$

3. $- \frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_r \varepsilon_u \varepsilon_v \delta^i{}_u \delta^j{}_r \delta^k{}_v \varepsilon_{kr}{}^{st} \omega_{st} \wedge \omega_{uv}$

$$= - \frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_j \varepsilon_i \varepsilon_v \varepsilon_{vj}{}^{st} \omega_{st} \wedge \omega_{iv}$$

$$= - \frac{1}{8} \varepsilon_v \varepsilon_{vj}{}^{st} \omega_{st} \wedge \omega_{iv}.$$

4. $\frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_r \varepsilon_u \varepsilon_v \delta^i{}_u \delta^j{}_v \delta^k{}_r \varepsilon_{kr}{}^{st} \omega_{st} \wedge \omega_{uv}$

$$= \frac{1}{8} \varepsilon_i \varepsilon_j \varepsilon_r \varepsilon_i \varepsilon_j \varepsilon_{rr}{}^{st} \omega_{st} \wedge \omega_{ij}$$

$$= 0.$$

5. $\frac{1}{8}\,\varepsilon_i\varepsilon_j\varepsilon_r\varepsilon_u\varepsilon_v\,\delta^i_v\,\delta^j_r\,\delta^k_u\,\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}$

$$= \frac{1}{8}\,\varepsilon_i\varepsilon_j\varepsilon_j\varepsilon_u\varepsilon_i\varepsilon_{uj}{}^{st}\omega_{st}\wedge\omega_{ui}$$

$$= \frac{1}{8}\,\varepsilon_u\varepsilon_{uj}{}^{st}\omega_{st}\wedge\omega_{ui}.$$

6. $-\frac{1}{8}\,\varepsilon_i\varepsilon_j\varepsilon_r\varepsilon_u\varepsilon_v\,\delta^i_v\,\delta^j_u\,\delta^k_r\,\varepsilon_{kr}{}^{st}\omega_{st}\wedge\omega_{uv}$

$$= -\frac{1}{8}\,\varepsilon_i\varepsilon_j\varepsilon_r\varepsilon_j\varepsilon_i\varepsilon_{rr}{}^{st}\omega_{st}\wedge\omega_{ji}$$

$$= 0.$$

So

$$1 + 2 = \frac{1}{8}\,\varepsilon_v\varepsilon_{vi}{}^{st}\omega_{st}\wedge\omega_{jv} - \frac{1}{8}\,\varepsilon_u\varepsilon_{ui}{}^{st}\omega_{st}\wedge\omega_{uj}$$

$$= -\frac{1}{8}\,\varepsilon_k\varepsilon_{ki}{}^{st}\omega_{st}\wedge\omega_{kj} - \frac{1}{8}\,\varepsilon_k\varepsilon_{ki}{}^{st}\omega_{st}\wedge\omega_{kj}$$

$$= \frac{1}{4}\,\varepsilon_k\varepsilon_{ik}{}^{st}\omega_{st}\wedge\omega_{kj}$$

$$= \frac{1}{2}\left(\frac{1}{2}\,\varepsilon_{ik}{}^{st}\omega_{st}\wedge\varepsilon_k\omega_{kj}\right)$$

$$= \frac{1}{2}\left((J\omega_\nabla)_{ik}\wedge\omega^k{}_j\right)$$

$$= \frac{1}{2}\,(J\omega_\nabla\wedge\omega_\nabla)_{ij}$$

while

$$3 + 5 = -\frac{1}{8}\,\varepsilon_v\varepsilon_{vj}{}^{st}\omega_{st}\wedge\omega_{iv} + \frac{1}{8}\,\varepsilon_u\varepsilon_{uj}{}^{st}\omega_{st}\wedge\omega_{ui}$$

$$= \frac{1}{8}\, \varepsilon_k \varepsilon_{kj}{}^{st} \omega_{ik} \wedge \omega_{st} + \frac{1}{8}\, \varepsilon_k \varepsilon_{kj}{}^{st} \omega_{ik} \wedge \omega_{st}$$

$$= \frac{1}{4}\, \varepsilon_k \varepsilon_{kj}{}^{st} \omega_{ik} \wedge \omega_{st}$$

$$= \frac{1}{2}\, (\omega_{ik} \wedge \frac{1}{2}\, \varepsilon_k \varepsilon_{kj}{}^{st} \omega_{st})$$

$$= \frac{1}{2}\, (\omega_{ik} \wedge \frac{1}{2}\, \varepsilon^{k}{}_{j}{}^{st} \omega_{st})$$

$$= \frac{1}{2}\, (\omega_{ik} \wedge (J\omega_\nabla)^{k}{}_{j})$$

$$= \frac{1}{2}\, (\omega_\nabla \wedge J\omega_\nabla)_{ij}.$$

Therefore

$$(1 + 2) + (3 + 5) = \frac{1}{2}\, (J\omega_\nabla \wedge \omega_\nabla + \omega_\nabla \wedge J\omega_\nabla)_{ij}.]$$

<u>Variant</u>

$$J(\omega_\nabla \wedge \omega_\nabla) = J(\frac{1}{2}\, [\omega_\nabla, \omega_\nabla])$$

$$= \frac{1}{2}\, J([\omega_\nabla, \omega_\nabla])$$

$$= \frac{1}{2}\, (\frac{1}{2}\, [J\omega_\nabla, \omega_\nabla] + \frac{1}{2}\, [\omega_\nabla, J\omega_\nabla])$$

$$= \frac{1}{2}\, (\frac{1}{2}\, (J\omega_\nabla \wedge \omega_\nabla + \omega_\nabla \wedge J\omega_\nabla) + \frac{1}{2}\, (\omega_\nabla \wedge J\omega_\nabla + J\omega_\nabla \wedge \omega_\nabla))$$

$$= \frac{1}{2}\, (J\omega_\nabla \wedge \omega_\nabla + \omega_\nabla \wedge J\omega_\nabla).$$

Returning to our g-connection ∇, write

$$\omega_\nabla = \omega_\nabla^+ + \omega_\nabla^-,$$

where

$$\begin{cases} \omega_\nabla^+ = P^+\omega_\nabla = \frac{1}{2}\,(\omega_\nabla - \sqrt{-1}\,J\omega_\nabla) \\ \omega_\nabla^- = P^-\omega_\nabla = \frac{1}{2}\,(\omega_\nabla + \sqrt{-1}\,J\omega_\nabla). \end{cases}$$

Decompose Ω_∇ analogously, thus

$$\Omega_\nabla = \Omega_\nabla^+ + \Omega_\nabla^-.$$

Then

$$\Omega_\nabla^+ = \frac{1}{2}\,(\Omega_\nabla - \sqrt{-1}\,J\Omega_\nabla)$$

$$= \frac{1}{2}\,(d\omega_\nabla + \omega_\nabla\wedge\omega_\nabla - \sqrt{-1}\,J(d\omega_\nabla + \omega_\nabla\wedge\omega_\nabla))$$

$$= \frac{1}{2}\,(d\omega_\nabla + \omega_\nabla\wedge\omega_\nabla - \sqrt{-1}\,(dJ\omega_\nabla + J(\omega_\nabla\wedge\omega_\nabla))$$

$$= \frac{1}{2}\,(d(\omega_\nabla - \sqrt{-1}\,J\omega_\nabla) + \omega_\nabla\wedge\omega_\nabla - \sqrt{-1}\,J(\omega_\nabla\wedge\omega_\nabla))$$

$$= \frac{1}{2}\,(d(\omega_\nabla - \sqrt{-1}\,J\omega_\nabla) + \frac{1}{2}\,\omega_\nabla\wedge\omega_\nabla - \frac{1}{2}\,J\omega_\nabla\wedge J\omega_\nabla$$

$$- \frac{\sqrt{-1}}{2}\,(J\omega_\nabla\wedge\omega_\nabla + \omega_\nabla\wedge J\omega_\nabla))$$

$$= \frac{1}{2}\,(d(\omega_\nabla - \sqrt{-1}\,J\omega_\nabla) + \frac{1}{2}\,(\omega_\nabla - \sqrt{-1}\,J\omega_\nabla)\wedge(\omega_\nabla - \sqrt{-1}\,J\omega_\nabla))$$

$$= d\left(\frac{1}{2}(\omega_\nabla - \sqrt{-1}\, J\omega_\nabla)\right) + \frac{1}{2}(\omega_\nabla - \sqrt{-1}\, J\omega_\nabla) \wedge \frac{1}{2}(\omega_\nabla - \sqrt{-1}\, J\omega_\nabla))$$

$$= d\omega_\nabla^+ + \omega_\nabla^+ \wedge \omega_\nabla^+.$$

Similarly

$$\Omega_\nabla^- = d\omega_\nabla^- + \omega_\nabla^- \wedge \omega_\nabla^-.$$

Remark: To interpret these relations, define complex g-connections $\nabla^\pm$ by

$$\nabla_X^\pm E_j = (\omega^\pm)^i{}_j(X) E_i.$$

Then

$$\begin{cases} \Omega_{\nabla^+} = \Omega_\nabla^+ \\ \Omega_{\nabla^-} = \Omega_\nabla^- . \end{cases}$$

<u>LEMMA</u> We have

$$\Omega^+{}_{ij}(\nabla) \wedge \theta^{ij} \;\; (= \Omega_{ij}(\nabla^+) \wedge \theta^{ij})$$

$$= \frac{1}{2}\left(\Omega_{ij}(\nabla) \wedge *(\omega^i \wedge \omega^j) \;-\; \sqrt{-1}\; \Omega_{ij}(\nabla) \wedge (\omega^i \wedge \omega^j)\right).$$

[In fact,

$$(J\Omega_\nabla)_{ij} \wedge *(\omega^i \wedge \omega^j)$$

$$= \frac{1}{4}\, \varepsilon_{ij}{}^{k\ell}\, \Omega_{k\ell}(\nabla) \wedge \varepsilon_i \varepsilon_j \varepsilon_{ijuv}\, \omega^u \wedge \omega^v$$

$$= \frac{1}{4}\, \varepsilon^{ijk\ell} \varepsilon_{ijuv}\, \Omega_{k\ell}(\nabla) \wedge \omega^u \wedge \omega^v$$

$$= \frac{1}{4}\,\varepsilon^{k\ell ij}\varepsilon_{uvij}\Omega_{k\ell}(\nabla)\wedge\omega^u\wedge\omega^v$$

$$= \frac{1}{4}\,\delta^{k\ell ij}{}_{uvij}\Omega_{k\ell}(\nabla)\wedge\omega^u\wedge\omega^v$$

$$= \frac{1}{4}\,\frac{(4-2)!}{(4-4)!}\,\delta^{k\ell}{}_{uv}\Omega_{k\ell}(\nabla)\wedge\omega^u\wedge\omega^v$$

$$= \frac{1}{2}\begin{vmatrix} \delta^k{}_u & \delta^k{}_v \\ \delta^\ell{}_u & \delta^\ell{}_v \end{vmatrix}\Omega_{k\ell}(\nabla)\wedge\omega^u\wedge\omega^v$$

$$= \frac{1}{2}\,(\delta^k{}_u\delta^\ell{}_v - \delta^k{}_v\delta^\ell{}_u)\Omega_{k\ell}(\nabla)\wedge\omega^u\wedge\omega^v$$

$$= \frac{1}{2}\,\delta^k{}_u\delta^\ell{}_v\Omega_{k\ell}(\nabla)\wedge\omega^u\wedge\omega^v$$

$$- \frac{1}{2}\,\delta^k{}_v\delta^\ell{}_u\Omega_{k\ell}(\nabla)\wedge\omega^u\wedge\omega^v$$

$$= \frac{1}{2}\,\Omega_{k\ell}(\nabla)\wedge\omega^k\wedge\omega^\ell - \frac{1}{2}\,\Omega_{k\ell}(\nabla)\wedge\omega^\ell\wedge\omega^k$$

$$= \Omega_{k\ell}(\nabla)\wedge\omega^k\wedge\omega^\ell = \Omega_{ij}(\nabla)\wedge\omega^i\wedge\omega^j.]$$

[Note: It is to be emphasized that here, ε_{ijuv} is the genuine lower Levi-Civita symbol and not its hybrid cousin used earlier.]

Rappel:

$$d\Theta^i(\nabla) + \omega^i{}_j\wedge\Theta^j(\nabla) = \Omega^i{}_j(\nabla)\wedge\omega^j.$$

Therefore

$$\Omega^{+}_{ij}(\nabla)\wedge\theta^{ij} = \frac{1}{2}\,\Omega_{ij}(\nabla)\wedge\theta^{ij}$$

$$+ \frac{\sqrt{-1}}{2}\,(d\Theta_{i}(\nabla) + \omega_{ij}\wedge\Theta^{j}(\nabla))\wedge\omega^{i}.$$

Now specialize and take $\nabla = \nabla^{g}$ -- then the conclusion is that

$$\Omega^{+}_{ij}\wedge\theta^{ij} = \frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij}.$$

Remark: The preceding considerations also imply that

$$\Omega^{-}_{ij}\wedge\theta^{ij} = \frac{1}{2}\,\Omega_{ij}\wedge\theta^{ij}.$$

<u>Section 55</u>: <u>The Selfdual Lagrangian</u> The assumptions and notation are those of the standard setup, subject now to the stipulation that n = 4, hence dim Σ = 3.

Consider

$$\theta^{ij}\wedge\Omega^{+}{}_{ij} \quad (= \frac{1}{2}\,\theta^{ij}\wedge\Omega_{ij}).$$

Write

$$\theta^{ij}\wedge\Omega^{+}{}_{ij} = 2\theta^{0a}\wedge\Omega^{+}{}_{0a} + \theta^{bc}\wedge\Omega^{+}{}_{bc}.$$

Since Ω^{+} is selfdual, we have

$$\sqrt{-1}\;\Omega^{+}{}_{0a} = \frac{1}{2}\,\varepsilon_{0a}{}^{k\ell}\Omega^{+}{}_{k\ell}$$

$$= \frac{1}{2}\,\underset{\sim}{\varepsilon}_{0ak\ell}\Omega^{+k\ell}.$$

But

$$\underset{\sim}{\varepsilon}_{0a0\ell} = \underset{\sim}{\varepsilon}_{0ak0} = 0$$

=>

$$\sqrt{-1}\;\Omega^{+}{}_{0a} = \frac{1}{2}\,\underset{\sim}{\varepsilon}_{0abc}\Omega^{+bc},$$

where

$$\underset{\sim}{\varepsilon}_{0abc} = \varepsilon_{0}\varepsilon^{0abc} = -\,\varepsilon^{0abc} = -\,\varepsilon_{abc} \quad (\varepsilon_{123} = 1).$$

Therefore

$$\theta^{ij}\wedge\Omega^{+}{}_{ij} = \sqrt{-1}\;\varepsilon_{abc}\Omega^{+bc}\wedge\theta^{0a} + \theta^{bc}\wedge\Omega^{+}{}_{bc}.$$

Observation:

$$0 = \iota_{E_0}(\Omega^{+bc}\wedge{*}\omega^{a})$$

$$= \iota_{E_0}\Omega^{+bc}\wedge *\omega^a + \Omega^{+bc}\wedge\iota_{E_0} *\omega^a$$

=>

$$\iota_{E_0}\Omega^{+bc}\wedge *\omega^a = -\Omega^{+bc}\wedge\iota_{E_0} *\omega^a$$

$$= -\Omega^{+bc}\wedge\iota_{\omega_0} *\omega^a$$

$$= \Omega^{+bc}\wedge\iota_{\omega^0} *\omega^a$$

$$= \Omega^{+bc}\wedge *(\omega^a\wedge\omega^0)$$

$$= -\Omega^{+bc}\wedge *(\omega^0\wedge\omega^a)$$

$$= -\Omega^{+bc}\wedge\theta^{0a}.$$

Consequently,

$$\theta^{ij}\wedge\Omega^{+}_{ij} = -\sqrt{-1}\,\epsilon_{abc}\iota_{E_0}\Omega^{+bc}\wedge *\omega^a + \theta^{bc}\wedge\Omega^{+}_{bc},$$

thus, on formal grounds,

$$\int_M \theta^{ij}\wedge\Omega^{+}_{ij}$$

$$= \int_{\underline{R}} dt \int_\Sigma i_t^* \iota_{\partial/\partial t}[-\sqrt{-1}\,\epsilon_{abc}\iota_{E_0}\Omega^{+bc}\wedge *\omega^a + \theta^{bc}\wedge\Omega^{+}_{bc}].$$

To explicate the integral over Σ, note first that

$$i_t^*\iota_{\partial/\partial t}[\theta^{bc}\wedge\Omega^{+}_{bc}] = N_t\bar{\Omega}^{+}_{bc}\wedge *(\bar{\omega}^b\wedge\bar{\omega}^c).$$

On the other hand, the calculation of

$$i_t^* \iota_{\partial/\partial t}[-\sqrt{-1}\,\varepsilon_{abc}\iota_{E_0}\Omega^{+bc}\wedge *\omega^a]$$

is trickier and hinges on a preliminary remark.

Define an element

$$\Phi\in\Lambda^0(M;T_0^2(M)\otimes \underline{C}) \quad (=\mathcal{D}_0^2(M;\underline{C}))$$

by

$$\Phi^{ij}=\Phi(\omega^i,\omega^j)=\omega^{+ij}(NE_0).$$

Then $(\nabla^+=(\nabla^g)^+)$

$$d^{\nabla^+}\Phi^{bc}=d\Phi^{bc}+\omega^{+b}{}_k\wedge\Phi^{kc}+\omega^{+c}{}_k\wedge\Phi^{bk}$$

$\Rightarrow$

$$d\Phi^{bc}-d^{\nabla^+}\Phi^{bc}=-\omega^{+b}{}_k\wedge\Phi^{kc}-\omega^{+c}{}_k\wedge\Phi^{bk}$$

$$=-\Phi^{bk}\omega^{+c}{}_k-\Phi^{kc}\omega^{+b}{}_k$$

$$=-\Phi^b{}_k\omega^{+ck}-\Phi^{kc}\omega^{+b}{}_k$$

$$=\Phi^b{}_k\omega^{+kc}-\Phi^{kc}\omega^{+b}{}_k.$$

Therefore

$$N\iota_{E_0}\Omega^{+bc}=N\iota_{E_0}(d\omega^{+bc}+\omega^{+b}{}_k\wedge\omega^{+kc})$$

$$=\iota_{NE_0}(d\omega^{+bc}+\omega^{+b}{}_k\wedge\omega^{+kc})$$

$$=\iota_{NE_0}d\omega^{+bc}+\iota_{NE_0}(\omega^{+b}{}_k\wedge\omega^{+kc}).$$

But

$$\iota_{NE_0}(\omega^{+b}{}_k \wedge \omega^{+kc})$$

$$= (\iota_{NE_0}\omega^{+b}{}_k)\wedge\omega^{+kc} - \omega^{+b}{}_k \wedge (\iota_{NE_0}\omega^{+kc})$$

$$= \omega^{+b}{}_k(NE_0)\omega^{+kc} - \omega^{+kc}(NE_0)\omega^{+b}{}_k$$

$$= \phi^b{}_k\omega^{+kc} - \phi^{kc}\omega^{+b}{}_k$$

=>

$$N\iota_{E_0}\Omega^{+bc} = \iota_{NE_0}d\omega^{+bc} + d\phi^{bc} - d^{\nabla^+}\phi^{bc}$$

$$= \iota_{NE_0}d\omega^{+bc} + d\iota_{NE_0}\omega^{+bc} - d^{\nabla^+}\iota_{NE_0}\omega^{+bc}$$

$$= L_{NE_0}\omega^{+bc} - d^{\nabla^+}\iota_{NE_0}\omega^{+bc}$$

$$= L_{\partial/\partial t}\omega^{+bc} - L_{\vec{N}}\omega^{+bc} - d^{\nabla^+}\iota_{NE_0}\omega^{+bc}.$$

Let $u,v = 1,2,3$ and write

$$*\omega^a = \frac{1}{2}\,\varepsilon_{a0uv}\omega^0\wedge\omega^u\wedge\omega^v$$

or still

$$*\omega^a = -\frac{1}{2}\,\varepsilon_{0auv}\omega^0\wedge\omega^u\wedge\omega^v.$$

Then

$$i_t^*\iota_{\partial/\partial t}[-\sqrt{-1}\,\varepsilon_{abc}\iota_{E_0}\Omega^{+bc}\wedge *\omega^a]$$

$$= - i_t^* \iota_{\partial/\partial t}[- \sqrt{-1}\ \varepsilon_{abc} {*\omega^a} \wedge \iota_{E_0} \Omega^{+bc}]$$

$$= - i_t^* \iota_{\partial/\partial t}[\sqrt{-1}\ \frac{1}{2}\ \varepsilon_{abc}\varepsilon_{0auv}\omega^0 \wedge \omega^u \wedge \omega^v \wedge \iota_{E_0} \Omega^{+bc}]$$

$$= - i_t^*[\sqrt{-1}\ \frac{1}{2}\ \varepsilon_{abc}\varepsilon_{0auv}(\iota_{\partial/\partial t}\omega^0) \wedge \omega^u \wedge \omega^v \wedge \iota_{E_0} \Omega^{+bc}]$$

$$- \sqrt{-1}\ \frac{1}{2}\ \varepsilon_{abc}\varepsilon_{0auv}\omega^0 \wedge \iota_{\partial/\partial t}(\omega^u \wedge \omega^v \wedge \iota_{E_0} \Omega^{+bc})]$$

$$= - \sqrt{-1}\ \frac{1}{2}\ \varepsilon_{abc}\varepsilon_{0auv}\bar{\omega}^u \wedge \bar{\omega}^v \wedge N_t i_t^* \iota_{E_0} \Omega^{+bc}$$

$$= - \sqrt{-1}\ \frac{1}{2}\ \varepsilon_{abc}\varepsilon_{0auv}\bar{\omega}^u \wedge \bar{\omega}^v \wedge i_t^*(N \iota_{E_0} \Omega^{+bc}).$$

But

$$\varepsilon_{abc}\varepsilon_{0auv} = \varepsilon_{abc}\varepsilon_{auv}$$

$$= \varepsilon_{bca}\varepsilon_{uva}$$

$$= \delta^{bc}{}_{uv}$$

$$= (\delta^b{}_u\delta^c{}_v - \delta^b{}_v\delta^c{}_u)$$

=>

$$\frac{1}{2}\ \varepsilon_{abc}\varepsilon_{0auv}\bar{\omega}^u \wedge \bar{\omega}^v = \bar{\omega}^b \wedge \bar{\omega}^c.$$

So finally

$$i_t^* \iota_{\partial/\partial t}[- \sqrt{-1}\ \varepsilon_{abc} \iota_{E_0} \Omega^{+bc} \wedge {*\omega^a}]$$

$$= - \sqrt{-1}\ (\dot{\bar{\omega}}^{+}{}_{bc} - L_{\vec{N}_t}\bar{\omega}^{+}{}_{bc} - i_t^* d^{\nabla^+} \iota_{NE_0}\omega^{+}{}_{bc}) \wedge (\bar{\omega}^{b} \wedge \bar{\omega}^{c})$$

or still,

$$- \sqrt{-1}\ (\dot{\bar{\omega}}^{+}{}_{ab} - L_{\vec{N}_t}\bar{\omega}^{+}{}_{ab} - i_t^* d^{\nabla^+} \iota_{NE_0}\omega^{+}{}_{ab}) \wedge (\bar{\omega}^{a} \wedge \bar{\omega}^{b}).$$

Remark: We could just as well have worked with

$$\theta^{ij} \wedge \Omega^{-}{}_{ij},$$

the upshot being that there would be a sign change, viz.

$$\int_M \theta^{ij} \wedge \Omega^{-}{}_{ij}$$

$$= \int_{\underline{R}} dt \int_\Sigma i_t^* \iota_{\partial/\partial t} [\sqrt{-1}\ \varepsilon_{abc} \iota_{E_0} \Omega^{-bc} \wedge *\omega^a + \theta^{bc} \wedge \Omega^{-}{}_{bc}].$$

This seemingly technical point has its uses and will come up again later on.

To make further progress, it will be necessary to take a closer look at $\Omega^{+}{}_{ab}$:

$$\Omega^{+}{}_{ab} = d\omega^{+}{}_{ab} + \omega^{+}{}_{ak} \wedge \omega^{+k}{}_{b}$$

$$= d\omega^{+}{}_{ab} + \omega^{+}{}_{ac} \wedge \omega^{+c}{}_{b} + \omega^{+}{}_{a0} \wedge \omega^{+0}{}_{b}$$

$$= d\omega^{+}{}_{ab} + \omega^{+}{}_{ac} \wedge \omega^{+c}{}_{b} + \omega^{+}{}_{0a} \wedge \omega^{+}{}_{0b}.$$

<u>LEMMA</u> We have

$$\omega^{+}{}_{0a} \wedge \omega^{+}{}_{0b} = \omega^{+}{}_{ac} \wedge \omega^{+c}{}_{b}.$$

[Let u,v = 1,2,3 -- then, since ω^{+} is selfdual,

$$\left[\begin{array}{l} \omega^{+}{}_{ac} = -\sqrt{-1}\ \varepsilon_{ac}{}^{0u}\omega^{+}{}_{0u} \\ \\ \omega^{+c}{}_{b} = -\sqrt{-1}\ \varepsilon^{c}{}_{b}{}^{0v}\omega^{+}{}_{0v} \end{array}\right.$$

=>

$$\omega^{+}{}_{ac}\wedge\omega^{+c}{}_{b} = -\ \varepsilon_{ac}{}^{0u}\varepsilon^{c}{}_{b}{}^{0v}\omega^{+}{}_{0u}\wedge\omega^{+}{}_{0v}.$$

But

$$\begin{aligned} \varepsilon_{ac}{}^{0u}\varepsilon^{c}{}_{b}{}^{0v} &= \varepsilon^{ac0u}\varepsilon^{cb0v} \\ &= \varepsilon^{0acu}\varepsilon^{0cbv} \\ &= -\ \varepsilon^{0auc}\varepsilon_{0bvc} \\ &= -\ \delta^{au}{}_{bv} \\ &= -\ (\delta^{a}{}_{b}\delta^{u}{}_{v} - \delta^{a}{}_{v}\delta^{u}{}_{b}) \end{aligned}$$

=>

$$\begin{aligned} \omega^{+}{}_{ac}\wedge\omega^{+c}{}_{b} &= \delta^{a}{}_{b}\delta^{u}{}_{v}\omega^{+}{}_{0u}\wedge\omega^{+}{}_{0v} - \delta^{a}{}_{v}\delta^{u}{}_{b}\omega^{+}{}_{0u}\wedge\omega^{+}{}_{0v} \\ &= \delta^{a}{}_{b}\omega^{+}{}_{0c}\wedge\omega^{+}{}_{0c} - \omega^{+}{}_{0b}\wedge\omega^{+}{}_{0a} \\ &= \omega^{+}{}_{0a}\wedge\omega^{+}{}_{0b}.] \end{aligned}$$

Application:

$$\Omega^{+}{}_{ab} = d\omega^{+}{}_{ab} + 2\omega^{+}{}_{ac}\wedge\omega^{+c}{}_{b}.$$

<u>LEMMA</u> We have

$$2i_t^*\omega^{+a}{}_b = \bar{\omega}^a{}_b - \sqrt{-1}\,\varepsilon_{abc}\bar{\omega}_{0c}.$$

[By definition,

$$2\omega^+ = \omega - \sqrt{-1}\,J\omega.$$

Thus

$$2\omega^{+i}{}_j = \omega^i{}_j - \frac{\sqrt{-1}}{2}\,\varepsilon^i{}_{jk}{}^\ell\omega^k{}_\ell$$

=>

$$2\omega^{+0}{}_c = \omega^0{}_c - \frac{\sqrt{-1}}{2}\,\varepsilon^0{}_{ck}{}^\ell\omega^k{}_\ell.$$

On the other hand, ω^+ is selfdual, hence

$$2\omega^{+ab} = -\sqrt{-1}\,\varepsilon^{ab}{}_{0c}2\omega^{+0c}$$

or still,

$$2\omega^{+a}{}_b = -\sqrt{-1}\,\varepsilon^a{}_{b0c}[\omega^0{}_c - \frac{\sqrt{-1}}{2}\,\varepsilon^0{}_{ck}{}^\ell\omega^k{}_\ell].$$

But

$$- \sqrt{-1}\,\varepsilon^a{}_{b0c}\omega^0{}_c = \sqrt{-1}\,\varepsilon^a{}_{b0c}\omega_{0c}$$

$$= \sqrt{-1}\,\varepsilon_0\varepsilon^{ab0c}\omega_{0c}$$

$$= \sqrt{-1}\,\varepsilon_0\varepsilon^{0abc}\omega_{0c}$$

$$= -\sqrt{-1}\,\varepsilon_{abc}\omega_{0c}.$$

And, in addition,

$$-\frac{1}{2}\,\varepsilon^{a}{}_{b0c}\,\varepsilon^{0}{}_{ck}{}^{\ell}\,\omega^{k}{}_{\ell}$$

$$= \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{k}\varepsilon^{0ck\ell}\omega^{k}{}_{\ell}$$

$$= \frac{1}{2}\,\varepsilon_{k}\varepsilon_{abc}\varepsilon_{ck\ell}\omega^{k}{}_{\ell}$$

$$= \frac{1}{2}\,\varepsilon_{k}\varepsilon_{abc}\varepsilon_{k\ell c}\omega^{k}{}_{\ell}$$

$$= \frac{1}{2}\,\varepsilon_{k}\delta^{ab}{}_{k\ell}\omega^{k}{}_{\ell}$$

$$= \frac{1}{2}\,\varepsilon_{k}(\delta^{a}{}_{k}\delta^{b}{}_{\ell} - \delta^{a}{}_{\ell}\delta^{b}{}_{k})\omega^{k}{}_{\ell}$$

$$= \frac{1}{2}\,(\omega^{a}{}_{b} - \omega^{b}{}_{a})$$

$$= \omega^{a}{}_{b}.]$$

[Note: By the same token,

$$2i^{*}_{t}\omega^{-a}{}_{b} = \bar{\omega}^{a}{}_{b} + \sqrt{-1}\,\varepsilon_{abc}\bar{\omega}_{0c}.]$$

Put

$$A^{a}{}_{b} = 2i^{*}_{t}\omega^{+a}{}_{b}.$$

Then

$$[A^a{}_b] \in \Lambda^1(\Sigma; \underline{so}(3,\underline{C}))$$

and the prescription

$$\nabla_X Y = < X, dY^a + A^a{}_b Y^b > E_a$$

defines a complex $\bar{g}$-connection A. Denoting by F the associated curvature, we have

$$F_{ab} = dA_{ab} + A_{ac} \wedge A^c{}_b$$

$$= 2i_t^*(d\omega^+{}_{ab} + 2\omega^+{}_{ac} \wedge \omega^{+c}{}_b)$$

$$= 2i_t^* \Omega^+{}_{ab}$$

$$= 2\bar{\Omega}^+{}_{ab}.$$

[Note: The proof of the preceding lemma is applicable to $\Omega^{+a}{}_b$, so

$$F_{ab} = 2i_t^* \Omega^+{}_{ab} = \bar{\Omega}_{ab} - \sqrt{-1}\, \epsilon_{abc} \bar{\Omega}_{0c}.]$$

Now write

$$i_t^* d^{\nabla^+} \iota_{NE_0} \omega^+{}_{ab} = d^A i_t^* \iota_{NE_0} \omega^+{}_{ab}$$

$$= \frac{1}{2}\, d^A Z_{ab},$$

where

$$Z_{ab} = 2N_t i_t^* \iota_{E_0} \omega^+{}_{ab}.$$

[Note: Accordingly,

$$Z_{ab} = N_t i_t^* (\omega_{ab}(E_0) - \sqrt{-1}\, \epsilon_{abc} \omega_{0c}(E_0))$$

$$= - \bar{Q}_{ab} - \sqrt{-1}\, \varepsilon_{abc} \bar{P}_c .]$$

<u>Details</u> The equality

$$i_t^* d^{\nabla^+} \iota_{NE_0} \omega^+{}_{ab} = d^A i_t^* \iota_{NE_0} \omega^+{}_{ab}$$

is not obvious. By definition,

$$d^{\nabla^+} \Phi_{ab} = d\Phi_{ab} + \omega^+{}_{ai} \wedge \Phi^i{}_b + \omega^+{}_{bi} \wedge \Phi_a{}^i ,$$

thus

$$i_t^* d^{\nabla^+} \iota_{NE_0} \omega^+{}_{ab}$$

$$= d i_t^* \Phi_{ab} + i_t^* \omega^+{}_{ai} \wedge i_t^* \Phi^i{}_b + i_t^* \omega^+{}_{bi} \wedge i_t^* \Phi_a{}^i$$

$$= dN_t i_t^* \omega^+{}_{ab}(E_0) + i_t^* \omega^+{}_{ai} \wedge N_t i_t^* \omega^{+i}{}_b (E_0) + i_t^* \omega^+{}_{bi} \wedge N_t i_t^* \omega^+{}_a{}^i (E_0) ,$$

whereas

$$d^A i_t^* \iota_{NE_0} \omega^+{}_{ab}$$

$$= dN_t i_t^* \omega^+{}_{ab}(E_0) + A_{ac} \wedge N_t i_t^* \omega^{+c}{}_b (E_0) + A_{bc} \wedge N_t i_t^* \omega^+{}_a{}^c (E_0) .$$

Write

$$\begin{cases} \omega^+{}_{ai} \wedge \omega^{+i}{}_b (E_0) = \omega^+{}_{a0} \wedge \omega^{+0}{}_b (E_0) + \omega^+{}_{ac} \wedge \omega^{+c}{}_b (E_0) \\ \omega^+{}_{bi} \wedge \omega^+{}_a{}^i (E_0) = \omega^+{}_{b0} \wedge \omega^+{}_a{}^0 (E_0) + \omega^+{}_{bc} \wedge \omega^+{}_a{}^c (E_0) . \end{cases}$$

Then

$$\begin{cases} i_t^*\omega^+{}_{ac}\wedge i_t^*\omega^{+c}{}_b(E_0) = \frac{1}{2}A_{ac}\wedge i_t^*\omega^{+c}{}_b(E_0) \\ i_t^*\omega^+{}_{bc}\wedge i_t^*\omega^{+\ c}{}_a(E_0) = \frac{1}{2}A_{bc}\wedge i_t^*\omega^{+\ c}{}_a(E_0). \end{cases}$$

Let u,v = 1,2,3 and r,s = 1,2,3:

$$\begin{cases} \sqrt{-1}\ \omega^+{}_{a0} = \frac{1}{2}\varepsilon_{auv}\omega^{+uv} \\ \sqrt{-1}\ \omega^{+0}{}_b = \frac{1}{2}\varepsilon_{brs}\omega^{+rs} \end{cases}$$

=>

$$\omega^+{}_{a0}\wedge\omega^{+0}{}_b(E_0) = \frac{1}{\sqrt{-1}}(\sqrt{-1}\ \omega^+{}_{a0})\wedge\frac{1}{\sqrt{-1}}(\sqrt{-1}\ \omega^{+0}{}_b(E_0))$$

$$= -\frac{1}{4}\delta^{auv}{}_{brs}\omega^{+uv}\wedge\omega^{+rs}(E_0).$$

But

$$\delta^{auv}{}_{brs} = \begin{vmatrix} \delta^a{}_b & \delta^a{}_r & \delta^a{}_s \\ \delta^u{}_b & \delta^u{}_r & \delta^u{}_s \\ \delta^v{}_b & \delta^v{}_r & \delta^v{}_s \end{vmatrix}$$

$$= \delta^a{}_b\delta^u{}_r\delta^v{}_s - \delta^a{}_b\delta^u{}_s\delta^v{}_r - \delta^a{}_r\delta^u{}_b\delta^v{}_s$$

$$+ \delta^a{}_r\delta^u{}_s\delta^v{}_b + \delta^a{}_s\delta^u{}_b\delta^v{}_r - \delta^a{}_s\delta^u{}_r\delta^v{}_b.$$

And

1. $\delta^a{}_b \delta^u{}_r \delta^v{}_s \omega^{+uv} \wedge \omega^{+rs}(E_0)$

$$= \delta^a{}_b \omega^{+uv} \wedge \omega^{+uv}(E_0).$$

2. $-\ \delta^a{}_b \delta^u{}_s \delta^v{}_r \omega^{+uv} \wedge \omega^{+rs}(E_0)$

$$= -\ \delta^a{}_b \omega^{+uv} \wedge \omega^{+vu}(E_0).$$

3. $-\ \delta^a{}_r \delta^u{}_b \delta^v{}_s \omega^{+uv} \wedge \omega^{+rs}(E_0)$

$$= -\ \omega^{+bv} \wedge \omega^{+av}(E_0)$$

$$= -\ \omega^+{}_{bv} \wedge \omega^+{}_a{}^v(E_0)$$

$$= -\ \omega^+{}_{bc} \wedge \omega^+{}_a{}^c(E_0).$$

4. $\delta^a{}_r \delta^u{}_s \delta^v{}_b \omega^{+uv} \wedge \omega^{+rs}(E_0)$

$$= \omega^{+ub} \wedge \omega^{+au}(E_0)$$

$$= -\ \omega^+{}_{bu} \wedge \omega^+{}_a{}^u(E_0)$$

$$= -\ \omega^+{}_{bc} \wedge \omega^+{}_a{}^c(E_0)$$

5. $\delta^a{}_s \delta^u{}_b \delta^v{}_r \omega^{+uv} \wedge \omega^{+rs}(E_0)$

$$= \omega^{+bv} \wedge \omega^{+va}(E_0)$$

$$= - \omega^+{}_{bv} \wedge \omega^+{}_a{}^v(E_0)$$

$$= - \omega^+{}_{bc} \wedge \omega^+{}_a{}^c(E_0).$$

6. $- \delta^a{}_s \delta^u{}_r \delta^v{}_b \omega^{+uv} \wedge \omega^{+rs}(E_0)$

$$= - \omega^{+ub} \wedge \omega^{+ua}(E_0)$$

$$= - \omega^+{}_{bu} \wedge \omega^+{}_a{}^u(E_0)$$

$$= - \omega^+{}_{bc} \wedge \omega^+{}_a{}^c(E_0).$$

So

$$- \frac{1}{4}\ (1 + 2)\ - \frac{1}{4}\ (3 + 4 + 5 + 6)$$

$$= - \frac{1}{4}\ \delta^a{}_b \omega^{+uv} \wedge \omega^{+uv}(E_0)\ + \frac{1}{4}\ \delta^a{}_b \omega^{+uv} \wedge \omega^{+vu}(E_0)$$

$$+ \omega^+{}_{bc} \wedge \omega^+{}_a{}^c(E_0).$$

Applying i^*_t to

$$\omega^+{}_{bc} \wedge \omega^+{}_a{}^c(E_0)$$

then gives

$$\frac{1}{2} A_{bc} \wedge i^*_t \omega^+{}_a{}^c(E_0).$$

There remains the contribution from

$$\overset{+}{\omega}_{b0}\wedge\overset{+}{\omega}{}^{0}{}_{a}(E_0)$$

or still,

$$-\,\overset{+}{\omega}_{b0}\wedge\overset{+}{\omega}{}^{0}{}_{a}(E_0)$$

or still,

$$\frac{1}{4}\,\delta^{buv}{}_{ars}\overset{+}{\omega}{}^{uv}\wedge\overset{+}{\omega}{}^{rs}(E_0).$$

Reverse the roles of a and b in the above to get:

$$\frac{1}{4}\,\delta^{b}{}_{a}\overset{+}{\omega}{}^{uv}\wedge\overset{+}{\omega}{}^{uv}(E_0)\;-\;\frac{1}{4}\,\delta^{b}{}_{a}\overset{+}{\omega}{}^{uv}\wedge\overset{+}{\omega}{}^{vu}(E_0)$$

$$+\;\overset{+}{\omega}_{ac}\wedge\overset{+}{\omega}{}^{c}{}_{b}(E_0).$$

The first line cancels with

$$-\,\frac{1}{4}\,\delta^{a}{}_{b}\overset{+}{\omega}{}^{uv}\wedge\overset{+}{\omega}{}^{uv}(E_0)\;+\;\frac{1}{4}\,\delta^{a}{}_{b}\overset{+}{\omega}{}^{uv}\wedge\overset{+}{\omega}{}^{vu}(E_0)$$

while the second, upon application of i_t^*, leads to

$$\frac{1}{2}\,A_{ac}\wedge i_t^*\overset{+}{\omega}{}^{c}{}_{b}(E_0).$$

Summary: We have

$$\int_M \theta^{ij}\wedge\overset{+}{\Omega}_{ij}$$

$$=\frac{1}{2}\int_{\underline{R}} dt \int_\Sigma -\sqrt{-1}\,[\dot{A}_{ab}-L_{\vec{N}_t}A_{ab}-d^{A}Z_{ab}]\wedge(\bar{\omega}^{a}\wedge\bar{\omega}^{b})$$

$$+\;N_tF_{ab}\wedge*(\bar{\omega}^{a}\wedge\bar{\omega}^{b}).$$

[Note: For the record,

$$\int_M \theta^{ij} \wedge \Omega^-{}_{ij}$$

$$= \frac{1}{2} \int_{\underline{R}} dt \int_\Sigma \sqrt{-1}\, [\dot{A}_{ab} - L_{\vec{N}_t} A_{ab} - d^A Z_{ab}] \wedge (\bar{\omega}^a \wedge \bar{\omega}^b)$$

$$+ N_t F_{ab} \wedge * (\bar{\omega}^a \wedge \bar{\omega}^b).$$

Here

$$\begin{cases} A_{ab} = \bar{\omega}_{ab} + \sqrt{-1}\, \varepsilon_{abc} \bar{\omega}_{0c} \\ Z_{ab} = - \bar{Q}_{ab} + \sqrt{-1}\, \varepsilon_{abc} \bar{P}_c \end{cases}$$

and

$$F_{ab} = dA_{ab} + A_{ac} \wedge A^c{}_b = 2\bar{\Omega}^-{}_{ab}.]$$

The preceding expression for

$$\int_M \theta^{ij} \wedge \Omega^+{}_{ij}$$

is not convenient for manipulation (no boundary terms have arisen thus far).

<u>LEMMA</u> We have

$$\frac{1}{2} \int_\Sigma - \sqrt{-1}\, [\dot{A}_{ab} - L_{\vec{N}_t} A_{ab} - d^A Z_{ab}] \wedge (\bar{\omega}^a \wedge \bar{\omega}^b)$$

$$= - \frac{\sqrt{-1}}{2} \frac{d}{dt} \int_\Sigma A_{ab} \wedge (\bar{\omega}^a \wedge \bar{\omega}^b) + \frac{\sqrt{-1}}{2} \int_\Sigma L_{\vec{N}_t} (A_{ab} \wedge (\bar{\omega}^a \wedge \bar{\omega}^b))$$

$$+ \frac{\sqrt{-1}}{2} \int_\Sigma d(Z_{ab}(\bar{\omega}^a \wedge \bar{\omega}^b))$$

$$+ \sqrt{-1} \int_\Sigma A_{ab} \wedge \dot{\bar{\omega}}^a \wedge \bar{\omega}^b - \sqrt{-1} \int_\Sigma A_{ab} \wedge L_{\vec{N}_t} \bar{\omega}^a \wedge \bar{\omega}^b$$

$$- \sqrt{-1} \int_\Sigma Z_{ab} d^A \bar{\omega}^a \wedge \bar{\omega}^b .$$

[Note:

$$\begin{bmatrix} d^A \bar{\omega}^a = d\bar{\omega}^a + A^a{}_c \wedge \bar{\omega}^c \\ d^A \bar{\omega}^b = d\bar{\omega}^b + A^b{}_d \wedge \bar{\omega}^d \end{bmatrix}$$

=>

$$d^A Z_{ab} \wedge \bar{\omega}^a \wedge \bar{\omega}^b$$

$$= (dZ_{ab} - A^c{}_a Z_{cb} - A^d{}_b Z_{ad}) \wedge \bar{\omega}^a \wedge \bar{\omega}^b$$

$$= dZ_{ab} \wedge \bar{\omega}^a \wedge \bar{\omega}^b - Z_{cb}(A^c{}_a \wedge \bar{\omega}^a) \wedge \bar{\omega}^b + Z_{ad}(A^d{}_b \wedge \bar{\omega}^b) \wedge \bar{\omega}^a$$

$$= dZ_{ab} \wedge \bar{\omega}^a \wedge \bar{\omega}^b - Z_{cb}(d^A \bar{\omega}^c - d\bar{\omega}^c) \wedge \bar{\omega}^b + Z_{ad}(d^A \bar{\omega}^d - d\bar{\omega}^d) \wedge \bar{\omega}^a$$

$$= dZ_{ab} \wedge \bar{\omega}^a \wedge \bar{\omega}^b - Z_{ab}(d^A \bar{\omega}^a - d\bar{\omega}^a) \wedge \bar{\omega}^b + Z_{ab}(d^A \bar{\omega}^b - d\bar{\omega}^b) \wedge \bar{\omega}^a$$

$$= dZ_{ab} \wedge \bar{\omega}^a \wedge \bar{\omega}^b + Z_{ab} d\bar{\omega}^a \wedge \bar{\omega}^b - Z_{ab} \bar{\omega}^a \wedge d\bar{\omega}^b$$

$$- Z_{ab} d^A \bar{\omega}^a \wedge \bar{\omega}^b + Z_{ab} d^A \bar{\omega}^b \wedge \bar{\omega}^a$$

$$= d(Z_{ab}(\bar{\omega}^a \wedge \bar{\omega}^b)) - Z_{ab} d^A \bar{\omega}^a \wedge \bar{\omega}^b + Z_{ba} d^A \bar{\omega}^a \wedge \bar{\omega}^b$$

$$= d(Z_{ab}(\bar{\omega}^a \wedge \bar{\omega}^b)) - 2Z_{ab} d^A \bar{\omega}^a \wedge \bar{\omega}^b .]$$

With the understanding that the expression

$$-\frac{\sqrt{-1}}{2} \frac{d}{dt} \int_\Sigma A_{ab} \wedge (\bar{\omega}^a \wedge \bar{\omega}^b)$$

is to be ignored, it follows that

$$\int_M \theta^{ij} \wedge \Omega^+{}_{ij}$$

equals

$$\int_{\underline{R}} dt \int_\Sigma [\sqrt{-1}\, A_{ab} \wedge \dot{\bar{\omega}}^a \wedge \bar{\omega}^b + \sqrt{-1}\, A_{ab} \wedge L_{\vec{N}_t} \bar{\omega}^a \wedge \bar{\omega}^b$$

$$- \sqrt{-1}\, Z_{ab} d^A \bar{\omega}^a \wedge \bar{\omega}^b + \frac{1}{2} N_t F_{ab} \wedge * (\bar{\omega}^a \wedge \bar{\omega}^b)].$$

Claim: There is a simplification, viz.

$$Z_{ab} d^A \bar{\omega}^a \wedge \bar{\omega}^b = 0.$$

To see this, write

$$Z_{ab} d^A \bar{\omega}^a \wedge \bar{\omega}^b$$

$$= Z_{ab}(d\bar{\omega}^a + A^a{}_c \wedge \bar{\omega}^c) \wedge \bar{\omega}^b$$

$$= Z_{ab}(d\bar{\omega}^a + (\bar{\omega}^a{}_c - \sqrt{-1}\, \varepsilon_{acd} \bar{\omega}_{0d}) \wedge \bar{\omega}^c) \wedge \bar{\omega}^b$$

$$= Z_{ab}(d\bar{\omega}^a + \bar{\omega}^a{}_c \wedge \bar{\omega}^c) \wedge \bar{\omega}^b - \sqrt{-1}\, Z_{ab} \varepsilon_{acd} \bar{\omega}_{0d} \wedge \bar{\omega}^c \wedge \bar{\omega}^b$$

$$= Z_{ab}\Theta^{a}(\bar{\nabla})\wedge\bar{\omega}^{b} - \sqrt{-1}\ Z_{ab}\varepsilon_{acd}\bar{\omega}_{0d}\wedge\bar{\omega}^{c}\wedge\bar{\omega}^{b}$$

$$= -\sqrt{-1}\ Z_{ab}\varepsilon_{dac}\bar{\omega}_{0d}\wedge\bar{\omega}^{c}\wedge\bar{\omega}^{b}$$

$$= -\sqrt{-1}\ Z_{ab}\varepsilon_{cad}\bar{\omega}_{0c}\wedge\bar{\omega}^{d}\wedge\bar{\omega}^{b}.$$

[Note:

$$\Theta^{a}(\bar{\nabla}) = 0 \quad (\bar{\nabla} = \nabla^{\bar{g}} = \nabla^{q_t}).]$$

$\bullet\ \varepsilon_{cad}\bar{\omega}_{0c}\wedge\bar{\omega}^{d}\wedge\bar{\omega}^{b}$

$$= \bar{\omega}_{0c}\wedge\varepsilon_{cad}\bar{\omega}^{d}\wedge\bar{\omega}^{b}$$

$$= \bar{\omega}_{0c}\wedge*(\bar{\omega}^{c}\wedge\bar{\omega}^{a})\wedge\bar{\omega}^{b}$$

$$= -\ \bar{\omega}_{0c}\wedge\bar{\omega}^{b}\wedge*(\bar{\omega}^{c}\wedge\bar{\omega}^{a})$$

$$= \bar{\omega}^{b}\wedge\bar{\omega}_{0c}\wedge*(\bar{\omega}^{c}\wedge\bar{\omega}^{a})$$

$$= -\ \bar{\omega}^{b}\wedge*(\bar{\omega}^{c}\wedge\bar{\omega}^{a})\wedge\bar{\omega}_{0c}$$

$$= -\ (-1)^{1(3-1)}\bar{\omega}^{b}\wedge*(\iota_{\bar{\omega}_{0c}}(\bar{\omega}^{c}\wedge\bar{\omega}^{a}))$$

$$= -\ \bar{\omega}^{b}\wedge*((\iota_{\bar{\omega}_{0c}}\bar{\omega}^{c})\bar{\omega}^{a} - (\iota_{\bar{\omega}_{0c}}\bar{\omega}^{a})\bar{\omega}^{c})$$

$$= -\ \bar{\omega}^{b}\wedge((\iota_{\bar{\omega}_{0c}}\bar{\omega}^{c})*\bar{\omega}^{a} - (\iota_{\bar{\omega}_{0c}}\bar{\omega}^{a})*\bar{\omega}^{c})$$

$$= - \bar{\omega}^b \wedge (q_t(\bar{\omega}_{0c}, \bar{\omega}^c) * \bar{\omega}^a - q_t(\bar{\omega}_{0c}, \bar{\omega}^a) * \bar{\omega}^c)$$

$$= - (q_t(\bar{\omega}_{0c}, \bar{\omega}^c) \bar{\omega}^b \wedge * \bar{\omega}^a - q_t(\bar{\omega}_{0c}, \bar{\omega}^a) \bar{\omega}^b \wedge * \bar{\omega}^c)$$

$$= (q_t(\bar{\omega}_{0c}, \bar{\omega}^a) q_t(\bar{\omega}^b, \bar{\omega}^c) - q_t(\bar{\omega}_{0c}, \bar{\omega}^c) q_t(\bar{\omega}^b, \bar{\omega}^a)) \mathrm{vol}_{q_t}$$

$$= (q_t(\bar{\omega}_{0c}, \bar{\omega}^a) \delta_{bc} - q_t(\bar{\omega}_{0c}, \bar{\omega}^c) \delta_{ab}) \mathrm{vol}_{q_t}$$

$$= (q_t(\bar{\omega}_{0b}, \bar{\omega}^a) - q_t(\bar{\omega}_{0c}, \bar{\omega}^c) \delta_{ab}) \mathrm{vol}_{q_t}.$$

Therefore

$$Z_{ab} \varepsilon_{cad} \bar{\omega}_{0c} \wedge \bar{\omega}^d \wedge \bar{\omega}^b$$

$$= Z_{ab}(q_t(\bar{\omega}_{0b}, \bar{\omega}^a) - q_t(\bar{\omega}_{0c}, \bar{\omega}^c) \delta_{ab}) \mathrm{vol}_{q_t}.$$

Bearing in mind that $Z_{ab} = - Z_{ba}$, take $a \neq b$ and consider

$$2Z_{ab} q_t(\bar{\omega}_{0b}, \bar{\omega}^a)$$

$$= Z_{ab} q_t(\bar{\omega}_{0b}, \bar{\omega}^a) + Z_{ab} q_t(\bar{\omega}_{0b}, \bar{\omega}^a)$$

$$= Z_{ab} q_t(\bar{\omega}_{0b}, \bar{\omega}^a) + Z_{ba} q_t(\bar{\omega}_{0a}, \bar{\omega}^b)$$

$$= Z_{ab}(q_t(\bar{\omega}_{0b}, \bar{\omega}^a) - q_t(\bar{\omega}_{0a}, \bar{\omega}^b)).$$

Write

$$\begin{cases} \bar{\omega}_{0a} = - \kappa_{ac} \bar{\omega}^c \\ \bar{\omega}_{0b} = - \kappa_{bd} \bar{\omega}^d. \end{cases}$$

Then

$$\begin{cases} q_t(\bar{\omega}_{0b},\bar{\omega}^a) = -\kappa_{ba} \\ q_t(\bar{\omega}_{0a},\bar{\omega}^b) = -\kappa_{ab} \end{cases}$$

=>

$$q_t(\bar{\omega}_{0b},\bar{\omega}^a) - q_t(\bar{\omega}_{0a},\bar{\omega}^b)$$

$$= -\kappa_{ba} + \kappa_{ab}$$

$$= -\kappa_{ab} + \kappa_{ab} = 0.$$

To recapitulate: Modulo the boundary term,

$$\int_M \theta^{ij}\wedge\Omega^+_{ij}$$

equals

$$\int_{\underline{R}} dt \int_\Sigma [\sqrt{-1}\, A_{ab}\wedge\dot{\bar{\omega}}^a\wedge\bar{\omega}^b - \sqrt{-1}\, A_{ab}\wedge L_{\vec{N}_t}\bar{\omega}^a\wedge\bar{\omega}^b$$

$$+ \frac{1}{2} N_t F_{ab}\wedge *(\bar{\omega}^a\wedge\bar{\omega}^b)].$$

[Note: Analogously,

$$\int_M \theta^{ij}\wedge\Omega^-_{ij}$$

equals

$$\int_{\underline{R}} dt \int_\Sigma [-\sqrt{-1}\, A_{ab}\wedge\dot{\bar{\omega}}^a\wedge\bar{\omega}^b + \sqrt{-1}\, A_{ab}\wedge L_{\vec{N}_t}\bar{\omega}^a\wedge\bar{\omega}^b$$

$$+ \frac{1}{2} N_t F_{ab}\wedge *(\bar{\omega}^a\wedge\bar{\omega}^b)].]$$

The theory (be it selfdual or antiselfdual) carries three external variables, namely

$$\begin{cases} N \in C^{\infty}_{>0}(\Sigma) \cup C^{\infty}_{<0}(\Sigma) \\ \vec{N} \in \mathcal{D}^1(\Sigma) \end{cases}$$

and

$$W = [W^a{}_b],$$

where $W^a{}_b \in C^{\infty}(\Sigma)$ and $W^a{}_b = -W^b{}_a$.

Given $(\vec{\omega},\vec{v};N,\vec{N},W)$, put

$$N\omega^a{}_0 = v^a - W^a{}_b\omega^b - L_{\vec{N}}\omega^a.$$

Definition:

<u>SD</u>: Let

$$\begin{cases} A_{ab} = \omega_{ab} - \sqrt{-1}\,\epsilon_{abc}\omega_{0c} \\ F_{ab} = dA_{ab} + A_{ac}\wedge A^c{}_b. \end{cases}$$

Then the <u>selfdual lagrangian</u> is the function

$$L^+ : T\underline{Q} \to \Lambda^3\Sigma \otimes \underline{C}$$

defined by the rule

$$L^+(\vec{\omega},\vec{v};N,\vec{N},W)$$

$$= \sqrt{-1}\,A_{ab}\wedge v^a\wedge\omega^b - \sqrt{-1}\,A_{ab}\wedge L_{\vec{N}}\omega^a\wedge\omega^b$$

$$+ \frac{1}{2} NF_{ab} \wedge *(\omega^a \wedge \omega^b).$$

<u>ASD</u>: Let

$$\begin{cases} A_{ab} = \omega_{ab} + \sqrt{-1}\, \varepsilon_{abc}\omega_{0c} \\ F_{ab} = dA_{ab} + A_{ac} \wedge A^c{}_b. \end{cases}$$

Then the <u>antiselfdual lagrangian</u> is the function

$$L^- : T\underline{Q} \to \Lambda^3 \Sigma \otimes \underline{C}$$

defined by the rule

$$L^-(\vec{\omega}, \vec{v}; N, \vec{N}, W)$$

$$= -\sqrt{-1}\, A_{ab} \wedge v^a \wedge \omega^b + \sqrt{-1}\, A_{ab} \wedge L_{\vec{N}} \omega^a \wedge \omega^b$$

$$+ \frac{1}{2} NF_{ab} \wedge *(\omega^a \wedge \omega^b).$$

[Note: The $\omega^a{}_b$ are the connection 1-forms of the metric connection ∇^q associated with q and, of course, the Hodge star is taken per q.]

To initiate the transition from $T\underline{Q}$ to $T^*\underline{Q}$, the usual procedure at this point would be to calculate the functional derivative

$$\frac{\delta L^{\pm}}{\delta \vec{v}}.$$

While possible, this is not totally straightforward and introduces certain technical complications which ultimately are irrelevant. Therefore it will be best to simply sidestep the issue and proceed directly to $T^*\underline{Q}$, where one can take advantage of its underlying symplectic structure.

Section 56: Two Canonical Transformations The assumptions and notation are those of the standard setup but with the restriction that $n = 4$.

Rappel:

$$H(\vec{\omega},\vec{p};N,\vec{N},W)$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge p_a + \int_\Sigma W^a{}_b \omega^b \wedge p_a + \int_\Sigma NE,$$

where

$$E(\vec{\omega},\vec{p}) = \frac{1}{2}\,[q(p_a,*\omega^b)q(p_b,*\omega^a) - \frac{p^2}{2} - S(q)]\mathrm{vol}_q.$$

Let $\underline{Q}_{\underline{C}}$ be the set of ordered complex coframes on Σ -- then each $\vec{\omega}\in\underline{Q}_{\underline{C}}$ gives rise to a complex metric q, viz.

$$q = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$$

and we write

$$\mathrm{vol}_q = \omega^1\wedge\omega^2\wedge\omega^3.$$

Put

$$T^*\underline{Q}_{\underline{C}} = \underline{Q}_{\underline{C}} \times \Lambda^2(\Sigma;T^0_1(\Sigma) \otimes \underline{C}).$$

[Note: Elements of $T^*\underline{Q}_{\underline{C}}$ are again denoted by $(\vec{\omega},\vec{p})$.]

Then the hamiltonian of complex general relativity is the function H above formally extended to $T^*\underline{Q}_{\underline{C}}$ by allowing $(\vec{\omega},\vec{p})$ to be complex.

Remark: The external variables $N,\vec{N},W^a{}_b$ are, at the beginning, real. However, in the formalities to follow, one can allow them to be complex. This does not change the earlier theory, which goes through unaltered. Still, at the end of the day, we shall return to the path $t \rightarrow (\vec{\omega}_t,\vec{p}_t)$ in the ADM sector of $T^*\underline{Q}$ and, of course, in this situation, the external variables $N_t, \vec{N}_t, \bar{Q}^a{}_b$ are real.

Define

$$T: T^*\underline{Q}_{\underline{C}} \to T^*\underline{Q}_{\underline{C}}$$

by

$$T(\vec{\omega},\vec{p}) = (\vec{\omega},\vec{p} - \sqrt{-1}\, d\vec{\omega}).$$

Then T is bijective.

[Note: Explicitly,

$$T^{-1}: T^*\underline{Q}_{\underline{C}} \to T^*\underline{Q}_{\underline{C}}$$

is given by

$$T^{-1}(\vec{\omega},\vec{p}) = (\vec{\omega},\vec{p} + \sqrt{-1}\, d\vec{\omega}).]$$

<u>LEMMA</u> T is a canonical transformation.

[It is a question of verifying that

$$\Omega(DT(\vec{\omega},\vec{p})(\alpha,\beta), DT(\vec{\omega},\vec{p})(\alpha',\beta')) = \Omega((\alpha,\beta),(\alpha',\beta'))$$

for all

$$\begin{cases} \alpha,\alpha' \in \Lambda^1(\Sigma; T_0^1(\Sigma) \otimes \underline{C}) \\ \beta,\beta' \in \Lambda^2(\Sigma; T_1^0(\Sigma) \otimes \underline{C}). \end{cases}$$

From the definitions

$$\begin{cases} DT(\vec{\omega},\vec{p})(\alpha,\beta) = (\alpha,\beta - \sqrt{-1}\, d\alpha) \\ DT(\vec{\omega},\vec{p})(\alpha',\beta') = (\alpha',\beta' - \sqrt{-1}\, d\alpha'). \end{cases}$$

And

$$\Omega((\alpha, \beta - \sqrt{-1}\, d\alpha),\ (\alpha', \beta' - \sqrt{-1}\, d\alpha'))$$

$$= \int_\Sigma (\alpha \wedge (\beta' - \sqrt{-1}\, d\alpha') - \alpha' \wedge (\beta - \sqrt{-1}\, d\alpha))$$

$$= \int_\Sigma (\alpha \wedge \beta' - \alpha' \wedge \beta)$$

$$+ \sqrt{-1} \int_\Sigma (\alpha' \wedge d\alpha - \alpha \wedge d\alpha')$$

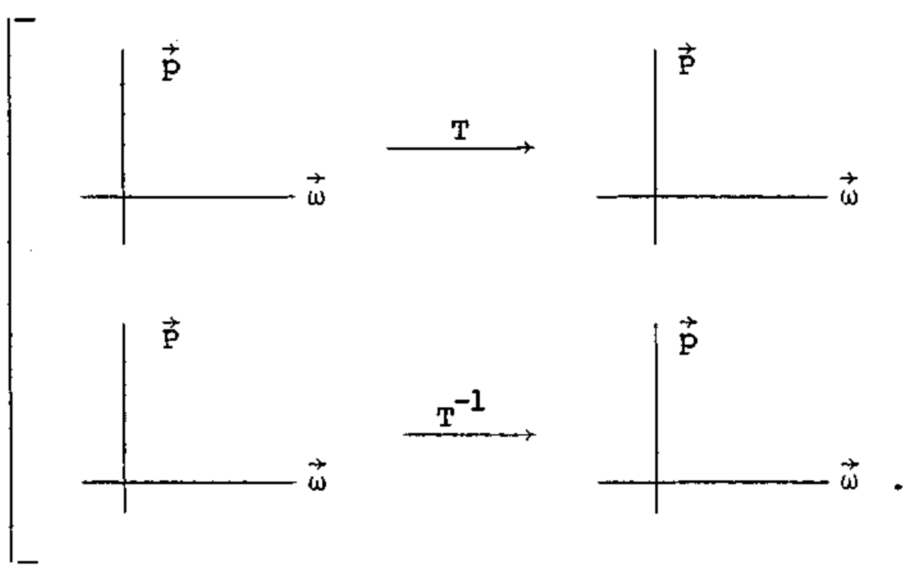
$$= \Omega((\alpha,\beta),\ (\alpha',\beta')) + \sqrt{-1} \int_\Sigma d(\alpha \wedge \alpha')$$

$$= \Omega((\alpha,\beta),\ (\alpha',\beta')).]$$

Let

$$\vec{P} = \vec{p} - \sqrt{-1}\, d\vec{\omega}.$$

So, schematically,



With this in mind, put

$$H_T = H \circ T^{-1}.$$

Then

$$H_T(\vec{\omega},\vec{P}) = H(\vec{\omega},\vec{P} + \sqrt{-1}\, d\vec{\omega})$$

and we shall now examine each of the terms figuring in the RHS.

The first of these is

$$\int_\Sigma L_{\vec{N}}\omega^a \wedge P_a + \sqrt{-1} \int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a.$$

Claim:

$$\int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a = 0.$$

[In fact,

$$d(L_{\vec{N}}\omega^a \wedge \omega^a) = dL_{\vec{N}}\omega^a \wedge \omega^a - L_{\vec{N}}\omega^a \wedge d\omega^a$$

=>

$$0 = \int_\Sigma d(L_{\vec{N}}\omega^a \wedge \omega^a) = \int_\Sigma dL_{\vec{N}}\omega^a \wedge \omega^a - \int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a$$

=>

$$\int_\Sigma dL_{\vec{N}}\omega^a \wedge \omega^a = \int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a$$

=>

$$\int_\Sigma L_{\vec{N}} d\omega^a \wedge \omega^a = \int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a.$$

But

$$0 = \int_\Sigma L_{\vec{N}}(\omega^a \wedge d\omega^a)$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a + \int_\Sigma \omega^a \wedge L_{\vec{N}} d\omega^a$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a + \int_\Sigma L_{\vec{N}} d\omega^a \wedge \omega^a.$$

Therefore

$$2 \int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a = 0$$

=>

$$\int_\Sigma L_{\vec{N}}\omega^a \wedge d\omega^a = 0,$$

as claimed.]

The second term is

$$\int_\Sigma W^a{}_b \omega^b \wedge (P_a + \sqrt{-1}\, d\omega_a),$$

which will be left as is.

It remains to consider

$$E(\vec{\omega}, \vec{P} + \sqrt{-1}\, d\vec{\omega}).$$

To begin with

$$q(P_a + \sqrt{-1}\, d\omega_a, *\omega^b) q(P_b + \sqrt{-1}\, d\omega_b, *\omega^a)$$

$$= q(P_a, *\omega^b) q(P_b, *\omega^a)$$

$$+ q(P_a, *\omega^b) q(\sqrt{-1}\, d\omega_b, *\omega^a) + q(P_b, *\omega^a) q(\sqrt{-1}\, d\omega_a, *\omega^b)$$

$$- q(d\omega_a, *\omega^b) q(d\omega_b, *\omega^a)$$

$$= q(P_a, *\omega^b) q(P_b, *\omega^a)$$

$$+ 2\sqrt{-1}\; q(P_a, *\omega^b) q(d\omega^b, *\omega^a)$$

$$- q(d\omega^a, *\omega^b) q(d\omega^b, *\omega^a).$$

Next

$$-\frac{1}{2} q(P_a + \sqrt{-1}\; d\omega_a, *\omega^a)^2$$

$$= -\frac{1}{2} q(P_a + \sqrt{-1}\; d\omega_a, *\omega^a) q(P_b + \sqrt{-1}\; d\omega_b, *\omega^b)$$

$$= -\frac{1}{2} q(P_a, *\omega^a) q(P_b, *\omega^b)$$

$$-\frac{\sqrt{-1}}{2} [q(P_a, *\omega^a) q(d\omega^b, *\omega^b) + q(P_b, *\omega^b) q(d\omega^a, *\omega^a)]$$

$$-\frac{1}{2} (\sqrt{-1})^2 q(d\omega^a, *\omega^a) q(d\omega^b, *\omega^b)$$

$$= -\frac{P^2}{2} - \sqrt{-1}\; P q(d\omega^a, *\omega^a)$$

$$+ \frac{1}{2} q(d\omega^a, *\omega^a) q(d\omega^b, *\omega^b),$$

where

$$P = q(P_a, *\omega^a).$$

Rappel: We have

$$S(q)\,\mathrm{vol}_q = -2d(\omega^a \wedge * d\omega^a)$$

$$+ \frac{1}{2} (d\omega^a \wedge \omega^a) \wedge * (d\omega^b \wedge \omega^b) - (d\omega^a \wedge \omega^b) \wedge * (d\omega^b \wedge \omega^a).$$

Claim:

1. The sum of

$$- q(d\omega^a, *\omega^b) q(d\omega^b, *\omega^a) \text{vol}_q$$

and

$$(d\omega^a \wedge \omega^b) \wedge * (d\omega^b \wedge \omega^a)$$

is zero.

2. The sum of

$$\frac{1}{2} q(d\omega^a, *\omega^a) q(d\omega^b, *\omega^b) \text{vol}_q$$

and

$$- \frac{1}{2} (d\omega^a \wedge \omega^a) \wedge * (d\omega^b \wedge \omega^b)$$

is zero.

[Consider, e.g., 1. Write

$$\begin{bmatrix} d\omega^a \wedge \omega^b = q(d\omega^a \wedge \omega^b, \text{vol}_q) \text{vol}_q \\ d\omega^b \wedge \omega^a = q(d\omega^b \wedge \omega^a, \text{vol}_q) \text{vol}_q. \end{bmatrix}$$

Then

$$(d\omega^a \wedge \omega^b) \wedge * (d\omega^b \wedge \omega^a)$$

$$= q(d\omega^a \wedge \omega^b, d\omega^b \wedge \omega^a) \text{vol}_q$$

$$= q(d\omega^a \wedge \omega^b, \text{vol}_q) q(d\omega^b \wedge \omega^a, \text{vol}_q) \text{vol}_q.$$

On the other hand,

$$q(d\omega^a,*\omega^b)q(d\omega^b,*\omega^a)\text{vol}_q$$

$$= q(d\omega^a,\iota_{\omega^b}\text{vol}_q)q(d\omega^b,\iota_{\omega^a}\text{vol}_q)\text{vol}_q$$

$$= q(\omega^b\wedge d\omega^a,\text{vol}_q)q(\omega^a\wedge d\omega^b,\text{vol}_q)\text{vol}_q$$

$$= q(d\omega^a\wedge\omega^b,\text{vol}_q)q(d\omega^b\wedge\omega^a,\text{vol}_q)\text{vol}_q.]$$

Summary: We have

$$H_T(\vec{\omega},\vec{P};N,\vec{N},W)$$

$$= \int_\Sigma L_{\vec{N}}\omega^a\wedge P_a + \int_\Sigma W^a{}_b\omega^b\wedge(P_a + \sqrt{-1}\, d\omega_a)$$

$$+ \int_\Sigma Nd(\omega^a\wedge *d\omega^a)$$

$$+ \int_\Sigma \frac{N}{2}\, [q(P_a,*\omega^b)q(P_b,*\omega^a)$$

$$+ 2\sqrt{-1}\, q(P_a,*\omega^b)q(d\omega^b,*\omega^a) - \frac{P^2}{2} - \sqrt{-1}\, Pq(d\omega^a,*\omega^a)]\text{vol}_q.$$

[Note:

$$d(N\wedge\omega^a\wedge *d\omega^a) = dN\wedge\omega^a\wedge *d\omega^a + Nd(\omega^a\wedge *d\omega^a)$$

=>

$$\int_\Sigma Nd(\omega^a\wedge *d\omega^a) = -\int_\Sigma dN\wedge\omega^a\wedge *d\omega^a$$

$$= -\int_\Sigma q(dN,\omega^c)\omega^c\wedge\omega^a\wedge *d\omega^a$$

$$= - \int_\Sigma q(dN,\omega^c)q(\omega^c\wedge\omega^a, d\omega^a)\mathrm{vol}_q.]$$

N.B. Write the constraint equations and the evolution equations in terms of H_T. Suppose that they are satisfied by the pair $(\vec{\omega}_t,\vec{P}_t)$ -- then $\mathrm{Ein}(g) = 0$.

At first glance, it appears that little has been gained by the foregoing procedure. However, the next step is to follow the canonical transformation $(\vec{\omega},\vec{p}) \rightarrow (\vec{\omega},\vec{P})$ by yet another and then the situation will simplify considerably.

Given $(\vec{\omega},\vec{P})$, let

$$A_{ab} = - \sqrt{-1}\ [q(P_c,\omega^a\wedge\omega^b)\omega^c - \frac{P}{2} *(\omega^a\wedge\omega^b)],$$

where

$$P = q(P_c,*\omega^c).$$

Reality Check On $(\vec{\omega}_t,\vec{P}_t)$, this definition of A_{ab} agrees with the one used in the last section, viz. (choosing the plus sign)

$$\bar{\omega}_{ab} + \sqrt{-1}\ \varepsilon_{abc}\bar{\omega}_{0c}.$$

Thus start by writing

$$- \sqrt{-1}\ [q_t(P_c,\bar{\omega}^a\wedge\bar{\omega}^b)\bar{\omega}^c - \frac{P}{2} *(\bar{\omega}^a\wedge\bar{\omega}^b)]$$

$$= - \sqrt{-1}\ [q_t(p_c - \sqrt{-1}\ d\bar{\omega}_c,\bar{\omega}^a\wedge\bar{\omega}^b)\bar{\omega}^c$$

$$- \frac{1}{2} q_t(p_c - \sqrt{-1}\ d\bar{\omega}_c,*\bar{\omega}^c) *(\bar{\omega}^a\wedge\bar{\omega}^b)]$$

$$= - \sqrt{-1}\ [q_t(p_c,\bar{\omega}^a\wedge\bar{\omega}^b)\bar{\omega}^c - \sqrt{-1}\ q_t(d\bar{\omega}_c,\bar{\omega}^a\wedge\bar{\omega}^b)\bar{\omega}^c$$

$$- \frac{1}{2} q_t(p_c,*\bar{\omega}^c) *(\bar{\omega}^a\wedge\bar{\omega}^b) + \frac{\sqrt{-1}}{2} q_t(d\bar{\omega}_c,*\bar{\omega}^c) *(\bar{\omega}^a\wedge\bar{\omega}^b)]$$

$$= - q_t(d\bar{\omega}_c, \bar{\omega}^a \wedge \bar{\omega}^b)\bar{\omega}^c + \frac{1}{2} q_t(d\bar{\omega}_c, *\bar{\omega}^c) * (\bar{\omega}^a \wedge \bar{\omega}^b)$$

$$- \sqrt{-1}\, q_t(p_c, \bar{\omega}^a \wedge \bar{\omega}^b)\bar{\omega}^c + \frac{\sqrt{-1}}{2} q_t(p_c, *\bar{\omega}^c) * (\bar{\omega}^a \wedge \bar{\omega}^b).$$

1. The $\bar{\omega}^a{}_b$ are the connection 1-forms per the metric connection $\bar{\nabla}$ $(= \nabla^{q_t})$, hence

$$\bar{\omega}_{ab} = \frac{1}{2} \left(q_t(d\bar{\omega}^a, \bar{\omega}^b \wedge \bar{\omega}^c)\bar{\omega}^c - q_t(d\bar{\omega}^b, \bar{\omega}^a \wedge \bar{\omega}^c)\bar{\omega}^c\right.$$

$$\left. - q_t(d\bar{\omega}_c, \bar{\omega}^a \wedge \bar{\omega}^b)\bar{\omega}^c\right)$$

=>

$$- q_t(d\bar{\omega}_c, \bar{\omega}^a \wedge \bar{\omega}^b)\bar{\omega}^c$$

$$= 2\bar{\omega}_{ab} - q_t(d\bar{\omega}^a, \bar{\omega}^b \wedge \bar{\omega}^c)\bar{\omega}^c + q_t(d\bar{\omega}^b, \bar{\omega}^a \wedge \bar{\omega}^c)\bar{\omega}^c.$$

On the other hand,

$$\bar{\omega}_{ab} = \iota_{E_b} d\bar{\omega}^a - \iota_{E_a} d\bar{\omega}^b - \frac{1}{2} \iota_{E_b}\iota_{E_a}(d\bar{\omega}_c \wedge \bar{\omega}^c).$$

•

$$\left[\begin{array}{l} \iota_{E_b} d\bar{\omega}^a = C^a{}_{bc}\bar{\omega}^c = q_t(d\bar{\omega}^a, \bar{\omega}^b \wedge \bar{\omega}^c)\bar{\omega}^c \\ \iota_{E_a} d\bar{\omega}^b = C^b{}_{ac}\bar{\omega}^c = q_t(d\bar{\omega}^b, \bar{\omega}^a \wedge \bar{\omega}^c)\bar{\omega}^c. \end{array}\right.$$

• $q_t(d\bar{\omega}_c, *\bar{\omega}^c)\,\mathrm{vol}_{q_t}$

$$= d\bar{\omega}_c \wedge **\bar{\omega}^c$$

$$= d\bar{\omega}_c \wedge (-1)^{1(3-1)} \bar{\omega}^c$$

$$= d\bar{\omega}_c \wedge \bar{\omega}^c$$

=>

$$\iota_{E_b} \iota_{E_a} (d\bar{\omega}_c \wedge \bar{\omega}^c)$$

$$= q_t(d\bar{\omega}_c, *\bar{\omega}^c)\, \iota_{E_b} \iota_{E_a} \mathrm{vol}_{q_t}$$

$$= q_t(d\bar{\omega}_c, *\bar{\omega}^c)\, \iota_{\bar{\omega}^b} \iota_{\bar{\omega}^a} \mathrm{vol}_{q_t}$$

$$= q_t(d\bar{\omega}_c, *\bar{\omega}^c)\, \iota_{\bar{\omega}^a \wedge \bar{\omega}^b} \mathrm{vol}_{q_t}$$

$$= q_t(d\bar{\omega}_c, *\bar{\omega}^c) * (\bar{\omega}^a \wedge \bar{\omega}^b).$$

Therefore

$$- q_t(d\bar{\omega}_c, \bar{\omega}^a \wedge \bar{\omega}^b)\bar{\omega}^c + \frac{1}{2} q_t(d\bar{\omega}_c, *\bar{\omega}^c) * (\bar{\omega}^a \wedge \bar{\omega}^b)$$

$$= 2\bar{\omega}_{ab} - q_t(d\bar{\omega}^a, \bar{\omega}^b \wedge \bar{\omega}^c)\bar{\omega}^c + q_t(d\bar{\omega}^b, \bar{\omega}^a \wedge \bar{\omega}^c)\bar{\omega}^c$$

$$- \bar{\omega}_{ab} + q_t(d\bar{\omega}^a, \bar{\omega}^b \wedge \bar{\omega}^c)\bar{\omega}^c - q_t(d\bar{\omega}^b, \bar{\omega}^a \wedge \bar{\omega}^c)\bar{\omega}^c$$

$$= \bar{\omega}_{ab}.$$

2. We have

$$\sqrt{-1}\ \varepsilon_{abc}\bar{\omega}_{0c} = -\sqrt{-1}\ \varepsilon_{abc} q_t(p_d, *\bar{\omega}^c)\bar{\omega}^d + \frac{\sqrt{-1}}{2} q_t(p_d, *\bar{\omega}^d)\varepsilon_{abc}\bar{\omega}^c$$

$$= -\sqrt{-1}\,\varepsilon_{abc}q_t(p_d,*\bar{\omega}^c)\bar{\omega}^d + \frac{\sqrt{-1}}{2}\,q_t(p_c,*\bar{\omega}^c)*(\bar{\omega}^a\wedge\bar{\omega}^b).$$

And

$$\varepsilon_{abc}q_t(p_d,*\bar{\omega}^c)\bar{\omega}^d$$

$$= \varepsilon_{abc}q_t(p_d,\ \frac{1}{2}\,\varepsilon_{cuv}\bar{\omega}^u\wedge\bar{\omega}^v)\bar{\omega}^d$$

$$= \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{cuv}q_t(p_d,\bar{\omega}^u\wedge\bar{\omega}^v)\bar{\omega}^d$$

$$= \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{uvc}q_t(p_d,\bar{\omega}^u\wedge\bar{\omega}^v)\bar{\omega}^d$$

$$= \frac{1}{2}\,\delta^{ab}{}_{uv}q_t(p_c,\bar{\omega}^u\wedge\bar{\omega}^v)\bar{\omega}^c$$

$$= \frac{1}{2}\,(\delta^a{}_u\delta^b{}_v - \delta^a{}_v\delta^b{}_u)q_t(p_c,\bar{\omega}^u\wedge\bar{\omega}^v)\bar{\omega}^c$$

$$= \frac{1}{2}\,q_t(p_c,\bar{\omega}^a\wedge\bar{\omega}^b)\bar{\omega}^c - \frac{1}{2}\,q_t(p_c,\bar{\omega}^b\wedge\bar{\omega}^a)\bar{\omega}^c$$

$$= q_t(p_c,\bar{\omega}^a\wedge\bar{\omega}^b)\bar{\omega}^c.$$

Put

$$A_c = \frac{\sqrt{-1}}{2}\,\varepsilon_{cuv}A_{uv}.$$

Then

$$A_{ab} = -\sqrt{-1}\,\varepsilon_{abc}A_c.$$

Indeed

$$- \sqrt{-1}\ \varepsilon_{abc} A_c = - \sqrt{-1}\ \varepsilon_{abc} \left(\frac{\sqrt{-1}}{2}\ \varepsilon_{cuv} \right) A_{uv}$$

$$= \frac{1}{2}\ \varepsilon_{abc} \varepsilon_{uvc} A_{uv}$$

$$= \frac{1}{2}\ \delta^{ab}{}_{uv} A_{uv}$$

$$= \frac{1}{2}\ (A_{ab} - A_{ba})$$

$$= A_{ab}.$$

<u>LEMMA</u> We have

$$\left[\begin{array}{l} A_a = q(P_b, *\omega_a)\omega^b - \frac{P}{2}\ \omega_a \\ \\ P_a = A_b \wedge *(\omega^b \wedge \omega_a). \end{array}\right.$$

[<u>Re A_a</u>: Write

$$A_a = \frac{\sqrt{-1}}{2}\ \varepsilon_{abc} A_{bc}$$

$$= \frac{\sqrt{-1}}{2}\ \varepsilon_{abc}\ [\ - \sqrt{-1}\ q(P_d, \omega^b \wedge \omega^c)\omega^d + \frac{\sqrt{-1}}{2}\ P*(\omega^b \wedge \omega^c)]$$

$$= q(P_d, \frac{1}{2}\ \varepsilon_{abc} \omega^b \wedge \omega^c)\omega^d - (\frac{P}{2})\ \frac{1}{2}\ \varepsilon_{abc} *(\omega^b \wedge \omega^c)$$

$$= q(P_b, *\omega_a)\omega^b - (\frac{P}{2})\ \frac{1}{2}\ \varepsilon_{abc} *(\omega^b \wedge \omega^c).$$

Then

$$\frac{1}{2}\,\varepsilon_{abc} * (\omega^b \wedge \omega^c) = \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{bcd}\omega^d$$

$$= \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{dbc}\omega^d$$

$$= \frac{1}{2}\,(2\delta^a{}_d\omega^d)$$

$$= \omega_a.$$

<u>Re P_a:</u> First

$$\iota_{\omega^b}(A_b \wedge *\omega^a) = \iota_{\omega^b}A_b \wedge *\omega^a - A_b \wedge \iota_{\omega^b} *\omega^a$$

$$= \iota_{\omega^b}A_b \wedge *\omega^a - A_b \wedge *(\omega^a \wedge \omega^b)$$

$$= \iota_{\omega^b}A_b \wedge *\omega^a + A_b \wedge *(\omega^b \wedge \omega^a)$$

=>

$$A_b \wedge *(\omega^b \wedge \omega^a) = \iota_{\omega^b}(A_b \wedge *\omega^a) - \iota_{\omega^b}A_b \wedge *\omega^a$$

$$= \iota_{\omega^b}q(A_b, \omega^a)\,\mathrm{vol}_q - q(A_b, \omega^b) *\omega^a$$

$$= q(A_b, \omega^a) *\omega^b - q(A_b, \omega^b) *\omega^a.$$

But

$$A_b = q(P_c, *\omega_b)\,\omega^c - \frac{P}{2}\,\omega_b$$

=>

$$\begin{cases} q(A_b, \omega^a) = q(P_a, *\omega_b) - \frac{P}{2}\,\delta^a{}_b \\ q(A_b, \omega^b) = q(P_b, *\omega_b) - (\frac{3}{2}\,P) \end{cases}$$

=>

$$A_b \wedge *(\omega^b \wedge \omega_a)$$

$$= q(P_a, *\omega_b) *\omega^b - (\frac{P}{2}\,\delta^a{}_b) *\omega^b$$

$$- q(P_b, *\omega_b) *\omega_a + (\frac{3}{2}\,P) *\omega_a$$

$$= P_a + (-\frac{1}{2}\,P - P + \frac{3}{2}\,P) *\omega_a$$

$$= P_a.]$$

Notation: Given $(\vec{\omega}, \vec{P})$, let

$$\begin{cases} Q^a = -\,*\omega^a \\ A_a = q(P_b, *\omega_a)\,\omega^b - \frac{P}{2}\,\omega_a \end{cases}$$

and put

$$\begin{cases} \vec{Q} = (Q^1, Q^2, Q^3) \\ \vec{A} = (A_1, A_2, A_3). \end{cases}$$

Set

$$T^* *\underline{Q}_{\underline{C}} = *\underline{Q}_{\underline{C}} \times \Lambda^1(\Sigma; T^0_1(\Sigma) \otimes \underline{C})$$

and equip it with the evident symplectic structure.

Define

$$S:T^*\underline{\underline{Q}}_{\underline{C}} \to T^*{}_*\underline{\underline{Q}}_{\underline{C}}$$

by

$$S(\vec{\omega},\vec{P}) = (\vec{Q},\vec{A}).$$

<u>SUBLEMMA</u> S is bijective.

[It is obvious that S is injective. To establish that S is surjective, fix $(\vec{Q},\vec{\alpha})\in T^*{}_*\underline{\underline{Q}}_{\underline{C}}$ and let

$$P_a = \alpha_b\wedge *(\omega^b\wedge\omega_a).$$

Then we claim that

$$S(\vec{\omega},\vec{P}) = (\vec{Q},\vec{\alpha}).$$

To see this, consider

$$q(\alpha_c\wedge *(\omega^c\wedge\omega_b),*\omega_a)\omega^b - \frac{P}{2}\,\omega_a.$$

- $\alpha_c\wedge *(\omega^c\wedge\omega_b)$

$$= q(\alpha_c,\omega_b)*\omega^c - q(\alpha_c,\omega^c)*\omega_b$$

=>

$$q(\alpha_c\wedge *(\omega^c\wedge\omega_b),*\omega_a)\omega^b$$

$$= q(\alpha_c,\omega_b)q(*\omega^c,*\omega_a)\omega^b - q(\alpha_c,\omega^c)q(*\omega_b,*\omega^a)\omega^b$$

$$= q(\alpha_a,\omega^b)\omega^b - q(\alpha_c,\omega^c)\omega^a.$$

$$\bullet - \frac{P}{2}\,\omega_a$$

$$= -\frac{1}{2}\,q(P_c, *\omega^c)\,\omega_a$$

$$= -\frac{1}{2}\,q(\alpha_d \wedge *(\omega^d \wedge \omega_c), *\omega^c)\,\omega_a$$

$$= -\frac{1}{2}\,q(q(\alpha_d, \omega_c) *\omega^d - q(\alpha_d, \omega^d) *\omega_c, *\omega^c)\,\omega_a$$

$$= -\frac{1}{2}\,q(\alpha_d, \omega^d)\,\omega_a + \frac{3}{2}\,q(\alpha_d, \omega^d)\,\omega_a$$

$$= q(\alpha_c, \omega^c)\,\omega_a.$$

Therefore

$$q(\alpha_c \wedge *(\omega^c \wedge \omega_b), *\omega_a)\,\omega^b - \frac{P}{2}\,\omega_a$$

$$= q(\alpha_a, \omega^b)\,\omega^b - q(\alpha_c, \omega^c)\,\omega^a + q(\alpha_c, \omega^c)\,\omega_a$$

$$= q(\alpha_a, \omega^b)\,\omega^b$$

$$= \alpha_a.]$$

<u>LEMMA</u> S is a canonical transformation.

It suffices to show that

$$\{\textstyle\int_\Sigma Q^a \wedge \alpha_a,\ \int_\Sigma A_b \wedge \beta^b\} = \int_\Sigma \alpha_c \wedge \beta^c$$

for all

$$\begin{cases} \alpha \in \Lambda^1(\Sigma; T^0_1(\Sigma) \otimes \underline{C}) \\ \beta \in \Lambda^2(\Sigma; T^1_0(\Sigma) \otimes \underline{C}). \end{cases}$$

Here the Poisson bracket on the left equals

$$\int_\Sigma \left[\frac{\delta}{\delta \vec{P}} \left(\int_\Sigma A_b \wedge \beta^b\right) \wedge \frac{\delta}{\delta \vec{\omega}} \left(\int_\Sigma Q^a \wedge \alpha_a\right) - \frac{\delta}{\delta \vec{P}} \left(\int_\Sigma Q^a \wedge \alpha_a\right) \wedge \frac{\delta}{\delta \vec{\omega}} \left(\int_\Sigma A_b \wedge \beta^b\right)\right]$$

or still,

$$\int_\Sigma \left[\frac{\delta}{\delta P_c} \left(\int_\Sigma A_b \wedge \beta^b\right) \wedge \frac{\delta}{\delta \omega^c} \left(\int_\Sigma Q^a \wedge \alpha_a\right) - \frac{\delta}{\delta P_c} \left(\int_\Sigma Q^a \wedge \alpha_a\right) \wedge \frac{\delta}{\delta \omega^c} \left(\int_\Sigma A_b \wedge \beta^b\right)\right].$$

But from the definitions, it is clear that

$$\frac{\delta}{\delta P_c} \left(\int_\Sigma Q^a \wedge \alpha_a\right) = 0,$$

which leaves

$$\int_\Sigma \left[\frac{\delta}{\delta P_c} \left(\int_\Sigma A_b \wedge \beta^b\right) \wedge \frac{\delta}{\delta \omega^c} \left(\int_\Sigma Q^a \wedge \alpha_a\right)\right].$$

- $\delta_c(Q^a \wedge \alpha_a)$

$$= \delta_c Q^a \wedge \alpha_a$$

$$= \delta_c(-*\omega^a) \wedge \alpha_a$$

$$= -\delta\omega^c \wedge \iota_{\omega^c} *\omega^a \wedge \alpha_a$$

$$= -\delta\omega^c \wedge *(\omega^a \wedge \omega^c) \wedge \alpha_a$$

=>

$$\frac{\delta}{\delta\omega^c}\left(\int_\Sigma Q^a\wedge\alpha_a\right) = - *(\omega^a\wedge\omega^c)\wedge\alpha_a.$$

- $\delta_c(A_b\wedge\beta^b)$

$$= \delta_c A_b\wedge\beta^b$$

$$= \delta_c(q(P_d, *\omega_b)\omega^d - \frac{P}{2}\,\omega_b)\wedge\beta^b$$

$$= q(\delta P_c, *\omega_b)\omega^c\wedge\beta^b - \frac{1}{2}\,q(\delta P_c, *\omega^c)\omega_b\wedge\beta^b$$

$$= (\iota_{\delta P_c} *\omega_b)\omega^c\wedge\beta^b - \frac{1}{2}\,(\iota_{\delta P_c} *\omega^c)\omega_b\wedge\beta^b$$

$$= *(\omega_b\wedge\delta P_c)\wedge\omega^c\wedge\beta^b - \frac{1}{2} *(\omega^c\wedge\delta P_c)\wedge\omega_b\wedge\beta^b$$

$$= \omega^c\wedge\beta^b\wedge *(\omega_b\wedge\delta P_c) - \frac{1}{2}\,\omega_b\wedge\beta^b\wedge *(\omega^c\wedge\delta P_c)$$

$$= \omega_b\wedge\delta P_c\wedge *(\omega^c\wedge\beta^b) - \frac{1}{2}\,\omega^c\wedge\delta P_c\wedge *(\omega_b\wedge\beta^b)$$

$$= \delta P_c\wedge q(\beta^b, *\omega^c)\omega_b - \frac{1}{2}\,\delta P_c\wedge q(\beta^b, *\omega_b)\omega^c$$

=>

$$\frac{\delta}{\delta P_c}\left(\int_\Sigma A_b\wedge\beta^b\right)$$

$$= q(\beta^b, *\omega^c)\omega_b - \frac{1}{2}\,q(\beta^b, *\omega_b)\omega^c.$$

Matters therefore reduce to consideration of

$$\int_\Sigma (q(\beta^b,*\omega^c)\omega_b - \frac{1}{2}q(\beta^b,*\omega_b)\omega^c)\wedge - *(\omega^a\wedge\omega^c)\wedge\alpha_a$$

or still,

$$\int_\Sigma \alpha_a\wedge*(\omega^c\wedge\omega^a)\wedge\gamma_c,$$

where

$$\gamma_c = \frac{1}{2}q(\beta^b,*\omega_b)\omega^c - q(\beta^b,*\omega^c)\omega_b.$$

To finish, one then has to prove that

$$*(\omega^c\wedge\omega^a)\wedge\gamma_c = \beta^a.$$

On purely algebraic grounds (cf. infra), there are unique complex 1-forms X_c that satisfy the equation

$$*(\omega^c\wedge\omega^a)\wedge X_c = \beta^a.$$

To compute them, begin by wedging both sides with ω^b:

$$\omega^b\wedge*(\omega^c\wedge\omega^a)\wedge X_c = \omega^b\wedge\beta^a.$$

• We have

$$0 = \iota_{\omega^a}(\omega^b\wedge*\omega^c\wedge X_c)$$

$$= \iota_{\omega^a}\omega^b\wedge*\omega^c\wedge X_c - \omega^b\wedge\iota_{\omega^a}(*\omega^c\wedge X_c)$$

$$= \delta_{ab}*\omega^c\wedge X_c - \omega^b\wedge\iota_{\omega^a}*\omega^c\wedge X_c - \omega^b\wedge*\omega^c\wedge\iota_{\omega^a}X_c$$

=>

$$\omega^b \wedge *(\omega^c \wedge \omega^a) \wedge X_c$$

$$= \omega^b \wedge \iota_{\omega^a} * \omega^c \wedge X_c$$

$$= \delta_{ab} * \omega^c \wedge X_c - \iota_{\omega^a} X_c \wedge \omega^b \wedge * \omega^c$$

$$= \delta_{ab} * \omega^c \wedge X_c - (\iota_{\omega^a} X_c) q(\omega^b, \omega^c) \text{vol}_q$$

$$= \delta_{ab} * \omega^c \wedge X_c - (\iota_{\omega^a} X_b) \text{vol}_q$$

$$= \delta_{ab} * \omega^c \wedge X_c - \iota_{\omega^a} (X_b \wedge \text{vol}_q) - X_b \wedge \iota_{\omega^a} \text{vol}_q$$

$$= \delta_{ab} * \omega^c \wedge X_c - X_b \wedge \iota_{\omega^a} \text{vol}_q$$

$$= \delta_{ab} * \omega^c \wedge X_c - X_b \wedge * \omega^a$$

$$= \delta_{ab} * \omega^c \wedge X_c - * \omega^a \wedge X_b.$$

Accordingly,

$$- * \omega^a \wedge X_b + \delta_{ab} * \omega^c \wedge X_c = \omega^b \wedge \beta^a$$

or still,

$$- q(X_b, \omega^a) + \delta_{ab} q(X_c, \omega^c) = q(\omega^b, * \beta^a).$$

Put

$$X = \sum_{c=1}^{3} q(X_c,\omega^c).$$

Then

$$\left[\begin{array}{l} X - q(X_1,\omega^1) = q(\omega^1,*\beta^1) \\ X - q(X_2,\omega^2) = q(\omega^2,*\beta^2) \\ X - q(X_3,\omega^3) = q(\omega^3,*\beta^3) \end{array}\right.$$

=>

$$3X - X = \sum_{c=1}^{3} q(\omega^c,*\beta^c)$$

=>

$$X = \frac{1}{2}\, q(\omega^c,*\beta^c)$$

=>

$$q(X_b,\omega^a) = \frac{1}{2}\,\delta_{ab} q(\omega^c,*\beta^c) - q(\omega^b,*\beta^a).$$

Therefore

$$\begin{aligned} X_c &= q(X_c,\omega^a)\,\omega^a \\ &= \frac{1}{2}\,\delta_{ac} q(\omega^b,*\beta^b)\,\omega^a - q(\omega^c,*\beta^a)\,\omega^a \\ &= \frac{1}{2}\, q(\beta^b,*\omega_b)\,\omega^c - q(\beta^b,*\omega^c)\,\omega^b \\ &= \gamma_c, \end{aligned}$$

which implies that

$$*(\omega^c \wedge \omega^a) \wedge \gamma_c = \beta^a.$$

<u>Details</u> The first thing to note is that by linear algebra, one can assume without loss of generality that

$$\beta^a = 0 \ (a = 1,2,3),$$

the point being to show that the only solution to

$$*(\omega^c \wedge \omega^a) \wedge X_c = 0$$

is the zero solution. This said, consider the system

$$\begin{cases} -\omega^3 \wedge X_2 + \omega^2 \wedge X_3 = 0 \\ \omega^3 \wedge X_1 - \omega^1 \wedge X_3 = 0 \\ -\omega^2 \wedge X_1 + \omega^1 \wedge X_2 = 0. \end{cases}$$

Write

$$\begin{cases} X_1 = X_{11}\omega^1 + X_{12}\omega^2 + X_{13}\omega^3 \\ X_2 = X_{21}\omega^1 + X_{22}\omega^2 + X_{23}\omega^3 \\ X_3 = X_{31}\omega^1 + X_{32}\omega^2 + X_{33}\omega^3. \end{cases}$$

Then

1. $-X_{21}\omega^3 \wedge \omega^1 - X_{22}\omega^3 \wedge \omega^2 + X_{31}\omega^2 \wedge \omega^1 + X_{33}\omega^2 \wedge \omega^3 = 0.$

2. $X_{11}\omega^3 \wedge \omega^1 + X_{12}\omega^3 \wedge \omega^2 - X_{32}\omega^1 \wedge \omega^2 - X_{33}\omega^1 \wedge \omega^3 = 0.$

3. $-X_{11}\omega^2 \wedge \omega^1 - X_{13}\omega^2 \wedge \omega^3 + X_{22}\omega^1 \wedge \omega^2 + X_{23}\omega^1 \wedge \omega^3 = 0.$

So

$$\left[\begin{array}{l} \omega^2\wedge 1 \Rightarrow X_{21} = 0 \\ \\ \omega^3\wedge 1 \Rightarrow X_{31} = 0, \end{array}\right. \left[\begin{array}{l} \omega^1\wedge 2 = 0 \Rightarrow X_{12} = 0 \\ \\ \omega^3\wedge 2 = 0 \Rightarrow X_{32} = 0, \end{array}\right. \left[\begin{array}{l} \omega^1\wedge 3 \Rightarrow X_{13} = 0 \\ \\ \omega^2\wedge 3 \Rightarrow X_{23} = 0. \end{array}\right.$$

Thus

$$\left[\begin{array}{l} X_1 = X_{11}\omega^1 \\ X_2 = X_{22}\omega^2 \\ X_3 = X_{33}\omega^3 \end{array}\right.$$

$\Rightarrow$

$$\left[\begin{array}{l} -\omega^3\wedge X_{22}\omega^2 + \omega^2\wedge X_{33}\omega^3 = 0 \\ \omega^3\wedge X_{11}\omega^1 - \omega^1\wedge X_{33}\omega^3 = 0 \\ -\omega^2\wedge X_{11}\omega^1 + \omega^1\wedge X_{22}\omega^2 = 0 \end{array}\right.$$

$\Rightarrow$

$$\left[\begin{array}{l} X_{22} + X_{33} = 0 \\ X_{11} + X_{33} = 0 \\ X_{11} + X_{22} = 0 \end{array}\right.$$

$\Rightarrow$

$$\left[\begin{array}{l} X_{22} = - X_{33} \\ \\ X_{22} = + X_{33} \end{array}\right.$$

=>

$$\left[\begin{array}{l} X_{22} = 0 \\ \\ X_{33} = 0 \end{array}\right. \quad \Rightarrow X_{11} = 0.$$

Interpretation of $\vec{A}$ Each triple

$$\vec{A} = (A_1, A_2, A_3)$$

determines an $\underline{s\ell}(2,\underline{C})$-valued 1-form on Σ. To explain this precisely, we need some preparation.

Rappel: Let

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$[\sigma_a, \sigma_b] = 2\sqrt{-1}\, \varepsilon_{abc}\sigma_c.$$

Let

$$T_1 = -\frac{1}{2}\sqrt{-1}\,\sigma_1,\ T_2 = -\frac{1}{2}\sqrt{-1}\,\sigma_2,\ T_3 = -\frac{1}{2}\sqrt{-1}\,\sigma_3.$$

Then

$$[T_a,T_b] = -\frac{1}{4}\,2\sqrt{-1}\;\varepsilon_{abc}\sigma_c$$

$$= \varepsilon_{abc}\left(-\frac{1}{2}\sqrt{-1}\right)\sigma_c$$

$$= \varepsilon_{abc}T_c.$$

Thus the set $\{T_1,T_2,T_3\}$ is a basis for $\underline{su}(2)$ (with structure constants ε_{abc}) which is orthonormal per the scalar product

$$< A,B > = -2\,\underline{tr}(AB).$$

Pass now to $\underline{s\ell}(2,\underline{C})$, the complexification of $\underline{su}(2)$. Let $\tau_a = \frac{1}{2}\sigma_a$ -- then the τ_a are a basis for $\underline{s\ell}(2,\underline{C})$ (viewed as a complex Lie algebra), the structure constants being $\sqrt{-1}\;\varepsilon_{abc}$:

$$[\tau_a,\tau_b] = \frac{1}{4}[\sigma_a,\sigma_b]$$

$$= \frac{\sqrt{-1}}{2}\varepsilon_{abc}\sigma_c$$

$$= \sqrt{-1}\;\varepsilon_{abc}\tau_c.$$

Given $\vec{A}$, the combination

$$A_1\tau_1 + A_2\tau_2 + A_3\tau_3$$

is an $\underline{s\ell}(2,\underline{C})$-valued 1-form on Σ, call it $\vec{A}$ again. The force term $\vec{F}$, i.e., the curvature of $\vec{A}$, is an $\underline{s\ell}(2,\underline{C})$-valued 2-form on Σ, viz.

$$\vec{F} = d\vec{A} + \vec{A}\wedge\vec{A}$$

$$= d\vec{A} + \frac{1}{2}[\vec{A},\vec{A}],$$

where

$$[\vec{A},\vec{A}] = [A_a\tau_a, A_b\tau_b]$$

$$= (A_a\wedge A_b)[\tau_a,\tau_b]$$

$$= \sqrt{-1}\,\varepsilon_{abc}(A_a\wedge A_b)\tau_c.$$

Therefore

$$F_c = dA_c + \frac{\sqrt{-1}}{2}\varepsilon_{abc}A_a\wedge A_b,$$

which is in agreement with the earlier definition of $\vec{F}$ as

$$d\vec{A} + \frac{\sqrt{-1}}{2}\vec{A}\overset{\times}{\wedge}\vec{A}.$$

<u>Section 57</u>: <u>Ashtekar's Hamiltonian</u> The assumptions and notation are those of the standard setup but with the restriction that n = 4.

As was established in the last section, there are canonical transformations T and S:

$$T^*\underline{Q}_{\underline{C}} \overset{T}{\to} T^*\underline{Q}_{\underline{C}} \overset{S}{\to} T^*\star\underline{Q}_{\underline{C}}.$$

Consequently,

$$H_{S \circ T} = H \circ (S \circ T)^{-1} = H \circ T^{-1} \circ S^{-1} = H_T \circ S^{-1}.$$

Here

$$H_{S \circ T}(\vec{Q},\vec{A}) = H_T(\vec{\omega},\vec{P}),$$

where

$$P_a = A_b \wedge \star(\omega^b \wedge \omega_a).$$

However, before we trace the effect of this change of variable, it will be best to review and reinforce our notation,

Recall that

$$\begin{cases} Q^a = -\star\omega^a \\ A_a = q(P_b,\star\omega_a)\omega^b - \frac{P}{2}\,\omega_a \end{cases}$$

and

$$\begin{cases} \vec{Q} = (Q^1,Q^2,Q^3) \\ \vec{A} = (A_1,A_2,A_3). \end{cases}$$

Therefore

$$d^A Q^a = dQ^a + A^a{}_b \wedge Q^b$$

$$= dQ^a - \sqrt{-1}\ \epsilon^a{}_{bc} A^c \wedge Q^b$$

=>

$$d^A \vec{Q} = d\vec{Q} + \sqrt{-1}\ \vec{A} \overset{\times}{\wedge} \vec{Q}.$$

Next put

$$\vec{F} = (F_1, F_2, F_3),$$

where

$$F_a = dA_a + \frac{\sqrt{-1}}{2}\,\epsilon_{abc} A_b \wedge A_c.$$

Then

$$\vec{F} = d\vec{A} + \frac{\sqrt{-1}}{2}\ \vec{A} \overset{\times}{\wedge} \vec{A}.$$

Finally let

$$Z = [Z^a{}_b] \quad (Z^a{}_b \in C^\infty(\Sigma;\underline{C}))$$

subject to $Z^a{}_b = -Z^b{}_a$ and write

$$\left[\begin{array}{l} Z_a = \dfrac{\sqrt{-1}}{2}\,\epsilon_{abc} Z_{bc} \\[2ex] \vec{Z} = (Z_1, Z_2, Z_3). \end{array}\right.$$

Remark: There is an issue of consistency present in the definition of $\vec{F}$. Thus a priori,

$$F_a = \frac{\sqrt{-1}}{2}\,\epsilon_{abc} F_{bc}$$

or still,

$$F_a = \frac{\sqrt{-1}}{2}\,\varepsilon_{abc}(dA_{bc} + A_{bd}\wedge A^d{}_c),$$

the implied assumption being that this reduces to

$$dA_a + \frac{\sqrt{-1}}{2}\,\varepsilon_{abc}A_b\wedge A_c.$$

- $\frac{\sqrt{-1}}{2}\,\varepsilon_{abc}dA_{bc}$

$$= d(\frac{\sqrt{-1}}{2}\,\varepsilon_{abc}A_{bc})$$

$$= dA_a.$$

- $\frac{\sqrt{-1}}{2}\,\varepsilon_{abc}A_{bd}\wedge A^d{}_c$

$$= \frac{\sqrt{-1}}{2}\,\varepsilon_{abc}(-\sqrt{-1}\,\varepsilon_{bdu}A_u\wedge -\sqrt{-1}\,\varepsilon_{dcv}A_v)$$

$$= -\frac{\sqrt{-1}}{2}\,\varepsilon_{abc}\varepsilon_{bdu}\varepsilon_{dcv}A_u\wedge A_v$$

$$= \frac{\sqrt{-1}}{2}\,\varepsilon_{abc}\varepsilon_{bud}\varepsilon_{cvd}A_u\wedge A_v$$

$$= \frac{\sqrt{-1}}{2}\,\varepsilon_{abc}\delta^{bu}{}_{cv}A_u\wedge A_v$$

$$= \frac{\sqrt{-1}}{2}\,\varepsilon_{abc}(\delta^b{}_c\delta^u{}_v - \delta^b{}_v\delta^u{}_c)A_u\wedge A_v$$

$$= \frac{\sqrt{-1}}{2}\,\varepsilon_{abc}(\delta^b{}_cA_u\wedge A_u - A_c\wedge A_b)$$

$$= \frac{\sqrt{-1}}{2}\,\epsilon_{abc}A_b \wedge A_c.$$

<u>FACT</u> We have

$$d(\vec{Z} \overset{\bullet}{\wedge} \vec{Q}) = d^A\vec{Z} \overset{\bullet}{\wedge} \vec{Q} + \vec{Z} \overset{\bullet}{\wedge} d^A\vec{Q}.$$

[For

$$d^A\vec{Z} \overset{\bullet}{\wedge} \vec{Q} = d^A Z_a \wedge Q^a$$

$$= (dZ_a - A^b{}_a \wedge Z_b) \wedge Q^a$$

$$= dZ_a \wedge Q^a + A^a{}_b \wedge Z_b \wedge Q^a$$

and

$$\vec{Z} \wedge d^A\vec{Q} = Z_a \wedge d^A Q^a$$

$$= Z_a \wedge (dQ^a + A^a{}_b \wedge Q^b)$$

$$= Z_a \wedge dQ^a + Z_a \wedge A^a{}_b \wedge Q^b$$

=>

$$d^A\vec{Z} \overset{\bullet}{\wedge} \vec{Q} + \vec{Z} \overset{\bullet}{\wedge} d^A\vec{Q}$$

$$= dZ_a \wedge Q^a + Z_a \wedge dQ^a$$

$$+ A^a{}_b \wedge Z_b \wedge Q^a + A^a{}_b \wedge Z_a \wedge Q^b$$

$$= d(Z_a \wedge Q^a)$$

$$+ A^a{}_b \wedge Z_b \wedge Q^a + A^b{}_a \wedge Z_b \wedge Q^a$$

$$= d(\vec{Z} \overset{\bullet}{\wedge} \vec{Q})$$

$$+ A^a{}_b \wedge Z_b \wedge Q^a - A^a{}_b \wedge Z_b \wedge Q^a$$

$$= d(\vec{Z} \overset{\bullet}{\wedge} \vec{Q}).]$$

Rappel:

$$H_T(\vec{\omega},\vec{P};N,\vec{N},W)$$

$$= \int_\Sigma L_{\vec{N}} \omega^a \wedge P_a + \int_\Sigma W^a{}_b \omega^b \wedge (P_a + \sqrt{-1}\, d\omega_a)$$

$$+ \int_\Sigma - q(dN,\omega^c) q(\omega^c \wedge \omega^a, d\omega^a) \mathrm{vol}_q$$

$$+ \int_\Sigma \frac{N}{2} [q(P_a, *\omega^b) q(P_b, *\omega^a)$$

$$+ 2\sqrt{-1}\, q(P_a, *\omega^b) q(d\omega^b, *\omega^a) - \frac{P^2}{2} - \sqrt{-1}\, Pq(d\omega^a, *\omega^a)] \mathrm{vol}_q.$$

We shall now make the change of variable $P_a \to A_b \wedge *(\omega^b \wedge \omega_a)$ in H_T and consider the various terms obtained thereby.

First

$$\int_\Sigma L_{\vec{N}} \omega^a \wedge P_a$$

$$= \int_\Sigma L_{\vec{N}} \omega^a \wedge A_b \wedge *(\omega^b \wedge \omega_a)$$

$$= \int_\Sigma - L_{\vec{N}} \omega^a \wedge *(\omega^b \wedge \omega_a) \wedge A_b$$

$$= \int_\Sigma *(\omega^b \wedge \omega_a) \wedge L_{\vec{N}} \omega^a \wedge A_b,$$

which we claim is equal to

$$\int_\Sigma L_{\vec{N}} \vec{Q} \mathbin{\dot{\wedge}} \vec{A}.$$

To see this, write

$$*\omega^b = \frac{1}{2} \varepsilon_{bcd} \omega^c \wedge \omega^d.$$

Then

$$L_{\vec{N}} *\omega^b = \varepsilon_{bcd} L_{\vec{N}} \omega^c \wedge \omega^d$$

=>

$$L_{\vec{N}} Q^b = - \varepsilon_{bcd} L_{\vec{N}} \omega^c \wedge \omega^d.$$

On the other hand,

$$*(\omega^b \wedge \omega^a) \wedge L_{\vec{N}} \omega^a$$

$$= \varepsilon_{bac} \omega^c \wedge L_{\vec{N}} \omega^a$$

$$= - \varepsilon_{bac} L_{\vec{N}} \omega^a \wedge \omega^c$$

$$= - \varepsilon_{bcd} L_{\vec{N}} \omega^c \wedge \omega^d.$$

Next

$$\int_\Sigma W^a{}_b \omega^b \wedge (P_a + \sqrt{-1}\, d\omega_a)$$

$$= \int_\Sigma W^a{}_b \omega^b \wedge (A_c \wedge *(\omega^c \wedge \omega_a) + \sqrt{-1}\, d\omega_a).$$

Put

$$Z_{ab} = -W_{ab} + \sqrt{-1}\,\varepsilon_{abc}W_c,$$

where

$$W_c = -q(dN,\omega^c).$$

The discussion then breaks into two parts:

1. $\int_\Sigma - Z_{ab}A_c \wedge *(\omega^c \wedge \omega^a) \wedge \omega^b.$

2. $\int_\Sigma - \sqrt{-1}\, Z_{ab} d\omega^a \wedge \omega^b.$

[Note: We shall hold

$$\sqrt{-1}\int_\Sigma \varepsilon_{abc}W_c(P_a + \sqrt{-1}\, d\omega_a) \wedge \omega^b$$

in abeyance for the time being.]

<u>LEMMA</u>

$$1 + 2 = \int_\Sigma \vec{Z} \dot{\wedge} d^A\vec{Q}.$$

[Note: Here, of course,

$$\vec{Z} \dot{\wedge} d^A\vec{Q} = Z_a \wedge d^A Q^a.]$$

• Write

$$- Z_{ab}A_c \wedge *(\omega^c \wedge \omega^a) \wedge \omega^b$$

$$= - Z_{ab}A_c \wedge \varepsilon_{cau}\omega^u \wedge \omega^b$$

$$= - (- \sqrt{-1})\,\varepsilon_{abv} Z_v A_c \wedge \varepsilon_{cau} \omega^u \wedge \omega^b$$

$$= \sqrt{-1}\ \varepsilon_{cau} \varepsilon_{abv} Z_v A_c \wedge \omega^u \wedge \omega^b$$

$$= \sqrt{-1}\ \varepsilon_{cua} \varepsilon_{vba} Z_v A_c \wedge \omega^u \wedge \omega^b$$

$$= \sqrt{-1}\ \delta^{cu}{}_{vb} Z_v A_c \wedge \omega^u \wedge \omega^b$$

$$= \sqrt{-1}\ (\delta^c{}_v \delta^u{}_b - \delta^c{}_b \delta^u{}_v) Z_v A_c \wedge \omega^u \wedge \omega^b$$

$$= \sqrt{-1}\ Z_c A_c \wedge \omega^b \wedge \omega^b - \sqrt{-1}\ Z_u A_b \wedge \omega^u \wedge \omega^b$$

$$= - \sqrt{-1}\ Z_u A_b \wedge \omega^u \wedge \omega^b$$

$$= - \sqrt{-1}\ Z_u A_c \wedge \omega^u \wedge \omega^c$$

$$= \sqrt{-1}\ Z_u A_c \wedge \omega^c \wedge \omega^u$$

$$= \sqrt{-1}\ Z_a A_c \wedge \omega^c \wedge \omega^a.$$

• Write

$$- \sqrt{-1}\ Z_a \varepsilon_{abc} A_c \wedge \Omega^b$$

$$= - \sqrt{-1}\ Z_a \varepsilon_{abc} A_c \wedge \left(- \frac{1}{2}\ \varepsilon_{buv} \omega^u \wedge \omega^v \right)$$

$$= \frac{\sqrt{-1}}{2}\ \varepsilon_{abc} \varepsilon_{buv} Z_a A_c \wedge \omega^u \wedge \omega^v$$

$$= \frac{\sqrt{-1}}{2}\ \varepsilon_{acb} \varepsilon_{vub} Z_a A_c \wedge \omega^u \wedge \omega^v$$

$$= \frac{\sqrt{-1}}{2}\, \delta^{ac}{}_{vu} Z_a A_c \wedge \omega^u \wedge \omega^v$$

$$= \frac{\sqrt{-1}}{2}\, (\delta^a{}_v \delta^c{}_u - \delta^a{}_u \delta^c{}_v) Z_a A_c \wedge \omega^u \wedge \omega^v$$

$$= \frac{\sqrt{-1}}{2}\, (Z_a A_c \wedge \omega^c \wedge \omega^a - Z_a A_c \wedge \omega^a \wedge \omega^c)$$

$$= \frac{\sqrt{-1}}{2}\, (Z_a A_c \wedge \omega^c \wedge \omega^a + Z_a A_c \wedge \omega^c \wedge \omega^a)$$

$$= \sqrt{-1}\, Z_a A_c \wedge \omega^c \wedge \omega^a.$$

Therefore

$$\int_\Sigma - Z_{ab} A_c \wedge * (\omega^c \wedge \omega^a) \wedge \omega^b$$

$$= \int_\Sigma - \sqrt{-1}\, Z_a \varepsilon_{abc} A_c \wedge \Omega^b.$$

As for the other term,

$$- \sqrt{-1}\, Z_{ab} d\omega^a \wedge \omega^b$$

$$= - \sqrt{-1}\, (- \sqrt{-1}) \varepsilon_{abc} Z_c d\omega^a \wedge \omega^b$$

$$= - \varepsilon_{abc} Z_c d\omega^a \wedge \omega^b$$

$$= - \varepsilon_{cba} Z_a d\omega^c \wedge \omega^b$$

$$= - \varepsilon_{bca} Z_a d\omega^b \wedge \omega^c$$

$$= - \varepsilon_{abc} Z_a d\omega^b \wedge \omega^c,$$

which we claim is the same as

$$Z_a dQ^a = - Z_a d{*}\omega^a.$$

Thus write

$$*\omega^a = \frac{1}{2} \varepsilon_{abc} \omega^b \wedge \omega^c.$$

Then

$$d{*}\omega^a = \frac{1}{2} \varepsilon_{abc} (d\omega^b \wedge \omega^c - \omega^b \wedge d\omega^c)$$

$$= \frac{1}{2} \varepsilon_{abc} d\omega^b \wedge \omega^c - \frac{1}{2} \varepsilon_{abc} d\omega^c \wedge \omega^b.$$

But

$$- \frac{1}{2} \varepsilon_{abc} d\omega^c \wedge \omega^b = - \frac{1}{2} \varepsilon_{acb} d\omega^b \wedge \omega^c$$

$$= \frac{1}{2} \varepsilon_{abc} d\omega^b \wedge \omega^c.$$

Therefore

$$d{*}\omega^a = \varepsilon_{abc} d\omega^b \wedge \omega^c$$

and the claim follows.

So, in recapitulation:

$$1 + 2 = \int_\Sigma \vec{Z} \dot{\wedge} d^A \vec{Q}.$$

Remark: The expression

$$- \bar{Q}_{ab} + \sqrt{-1}\, \varepsilon_{abc} \bar{P}_c \quad (\bar{P}_c = - q_t (dN_t, \overline{\omega}^c))$$

appeared earlier during the course of the lagrangian analysis.

LEMMA We have

$$\int_\Sigma \frac{N}{2} [q(P_a,*\omega^b)q(P_b,*\omega^a)$$

$$+ 2\sqrt{-1}\, q(P_a,*\omega^b)q(d\omega^b,*\omega^a) - \frac{P^2}{2} - \sqrt{-1}\, Pq(d\omega^a,*\omega^a)]\mathrm{vol}_q$$

$$= \int_\Sigma -\sqrt{-1}\, N\vec{F} \overset{\bullet}{\wedge} *\vec{Q} + \sqrt{-1}\int_\Sigma q(dN,\omega^a)P_b\wedge*(\omega^a\wedge\omega^b).$$

[From the definitions,

$$-\sqrt{-1}\, N\vec{F} \overset{\bullet}{\wedge} *\vec{Q}$$

$$= -\sqrt{-1}\, NF_a\wedge *Q^a$$

$$= -\sqrt{-1}\, NF_a\wedge *(-*\omega^a)$$

$$= \sqrt{-1}\, NF_a\wedge\omega^a$$

$$= \sqrt{-1}\, N(dA_a + \frac{\sqrt{-1}}{2}\varepsilon_{abc}A_b\wedge A_c)\wedge\omega^a$$

$$= \sqrt{-1}\, NdA_a\wedge\omega^a - \frac{N}{2}\varepsilon_{abc}A_b\wedge A_c\wedge\omega^a.$$

And

$$-\frac{N}{2}\varepsilon_{abc}A_b\wedge A_c\wedge\omega^a$$

$$= -\frac{N}{2}\varepsilon_{abc}(q(P_u,*\omega^b)\omega^u - \frac{P}{2}\omega^b)\wedge(q(P_v,*\omega^c)\omega^v - \frac{P}{2}\omega^c)\wedge\omega^a$$

$$= -\frac{N}{2}\varepsilon_{abc}[q(P_u,*\omega^b)q(P_v,*\omega^c)\omega^u\wedge\omega^v$$

$$-\frac{P}{2}\,q(P_u,*\omega^b)\,\omega^u\wedge\omega^c - \frac{P}{2}\,q(P_v,*\omega^c)\,\omega^b\wedge\omega^v$$

$$+\frac{P^2}{4}\,\omega^b\wedge\omega^c]\wedge\omega^a.$$

- $-\varepsilon_{abc}q(P_u,*\omega^b)q(P_v,*\omega^c)\,\omega^u\wedge\omega^v\wedge\omega^a$

$$= -q(P_u,*\omega^b)q(P_v,*\omega^c)\,\omega^u\wedge\omega^v\wedge\varepsilon_{bca}\omega^a$$

$$= -q(P_u,*\omega^b)q(P_v,*\omega^c)\,\omega^u\wedge\omega^v\wedge*(\omega^b\wedge\omega^c)$$

$$= -q(P_u,*\omega^b)q(P_v,*\omega^c)\,q(\omega^u\wedge\omega^v,\omega^b\wedge\omega^c)\,\mathrm{vol}_q$$

$$= -q(P_u,*\omega^b)q(P_v,*\omega^c)\,q(\iota_{\omega_b}(\omega^u\wedge\omega^v),\omega_c)\,\mathrm{vol}_q$$

$$= -q(P_u,*\omega^b)q(P_v,*\omega^c)\,q(\delta^u{}_b\omega^v - \omega^u\delta^v{}_b,\omega_c)\,\mathrm{vol}_q$$

$$= -q(P_u,*\omega^b)q(P_v,*\omega^c)\,(\delta^u{}_b\delta^v{}_c - \delta^v{}_b\delta^u{}_c)\,\mathrm{vol}_q$$

$$= -q(P_u,*\omega^b)q(P_v,*\omega^c)\,\delta^u{}_b\delta^v{}_c\mathrm{vol}_q$$

$$+ q(P_u,*\omega^b)q(P_v,*\omega^c)\,\delta^v{}_b\delta^u{}_c\mathrm{vol}_q$$

$$= (q(P_c,*\omega^b)q(P_b,*\omega^c) - P^2)\,\mathrm{vol}_q$$

$$= (q(P_a,*\omega^b)q(P_b,*\omega^a) - P^2)\,\mathrm{vol}_q.$$

$$\bullet\ \varepsilon_{abc} \frac{P}{2} q(P_u, *\omega^b) \omega^u \wedge \omega^c \wedge \omega^a$$

$$+ \varepsilon_{abc} \frac{P}{2} q(P_v, *\omega^c) \omega^b \wedge \omega^v \wedge \omega^a$$

$$= Pq(P_u, *\omega^b) \omega^u \wedge \frac{1}{2} \varepsilon_{bac} \omega^a \wedge \omega^c$$

$$+ Pq(P_v, *\omega^c) \omega^v \wedge \frac{1}{2} \varepsilon_{cab} \omega^a \wedge \omega^b$$

$$= Pq(P_u, *\omega^b) \omega^u \wedge *\omega^b + Pq(P_v, *\omega^c) \omega^v \wedge *\omega^c$$

$$= Pq(P_u, *\omega^b) \delta^u{}_b \mathrm{vol}_q + Pq(P_v, *\omega^c) \delta^v{}_c \mathrm{vol}_q$$

$$= (P^2 + P^2) \mathrm{vol}_q .$$

$$\bullet\ - \varepsilon_{abc} \frac{P^2}{4} \omega^b \wedge \omega^c \wedge \omega^a$$

$$= - \frac{P^2}{4} \varepsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c$$

$$= - \frac{P^2}{4}\, 6\ \mathrm{vol}_q$$

$$= - \frac{3}{2} P^2 \mathrm{vol}_q .$$

Therefore

$$\int_\Sigma - \frac{N}{2} \varepsilon_{abc} A_b \wedge A_c \wedge \omega^a$$

$$= \int_\Sigma \frac{N}{2}\ [q(P_a, *\omega^b) q(P_b, *\omega^a)\ - \frac{P^2}{2}] \text{vol}_q.$$

Next

$$0 = \int_\Sigma d(NA_a \wedge \omega^a)$$

$$= \int_\Sigma dN \wedge A_a \wedge \omega^a + \int_\Sigma Nd(A_a \wedge \omega^a)$$

$$= \int_\Sigma dN \wedge A_a \wedge \omega^a + \int_\Sigma N(dA_a \wedge \omega^a - A_a \wedge d\omega^a)$$

=>

$$\int_\Sigma NdA_a \wedge \omega^a = \int_\Sigma NA_a \wedge d\omega^a - \int_\Sigma dN \wedge A_a \wedge \omega^a.$$

- $NA_a \wedge d\omega^a$

$$= N(q(P_b, *\omega_a)\omega^b - \frac{P}{2}\ \omega_a) \wedge d\omega^a$$

$$= N(q(P_b, *\omega_a)\omega^b \wedge d\omega^a - \frac{P}{2}\ \omega_a \wedge d\omega^a)$$

$$= N(q(P_b, *\omega_a) q(\omega^b \wedge d\omega^a, \text{vol}_q)$$

$$- \frac{P}{2}\ q(\omega_a \wedge d\omega^a, \text{vol}_q)) \text{vol}_q$$

$$= N(q(P_b, *\omega_a) q(d\omega^a, \iota_{\omega^b} \text{vol}_q)$$

$$- \frac{P}{2}\ q(d\omega^a, \iota_{\omega^a} \text{vol}_q)) \text{vol}_q$$

$$= N(q(P_b, *\omega_a) q(d\omega^a, *\omega^b)\ - \frac{P}{2}\ q(d\omega^a, *\omega^a)) \text{vol}_q$$

$$= N(q(P_a,*\omega^b)q(d\omega^b,*\omega^a) - \frac{P}{2}q(d\omega^a,*\omega^a))\text{vol}_q.$$

• $- dN\wedge A_a\wedge\omega^a$

$$= - q(dN,\omega^c)\omega^c\wedge A_a\wedge\omega^a.$$

But

$$q(dN,\omega^a)P_b\wedge*(\omega^a\wedge\omega^b)$$

$$= q(dN,\omega^a)A_c\wedge*(\omega^c\wedge\omega^b)\wedge*(\omega^a\wedge\omega^b)$$

$$= q(dN,\omega^a)A_c\wedge\varepsilon_{cbu}\omega^u\wedge\varepsilon_{abv}\omega^v$$

$$= q(dN,\omega^a)A_c\wedge\varepsilon_{cbu}\varepsilon_{abv}\omega^u\wedge\omega^v$$

$$= q(dN,\omega^a)A_c\wedge\varepsilon_{cub}\varepsilon_{avb}\omega^u\wedge\omega^v$$

$$= q(dN,\omega^a)A_c\wedge\delta^{cu}{}_{av}\omega^u\wedge\omega^v$$

$$= q(dN,\omega^a)A_c\wedge(\delta^c{}_a\delta^u{}_v - \delta^c{}_v\delta^u{}_a)\omega^u\wedge\omega^v$$

$$= - q(dN,\omega^a)A_c\wedge\delta^c{}_v\delta^u{}_a\omega^u\wedge\omega^v$$

$$= - q(dN,\omega^a)A_c\wedge\omega^a\wedge\omega^c$$

$$= q(dN,\omega^a)\omega^a\wedge A_c\wedge\omega^c$$

$$= q(dN,\omega^c)\omega^c\wedge A_a\wedge\omega^a$$

=>

$$- dN \wedge A_a \wedge \omega^a$$

$$= - q(dN, \omega^a) P_b \wedge *(\omega^a \wedge \omega^b).$$

Therefore

$$\int_\Sigma \sqrt{-1}\, N dA_a \wedge \omega^a$$

$$= \int_\Sigma \frac{N}{2} [2\sqrt{-1}\, q(P_a, *\omega^b) q(d\omega^b, *\omega^a) - \sqrt{-1}\, P q(d\omega^a, *\omega^a)] \mathrm{vol}_q$$

$$- \sqrt{-1} \int_\Sigma q(dN, \omega^a) P_b \wedge *(\omega^a \wedge \omega^b).]$$

Earlier we had set aside

$$\sqrt{-1} \int_\Sigma \varepsilon_{abc} W_c (P_a + \sqrt{-1}\, d\omega_a) \wedge \omega^b,$$

where

$$W_c = - q(dN, \omega^c).$$

Since

$$\sqrt{-1}\, q(dN, \omega^a) P_b \wedge *(\omega^a \wedge \omega^b)$$

$$= \sqrt{-1}\, \varepsilon_{abc} q(dN, \omega^a) P_b \wedge \omega^c$$

$$= \sqrt{-1}\, \varepsilon_{cba} q(dN, \omega^c) P_b \wedge \omega^a$$

$$= \sqrt{-1}\, \varepsilon_{cab} q(dN, \omega^c) P_a \wedge \omega^b$$

$$= \sqrt{-1}\ \varepsilon_{abc} q(dN, \omega^c) P_a \wedge \omega^b,$$

it follows that

$$\sqrt{-1} \int_\Sigma \varepsilon_{abc} W_c P_a \wedge \omega^b$$

cancels with

$$\sqrt{-1} \int_\Sigma q(dN, \omega^a) P_b \wedge * (\omega^a \wedge \omega^b).$$

What remains, viz.

$$- \int_\Sigma \varepsilon_{abc} W_c d\omega^a \wedge \omega^b,$$

cancels with

$$\int_\Sigma - q(dN, \omega^c) q(\omega^c \wedge \omega^a, d\omega^a) \mathrm{vol}_q$$

$$= \int_\Sigma W_c q(\omega^c \wedge \omega^a, d\omega^a) \mathrm{vol}_q.$$

Indeed

$$- \int_\Sigma \varepsilon_{abc} W_c d\omega^a \wedge \omega^b$$

$$= \int_\Sigma W_c d\omega^a \wedge \varepsilon_{acb} \omega^b$$

$$= \int_\Sigma W_c d\omega^a \wedge * (\omega^a \wedge \omega^c)$$

$$= \int_\Sigma W_c q(\omega^a \wedge \omega^c, d\omega^a) \mathrm{vol}_q$$

$$= - \int_\Sigma W_c q(\omega^c \wedge \omega^a, d\omega^a) \mathrm{vol}_q.$$

Definition: The Ashtekar hamiltonian is the function

$$H : T^* * \underline{Q}_{\underline{C}} \to \underline{C}$$

defined by the prescription

$$H(\vec{Q},\vec{A};N,\vec{N},\vec{Z})$$

$$= \int_\Sigma L_{\vec{N}}\vec{Q} \mathbin{\dot{\wedge}} \vec{A} + \int_\Sigma \vec{Z} \mathbin{\dot{\wedge}} d^A\vec{Q} + \int_\Sigma - \sqrt{-1}\, N\vec{F} \mathbin{\dot{\wedge}} *\vec{Q}.$$

The constraints of the theory are encoded in the demand that

$$\frac{\delta H}{\delta N} = 0, \quad \frac{\delta H}{\delta \vec{N}} = 0, \quad \frac{\delta H}{\delta \vec{Z}} = 0.$$

We shall deal with the first and second of these later on. As for the third, it is clear that

$$\frac{\delta H}{\delta \vec{Z}} = d^A\vec{Q}.$$

Rappel: The ADM sector of $T^*\underline{\underline{Q}}_C$ consists of the pairs $(\vec{\omega},\vec{p})$ for which

$$\omega^a \wedge p_b = \omega^b \wedge p_a.$$

The image of the ADM sector of $T^*\underline{\underline{Q}}_C$ under T is the set of pairs $(\vec{\omega},\vec{P})$ such that

$$\omega^b \wedge P_a - \omega^a \wedge P_b + \sqrt{-1}\, d(\omega^a \wedge \omega^b) = 0.$$

E.g.:

$$\omega^a \wedge p_b = \omega^b \wedge p_a$$

=>

$$\omega^b \wedge (p_a - \sqrt{-1}\, d\omega_a) - \omega^a \wedge (p_b - \sqrt{-1}\, d\omega_b) + \sqrt{-1}\, d(\omega^a \wedge \omega^b)$$

$$= \omega^b \wedge p_a - \omega^a \wedge p_b + \sqrt{-1}\, (- d\omega^a \wedge \omega^b + \omega^a \wedge d\omega^b) + \sqrt{-1}\, d(\omega^a \wedge \omega^b)$$

$$= \sqrt{-1}\ (-\, d\omega^a \wedge \omega^b + \omega^a \wedge d\omega^b) + \sqrt{-1}\ (d\omega^a \wedge \omega^b - \omega^a \wedge d\omega^b)$$

$$= 0.$$

The image of the ADM sector of $T^*\underline{\underline{Q}}_C$ under $S \circ T$ is the set of pairs $(\vec{Q},\vec{A})$ such that

$$d^A\vec{Q} = 0.$$

E.g.:

$$\omega^c \wedge P_b - \omega^b \wedge P_c + \sqrt{-1}\ d(\omega^b \wedge \omega^c) = 0$$

$\Rightarrow$

$$d^A Q^a = dQ^a - \sqrt{-1}\ \varepsilon_{abc} A^c \wedge Q^b$$

$$= -\, d{*}\omega^a - \sqrt{-1}\ \varepsilon_{abc}(q(P_d, {*}\omega_c)\omega^d - \frac{P}{2}\,\omega_c) \wedge -\, {*}\omega^b$$

$$= -\ \varepsilon_{abc} d\omega^b \wedge \omega^c$$

$$+ \sqrt{-1}\ \varepsilon_{abc} q(P_d, {*}\omega_c) q(\omega^d, \omega^b) \mathrm{vol}_q$$

$$- \sqrt{-1}\ \varepsilon_{abc}\ \frac{P}{2}\ q(\omega^c, \omega^b) \mathrm{vol}_q$$

$$= -\ \varepsilon_{abc} d\omega^b \wedge \omega^c$$

$$+ \sqrt{-1}\ \varepsilon_{abc} q(P_b, {*}\omega_c) \mathrm{vol}_q - \sqrt{-1}\ \varepsilon_{abb}\ \frac{P}{2}\ \mathrm{vol}_q$$

$$= -\ \varepsilon_{abc} d\omega^b \wedge \omega^c + \sqrt{-1}\ \varepsilon_{abc} q(P_b, {*}\omega_c) \mathrm{vol}_q$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \sqrt{-1}\, \varepsilon_{abc} \omega^c \wedge P_b$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{\sqrt{-1}}{2} \varepsilon_{abc} \omega^c \wedge P_b + \frac{\sqrt{-1}}{2} \varepsilon_{abc} \omega^c \wedge P_b$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{\sqrt{-1}}{2} \varepsilon_{abc} \omega^c \wedge P_b + \frac{\sqrt{-1}}{2} \varepsilon_{acb} \omega^b \wedge P_c$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{\sqrt{-1}}{2} \varepsilon_{abc} \omega^c \wedge P_b - \frac{\sqrt{-1}}{2} \varepsilon_{abc} \omega^b \wedge P_c$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{\sqrt{-1}}{2} \varepsilon_{abc} (- \sqrt{-1}\, d(\omega^b \wedge \omega^c))$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{1}{2} \varepsilon_{abc} d(\omega^b \wedge \omega^c)$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{1}{2} \varepsilon_{abc} (d\omega^b \wedge \omega^c - \omega^b \wedge d\omega^c)$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{1}{2} \varepsilon_{abc} d\omega^b \wedge \omega^c - \frac{1}{2} \varepsilon_{acb} \omega^c \wedge d\omega^b$$

$$= - \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{1}{2} \varepsilon_{abc} d\omega^b \wedge \omega^c + \frac{1}{2} \varepsilon_{abc} d\omega^b \wedge \omega^c$$

$$= 0.$$

We have

$$T^*\underline{Q} \supset T^*\underline{Q}_{\underline{C}} \xrightarrow{T} T^*\underline{Q}_{\underline{C}} \xrightarrow{S} T^*{*}\underline{Q}_{\underline{C}}.$$

The path $t \to (\vec{\omega}_t, \vec{p}_t)$ lies in the ADM sector of $T^*\underline{Q}$ and $\mathrm{Ein}(g) = 0$ provided the constraint equations and the evolution equations are satisfied by the pair $(\vec{\omega}_t, \vec{p}_t)$. The path $t \to T(\vec{\omega}_t, \vec{p}_t)$ $(= (\vec{\omega}_t, \vec{P}_t))$ lies in the image under T of the

ADM sector of $T^*\underline{\underline{Q}}_C$ and, since T is canonical, $\mathrm{Ein}(g) = 0$ provided the constraint equations are satisfied by the pair $(\vec{\omega}_t,\vec{P}_t)$. Finally, the path $t \to S \circ T(\vec{\omega}_t,\vec{p}_t)$ $(= (\vec{Q}_t,\vec{A}_t))$ lies in the image under $S \circ T$ of the ADM sector of $T^*\underline{\underline{Q}}_C$ and, since $S \circ T$ is canonical, $\mathrm{Ein}(g) = 0$ provided the constraint equations and the evolution equations are satisfied by the pair $(\vec{Q}_t,\vec{A}_t)$.

<u>N.B.</u> The constraint equations and the evolution equations per $(\vec{Q}_t,\vec{A}_t)$ are explicated in the ensuing sections.

Section 58: Evolution in the Ashtekar Picture The assumptions and notation are those of the standard setup but with the restriction that n = 4.

Let

$$H = H(\vec{Q},\vec{A};N,\vec{N},\vec{Z})$$

$$= \int_\Sigma L_{\vec{N}}\vec{Q} \overset{\bullet}{\wedge} \vec{A} + \int_\Sigma \vec{Z} \overset{\bullet}{\wedge} d^A\vec{Q} + \int_\Sigma - \sqrt{-1}\, N\vec{F} \overset{\bullet}{\wedge} *\vec{Q}.$$

Objective: Compute the functional derivatives

$$\left[\begin{array}{l} \dfrac{\delta H}{\delta \vec{A}} \\[2em] \dfrac{\delta H}{\delta \vec{Q}} \end{array}\right.$$

and hence determine the equations of motion

$$\left[\begin{array}{l} \dot{\vec{Q}} = \dfrac{\delta H}{\delta \vec{A}} \\[2em] \dot{\vec{A}} = - \dfrac{\delta H}{\delta \vec{Q}}\ . \end{array}\right.$$

Calculation of $\dfrac{\delta H}{\delta \vec{A}}$:

1. Consider

$$\delta_a(L_{\vec{N}}\vec{Q} \overset{\bullet}{\wedge} \vec{A}).$$

Thus

$$\delta_a(L_{\vec{N}}Q^b \wedge A_b) = L_{\vec{N}}Q^b \wedge \delta_a A_b$$

$$= L_{\vec{N}}Q^a \wedge \delta A_a.$$

Therefore

$$\frac{\delta}{\delta A_a}\ [\ \int_\Sigma\ L_{\vec{N}}\vec{Q}\ \dot{\wedge}\ \vec{A}] = L_{\vec{N}}Q^a.$$

2. Consider

$$\delta_a(\vec{Z}\ \dot{\wedge}\ d^A\vec{Q}).$$

Thus

$$\delta_a(Z_b\wedge d^A Q^b) = \delta_a(Z_b\wedge(dQ^b - \sqrt{-1}\ \varepsilon^b{}_{cd}A^d\wedge Q^c))$$

$$= -\sqrt{-1}\ Z_b\wedge\delta_a(\varepsilon^b{}_{cd}A^d)\wedge Q^c$$

$$= -\sqrt{-1}\ Z_b\wedge\varepsilon^b{}_{ca}\delta A_a\wedge Q^c$$

$$= -\sqrt{-1}\ \varepsilon_{bca}Z_b\wedge Q^c\wedge\delta A_a$$

$$= -\sqrt{-1}\ \varepsilon_{abc}Z_b\wedge Q^c\wedge\delta A_a$$

$$= -\sqrt{-1}\ (\vec{Z}\ \overset{\times}{\wedge}\ \vec{Q})_a\wedge\delta A_a.$$

Therefore

$$\frac{\delta}{\delta A_a}\ [\ \int_\Sigma\ \vec{Z}\ \dot{\wedge}\ d^A\vec{Q}] = -\sqrt{-1}\ (\vec{Z}\ \overset{\times}{\wedge}\ \vec{Q})_a.$$

3. Consider

$$\delta_a(-\sqrt{-1}\ N\vec{F}\ \dot{\wedge}\ *\vec{Q}).$$

Thus

$$\delta_a(-\sqrt{-1}\ NF_b\wedge *Q^b)$$

$$= -\sqrt{-1}\ (N*Q^b\wedge\delta_a(dA_b + \frac{\sqrt{-1}}{2}\ (\vec{A}\overset{\times}{\wedge}\vec{A})_b)$$

$$= -\sqrt{-1}\ (N*Q^a\wedge d\delta A_a + \frac{\sqrt{-1}}{2}\ N*Q^b\wedge\delta_a(\vec{A}\overset{\times}{\wedge}\vec{A})_b).$$

- $d(N*Q^a\wedge\delta A_a)$

$$= d(N*Q^a)\wedge\delta A_a - N*Q^a\wedge d\delta A_a$$

=>

$$N*Q^a\wedge d\delta A_a$$

$$= -\ d(N*Q^a\wedge\delta A_a)\ +\ d(N*Q^a)\wedge\delta A_a.$$

- $\frac{\sqrt{-1}}{2}\ \delta_a(\vec{A}\overset{\times}{\wedge}\vec{A})_b$

$$= \frac{\sqrt{-1}}{2}\ \delta_a(\varepsilon_{bcd}A_c\wedge A_d)$$

$$= \frac{\sqrt{-1}}{2}\ (\varepsilon_{bad}\delta A_a\wedge A_d + \varepsilon_{bca}A_c\wedge\delta A_a)$$

$$= \frac{\sqrt{-1}}{2}\ (-\ \varepsilon_{bad}A_d\wedge\delta A_a + \varepsilon_{bca}A_c\wedge\delta A_a)$$

$$= \frac{\sqrt{-1}}{2}\ (-\ \varepsilon_{bad}A_d + \varepsilon_{bca}A_c)\wedge\delta A_a$$

$$= \frac{\sqrt{-1}}{2}\ (\varepsilon_{bda}A_d + \varepsilon_{bca}A_c)\wedge\delta A_a$$

$$= \frac{\sqrt{-1}}{2} (\varepsilon_{bca} A_c + \varepsilon_{bca} A_c) \wedge \delta A_a$$

$$= \sqrt{-1}\ \varepsilon_{bca} A_c \wedge \delta A_a$$

=>

$$\frac{\sqrt{-1}}{2} N{*}Q^b \wedge \delta_a (\vec{A} \overset{\times}{\wedge} \vec{A})_b$$

$$= N{*}Q^b \wedge \sqrt{-1}\ \varepsilon_{bca} A_c \wedge \delta A_a$$

$$= - \sqrt{-1}\ \varepsilon_{bca} A_c \wedge N{*}Q^b \wedge \delta A_a$$

$$= - \sqrt{-1}\ \varepsilon^a{}_{bc} A^c \wedge N{*}Q^b \wedge \delta A_a .$$

Therefore

$$\frac{\delta}{\delta A_a} [\ \textstyle\int_\Sigma - \sqrt{-1}\ N\vec{F} \overset{\cdot}{\wedge} {*}\vec{Q}]$$

$$= - \sqrt{-1}\ (d(N{*}Q^a) - \sqrt{-1}\ \varepsilon^a{}_{bc} A^c \wedge (N{*}Q^b))$$

$$= - \sqrt{-1}\ d^A (N{*}Q^a) .$$

Combining 1, 2, and 3 then gives

$$\frac{\delta H}{\delta \vec{A}} = L_{\vec{N}} \vec{Q} - \sqrt{-1}\ \vec{Z} \overset{\times}{\wedge} \vec{Q} - \sqrt{-1}\ d^A (N{*}\vec{Q}) .$$

<u>Calculation of $\frac{\delta H}{\delta \vec{Q}}$:</u>

1. Consider

$$\delta_a(\mathcal{L}_{\vec{N}}\vec{Q} \dot{\wedge} \vec{A}).$$

Thus

$$\delta_a(\mathcal{L}_{\vec{N}}Q^b \wedge A_b) = \delta_a \mathcal{L}_{\vec{N}}Q^b \wedge A_b$$

$$= \mathcal{L}_{\vec{N}}\delta Q^a \wedge A_a$$

$$= -\ \delta Q^a \wedge \mathcal{L}_{\vec{N}}A_a + \mathcal{L}_{\vec{N}}(\delta Q^a \wedge A_a).$$

Therefore

$$\frac{\delta}{\delta Q^a}\ [\ \int_\Sigma \mathcal{L}_{\vec{N}}\vec{Q} \dot{\wedge} \vec{A}] = -\ \mathcal{L}_{\vec{N}}A_a.$$

2. Consider

$$\delta_a(\vec{Z} \dot{\wedge} d^A\vec{Q}).$$

Thus

$$\delta_a(Z_b \wedge d^A Q^b) = \delta_a(d^A Q^b \wedge Z_b)$$

$$= \delta_a((dQ^b - \sqrt{-1}\ \varepsilon^b{}_{cd}A^d \wedge Q^c) \wedge Z_b)$$

$$= d\delta Q^a \wedge Z_a - \sqrt{-1}\ \varepsilon^b{}_{ac}A^c \wedge \delta Q^a \wedge Z_b$$

$$= -\ \delta Q^a \wedge dZ_a - \delta Q^a \wedge \sqrt{-1}\ \varepsilon^b{}_{ac}A^c \wedge Z_b$$

$$+ d(\delta Q^a \wedge Z_a)$$

$$= \delta Q^a \wedge (- dZ_a - \sqrt{-1}\, \varepsilon^b{}_{ac} A^c \wedge Z_b)$$

$$+ d(\delta Q^a \wedge Z_a)$$

$$= \delta Q^a \wedge (- dZ_a + \sqrt{-1}\, \varepsilon_{abc} A^c \wedge Z_b)$$

$$+ d(\delta Q^a \wedge Z_a).$$

Therefore

$$\frac{\delta}{\delta Q^a} [\int_\Sigma \vec{Z} \overset{\bullet}{\wedge} d^A \vec{Q}]$$

$$= - (dZ_a - \sqrt{-1}\, \varepsilon_{abc} A^c \wedge Z_b)$$

$$= - d^A Z_a.$$

3. Consider

$$\delta_a (- \sqrt{-1}\, N\vec{F} \overset{\bullet}{\wedge} *\vec{Q}).$$

Thus

$$\delta_a (- \sqrt{-1}\, N\vec{F}_b \wedge *Q^b) = \delta_a (- \sqrt{-1}\, NF_b \wedge - \omega^b)$$

$$= \sqrt{-1}\, \delta_a (\omega^b \wedge NF_b).$$

<u>LEMMA</u> We have

$$- \frac{\delta}{\delta Q^a} [\int_\Sigma \omega^b \wedge NF_b] \quad (= \frac{\delta}{\delta * \omega^a} [\int_\Sigma \omega^b \wedge NF_b])$$

$$= \frac{1}{2} q(NF_c, *\omega^c)\omega^a - q(\omega^a, *NF_b)\omega^b.$$

[Let $\beta_a = NF_a$ and

$$\gamma_a = \frac{1}{2} q(\beta_c, *\omega^c)\omega^a - q(\omega^a, *\beta_b)\omega^b.$$

Then (see the end of Section 56)

$$\beta_a = *(\omega^b \wedge \omega^a) \wedge \gamma_b$$

=>

$$\omega^a \wedge \beta_a = \omega^a \wedge *(\omega^b \wedge \omega^a) \wedge \gamma_b$$

$$= \omega^a \wedge \epsilon_{bac}\omega^c \wedge \gamma_b$$

$$= \epsilon_{bac}(\omega^a \wedge \omega^c) \wedge \gamma_b$$

$$= 2*\omega^b \wedge \gamma_b$$

=>

$$\frac{\delta}{\delta *\omega^a} [\textstyle\int_\Sigma \omega^b \wedge \beta_b]$$

$$= \frac{\delta}{\delta *\omega^a} [\textstyle\int_\Sigma 2*\omega^b \wedge \gamma_b].$$

Now take δ_a per $*\omega^a$ and not $- *\omega^a$ ($= Q^a$) -- then

$$2\delta_a(*\omega^b \wedge \gamma_b) = 2\delta_a *\omega^b \wedge \gamma_b + 2*\omega^b \wedge \delta_a \gamma_b$$

$$= 2\delta *\omega^a \wedge \gamma_a + 2*\omega^b \wedge \delta_a \gamma_b.$$

But

$$0 = \delta_a \beta_c = \delta_a(*(\omega^b \wedge \omega^c) \wedge \gamma_b)$$

$$= \delta_a(*(\omega^b \wedge \omega^c)) \wedge \gamma_b + *(\omega^b \wedge \omega^c) \wedge \delta_a \gamma_b.$$

Therefore

$$2*\omega^b \wedge \delta_a \gamma_b = \omega^c \wedge *(\omega^b \wedge \omega^c) \wedge \delta_a \gamma_b$$

$$= - \omega^c \wedge \delta_a(*(\omega^b \wedge \omega^c)) \wedge \gamma_b$$

$$= - \omega^c \wedge \varepsilon_{bcd} \delta_a \omega^d \wedge \gamma_b$$

$$= \delta_a \omega^d \wedge \varepsilon_{bcd} \omega^c \wedge \gamma_b$$

$$= - \delta_a \omega^d \wedge \varepsilon_{bdc} \omega^c \wedge \gamma_b$$

$$= - \delta_a \omega^d \wedge *(\omega^b \wedge \omega^d) \wedge \gamma_b$$

$$= - \delta_a \omega^d \wedge \beta_d$$

$$= - \delta_a \omega^b \wedge \beta_b$$

$\Rightarrow$

$$2\delta_a(*\omega^b \wedge \gamma_b) = 2\delta *\omega^a \wedge \gamma_a - \delta_a \omega^b \wedge \beta_b$$

$$= 2\delta *\omega^a \wedge \gamma_a - \delta_a \omega^b \wedge \beta_b - \omega^b \wedge \delta_a \beta_b$$

$$= 2\delta *\omega^a \wedge \gamma_a - \delta_a(\omega^b \wedge \beta_b)$$

=>

$$\delta_a(\omega^b \wedge \beta_b) = 2\delta_a(*\omega^b \wedge \gamma_b)$$

$$= 2\delta *\omega^a \wedge \gamma_a - \delta_a(\omega^b \wedge \beta_b)$$

=>

$$\delta_a(\omega^b \wedge \beta_b) = \delta *\omega^a \wedge \gamma_a$$

=>

$$\frac{\delta}{\delta *\omega^a} [\int_\Sigma \omega^b \wedge \beta_b] = \gamma_a.$$

I.e.:

$$\frac{\delta}{\delta *\omega^a} [\int_\Sigma \omega^b \wedge NF_b]$$

$$= \frac{1}{2} q(NF_c, *\omega^c)\omega^a - q(\omega^a, *NF_b)\omega^b.]$$

Notation: Put

$$F = q(F_{uv}, \omega^u \wedge \omega^v).$$

Then

$$\frac{1}{2} q(NF_c, *\omega^c)\omega^a$$

$$= -\frac{1}{2} q(NF_c, *\omega^c) *Q^a$$

$$= -\frac{1}{2} Nq(\frac{\sqrt{-1}}{2} \varepsilon_{cuv} F_{uv}, *\omega^c) *Q^a$$

$$= -\frac{\sqrt{-1}}{4}\,Nq(F_{uv},\varepsilon_{cuv}*\omega^c)*Q^a$$

$$= -\frac{\sqrt{-1}}{4}\,Nq(F_{uv},\varepsilon_{uvc}*\omega^c)*Q^a$$

$$= -\frac{\sqrt{-1}}{4}\,Nq(F_{uv},\omega^u\wedge\omega^v)*Q^a$$

$$= -\frac{\sqrt{-1}}{4}\,NF*Q^a.$$

Notation: Put

$$(\overrightarrow{\text{Ric}}\ F)_a = \iota_{\omega^b}F_{ba}.$$

Then

$$(\overrightarrow{\text{Ric}}\ F)_a = -\sqrt{-1}\ \varepsilon_{cba}\iota_{\omega^b}F_c$$

$$= \sqrt{-1}\ \varepsilon_{abc}\iota_{\omega^b}F_c$$

$\Rightarrow$

$$\sqrt{-1}\ (\overrightarrow{\text{Ric}}\ F)_a = -\varepsilon_{abc}\iota_{\omega^b}F_c$$

$$= -\varepsilon_{abc}q(\omega^u,\iota_{\omega^b}F_c)\omega^u$$

$$= -\varepsilon_{abc}q(\omega^b\wedge\omega^u,F_c)\omega^u$$

$$= -\varepsilon_{abc}\varepsilon_{buv}q(*\omega^v,F_c)\omega^u$$

$$= \varepsilon_{acb}\varepsilon_{uvb}q(*\omega^v,F_c)\omega^u$$

$$= \delta^{ac}{}_{uv} q(*\omega^v, F_c)\omega^u$$

$$= (\delta^a{}_u \delta^c{}_v - \delta^a{}_v \delta^c{}_u) q(*\omega^v, F_c)\omega^u$$

$$= q(*\omega^c, F_c)\omega^a - q(\omega^a, *F_c)\omega^c$$

=>

$$\sqrt{-1}\, N(\overrightarrow{\mathrm{Ric}}\ F)_a$$

$$= q(NF_c, *\omega^c)\omega^a - q(\omega^a, *NF_b)\omega^b$$

=>

$$\sqrt{-1}\, N(\overrightarrow{\mathrm{Ric}}\ F)_a - \frac{1}{2} q(NF_c, *\omega^c)\omega^a$$

$$= \frac{1}{2} q(NF_c, *\omega^c)\omega^a - q(\omega^a, *NF_b)\omega^b$$

=>

$$\frac{\delta}{\delta *\omega^a} [\ \int_\Sigma \omega^b \wedge NF_b]$$

$$= \sqrt{-1}\, N(\overrightarrow{\mathrm{Ric}}\ F)_a + \frac{\sqrt{-1}}{4} NF*Q^a$$

=>

$$\frac{\delta}{\delta Q^a} [\ \int_\Sigma - \sqrt{-1}\, N\vec{F} \stackrel{\bullet}{\wedge} *\vec{Q}] = N(\overrightarrow{\mathrm{Ric}}\ F)_a + \frac{1}{4} NF*Q^a.$$

Combining 1, 2, and 3 then gives

$$\frac{\delta H}{\delta \vec{Q}} = - L_{\vec{N}} \vec{A} - d^A \vec{Z} + N(\overrightarrow{\mathrm{Ric}}\ F) + \frac{1}{4} NF*\vec{Q}.$$

Definition: The relations

$$\begin{cases} \dot{\vec{Q}} = L_{\vec{N}}\vec{Q} - \sqrt{-1}\,\vec{Z} \overset{\times}{\wedge} \vec{Q} - \sqrt{-1}\, d^{A}(N*\vec{Q}) \\ \dot{\vec{A}} = L_{\vec{N}}\vec{A} + d^{A}\vec{Z} - N(\overrightarrow{\mathrm{Ric}}\ F) - \frac{1}{4} NF*\vec{Q} \end{cases}$$

are the Ashtekar equations of motion.

Reality Check Along the path $t \to (\vec{\omega}_t, \vec{p}_t)$, we have

$$\dot{\bar{\omega}}^{a} = N_t \bar{\omega}^{a}{}_{0} + \bar{Q}^{a}{}_{b}\bar{\omega}^{b} + L_{\vec{N}_t}\bar{\omega}^{a}.$$

Write

$$*\bar{\omega}^{a} = \frac{1}{2}\,\varepsilon_{abc}\bar{\omega}^{b}\wedge\bar{\omega}^{c}.$$

Then

$$\begin{aligned} \frac{d}{dt} *\bar{\omega}^{a} &= \frac{1}{2}\,\varepsilon_{abc}\dot{\bar{\omega}}^{b}\wedge\bar{\omega}^{c} + \frac{1}{2}\,\varepsilon_{abc}\bar{\omega}^{b}\wedge\dot{\bar{\omega}}^{c} \\ &= \frac{1}{2}\,\varepsilon_{abc}\dot{\bar{\omega}}^{b}\wedge\bar{\omega}^{c} + \frac{1}{2}\,\varepsilon_{acb}\bar{\omega}^{c}\wedge\dot{\bar{\omega}}^{b} \\ &= \frac{1}{2}\,\varepsilon_{abc}\dot{\bar{\omega}}^{b}\wedge\bar{\omega}^{c} - \frac{1}{2}\,\varepsilon_{abc}\bar{\omega}^{c}\wedge\dot{\bar{\omega}}^{b} \\ &= \frac{1}{2}\,\varepsilon_{abc}\dot{\bar{\omega}}^{b}\wedge\bar{\omega}^{c} + \frac{1}{2}\,\varepsilon_{abc}\dot{\bar{\omega}}^{b}\wedge\bar{\omega}^{c} \\ &= \varepsilon_{abc}\dot{\bar{\omega}}^{b}\wedge\bar{\omega}^{c} \\ &= \varepsilon_{abc}(N_t\bar{\omega}^{b}{}_{0} + \bar{Q}^{b}{}_{d}\bar{\omega}^{d} + L_{\vec{N}_t}\bar{\omega}^{b})\wedge\bar{\omega}^{c}. \end{aligned}$$

On the other hand,

$$\dot{\vec{Q}} = L_{\vec{N}}\vec{Q} - \sqrt{-1}\ \vec{Z} \overset{\times}{\wedge} \vec{Q} - \sqrt{-1}\ d^{A}(N*\vec{Q})$$

=>

$$\frac{d}{dt} - *\bar{\omega}^{a} = L_{\vec{N}_t} - *\bar{\omega}_{a} - \sqrt{-1}\ \varepsilon_{abc}Z_{b}\wedge - *\bar{\omega}^{c}$$

$$- \sqrt{-1}\ (- d(N_t\bar{\omega}^{a}) - \sqrt{-1}\ \varepsilon^{a}{}_{bc}A^{c}\wedge - (N_t\bar{\omega}^{b}))$$

=>

$$\frac{d}{dt} *\bar{\omega}^{a} = L_{\vec{N}_t} *\bar{\omega}_{a} + \sqrt{-1}\ \varepsilon_{abc}Z_{b}\wedge - *\bar{\omega}^{c}$$

$$+ \sqrt{-1}\ (- d(N_t\bar{\omega}^{a}) - \sqrt{-1}\ \varepsilon^{a}{}_{bc}A^{c}\wedge - (N_t\bar{\omega}^{b})).$$

And this expression for $\frac{d}{dt} *\bar{\omega}^{a}$ had better agree with the one given above (in particular, the imaginary terms must vanish, our data being real).

- $L_{\vec{N}_t} *\bar{\omega}_{a}$

$$= \frac{1}{2}\varepsilon_{abc}L_{\vec{N}_t}\bar{\omega}^{b}\wedge\bar{\omega}^{c} + \frac{1}{2}\varepsilon_{abc}\bar{\omega}^{b}\wedge L_{\vec{N}_t}\bar{\omega}^{c}$$

$$= \frac{1}{2}\varepsilon_{abc}L_{\vec{N}_t}\bar{\omega}^{b}\wedge\bar{\omega}^{c} + \frac{1}{2}\varepsilon_{acb}\bar{\omega}^{c}\wedge L_{\vec{N}_t}\bar{\omega}^{b}$$

$$= \frac{1}{2}\varepsilon_{abc}L_{\vec{N}_t}\bar{\omega}^{b}\wedge\bar{\omega}^{c} - \frac{1}{2}\varepsilon_{abc}\bar{\omega}^{c}\wedge L_{\vec{N}_t}\bar{\omega}^{b}$$

$$= \frac{1}{2}\varepsilon_{abc}L_{\vec{N}_t}\bar{\omega}^{b}\wedge\bar{\omega}^{c} + \frac{1}{2}\varepsilon_{abc}L_{\vec{N}_t}\bar{\omega}^{b}\wedge\bar{\omega}^{c}$$

$$= \varepsilon_{abc} L_{\vec{N}_t} \bar{\omega}^b \wedge \bar{\omega}^c .$$

The Lie derivative terms thus match up. To compare the rotational terms, recall that

$$Z_{ab} = -\bar{Q}_{ab} + \sqrt{-1}\ \varepsilon_{abc}\bar{P}_c$$

=>

$$Z_a = \frac{\sqrt{-1}}{2}\ \varepsilon_{auv} Z_{uv}$$

$$= \frac{\sqrt{-1}}{2}\ \varepsilon_{auv}(-\ \bar{Q}_{uv} + \sqrt{-1}\ \varepsilon_{uvc}\bar{P}_c)$$

$$= -\ \frac{\sqrt{-1}}{2}\ \varepsilon_{auv}\bar{Q}_{uv} - \frac{1}{2}\ \varepsilon_{auv}\varepsilon_{uvc}\bar{P}_c$$

$$= -\ \frac{\sqrt{-1}}{2}\ \varepsilon_{auv}\bar{Q}_{uv} - \frac{1}{2}\ \varepsilon_{auv}\varepsilon_{cuv}\bar{P}_c$$

$$= -\ \frac{\sqrt{-1}}{2}\ \varepsilon_{auv}\bar{Q}_{uv} - \frac{1}{2}\ (2\delta^a{}_c)\bar{P}_c$$

$$= -\ \frac{\sqrt{-1}}{2}\ \varepsilon_{auv}\bar{Q}_{uv} - \bar{P}_a$$

$$= -\ \frac{\sqrt{-1}}{2}\ \varepsilon_{auv}\bar{Q}_{uv} + q_t(dN_t,\bar{\omega}^a)$$

=>

$$-\ \sqrt{-1}\ \varepsilon_{abc} Z_b \wedge * \bar{\omega}^c$$

$$= -\ \sqrt{-1}\ \varepsilon_{abc}(-\ \frac{\sqrt{-1}}{2}\ \varepsilon_{buv}\bar{Q}_{uv} + q_t(dN_t,\bar{\omega}^b)) \wedge * \bar{\omega}^c .$$

$$\bullet\ -\sqrt{-1}\ \varepsilon_{abc}\left(-\frac{\sqrt{-1}}{2}\right)\varepsilon_{buv}\bar{Q}_{uv}\wedge *\bar{\omega}^{c}$$

$$= -\frac{1}{2}\ \varepsilon_{abc}\varepsilon_{buv}\bar{Q}_{uv}\wedge *\bar{\omega}^{c}$$

$$= \frac{1}{2}\ \varepsilon_{acb}\varepsilon_{uvb}\bar{Q}_{uv}\wedge *\bar{\omega}^{c}$$

$$= \frac{1}{2}\ \delta^{ac}{}_{uv}\bar{Q}_{uv}\wedge *\bar{\omega}^{c}$$

$$= \frac{1}{2}\ (\delta^{a}{}_{u}\delta^{c}{}_{v} - \delta^{a}{}_{v}\delta^{c}{}_{u})\bar{Q}_{uv}\wedge *\bar{\omega}^{c}$$

$$= \frac{1}{2}\ (\bar{Q}_{ac}\wedge *\bar{\omega}^{c} - \bar{Q}_{ca}\wedge *\bar{\omega}^{c})$$

$$= \frac{1}{2}\ (\bar{Q}_{ac}\wedge *\bar{\omega}^{c} + \bar{Q}_{ac}\wedge *\bar{\omega}^{c})$$

$$= \bar{Q}_{ac}\wedge *\bar{\omega}^{c}.$$

$$\bullet\ \varepsilon_{abc}\bar{Q}^{b}{}_{d}\bar{\omega}^{d}\wedge\bar{\omega}^{c}$$

$$= \varepsilon_{abc}\bar{Q}_{bd}\bar{\omega}^{d}\wedge\bar{\omega}^{c}$$

$$= \varepsilon_{abc}\varepsilon_{dcu}\bar{Q}_{bd}\wedge *\bar{\omega}^{u}$$

$$= -\ \varepsilon_{abc}\varepsilon_{duc}\bar{Q}_{bd}\wedge *\bar{\omega}^{u}$$

$$= -\ \delta^{ab}{}_{du}\bar{Q}_{bd}\wedge *\bar{\omega}^{u}$$

$$= -\ (\delta^a{}_d\delta^b{}_u - \delta^a{}_u\delta^b{}_d)\bar{Q}_{bd}\wedge *\bar{\omega}^u$$

$$= -\ \bar{Q}_{ba}\wedge *\bar{\omega}^b + \bar{Q}_{bb}\wedge *\bar{\omega}^a$$

$$= -\ \bar{Q}_{ba}\wedge *\bar{\omega}^b$$

$$= -\ \bar{Q}_{ca}\wedge *\bar{\omega}^c$$

$$= \bar{Q}_{ac}\wedge *\bar{\omega}^c.$$

The rotational terms are thereby accounted for. Next

$$-\ \sqrt{-1}\ \sqrt{-1}\ \varepsilon^a{}_{bc}A^c\wedge -\ (N_t\bar{\omega}^b) = -\ \varepsilon_{abc}\left(\frac{\sqrt{-1}}{2}\ \varepsilon_{cuv}\bar{\omega}_{uv} + \frac{\sqrt{-1}}{2}\ \varepsilon_{cuv}\sqrt{-1}\ \varepsilon_{duv}\bar{\omega}_{0d}\right)\wedge N_t\bar{\omega}^b$$

$$= -\ \varepsilon_{abc}\left(\frac{\sqrt{-1}}{2}\ \varepsilon_{cuv}\bar{\omega}_{uv} - \frac{1}{2}\ (2\delta^c{}_d)\bar{\omega}_{0d}\right)\wedge N_t\bar{\omega}^b$$

$$= -\ \frac{\sqrt{-1}}{2}\ \varepsilon_{abc}\varepsilon_{uvc}\bar{\omega}_{uv}\wedge N_t\bar{\omega}^b + \varepsilon_{abc}\bar{\omega}_{0c}\wedge N_t\bar{\omega}^b$$

$$= -\ \frac{\sqrt{-1}}{2}\ \delta^{ab}{}_{uv}\bar{\omega}_{uv}\wedge N_t\bar{\omega}^b + \varepsilon_{acb}N_t\bar{\omega}_{0b}\wedge\bar{\omega}^c$$

$$= -\ \frac{\sqrt{-1}}{2}\ (\delta^a{}_u\delta^b{}_v - \delta^a{}_v\delta^b{}_u)\bar{\omega}_{uv}\wedge N_t\bar{\omega}^b - \varepsilon_{abc}N_t\bar{\omega}_{0b}\wedge\bar{\omega}^c$$

$$= -\ \frac{\sqrt{-1}}{2}\ (\bar{\omega}_{ab} - \bar{\omega}_{ba})\wedge N_t\bar{\omega}^b + \varepsilon_{abc}N_t\bar{\omega}_{b0}\wedge\bar{\omega}^c$$

$$= -\ \sqrt{-1}\ N_t\bar{\omega}^a{}_b\wedge\bar{\omega}^b + \varepsilon_{abc}N_t\bar{\omega}^b{}_0\wedge\bar{\omega}^c.$$

In view of this, all that remains is to show that the imaginary terms add up to zero:

$$- \sqrt{-1}\ (\varepsilon_{abc} q_t (dN_t, \bar{\omega}^b) \wedge * \bar{\omega}^c + d(N_t \bar{\omega}^a) + N_t \bar{\omega}^a{}_b \wedge \bar{\omega}^b) = 0.$$

1. $\varepsilon_{abc} q_t (dN_t, \bar{\omega}^b) \wedge * \bar{\omega}^c$

$$= \frac{1}{2}\, \varepsilon_{abc} \varepsilon_{cuv} q_t (dN_t, \bar{\omega}^b) \wedge \bar{\omega}^u \wedge \bar{\omega}^v$$

$$= \frac{1}{2}\, \varepsilon_{abc} \varepsilon_{uvc} q_t (dN_t, \bar{\omega}^b) \wedge \bar{\omega}^u \wedge \bar{\omega}^v$$

$$= \frac{1}{2}\, \delta^{ab}{}_{uv} q_t (dN_t, \bar{\omega}^b) \wedge \bar{\omega}^u \wedge \bar{\omega}^v$$

$$= \frac{1}{2}\, (\delta^a{}_u \delta^b{}_v - \delta^a{}_v \delta^b{}_u) q_t (dN_t, \bar{\omega}^b) \wedge \bar{\omega}^u \wedge \bar{\omega}^v$$

$$= \frac{1}{2}\, q_t (dN_t, \bar{\omega}^b) \wedge (\bar{\omega}^a \wedge \bar{\omega}^b - \bar{\omega}^b \wedge \bar{\omega}^a)$$

$$= q_t (dN_t, \bar{\omega}^b) \bar{\omega}^a \wedge \bar{\omega}^b.$$

2. $d(N_t \bar{\omega}^a) = dN_t \wedge \bar{\omega}^a + N_t \wedge d\bar{\omega}^a$

$$= q_t (dN_t, \bar{\omega}^b) \bar{\omega}^b \wedge \bar{\omega}^a + N_t \wedge - \bar{\omega}^a{}_b \wedge \bar{\omega}^b$$

$$= - q_t (dN_t, \bar{\omega}^b) \bar{\omega}^a \wedge \bar{\omega}^b - N_t \bar{\omega}^a{}_b \wedge \bar{\omega}^b.$$

3. $N_t \bar{\omega}^a{}_b \wedge \bar{\omega}^b.$

Therefore

$$1 + 2 + 3 = 0.$$

The evolution equation for

$$\frac{d}{dt} \vec{A}_t$$

is, of course, complex (even though $(\vec{\omega}_t, \vec{p}_t)$ is real), hence breaks up into real and imaginary parts. As will be shown below, its real part admits a simple interpretation (but its imaginary part appears to be less amenable to explicit recognition).

Let μ, ν be indices that run between 1 and 3 and work locally.

Rappel: We have

$$\dot{\kappa}_{\mu\nu} = L_{\vec{N}_t} \kappa_{\mu\nu} + 2N_t (\kappa_t * \kappa_t)_{\mu\nu}$$

$$- N_t K_t \kappa_{\mu\nu} - N_t \mathrm{Ric}(q_t)_{\mu\nu} + \bar{\bar{\nabla}}_\mu \bar{\nabla}_\nu N_t$$

$$+ \frac{1}{4} N_t (S(q_t) - [\kappa_t, \kappa_t]_{q_t} + K_t^2)(q_t)_{\mu\nu}.$$

[Note: As usual, $\kappa_{\mu\nu} = (\kappa_t)_{\mu\nu}$.]

Write

$$\bar{\omega}_{0a} = - \kappa_{ab} \bar{\omega}^b$$

$$= - \kappa_t (E_a, E_b) \bar{\omega}^b$$

$$= - \kappa_t (e^\mu{}_a \frac{\partial}{\partial x^\mu}, e^\nu{}_b \frac{\partial}{\partial x^\nu}) \bar{\omega}^b$$

$$= - e^\mu{}_a e^\nu{}_b \kappa_{\mu\nu} \bar{\omega}^b$$

$$= - e^\mu{}_a e^\nu{}_b e^b{}_{\nu'} \kappa_{\mu\nu} dx^{\nu'}$$

$$= - e^\mu{}_a \delta^\nu{}_{\nu'} \kappa_{\mu\nu} dx^{\nu'}$$

$$= - e^{\mu}{}_{a}\kappa_{\mu\nu}dx^{\nu}$$

$$= - \kappa_{\mu\nu}e^{\mu}{}_{a}dx^{\nu}.$$

Then

$$\dot{\bar{\omega}}_{0a} = - (\kappa_{\mu\nu}e^{\mu}{}_{a})^{\cdot}dx^{\nu},$$

where

$$(e^{\mu}{}_{a})^{\cdot} = - e^{\mu}{}_{b}\dot{e}^{b}{}_{\mu'}e^{\mu'}{}_{a}.$$

<u>LEMMA</u> We have

$$\dot{\bar{\omega}}_{0a} = L_{\vec{N}_t}\omega_{0a} - d^{\nabla^{q_t}}q_t(dN_t,\bar{\omega}_a) + \bar{Q}_{ac}\bar{\omega}_{0c}$$

$$+ N_t(\text{Ric}(q_t)_{ab} - (\kappa_t*\kappa_t)_{ab} + K_t\kappa_{ab})\bar{\omega}^b$$

$$- \frac{1}{4}N_t(S(q_t) - [\kappa_t,\kappa_t]_{q_t} + K_t^2)\bar{\omega}^a.$$

The point now is that the equation for $\dot{\bar{\omega}}_{0a}$ is the negative of the real part of the equation for $\dot{A}_a$.

To verify this, start from the fact that

$$A_a(= (\vec{A}_t)_a) = \frac{\sqrt{-1}}{2}\varepsilon_{abc}\bar{\omega}_{bc} - \bar{\omega}_{0a}.$$

Taking the real part of $L_{\vec{N}_t}A_a$ thus gives $- L_{\vec{N}_t}\bar{\omega}_{0a}$. To see where

$$- d^{\nabla^{q_t}} q_t(dN_t, \bar{\omega}_a) + \bar{Q}_{ac}\bar{\omega}_{0c}$$

comes from, write

$$dZ_a - \sqrt{-1}\,\varepsilon_{abc}A^c \wedge Z_b$$

$$= - \frac{\sqrt{-1}}{2}\,\varepsilon_{auv} d\bar{Q}_{uv} + dq_t(dN_t, \bar{\omega}_a)$$

$$- \sqrt{-1}\,\varepsilon_{abc}\left(\frac{\sqrt{-1}}{2}\,\varepsilon_{cuv}\bar{\omega}_{uv} - \bar{\omega}_{0c}\right) \wedge \left(- \frac{\sqrt{-1}}{2}\,\varepsilon_{buv}\bar{Q}_{uv} + q_t(dN_t, \bar{\omega}_b)\right).$$

The real part of this is

$$dq_t(dN_t, \bar{\omega}_a) + \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{uvc}\bar{\omega}_{uv} q_t(dN_t, \bar{\omega}_b) - \frac{1}{2}\,\varepsilon_{acb}\varepsilon_{uvb}\bar{Q}_{uv}\bar{\omega}_{0c}$$

or still,

$$dq_t(dN_t, \bar{\omega}_a) + \frac{1}{2}\,\delta^{ab}{}_{uv}\bar{\omega}_{uv} q_t(dN_t, \bar{\omega}_b) - \frac{1}{2}\,\delta^{ac}{}_{uv}\bar{Q}_{uv}\bar{\omega}_{0c}$$

or still,

$$dq_t(dN_t, \bar{\omega}_a) + \bar{\omega}_{ab} q_t(dN_t, \bar{\omega}_b) - \bar{Q}_{ac}\bar{\omega}_{0c}$$

or still,

$$d^{\nabla^{q_t}} q_t(dN_t, \bar{\omega}_a) - \bar{Q}_{ac}\bar{\omega}_{0c}.$$

The remaining terms can be identified in the same straightforward fashion, so the details will be omitted.

<u>Section 59:</u> <u>The Constraint Analysis</u> The assumptions and notation are those of the standard setup but with the restriction that n = 4.

Rappel:

$$H(\vec{Q},\vec{A};N,\vec{N},\vec{Z})$$

$$= \int_\Sigma L_{\vec{N}}\vec{Q} \dot{\wedge} \vec{A} + \int_\Sigma \vec{Z} \dot{\wedge} d^A\vec{Q} + \int_\Sigma - \sqrt{-1}\, N\vec{F} \dot{\wedge} *\vec{Q}.$$

Definition: The <u>physical phase space</u> of the theory (a.k.a. the <u>constraint surface</u> of the theory) is the subset $\text{Con}_{*\underline{Q}_{\underline{C}}}$ of $T^* * \underline{Q}_{\underline{C}}$ whose elements are the points $(\vec{Q},\vec{A})$ such that simultaneously

$$\frac{\delta H}{\delta N} = 0, \quad \frac{\delta H}{\delta \vec{N}} = 0, \quad \frac{\delta H}{\delta \vec{Z}} = 0.$$

$\underline{\frac{\delta H}{\delta N}}$: We have

$$\frac{\delta H}{\delta N} = - \sqrt{-1}\, \vec{F} \dot{\wedge} *\vec{Q}$$

$$= - \sqrt{-1}\, (F_a \wedge - **\omega^a)$$

$$= \sqrt{-1}\, (F_a \wedge \omega^a)$$

$$= \sqrt{-1}\, (\frac{\sqrt{-1}}{2}\, \varepsilon_{abc} F_{bc} \wedge \omega^a)$$

$$= - \frac{1}{2}\, F_{bc} \wedge \varepsilon_{abc} \omega^a$$

$$= - \frac{1}{2}\, F_{bc} \wedge \varepsilon_{bca} \omega^a$$

$$= -\frac{1}{2} F_{bc} \wedge *(\omega^b \wedge \omega^c)$$

$$= -\frac{1}{2} q(F_{bc}, \omega^b \wedge \omega^c) \text{vol}_q$$

$$= -\frac{F}{2} \text{vol}_q.$$

$\underline{\frac{\delta H}{\delta N^a}}$: We have

$$\delta_a [L_{\vec{N}} Q^b \wedge A_b]$$

$$= L_{(\delta N^a) E_a} Q^b \wedge A_b.$$

Write

$$L_{(\delta N^a) E_a} Q^b \wedge A_b$$

$$= (\iota_{(\delta N^a) E_a} \circ d + d \circ \iota_{(\delta N^a) E_a}) Q^b \wedge A_b$$

$$= \delta N^a (\iota_{E_a} dQ^b \wedge A_b) + d(\delta N^a \iota_{E_a} Q^b) \wedge A_b.$$

But

$$d((\delta N^a \iota_{E_a} Q^b) \wedge A_b)$$

$$= d(\delta N^a \iota_{E_a} Q^b) \wedge A_b - \delta N^a \iota_{E_a} Q^b \wedge dA_b$$

$\Rightarrow$

$$d(\delta N^a \iota_{E_a} Q^b) \wedge A_b = \delta N^a \iota_{E_a} Q^b \wedge dA_b$$

$$+ d((\delta N^a \iota_{E_a} Q^b) \wedge A_b).$$

Since

$$\int_\Sigma d((\delta N^a \iota_{E_a} Q^b) \wedge A_b) = 0,$$

it follows that

$$\frac{\delta H}{\delta N^a} = \iota_{E_a} dQ^b \wedge A_b + \iota_{E_a} Q^b \wedge dA_b.$$

Some additional manipulation of this formula will prove to be convenient. First

$$dA_b = F_b - \frac{\sqrt{-1}}{2} \varepsilon_{buv} A_u \wedge A_v, \text{ so}$$

$$\iota_{E_a} Q^b \wedge dA_b$$

$$= \iota_{E_a} Q^b \wedge F_b - \frac{\sqrt{-1}}{2} \varepsilon_{buv} \iota_{E_a} Q^b \wedge A_u \wedge A_v.$$

But

$$0 = \iota_{E_a} (Q^b \wedge A_u \wedge A_v)$$

$$= \iota_{E_a} Q^b \wedge A_u \wedge A_v$$

$$+ Q^b \wedge \iota_{E_a} A_u \wedge A_v - Q^b \wedge A_u \wedge \iota_{E_a} A_v$$

=>

$$- \frac{\sqrt{-1}}{2} \varepsilon_{buv} \iota_{E_a} Q^b \wedge A_u \wedge A_v$$

$$= - \frac{\sqrt{-1}}{2} [\varepsilon_{buv} Q^b \wedge A_u \wedge \iota_{E_a} A_v$$

$$- \varepsilon_{buv} Q^b \wedge \iota_{E_a} A_u \wedge A_v]$$

$$= - \frac{\sqrt{-1}}{2} [\varepsilon_{buv} (\iota_{E_a} A_v) Q^b \wedge A_u$$

$$- \varepsilon_{buv} (\iota_{E_a} A_u) Q^b \wedge A_v]$$

$$= - \frac{\sqrt{-1}}{2} [\varepsilon_{buv} (\iota_{E_a} A_v) Q^b \wedge A_u$$

$$- \varepsilon_{bvu} (\iota_{E_a} A_v) Q^b \wedge A_u]$$

$$= - \frac{\sqrt{-1}}{2} [\varepsilon_{buv} (\iota_{E_a} A_v) Q^b \wedge A_u$$

$$+ \varepsilon_{buv} (\iota_{E_a} A_v) Q^b \wedge A_u]$$

$$= (\iota_{E_a} A_v)(- \sqrt{-1}\, \varepsilon_{buv} Q^b \wedge A_u)$$

$$= (\iota_{E_a} A_v)(- \sqrt{-1}\, \varepsilon_{buv} A_u \wedge Q^b)$$

=>

$$\iota_{E_a} Q^b \wedge dA_b$$

$$= F_b \wedge \iota_{E_a} Q^b + (\iota_{E_a} A_v)(- \sqrt{-1}\, \varepsilon^v{}_{bu} A_u \wedge Q^b)$$

$$= F_b \wedge \iota_{E_a} Q^b + (\iota_{E_a} A_v)(d^A Q^v - dQ^v)$$

$$= F_b \wedge \iota_{E_a} Q^b + (\iota_{E_a} A_b)(d^A Q^b - dQ^b).$$

On the other hand,

$$0 = \iota_{E_a}(dQ^b \wedge A_b)$$

$$= \iota_{E_a} dQ^b \wedge A_b - dQ^b \wedge \iota_{E_a} A_b$$

=>

$$\iota_{E_a} dQ^b \wedge A_b = dQ^b \wedge \iota_{E_a} A_b$$

$$= (\iota_{E_a} A_b) dQ^b.$$

Therefore

$$\frac{\delta H}{\delta N^a} = \iota_{E_a} dQ^b \wedge A_b + \iota_{E_a} Q^b \wedge dA_b$$

$$= (\iota_{E_a} A_b) dQ^b + F_b \wedge \iota_{E_a} Q^b + (\iota_{E_a} A_b)(d^A Q^b - dQ^b)$$

$$= \iota_{E_a} A_b \wedge d^A Q^b + F_b \wedge \iota_{E_a} Q^b.$$

One can go further. In fact,

$$\iota_{E_a} Q^b = \iota_{E_a} - *\omega^b$$

$$= - \iota_{E_a} *\omega^b$$

$$= - *(\omega^b \wedge \omega^a)$$

$$= - \varepsilon_{bac}\omega^c$$

$$= \varepsilon_{abc}\omega^c$$

$$= - \varepsilon_{abc}*Q^c$$

$\Rightarrow$

$$F_b \wedge \iota_{E_a} Q^b = - \varepsilon_{abc} F_b \wedge *Q^c$$

$$= - (\vec{F} \overset{\times}{\wedge} *\vec{Q})_a.$$

LEMMA We have

$$- (\vec{F} \overset{\times}{\wedge} *\vec{Q})_a = - \sqrt{-1}\ (\overrightarrow{\text{Ric}\ F} \overset{\times}{\wedge} \vec{Q})_a.$$

[Start from the LHS -- then

$$- \varepsilon_{abc} F_b \wedge *Q^c$$

$$= - \varepsilon_{abc} q(F_b, Q^c) \text{vol}_q$$

$$= - \varepsilon_{abc} q(\frac{\sqrt{-1}}{2}\ \varepsilon_{buv} F_{uv}, Q^c) \text{vol}_q$$

$$= - \varepsilon_{abc} q(\frac{\sqrt{-1}}{2}\ \varepsilon_{buv} F_{uv}, - \frac{1}{2}\ \varepsilon_{crs} \omega^r \wedge \omega^s) \text{vol}_q$$

$$= \frac{\sqrt{-1}}{4}\ \varepsilon_{abc} \varepsilon_{buv} \varepsilon_{crs} q(F_{uv}, \omega^r \wedge \omega^s) \text{vol}_q$$

$$= \frac{\sqrt{-1}}{4}\,\varepsilon_{abc}\delta^{buv}{}_{crs}q(F_{uv},\omega^r\wedge\omega^s)\,\mathrm{vol}_q.$$

But

$$\delta^{buv}{}_{crs} = \begin{vmatrix} \delta^b{}_c & \delta^b{}_r & \delta^b{}_s \\ \delta^u{}_c & \delta^u{}_r & \delta^u{}_s \\ \delta^v{}_c & \delta^v{}_r & \delta^v{}_s \end{vmatrix}$$

$$= \delta^b{}_c\delta^u{}_r\delta^v{}_s - \delta^b{}_c\delta^u{}_s\delta^v{}_r - \delta^b{}_r\delta^u{}_c\delta^v{}_s$$

$$+ \delta^b{}_r\delta^u{}_s\delta^v{}_c + \delta^b{}_s\delta^u{}_c\delta^v{}_r - \delta^b{}_s\delta^u{}_r\delta^v{}_c.$$

And

1. $\delta^b{}_c\delta^u{}_r\delta^v{}_s q(F_{uv},\omega^r\wedge\omega^s)$

$$= \delta^b{}_c q(F_{uv},\omega^u\wedge\omega^v)$$

=>

$$\varepsilon_{abc}\delta^b{}_c q(F_{uv},\omega^u\wedge\omega^v)$$

$$= \varepsilon_{abb}q(F_{uv},\omega^u\wedge\omega^v) = 0.$$

2. $-\ \delta^b{}_c\delta^u{}_s\delta^v{}_r q(F_{uv},\omega^r\wedge\omega^s)$

$$= -\ \delta^b{}_c q(F_{uv},\omega^v\wedge\omega^u)$$

=>

$$- \varepsilon_{abc}\delta^{b}{}_{c}q(F_{uv},\omega^{v}\wedge\omega^{u})$$

$$= - \varepsilon_{abb}q(F_{uv},\omega^{v}\wedge\omega^{u}) = 0.$$

3. $$- \delta^{b}{}_{r}\delta^{u}{}_{c}\delta^{v}{}_{s}q(F_{uv},\omega^{r}\wedge\omega^{s})$$

$$= - q(F_{cv},\omega^{b}\wedge\omega^{v})$$

$$= q(F_{uc},\omega^{b}\wedge\omega^{u}).$$

4. $$\delta^{b}{}_{r}\delta^{u}{}_{s}\delta^{v}{}_{c}q(F_{uv},\omega^{r}\wedge\omega^{s})$$

$$= q(F_{uc},\omega^{b}\wedge\omega^{u}).$$

5. $$\delta^{b}{}_{s}\delta^{u}{}_{c}\delta^{v}{}_{r}q(F_{uv},\omega^{r}\wedge\omega^{s})$$

$$= q(F_{cv},\omega^{v}\wedge\omega^{b})$$

$$= q(F_{uc},\omega^{b}\wedge\omega^{u}).$$

6. $$- \delta^{b}{}_{s}\delta^{u}{}_{r}\delta^{v}{}_{c}q(F_{uv},\omega^{r}\wedge\omega^{s})$$

$$= - q(F_{uc},\omega^{u}\wedge\omega^{b})$$

$$= q(F_{uc},\omega^{b}\wedge\omega^{u}).$$

Consequently,

$$\frac{\sqrt{-1}}{4}\,\varepsilon_{abc}\delta^{buv}{}_{crs}\,q(F_{uv},\omega^{r}\wedge\omega^{s})\,\mathrm{vol}_q$$

$$= \frac{\sqrt{-1}}{4}\,\varepsilon_{abc}(4q(F_{uc},\omega^{b}\wedge\omega^{u}))\,\mathrm{vol}_q$$

$$= -\sqrt{-1}\,\varepsilon_{abc}\,q(F_{dc},\omega^{d}\wedge\omega^{b})\,\mathrm{vol}_q$$

$$= -\sqrt{-1}\,\varepsilon_{abc}\,q(\iota_{\omega^{d}}F_{dc},\omega^{b})\,\mathrm{vol}_q$$

$$= -\sqrt{-1}\,\varepsilon_{abc}\,q((\overrightarrow{\mathrm{Ric}}\;F)_{c},\omega^{b})\,\mathrm{vol}_q$$

$$= -\sqrt{-1}\,\varepsilon_{acb}\,q((\overrightarrow{\mathrm{Ric}}\;F)_{b},\omega^{c})\,\mathrm{vol}_q$$

$$= \sqrt{-1}\,\varepsilon_{abc}\,q((\overrightarrow{\mathrm{Ric}}\;F)_{b},\omega^{c})\,\mathrm{vol}_q$$

$$= \sqrt{-1}\,\varepsilon_{abc}\,q((\overrightarrow{\mathrm{Ric}}\;F)_{b},**\omega^{c})\,\mathrm{vol}_q$$

$$= \sqrt{-1}\,\varepsilon_{abc}(\overrightarrow{\mathrm{Ric}}\;F)_{b}\wedge*\omega^{c}$$

$$= -\sqrt{-1}\,\varepsilon_{abc}(\overrightarrow{\mathrm{Ric}}\;F)_{b}\wedge -*\omega^{c}$$

$$= -\sqrt{-1}\,\varepsilon_{abc}(\overrightarrow{\mathrm{Ric}}\;F)_{b}\wedge\Omega^{c}$$

$$= -\sqrt{-1}\,(\overrightarrow{\mathrm{Ric}}\;F\overset{\times}{\wedge}\vec{\Omega})_{a}.]$$

[Note: There is another way to write $(\vec{F} \overset{\times}{\wedge} *\vec{Q})_a$ which we shall use below. Thus, since F_a and Q^a are 2-forms,

$$\vec{F} \overset{\bullet}{\wedge} \vec{Q} = 0$$

=>

$$0 = \iota_{E_a}(F_b \wedge Q^b)$$

$$= \iota_{E_a} F_b \wedge Q^b + F_b \wedge \iota_{E_a} Q^b$$

$$= \iota_{E_a} F_b \wedge Q^b + F_b \wedge \iota_{E_a}(- *\omega^b)$$

=>

$$\iota_{E_a} \vec{F} \overset{\bullet}{\wedge} \vec{Q} = F_b \wedge \iota_{E_a} *\omega^b$$

$$= F_b \wedge *(\omega^b \wedge \omega^a)$$

$$= F_b \wedge \varepsilon_{bac} \omega^c$$

$$= - F_b \wedge \varepsilon_{bac} *Q^c$$

$$= - \varepsilon_{bac} F_b \wedge *Q^c$$

$$= \varepsilon_{abc} F_b \wedge *Q^c$$

$$= (\vec{F} \overset{\times}{\wedge} *\vec{Q})_a .]$$

Therefore

$$\frac{\delta H}{\delta N^a} = \iota_{E_a} A_b \wedge d^A Q^b - \sqrt{-1}\ (\overrightarrow{\mathrm{Ric}}\ F \overset{\times}{\wedge} \vec{Q})_a .$$

Definition:

$$H_D(\vec{N}) = \int_\Sigma L_{\vec{N}} \vec{Q} \overset{\bullet}{\wedge} \vec{A}$$

is the integrated <u>diffeomorphism constraint</u>;

$$H_R(\vec{Z}) = \int_\Sigma \vec{Z} \overset{\bullet}{\wedge} d^A \vec{Q}$$

is the integrated <u>rotational constraint</u>;

$$H_H(N) = \int_\Sigma - \sqrt{-1}\ N\vec{F} \overset{\bullet}{\wedge} *\vec{Q}$$

is the integrated <u>hamiltonian constraint</u>.

Remark: The preceding considerations imply that

$$H_D(\vec{N}) = \int_\Sigma [\vec{A}(\vec{N}) d^A \vec{Q} - \sqrt{-1}\ \vec{N}\cdot(\overrightarrow{\mathrm{Ric}}\ F \overset{\times}{\wedge} \vec{Q})]$$

and

$$H_H(N) = \int_\Sigma N(-\frac{F}{2})\mathrm{vol}_q .$$

[Note: Here

$$\vec{A}(\vec{N}) d^A \vec{Q}$$

$$= A_b(\vec{N}) d^A Q^b$$

$$= A_b(N^a E_a) d^A Q^b$$

$$= N^a \iota_{E_a} A_b \wedge d^A Q^b .$$

Incidentally, in the subset of $T^*{}_*\underline{Q}_{\underline{C}}$ where $d^A\vec{Q} = 0$, $H_D(\vec{N})$ reduces to

$$\int_\Sigma - \sqrt{-1}\ \vec{N}\cdot(\overrightarrow{\mathrm{Ric}}\ F \overset{\times}{\wedge} \vec{Q})$$

$$= \int_\Sigma - \iota_{\vec{N}} \vec{F} \overset{\bullet}{\wedge} \vec{Q}$$

$$\equiv \bar{H}_D(\vec{N}).]$$

Therefore

$$H = H_D + H_R + H_H$$

and we have

1. $\{H_D(\vec{N}_1), H_D(\vec{N}_2)\} = H_D([\vec{N}_1, \vec{N}_2]);$

2. $\{H_D(\vec{N}), H_R(\vec{Z})\} = H_R(L_{\vec{N}}\vec{Z});$

3. $\{H_D(\vec{N}), H_H(N)\} = H_H(L_{\vec{N}}N);$

4. $\{H_R(\vec{Z}_1), H_R(\vec{Z}_2)\} = -\sqrt{-1}\ H_R(\vec{Z}_1 \times \vec{Z}_2);$

5. $\{H_R(\vec{Z}), H_H(N)\} = 0;$

6. $\{H_H(N_1), H_H(N_2)\}$

$$= H_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)$$

$$- H_R(\vec{A}(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)).$$

Remark: A <u>constraint</u> is a function $f: T^*{}_*\underline{Q}_{\underline{C}} \to \underline{C}$ such that $f|\mathrm{Con}_{*\underline{Q}_{\underline{C}}} = 0$.

Thus, by construction, $H_D(\vec{N})$, $H_R(\vec{Z})$, and $H_H(N)$ are constraints, these being termed <u>primary</u>. The foregoing relations then imply that the Poisson bracket of two primary constraints is a constraint.

Items 1 and 3 are established in the usual way, so we shall concentrate on Items 2, 4, 5, and 6.

<u>Ad 2</u>: We have

$$\{H_D(\vec{N}), H_R(\vec{Z})\}$$

$$= \int_\Sigma \left[\frac{\delta H_R(\vec{Z})}{\delta \vec{A}} \wedge \frac{\delta H_D(\vec{N})}{\delta \vec{Q}} - \frac{\delta H_D(\vec{N})}{\delta \vec{A}} \wedge \frac{\delta H_R(\vec{Z})}{\delta \vec{Q}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_R(\vec{Z})}{\delta A_a} \wedge \frac{\delta H_D(\vec{N})}{\delta Q^a} - \frac{\delta H_D(\vec{N})}{\delta A_a} \wedge \frac{\delta H_R(\vec{Z})}{\delta Q^a} \right]$$

$$= \int_\Sigma [L_{\vec{N}} A_a \wedge \sqrt{-1}\ (\vec{Z} \overset{\times}{\wedge} \vec{Q})_a + L_{\vec{N}} Q^a \wedge d^A Z_a].$$

Consider first

$$\int_\Sigma L_{\vec{N}} Q^a \wedge d^A Z_a.$$

Thus

$$L_{\vec{N}} Q^a \wedge d^A Z_a$$

$$= L_{\vec{N}} Q^a \wedge (dZ_a + \sqrt{-1}\ (\vec{A} \overset{\times}{\wedge} \vec{Z})_a)$$

$$= L_{\vec{N}} Q^a \wedge dZ_a + \sqrt{-1}\ L_{\vec{N}} Q^a \wedge (\vec{A} \overset{\times}{\wedge} \vec{Z})_a.$$

But

$$d(L_{\vec{N}}Q^a \wedge Z_a)$$

$$= dL_{\vec{N}}Q^a \wedge Z_a + L_{\vec{N}}Q_a \wedge dZ_a$$

$$= L_{\vec{N}}dQ^a \wedge Z_a + L_{\vec{N}}Q_a \wedge dZ_a$$

=>

$$0 = \int_\Sigma d(L_{\vec{N}}Q^a \wedge Z_a)$$

$$= \int_\Sigma L_{\vec{N}}dQ^a \wedge Z_a + \int_\Sigma L_{\vec{N}}Q_a \wedge dZ_a$$

=>

$$\int_\Sigma L_{\vec{N}}Q^a \wedge dZ_a = - \int_\Sigma L_{\vec{N}}dQ^a \wedge Z_a.$$

And

$$0 = \int_\Sigma L_{\vec{N}}(dQ^a \wedge Z_a)$$

$$= \int_\Sigma L_{\vec{N}}dQ^a \wedge Z_a + \int_\Sigma dQ^a \wedge L_{\vec{N}}Z_a$$

=>

$$- \int_\Sigma L_{\vec{N}}dQ^a \wedge Z_a = \int_\Sigma dQ^a \wedge L_{\vec{N}}Z_a$$

$$= \int_\Sigma L_{\vec{N}}Z_a \wedge dQ^a.$$

Let us now turn to

$$\int_\Sigma \sqrt{-1}\ L_{\vec{N}} Q^a \wedge (\vec{A} \overset{\times}{\wedge} \vec{Z})_a$$

or still,

$$\int_\Sigma -\sqrt{-1}\ Q^a \wedge L_{\vec{N}} (\vec{A} \overset{\times}{\wedge} \vec{Z})_a,$$

$Q^a \wedge (\vec{A} \overset{\times}{\wedge} \vec{Z})_a$ being a 3-form. Write

$$-\sqrt{-1}\ Q^a \wedge L_{\vec{N}} (\vec{A} \overset{\times}{\wedge} \vec{Z})_a$$

$$= -\sqrt{-1}\ Q^a \wedge (L_{\vec{N}} (\varepsilon_{abc} A_b \wedge Z_c))$$

$$= -\sqrt{-1}\ Q^a \wedge \varepsilon_{abc} L_{\vec{N}} A_b \wedge Z_c$$

$$-\sqrt{-1}\ Q^a \wedge \varepsilon_{abc} A_b \wedge L_{\vec{N}} Z_c.$$

Rewrite the second term as

$$L_{\vec{N}} Z_c \wedge (-\sqrt{-1}\ \varepsilon_{abc} A_b \wedge Q^a)$$

or still,

$$L_{\vec{N}} Z_c \wedge (-\sqrt{-1}\ \varepsilon_{cab} A_b \wedge Q^a)$$

or still,

$$L_{\vec{N}} Z_c \wedge (\sqrt{-1}\ \varepsilon_{cba} A_b \wedge Q^a)$$

or still,

$$L_{\vec{N}} Z_a \wedge \sqrt{-1}\ (\vec{A} \overset{\times}{\wedge} \vec{Q})_a.$$

Therefore

$$\int_\Sigma L_{\vec{N}} Q^a \wedge d^A Z_a$$

$$= \int_\Sigma L_{\vec{N}} Z_a \wedge dQ^a + \int_\Sigma L_{\vec{N}} Z_a \wedge \sqrt{-1}\ (\vec{A} \overset{\times}{\wedge} \vec{Q})_a$$

$$+ \int_\Sigma - \sqrt{-1}\ Q^a \wedge \varepsilon_{abc} L_{\vec{N}} A_b \wedge Z_c .$$

It remains to compare

$$- \sqrt{-1}\ Q^a \wedge \varepsilon_{abc} L_{\vec{N}} A_b \wedge Z_c$$

with

$$L_{\vec{N}} A_a \wedge \sqrt{-1}\ (\vec{Z} \overset{\times}{\wedge} \vec{Q})_a$$

or

$$- \sqrt{-1}\ \varepsilon_{abc} Q^a \wedge L_{\vec{N}} A_b \wedge Z_c$$

with

$$\sqrt{-1}\ \varepsilon_{abc} Q^c \wedge L_{\vec{N}} A_a \wedge Z_b .$$

In the last line, change

$$\left[\begin{array}{l} c \rightarrow a \\ a \rightarrow b \\ b \rightarrow c \end{array}\right.$$

to get

$$\sqrt{-1}\ \varepsilon_{bca} Q^a \wedge L_{\vec{N}} A_b \wedge Z_c$$

$$= \sqrt{-1}\ \varepsilon_{abc} Q^a \wedge L_{\vec{N}} A_b \wedge Z_c .$$

The terms in question thus cancel, leaving

$$\int_\Sigma L_{\vec{N}} Z_a \wedge (dQ^a + \sqrt{-1}\ (\vec{A} \overset{\times}{\wedge} \vec{Q})_a)$$

$$= \int_\Sigma L_{\vec{N}} \vec{Z} \overset{\cdot}{\wedge} d^A\vec{Q}$$

$$= H_R(L_{\vec{N}}\vec{Z}).$$

<u>Ad 4</u>: We have

$$\{H_R(\vec{Z}_1), H_R(\vec{Z}_2)\}$$

$$= \int_\Sigma \left[\frac{\delta H_R(\vec{Z}_2)}{\delta \vec{A}} \wedge \frac{\delta H_R(\vec{Z}_1)}{\delta \vec{Q}} - \frac{\delta H_R(\vec{Z}_1)}{\delta \vec{A}} \wedge \frac{\delta H_R(\vec{Z}_2)}{\delta \vec{Q}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_R(\vec{Z}_2)}{\delta A_a} \wedge \frac{\delta H_R(\vec{Z}_1)}{\delta Q^a} - \frac{\delta H_R(\vec{Z}_1)}{\delta A_a} \wedge \frac{\delta H_R(\vec{Z}_2)}{\delta Q^a} \right]$$

$$= \sqrt{-1} \int_\Sigma [d^A\vec{Z}_1 \overset{\cdot}{\wedge} (\vec{Z}_2 \overset{\times}{\wedge} \vec{Q}) - d^A\vec{Z}_2 \overset{\cdot}{\wedge} (\vec{Z}_1 \overset{\times}{\wedge} \vec{Q})]$$

$$= \sqrt{-1} \int_\Sigma [(d^A\vec{Z}_1 \times \vec{Z}_2) \overset{\cdot}{\wedge} \vec{Q} - (d^A\vec{Z}_2 \times \vec{Z}_1) \overset{\cdot}{\wedge} \vec{Q}]$$

$$= \sqrt{-1} \int_\Sigma [(d^A\vec{Z}_1 \times \vec{Z}_2 + \vec{Z}_1 \times d^A\vec{Z}_2) \overset{\cdot}{\wedge} \vec{Q}].$$

But

$$d((\vec{Z}_1 \times \vec{Z}_2) \overset{\cdot}{\wedge} \vec{Q})$$

$$= d^A(\vec{Z}_1 \times \vec{Z}_2) \overset{\cdot}{\wedge} \vec{Q} + (\vec{Z}_1 \times \vec{Z}_2) \overset{\cdot}{\wedge} d^A\vec{Q}$$

$$= (d^A\vec{Z}_1 \times \vec{Z}_2) \overset{\bullet}{\wedge} \vec{Q} + (\vec{Z}_1 \times d^A\vec{Z}_2) \overset{\bullet}{\wedge} \vec{Q} + (\vec{Z}_1 \times \vec{Z}_2) \overset{\bullet}{\wedge} d^A\vec{Q}$$

=>

$$(d^A\vec{Z}_1 \times \vec{Z}_2 + \vec{Z}_1 \times d^A\vec{Z}_2) \overset{\bullet}{\wedge} \vec{Q}$$

$$= - (\vec{Z}_1 \times \vec{Z}_2) \overset{\bullet}{\wedge} d^A\vec{Q} + d((\vec{Z}_1 \times \vec{Z}_2) \overset{\bullet}{\wedge} \vec{Q}).$$

Therefore

$$\{H_R(\vec{Z}_1), H_R(\vec{Z}_2)\} = - \sqrt{-1}\, H_R(\vec{Z}_1 \times \vec{Z}_2).$$

<u>Ad 5</u>: We have

$$\{H_R(\vec{Z}), H_H(N)\}$$

$$= \int_\Sigma \left[\frac{\delta H_H(N)}{\delta\vec{A}} \wedge \frac{\delta H_R(\vec{Z})}{\delta\vec{Q}} - \frac{\delta H_R(\vec{Z})}{\delta\vec{A}} \wedge \frac{\delta H_H(N)}{\delta\vec{Q}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_H(N)}{\delta A_a} \wedge \frac{\delta H_R(\vec{Z})}{\delta Q^a} - \frac{\delta H_R(\vec{Z})}{\delta A_a} \wedge \frac{\delta H_H(N)}{\delta Q^a} \right]$$

$$= \int_\Sigma [\sqrt{-1}\, d^A (N{*}Q^a) \wedge d^A Z_a$$

$$+ \sqrt{-1}\, (\vec{Z} \overset{\times}{\wedge} \vec{Q})_a \wedge (N(\overrightarrow{\text{Ric}}\ F)_a + \frac{1}{4} NF{*}Q^a)].$$

Write

$$\sqrt{-1}\, d^A (N{*}Q^a) \wedge d^A Z_a$$

$$= \sqrt{-1}\, d^A Z_a \wedge d^A (N{*}Q^a)$$

$$= - \sqrt{-1}\, (dZ_a + A^a{}_b \wedge Z_b) \wedge (dN \wedge \omega^a + N d\omega^a + NA^a{}_c \wedge \omega^c).$$

• $d(Z_a \wedge dN \wedge \omega^a) = dZ_a \wedge dN \wedge \omega^a$

$+ Z_a \wedge d^2 N \wedge \omega^a - Z_a \wedge dN \wedge d\omega^a$

=>

$$\int_\Sigma dZ_a \wedge dN \wedge \omega^a = \int_\Sigma Z_a \wedge dN \wedge d\omega^a.$$

• $d(Z_a \wedge N \wedge d\omega^a) = dZ_a \wedge N \wedge d\omega^a$

$+ Z_a \wedge dN \wedge d\omega^a + Z_a \wedge N \wedge d^2\omega^a$

=>

$$\int_\Sigma dZ_a \wedge N \wedge d\omega^a = - \int_\Sigma Z_a \wedge dN \wedge d\omega^a.$$

Matters thus reduce to consideration of

$$dZ_a \wedge N \wedge A^a{}_c \wedge \omega^c$$

and

1. $A^a{}_b \wedge Z_b \wedge dN \wedge \omega^a$
2. $A^a{}_b \wedge Z_b \wedge N \wedge d\omega^a$
3. $A^a{}_b \wedge Z_b \wedge N \wedge A^a{}_c \wedge \omega^c$.

• $d(Z_a \wedge N \wedge A^a{}_c \wedge \omega^c)$

$= dZ_a \wedge N \wedge A^a{}_c \wedge \omega^c + Z_a \wedge dN \wedge A^a{}_c \wedge \omega^c$

$+ Z_a \wedge N \wedge dA^a{}_c \wedge \omega^c - Z_a \wedge N \wedge A^a{}_c \wedge d\omega^c.$

1. $-Z_a \wedge dN \wedge A^a{}_c \wedge \omega^c$

$$= A^a{}_c \wedge Z_a \wedge dN \wedge \omega^c$$

$$= A^c{}_a \wedge Z_c \wedge dN \wedge \omega^a$$

$$= A^b{}_a \wedge Z_b \wedge dN \wedge \omega^a$$

$$= -A^a{}_b \wedge Z_b \wedge dN \wedge \omega^a.$$

2. $Z_a \wedge N \wedge A^a{}_c \wedge d\omega^c$

$$= A^a{}_c \wedge Z_a \wedge N \wedge d\omega^c$$

$$= A^c{}_a \wedge Z_c \wedge N \wedge d\omega^a$$

$$= A^b{}_a \wedge Z_b \wedge N \wedge d\omega^a$$

$$= -A^a{}_b \wedge Z_b \wedge N \wedge d\omega^a.$$

There remains $-\sqrt{-1}$ times

$$A^a{}_b \wedge Z_b \wedge N \wedge A^a{}_c \wedge \omega^c - Z_a \wedge N \wedge dA^a{}_c \wedge \omega^c$$

$$= A^b{}_a \wedge Z_a \wedge N \wedge A^b{}_c \wedge \omega^c - Z_a \wedge N \wedge dA^a{}_b \wedge \omega^b$$

$$= A^c{}_a \wedge Z_a \wedge N \wedge A^c{}_b \wedge \omega^b - Z_a \wedge N \wedge dA^a{}_b \wedge \omega^b$$

$$= Z_a \wedge N \wedge (-dA^a{}_b + A^c{}_a \wedge A^c{}_b) \wedge \omega^b$$

$$= Z_a \wedge N \wedge (-\, dA^a{}_b - A^a{}_c \wedge A^c{}_b) \wedge \omega^b$$

$$= Z_a \wedge N \wedge -\, F_{ab} \wedge \omega^b$$

or still,

$$\sqrt{-1}\; N(Z_a \wedge F_{ab} \wedge \omega^b)$$

$$= \sqrt{-1}\; N(Z_a \wedge -\, \sqrt{-1}\; \varepsilon_{abc} F_c \wedge \omega^b)$$

$$= N\varepsilon_{abc} Z_a \wedge F_c \wedge \omega^b$$

$$= N\varepsilon_{bac} Z_b \wedge F_c \wedge \omega^a$$

$$= -\, N\varepsilon_{abc} Z_b \wedge F_c \wedge \omega^a$$

$$= N\varepsilon_{abc} Z_b \wedge F_c \wedge -\, \omega^a$$

$$= N\varepsilon_{abc} Z_b \wedge F_c \wedge * Q^a$$

$$= N(\vec{Z} \overset{\times}{\wedge} \vec{F})_a \wedge * Q^a$$

$$= N(\vec{Z} \overset{\times}{\wedge} \vec{F}) \overset{\bullet}{\wedge} * \vec{Q}$$

$$= N(\vec{Z} \overset{\bullet}{\wedge} (\vec{F} \overset{\times}{\wedge} * \vec{Q}))$$

$$= N((\vec{F} \overset{\times}{\wedge} * \vec{Q}) \overset{\bullet}{\wedge} \vec{Z}).$$

This has now to be combined with

$$\sqrt{-1}\; (\vec{Z} \overset{\times}{\wedge} \vec{Q})_a \wedge (N(\overrightarrow{\mathrm{Ric}}\; F)_a + \frac{1}{4} NF * Q^a).$$

Write

$$\sqrt{-1}\ (\vec{Z} \overset{\times}{\wedge} \vec{Q})_a \wedge N(\overrightarrow{\mathrm{Ric}}\ F)_a$$

$$= N\ \sqrt{-1}\ (\vec{Z} \overset{\times}{\wedge} \vec{Q})\ \dot{\wedge}\ \overrightarrow{\mathrm{Ric}}\ F$$

$$= -\ N\ \sqrt{-1}\ (\vec{Q} \overset{\times}{\wedge} \vec{Z})\ \dot{\wedge}\ \overrightarrow{\mathrm{Ric}}\ F$$

$$= -\ N\ \sqrt{-1}\ \overrightarrow{\mathrm{Ric}}\ F\ \dot{\wedge}\ (\vec{Q} \overset{\times}{\wedge} \vec{Z})$$

$$= -\ N\ \sqrt{-1}\ (\overrightarrow{\mathrm{Ric}}\ F \overset{\times}{\wedge} \vec{Q})\ \dot{\wedge}\ \vec{Z}.$$

Then

$$N((\vec{F} \overset{\times}{\wedge} *\vec{Q})\ \dot{\wedge}\ \vec{Z})\ -\ N\ \sqrt{-1}\ (\overrightarrow{\mathrm{Ric}}\ F \overset{\times}{\wedge} \vec{Q})\ \dot{\wedge}\ \vec{Z}$$

$$= N(\vec{F} \overset{\times}{\wedge} *\vec{Q} - \sqrt{-1}\ (\overrightarrow{\mathrm{Ric}}\ F \overset{\times}{\wedge} \vec{Q}))\ \dot{\wedge}\ \vec{Z}$$

$$= 0.$$

Finally

$$\sqrt{-1}\ (\vec{Z} \overset{\times}{\wedge} \vec{Q})_a \wedge \frac{1}{4}\ NF*Q^a$$

$$= \frac{\sqrt{-1}}{4}\ NF\varepsilon_{abc} Z_b \wedge Q_c \wedge *Q^a$$

$$= \frac{\sqrt{-1}}{4}\ NF\varepsilon_{abc} Z_b \wedge -\ *\omega_c \wedge -\ **\omega^a$$

$$= \frac{\sqrt{-1}}{4}\ NF\varepsilon_{abc} Z_b \wedge \omega^a \wedge *\omega_c$$

$$= \frac{\sqrt{-1}}{4}\ NF\varepsilon_{abc} Z_b\, q(\omega^a, \omega_c)\, \mathrm{vol}_q$$

$$= \frac{\sqrt{-1}}{4} NF\varepsilon_{abc}Z_b\delta^a{}_c vol_q$$

$$= \frac{\sqrt{-1}}{4} NF\varepsilon_{aba}Z_b vol_q$$

$$= 0.$$

Therefore

$$\{H_R(\vec{Z}), H_H(N)\} = 0.$$

Ad 6: We have

$$\{H_H(N_1), H_H(N_2)\}$$

$$= \int_\Sigma \left[\frac{\delta H_H(N_2)}{\delta\vec{A}} \wedge \frac{\delta H_H(N_1)}{\delta\vec{Q}} - \frac{\delta H_H(N_1)}{\delta\vec{A}} \wedge \frac{\delta H_H(N_2)}{\delta\vec{Q}} \right]$$

$$= \int_\Sigma \left[\frac{\delta H_H(N_2)}{\delta A_a} \wedge \frac{\delta H_H(N_1)}{\delta Q^a} - \frac{\delta H_H(N_1)}{\delta A_a} \wedge \frac{\delta H_H(N_2)}{\delta Q^a} \right]$$

$$= \int_\Sigma [- \sqrt{-1}\, N_1((\overrightarrow{Ric}\ F)_a + \frac{1}{4} F{*}Q^a) \wedge d^A(N_2{*}Q^a)$$

$$+ \sqrt{-1}\, N_2((\overrightarrow{Ric}\ F)_a + \frac{1}{4} F{*}Q^a) \wedge d^A(N_1{*}Q^a)].$$

Write

$$- \sqrt{-1}\, N_1(\frac{1}{4} F{*}Q^a) \wedge d^A(N_2{*}Q^a)$$

$$= - \sqrt{-1}\, N_1(\frac{1}{4} F{*}\vec{Q})\ \dot{\wedge}\ d^A(N_2{*}\vec{Q})$$

$$= - \sqrt{-1}\, N_1(\frac{1}{4} F{*}\vec{Q})\ \dot{\wedge}\ (d(N_2{*}\vec{Q}) + \sqrt{-1}\,\vec{A}\ \overset{\times}{\wedge}\ N_2{*}\vec{Q})$$

$$= - \sqrt{-1}\ (\frac{1}{4} F{*}\vec{Q})\ \dot{\wedge}\ (N_1 dN_2 \wedge {*}\vec{Q})$$

$$- \sqrt{-1}\ (\tfrac{1}{4}\ F{*}\vec{Q})\ \dot{\wedge}\ (N_1N_2\wedge d{*}\vec{Q} + \sqrt{-1}\ \vec{A}\ \overset{\times}{\wedge}\ N_1N_2{*}\vec{Q}).$$

By the same token,

$$\sqrt{-1}\ N_2(\tfrac{1}{4}\ F{*}Q^a)\wedge d^A(N_1{*}Q^a)$$

$$= \sqrt{-1}\ (\tfrac{1}{4}\ F{*}\vec{Q})\ \dot{\wedge}\ (N_2dN_1\wedge{*}\vec{Q})$$

$$+ \sqrt{-1}\ (\tfrac{1}{4}\ F{*}\vec{Q})\ \dot{\wedge}\ (N_2N_1\wedge d{*}\vec{Q} + \sqrt{-1}\ \vec{A}\ \overset{\times}{\wedge}\ N_2N_1{*}\vec{Q}).$$

Combining terms thus gives

$$\sqrt{-1}\ (\tfrac{1}{4}\ F{*}\vec{Q})\ \dot{\wedge}\ (N_2dN_1 - N_1dN_2)\wedge{*}\vec{Q},$$

which, of course, is equal to zero. Next

$$- \sqrt{-1}\ N_1(\overrightarrow{\text{Ric}}\ F)_a\wedge d^A(N_2{*}Q^a)$$

$$= - \sqrt{-1}\ N_1\overrightarrow{\text{Ric}}\ F\ \dot{\wedge}\ d^A(N_2{*}\vec{Q})$$

$$= - \sqrt{-1}\ N_1\overrightarrow{\text{Ric}}\ F\ \dot{\wedge}\ (dN_2\wedge{*}\vec{Q} + N_2\wedge d{*}\vec{Q} + \sqrt{-1}\ \vec{A}\ \overset{\times}{\wedge}\ N_2{*}\vec{Q}).$$

Now change the sign, switch the roles of N_1 and N_2, and add -- then we get

$$\sqrt{-1}\ \overrightarrow{\text{Ric}}\ F\ \dot{\wedge}\ (N_2dN_1 - N_1dN_2)\wedge{*}\vec{Q}$$

or still,

$$\sqrt{-1}\ (\overrightarrow{\text{Ric}}\ F\ \dot{\wedge}\ {*}\vec{Q})\wedge(N_1dN_2 - N_2dN_1).$$

Put, for the moment,

$$\begin{cases} \vec{\alpha} = \overrightarrow{\text{Ric}}\ F \\ \beta = N_1dN_2 - N_2dN_1. \end{cases}$$

Then we claim that

$$(\vec{\alpha} \overset{\cdot}{\wedge} *\vec{Q}) \wedge \beta = (\vec{\alpha} \overset{\times}{\wedge} \vec{Q}) \overset{\cdot}{\wedge} q(\beta, *\vec{Q}).$$

To establish this, note that the LHS equals

$$-\alpha_a \wedge \omega^a \wedge \beta.$$

On the other hand, the RHS equals

$$(\vec{\alpha} \overset{\times}{\wedge} \vec{Q})_a \wedge q(\beta, *Q^a)$$

$$= \varepsilon_{abc} \alpha_b \wedge Q^c \wedge q(\beta, *Q^a).$$

It will be simplest to work from left to right. So let

$$\beta = q(\beta, \omega^b)\omega^b.$$

Then

$$-\alpha_a \wedge \omega^a \wedge \beta$$

$$= -\alpha_a \wedge \omega^a \wedge q(\beta, \omega^b)\omega^b$$

$$= -\alpha_a \wedge \omega^a \wedge \omega^b \wedge q(\beta, \omega^b)$$

$$= -\alpha_a \wedge \varepsilon_{abc} *\omega^c \wedge q(\beta, \omega^b)$$

$$= \alpha_a \wedge \varepsilon_{abc} Q^c \wedge q(\beta, \omega^b)$$

$$= \alpha_b \wedge \varepsilon_{bac} Q^c \wedge q(\beta, \omega^a)$$

$$= \varepsilon_{bac} \alpha_b \wedge Q^c \wedge q(\beta, \omega^a)$$

$$= - \varepsilon_{abc}\alpha_b \wedge \Omega^c \wedge q(\beta, \omega^a)$$

$$= \varepsilon_{abc}\alpha_b \wedge \Omega^c \wedge q(\beta, - \omega^a)$$

$$= \varepsilon_{abc}\alpha_b \wedge \Omega^c \wedge q(\beta, *\Omega^a).$$

Hence the claim. In summary,

$$\sqrt{-1}\ (\overrightarrow{\text{Ric}}\ F \overset{\bullet}{\wedge} *\vec{\Omega}) \wedge (N_1 dN_2 - N_2 dN_1)$$

$$= \sqrt{-1}\ (\overrightarrow{\text{Ric}}\ F \overset{\times}{\wedge} \vec{\Omega}) \overset{\bullet}{\wedge} q(N_1 dN_2 - N_2 dN_1, *\vec{\Omega}).$$

But

$$N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1$$

$$= q(N_1 dN_2 - N_2 dN_1, \omega^a) E_a$$

$$= - q(N_1 dN_2 - N_2 dN_1, *\Omega^a) E_a.$$

And

$$\int_\Sigma - \sqrt{-1}\ (N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1) \cdot (\overrightarrow{\text{Ric}}\ F \overset{\times}{\wedge} \vec{\Omega})$$

$$= H_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)$$

$$- \int_\Sigma \vec{A}(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1) d^A \vec{\Omega}$$

$$= H_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)$$

$$- H_R(\vec{A}(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1)).$$

<u>Section 60</u>: <u>Densitized Variables</u> The assumptions and notation are those of the standard setup but with the restriction that $n = 4$.

The Ashtekar hamiltonian

$$H(\vec{Q},\vec{A};N,\vec{N},\vec{Z})$$

$$= \int_\Sigma L_{\vec{N}}\vec{Q} \,\dot{\wedge}\, \vec{A} + \int_\Sigma \vec{Z} \,\dot{\wedge}\, d^{A}\vec{Q} + \int_\Sigma - \sqrt{-1}\, N\vec{F} \,\dot{\wedge}\, *\vec{Q}$$

is globally defined but this is not the case of its traditional counterpart which is only defined locally.

Let x^1,x^2,x^3 be coordinates on Σ consistent with the underlying orientation of Σ.

[Note: If the domain of x^1,x^2,x^3 is U, then, for economy of notation, we shall pretend in what follows that $U = \Sigma$.]

Convention: μ,ν and $\alpha,\beta,\gamma,\delta$ are coordinate indices that run between 1 and 3.

<u>Local Formulas</u>

1. $\dfrac{\partial}{\partial x^\mu} = e^a{}_\mu E_a$ & $E_a = e^\mu{}_a \dfrac{\partial}{\partial x^\mu}$.

2. $e^\mu{}_a e^a{}_\nu = \delta^\mu{}_\nu$ & $e^a{}_\mu e^\mu{}_b = \delta^a{}_b$.

3. $q_{\mu\nu} = e^a{}_\mu e^a{}_\nu$ & $q^{\mu\nu} = e^\mu{}_a e^\nu{}_a$.

<u>LEMMA</u> We have

$$\det[q_{\mu\nu}] = \det[e^a{}_\mu]\det[e^a{}_\nu].$$

[In fact,

$$e^{a}{}_{\mu} = q_{\mu\nu}e^{\nu}{}_{a}.]$$

Abusing the notation, let

$$\sqrt{q} = \det[e^{a}{}_{\mu}]$$

and then put

$$E^{\mu}{}_{a} = \sqrt{q}\, e^{\mu}{}_{a}.$$

[Note: Accordingly,

$$E^{\mu}{}_{a}E^{\nu}{}_{a} = (\det q)e^{\mu}{}_{a}e^{\nu}{}_{a}$$

$$= (\det q)q^{\mu\nu}.]$$

<u>LEMMA</u> We have

$$\varepsilon_{\alpha\beta\gamma}(*\omega^{a})_{\alpha\beta} = 2E^{\gamma}{}_{a}.$$

[Write

$$*\omega^{a} = \frac{1}{2}\,\varepsilon_{abc}\omega^{b}\wedge\omega^{c}.$$

Then

$$\varepsilon_{\alpha\beta\gamma}(*\omega^{a})_{\alpha\beta} = \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma}\varepsilon_{abc}(\omega^{b}\wedge\omega^{c})_{\alpha\beta}.$$

But

$$\begin{bmatrix} \omega^{b} = e^{b}{}_{\mu}dx^{\mu} \\ \omega^{c} = e^{c}{}_{\nu}dx^{\nu} \end{bmatrix}$$

=>

$$\omega^b \wedge \omega^c = e^b{}_\mu e^c{}_\nu dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2} (\omega^b \wedge \omega^c)_{\mu\nu} dx^\mu \wedge dx^\nu .$$

Therefore

$$\varepsilon_{\alpha\beta\gamma}(*\omega^a)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}(\varepsilon_{abc} e^b{}_\alpha e^c{}_\beta)$$

$$= \varepsilon_{\alpha\beta\gamma}(\varepsilon_{bca} e^b{}_\alpha e^c{}_\beta) .$$

Let $A = [e^a{}_\alpha]$ — then

$$\varepsilon_{bca} \det A = \varepsilon_{\alpha\beta\gamma} e^b{}_\alpha e^c{}_\beta e^a{}_\gamma$$

=>

$$e^{a'}{}_\gamma \cdot \frac{1}{2 \det A} \varepsilon_{\alpha\beta\gamma}(\varepsilon_{bca''} e^b{}_\alpha e^c{}_\beta)$$

$$= \frac{1}{2 \det A} \varepsilon_{bca''}(\varepsilon_{\alpha\beta\gamma} e^b{}_\alpha e^c{}_\beta e^{a'}{}_\gamma)$$

$$= \frac{1}{2 \det A} \varepsilon_{bca''}(\varepsilon_{bca'} \det A)$$

$$= \frac{1}{2} \varepsilon_{bca'} \varepsilon_{bca''} = \delta^{a'}{}_{a''}$$

=>

$$e^\gamma{}_a = (A^{-1})^\gamma{}_a = \frac{1}{2 \det A} \varepsilon_{\alpha\beta\gamma}(\varepsilon_{bca} e^b{}_\alpha e^c{}_\beta)$$

=>

$$\varepsilon_{\alpha\beta\gamma}(*\omega^a)_{\alpha\beta} = 2(\det A) e^\gamma{}_a$$

$$= 2\sqrt{q}\, e^{\gamma}{}_{a}$$

$$= 2E^{\gamma}{}_{a}.]$$

We are now in a position to discuss the local version of H.

<u>Analysis of $H_D(\vec{N})$:</u> In the literature, it is customary to restrict attention to $\bar{H}_D(\vec{N})$ which, by definition, is

$$\int_\Sigma - \iota_{\vec{N}} \vec{F} \stackrel{\bullet}{\wedge} \vec{Q}.$$

Here

$$- \iota_{\vec{N}} \vec{F} \stackrel{\bullet}{\wedge} \vec{Q} = - \iota_{\vec{N}} F^a \wedge Q^a.$$

Write

$$\vec{N} = N^a E_a = N^\alpha \frac{\partial}{\partial x^\alpha}$$

and

$$F^a = \frac{1}{2} F^a{}_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

Then

$$\begin{aligned} &\iota_{\vec{N}} (dx^\alpha \wedge dx^\beta) \\ &= (\iota_{\vec{N}} dx^\alpha) \wedge dx^\beta - (\iota_{\vec{N}} dx^\beta) \wedge dx^\alpha \\ &= dx^\alpha(\vec{N})\, dx^\beta - dx^\beta(\vec{N})\, dx^\alpha \\ &= N^\alpha dx^\beta - N^\beta dx^\alpha. \end{aligned}$$

And

$$- F^{a}{}_{\alpha\beta} N^{\beta} dx^{\alpha} = - F^{a}{}_{\beta\alpha} N^{\alpha} dx^{\beta}$$

$$= F^{a}{}_{\alpha\beta} N^{\alpha} dx^{\beta}$$

=>

$$\iota_{\vec{N}} F^{a} = N^{\alpha} F^{a}{}_{\alpha\beta} dx^{\beta}.$$

Write

$$Q^{a} = \frac{1}{2} Q^{a}{}_{\gamma\delta} dx^{\gamma} \wedge dx^{\delta}.$$

Then

$$dx^{\beta} \wedge dx^{\gamma} \wedge dx^{\delta} = \varepsilon_{\beta\gamma\delta} d^{3}x$$

$$= \varepsilon_{\gamma\delta\beta} d^{3}x.$$

And

$$\frac{1}{2} \varepsilon_{\gamma\delta\beta} Q^{a}{}_{\gamma\delta} = - E^{\beta}{}_{a}$$

=>

$$\iota_{\vec{N}} F^{a} \wedge Q^{a} = - N^{\alpha} F^{a}{}_{\alpha\beta} E^{\beta}{}_{a} d^{3}x.$$

Therefore

$$\vec{H}_{D}(\vec{N}) = \int_{\Sigma} - \iota_{\vec{N}} \vec{F} \stackrel{\cdot}{\wedge} \vec{Q}$$

$$= \int_{\Sigma} N^{\alpha} F^{a}{}_{\alpha\beta} E^{\beta}{}_{a} d^{3}x.$$

Analysis of $\mathcal{H}_R(\vec{Z})$: By definition,

$$\int_\Sigma \vec{Z} \dot{\wedge} d^A\vec{Q} = \int_\Sigma Z^a(d^A Q^a),$$

where

$$d^A Q^a = dQ^a - \sqrt{-1}\,\epsilon_{abc}A^c \wedge Q^b.$$

Write

$$Q^a = \frac{1}{2} Q^a{}_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

Then

$$dQ^a = \frac{1}{2} \frac{\partial Q^a{}_{\alpha\beta}}{\partial x^\gamma} dx^\gamma \wedge dx^\alpha \wedge dx^\beta$$

$$= \frac{1}{2} \frac{\partial Q^a{}_{\alpha\beta}}{\partial x^\gamma} \epsilon_{\gamma\alpha\beta} d^3x$$

$$= \partial_\gamma (\frac{1}{2} \epsilon_{\alpha\beta\gamma} Q^a{}_{\alpha\beta}) d^3x$$

$$= - \partial_\gamma E^\gamma{}_a d^3x$$

$$= - \partial_\alpha E^\alpha{}_a d^3x.$$

Write

$$A^c = A^c{}_\gamma dx^\gamma.$$

Then

$$- \sqrt{-1}\,\epsilon_{abc} A^c \wedge Q^b$$

$$= - \sqrt{-1}\,\epsilon_{abc} A^c{}_\gamma \frac{1}{2} Q^b{}_{\alpha\beta} dx^\gamma \wedge dx^\alpha \wedge dx^\beta$$

$$= -\sqrt{-1}\ \varepsilon_{abc}A^{c}{}_{\gamma}\left(\frac{1}{2}\ \varepsilon_{\alpha\beta\gamma}\Omega^{b}{}_{\alpha\beta}\right)d^{3}x$$

$$= \sqrt{-1}\ \varepsilon_{abc}A^{c}{}_{\gamma}E^{\gamma}{}_{b}d^{3}x$$

$$= \sqrt{-1}\ \varepsilon_{acb}A^{b}{}_{\gamma}E^{\gamma}{}_{c}d^{3}x$$

$$= -\sqrt{-1}\ \varepsilon_{abc}A^{b}{}_{\alpha}E^{\alpha}{}_{c}d^{3}x.$$

Therefore

$$H_{R}(\vec{Z}) = \int_{\Sigma} \vec{Z}\ \dot{\wedge}\ d^{A}\vec{Q}$$

$$= -\int_{\Sigma} Z^{a}(\partial_{\alpha}E^{\alpha}{}_{a} + \sqrt{-1}\ \varepsilon_{abc}A^{b}{}_{\alpha}E^{\alpha}{}_{c})d^{3}x.$$

Analysis of $H_{H}(N)$: To discuss

$$\int_{\Sigma} -\sqrt{-1}\ N\vec{F}\ \dot{\wedge}\ *\vec{Q},$$

note that

$$-\vec{F}\ \dot{\wedge}\ *\vec{Q} = F^{c}\wedge\omega^{c}$$

and then write

$$\left[\begin{array}{l} F^{c} = \frac{1}{2}\ F^{c}{}_{\alpha\beta}dx^{\alpha}\wedge dx^{\beta} \\ \\ \omega^{c} = e^{c}{}_{\gamma}dx^{\gamma}, \end{array}\right.$$

thus reducing matters to consideration of

$$\frac{1}{2}\ F^{c}{}_{\alpha\beta}e^{c}{}_{\gamma}dx^{\alpha}\wedge dx^{\beta}\wedge dx^{\gamma}$$

$$= \frac{1}{2} F^{c}{}_{\alpha\beta}\varepsilon_{\alpha\beta\gamma}e^{c}{}_{\gamma}d^{3}x.$$

But

$$e^{c}{}_{\gamma} = \frac{1}{2\,\det[e^{\gamma}{}_{c}]}\ \varepsilon_{cab}\varepsilon_{\gamma\mu\nu}e^{\mu}{}_{a}e^{\nu}{}_{b}$$

$$= \frac{1}{2\sqrt{q}}\varepsilon_{cab}\varepsilon_{\gamma\mu\nu}E^{\mu}{}_{a}E^{\nu}{}_{b}.$$

And

$$\varepsilon_{\alpha\beta\gamma}\varepsilon_{\gamma\mu\nu}E^{\mu}{}_{a}E^{\nu}{}_{b}$$

$$= \varepsilon_{\alpha\beta\gamma}\varepsilon_{\mu\nu\gamma}E^{\mu}{}_{a}E^{\nu}{}_{b}$$

$$= \delta^{\alpha\beta}{}_{\mu\nu}E^{\mu}{}_{a}E^{\nu}{}_{b}$$

$$= (\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu})E^{\mu}{}_{a}E^{\nu}{}_{b}$$

$$= E^{\alpha}{}_{a}E^{\beta}{}_{b} - E^{\beta}{}_{a}E^{\alpha}{}_{b}$$

=>

$$\varepsilon_{\alpha\beta\gamma}e^{c}{}_{\gamma} = \frac{1}{2\sqrt{q}}(\varepsilon_{cab}E^{\alpha}{}_{a}E^{\beta}{}_{b} - \varepsilon_{cab}E^{\beta}{}_{a}E^{\alpha}{}_{b})$$

$$= \frac{1}{2\sqrt{q}}(\varepsilon_{cab}E^{\alpha}{}_{a}E^{\beta}{}_{b} - \varepsilon_{cba}E^{\beta}{}_{b}E^{\alpha}{}_{a})$$

$$= \frac{1}{2\sqrt{q}}(\varepsilon_{cab}E^{\alpha}{}_{a}E^{\beta}{}_{b} + \varepsilon_{cab}E^{\alpha}{}_{a}E^{\beta}{}_{b})$$

$$= \frac{1}{\sqrt{q}}\varepsilon_{abc}E^{\alpha}{}_{a}E^{\beta}{}_{b}.$$

Therefore

$$\mathcal{H}_H(N) = \int_\Sigma -\sqrt{-1}\, N\vec{F} \dot{\wedge} *\vec{Q}$$

$$= \int_\Sigma \sqrt{-1}\, \frac{N}{2}\, \epsilon_{abc} E^\alpha{}_a E^\beta{}_b F^c{}_{\alpha\beta} \frac{d^3x}{\sqrt{q}} .$$

Summary: We have

$\underline{D}$: $\bar{\mathcal{H}}_D(\vec{N}) = \int_\Sigma N^\alpha F^a{}_{\alpha\beta} E^\beta{}_a d^3x.$

$\underline{R}$: $\mathcal{H}_R(\vec{Z}) = -\int_\Sigma Z^a(\partial_\alpha E^\alpha{}_a + \sqrt{-1}\, \epsilon_{abc} A^b{}_\alpha E^\alpha{}_c) d^3x.$

$\underline{H}$: $\mathcal{H}_H(N) = \int_\Sigma \sqrt{-1}\, \frac{N}{2}\, \epsilon_{abc} E^\alpha{}_a E^\beta{}_b F^c{}_{\alpha\beta} \frac{d^3x}{\sqrt{q}} .$

Remark: From the definitions,

$$F^c{}_{\alpha\beta} = \partial_\alpha A^c{}_\beta - \partial_\beta A^c{}_\alpha + \sqrt{-1}\, \epsilon_{abc} A^a{}_\alpha A^b{}_\beta .$$

Section 61: Rescaling the Theory The assumptions and notation are those of the standard setup but with the restriction that n = 4.

Fix a nonzero complex number ι (the Immirizi parameter). Define

$$T_\iota : T^*\underline{Q}_{\underline{C}} \to T^*\underline{Q}_{\underline{C}}$$

by

$$T_\iota(\vec{\omega},\vec{p}) = (\vec{\omega},\vec{p} - \iota d\vec{\omega}).$$

Then T_ι is bijective.

[Note: Explicitly,

$$T_\iota^{-1} : T^*\underline{Q}_{\underline{C}} \to T^*\underline{Q}_{\underline{C}}$$

is given by

$$T_\iota^{-1}(\vec{\omega},\vec{p}) = (\vec{\omega},\vec{p} + \iota d\vec{\omega}).]$$

N.B. The Ashtekar theory is the case $\iota = \sqrt{-1}$.

LEMMA T_ι is a canonical transformation.

Remark: If ι is real, then T_ι restricts to a canonical transformation

$$T^*\underline{Q} \to T^*\underline{Q}.$$

Proceeding as before, put

$$H_{T_\iota} = H \circ T_\iota^{-1}.$$

Then

$$H_{T_\iota}(\vec{\omega},\vec{P}) = H(\vec{\omega},\vec{P} + \iota d\vec{\omega})$$

$$= \int_\Sigma L_{\vec{N}}\omega^a \wedge P_a + \int_\Sigma W^a{}_b \omega^b \wedge (P_a + \iota d\omega_a)$$

$$+ \int_\Sigma N E(\vec{\omega}, \vec{P} + \iota d\vec{\omega}) .$$

And

$$E(\vec{\omega}, \vec{P} + \iota d\vec{\omega})$$

$$= \frac{1}{2} [q(P_a + \iota d\omega_a, *\omega^b) q(P_b + \iota d\omega_b, *\omega^a)$$

$$- \frac{1}{2} q(P_a + \iota d\omega_a, *\omega^a)^2 - S(q)] \mathrm{vol}_q .$$

• $q(P_a + \iota d\omega_a, *\omega^b) q(P_b + \iota d\omega_b, *\omega^a)$

$$= q(P_a, *\omega^b) q(P_b, *\omega^a)$$

$$+ 2\iota q(P_a, *\omega^b) q(d\omega^b, *\omega^a) + \iota^2 q(d\omega^a, *\omega^b) q(d\omega^b, *\omega^a) .$$

• $- \frac{1}{2} q(P_a + \iota d\omega_a, *\omega^a)^2$

$$= - \frac{1}{2} q(P_a + \iota d\omega_a, *\omega^a) q(P_b + \iota d\omega_b, *\omega^b)$$

$$= - \frac{P^2}{2} - \iota P q(d\omega^a, *\omega^a)$$

$$- \frac{1}{2} \iota^2 q(d\omega^a, *\omega^a) q(d\omega^b, *\omega^b) ,$$

where

$$P = q(P_a, *\omega^a) .$$

$$\bullet - S(q)\,\mathrm{vol}_q = 2d(\omega^a \wedge * d\omega^a)$$

$$- \frac{1}{2}(d\omega^a \wedge \omega^a) \wedge * (d\omega^b \wedge \omega^b) + (d\omega^a \wedge \omega^b) \wedge * (d\omega^b \wedge \omega^a)$$

$$= 2d(\omega^a \wedge * d\omega^a)$$

$$- \frac{1}{2} q(d\omega^a, *\omega^a) q(d\omega^b, *\omega^b)\,\mathrm{vol}_q + q(d\omega^a, *\omega^b) q(d\omega^b, *\omega^a)\,\mathrm{vol}_q.$$

Therefore

$$E(\vec{\omega}, \vec{P} + \iota d\vec{\omega})$$

$$= \frac{1}{2}[q(P_a, *\omega^b) q(P_b, *\omega^a)$$

$$+ 2\iota q(P_a, *\omega^b) q(d\omega^b, *\omega^a) - \frac{P^2}{2} - \iota P q(d\omega^a, *\omega^a)$$

$$+ (\iota^2+1) q(d\omega^a, *\omega^b) q(d\omega^b, *\omega^a) - \frac{(\iota^2+1)}{2} q(d\omega^a, *\omega^a) q(d\omega^b, *\omega^b)]\,\mathrm{vol}_q$$

$$+ d(\omega^a \wedge * d\omega^a).$$

Now set

$$H_\iota = H_{T_\iota} \circ S^{-1}$$

so that

$$H_\iota(\vec{Q}, \vec{A}) = H_{T_\iota}(\vec{\omega}, \vec{P}),$$

where

$$P_a = A_b \wedge * (\omega^b \wedge \omega_a).$$

To continue, it will be necessary to introduce some notation that reflects the presence of ι.

Thus given $(\vec{\omega},\vec{P})$, let

$$A_{ab} = \frac{1}{\iota}\,[q(P_c,\omega^a\wedge\omega^b)\omega^c - \frac{P}{2}\,*(\omega^a\wedge\omega^b)].$$

Put

$$A_c = \frac{\iota}{2}\,\varepsilon_{cuv}A_{uv}.$$

Then

$$\varepsilon_{abc}A_c = \varepsilon_{abc}(\frac{\iota}{2}\,\varepsilon_{cuv}A_{uv})$$

$$= \iota A_{ab}.$$

And again

$$\begin{cases} A_a = q(P_b,*\omega_a)\omega^b - \frac{P}{2}\,\omega_a \\ P_a = A_b\wedge *(\omega^b\wedge\omega_a). \end{cases}$$

- $d^{A,\iota}Q^a = dQ^a + A^a{}_b\wedge Q^b$

$$= dQ^a + \frac{1}{\iota}\,\varepsilon^a{}_{bc}A^c\wedge Q^b$$

=>

$$d^{A,\iota}\vec{Q} = d\vec{Q} - \frac{1}{\iota}\,\vec{A}\overset{\times}{\wedge}\vec{Q}.$$

- $F_a = \frac{\iota}{2}\,\varepsilon_{abc}F_{bc}$

$$= \frac{1}{2}\,\varepsilon_{abc}(dA_{bc} + A_{bd}\wedge A^{d}{}_{c})$$

$$= dA_{a} - \frac{1}{2\iota}\,\varepsilon_{abc}A_{b}\wedge A_{c}$$

$\Rightarrow$

$$\vec{F} = d\vec{A} - \frac{1}{2\iota}\,\vec{A} \overset{\times}{\wedge} \vec{A}.$$

Computation of H_{ι} This is simply a matter of replacing P_a by $A_b\wedge *(\omega^b\wedge\omega^a)$ in the foregoing expression for H_{T_ι} and keeping track of the terms obtained thereby. Fortunately most of the work has already been carried out during the course of deriving the Ashtekar hamiltonian, hence there is no point in repeating the details.

First

$$\int_{\Sigma} L_{\vec{N}}\omega^{a}\wedge P_{a}$$

does not involve ι and is equal to

$$\int_{\Sigma} L_{\vec{N}}\vec{Q} \overset{\bullet}{\wedge} \vec{A}.$$

To discuss

$$\int_{\Sigma} W^{a}{}_{b}\omega^{b}\wedge(P_{a} + \iota d\omega_{a})$$

$$= \int_{\Sigma} W^{a}{}_{b}\omega^{b}\wedge(A_{c}\wedge *(\omega^{c}\wedge\omega_{a}) + \iota d\omega_{a}),$$

put

$$Z_{ab} = - W_{ab} + \iota\varepsilon_{abc}W_{c},$$

where

$$W_c = - q(dN,\omega^c).$$

Setting aside

$$\iota \int_\Sigma \varepsilon_{abc} W_c (P_a + \iota d\omega_a)\wedge\omega^b,$$

we have:

1. $\int_\Sigma - Z_{ab} A_c \wedge * (\omega^c \wedge \omega^a)\wedge\omega^b$

$$= \int_\Sigma Z_a \wedge \frac{1}{\iota} \varepsilon^a{}_{bc} A^c \wedge Q^b.$$

2. $\int_\Sigma - Z_{ab} (\iota d\omega^a)\wedge\omega^b$

$$= \int_\Sigma Z_a \wedge dQ^a.$$

Therefore

$$1 + 2 = \int_\Sigma \vec{Z} \overset{\bullet}{\wedge} d^{A,\iota}\vec{Q}.$$

Finally

$$- \iota N\vec{F} \overset{\bullet}{\wedge} *\vec{Q}$$

$$= \iota N(dA_a - \frac{1}{2\iota} \varepsilon_{abc} A_b \wedge A_c)\wedge\omega^a$$

$$= \iota N dA_a \wedge\omega^a - \frac{N}{2} \varepsilon_{abc} A_b \wedge A_c \wedge\omega^a.$$

But

- $\int_\Sigma - \frac{N}{2} \varepsilon_{abc} A_b \wedge A_c \wedge \omega^a$

$$= \int_\Sigma \frac{N}{2} [q(P_a,*\omega^b)q(P_b,*\omega^a) - \frac{P^2}{2}]\mathrm{vol}_q.$$

• $\int_\Sigma \iota N dA_a \wedge \omega^a$

$$= \iota \int_\Sigma NA_a \wedge d\omega^a - \iota \int_\Sigma dN \wedge A_a \wedge \omega^a$$

$$= \iota \int_\Sigma N[q(P_a,*\omega^b)q(d\omega^b,*\omega^a) - \frac{P}{2} q(d\omega^a,*\omega^a)]\mathrm{vol}_q$$

$$- \iota \int_\Sigma q(dN,\omega^a)P_b \wedge *(\omega^a \wedge \omega^b).$$

Thus it follows that

$$\int_\Sigma \frac{N}{2} [q(P_a,*\omega^b)q(P_b,*\omega^a)$$

$$+ 2\iota q(P_a,*\omega^b)q(d\omega^b,*\omega^a) - \frac{P^2}{2} - \iota P q(d\omega^a,*\omega^a)]\mathrm{vol}_q$$

$$= \int_\Sigma - \iota N\vec{F} \overset{\cdot}{\wedge} *\vec{Q} + \iota \int_\Sigma q(dN,\omega^a)P_b \wedge *(\omega^a \wedge \omega^b).$$

To eliminate

$$\iota \int_\Sigma q(dN,\omega^a)P_b \wedge *(\omega^a \wedge \omega^b),$$

reintroduce

$$\iota \int_\Sigma \varepsilon_{abc} W_c (P_a + \iota d\omega_a) \wedge \omega^b$$

$$= \iota \int_\Sigma \varepsilon_{abc} W_c P_a \wedge \omega^b + \iota^2 \int_\Sigma \varepsilon_{abc} W_c d\omega_a \wedge \omega^b$$

$$= - \iota \int_\Sigma q(dN,\omega^a)P_b \wedge *(\omega^a \wedge \omega^b) + \iota^2 \int_\Sigma \varepsilon_{abc} W_c d\omega_a \wedge \omega^b,$$

which leaves

$$\iota^2 \int_\Sigma \varepsilon_{abc} W_c d\omega_a \wedge \omega^b$$

or still,

$$\iota^2 \int_\Sigma - q(dN,\omega^c) q(\omega^c \wedge \omega^a, d\omega^a) \mathrm{vol}_q$$

or still,

$$\iota^2 \int_\Sigma N d(\omega^a \wedge * d\omega^a).$$

Definition: The <u>ι-modification</u> of the Ashtekar hamiltonian is the function

$$H_\iota : T^* * \underline{Q}_{\underline{C}} \to \underline{C}$$

defined by the prescription

$$H_\iota(\vec{Q},\vec{A};N,\vec{N},\vec{Z})$$

$$= \int_\Sigma L_{\vec{N}} \vec{Q} \overset{\cdot}{\wedge} \vec{A} + \int_\Sigma \vec{Z} \overset{\cdot}{\wedge} d^{A,\iota}\vec{Q} + \int_\Sigma - \iota N\vec{F} \overset{\cdot}{\wedge} *\vec{Q}$$

$$+ (\iota^2+1) \int_\Sigma - \frac{N}{2} S(q) \mathrm{vol}_q.$$

[Note: $H_{\sqrt{-1}}$ is the Ashtekar hamiltonian.]

Remark: If ι is real, then the theory restricts to a theory on $T^* * \underline{Q}$.

<u>LEMMA</u> We have

$$\frac{\delta}{\delta Q^a} [\int_\Sigma - \frac{N}{2} S(q) \mathrm{vol}_q]$$

$$= - N(\mathrm{Ric}_a - \frac{1}{2} S(q)\omega^a)$$

$$+ (\nabla_a dN - (\Delta_q N)\omega^a).$$

[Recall that

$$\frac{\delta}{\delta\omega^a} [\int_\Sigma - \frac{N}{2} S(q)\mathrm{vol}_q]$$

$$= N*(\mathrm{Ric}_a - \frac{1}{2} S(q)\omega^a)$$

$$- *(\nabla_a dN - (\Delta_q N)\omega^a).]$$

Using the lemma and the fact that

$$\frac{\delta}{\delta A_a} [\int_\Sigma - \frac{N}{2} S(q)\mathrm{vol}_q] = 0,$$

one can write down the ι-modified equations of motion and the ι-modified Poisson bracket structure, a task that will be left to the reader as an exercise ad libitum.

N.B.

$$\frac{\delta H_\iota}{\delta N} = \frac{1}{2} (\iota^2 F - (\iota^2+1)S(q))\mathrm{vol}_q.$$

$$\frac{\delta H_\iota}{\delta N^a} = \iota_{E_a} A_b \wedge d^{A,\iota} Q^b - \iota(\mathrm{Ric}\ \vec{F} \overset{\times}{\wedge} \vec{Q})_a.$$

$$\frac{\delta H_\iota}{\delta Z_a} = d^{A,\iota} Q^a.$$

The local expressions for

$$\bar{H}_D(\vec{N}),\ H_R(\vec{Z}),\ H_H(N)$$

can be repackaged so as to give local expressions for

$$\bar{H}_{\iota,D}(\vec{N}),\ H_{\iota,R}(\vec{Z}),\ H_{\iota,H}(N).$$

This is completely obvious but, due to the presence of the potential

$$\int_\Sigma -\frac{N}{2} S(q)\mathrm{vol}_q,$$

an additional term is present in $H_{\iota,H}(N)$ which has to be isolated.

Notation: Given q, let $\omega^a{}_b$ be the connection 1-forms per the metric connection ∇^q. Write, as usual,

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

and put

$$\Omega^a = \frac{1}{2}\,\varepsilon^a{}_{bc}\Omega_{bc}.$$

- $\varepsilon_{abc}E^\alpha{}_a E^\beta{}_b \Omega_c \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$

$$= \varepsilon_{abc}E^\alpha{}_a E^\beta{}_b \frac{1}{2}\,\varepsilon_{cuv}\Omega_{uv} \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

$$= \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{uvc}E^\alpha{}_a E^\beta{}_b \Omega_{uv} \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

$$= \frac{1}{2}\,\delta^{ab}{}_{uv}E^\alpha{}_a E^\beta{}_b \Omega_{uv} \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

$$= \frac{1}{2}\,(\delta^a{}_u\delta^b{}_v - \delta^a{}_v\delta^b{}_u)E^\alpha{}_a E^\beta{}_b \Omega_{uv} \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

$$= E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega_{ab}\ \left(\frac{\partial}{\partial x^{\alpha}}, \frac{\partial}{\partial x^{\beta}}\right).$$

Working locally, write

$$S(q)\,\mathrm{vol}_q = *(\omega^{a}\wedge\omega^{b})\wedge\Omega_{ab}$$

$$= (\varepsilon_{abc}\omega^{c})\wedge \frac{1}{2}\,\Omega^{ab}{}_{\alpha\beta}dx^{\alpha}\wedge dx^{\beta}$$

$$= (\varepsilon_{abc}e^{c}{}_{\mu}dx^{\mu})\wedge \frac{1}{2}\,\Omega^{ab}{}_{\alpha\beta}dx^{\alpha}\wedge dx^{\beta}$$

$$= \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{\mu\alpha\beta}e^{c}{}_{\mu}\Omega^{ab}{}_{\alpha\beta}d^{3}x$$

$$= \frac{1}{2\sqrt{q}}\,\varepsilon_{abc}\varepsilon_{\mu\alpha\beta}e^{c}{}_{\mu}\Omega^{ab}{}_{\alpha\beta}\ \sqrt{q}\,d^{3}x$$

$$= \frac{1}{2\sqrt{q}}\,\varepsilon_{abc}\varepsilon_{\mu\alpha\beta}e^{c}{}_{\mu}\Omega^{ab}{}_{\alpha\beta}\mathrm{vol}_q$$

=>

$$\sqrt{q}\,S(q) = \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{\mu\alpha\beta}e^{c}{}_{\mu}\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{2}\,\varepsilon_{abc}\varepsilon_{\mu\alpha\beta}\ \left(\frac{1}{2\sqrt{q}}\,\varepsilon_{cuv}\varepsilon_{\mu\gamma\delta}E^{\gamma}{}_{u}E^{\delta}{}_{v}\right)\Omega^{ab}{}_{\alpha\beta}$$

=>

$$(\det q)S(q) = \frac{1}{4}\,\varepsilon_{abc}\varepsilon_{\mu\alpha\beta}\varepsilon_{cuv}\varepsilon_{\mu\gamma\delta}E^{\gamma}{}_{u}E^{\delta}{}_{v}\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,\varepsilon_{abc}\varepsilon_{\gamma\alpha\beta}\varepsilon_{cuv}\varepsilon_{\gamma\mu\nu}E^{\mu}{}_{u}E^{\nu}{}_{v}\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,\varepsilon_{abc}\varepsilon_{uvc}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\mu\nu\gamma}E^{\mu}{}_{u}E^{\nu}{}_{v}\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,\varepsilon_{abc}\varepsilon_{uvc}\delta^{\alpha\beta}{}_{\mu\nu}E^{\mu}{}_{u}E^{\nu}{}_{v}\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,\varepsilon_{abc}\varepsilon_{uvc}(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu})E^{\mu}{}_{u}E^{\nu}{}_{v}\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,\varepsilon_{abc}\varepsilon_{uvc}(E^{\alpha}{}_{u}E^{\beta}{}_{v} - E^{\beta}{}_{u}E^{\alpha}{}_{v})\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,\delta^{ab}{}_{uv}(E^{\alpha}{}_{u}E^{\beta}{}_{v} - E^{\beta}{}_{u}E^{\alpha}{}_{v})\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,(\delta^{a}{}_{u}\delta^{b}{}_{v} - \delta^{a}{}_{v}\delta^{b}{}_{u})(E^{\alpha}{}_{u}E^{\beta}{}_{v} - E^{\beta}{}_{u}E^{\alpha}{}_{v})\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{4}\,(2E^{\alpha}{}_{a}E^{\beta}{}_{b} - 2E^{\alpha}{}_{b}E^{\beta}{}_{a})\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{2}\,E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega^{ab}{}_{\alpha\beta} - \frac{1}{2}\,E^{\alpha}{}_{b}E^{\beta}{}_{a}\Omega^{ab}{}_{\alpha\beta}$$

$$= \frac{1}{2}\,E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega^{ab}{}_{\alpha\beta} - \frac{1}{2}\,E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega^{ba}{}_{\alpha\beta}$$

$$= \frac{1}{2}\,E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega^{ab}{}_{\alpha\beta} + \frac{1}{2}\,E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega^{ab}{}_{\alpha\beta}$$

$$= E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega^{ab}{}_{\alpha\beta}$$

$$= E^{\alpha}{}_{a}E^{\beta}{}_{b}\Omega_{ab}\left(\frac{\partial}{\partial x^{\alpha}}, \frac{\partial}{\partial x^{\beta}}\right).$$

Therefore

$$S(q)\mathrm{vol}_q = S(q)\sqrt{q}\;d^3x$$

$$= (\det q) S(q) \frac{d^3x}{\sqrt{q}}$$

$$= E^{\alpha}{}_{a} E^{\beta}{}_{b} \Omega_{ab} \left(\frac{\partial}{\partial x^{\alpha}}, \frac{\partial}{\partial x^{\beta}}\right) \frac{d^3x}{\sqrt{q}}$$

$$= \varepsilon_{abc} E^{\alpha}{}_{a} E^{\beta}{}_{b} \Omega^{c}{}_{\alpha\beta} \frac{d^3x}{\sqrt{q}} \,.$$

And then

$$\mathcal{H}_{\iota,H}(N) = \int_{\Sigma} \iota \frac{N}{2} \varepsilon_{abc} E^{\alpha}{}_{a} E^{\beta}{}_{b} F^{c}{}_{\alpha\beta} \frac{d^3x}{\sqrt{q}}$$

$$- (\iota^2+1) \int_{\Sigma} \frac{N}{2} \varepsilon_{abc} E^{\alpha}{}_{a} E^{\beta}{}_{b} \Omega^{c}{}_{\alpha\beta} \frac{d^3x}{\sqrt{q}}$$

$$= \int_{\Sigma} \frac{N}{2} \varepsilon_{abc} E^{\alpha}{}_{a} E^{\beta}{}_{b} (\iota F^{c}{}_{\alpha\beta} - (\iota^2+1) \Omega^{c}{}_{\alpha\beta}) \frac{d^3x}{\sqrt{q}} \,.$$

Reconciliation In the literature, one will find a different formula for $\mathcal{H}_{\iota,H}(N)$. To explain this, consider the path $t \to (\vec{\omega}_t, \vec{p}_t)$ and suppose that the constraint

$$\frac{1}{2} (S(q_t) + K_t^2 - [\kappa_t, \kappa_t]_{q_t}) = 0$$

is in force. Bearing in mind that $(\kappa_t)_{ab} = \kappa_{ab}$, write

$$\kappa_{ab} = \kappa(E_a, E_b)$$

$$= \kappa\left(e^{\mu}{}_{a} \frac{\partial}{\partial x^{\mu}}, e^{\nu}{}_{b} \frac{\partial}{\partial x^{\nu}}\right)$$

$$= e^{\mu}{}_{a} e^{\nu}{}_{b} \kappa_{\mu\nu}$$

$$= e^{\mu}{}_a \kappa_{\mu\nu} e^{\nu}{}_b$$

$$= \frac{(\sqrt{q_t}\; e^{\mu}{}_a)\kappa_{\mu\nu}(\sqrt{q_t}\; e^{\nu}{}_b)}{\det\, q_t}$$

$$= \frac{E^{\mu}{}_a \kappa_{\mu\nu} E^{\nu}{}_b}{\det\, q_t}$$

$$= \frac{1}{\sqrt{q_t}}\, E^{\mu}{}_a K^b{}_{\mu},$$

where we have put

$$K^b{}_{\mu} = \frac{1}{\sqrt{q_t}}\, \kappa_{\mu\nu} E^{\nu}{}_b.$$

Therefore

$$S(q_t) = [\kappa_t, \kappa_t]_{q_t} - K_t^2$$

$$= \kappa_{ab}\kappa_{ba} - \kappa_{aa}\kappa_{bb}$$

$$= \frac{1}{\det\, q_t}\, [E^{\mu}{}_a K^b{}_{\mu} E^{\nu}{}_b K^a{}_{\nu} - E^{\mu}{}_a K^a{}_{\mu} E^{\nu}{}_b K^b{}_{\nu}]$$

$$= \frac{1}{\det\, q_t}\, E^{\mu}{}_a E^{\nu}{}_b (K^a{}_{\nu} K^b{}_{\mu} - K^a{}_{\mu} K^b{}_{\nu})$$

or still,

$$(\det\, q_t) S(q_t) = E^{\mu}{}_a E^{\nu}{}_b (K^a{}_{\nu} K^b{}_{\mu} - K^a{}_{\mu} K^b{}_{\nu})$$

from which

$$(\iota^2 + 1) \int_{\Sigma} - \frac{N}{2}\, S(q_t) \mathrm{vol}_{q_t}$$

$$= (\iota^2+1) \int_\Sigma - \frac{N}{2} (\det q_t) S(q_t) \frac{d^3x}{\sqrt{q_t}}$$

$$= (\iota^2+1) \int_\Sigma - \frac{N}{2} E^\mu{}_a E^\nu{}_b (K^a{}_\nu K^b{}_\mu - K^a{}_\mu K^b{}_\nu) \frac{d^3x}{\sqrt{q_t}}$$

$$= (\iota^2+1) \int_\Sigma \frac{N}{2} E^\mu{}_a E^\nu{}_b (K^a{}_\mu K^b{}_\nu - K^a{}_\nu K^b{}_\mu) \frac{d^3x}{\sqrt{q_t}} .$$

So, under the above assumptions,

$$H_{\iota,H}(N) = \int_\Sigma \iota \frac{N}{2} \varepsilon_{abc} E^\alpha{}_a E^\beta{}_b F^c{}_{\alpha\beta} \frac{d^3x}{\sqrt{q_t}}$$

$$+ (\iota^2+1) \int_\Sigma N E^\mu{}_a E^\nu{}_b K^a{}_{[\mu} K^b{}_{\nu]} \frac{d^3x}{\sqrt{q_t}} .$$

Section 62: Asymptotic Flatness In the metric theory, take $M = \underline{R} \times \Sigma$ ($\dim M = n > 2$) and recall:

Constraint Equations These are the relations

$$\begin{cases} ([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2 - S(q_t)) \otimes |q_t|^{1/2} = 0 \\ \operatorname{div}_{q_t} p_t = 0. \end{cases}$$

Evolution Equations These are the relations

$$\dot{q}_t = 2N_t(\pi_t^{\flat} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t^{\flat}) q_t) + L_{\vec{N}_t} q_t$$

and

$$\begin{aligned} \dot{p}_t = &- 2N_t(\pi_t * \pi_t - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)\pi_t) \otimes |q_t|^{1/2} \\ &+ \frac{N_t}{2} ([\pi_t, \pi_t]_{q_t} - \frac{1}{n-2} \operatorname{tr}_{q_t}(\pi_t)^2) q_t^{\#} \otimes |q_t|^{1/2} \\ &- N_t \operatorname{Ein}(q_t)^{\#} \otimes |q_t|^{1/2} \\ &+ (H_{N_t} - (\Delta_{q_t} N_t) q_t)^{\#} \otimes |q_t|^{1/2} + L_{\vec{N}_t} p_t. \end{aligned}$$

THEOREM $\operatorname{Ein}(g) = 0$ iff the constraint equations and the evolution equations are satisfied by the pair (q_t, p_t).

For this, we assumed that Σ was compact. But actually compactness played no role at all in the proof which was purely algebraic.

Q: So where does compactness play a role?

A: In the hamiltonian formulation of the dynamics.

N.B. The point is that this interpretation hinges on the calculation of certain functional derivatives and the formulas derived thereby depend on ignoring all boundary terms. While permissible if Σ is compact, in the noncompact case the boundary terms have to be taken into account.

To minimize technicalities, we shall assume that $M = \underline{R}^4 = \underline{R} \times \underline{R}^3$, thus now $\Sigma = \underline{R}^3$. The strategy then is to consider a certain class of riemannian structures on $\underline{R}^3$ which is sufficiently broad to cover the standard examples but sufficiently restrictive to give a sensible theory.

[Note: For the sake of simplicity, I shall pass in silence on the role of covariance in the theory.]

Notation: Put

$$r = [x^i x^j \delta_{ij}]^{1/2} \ (= |x|).$$

Parity Let $\rho \in C^\infty(\underline{S}^2)$ -- then ρ determines a radially constant function $\tilde{\rho}$ on $\underline{R}^3 - \{0\}$:

$$\tilde{\rho}(x) = \rho(\frac{x}{r}).$$

If the parity of ρ is even (odd), then $\tilde{\rho}$ is even (odd).

[Note: The antipodal map on $\underline{S}^2$ sends p to -p. In terms of the azimuthal angle θ and the polar angle ϕ, it is the arrow

$$(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

$$\rightarrow (\cos(\theta+\pi)\sin(\pi-\phi), \sin(\theta+\pi)\sin(\pi-\phi), \cos(\pi-\phi)).]$$

SUBLEMMA If the parity of ρ is even (odd), then $\partial_k\tilde{\rho}$ is odd (even) $(k = 1,2,3)$.

[Note: $\tilde{\rho}$ is homogeneous of degree 0, hence $\partial_k\tilde{\rho}$ is homogeneous of degree -1. But then $r(\partial_k\tilde{\rho})$ is homogeneous of degree 0, thus $\exists\ \rho_k \in C^\infty(\underline{S}^2)$:

$$r(\partial_k\tilde{\rho})\Big|_x = \rho_k(\frac{x}{r})$$

or still,

$$\partial_k\tilde{\rho}\Big|_x = \frac{1}{r}\,\rho_k(\frac{x}{r}).]$$

Notation:

$O^+(\frac{1}{r^\varepsilon})$ stands for an even function which is $O(\frac{1}{r^\varepsilon})$ $(\varepsilon \geq 0)$.

$O^-(\frac{1}{r^\varepsilon})$ stands for an odd function which is $O(\frac{1}{r^\varepsilon})$ $(\varepsilon \geq 0)$.

[Note: In either case, $\varepsilon = 0$ is admitted, so $O^+(1)$ $(O^-(1))$ represents a bounded even (odd) function. In particular: If $\rho \in C^\infty(\underline{S}^2)$ and is of even (odd) parity, then $\tilde{\rho} = O^+(1)$ $(O^-(1))$.]

Example: Let $\rho \in C^\infty(\underline{S}^2)$.

- If the parity of ρ is even, then

$$\partial_k\tilde{\rho} = O^-(\frac{1}{r}).$$

- If the parity of ρ is odd, then

$$\partial_k\tilde{\rho} = O^+(\frac{1}{r}).$$

Example: Let $\rho \in C^\infty(\underline{S}^2)$.

- If the parity of ρ is even, then

$$\partial_k(\frac{\tilde{\rho}}{r}) = O^-(\frac{1}{r^2}).$$

- If the parity of ρ is odd, then

$$\partial_k(\frac{\tilde{\rho}}{r}) = O^+(\frac{1}{r^2}).$$

<u>Integrals</u> If $f = O(\frac{1}{r^{3+\delta}})$ $(\delta > 0)$, then

$$\int_{\underline{R}^3} |f| d^3x$$

$$= \int_0^\infty \int_{\underline{S}^2} |f(rp)| r^2 d\Omega(p) dr$$

$$< \infty,$$

hence f is Lebesgue integrable, so

$$\int_{\underline{R}^3} f d^3x = \lim_{R \to \infty} \int_{\underline{D}^3(R)} f d^3x.$$

In general, however, our integrals will be improper, i.e., by

$$\int_{\underline{R}^3} f d^3x$$

we shall simply understand

$$\lim_{R \to \infty} \int_{\underline{D}^3(R)} f d^3x.$$

Accordingly, if f is odd, then

$$\int_{\underline{R}^3} f d^3x = 0.$$

Notation: Let $f \in C^\infty(\underline{R}^3)$ — then we write

$$f = O^{\infty}(\frac{1}{r^{\varepsilon}})$$

provided f is $O(\frac{1}{r^{\varepsilon}})$ and its partial derivatives of order m are $O(\frac{1}{r^{m+\varepsilon}})$ $(m = 1,2,\ldots)$.

[Note: Here ε is nonnegative. E.g.: Let $\rho \in C^{\infty}(\underline{S}^2)$ — then $\tilde{\rho} = O^{\infty}(1)$, meaning that $\tilde{\rho} = O(1)$, $\partial_i \tilde{\rho} = O(\frac{1}{r})$, $\partial_i \partial_j \tilde{\rho} = O(\frac{1}{r^2})$ etc.]

Example: If for large r,

$$f = \frac{\sin(r^4)}{r^2},$$

then

$$f = O^{+}(\frac{1}{r^2})$$

but its partial derivatives of every order blow up at infinity.

Observation: If $f_1 = O^{\infty}(\frac{1}{r^{\varepsilon_1}})$ and $f_2 = O^{\infty}(\frac{1}{r^{\varepsilon_2}})$, then $f_1 f_2 = O^{\infty}(\frac{1}{r^{\varepsilon_1+\varepsilon_2}})$.

Let $\mathcal{S}_{2,\infty}$ stand for the set of 2-covariant symmetric tensors in $\underline{R}^3$ with the following property: Given s, $\exists$

$$\begin{bmatrix} \sigma_{ij} \in C^{\infty}(\underline{S}^2) & (\sigma_{ij} = \sigma_{ji}) \\ \mu_{ij} \in C^{\infty}(\underline{R}^3) & (\mu_{ij} = \mu_{ji}) \end{bmatrix}$$

such that for $r >> 0$,

$$s_{ij}(x) = \frac{1}{r}\,\sigma_{ij}(\frac{x}{r}) + \mu_{ij}(x),$$

where

$$\sigma_{ij}(-p) = \sigma_{ij}(p) \quad (p\in\underline{S}^2)$$

and

$$\mu_{ij} = O^{\infty}(\frac{1}{r^{1+\delta}}) \quad (0 < \delta \leq 1).$$

Definition: Let η be the usual flat metric on $\underline{R}^3$ and let q be a riemannian structure on $\underline{R}^3$ — then q is said to be asymtotically flat provided $q - \eta\in S_{2,\infty}$.

Notation: Q_∞ is the set of asymtotically flat riemannian structures on $\underline{R}^3$.

Example: If for $r >> 0$,

$$q_{ij}(x) = \eta_{ij} + m\frac{x^i x^j}{r^3} \quad (m > 0),$$

then $q\in Q_\infty$.

LEMMA Let $q\in Q_\infty$ and $s\in S_{2,\infty}$ — then $q + \varepsilon s\in Q_\infty$ for ε sufficiently small.

[This is certainly true on compact sets, in particular on the $\underline{D}^3(R)$. As for the situation at infinity, one has only to show that $q + \varepsilon s$ is nonsingular provided $|\varepsilon| << 1$. Indeed, $q + \varepsilon s \to \eta$ as $|x| \to \infty$ and the property of being positive definite is closed in the set of nonsingular symmetric 3-by-3 matrices. Fix positive constants C and D such that

$$\begin{cases} ||q(x) - \eta(x)||_{OP} \leq \frac{C}{|x|} \\ \\ ||s(x)||_{OP} \leq \frac{D}{|x|} \end{cases} \quad (|x| \geq 1).$$

Then

$$|x| \geq 1 \Rightarrow$$

$$||q(x) + \epsilon s(x) - \eta(x)||_{OP}$$

$$\leq ||q(x) - \eta(x)||_{OP} + |\epsilon|\cdot||s(x)||_{OP}$$

$$\leq \frac{C+|\epsilon|D}{|x|} < \frac{C+D}{|x|} \quad (|\epsilon| < 1).$$

Choose R > 1:

$$|x| \geq R \Rightarrow \frac{C+D}{R} < 1$$

$\Rightarrow$

$$||q(x) + \epsilon s(x) - \eta(x)||_{OP} < 1.$$

Therefore $q(x) + \epsilon s(x)$ is nonsingular. Now restrict ϵ so that it also works on $\underline{D}^3(R)$.]

[Note: Thus, on formal grounds, the tangent space to Q_∞ at q is $S_{2,\infty}$, i.e.,

$$T_q Q_\infty = S_{2,\infty}.]$$

<u>LEMMA</u> Let $q \in Q_\infty$ — then

$$q^{ij} = \eta_{ij} + O(\frac{1}{r}).$$

[In fact, the map

$$\begin{cases} \underline{GL}(3,\underline{R}) \rightarrow \underline{GL}(3,\underline{R}) \\ A \rightarrow A^{-1} \end{cases}$$

is C^1, thus is Lipschitz in a neighborhood of the identity.]

[Note: One can be more precise, viz. for $r >> 0$,

$$q^{ij}(x) = \eta_{ij} - \frac{1}{r}\sigma_{ij}(\frac{x}{r}) + O(\frac{1}{r^{1+\delta}}).]$$

Connection Coefficients Let $q \in Q_\infty$ -- then, per the metric connection,

$$\Gamma^k{}_{ij} = \frac{1}{2} q^{k\ell}(q_{\ell i,j} + q_{\ell j,i} - q_{ij,\ell}).$$

Therefore

$$\Gamma^k{}_{ij} = \frac{1}{2} (\eta_{k\ell} + O(\frac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))$$

$$= O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}).$$

Miscellaneous Estimates Let $q \in Q_\infty$.

- $\det q = 1 + O(\frac{1}{r})$.

- $\sqrt{\det q} = 1 + O(\frac{1}{r})$.

[Explicitly,

$$\left[\begin{array}{l} \det q\Big|_x = 1 + \frac{1}{r}\sum_{i=1}^{3}\sigma_{ii}(\frac{x}{r}) + O(\frac{1}{r^{1+\delta}}) \\ \qquad\qquad\qquad\qquad\qquad\qquad (r >> 0) \\ \sqrt{\det q}\Big|_x = 1 + \frac{1}{2r}\sum_{i=1}^{3}\sigma_{ii}(\frac{x}{r}) + O(\frac{1}{r^{1+\delta}}).] \end{array}\right.$$

LEMMA Let $q \in Q_\infty$ -- then

$$\partial_k q^{ij} = O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}).$$

[For

$$\partial_k q^{ij} = - q^{iu}\partial_k q_{uv} q^{vj}$$

$$= - (\eta_{iu} + O(\frac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))(\eta_{vj} + O(\frac{1}{r}))$$

$$= O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}).]$$

[Note: Iteration of this procedure shows that the partial derivatives of q^{ij} of order $m > 1$ are $O(\frac{1}{r^{m+1}})$.]

Let $S_d^{2,\infty}$ stand for the set of 2-contravariant symmetric tensor densities on $\underline{R}^3$ with the following property: Given $\Lambda = \lambda d^3x$, $\exists$

$$\begin{cases} \tau^{ij} \in C^\infty(\underline{S}^2) & (\tau^{ij} = \tau^{ji}) \\ \nu^{ij} \in C^\infty(\underline{R}^3) & (\nu^{ij} = \nu^{ji}) \end{cases}$$

such that for $r >> 0$,

$$\lambda^{ij}(x) = \frac{1}{r^2}\tau^{ij}(\frac{x}{r}) + \nu^{ij}(x),$$

where

$$\tau^{ij}(-p) = -\tau^{ij}(p) \quad (p \in \underline{S}^2)$$

and

$$\nu^{ij} \in O^{\infty}\left(\frac{1}{r^{2+\delta}}\right) \quad (0 < \delta \le 1).$$

Define

$$< \, , \, >: S_{2,\infty} \times S_d^{2,\infty} \to \underline{R}$$

by

$$<s,\Lambda> = \int_{\underline{R}^3} \lambda^{ij} s_{ij} d^3x.$$

[Note: This integral is finite. Thus fix $R_0 >> 0$ -- then for $R > R_0$,

$$\int_{\underline{D}^3(R)} = \int_{R \ge |x| \ge R_0} + \int_{\underline{D}^3(R_0)}$$

and

$$\int_{R \ge |x| \ge R_0} \lambda^{ij}(x) s_{ij}(x) d^3x$$

$$= \int_{R \ge |x| \ge R_0} \left(\frac{1}{r^2} \tau^{ij}\left(\frac{x}{r}\right) + O\left(\frac{1}{r^{2+\delta}}\right)\right)\left(\frac{1}{r} \sigma_{ij}\left(\frac{x}{r}\right) + O\left(\frac{1}{r^{1+\delta}}\right)\right) d^3x$$

$$= \int_{R \ge |x| \ge R_0} \left(\frac{1}{r^3} \tau^{ij}\left(\frac{x}{r}\right) \sigma_{ij}\left(\frac{x}{r}\right) + O\left(\frac{1}{r^{3+\delta}}\right)\right) d^3x$$

$$= \int_{R \ge |x| \ge R_0} \frac{1}{r^3} \tilde{\tau}^{ij}(x) \tilde{\sigma}_{ij}(x) d^3x$$

$$+ \int_{R \ge |x| \ge R_0} O\left(\frac{1}{r^{3+\delta}}\right) d^3x$$

$$= 0 + \int_{R \ge |x| \ge R_0} O\left(\frac{1}{r^{3+\delta}}\right) d^3x,$$

the parity of $\tau^{ij}\sigma_{ij}$ being odd.]

Put

$$\Gamma = Q_\infty \times S_d^{2,\infty}.$$

Then

$$T_{(q,\Lambda)}\Gamma = S_{2,\infty} \times S_d^{2,\infty}$$

and the map

$$\Omega_{(q,\Lambda)}: T_{(q,\Lambda)}\Gamma \times T_{(q,\Lambda)}\Gamma \to \underline{R}$$

defined by the prescription

$$\Omega_{(q,\Lambda)}((s_1,\Lambda_1),(s_2,\Lambda_2)) = \langle s_1,\Lambda_2\rangle - \langle s_2,\Lambda_1\rangle$$

serves to equip Γ with a globally constant symplectic structure.

The hamiltonian $H:\Gamma \to \underline{R}$ of the metric theory depends on external variables $N,\vec{N}$:

$$H(q,\Lambda;N,\vec{N}) = \int_{\underline{R}^3} - 2\,\mathrm{div}_q\,\Lambda(\vec{N})$$

$$+ \int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\,\mathrm{tr}_q(s)^2 - S(q))\sqrt{q}\,d^3x$$

if $\Lambda = s^\# \otimes |q|^{1/2}$. However, there is a difficulty in that neither integral will be convergent unless conditions are imposed on N and $\vec{N}$.

Assumption:

$$\left[\begin{array}{l} N(x) = \psi(\frac{x}{r}) + O^\infty(\frac{1}{r^\varepsilon}) \\ \\ N^i(x) = \psi^i(\frac{x}{r}) + O^\infty(\frac{1}{r^\varepsilon}), \end{array}\right. \qquad (\varepsilon > 0)$$

where ψ and ψ^i are C^∞ functions on $\underline{S}^2$ of odd parity.

[Note: These are, by definition, the standard conditions on N and $\vec{N}$.]

LEMMA If N and $\vec{N}$ satisfy the standard conditions, then the integrals defining

$$H(q,\Lambda;N,\vec{N})$$

are convergent.

While elementary, it will be safest to run through the particulars.

Convention: In the sequel, we shall sometimes write ho when it is a question of terms that are $O(\frac{1}{r^{3+\delta}})$ $(\delta > 0)$.

To deal with

$$\int_{\underline{R}^3} \text{div}_q \, \Lambda(\vec{N})$$

amounts to dealing with

$$\int_{\underline{R}^3} (\text{div}_q \, s)_i N^i \sqrt{q} \, d^3x,$$

where, as will be recalled,

$$(\text{div}_q \, s)_i = q^{jk} \nabla_j s_{ik}.$$

Put $\lambda = s^{\#}\sqrt{q}$ -- then

$$(\text{div}_q \, s)_i N^i \sqrt{q} = (\text{div}_q \frac{\lambda^{\flat}}{\sqrt{q}})_i N^i \sqrt{q}$$

$$= q^{jk} \nabla_j (\frac{\lambda^{\flat}}{\sqrt{q}})_{ik} N^i \sqrt{q}$$

$$= q^{jk} \frac{1}{\sqrt{q}} (\nabla_j \lambda_{ik}) N^i \sqrt{q}$$

$$= q^{jk} (\nabla_j \lambda_{ik}) N^i$$

$$= q^{jk} (\nabla_j q_{ii'} q_{kk'} \lambda^{i'k'}) N^i$$

$$= q^{jk} q_{kk'} q_{ii'} (\nabla_j \lambda^{i'k'}) N^i$$

$$= \delta^j{}_{k'} q_{ii'} (\nabla_j \lambda^{i'k'}) N^i$$

$$= q_{ii'} (\nabla_j \lambda^{i'j}) N^i$$

$$= q_{ij} (\nabla_k \lambda^{jk}) N^i .$$

- $q_{ij} = \eta_{ij} + O(\frac{1}{r})$.

- $\nabla_k \lambda^{jk} = \partial_k \lambda^{jk} + \Gamma^j{}_{k\ell} \lambda^{\ell k} + \Gamma^k{}_{k\ell} \lambda^{j\ell}$

$$= \partial_k \lambda^{jk} + O(\frac{1}{r^4}).$$

- $N^i = O(1)$.

=>

$$q_{ij} (\nabla_k \lambda^{jk}) N^i$$

$$= (\eta_{ij} + O(\frac{1}{r})) (\partial_k \lambda^{jk} + O(\frac{1}{r^4})) N^i$$

$$= \eta_{ij}(\partial_k \lambda^{jk})N^i + O(\frac{1}{r})(\partial_k \lambda^{jk})N^i + O(\frac{1}{r^4}).$$

The issue of integrability thus becomes that of

$$\eta_{ij}(\partial_k \lambda^{jk})N^i + O(\frac{1}{r})(\partial_k \lambda^{jk})N^i.$$

• $\partial_k \lambda^{jk} = \partial_k(\frac{1}{r^2}\tilde{\tau}^{jk}) + \partial_k \nu^{jk}$

$$= -\frac{2x_k}{r^4}\tilde{\tau}^{jk} + \frac{1}{r^2}\partial_k \tilde{\tau}^{jk} + O(\frac{1}{r^{3+\delta}})$$

$$= O^+(\frac{1}{r^3}) + \text{ho}.$$

This reduces matters to consideration of

$$\eta_{ij}O^+(\frac{1}{r^3})N^i = \eta_{ij}O^+(\frac{1}{r^3})(\tilde{\psi}^i + O(\frac{1}{r^\varepsilon}))$$

or still, to

$$\eta_{ij}O^+(\frac{1}{r^3})\tilde{\psi}^i,$$

which is $O^-(\frac{1}{r^3})$.

Therefore the integral

$$\int_{\underline{R}^3} - 2\,\text{div}_q \Lambda(\vec{N})$$

is convergent.

To discuss the integral

$$\int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\text{tr}_q(s)^2 - S(q))\sqrt{q}\, d^3x,$$

start by writing

$$[s,s]_q = s^{ij}s_{ij}$$

$$= \frac{\lambda^{ij}}{\sqrt{q}} s_{ij}$$

$$= \frac{\lambda^{ij}}{\sqrt{q}} q_{ik}q_{j\ell}s^{k\ell}$$

$$= \frac{\lambda^{ij}}{\sqrt{q}} q_{ik}q_{j\ell} \frac{\lambda^{k\ell}}{\sqrt{q}}$$

$$= \frac{1}{\sqrt{q}} \frac{1}{\sqrt{q}} q_{ik}q_{j\ell}\lambda^{ij}\lambda^{k\ell}.$$

Then

$$N[s,s]_q\sqrt{q}$$

$$= \frac{N}{\sqrt{q}} q_{ik}q_{j\ell}\lambda^{ij}\lambda^{k\ell}$$

$$= O(1)\lambda^{ij}\lambda^{k\ell}$$

$$= O(\frac{1}{r^4}).$$

Next

$$tr_q(s)^2 = (q^{ij}s_{ij})^2$$

$$= (q^{ij}q_{ik}q_{j\ell}s^{k\ell})^2$$

$$= (\delta^j{}_k q_{j\ell} s^{k\ell})^2$$

$$= (q_{j\ell} s^{j\ell})^2$$

$$= (q_{j\ell} \frac{\lambda^{j\ell}}{\sqrt{q}})^2$$

$$= (q_{ij} \frac{\lambda^{ij}}{\sqrt{q}}) (q_{k\ell} \frac{\lambda^{k\ell}}{\sqrt{q}})$$

$$= \frac{1}{\sqrt{q}} \frac{1}{\sqrt{q}} q_{ij} q_{k\ell} \lambda^{ij} \lambda^{k\ell}$$

=>

$$N tr_q(s)^2 \sqrt{q}$$

$$= \frac{N}{\sqrt{q}} q_{ij} q_{k\ell} \lambda^{ij} \lambda^{k\ell}$$

$$= O(1) \lambda^{ij} \lambda^{k\ell}$$

$$= O(\frac{1}{r^4}).$$

Finally

$$S(q) = q^{j\ell} R^i{}_{ji\ell}.$$

And

$$R^i{}_{ji\ell} = \Gamma^i{}_{\ell j,i} - \Gamma^i{}_{ij,\ell} + \Gamma^a{}_{\ell j} \Gamma^i{}_{ia} - \Gamma^a{}_{ij} \Gamma^i{}_{\ell a}$$

$$= \Gamma^i{}_{\ell j,i} - \Gamma^i{}_{ij,\ell} + O(\frac{1}{r^4})$$

$$= O^+(\frac{1}{r^3}) + \text{ho}.$$

But then

$$NS(q)\sqrt{q} = (\tilde{\psi} + O(\frac{1}{r^{\varepsilon}}))(O^{+}(\frac{1}{r^3}) + ho)(1 + O(\frac{1}{r}))$$

$$= O^{-}(\frac{1}{r^3}) + ho.$$

Therefore the integral

$$\int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\,tr_q(s)^2 - S(q))\sqrt{q}\; d^3x$$

is convergent.

Maintaining the assumption that N and $\vec{N}$ are subject to the standard conditions, if we ignore the boundary terms, then

$$\frac{\delta H}{\delta q} = 2N(s*s - \frac{1}{2}\,tr_q(s)s)^{\#} \otimes |q|^{1/2}$$

$$- \frac{N}{2}\,([s,s]_q - \frac{1}{2}\,tr_q(s)^2)q^{\#} \otimes |q|^{1/2}$$

$$+ N\,Ein(q)^{\#} \otimes |q|^{1/2}$$

$$- (H_N - (\Delta_q N)q)^{\#} \otimes |q|^{1/2} - L_{\vec{N}}\Lambda$$

and

$$\frac{\delta H}{\delta \Lambda} = 2N(s - \frac{1}{2}\,tr_q(s)q) + L_{\vec{N}}q.$$

[Note: These formulas imply that

$$\frac{\delta H}{\delta q} \in S_d^{2,\infty} \text{ and } \frac{\delta H}{\delta \Lambda} \in S_{2,\infty}.]$$

To justify the foregoing, one has to identify the boundary terms and show that they make no contribution.

Surface Integrals Working in $\underline{R}^n$, let

$$\begin{cases} \underline{D}^n(R) = \{x: \sum_{i=1}^{n} (x^i)^2 \leq R\} \\ \underline{S}^{n-1}(R) = \{x: \sum_{i=1}^{n} (x^i)^2 = R\}. \end{cases}$$

Equip $\underline{R}^n$ with its usual riemannian structure and view $\underline{S}^{n-1}(R)$ as a riemannian submanifold — then the volume form on $\underline{S}^{n-1}(R)$ is the pullback of the (n-1)-form

$$\omega_R^{n-1} = \frac{1}{R} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n$$

on $\underline{R}^n - \{0\}$. E.g.: When $n = 3$,

$$\omega_R^2 = \frac{1}{R} (x^1 dx^2 \wedge dx^3 - x^2 dx^1 \wedge dx^3 + x^3 dx^1 \wedge dx^2).$$

The exterior unit normal to $\underline{S}^{n-1}(R)$, considered as the boundary of $\underline{D}^n(R)$, is

$$\underline{n}\Big|_x = \frac{1}{R} (x^1 \frac{\partial}{\partial x^1} + \cdots + x^n \frac{\partial}{\partial x^n})$$

and the divergence theorem says that

$$\int_{\underline{D}^n(R)} (\text{div } X) d^n x = \int_{\underline{S}^{n-1}(R)} (X \cdot \underline{n}) \omega_R^{n-1}.$$

[Note: Take $n = 3$ and define

$$\iota_R:]0,2\pi[\times]0,\pi[\rightarrow \underline{S}^2(R)$$

by

$$\iota_R(\theta,\phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi).$$

Then

$$\int_{\underline{S}^2(R)} f\omega_R^2 = R^2 \int_0^{2\pi} \int_0^{\pi} f \circ \iota_R \sin\phi \, d\phi \, d\theta.$$

Therefore

$$\int_{\underline{S}^2(R)} (X\cdot\underline{n})\,\omega_R^2 = \int_{\underline{S}^2(R)} (\frac{1}{R}\, x^i X^i)\,\omega_R^2$$

$$= \frac{R^2}{R} \int_0^{2\pi}\int_0^{\pi}(R\cos\theta\sin\phi\, X^1 \circ \iota_R + R\sin\theta\sin\phi\, X^2 \circ \iota_R + R\cos\phi\, X^3 \circ \iota_R)\sin\phi\, d\phi\, d\theta$$

$$= R^2 \int_0^{2\pi}\int_0^{\pi}(\cos\theta\sin\phi\, X^1 \circ \iota_R + \sin\theta\sin\phi\, X^2 \circ \iota_R + \cos\phi\, X^3 \circ \iota_R)\sin\phi\, d\phi\, d\theta.$$

So, if

$$\int_{\underline{R}^3} (\text{div } X)\, d^3x$$

is defined to be

$$\lim_{R\to\infty} \int_{\underline{D}^3(R)} (\text{div } X)\, d^3x$$

and if

$$X = O(\frac{1}{R^{2+\delta}}) \quad (\delta > 0),$$

then

$$\lim_{R\to\infty} \int_{\underline{S}^2(R)} (X\cdot\underline{n})\,\omega_R^2 = 0.$$

However the weaker assumption that

$$X = O(\frac{1}{R^2})$$

does not guarantee that

$$\int_{\underline{R}^3} (\text{div } X)\, d^3x$$

exists: Without additional data, the conclusion is merely that

$$\int_{\underline{S}^2(R)} (X\cdot\underline{n})\omega_R^2 = O(1).$$

To appreciate the point, consider

$$X = \frac{\sin r}{r^3}\left(x^i \frac{\partial}{\partial x^i}\right) \quad (r >> 0).]$$

Later on, it will be necessary to differentiate under the integral sign, a process that requires some backup. Here is one such result, tailored for improper integrals.

Criterion Suppose given $f(x,t)$ $(x\in\underline{R}^3, t\in[-a,a])$. Make the following assumptions.

1. f is a continuous function of (x,t).
2. $\frac{\partial f}{\partial t}$ is a continuous function of (x,t).
3. $\int_{\underline{R}^3} f(x,t)d^3x$ exists and is a continuous function of t.
4. $\int_{\underline{R}^3} \frac{\partial f}{\partial t}(x,t)d^3x$ exists and is a continuous function of t.
5. $\exists\, M > 0: \forall\, R,$

$$M \geq \left|\int_{\underline{D}^3(R)} \frac{\partial f}{\partial t}(x,t)d^3x\right| (-a \leq t \leq a).$$

Then

$$\frac{d}{dt}\left[\int_{\underline{R}^3} f(x,t)d^3x\right] = \int_{\underline{R}^3} \frac{\partial f}{\partial t}(x,t)d^3x.$$

[Choose $R_n: R_n < R_{n+1}$ & $\lim R_n = \infty$:

$$\int_{-a}^{t} \int_{\underline{R}^3} \frac{\partial f}{\partial t'}(x,t')d^3x\, dt'$$

$$= \int_{-a}^{t} \lim_{n\to\infty} \int_{\underline{D}^3(R_n)} \frac{\partial f}{\partial t'} (x,t')d^3x\, dt'$$

$$= \lim_{n\to\infty} \int_{-a}^{t} \int_{\underline{D}^3(R_n)} \frac{\partial f}{\partial t'} (x,t')d^3x\, dt' \quad \text{(dominated convergence)}$$

$$= \lim_{n\to\infty} \int_{\underline{D}^3(R_n)} \int_{-a}^{t} \frac{\partial f}{\partial t'} (x,t')dt'\, d^3x \quad \text{(Fubini)}$$

$$= \lim_{n\to\infty} \int_{\underline{D}^3(R_n)} (f(x,t) - f(x,-a))d^3x$$

$$= \int_{\underline{R}^3} f(x,t)d^3x - \int_{\underline{R}^3} f(x,-a)d^3x$$

=>

$$\frac{d}{dt} [\int_{\underline{R}^3} f(x,t)d^3x]$$

$$= \frac{d}{dt} [\int_{-a}^{t} \int_{\underline{R}^3} \frac{\partial f}{\partial t'} (x,t')d^3x\, dt']$$

$$= \int_{\underline{R}^3} \frac{\partial f}{\partial t} (x,t)d^3x.]$$

Rappel:

$$H(q,\Lambda;N,\vec{N}) = \int_{\underline{R}^3} - 2\text{div}_q\, \Lambda(\vec{N})$$

$$+ \int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\, \text{tr}_q(s)^2 - S(q))\sqrt{q}\, d^3x.$$

The computation of

$$\frac{\delta}{\delta q} [\int_{\underline{R}^3} - 2\text{div}_q\, \Lambda(\vec{N})]$$

and

$$\frac{\delta}{\delta\Lambda}\left[\int_{\underline{R}^3} - 2\mathrm{div}_q\, \Lambda(\vec{N})\right]$$

depends on rewriting

$$\int_{\underline{R}^3} -2\mathrm{div}_q\, \Lambda(\vec{N})$$

as

$$\int_{\underline{R}^3} \Lambda(L_{\vec{N}}q)$$

and this is where an integration by parts creeps in.

<u>LEMMA</u> The integral

$$\int_{\underline{R}^3} \Lambda(L_{\vec{N}}q)$$

is convergent.

[We have

$$\int_{\underline{R}^3} \Lambda(L_{\vec{N}}q) = \int_{\underline{R}^3} s^{\#}(L_{\vec{N}}q)\sqrt{q}\, d^3x$$

$$= \int_{\underline{R}^3} s^{ij}(N_{i;j} + N_{j;i})\sqrt{q}\, d^3x.$$

Since $s^{ij} = s^{ji}$, it suffices to consider

$$\int_{\underline{R}^3} s^{ij}N_{i;j}\sqrt{q}\, d^3x.$$

Write

$$s^{ij}N_{i;j}\sqrt{q}$$

$$= \frac{\lambda^{ij}}{\sqrt{q}} N_{i;j}\sqrt{q}$$

$$= \lambda^{ij}N_{i;j}$$

$$= \lambda^{ij}\nabla_j q_{ik}N^k$$

$$= \lambda^{ij}q_{ik}\nabla_j N^k$$

$$= \lambda^{ij}q_{ik}(\partial_j N^k + \Gamma^k{}_{j\ell}N^\ell).$$

Then

- $\lambda^{ij}q_{ik}\partial_j N^k$

$$= (O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))(\eta_{ik} + O(\frac{1}{r}))(O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\epsilon}}))$$

$$= O^-(\frac{1}{r^3}) + \text{ho}.$$

- $\lambda^{ij}q_{ik}\Gamma^k{}_{j\ell}N^\ell$

$$= O(\frac{1}{r^2})(\eta_{ik} + O(\frac{1}{r}))O(\frac{1}{r^2})O(1)$$

$$= O(\frac{1}{r^4}).]$$

The boundary term that figures in the passage from

$$\int_{\underline{R}^3} - 2\text{div}_q\, \Lambda(\vec{N})$$

to

$$\int_{\underline{R}^3} \Lambda(L_{\vec{N}}q)$$

arises from the identity

$$N_{i;j}s^{ij} = -N_i\nabla_j s^{ij} + \nabla_j(N_i s^{ij}).$$

Here

$$\nabla_j(N_i s^{ij}) = X^j_{;j},$$

where

$$X^j = s^{ij}N_i.$$

We then want to argue that

$$\int_{\underline{R}^3} (\mathrm{div}_q\, X)\mathrm{vol}_q = 0.$$

For this purpose, write

$$\int_{\underline{R}^3} (\mathrm{div}_q\, X)\mathrm{vol}_q$$

$$= \int_{\underline{R}^3} \left(\frac{1}{\sqrt{q}}\frac{\partial(\sqrt{q}\,X^j)}{\partial x^j}\right)\sqrt{q}\, d^3x$$

$$= \int_{\underline{R}^3} \frac{\partial(\sqrt{q}\,X^j)}{\partial x^j}\, d^3x$$

$$= \lim_{R\to\infty} \int_{\underline{D}^3(R)} (\mathrm{div}\,\sqrt{q}\,X)\,d^3x$$

$$= \lim_{R\to\infty} \int_{\underline{S}^2(R)} (\sqrt{q}\,X\cdot\underline{n})\,\omega_R^2.$$

If

$$\sqrt{q}\,X = O\left(\frac{1}{R^{2+\delta}}\right),$$

then

$$\int_{\underline{R}^3} (\mathrm{div}_q\, X)\mathrm{vol}_q = 0$$

and we are done. But we don't quite have this. To see what we do have, note that for $R >> 0$,

$$\sqrt{q}\, X^j = \sqrt{q}\, s^{ij}N_i$$

$$= \sqrt{q}\, \left(\frac{\lambda^{ij}}{\sqrt{q}}\right)N_i$$

$$= \lambda^{ij}N_i$$

$$= \lambda^{ij}q_{ik}N^k$$

$$= \left(\frac{1}{R^2}\, \tilde{\tau}^{ij} + O\left(\frac{1}{R^{2+\delta}}\right)\right)\left(\eta_{ik} + \frac{1}{R}\, \tilde{\sigma}_{ik} + O\left(\frac{1}{R^{1+\delta}}\right)\right)\left(\tilde{\psi}^k + O\left(\frac{1}{R^{\varepsilon}}\right)\right)$$

$$= \frac{1}{R^2}\, \tilde{\tau}^{ij}\eta_{ik}\tilde{\psi}^k + O\left(\frac{1}{R^{2+c}}\right) \quad (c > 0).$$

Accordingly, it remains to examine

$$R^2 \int_0^{2\pi}\int_0^{\pi} \frac{1}{R^2}\, (1+2+3)\sin\phi\, d\phi\, d\theta,$$

where

$$\begin{cases} 1 = \cos\theta\, \sin\phi\, \tau^{i1}(\theta,\phi)\eta_{ik}\psi^k(\theta,\phi) \\ 2 = \sin\theta\, \sin\phi\, \tau^{i2}(\theta,\phi)\eta_{ik}\psi^k(\theta,\phi) \\ 3 = \cos\phi\, \tau^{i3}(\theta,\phi)\eta_{ik}\psi^k(\theta,\phi). \end{cases}$$

But since the parity of 1,2,3 is odd, the integral vanishes, thus

$$\int_{\underline{R}^3} (\mathrm{div}_q X)\mathrm{vol}_q = 0.$$

The functional derivative of

$$\int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\mathrm{tr}_q(s)^2 - S(q))\sqrt{q}\, d^3x$$

w.r.t. Λ does not involve a boundary term. As for the functional derivative of

$$\int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\mathrm{tr}_q(s)^2 - S(q))\sqrt{q}\, d^3x$$

w.r.t. q, a boundary term is encountered only in the computation of

$$\frac{\delta}{\delta q}\left[\int_{\underline{R}^3} - NS(q)\sqrt{q}\, d^3x\right].$$

We have

$$\frac{d}{d\varepsilon}\left[\int_{\underline{R}^3} - NS(q + \varepsilon\delta q)\sqrt{q + \varepsilon\delta q}\, d^3x\right]\Big|_{\varepsilon=0}$$

$$= \int_{\underline{R}^3} - N\frac{d}{d\varepsilon}\left[S(q + \varepsilon\delta q)\sqrt{q + \varepsilon\delta q}\right]\Big|_{\varepsilon=0} d^3x,$$

where $\delta q \in S_{2,\infty}$. But

$$\int_{\underline{R}^3} - N\frac{d}{d\varepsilon}\left[S(q + \varepsilon\delta q)\sqrt{q + \varepsilon\delta q}\right]\Big|_{\varepsilon=0} d^3x$$

$$= \int_{\underline{R}^3} N[\Delta_q \mathrm{tr}_q(\delta q) + \delta_q \mathrm{div}_q\, \delta q]\sqrt{q}\, d^3x$$

$$+ \int_{\underline{R}^3} Nq[{}^0_2](\mathrm{Ein}(q), \delta q)\sqrt{q}\, d^3x.$$

Since both integrals are convergent (cf. infra), this makes sense.

That the second integral is convergent is easy to see: In fact,

$$Nq[{}^0_2](\mathrm{Ein}(q),\delta q)\sqrt{q}$$

$$= N\ \mathrm{Ein}(q)_{ij}(\delta q)^{ij}\sqrt{q}$$

$$= N\ \mathrm{Ein}(q)_{ij}q^{ik}q^{j\ell}(\delta q)_{k\ell}\sqrt{q}$$

$$= O(1)O(\frac{1}{r^3})(\eta_{ik} + O(\frac{1}{r}))(\eta_{j\ell} + O(\frac{1}{r}))O(\frac{1}{r})(1 + O(\frac{1}{r}))$$

$$= O(\frac{1}{r^4}).$$

[Note: No additional manipulation is needed for the second integral (it contributes directly to $\frac{\delta H}{\delta q}$).]

Notation: Put

$$(dN\cdot\delta q)_i = (dN)_j\delta q^j{}_i.$$

Identity We have

$$N[\Delta_q \mathrm{tr}_q(\delta q) + \delta_q \mathrm{div}_q\ \delta q]$$

$$= -\ [H_N - (\Delta_q N)q,\delta q]_q$$

$$-\ \delta_q(N(d\mathrm{tr}_q(\delta q) - \mathrm{div}_q\ \delta q))$$

$$-\ \delta_q(dN\cdot\delta q - \mathrm{tr}_q(\delta q)dN).$$

The integral

$$\int_{\underline{R}^3} -\ [H_N - (\Delta_q N)q,\delta q]_q\sqrt{q}\ d^3x$$

is convergent and leads to the remaining term in the expression for $\frac{\delta H}{\delta q}$.

<u>Details</u> Write

$$[H_N, \delta q]_q \sqrt{q}$$

$$= (H_N)_{ij} (\delta q)^{ij} \sqrt{q}$$

$$= (\partial_i \partial_j N - \Gamma^a{}_{ij} \partial_a N) q^{ik} q^{j\ell} (\delta q)_{k\ell} \sqrt{q}.$$

• $\partial_i \partial_j N q^{ik} q^{j\ell} (\delta q)_{k\ell} \sqrt{q}$

$$= (O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\varepsilon}}))(\eta_{ik} + O(\frac{1}{r}))(\eta_{j\ell} + O(\frac{1}{r}))$$

$$\times (O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}}))(1 + O(\frac{1}{r}))$$

$$= O^-(\frac{1}{r^3}) + \text{ho}.$$

• $\Gamma^a{}_{ij} \partial_a N q^{ik} q^{j\ell} (\delta q)_{k\ell} \sqrt{q}$

$$= (O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))(O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\varepsilon}}))(\eta_{ik} + O(\frac{1}{r}))(\eta_{j\ell} + O(\frac{1}{r}))$$

$$\times (O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}}))(1 + O(\frac{1}{r}))$$

$$= O(\frac{1}{r^4}).$$

[Note: The discussion of

$$[(\Delta_q N) q, \delta q]_q \sqrt{q}$$

is analogous.]

Therefore, to finish up, it has to be shown that

$$\begin{cases} \int_{\underline{R}^3} \delta_q(N(d\,tr_q(\delta q) - div_q\ \delta q))\sqrt{q}\ d^3x = 0 \\ \\ \int_{\underline{R}^3} \delta_q(dN\cdot\delta q - tr_q(\delta q)dN)\sqrt{q}\ d^3x = 0. \end{cases}$$

Rappel: Let $\omega = f_i dx^i$ -- then

$$\delta_q\omega = -\frac{1}{\sqrt{q}}\frac{\partial}{\partial x^i}(\sqrt{q}\ q^{ij}f_j).$$

Because of this, each integral is an ordinary divergence, hence it suffices to consider

$$X^i = \sqrt{q}\ q^{ij}f_j,$$

where

$$\begin{cases} f_j = N\frac{\partial}{\partial x^j}tr_q(\delta q),\ f_j = N(div_q\ \delta q)_j \\ \\ f_j = (dN\cdot\delta q)_j\ ,\ f_j = tr_q(\delta q)\ \frac{\partial N}{\partial x^j}. \end{cases}$$

<u>N.B.</u>

$$\sqrt{q}\ q^{ij} = \eta_{ij} + O(\frac{1}{r}).$$

- $N\frac{\partial}{\partial x^j}tr_q(\delta q)$

$$= N\frac{\partial}{\partial x^j}(q^{k\ell}\delta q_{k\ell})$$

$$= N(\frac{\partial}{\partial x^j}q^{k\ell})\delta q_{k\ell} + Nq^{k\ell}\frac{\partial}{\partial x^j}\delta q_{k\ell}$$

$$= O(1)O(\frac{1}{r^2})O(\frac{1}{r}) + Nq^{k\ell}\frac{\partial}{\partial x^j}\delta q_{k\ell}$$

$$= O(\frac{1}{r^3}) + Nq^{k\ell}\frac{\partial}{\partial x^j}\delta q_{k\ell}.$$

And

$$Nq^{k\ell}\frac{\partial}{\partial x^j}\delta q_{k\ell}$$

$$= (\tilde{\psi} + O(\frac{1}{r^\varepsilon}))(\eta_{k\ell} + O(\frac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))$$

$$= O^+(\frac{1}{r^2}) + O(\frac{1}{r^{2+c}}) \quad (c > 0).$$

• $N(\mathrm{div}_q\, \delta q)_j$

$$= Nq^{k\ell}\nabla_k \delta q_{j\ell}$$

$$= Nq^{k\ell}[\partial_k \delta q_{j\ell} - \Gamma^a{}_{kj}\delta q_{a\ell} - \Gamma^a{}_{k\ell}\delta q_{ja}]$$

$$= Nq^{k\ell}\partial_k \delta q_{j\ell}$$

$$- Nq^{k\ell}[\Gamma^a{}_{kj}\delta q_{a\ell} + \Gamma^a{}_{k\ell}\delta q_{ja}]$$

$$= O^+(\frac{1}{r^2}) + O(\frac{1}{r^{2+c}}) \quad (c > 0)$$

$$- Nq^{k\ell}[\Gamma^a{}_{kj}\delta q_{a\ell} + \Gamma^a{}_{k\ell}\delta q_{ja}].$$

And

$$Nq^{k\ell}[\Gamma^a{}_{kj}\delta q_{a\ell} + \Gamma^a{}_{k\ell}\delta q_{ja}]$$

$$= O(1)(\eta_{k\ell} + O(\tfrac{1}{r}))O(\tfrac{1}{r^2})O(\tfrac{1}{r})$$

$$= O(\tfrac{1}{r^3}).$$

- $(dN\cdot\delta q)_j$

$$= (dN)_i \delta q^i{}_j$$

$$= (\partial_i N)\delta q^i{}_j$$

$$= (\partial_i N) q^{ik}\delta q_{kj}$$

$$= (O^+(\tfrac{1}{r}) + O(\tfrac{1}{r^{1+\varepsilon}}))(\eta_{ik} + O(\tfrac{1}{r}))(O^+(\tfrac{1}{r}) + O(\tfrac{1}{r^{1+\delta}}))$$

$$= O^+(\tfrac{1}{r^2}) + O(\tfrac{1}{r^{2+c}}) \quad (c > 0).$$

- $\mathrm{tr}_q(\delta q)\ \dfrac{\partial N}{\partial x^j}$

$$= q^{k\ell}\delta q_{k\ell}\frac{\partial N}{\partial x^j}$$

$$= (\eta_{k\ell} + O(\tfrac{1}{r}))(O^+(\tfrac{1}{r}) + O(\tfrac{1}{r^{1+\delta}}))(O^+(\tfrac{1}{r}) + O(\tfrac{1}{r^{1+\varepsilon}}))$$

$$= O^+(\tfrac{1}{r^2}) + O(\tfrac{1}{r^{2+c}}) \quad (c > 0).$$

Conclusion: The potentially troublesome part of X^i is $O^+(\frac{1}{r^2})$ which, when multiplied by x^i, integrates to zero over $\underline{S}^2(R)$.

<u>Poisson Brackets</u> Put

$$\mathcal{H}_D(\vec{N}) = \int_{\underline{R}^3} -2\mathrm{div}_q\, \Lambda(\vec{N})$$

and

$$\mathcal{H}_H(N) = \int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\,\mathrm{tr}_q(s)^2 - S(q))\sqrt{q}\, d^3x.$$

Therefore

$$\mathcal{H} = \mathcal{H}_D + \mathcal{H}_H$$

and we have:

1. $\{\mathcal{H}_D(\vec{N}_1), \mathcal{H}_D(\vec{N}_2)\} = \mathcal{H}_D([\vec{N}_1, \vec{N}_2])$;

2. $\{\mathcal{H}_D(\vec{N}), \mathcal{H}_H(N)\} = \mathcal{H}_H(L_{\vec{N}}N)$;

3. $\{\mathcal{H}_H(N_1), \mathcal{H}_H(N_2)\}$

$$= \mathcal{H}_D(N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1).$$

<u>N.B.</u> Tacitly, $\vec{N}$, N, $\vec{N}_1$, $\vec{N}_2$, N_1, N_2 are subject to the standard conditions. To ensure consistency, one then has to check that

$$[\vec{N}_1, \vec{N}_2],\; L_{\vec{N}}N, \text{ and } N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1$$

also satisfy the standard conditions, which is straightforward (they all have the form $O^\infty(\frac{1}{r^\varepsilon})$ $(\varepsilon > 0)$).

[Note: In this context, the gradient depends on q, i.e., grad = grad_q.]

Each of the three computations leads to a boundary term, ignorable in the case of a compact Σ but, of course, not in general.

To illustrate, consider the derivation of the relation

$$\{H_D(\vec{N}), H_H(N)\} = H_H(L_{\vec{N}} N).$$

Here the boundary term is

$$-\int_{\underline{R}^3} L_{\vec{N}}(NE\mathrm{vol}_q),$$

where

$$E = [s,s]_q - \frac{1}{2}\,\mathrm{tr}_q(s)^2 - S(q).$$

This said, write

$$\int_{\underline{R}^3} L_{\vec{N}}(NE\mathrm{vol}_q)$$

$$= \int_{\underline{R}^3} d(\iota_{\vec{N}}(NE\mathrm{vol}_q))$$

$$= \int_{\underline{R}^3} d(NE\iota_{\vec{N}}\mathrm{vol}_q)$$

$$= \int_{\underline{R}^3} d(\iota_{NE\vec{N}}\mathrm{vol}_q)$$

$$= \int_{\underline{R}^3} d(\iota_{NE\vec{N}}\sqrt{q}\; d^3x)$$

$$= \int_{\underline{R}^3} d(\iota_{\sqrt{q}\,NE\vec{N}} d^3x)$$

$$= \int_{\underline{R}^3} (\mathrm{div}\; X)d^3x,$$

the vector field

$$X = X^i \frac{\partial}{\partial x^i} \in \mathcal{D}^1(\underline{R}^3)$$

being given by

$$X^i = \sqrt{q}\, NEN^i.$$

But, on the basis of earlier work,

$$\begin{cases} \sqrt{q}\, N[s,s]_q N^i = O(\frac{1}{r^4}) \\ \sqrt{q}\, N\mathrm{tr}_q(s)^2 N^i = O(\frac{1}{r^4}) \\ \sqrt{q}\, NS(q)N^i = O(\frac{1}{r^3}). \end{cases}$$

Therefore

$$X = O(\frac{1}{r^3})$$

$$\Rightarrow$$

$$\int_{\underline{R}^3} (\mathrm{div}\, X)d^3x = 0.$$

- Denote by Con_D the subset of Γ consisting of those pairs (q,Λ) such that

$$\mathrm{div}_q\, s = 0.$$

- Denote by Con_H the subset of Γ consisting of those pairs (q,Λ) such that

$$[s,s]_q - \frac{1}{2}\mathrm{tr}_q(s)^2 - S(q) = 0.$$

[Note: Here, as always, $\Lambda = s^\# \otimes |q|^{1/2}$.]

Put

$$\mathrm{Con}_{Q_\infty} = \mathrm{Con}_D \cap \mathrm{Con}_H \subset \Gamma.$$

Definition: A <u>constraint</u> is a function $f:\Gamma \to \underline{R}$ such that $f|\mathrm{Con}_{Q_\infty} = 0$.

Therefore

$$\left[\begin{array}{l} \mathcal{H}_D(\vec{N}) \\ \\ \mathcal{H}_H(N) \end{array}\right.$$

are constraints, these being termed <u>primary</u>. Since the Poisson bracket of two primary constraints is a constraint, our system is first class.

Section 63: The Integrals of Motion-Energy and Center of Mass

The assumptions and notation are those of Section 62.

Rappel:

$$H = H_D + H_H,$$

where

$$H_D(\vec{N}) = \int_{\underline{R}^3} - 2\mathrm{div}_q \, \Lambda(\vec{N})$$

and

$$H_H(N) = \int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\,\mathrm{tr}_q(s)^2 - S(q))\sqrt{q}\, d^3x.$$

Needless to say, $\vec{N}$ and N are subject to the standard conditons. However, in order to formulate the definition of energy, linear momentum, angular momentum, and center of mass, the standard conditions are too restrictive, thus must be relaxed.

In this section, we shall deal with $H_H(N)$ and suppose that

$$N = A + B_1x^1 + B_2x^2 + B_3x^3 + sc,$$

where A and B_1, B_2, B_3 are constants and sc stands for a function which satisfies the standard conditions.

Problem: Determine whether the integral defining $H_H(N)$ is convergent or not.

Since this is the case of $H_H(sc)$, it suffices to consider the matter when $N = A + Bx^b$ $(b = 1,2,3)$.

First,

$$\int_{\underline{R}^3} N[s,s]_q\sqrt{q}\, d^3x$$

is convergent, as is

$$\int_{\underline{R}^3} - \frac{N}{2}\,\mathrm{tr}_q(s)^2\sqrt{q}\, d^3x.$$

E.g.:

$$N[s,s]_q\sqrt{q}$$

$$= \frac{N}{\sqrt{q}}\, q_{ik}q_{j\ell}\lambda^{ij}\lambda^{k\ell}$$

$$= (A + Bx^b)(1 + O(\tfrac{1}{r}))(\eta_{ik} + O(\tfrac{1}{r}))(\eta_{j\ell} + O(\tfrac{1}{r}))$$

$$\times\ (\frac{\tilde{\tau}^{ij}}{r^2} + O(\frac{1}{r^{2+\delta}}))(\frac{\tilde{\tau}^{k\ell}}{r^2} + O(\frac{1}{r^{2+\delta}}))$$

$$= AO(\frac{1}{r^4}) + Bx^b O^+(\frac{1}{r^4}) + \cdots$$

$$= AO(\frac{1}{r^4}) + BO^-(\frac{1}{r^3}) + \cdots .$$

There remains

$$\int_{\underline{R}^3} - NS(q)\sqrt{q}\, d^3x.$$

Write

$$S(q) = q^{j\ell}R^i{}_{ji\ell}$$

$$= q^{j\ell}(\Gamma^i{}_{\ell j,i} - \Gamma^i{}_{ij,\ell} + \Gamma^a{}_{\ell j}\Gamma^i{}_{ia} - \Gamma^a{}_{ij}\Gamma^i{}_{\ell a}).$$

Then

$$Nq^{j\ell}(\Gamma^a{}_{\ell j}\Gamma^i{}_{ia} - \Gamma^a{}_{ij}\Gamma^i{}_{\ell a})\sqrt{q}$$

$$= (A + Bx^b)(\eta_{j\ell} + O(\tfrac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))^2(1 + O(\tfrac{1}{r}))$$

$$= AO(\frac{1}{r^4}) + Bx^b O^+(\frac{1}{r^4}) + \cdots$$

$$= AO(\frac{1}{r^4}) + BO^-(\frac{1}{r^3}) + \cdots .$$

Accordingly, the convergence of the integral defining $H_H(N)$ hinges on the behavior of

$$- Nq^{j\ell}(\Gamma^i{}_{\ell j,i} - \Gamma^i{}_{ij,\ell})\sqrt{q}.$$

- $q^{j\ell}\Gamma^i{}_{\ell j,i}$

$$= q^{j\ell}\partial_i[\frac{1}{2} q^{ik}(q_{k\ell,j} + q_{kj,\ell} - q_{\ell j,k})]$$

$$= \frac{1}{2} q^{j\ell}[(\partial_i q^{ik})(q_{k\ell,j} + q_{kj,\ell} - q_{\ell j,k})$$

$$+ q^{ik}(q_{k\ell,j,i} + q_{kj,\ell,i} - q_{\ell j,k,i})].$$

- $- q^{j\ell}\Gamma^i{}_{ij,\ell}$

$$= - q^{j\ell}\partial_\ell[\frac{1}{2} q^{ik}(q_{ki,j} + q_{kj,i} - q_{ij,k})]$$

$$= - \frac{1}{2} q^{j\ell}[(\partial_\ell q^{ik})(q_{ki,j} + q_{kj,i} - q_{ij,k})$$

$$+ q^{ik}(q_{ki,j,\ell} + q_{kj,i,\ell} - q_{ij,k,\ell})].$$

The integral of a term involving $\partial_i q^{ik}$ or $\partial_\ell q^{ik}$ is convergent. E.g.:

$$Nq^{j\ell}(\partial_i q^{ik})q_{k\ell,j}\sqrt{q}$$

$$= (A + Bx^b)(\eta_{j\ell} + O(\frac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))$$

$$\times (O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))(1 + O(\frac{1}{r}))$$

$$= AO\left(\frac{1}{r^4}\right) + Bx^b O^+\left(\frac{1}{r^4}\right) + \cdots$$

$$= AO\left(\frac{1}{r^4}\right) + BO^-\left(\frac{1}{r^3}\right) + \cdots .$$

This leaves $\frac{1}{2}$ times

$$q^{j\ell}q^{ik}q_{k\ell,j,i} + q^{j\ell}q^{ik}q_{kj,\ell,i} - q^{j\ell}q^{ik}q_{\ell j,k,i}$$

$$- q^{j\ell}q^{ik}q_{ki,j,\ell} - q^{j\ell}q^{ik}q_{kj,i,\ell} + q^{j\ell}q^{ik}q_{ij,k,\ell}$$

or still, $\frac{1}{2}$ times

$$q^{j\ell}q^{ik}q_{k\ell,j,i} + q^{j\ell}q^{ik}q_{ij,k,\ell}$$

$$- q^{j\ell}q^{ik}q_{\ell j,k,i} - q^{j\ell}q^{ik}q_{ki,j,\ell}.$$

- $q^{j\ell}q^{ik}q_{ij,k,\ell}$

$$= q^{k\ell}q^{ij}q_{ik,j,\ell}$$

$$= q^{ki}q^{\ell j}q_{\ell k,j,i}$$

$$= q^{j\ell}q^{ik}q_{k\ell,j,i}.$$

- $q^{j\ell}q^{ik}q_{ki,j,\ell}$

$$= q^{k\ell}q^{ij}q_{ji,k,\ell}$$

$$= q^{ki}q^{\ell j}q_{j\ell,k,i}$$

$$= q^{j\ell}q^{ik}q_{\ell j,k,i}.$$

Thus things simplify to

$$q^{j\ell}q^{ik}(q_{ij,k,\ell} - q_{ki,j,\ell}).$$

But

$$\left[\begin{array}{l} q^{j\ell}q^{ik}q_{ij,k,\ell} = q^{k\ell}q^{ij}q_{ik,j,\ell} = q^{ij}q^{k\ell}q_{ik,j,\ell} \\ \\ q^{j\ell}q^{ik}q_{ki,j,\ell} = q^{k\ell}q^{ij}q_{ji,k,\ell} = q^{ij}q^{k\ell}q_{ij,k,\ell}. \end{array}\right.$$

So we are left with

$$q^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell}).$$

Write

$$\begin{aligned} &\partial_\ell(Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q}) \\ &\quad = (\partial_\ell N)q^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q} \\ &\qquad + N\partial_\ell(q^{ij}q^{k\ell}\sqrt{q})(q_{ik,j} - q_{ij,k}) \\ &\qquad + Nq^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell})\sqrt{q} \end{aligned}$$

or, for later convenience,

$$\begin{aligned} &(\partial_\ell N)q^{ij}q^{k\ell}(\partial_j(q_{ik} - \eta_{ik}) - \partial_k(q_{ij} - \eta_{ij}))\sqrt{q} \\ &\qquad + N\partial_\ell(q^{ij}q^{k\ell}\sqrt{q})(q_{ik,j} - q_{ij,k}) \\ &\qquad + Nq^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell})\sqrt{q}. \end{aligned}$$

Therefore

$$Nq^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell})\sqrt{q}$$

$$= \partial_\ell(Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q})$$

$$- N\partial_\ell(q^{ij}q^{k\ell}\sqrt{q})(q_{ik,j} - q_{ij,k})$$

$$- (\partial_\ell N)q^{ij}q^{k\ell}(\partial_j(q_{ik} - \eta_{ik}) - \partial_k(q_{ij} - \eta_{ij}))\sqrt{q}$$

$$= \partial_\ell(Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q})$$

$$- N\partial_\ell(q^{ij}q^{k\ell}\sqrt{q})(q_{ik,j} - q_{ij,k})$$

$$+ (\partial_\ell N)q^{ij}q^{k\ell}(\partial_k(q_{ij} - \eta_{ij}) - \partial_j(q_{ik} - \eta_{ik}))\sqrt{q}.$$

• $\partial_\ell((\partial_k N)q^{ij}q^{k\ell}(q_{ij} - \eta_{ij})\sqrt{q})$

$$- \partial_k((\partial_\ell N)q^{ij}q^{k\ell}\sqrt{q})(q_{ij} - \eta_{ij})$$

$$= (\partial_\ell\partial_k N)(q^{ij}q^{k\ell}\sqrt{q})(q_{ij} - \eta_{ij})$$

$$+ (\partial_k N)\partial_\ell(q^{ij}q^{k\ell}\sqrt{q})(q_{ij} - \eta_{ij})$$

$$+ (\partial_k N)(q^{ij}q^{k\ell}\sqrt{q})\partial_\ell(q_{ij} - \eta_{ij})$$

$$- (\partial_k\partial_\ell N)(q^{ij}q^{k\ell}\sqrt{q})(q_{ij} - \eta_{ij})$$

$$- (\partial_\ell N)\partial_k(q^{ij}q^{k\ell}\sqrt{q})(q_{ij} - \eta_{ij})$$

$$= (\partial_\ell N) q^{ij} q^{k\ell} \partial_k (q_{ij} - \eta_{ij}) \sqrt{q}.$$

$$\bullet \; - \partial_j ((\partial_\ell N) q^{ij} q^{k\ell} (q_{ik} - \eta_{ik}) \sqrt{q})$$

$$+ \partial_j ((\partial_\ell N) q^{ij} q^{k\ell} \sqrt{q}) (q_{ik} - \eta_{ik})$$

$$= - (\partial_j \partial_\ell N) (q^{ij} q^{k\ell} \sqrt{q}) (q_{ik} - \eta_{ik})$$

$$- (\partial_\ell N) \partial_j (q^{ij} q^{k\ell} \sqrt{q}) (q_{ik} - \eta_{ik})$$

$$- (\partial_\ell N) (q^{ij} q^{k\ell} \sqrt{q}) \partial_j (q_{ik} - \eta_{ik})$$

$$+ (\partial_j \partial_\ell N) (q^{ij} q^{k\ell} \sqrt{q}) (q_{ik} - \eta_{ik})$$

$$+ (\partial_\ell N) \partial_j (q^{ij} q^{k\ell} \sqrt{q}) (q_{ik} - \eta_{ik})$$

$$= - (\partial_\ell N) q^{ij} q^{k\ell} \partial_j (q_{ik} - \eta_{ik}) \sqrt{q}.$$

These relations then imply that

$$N q^{ij} q^{k\ell} (q_{ik,j,\ell} - q_{ij,k,\ell}) \sqrt{q}$$

$$= \partial_\ell (N q^{ij} q^{k\ell} (q_{ik,j} - q_{ij,k}) \sqrt{q})$$

$$- N \partial_\ell (q^{ij} q^{k\ell} \sqrt{q}) (q_{ik,j} - q_{ij,k})$$

$$+ \partial_\ell ((\partial_k N) q^{ij} q^{k\ell} (q_{ij} - \eta_{ij}) \sqrt{q})$$

$$- \partial_k ((\partial_\ell N) q^{ij} q^{k\ell} \sqrt{q}) (q_{ij} - \eta_{ij})$$

$$- \partial_j((\partial_\ell N)q^{ij}q^{k\ell}(q_{ik} - \eta_{ik})\sqrt{q})$$

$$+ \partial_j((\partial_\ell N)q^{ij}q^{k\ell}\sqrt{q})(q_{ik} - \eta_{ik}).$$

• $- \partial_j((\partial_\ell N)q^{ij}q^{k\ell}(q_{ik} - \eta_{ik})\sqrt{q})$

$$= - \partial_\ell((\partial_j N)q^{i\ell}q^{kj}(q_{ik} - \eta_{ik})\sqrt{q})$$

$$= - \partial_\ell((\partial_j N)q^{k\ell}q^{ij}(q_{ki} - \eta_{ki})\sqrt{q})$$

$$= - \partial_\ell((\partial_j N)q^{ij}q^{k\ell}(q_{ik} - \eta_{ik})\sqrt{q}).$$

• $\partial_j((\partial_\ell N)q^{ij}q^{k\ell}\sqrt{q})(q_{ik} - \eta_{ik})$

$$= (\partial_j\partial_\ell N)q^{ij}q^{k\ell}\sqrt{q}(q_{ik} - \eta_{ik})$$

$$+ (\partial_\ell N)\partial_j(q^{ij}q^{k\ell}\sqrt{q})(q_{ik} - \eta_{ik})$$

$$= (\partial_k\partial_\ell N)q^{ik}q^{j\ell}\sqrt{q}(q_{ij} - \eta_{ij})$$

$$+ (\partial_j N)\partial_\ell(q^{i\ell}q^{kj}\sqrt{q})(q_{ik} - \eta_{ik})$$

$$= (\partial_k\partial_\ell N)q^{ik}q^{j\ell}\sqrt{q}(q_{ij} - \eta_{ij})$$

$$+ (\partial_j N)\partial_\ell(q^{ij}q^{k\ell}\sqrt{q})(q_{ik} - \eta_{ik}).$$

Therefore

$$Nq^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell})\sqrt{q}$$

$$= \partial_\ell (N q^{ij} q^{k\ell} (q_{ik,j} - q_{ij,k}) \sqrt{q}$$

$$+ (\partial_k N) q^{ij} q^{k\ell} (q_{ij} - \eta_{ij}) \sqrt{q} - (\partial_j N) q^{ij} q^{k\ell} (q_{ik} - \eta_{ik}) \sqrt{q})$$

$$+ (\partial_k \partial_\ell N)(q^{ik} q^{j\ell} \sqrt{q}(q_{ij} - \eta_{ij}) - q^{ij} q^{k\ell} \sqrt{q}(q_{ij} - \eta_{ij}))$$

$$+ \partial_\ell (q^{ij} q^{k\ell} \sqrt{q})(- N(q_{ik,j} - q_{ij,k})$$

$$- (\partial_k N)(q_{ij} - \eta_{ij}) + (\partial_j N)(q_{ik} - \eta_{ik})).$$

If

$$N = A + Bx^b + sc,$$

then the integrals

$$\left[\begin{array}{l} \int_{\underline{R}^3} (\partial_k \partial_\ell N)(\ldots) d^3x \\ \\ \int_{\underline{R}^3} \partial_\ell (q^{ij} q^{k\ell} \sqrt{q})(\ldots) d^3x \end{array}\right.$$

are convergent.

<u>Details</u> To discuss the second integral, write

$$\partial_\ell (q^{ij} q^{k\ell} \sqrt{q})$$

$$= (\partial_\ell q^{ij}) q^{k\ell} \sqrt{q} + q^{ij} (\partial_\ell q^{k\ell}) \sqrt{q} + q^{ij} q^{k\ell} \partial_\ell \sqrt{q}$$

$$= (O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))(\eta_{k\ell} + O(\frac{1}{r}))(1 + O(\frac{1}{r}))$$

$$+ (\eta_{ij} + O(\frac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))(1 + O(\frac{1}{r}))$$

$$+ (\eta_{ij} + O(\frac{1}{r}))(\eta_{k\ell} + O(\frac{1}{r}))\partial_\ell\sqrt{q}$$

$$= O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}})$$

$$+ (\eta_{ij} + O(\frac{1}{r}))(\eta_{k\ell} + O(\frac{1}{r}))\partial_\ell\sqrt{q}.$$

• $\partial_\ell \det q = (\det q) q^{ij}\partial_\ell q_{ij}$

$$= (1 + O(\frac{1}{r}))(\eta_{ij} + O(\frac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))$$

$$= O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}})$$

=>

$$\partial_\ell\sqrt{q} = \partial_\ell(\det q)^{1/2}$$

$$= \frac{1}{2}\frac{1}{\sqrt{q}}\partial_\ell \det q$$

$$= \frac{1}{2}(1 + O(\frac{1}{r}))(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))$$

$$= O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}).$$

Thus

$$\partial_\ell(q^{ij}q^{k\ell}\sqrt{q}) = O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}).$$

1. Suppose that $N = sc$ — then

$$N(q_{ik,j} - q_{ij,k}) = O(\frac{1}{r^2})$$

and

$$\left[\begin{array}{l} (\partial_k N)(q_{ij} - \eta_{ij}) \\ \\ (\partial_j N)(q_{ik} - \eta_{ik}) \end{array}\right. = O(\frac{1}{r^2}).$$

So in this case parity plays no role.

2. Suppose that $N = A + Bx^b$ -- then

$$N(q_{ik,j} - q_{ij,k})$$

$$= (A + Bx^b)(O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))$$

$$= AO(\frac{1}{r^2}) + BO^+(\frac{1}{r}) + \cdots$$

and

$$\left[\begin{array}{l} (\partial_k N)(q_{ij} - \eta_{ij}) = B\delta^b{}_k(O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}})) \\ \\ (\partial_j N)(q_{ik} - \eta_{ik}) = B\delta^b{}_j(O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}})). \end{array}\right.$$

So in this case parity is crucial.

Notation: Let

$$X = X^\ell \frac{\partial}{\partial x^\ell},$$

where

$$X^\ell = Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q}$$

$$+ (\partial_k N)q^{ij}q^{k\ell}(q_{ij} - \eta_{ij})\sqrt{q} - (\partial_j N)q^{ij}q^{k\ell}(q_{ik} - \eta_{ik})\sqrt{q}.$$

Then

$$\mathrm{div}\ X = \partial_\ell X^\ell$$

and we have

$$\int_{\underline{R}^3} (\mathrm{div}\ X) d^3x = \lim_{R \to \infty} \int_{\underline{D}^3(R)} (\mathrm{div}\ X) d^3x$$

$$= \lim_{R \to \infty} \int_{\underline{S}^2(R)} (X \cdot \underline{n}) \omega_R^2.$$

Observation: If N = sc, then

$$\int_{\underline{R}^3} (\mathrm{div}\ X) d^3x = 0.$$

[The terms that might cause trouble are $O^+(\frac{1}{r^2})$ but, before carrying out the integration, they must be multiplied by a function of odd parity.]

Assume next that N = 1, hence

$$X^\ell = q^{ij} q^{k\ell} (q_{ik,j} - q_{ij,k}) \sqrt{q}.$$

Write

- $q^{ij} q^{k\ell} q_{ik,j} \sqrt{q}$

$$= (\eta_{ij} + O(\tfrac{1}{r}))(\eta_{k\ell} + O(\tfrac{1}{r})) q_{ik,j} \sqrt{q}$$

$$= \eta_{ij}\eta_{k\ell} q_{ik,j} \sqrt{q} + O(\tfrac{1}{r}) q_{ik,j} \sqrt{q} + O(\tfrac{1}{r^2}) q_{ik,j} \sqrt{q}$$

$$= \eta_{ij}\eta_{k\ell} q_{ik,j} \sqrt{q} + O(\tfrac{1}{r}) O(\tfrac{1}{r^2}) O(1) + O(\tfrac{1}{r^2}) O(\tfrac{1}{r^2}) O(1)$$

$$= \eta_{ij}\eta_{k\ell} q_{ik,j} \sqrt{q} + O(\tfrac{1}{r^3})$$

$$= q_{i\ell,i}\sqrt{q} + O(\frac{1}{r^3}).$$

• $- q^{ij}q^{k\ell}q_{ij,k}\sqrt{q}$

$$= - (\eta_{ij} + O(\frac{1}{r}))(\eta_{k\ell} + O(\frac{1}{r}))q_{ij,k}\sqrt{q}$$

$$= - \eta_{ij}\eta_{k\ell}q_{ij,k}\sqrt{q} + O(\frac{1}{r^3})$$

$$= - q_{ii,\ell}\sqrt{q} + O(\frac{1}{r^3}).$$

The integral of $O(\frac{1}{r^3})$ over $\underline{S}^2(R)$ vanishes in the limit, thus we need only consider

$$R^2 \int_0^{2\pi}\int_0^{\pi} (\cos\theta \sin\phi\ (q_{i1,i} - q_{ii,1})\circ\iota_R$$

$$+ \sin\theta\sin\phi\ (q_{i2,i} - q_{ii,2})\circ\iota_R + \cos\phi\ (q_{i3,i} - q_{ii,3})\circ\iota_R)\sqrt{q}\circ\iota_R \sin\phi\, d\phi\, d\theta.$$

From the definitions,

$$q_{i\ell,i} - q_{ii,\ell}$$

$$= \partial_i(\frac{\tilde{\sigma}_{i\ell}}{r}) - \partial_\ell(\frac{\tilde{\sigma}_{ii}}{r}) + O(\frac{1}{r^{2+\delta}}).$$

But

$$\partial_i(\frac{\tilde{\sigma}_{i\ell}}{r}) - \partial_\ell(\frac{\tilde{\sigma}_{ii}}{r})$$

is homogeneous of degree -2, so

$$r^2(\partial_i(\frac{\tilde{\sigma}_{i\ell}}{r}) - \partial_\ell(\frac{\tilde{\sigma}_{ii}}{r}))$$

is homogeneous of degree 0 and is therefore the radial extension of a function $F_\ell \in C^\infty(\underline{S}^2)$. Consequently, the dependence on R in the integral

$$\int_0^{2\pi}\int_0^{\pi} (\cos\theta \sin\phi\, F_1(\theta,\phi) + \sin\theta \sin\phi\, F_2(\theta,\phi) + \cos\phi\, F_3(\theta,\phi))\sqrt{q}\circ\iota_R \sin\phi\, d\phi\, d\theta$$

resides solely in $\sqrt{q}\circ\iota_R$. Since $\sqrt{q} = 1 + O(\frac{1}{r})$, it follows that

$$\lim_{R\to\infty} \int_0^{2\pi}\int_0^{\pi} (\ldots)\sqrt{q}\circ\iota_R \sin\phi\, d\phi\, d\theta$$

exists, the traditional notation for this being the symbol

$$\int_{\underline{S}^2(\infty)} (q_{i\ell,i} - q_{ii,\ell})\Omega_\infty^\ell.$$

<u>N.B.</u> What the analysis really shows is:

$$\int_{\underline{R}^3} (q^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q})_{,\ell}d^3x$$

$$= \lim_{R\to\infty} \int_{\underline{D}^3(R)} (q^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q})_{,\ell}d^3x$$

$$= \lim_{R\to\infty} \int_{\underline{S}^2(R)} (q_{i\ell,i} - q_{ii,\ell})\Omega_R^\ell,$$

where

$$\Omega_R^\ell = \frac{x^\ell}{R}\,\omega_R^2.$$

Definition: The <u>energy</u> is the function

$$P^0 : Q_\infty \to \underline{R}$$

given by the prescription

$$P^0(q) = \int_{\underline{S}^2(\infty)} (q_{i\ell,i} - q_{ii,\ell})\Omega_\infty^\ell.$$

Example: If for $r >> 0$,

$$q_{ij} = \eta_{ij} + m \frac{x^i x^j}{r^3} \quad (m > 0),$$

then

$$P^0(q) = 8\pi m.$$

[Set $m = 1$ and, to facilitate the computation, use x,y,z instead of x^1,x^2,x^3.

1. $\partial_x(\frac{x^2}{r^3}) + \partial_y(\frac{yx}{r^3}) + \partial_z(\frac{zx}{r^3}) = \frac{x}{r^3}.$

2. $\partial_x(\frac{xy}{r^3}) + \partial_y(\frac{y^2}{r^3}) + \partial_z(\frac{zy}{r^3}) = \frac{y}{r^3}.$

3. $\partial_x(\frac{xz}{r^3}) + \partial_y(\frac{yz}{r^3}) + \partial_z(\frac{z^2}{r^3}) = \frac{z}{r^3}.$

4. $\partial_x(\frac{x^2}{r^3} + \frac{y^2}{r^3} + \frac{z^2}{r^3}) = -\frac{x}{r^3}.$

5. $\partial_y(\frac{x^2}{r^3} + \frac{y^2}{r^3} + \frac{z^2}{r^3}) = -\frac{y}{r^3}.$

6. $\partial_z(\frac{x^2}{r^3} + \frac{y^2}{r^3} + \frac{z^2}{r^3}) = -\frac{z}{r^3}.$

$\Rightarrow$

$$\left[\begin{array}{l} 1-4 = 2\,\frac{x}{r^3}. \\ 2-5 = 2\,\frac{y}{r^3}. \\ 3-6 = 2\,\frac{z}{r^3}. \end{array}\right.$$

Take R > > 0 -- then

$$R^2 \int_0^{2\pi} \int_0^{\pi} (\cos\theta \sin\phi \, (\frac{2R \cos\theta \sin\phi}{R^3})$$

$$+ \sin\theta \sin\phi \, (\frac{2R \sin\theta \sin\phi}{R^3}) + \cos\phi \, (\frac{2R \cos\phi}{R^3}))\sin\phi \, d\phi \, d\theta$$

$$= 2\int_0^{2\pi} \int_0^{\pi} ((\cos\theta \sin\phi)^2 + (\sin\theta \sin\phi)^2 + \cos^2\phi)\sin\phi \, d\phi \, d\theta$$

$$= 2\int_0^{2\pi} (\int_0^{\pi} \sin\phi \, d\phi)d\theta$$

$$= 8\pi.]$$

<u>LEMMA</u> We have

$$\frac{d}{d\varepsilon} P^0(q + \varepsilon\delta q)\Big|_{\varepsilon=0}$$

$$= \int_{\underline{R}^3} \delta_q(dtr_q(\delta q) - div_q \, \delta q)\sqrt{q} \, d^3x.$$

Suppose now that $N = x^b$ — then

$$X^\ell = x^b q^{ij} q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q}$$

$$+ \delta^b{}_k q^{ij} q^{k\ell}(q_{ij} - \eta_{ij})\sqrt{q} - \delta^b{}_j q^{ij} q^{k\ell}(q_{ik} - \eta_{ik})\sqrt{q}$$

$$= x^b q^{ij} q^{k\ell}(q_{ik,j} - q_{ij,k})\sqrt{q}$$

$$+ q^{ij} q^{b\ell}(q_{ij} - \eta_{ij})\sqrt{q} - q^{ib} q^{k\ell}(q_{ik} - \eta_{ik})\sqrt{q}.$$

Unfortunately, for arbitrary q, the integral

$$\int_{\underline{R}^3} (\operatorname{div} X)\, d^3x$$

is divergent. However, if q is suitably restricted, then, as we shall see, convergence is guaranteed.

Definition: Let $q \in Q_\infty$ — then q is said to satisfy <u>condition *</u> if for $r >> 0$,

$$q_{ij}(x) = \eta_{ij} + \frac{1}{r}\sigma_{ij}\left(\frac{x}{r}\right) + \frac{1}{r^2}\sigma^*_{ij}\left(\frac{x}{r}\right) + \mu_{ij}(x),$$

where $\sigma_{ij}, \sigma^*_{ij} \in C^\infty(\underline{S}^2)$, σ_{ij} is of even parity, and

$$\mu_{ij} = O^\infty\left(\frac{1}{r^{2+\delta}}\right) \quad (0 < \delta \le 1).$$

[Note: Here it is understood that

$$\sigma_{ij} = \sigma_{ji}, \ \sigma^*_{ij} = \sigma^*_{ji}, \ \mu_{ij} = \mu_{ji}.$$

Observe too that

$$\partial_k\left(\frac{1}{r}\tilde{\sigma}_{ij}\right)$$

is odd and homogeneous of degree −2 while

$$\partial_k\left(\frac{1}{r^2}\tilde{\sigma}^*_{ij}\right)$$

is homogeneous of degree −3 ($\tilde{\sigma}^*_{ij}$ is not subject to a parity assumption).]

Notation: Q^*_∞ is the subset of Q_∞ consisting of those q which satisfy condition *.

Remark: Let $q \in Q^*_\infty$ — then for $r >> 0$,

$$q^{ij}(x) = \eta_{ij} - \frac{1}{r}\,\sigma_{ij}\left(\frac{x}{r}\right) - \frac{1}{r^2}\,\sigma^*_{ij}\left(\frac{x}{r}\right) + O\left(\frac{1}{r^{2+\delta}}\right).$$

<u>LEMMA</u> $\forall\ q \in Q^*_\infty$, the integral

$$\int_{\underline{R}^3} (\text{div } X)\, d^3x$$

is convergent.

It will be enough to consider

$$\text{I:}\quad x^b q^{ij} q^{k\ell} q_{ik,j}\sqrt{q}$$

and

$$\text{II:}\quad q^{ij} q^{b\ell}(q_{ij} - \eta_{ij})\sqrt{q}.$$

As usual, pass from $\underline{D}^3(R)$ to $\underline{S}^2(R)$.

<u>Ad</u> I: Write

$$x^b q^{ij} q^{k\ell} q_{ik,j}\sqrt{q}$$

$$= x^b\left(\eta_{ij} - \frac{1}{r}\,\tilde{\sigma}_{ij} - \frac{1}{r^2}\,\tilde{\sigma}^*_{ij} + O\left(\frac{1}{r^{2+\delta}}\right)\right)$$

$$\times\ \left(\eta_{k\ell} - \frac{1}{r}\,\tilde{\sigma}_{k\ell} - \frac{1}{r^2}\,\tilde{\sigma}^*_{k\ell} + O\left(\frac{1}{r^{2+\delta}}\right)\right)$$

$$\times\ \left(\partial_j\left(\frac{1}{r}\,\tilde{\sigma}_{ik}\right) + \partial_j\left(\frac{1}{r^2}\,\tilde{\sigma}^*_{ik}\right) + \partial_j \mu_{ik}\right)\sqrt{q}.$$

When expanded, there is a total of 48 terms but not all of them need be considered individually provided we first do some judicious regrouping. To this end, start by writing

$$x^b(\eta_{ij} - \frac{1}{r}\tilde{\sigma}_{ij} - \frac{1}{r^2}\tilde{\sigma}^*_{ij} + O(\frac{1}{r^{2+\delta}}))$$

$$= x^b\eta_{ij} - \frac{x^b}{r}\tilde{\sigma}_{ij} + O(\frac{1}{r})$$

and

$$\partial_j(\frac{1}{r}\tilde{\sigma}_{ik}) + \partial_j(\frac{1}{r^2}\tilde{\sigma}^*_{ik}) + \partial_j\mu_{ik} = O(\frac{1}{r^2}).$$

Then

$$O(\frac{1}{r})(\eta_{k\ell} - \frac{1}{r}\tilde{\sigma}_{k\ell} - \frac{1}{r^2}\tilde{\sigma}^*_{k\ell} + O(\frac{1}{r^{2+\delta}}))O(\frac{1}{r^2})\sqrt{q}$$

$$= O(\frac{1}{r^3}).$$

- $$x^b\eta_{ij}(\eta_{k\ell} - \frac{1}{r}\tilde{\sigma}_{k\ell} - \frac{1}{r^2}\tilde{\sigma}^*_{k\ell} + O(\frac{1}{r^{2+\delta}}))O(\frac{1}{r^2})\sqrt{q}$$

$$= x^b\eta_{ij}(\eta_{k\ell} - \frac{1}{r}\tilde{\sigma}_{k\ell})O(\frac{1}{r^2})\sqrt{q} + O(\frac{1}{r^3}).$$

- $$-\frac{x^b}{r}\tilde{\sigma}_{ij}(\eta_{k\ell} - \frac{1}{r}\tilde{\sigma}_{k\ell} - \frac{1}{r^2}\tilde{\sigma}^*_{k\ell} + O(\frac{1}{r^{2+\sigma}}))O(\frac{1}{r^2})\sqrt{q}$$

$$= -\frac{x^b}{r}\tilde{\sigma}_{ij}\eta_{k\ell}O(\frac{1}{r^2})\sqrt{q} + O(\frac{1}{r^3}).$$

Bearing in mind that

$$O(\frac{1}{r^2}) = \partial_j(\frac{1}{r}\tilde{\sigma}_{ik}) + \partial_j(\frac{1}{r^2}\tilde{\sigma}^*_{ik}) + \partial_j\mu_{ik},$$

there remains

1. $x^b\eta_{ij}\eta_{k\ell}\partial_j(\frac{1}{r}\tilde{\sigma}_{ik})\sqrt{q}$

2. $x^b \eta_{ij} \eta_{k\ell} \partial_j (\frac{1}{r^2} \tilde{\sigma}^*_{ik}) \sqrt{q}$

3. $x^b \eta_{ij} \eta_{k\ell} \partial_j \mu_{ik} \sqrt{q}$

4. $- x^b \eta_{ij} (\frac{1}{r} \tilde{\sigma}_{k\ell}) \partial_j (\frac{1}{r} \tilde{\sigma}_{ik}) \sqrt{q}$

5. $- x^b \eta_{ij} (\frac{1}{r} \tilde{\sigma}_{k\ell}) \partial_j (\frac{1}{r^2} \tilde{\sigma}^*_{ik}) \sqrt{q}$

6. $- x^b \eta_{ij} (\frac{1}{r} \tilde{\sigma}_{k\ell}) \partial_j \mu_{ik} \sqrt{q}$

7. $- \frac{x^b}{r} \tilde{\sigma}_{ij} \eta_{k\ell} \partial_j (\frac{1}{r} \tilde{\sigma}_{ik}) \sqrt{q}$

8. $- \frac{x^b}{r} \tilde{\sigma}_{ij} \eta_{k\ell} \partial_j (\frac{1}{r^2} \tilde{\sigma}^*_{ik}) \sqrt{q}$

9. $- \frac{x^b}{r} \tilde{\sigma}_{ij} \eta_{k\ell} \partial_j \mu_{ik} \sqrt{q}$.

Since $\sqrt{q} = O(1)$ and

$$\partial_j \mu_{ik} = O(\frac{1}{r^{3+\delta}}),$$

Items 3, 6, and 9 are, respectively,

$$O(\frac{1}{r^{2+\delta}}),\ O(\frac{1}{r^{3+\delta}}),\ O(\frac{1}{r^{3+\delta}}).$$

Rappel:

$$\sqrt{q} = 1 + O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}}).$$

Item 1:

$$x^b \eta_{ij} \eta_{k\ell} \partial_j (\frac{1}{r} \tilde{\sigma}_{ik}) \sqrt{q}$$

$$= O^+(\frac{1}{r})\sqrt{q}$$

$$= O^+(\frac{1}{r}) + O^+(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}).$$

Item 2:

$$x^b \eta_{ij}\eta_{k\ell}\partial_j(\frac{1}{r^2}\tilde{\sigma}^*_{ik})\sqrt{q}$$

$$= x^b \eta_{ij}\eta_{k\ell}\,\frac{1}{r^3}(r^3\partial_j(\frac{1}{r^2}\tilde{\sigma}^*_{ik}))\sqrt{q}$$

$$= x^b \eta_{ij}\eta_{k\ell}\,\frac{1}{r^3}(r^3\partial_j(\frac{1}{r^2}\tilde{\sigma}^*_{ik})) + O(\frac{1}{r^3}).$$

Item 4:

$$- x^b \eta_{ij}(\frac{1}{r}\tilde{\sigma}_{k\ell})\partial_j(\frac{1}{r}\tilde{\sigma}_{ik})\sqrt{q}$$

$$= O^+(\frac{1}{r^2})\sqrt{q}$$

$$= O^+(\frac{1}{r^2}) + O(\frac{1}{r^3}).$$

Item 5:

$$- x^b \eta_{ij}(\frac{1}{r}\tilde{\sigma}_{k\ell})\partial_j(\frac{1}{r^2}\tilde{\sigma}^*_{ik})\sqrt{q}$$

$$= O(\frac{1}{r^3}).$$

Item 7:

$$- \frac{x^b}{r}\tilde{\sigma}_{ij}\eta_{k\ell}\partial_j(\frac{1}{r}\tilde{\sigma}_{ik})\sqrt{q}$$

$$= O^{+}(\frac{1}{r^2})\sqrt{q}$$

$$= O^{+}(\frac{1}{r^2}) + O(\frac{1}{r^3}).$$

Item 8:

$$-\frac{x^b}{r}\,\tilde{\sigma}_{ij}\eta_{k\ell}\partial_j(\frac{1}{r^2}\,\tilde{\sigma}^*_{ik})\sqrt{q}$$

$$= O(\frac{1}{r^3}).$$

On the basis of the foregoing, it is clear that only Item 2 has the potential to make a finite nonzero contribution to

$$\int_{\underline{R}^3} (\text{div } X)d^3x.$$

<u>Ad</u> II: Write

$$q^{ij}q^{b\ell}(q_{ij} - \eta_{ij})\sqrt{q}$$

$$= (\eta_{ij} - \frac{1}{r}\,\tilde{\sigma}_{ij} - \frac{1}{r^2}\,\tilde{\sigma}^*_{ij} + O(\frac{1}{r^{2+\delta}}))$$

$$\times\ (\eta_{b\ell} - \frac{1}{r}\,\tilde{\sigma}_{b\ell} - \frac{1}{r^2}\,\tilde{\sigma}^*_{b\ell} + O(\frac{1}{r^{2+\delta}}))$$

$$\times\ (\frac{1}{r}\,\tilde{\sigma}_{ij} + \frac{1}{r^2}\,\tilde{\sigma}^*_{ij} + \mu_{ij})\sqrt{q}$$

$$= (\eta_{ij} - \frac{1}{r}\,\tilde{\sigma}_{ij} + O(\frac{1}{r^2}))$$

$$\times\ (\eta_{b\ell} - \frac{1}{r}\,\tilde{\sigma}_{b\ell} + O(\frac{1}{r^2}))$$

$$\times \left(\frac{1}{r}\tilde{\sigma}_{ij} + \frac{1}{r^2}\tilde{\sigma}^*_{ij} + O\left(\frac{1}{r^{2+\delta}}\right)\right)\sqrt{q}$$

$$= \left(\eta_{ij}\eta_{b\ell} - \eta_{ij}\frac{1}{r}\tilde{\sigma}_{b\ell} - \eta_{b\ell}\frac{1}{r}\tilde{\sigma}_{ij} + O\left(\frac{1}{r^2}\right)\right)$$

$$\times \left(\frac{1}{r}\tilde{\sigma}_{ij} + \frac{1}{r^2}\tilde{\sigma}^*_{ij} + O\left(\frac{1}{r^{2+\delta}}\right)\right)\sqrt{q}.$$

The relevant terms are then:

1. $\eta_{ij}\eta_{b\ell}\frac{1}{r}\tilde{\sigma}_{ij}\sqrt{q}$

2. $\eta_{ij}\eta_{b\ell}\frac{1}{r^2}\tilde{\sigma}^*_{ij}\sqrt{q}$

3. $-\eta_{ij}\frac{1}{r^2}\tilde{\sigma}_{b\ell}\tilde{\sigma}_{ij}\sqrt{q}$

4. $-\eta_{b\ell}\frac{1}{r^2}\tilde{\sigma}_{ij}\tilde{\sigma}_{ij}\sqrt{q}.$

And of these, only Item 2 is germane in that it might make a finite nonzero contribution to

$$\int_{\underline{R}^3} (\operatorname{div} X)d^3x.$$

Definition: The <u>center of mass</u> J^0 is the triple

$$(J^{01},J^{02},J^{03}),$$

where for $b = 1,2,3$,

$$J^{0b}:Q^*_\infty \to \underline{R}$$

sends q to

$$\int_{\underline{S}^2(\infty)} (x^b q^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})$$

$$+ q^{ij}q^{b\ell}(q_{ij} - \eta_{ij}) - q^{ib}q^{k\ell}(q_{ik} - \eta_{ik}))\Omega_\infty^\ell.$$

Exercise: Compute $J^{0b}(q)$, where for $r >> 0$,

$$q_{ij} = \eta_{ij} + m \frac{x^i x^j}{r^3} \quad (m > 0).$$

N.B. Let $N = A + Bx^b + sc$ — then for arbitrary $q \in Q_\infty$, the preceding investigation isolates the potentially divergent part of

$$\int_{\underline{R}^3} - NS(q)\sqrt{q}\, d^3x$$

as a limit of surface integrals, namely

$$\int_{\underline{S}^2(\infty)} (- Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})$$

$$+ N_{,j}q^{ij}q^{k\ell}(q_{ik} - \eta_{ik}) - N_{,k}q^{ij}q^{k\ell}(q_{ij} - \eta_{ij}))\Omega_\infty^\ell.$$

Scholium: On Con_H (hence too on Con_{Q_∞}),

$$\int_{\underline{S}^2(\infty)} (- Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})$$

$$+ N_{,j}q^{ij}q^{k\ell}(q_{ik} - \eta_{ik}) - N_{,k}q^{ij}q^{k\ell}(q_{ij} - \eta_{ij}))\Omega_\infty^\ell$$

is finite.

[If $(q,\Lambda) \in \text{Con}_H$, then

$$NS(q) = N[s,s]_q - \frac{N}{2}\text{tr}_q(s)^2.$$

And, as we have seen earlier, the integrals

$$\left[\begin{array}{l} \int_{\underline{R}^3} N[s,s]_q\sqrt{q}\, d^3x \\ \int_{\underline{R}^3} - \frac{N}{2}\text{tr}_q(s)^2\sqrt{q}\, d^3x \end{array}\right.$$

are convergent, thus the same is true of

$$\int_{\underline{R}^3} - NS(q)\sqrt{q}\, d^3x.]$$

Recall now that

$$\int_{\underline{R}^3} N[\Delta_q tr_q(\delta q) + \delta_q div_q\, \delta q]\sqrt{q}\, d^3x$$

$$= \int_{\underline{R}^3} - [H_N - (\Delta_q N)q, \delta q]_q \sqrt{q}\, d^3x$$

$$+ \int_{\underline{R}^3} - \delta_q(N(dtr_q(\delta q) - div_q\, \delta q))\sqrt{q}\, d^3x$$

$$+ \int_{\underline{R}^3} - \delta_q(dN\cdot\delta q - tr_q(\delta q)dN)\sqrt{q}\, d^3x.$$

But

$$N[\Delta_q tr_q(\delta q) + \delta_q div_q\, \delta q]$$

$$= - Nq^{ij}q^{k\ell}(\delta q_{ik;j;\ell} - \delta q_{ij;k;\ell})$$

=>

$$\int_{\underline{R}^3} N[\Delta_q tr_q(\delta q) + \delta_q div_q\, \delta q]\sqrt{q}\, d^3x$$

$$= \int_{\underline{R}^3} - [H_N - (\Delta_q N)q, \delta q]_q \sqrt{q}\, d^3x$$

$$+ \int_{\underline{S}^2(\infty)} - Nq^{ij}q^{k\ell}(\delta q_{ik;j} - \delta q_{ij;k})\Omega_\infty^\ell$$

$$+ \int_{\underline{S}^2(\infty)} N_{;\ell}q^{ij}q^{k\ell}\delta q_{ik}\Omega_\infty^j - \int_{\underline{S}^2(\infty)} N_{;\ell}q^{ij}q^{k\ell}\delta q_{ij}\Omega_\infty^k.$$

And (see below)

- $\int_{\underline{S}^2(\infty)} - Nq^{ij}q^{k\ell}(\delta q_{ik;j} - \delta q_{ij;k})\Omega_\infty^\ell$

$$= \int_{\underline{R}^3} - \delta_q(N(dtr_q(\delta q) - div_q\ \delta q))\sqrt{q}\ d^3x.$$

- $\int_{\underline{S}^2(\infty)} N_{;\ell}q^{ij}q^{k\ell}\delta q_{ik}\Omega_\infty^j - \int_{\underline{S}^2(\infty)} N_{;\ell}q^{ij}q^{k\ell}\delta q_{ij}\Omega_\infty^k$

$$= \int_{\underline{R}^3} - \delta q(dN\cdot\delta q - tr_q(\delta q)dN)\sqrt{q}\ d^3x.$$

[Note:

- Formally, the variation of

$$\int_{\underline{S}^2(\infty)} - Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})\Omega_\infty^\ell$$

is equal to

$$\int_{\underline{S}^2(\infty)} - Nq^{ij}q^{k\ell}(\delta q_{ik;j} - \delta q_{ij;k})\Omega_\infty^\ell.$$

- Formally, the variation of

$$\int_{\underline{S}^2(\infty)} (N_{,j}q^{ij}q^{k\ell}(q_{ik} - \eta_{ik}) - N_{,k}q^{ij}q^{k\ell}(q_{ij} - \eta_{ij}))\Omega_\infty^\ell$$

is equal to

$$\int_{\underline{S}^2(\infty)} N_{;\ell}q^{ij}q^{k\ell}\delta q_{ik}\Omega_\infty^j - \int_{\underline{S}^2(\infty)} N_{;\ell}q^{ij}q^{k\ell}\delta q_{ij}\Omega_\infty^k.]$$

<u>Details</u> While the integrals may very well be infinite, let us manipulate them as if they were finite. So, for example,

$$\int_{\underline{S}^2(\infty)} Nq^{ij}q^{k\ell}(\delta q_{ik;j} - \delta q_{ij;k})\Omega_\infty^\ell$$

$$= \int_{\underline{R}^3} (Nq^{ij}q^{k\ell}(\delta q_{ik;j} - \delta q_{ij;k})\sqrt{q})_{,\ell} d^3x$$

$$= \int_{\underline{R}^3} (\sqrt{q}\, N(q^{\ell k}\nabla_j \delta q_k{}^j - q^{\ell k}\nabla_k(q^{ij}\delta q_{ij})))_{,\ell} d^3x$$

$$= \int_{\underline{R}^3} \frac{1}{\sqrt{q}} (\sqrt{q}\, Nq^{\ell k}((\mathrm{div}_q\ \delta q)_k - (\mathrm{dtr}_q(\delta q))_k))_{,\ell} \sqrt{q}\ d^3x$$

$$= \int_{\underline{R}^3} -\frac{1}{\sqrt{q}} (\sqrt{q}\, Nq^{\ell k}((\mathrm{dtr}_q(\delta q))_k - (\mathrm{div}_q\ \delta q)_k))_{,\ell} \sqrt{q}\ d^3x$$

$$= \int_{\underline{R}^3} \delta_q(N(\mathrm{dtr}_q(\delta q) - \mathrm{div}_q\ \delta q))\sqrt{q}\ d^3x.$$

<u>LEMMA</u> Suppose that $N = A + Bx^b + sc$ — then $\forall\ q \in Q_\infty$, the integral

$$\int_{\underline{R}^3} Nq[{}^0_2](\mathrm{Ein}(q), \delta q)\sqrt{q}\ d^3x$$

is convergent.

[The case when $N = A + sc$ was dispatched in the last section, thus it suffices to take $N = x^b$. Write

$$Nq[{}^0_2](\mathrm{Ein}(q), \delta q)\sqrt{q}$$

$$= x^b\ \mathrm{Ein}(q)_{ij}(\delta q)^{ij}\sqrt{q}$$

$$= x^b\ \mathrm{Ein}(q)_{ij} q^{ik} q^{j\ell}(\delta q)_{k\ell}\sqrt{q}$$

$$= x^b(O^+(\frac{1}{r^3}) + O(\frac{1}{r^{3+\delta}}))(\eta_{ik} + O(\frac{1}{r}))(\eta_{j\ell} + O(\frac{1}{r}))$$

$$\times\ (O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}}))(1 + O(\frac{1}{r}))$$

$$= x^{b}(O^{+}(\frac{1}{r^{3}}) + O(\frac{1}{r^{3+\delta}}))(O^{+}(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}}))(C + O(\frac{1}{r}))$$

$$= (O^{-}(\frac{1}{r^{2}}) + O(\frac{1}{r^{2+\delta}}))(O^{+}(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}}))(C + O(\frac{1}{r}))$$

$$= (O^{-}(\frac{1}{r^{3}}) + O(\frac{1}{r^{3+\delta}}))(C + O(\frac{1}{r}))$$

$$= O^{-}(\frac{1}{r^{3}}) + \text{ho.}]$$

Section 64: The Integrals of Motion-Linear and Angular Momentum

The assumptions and notation are those of Section 62.

Rappel: If $\vec{N}$ satisfies the standard conditions, then the integral

$$\int_{\underline{R}^3} -2\mathrm{div}_q \, \Lambda(\vec{N})$$

defining $H_D(\vec{N})$ is convergent and equals

$$\int_{\underline{R}^3} \Lambda(L_{\vec{N}} q).$$

[Note: Recall that the boundary term implicit in this relation necessarily vanishes.]

Suppose now that

$$\vec{N} = \vec{A} + B\vec{r} + \overrightarrow{sc}.$$

Here

$$\vec{A} \in \underline{R}^3, \ B \in \underline{so}(3),$$

and $\overrightarrow{sc}$ stands for a vector field satisfying the standard conditions, so

$$N^i(x) = A^i + \sum_{j=1}^{3} B^i{}_j x^j + \psi^i(\frac{x}{r}) + O^\infty(\frac{1}{r^\varepsilon}),$$

where A^i, $B^i{}_j$ $(= - B^j{}_i)$ are constants, ψ^i is a C^∞ function on $\underline{S}^2$ of odd parity, and $\varepsilon > 0$.

Problem: Determine whether the integral defining $H_D(\vec{N})$ is convergent or not.

To isolate the issues, drop the standard conditions and assume only that $\vec{N} = \vec{A} + B\vec{r}$.

On formal grounds,

$$\int_{\underline{R}^3} -2\mathrm{div}_q \, \Lambda(\vec{N}) + 2 \int_{\underline{R}^3} \nabla_j (N_i s^{ij}) \sqrt{q} \, d^3x$$

$$= \int_{\underline{R}^3} \Lambda(L_{\vec{N}} q).$$

<u>LEMMA</u> The integral

$$\int_{\underline{R}^3} \Lambda(L_{\vec{N}} q)$$

is convergent.

[We have

$$\int_{\underline{R}^3} \Lambda(L_{\vec{N}} q) = \int_{\underline{R}^3} s^{\#}(L_{\vec{N}} q)\sqrt{q}\, d^3x$$

$$= \int_{\underline{R}^3} s^{ij}(N_{i;j} + N_{j;i})\sqrt{q}\, d^3x$$

$$= \int_{\underline{R}^3} \lambda^{ij}(\nabla_j q_{ik} N^k + \nabla_i q_{jk} N^k) d^3x$$

$$= \int_{\underline{R}^3} \lambda^{ij}(q_{ik} \nabla_j N^k + q_{jk} \nabla_i N^k) d^3x$$

$$= \int_{\underline{R}^3} \lambda^{ij}(q_{ik} \partial_j N^k + q_{jk} \partial_i N^k) d^3x$$

$$+ \int_{\underline{R}^3} \lambda^{ij}(q_{ik} \Gamma^k{}_{j\ell} N^\ell + q_{jk} \Gamma^k{}_{i\ell} N^\ell) d^3x.$$

Then

- $\lambda^{ij}(q_{ik}\partial_j N^k + q_{jk}\partial_i N^k)$

$$= \lambda^{ij}(q_{ik} B^k{}_j + q_{jk} B^k{}_i)$$

$$= \lambda^{ij} q_{ik} B^k{}_j + \lambda^{ij} q_{jk} B^k{}_i$$

$$= 2\lambda^{ij} q_{ik} B^k{}_j .$$

And

$$\lambda^{ij} q_{ik} B^k{}_j$$

$$= \left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right)\left(\eta_{ik} + \frac{1}{r}\tilde{\sigma}_{ik} + \mu_{ik}\right) B^k{}_j$$

$$= \left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right)\eta_{ik} B^k{}_j + \frac{1}{r^3}\tilde{\tau}^{ij}\tilde{\sigma}_{ik} B^k{}_j + \frac{1}{r}\nu^{ij}\tilde{\sigma}_{ik} B^k{}_j$$

$$+ \left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right)\mu_{ik} B^k{}_j$$

$$= \left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right)\eta_{ik} B^k{}_j + O^-\left(\frac{1}{r^3}\right) + \text{ho.}$$

But

$$\left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right)\eta_{ik} B^k{}_j$$

$$= \left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right) B^i{}_j$$

$$= -\left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right) B^j{}_i$$

$$= -\left(\frac{1}{r^2}\tilde{\tau}^{ji} + \nu^{ji}\right) B^j{}_i$$

$$= -\left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right) B^i{}_j$$

$\Rightarrow$

$$\left(\frac{1}{r^2}\tilde{\tau}^{ij} + \nu^{ij}\right)\eta_{ik}B^k{}_j = 0.$$

Therefore

$$\lambda^{ij}q_{ik}B^k{}_j = O^-\left(\frac{1}{r^3}\right) + \text{ho.}$$

- $\lambda^{ij}(q_{ik}\Gamma^k{}_{j\ell}N^\ell + q_{jk}\Gamma^k{}_{i\ell}N^\ell)$

$$= \lambda^{ij}q_{ik}\Gamma^k{}_{j\ell}N^\ell + \lambda^{ij}q_{jk}\Gamma^k{}_{i\ell}N^\ell$$

$$= 2\lambda^{ij}q_{ik}\Gamma^k{}_{j\ell}N^\ell$$

$$= 2\lambda^{ij}q_{ik}\Gamma^k{}_{j\ell}(A^\ell + B^\ell{}_{\ell'}x^{\ell'})$$

$$= O\left(\frac{1}{r^4}\right) + \lambda^{ij}q_{ik}\Gamma^k{}_{j\ell}O^-(r).$$

And

$$\lambda^{ij}q_{ik}\Gamma^k{}_{j\ell}O^-(r)$$

$$= \left(O^-\left(\frac{1}{r^2}\right) + O\left(\frac{1}{r^{2+\delta}}\right)\right)\left(\eta_{ik} + O\left(\frac{1}{r}\right)\right)\left(O^-\left(\frac{1}{r^2}\right) + O\left(\frac{1}{r^{2+\delta}}\right)\right)O^-(r)$$

$$= O^-\left(\frac{1}{r^3}\right) + \text{ho.}]$$

Application: Let $\vec{N} = \vec{A} + B\vec{r}$ -- then the sum

$$\int_{\underline{R}^3} - 2\text{div}_q\, \Lambda(\vec{N}) + 2\int_{\underline{R}^3} \nabla_j(N_i s^{ij})\sqrt{q}\, d^3x$$

is convergent.

[Note: It is not claimed that the individual constituents are convergent.]

N.B. We have

$$\frac{d}{d\varepsilon}\left[\int_{\underline{R}^3} \Lambda(L_{\vec{N}}(q + \varepsilon\delta q))\right]\Big|_{\varepsilon=0}$$

$$= -\int_{\underline{R}^3} (L_{\vec{N}}\Lambda)(\delta q) + \int_{\underline{R}^3} \operatorname{div}_q(s^{\#}(\delta q)\vec{N})\,\mathrm{vol}_q.$$

And

$$\int_{\underline{R}^3} \operatorname{div}_q(s^{\#}(\delta q)\vec{N})\,\mathrm{vol}_q$$

$$= \int_{\underline{R}^3} \left(\frac{1}{\sqrt{q}}\,\partial_\ell(\sqrt{q}\; s^{\#}(\delta q)N^\ell)\right)\sqrt{q}\; d^3x$$

$$= \int_{\underline{R}^3} \partial_\ell(\sqrt{q}\; s^{\#}(\delta q)N^\ell)\,d^3x$$

$$= \lim_{R\to\infty}\int_{\underline{D}^3(R)} \operatorname{div}(\sqrt{q}\; s^{\#}(\delta q)\vec{N})\,d^3x$$

$$= \lim_{R\to\infty}\int_{\underline{S}^2(R)} (\sqrt{q}\; s^{\#}(\delta q)\vec{N}\cdot\underline{n})\,\omega_R^2$$

$$= 0.$$

To see this, write

$$\sqrt{q}\; s^{\#}(\delta q)N^\ell$$

$$= \lambda^{ij}\delta q_{ij}N^\ell$$

$$= \left(O^-\left(\frac{1}{r^2}\right) + O\left(\frac{1}{r^{2+\delta}}\right)\right)\left(O^+\left(\frac{1}{r}\right) + O\left(\frac{1}{r^{1+\delta}}\right)\right)(A^\ell + B^\ell{}_{\ell'}x^{\ell'})$$

$$= O(\frac{1}{r^3}) + (O^-(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}))(O^+(\frac{1}{r}) + O(\frac{1}{r^{1+\delta}}))B^\ell{}_{\ell'}x^{\ell'}$$

$$= O(\frac{1}{r^3}) + O^-(\frac{1}{r^2})O^+(\frac{1}{r})B^\ell{}_{\ell'}x^{\ell'} + O(\frac{1}{r^{2+\delta}})$$

$$= O(\frac{1}{r^3}) + O^+(\frac{1}{r^2}) + O(\frac{1}{r^{2+\delta}}).$$

LEMMA Suppose that $\vec{N} = \vec{A}$ — then the integral

$$\int_{\underline{R}^3} \nabla_j(N_i s^{ij})\sqrt{q}\, d^3x$$

is convergent.

[In fact,

$$\nabla_j(N_i s^{ij})\sqrt{q}$$

$$= \partial_j(N_i s^{ij}\sqrt{q})$$

$$= \partial_j(N_i \lambda^{ij})$$

$$= \partial_j(\lambda^{ij} q_{ik} N^k)$$

$$= \partial_j(\lambda^{ij} q_{ik} A^k).$$

But on $\underline{S}^2(R)$ $(R >> 0)$,

$$\lambda^{ij} q_{ik} A^k$$

$$= (\frac{1}{R^2}\tilde{\tau}^{ij} + O(\frac{1}{R^{2+\delta}}))(\eta_{ik} + O(\frac{1}{R}))A_k$$

$$= \frac{1}{R^2} \tilde{\tau}^{ij} \eta_{ik} A_k + O(\frac{1}{R^{2+c}}) \quad (c > 0).$$

And

$$\int_0^{2\pi} \int_0^{\pi} (\cos\theta \sin\phi\, \tau^{i1}(\theta,\phi) \eta_{ik} A_k$$

$$+ \sin\theta \sin\phi\, \tau^{i2}(\theta,\phi) \eta_{ik} A_k + \cos\phi\, \tau^{i3}(\theta,\phi) \eta_{ik} A^k) \sin\phi \, d\phi \, d\theta$$

is independent of R.]

Consequently, the integral

$$\int_{\underline{R}^3} - 2\mathrm{div}_q \, \Lambda(\vec{A})$$

is convergent.

Heuristics To motivate the next definition, take

$$\vec{A} = \begin{cases} (1,0,0) \\ (0,1,0) \\ (0,0,1). \end{cases}$$

To be specific, work with (1,0,0) — then

$$N_i = q_{ik} N^k = q_{i1}$$

=>

$$\nabla_j (N_i s^{ij}) \sqrt{q}$$

$$= \partial_j (\lambda^{ij} q_{i1}).$$

And

$$\lambda^{ij} q_{i1} = \lambda^{ij} (\eta_{i1} + O(\frac{1}{r}))$$

$$= \lambda^{1j} + O(\frac{1}{r^3}).$$

Therefore

$$\lim_{R \to \infty} \int_{\underline{S}^2(R)} \lambda^{1j} \Omega^j_R$$

exists and equals

$$\int_{\underline{R}^3} \nabla_j (q_{i1} s^{ij}) \sqrt{q}\, d^3x.$$

Definition: The <u>linear momentum</u> is the triple

$$(P^1, P^2, P^3),$$

where for b = 1,2,3,

$$P^b : S_d^{2,\infty} \to \underline{R}$$

sends Λ to

$$2\int_{\underline{S}^2(\infty)} \lambda^{b\ell} \Omega^{\ell}_{\infty}.$$

[Note: In view of what has been said above, the integral defining P^b is convergent.]

If $\vec{N} = B\vec{r}$, then, in general, the integral

$$\int_{\underline{R}^3} \nabla_j (N_i s^{ij}) \sqrt{q}\, d^3x$$

is divergent (however, it will be convergent if $(q,\Lambda) \in \text{Con}_D$).

Notation: Let $\bar{S}_d^{2,\infty}$ stand for the subset of $S_d^{2,\infty}$ consisting of those $\Lambda = \lambda d^3x$ such that for $r >> 0$,

$$\lambda^{ij}(x) = \frac{1}{r^2}\,\tau^{ij}(\frac{x}{r}) + \frac{1}{r^3}\,\bar{\tau}^{ij}(\frac{x}{r}) + \nu^{ij}(x),$$

where $\tau^{ij},\bar{\tau}^{ij}\in C^\infty(\underline{S}^2)$, τ^{ij} is of odd parity, and

$$\nu^{ij} = O^\infty(\frac{1}{r^{3+\delta}}) \quad (0 < \delta \leq 1).$$

[Note: Tacitly,

$$\tau^{ij} = \tau^{ji},\ \bar{\tau}^{ij} = \bar{\tau}^{ji},\ \nu^{ij} = \nu^{ji}.]$$

<u>LEMMA</u> Suppose that $\vec{N} = B\vec{r}$ -- then $\forall\ \Lambda\in\bar{S}_d^{2,\infty}$, the integral

$$\int_{\underline{R}^3} \nabla_j(N_i s^{ij})\sqrt{q}\, d^3x$$

is convergent.

[We have

$$\nabla_j(N_i s^{ij})\sqrt{q}$$

$$= \partial_j(\lambda^{ij} q_{ik} B^k{}_\ell x^\ell).$$

But on $\underline{S}^2(R)$ $(R >> 0)$,

$$\lambda^{ij} q_{ik} B^k{}_\ell x^\ell$$

$$= (\frac{1}{R^2}\,\tilde{\tau}^{ij} + \frac{1}{R^3}\,\tilde{\bar{\tau}}^{ij} + O(\frac{1}{R^{3+\delta}}))(\eta_{ik} + \frac{1}{R}\,\tilde{\sigma}_{ik} + O(\frac{1}{R^{1+\delta}}))B^k{}_\ell x^\ell.$$

Obviously,

$$(\frac{1}{R^2}\,\tilde{\tau}^{ij} + \frac{1}{R^3}\,\tilde{\bar{\tau}}^{ij})O(\frac{1}{R^{1+\delta}})B^k{}_\ell x^\ell$$

$$+ O(\frac{1}{R^{3+\delta}})(\eta_{ik} + \frac{1}{R}\,\tilde{\sigma}_{ik} + O(\frac{1}{R^{1+\delta}}))B^k{}_\ell x^\ell$$

$$= O(\frac{1}{R^{2+c}}) \quad (c > 0),$$

which leaves

1. $\frac{1}{R^2} \tilde{\tau}^{ij} \eta_{ik} B^k{}_\ell x^\ell$

2. $\frac{1}{R^3} \tilde{\bar{\tau}}^{ij} \eta_{ik} B^k{}_\ell x^\ell$

3. $\frac{1}{R^2} \tilde{\tau}^{ij} \frac{1}{R} \tilde{\sigma}_{ik} B^k{}_\ell x^\ell$

4. $\frac{1}{R^3} \tilde{\bar{\tau}}^{ij} \frac{1}{R} \tilde{\sigma}_{ik} B^k{}_\ell x^\ell$.

Bearing in mind that $\tau^{ij}, \bar{\tau}^{ij}$ are functions of the angular variables alone, it remains only to note that Items 1 and 3 are even while Item 4 is $O(\frac{1}{R^3})$.]

Consequently, the integral

$$\int_{\underline{R}^3} - 2\text{div}_q \, \Lambda(B\vec{r})$$

is convergent provided $\Lambda \in \bar{S}_d^{2,\infty}$.

Rappel: The canonical basis for $\underline{so}(3)$ is

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$X\cdot\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x^3 \\ x^2 \end{bmatrix} \longleftrightarrow (0, -x^3, x^2)$$

$$Y\cdot\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} x^3 \\ 0 \\ -x^1 \end{bmatrix} \longleftrightarrow (x^3, 0, -x^1)$$

$$Z\cdot\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} -x^2 \\ x^1 \\ 0 \end{bmatrix} \longleftrightarrow (-x^2, x^1, 0).$$

In addition,

$$[X,Y] = Z,\ [Y,Z] = X,\ [Z,X] = Y$$

$\Rightarrow$

$$[\bar{X},\bar{Y}] = -\bar{Z},\ [\bar{Y},\bar{Z}] = -\bar{X},\ [\bar{Z},\bar{X}] = -\bar{Y}$$

if

$$\bar{X} = -X,\ \bar{Y} = -Y,\ \bar{Z} = -Z.$$

<u>Heuristics</u> To motivate the next definition, take

$$B\vec{r} = \begin{cases} (0, -x^3, x^2) = x^2 \dfrac{\partial}{\partial x^3} - x^3 \dfrac{\partial}{\partial x^2} \\ (x^3, 0, -x^1) = x^3 \dfrac{\partial}{\partial x^1} - x^1 \dfrac{\partial}{\partial x^3} \\ (-x^2, x^1, 0) = x^1 \dfrac{\partial}{\partial x^2} - x^2 \dfrac{\partial}{\partial x^1}\ . \end{cases}$$

To be specific, work with $(0, -x^3, x^2)$ and restrict Λ to $\bar{S}_d^{2,\infty}$ — then

$$N_i = q_{ik}N^k = q_{i3}x^2 - q_{i2}x^3$$

=>

$$\nabla_j(N_i s^{ij})\sqrt{q}$$

$$= \partial_j(\lambda^{ij}(q_{i3}x^2 - q_{i2}x^3)).$$

And

$$\lambda^{ij}(q_{i3}x^2 - q_{i2}x^3)$$

$$= \lambda^{ij}(\eta_{i3}x^2 - \eta_{i2}x^3) + \cdots$$

$$= x^2\lambda^{3j} - x^3\lambda^{2j} + \cdots .$$

Therefore

$$\lim_{R \to \infty} \int_{\underline{S}^2(R)} (x^2\lambda^{3j} - x^3\lambda^{2j})\Omega_R^j$$

exists and equals

$$\int_{\underline{R}^3} \nabla_j((q_{i3}x^2 - q_{i2}x^3)s^{ij})\sqrt{q}\, d^3x.$$

[Note: The proof of the preceding lemma shows that

$$\lim_{R \to \infty} \int_{\underline{S}^2(R)} \ldots\Omega_R^j = 0.]$$

Definition: The angular momentum is the triple

$$(J^1, J^2, J^3),$$

where for $b = 1,2,3,$

$$J^b : \bar{S}_d^{2,\infty} \to \underline{R}$$

sends Λ to

$$2\int_{\underline{S}^2(\infty)} \varepsilon_{bjk} x^j \lambda^{k\ell} \Omega_\infty^\ell .$$

[Note: In view of what has been said above, the integral defining J^b is convergent.]

Example: Suppose that

$$\lambda^{11} = -\frac{2}{r^3}\frac{x^1x^2}{r^2} + O^\infty\left(\frac{1}{r^{3+\delta}}\right)$$

$$\lambda^{22} = \frac{2}{r^3}\frac{x^1x^2}{r^2} + O^\infty\left(\frac{1}{r^{3+\delta}}\right)$$

$$\lambda^{33} = 0 + O^\infty\left(\frac{1}{r^{3+\delta}}\right)$$

$$\lambda^{12} = \frac{1}{r^3}\left(\frac{x^1x^1}{r^2} - \frac{x^2x^2}{r^2}\right) + O^\infty\left(\frac{1}{r^{3+\delta}}\right)$$

$$\lambda^{13} = -\frac{1}{r^3}\frac{x^2x^3}{r^2} + O^\infty\left(\frac{1}{r^{3+\delta}}\right)$$

$$\lambda^{23} = \frac{1}{r^3}\frac{x^1x^3}{r^2} + O^\infty\left(\frac{1}{r^{3+\delta}}\right).$$

Then $\Lambda\in\bar{S}_d^{2,\infty}$ (here, $\tau^{ij} = 0$) and we claim that

$$J^1(\Lambda) = 0,\ J^2(\Lambda) = 0,\ J^3(\Lambda) = \frac{16\pi}{3} .$$

Consider first $J^1(\Lambda)$ which, by definition, is

$$2\int_{\underline{S}^2(\infty)} (x^2\lambda^{3\ell} - x^3\lambda^{2\ell})\Omega_\infty^\ell$$

$$= 2 \lim_{R\to\infty} \int_{\underline{S}^2(R)} (x^2\lambda^{3\ell} - x^3\lambda^{2\ell})\frac{x^\ell}{R}\,\omega_R^2.$$

Dropping the 2 and setting aside the $O^\infty(\frac{1}{r^{3+\delta}})$ (as they will not contribute), we have

1. $R^2 \int_0^{2\pi}\int_0^\pi (\cos\theta \sin\phi)[(R\sin\theta\sin\phi)(-\frac{1}{R^3})\sin\theta\sin\phi\cos\phi$

   $-(R\cos\phi)(\frac{1}{R^3})((\cos\theta\sin\phi)^2 - (\sin\theta\sin\phi)^2]\sin\phi\, d\phi\, d\theta$

2. $R^2 \int_0^{2\pi}\int_0^\pi (\sin\theta\sin\phi)[(R\sin\theta\sin\phi)(\frac{1}{R^3})\cos\theta\sin\phi\cos\phi$

   $-(R\cos\phi)(\frac{2}{R^3})\cos\theta\sin\phi\sin\theta\sin\phi]\sin\phi\, d\phi\, d\theta$

3. $R^2 \int_0^{2\pi}\int_0^\pi (\cos\phi)[(R\sin\theta\sin\phi)0$

   $-(R\cos\phi)(\frac{1}{R^3})\cos\theta\sin\phi\cos\phi]\sin\phi\, d\phi\, d\theta$

or still,

1. $\int_0^{2\pi}\int_0^\pi (\cos\theta\sin\phi)[-\sin^2\theta\sin^2\phi\cos\phi$

   $-\cos\phi\sin^2\phi(\cos^2\theta - \sin^2\theta)]\sin\phi\, d\phi\, d\theta$

2. $\int_0^{2\pi}\int_0^\pi (\sin\theta\sin\phi)[\sin\theta\cos\theta\sin^2\phi\cos\phi$

   $-2\sin\theta\cos\theta\sin^2\phi\cos\phi]\sin\phi\, d\phi\, d\theta$

3. $\int_0^{2\pi}\int_0^\pi -\cos\theta\sin^2\phi\cos^2\phi\, d\phi\, d\theta$

or still,

1. $\int_0^{2\pi}\int_0^{\pi} - \sin^4\phi \cos\phi \cos^3\theta \, d\phi \, d\theta$

2. $\int_0^{2\pi}\int_0^{\pi} [\sin^4\phi \cos\phi \sin^2\theta \cos\theta$

$- 2 \sin^4\phi \cos\phi \sin^2\theta \cos\theta] d\phi \, d\theta$

3. $\int_0^{2\pi}\int_0^{\pi} - \cos\theta \sin^2\phi \cos^3\phi \, d\phi \, d\theta$

=>

1 + 2 + 3

$= - \int_0^{2\pi}\int_0^{\pi} [\sin^4\phi \cos\phi \cos^3\theta$

$+ \sin^4\phi \cos\phi \sin^2\theta \cos\theta + \sin^2\phi \cos^3\phi \cos\theta] d\phi \, d\theta$

$= - \int_0^{2\pi}\int_0^{\pi} [\sin^4\phi \cos\phi \cos^3\theta$

$+ \sin^4\phi \cos\phi (1 - \cos^2\theta) \cos\theta$

$+ \sin^2\phi \cos^3\phi \cos\theta] d\phi \, d\theta$

$= - \int_0^{2\pi}\int_0^{\pi} [\sin^4\phi \cos\phi \cos\theta + \sin^2\phi \cos^3\phi \cos\theta] d\phi \, d\theta$

$= - \int_0^{2\pi}\int_0^{\pi} [\sin^2\phi (1 - \cos^2\phi) \cos\phi \cos\theta$

$+ \sin^2\phi \cos^3\phi \cos\theta] d\phi \, d\theta$

$$= - \left(\int_0^{2\pi} \cos\theta \, d\theta\right)\left(\int_0^{\pi} \sin^2\phi \cos\phi \, d\phi\right)$$

$$= 0.$$

Analogously,

$$J^2(\Lambda) = 0.$$

Turning to $J^3(\Lambda)$, insertion of the data leads to

$$J^3(\Lambda) = 2\int_0^{2\pi} d\theta \int_0^{\pi} \sin^3\phi \, d\phi$$

$$= 8\pi \int_0^{\pi/2} \sin^3\phi \, d\phi$$

$$= 8\pi \cdot 4B(2,2)$$

$$= 8\pi \cdot \frac{4}{\Gamma(4)}$$

$$= \frac{32\pi}{3!} = \frac{16\pi}{3} .$$

Section 65: Modifying the Hamiltonian The assumptions and notation are those of Section 62.

Definition: A lapse $N \in C^\infty(\underline{R}^3)$ is said to be asymptotic if

$$N = A + B_1 x^1 + B_2 x^2 + B_3 x^3 + sc,$$

where A and B_1, B_2, B_3 are constants.

Definition: A shift $\vec{N} \in \mathcal{D}^1(\underline{R}^3)$ is said to be asymptotic if

$$\vec{N} = \vec{A} + B\vec{r} + \vec{sc},$$

where $\vec{A} \in \underline{R}^3$ and $B \in \underline{so}(3)$.

N.B. Recall that sc and $\vec{sc}$ are short for the standard conditions.

Suppose that N = sc and $\vec{N} = \vec{sc}$ — then

$$H(q,\Lambda;N,\vec{N}) = \int_{\underline{R}^3} - 2\text{div}_q \Lambda(\vec{N})$$

$$+ \int_{\underline{R}^3} N([s,s]_q - \frac{1}{2}\text{tr}_q(s)^2 - S(q))\sqrt{q}\, d^3x$$

if $\Lambda = s^\# \otimes |q|^{1/2}$. Furthermore, the functional derivatives $\frac{\delta H}{\delta q}$ and $\frac{\delta H}{\delta \Lambda}$ exist and satisfy what we shall term the ADM relations, i.e.,

$$\frac{\delta H}{\delta q} = 2N(s*s - \frac{1}{2}\text{tr}_q(s)s)^\# \otimes |q|^{1/2}$$

$$- \frac{N}{2}([s,s]_q - \frac{1}{2}\text{tr}_q(s)^2)q^\# \otimes |q|^{1/2}$$

$$+ N\,\text{Ein}(q)^\# \otimes |q|^{1/2}$$

$$- (H_N - (\Delta_q N)q)^\# \otimes |q|^{1/2} - L_{\vec{N}}\Lambda$$

and

$$\frac{\delta H}{\delta \Lambda} = 2N(s - \frac{1}{2}\,\mathrm{tr}_q(s)q) + L_{\vec{N}}q.$$

However, for an arbitrary asymptotic lapse or shift, the boundary terms come into play and the ADM relations break down. To restore them, it is necessary to modify the definition of H.

[Note: Implicit in this is the functional differentiability of the modification.]

Example: Consider the situation when $N = 1$ and $\vec{N} = \vec{sc}$. Define

$$H_{RT}:\Gamma \rightarrow \underline{R}$$

by

$$H_{RT}(q,\Lambda) = H(q,\Lambda) + P^0(q)$$

or still,

$$H_{RT}(q,\Lambda) = H(q,\Lambda) + \int_{\underline{S}^2(\infty)} (q_{i\ell,i} - q_{ii,\ell})\Omega_\infty^\ell.$$

Then H_{RT} is functionally differentiable and satisfies the ADM relations.

Example: Consider the situation when $N = sc$ and $\vec{N} = (\delta^b{}_1, \delta^b{}_2, \delta^b{}_3)$ $(b = 1,2,3)$. Define

$$H_{RT}:\Gamma \rightarrow \underline{R}$$

by

$$H_{RT}(q,\Lambda) = H(q,\Lambda) + P^b(\Lambda)$$

or still,

$$H_{RT}(q,\Lambda) = H(q,\Lambda) + 2\int_{\underline{S}^2(\infty)} \lambda^{b\ell}\Omega_\infty^\ell.$$

Then H_{RT} is functionally differentiable and satisfies the ADM relations.

Definition: The Regge-Teitelboim modification of the hamiltonian is the function

$$H_{RT}:\Gamma \rightarrow \underline{R}$$

defined by the prescription

$$H_{RT}(q,\Lambda;N,\vec{N}) = \int_{\underline{R}^3} \Lambda(L_{\vec{N}} q)$$

$$+ H_H(N)(q,\Lambda)$$

$$+ \int_{\underline{S}^2(\infty)} (Nq^{ij}q^{k\ell}(q_{ik,j} - q_{ij,k})$$

$$- N_{,j}q^{ij}q^{k\ell}(q_{ik} - \eta_{ik}) + N_{,k}q^{ij}q^{k\ell}(q_{ij} - \eta_{ij}))\Omega^{\ell}_{\infty}.$$

[Note: Here, of course, N and $\vec{N}$ are asymptotic.]

THEOREM H_{RT} is functionally differentiable and satisfies the ADM relations.

[This follows from what has been said in Sections 63 and 64.]

Remark: If $N = sc$ and $\vec{N} = \overrightarrow{sc}$, then $H_{RT} = H$ and $H|Con_{Q_\infty} = 0$. But for arbitrary asymptotic N and $\vec{N}$, $H_{RT}|Con_{Q_\infty} \neq 0$. E.g.: Take $N = 1$ and suppose that $\vec{N} = \overrightarrow{sc}$ -- then $H_{RT}|Con_{Q_\infty} = P^0$.

LEMMA Suppose that N_1, N_2, $\vec{N}_1$, $\vec{N}_2$ are asymptotic — then

$$\left[\begin{array}{l} L_{\vec{N}_1} N_2 \\ \\ L_{\vec{N}_2} N_1 \end{array}\right.$$

and

$$[\vec{N}_1,\vec{N}_2], \quad \left[\begin{array}{l} N_1 \text{ grad } N_2 \\ \\ N_2 \text{ grad } N_1 \end{array}\right.$$

are asymptotic.

[Note: If $N_1 = sc$, $\vec{N}_1 = \overrightarrow{sc}$, then the resulting entities also satisfy the standard conditions (and ditto if instead $N_2 = sc$, $\vec{N}_2 = \overrightarrow{sc}$). Let us also remind ourselves that grad refers to grad_q $(q \in Q_\infty)$.]

<u>THEOREM</u> We have

$$\{H_{RT}(N_1,\vec{N}_1), H_{RT}(N_2,\vec{N}_2)\}$$

$$= H_{RT}(L_{\vec{N}_1} N_2 - L_{\vec{N}_2} N_1, [\vec{N}_1,\vec{N}_2] + N_1 \text{ grad } N_2 - N_2 \text{ grad } N_1).$$

Remark: In general, the Poisson bracket

$$\{H_{RT}(N_1,\vec{N}_1), H_{RT}(N_2,\vec{N}_2)\}$$

does not vanish on Con_{Q_∞}, hence is not a constraint (but this will be true if either $N_1 = sc$, $\vec{N}_1 = \overrightarrow{sc}$ or $N_2 = sc$, $\vec{N}_2 = \overrightarrow{sc}$).

Section 66: The Poincaré Structure The assumptions and notation are those of Section 62.

Definition:

$$H_{RT}(1,\vec{0}) \text{ — generator of time translations.}$$

Definition:

$$\left[\begin{array}{l} H_{RT}(0,(\delta^b{}_1,\delta^b{}_2,\delta^b{}_3)) \text{ — generators of space translations} \\ \\ H_{RT}(x^b,\vec{0}) \text{ — generators of boosts} \\ \\ H_{RT}(0,(\varepsilon_{bj1}x^j,\varepsilon_{bj2}x^j,\varepsilon_{bj3}x^j)) \text{ — generators of rotations.} \end{array}\right.$$

[Note: In each case, $b = 1,2,3$.]

The objective now will be to compute all of the Poisson brackets amongst these 10 entities.

For use below, recall the following points.

- Given $X,Y\in\mathcal{D}^1(\underline{R}^3)$,

$$[X,Y] = (X^iY^j{}_{,i} - Y^iX^j{}_{,i})\frac{\partial}{\partial x^j},$$

thus

$$[X,Y]^j = X^i\partial_iY^j - Y^i\partial_iX^j.$$

- Given $f\in C^\infty(\underline{R}^3)$ and $q\in Q_\infty$,

$$\text{grad } f \ (= \text{grad}_q\ f) = (\frac{\partial f}{\partial x^i}q^{ij})\frac{\partial}{\partial y^j},$$

thus

$$(\text{grad } f)^j = (\partial_i f)q^{ij}.$$

Time Translation/Space Translation: We have

$$\{H_{RT}(1,\vec{0}), H_{RT}(0,(\delta^b{}_1,\delta^b{}_2,\delta^b{}_3))\}$$

$$= H_{RT}(- L_{(\delta^b{}_1,\delta^b{}_2,\delta^b{}_3)} 1,\vec{0}) = 0.$$

Time Translation/Boost: We have

$$\{H_{RT}(1,\vec{0}), H_{RT}(x^b,\vec{0})\}$$

$$= H_{RT}(0, \text{grad } x^b)$$

$$= H_{RT}(0, \delta^b{}_i \eta_{ij} \frac{\partial}{\partial y^j} + \vec{sc})$$

$$= H_{RT}(0,(\delta^b{}_1,\delta^b{}_2,\delta^b{}_3)) + H(0,\vec{sc}).$$

Time Translation/Rotation: We have

$$\{H_{RT}(1,\vec{0}), H_{RT}(0,(\epsilon_{bj1}x^j,\epsilon_{bj2}x^j,\epsilon_{bj3}x^j))\}$$

$$= H_{RT}(- L_{(\epsilon_{bj1}x^j,\epsilon_{bj2}x^j,\epsilon_{bj3}x^j)} 1,\vec{0}) = 0.$$

Boost/Boost: We have

$$\{H_{RT}(x^{b'},\vec{0}), H_{RT}(x^{b''},\vec{0})\}$$

$$= H_{RT}(0, x^{b'} \text{ grad } x^{b''} - x^{b''} \text{ grad } x^{b'}).$$

Take, for example, b' = 1, b" = 2 — then

$$x^1 \text{ grad } x^2 - x^2 \text{ grad } x^1$$

$$= (-x^2, x^1, 0) + \vec{sc}.$$

Therefore

$$\{H_{RT}(x^1, \vec{0}), H_{RT}(x^2, \vec{0})\}$$

$$= H_{RT}(0, (\varepsilon_{3j1}x^j, \varepsilon_{3j2}x^j, \varepsilon_{3j3}x^j)) + H(0, \vec{sc}).$$

In general:

$$\{H_{RT}(x^{b'}, \vec{0}), H_{RT}(x^{b''}, \vec{0})\}$$

$$= \varepsilon_{b'b''c}H_{RT}(0, (\varepsilon_{cj1}x^j, \varepsilon_{cj2}x^j, \varepsilon_{cj3}x^j)) + H(0, \vec{sc}).$$

<u>Boost/Space Translation</u>: We have

$$\{H_{RT}(x^{b'}, \vec{0}), H_{RT}(0, (\delta^{b''}{}_1, \delta^{b''}{}_2, \delta^{b''}{}_3))\}$$

$$= H_{RT}(-L_{(\delta^{b''}{}_1, \delta^{b''}{}_2, \delta^{b''}{}_3)} x^{b'}, \vec{0})$$

$$= -H_{RT}(\delta^{b''}{}_c \partial_c x^{b'}, \vec{0})$$

$$= -H_{RT}(\delta^{b''}{}_c \delta^{b'}{}_c, \vec{0})$$

$$= -\delta_{b'b''}H_{RT}(1, \vec{0}).$$

Boost/Rotation: We have

$$\{\mathcal{H}_{RT}(x^{b'},\vec{0}),\mathcal{H}_{RT}(0,(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j))\}$$

$$= \mathcal{H}_{RT}(-\mathcal{L}_{(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j)}x^{b'},\vec{0}).$$

Take, for example, b' = 1, b" = 2 — then

$$\mathcal{L}_{(\varepsilon_{2j1}x^j,\varepsilon_{2j2}x^j,\varepsilon_{2j3}x^j)}x^1$$

$$= \mathcal{L}_{(x^3,0,\ -x^1)}x^1$$

$$= (x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3})x^1$$

$$= x^3.$$

Therefore

$$\{\mathcal{H}_{RT}(x^1,\vec{0}),\mathcal{H}_{RT}(0,(\varepsilon_{2j1}x^j,\varepsilon_{2j2}x^j,\varepsilon_{2j3}x^j))\}$$

$$= -\mathcal{H}_{RT}(x^3,\vec{0}).$$

In general:

$$\{\mathcal{H}_{RT}(x^{b'},\vec{0}),\mathcal{H}_{RT}(0,(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j))\}$$

$$= -\varepsilon_{b'b''c}\mathcal{H}_{RT}(x^c,\vec{0}).$$

Space Translation/Space Translation: We have

$$\{\mathcal{H}_{RT}(0,(\delta^{b'}{}_1,\delta^{b'}{}_2,\delta^{b'}{}_3)),\mathcal{H}_{RT}(0,(\delta^{b''}{}_1,\delta^{b''}{}_2,\delta^{b''}{}_3))\}$$

$$= H_{RT}(0,[(\delta^{b'}{}_1,\delta^{b'}{}_2,\delta^{b'}{}_3),(\delta^{b''}{}_1,\delta^{b''}{}_2,\delta^{b''}{}_3)])$$

$$= 0.$$

<u>Space Translation/Rotation:</u> We have

$$\{H_{RT}(0,(\delta^{b'}{}_1,\delta^{b'}{}_2,\delta^{b'}{}_3)),H_{RT}(0,(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j))\}$$

$$= H_{RT}(0,[(\delta^{b'}{}_1,\delta^{b'}{}_2,\delta^{b'}{}_3),(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j)]).$$

Take, for example, b' = 2, b" = 3 — then

$$[(\delta^2{}_1,\delta^2{}_2,\delta^2{}_3),(\varepsilon_{3j1}x^j,\varepsilon_{3j2}x^j,\varepsilon_{3j3}x^j)]$$

$$= [(0,1,0),(-x^2,x^1,0)]$$

$$= (-1,0,0)$$

$$= -(\delta^1{}_1,\delta^1{}_2,\delta^1{}_3).$$

Therefore

$$\{H_{RT}(0,(\delta^2{}_1,\delta^2{}_2,\delta^2{}_3)),H_{RT}(0,(\varepsilon_{3j1}x^j,\varepsilon_{3j2}x^j,\varepsilon_{3j3}x^j))\}$$

$$= -H_{RT}(0,(\delta^1{}_1,\delta^1{}_2,\delta^1{}_3)).$$

In general:

$$\{H_{RT}(0,(\delta^{b'}{}_1,\delta^{b'}{}_2,\delta^{b'}{}_3)),H_{RT}(0,(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j))\}$$

$$= -\varepsilon_{b'b''c}H_{RT}(0,(\delta^c{}_1,\delta^c{}_2,\delta^c{}_3)).$$

<u>Rotation/Rotation</u>: We have

$$\{H_{RT}(0,(\varepsilon_{b'j1}x^j,\varepsilon_{b'j2}x^j,\varepsilon_{b'j3}x^j)),H_{RT}(0,(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j))\}$$

$$= H_{RT}(0,[(\varepsilon_{b'j1}x^j,\varepsilon_{b'j2}x^j,\varepsilon_{b'j3}x^j),(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j)]).$$

Take, for example, $b' = 1$, $b'' = 2$ -- then

$$[(\varepsilon_{1j1}x^j,\varepsilon_{1j2}x^j,\varepsilon_{1j3}x^j),(\varepsilon_{2j1}x^j,\varepsilon_{2j2}x^j,\varepsilon_{2j3}x^j)]$$

$$= [(0, -x^3,x^2),(x^3,0, -x^1)]$$

$$= (x^2, -x^1,0).$$

Therefore

$$\{H_{RT}(0,(\varepsilon_{1j1}x^j,\varepsilon_{1j2}x^j,\varepsilon_{1j3}x^j)),H_{RT}(0,(\varepsilon_{2j1}x^j,\varepsilon_{2j2}x^j,\varepsilon_{2j3}x^j))\}$$

$$= - H_{RT}(0,(\varepsilon_{3j1}x^j,\varepsilon_{3j2}x^j,\varepsilon_{3j3}x^j)).$$

In general:

$$\{H_{RT}(0,(\varepsilon_{b'j1}x^j,\varepsilon_{b'j2}x^j,\varepsilon_{b'j3}x^j)),H_{RT}(0,(\varepsilon_{b''j1}x^j,\varepsilon_{b''j2}x^j,\varepsilon_{b''j3}x^j))\}$$

$$= - \varepsilon_{b'b''c}H_{RT}(0,(\varepsilon_{cj1}x^j,\varepsilon_{cj2}x^j,\varepsilon_{cj3}x^j)).$$

Rappel: Let $\underline{g}$ be the Lie algebra of the Poincaré group -- then $\dim \underline{g} = 10$ and admits a basis

- P^0 — generator of time translations
- P^1,P^2,P^3 — generators of space translations
- N^1,N^2,N^3 — generators of boosts
- J^1,J^2,J^3 — generators of rotations

subject to the following commutation relations:

$$[P^0,P^b] = 0,\ [P^0,N^b] = P^b,\ [P^0,J^b] = 0,$$

$$[N^{b'},N^{b''}] = \varepsilon_{b'b''c}J^c,\ [N^{b'},P^{b''}] = -\ \delta_{b'b''}P^0,$$

$$[N^{b'},J^{b''}] = -\ \varepsilon_{b'b''c}N^c,$$

$$[P^{b'},P^{b''}] = 0,\ [P^{b'},J^{b''}] = -\ \varepsilon_{b'b''c}P^c,$$

$$[J^{b'},J^{b''}] = -\ \varepsilon_{b'b''c}J^c.$$

The formulas for

- Time Translation/Boost
- Boost/Boost

each contain a term of the form $\mathcal{H}(0,\overrightarrow{sc})$, which somewhat spoils what otherwise would be a very pretty picture. Still,

$$\mathcal{H}(0,\overrightarrow{sc})\,|\text{Con}_D = 0.$$

Therefore, upon restriction to Con_D, the Poisson brackets derived above have exactly the same structure as the commutation relations of $\underline{g}$.

Section 67: Function Spaces In $\underline{R}^n$, Sobolev space theory is standard fare but weighted Sobolev space theory is less so. Since it is the latter which will be needed for the applications, a brief account seems appropriate.

[Note: In what follows, it will be assumed that $n \geq 3$ ($n = 3$ being the case of ultimate interest).]

Notation: Let

$$\sigma(x) = (1 + |x|^2)^{1/2} \quad (x \in \underline{R}^n),$$

i.e., let

$$\sigma = (1 + r^2)^{1/2}.$$

Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, write

$$\partial^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}$$

and put

$$|\alpha| = \sum_1^n \alpha_i.$$

Definition: Let $k \in \underline{Z}_{\geq 0}$, $\delta \in \underline{R}$ — then by C_δ^k we understand the Banach space consisting of those functions $f: \underline{R}^n \rightarrow \underline{R}$ of class C^k such that

$$||f||_{C_\delta^k} = \sum_{|\alpha| \leq k} \sup_{\underline{R}^n} \sigma^{\delta + |\alpha|} |\partial^\alpha f| < \infty.$$

[Note: The indexing has been arranged so that

$$\partial^\alpha f = O\left(\frac{1}{r^{|\alpha| + \delta}}\right) \quad (|\alpha| \leq k).]$$

Example: Take $n = 3$ and suppose that q is an asymptotically flat riemannian structure on $\underline{R}^3$ — then

$$q_{ij} - \eta_{ij} \in C_1^k \ \forall \ k \geq 0.$$

LEMMA Pointwise multiplication induces a continuous bilinear map

$$C_{\delta_1}^k \times C_{\delta_2}^k \rightarrow C_{\delta_1+\delta_2}^k.$$

Definition: Let $k \in \underline{Z}_{\geq 0}$, $\delta \in \underline{R}$ — then by W_δ^k we understand the Hilbert space consisting of those locally integrable functions $f:\underline{R}^n \rightarrow \underline{R}$ possessing locally integrable distributional derivatives up to order k such that

$$||f||_{W_\delta^k} = [\sum_{|\alpha|\leq k} \int_{\underline{R}^n} \sigma^{2(\delta+|\alpha|)} |\partial^\alpha f|^2 \, d^n x]^{1/2} < \infty.$$

[Note: The inner product in W_δ^k is

$$< f_1, f_2 >_{W_\delta^k} = \sum_{|\alpha|\leq k} \int_{\underline{R}^n} \sigma^{2(\delta+|\alpha|)} (\partial^\alpha f_1)(\partial^\alpha f_2) d^n x.]$$

N.B. $C_c^\infty(\underline{R}^n)$ is dense in W_δ^k.

Example: Suppose that $f \in W_{-1}^1$ — then the partial derivatives $\partial_i f$ are square integrable.

Example: Let $c \in \underline{R}$ — then $\sigma^c \in W_\delta^k \Leftrightarrow c < -(\delta + \frac{n}{2})$. In particular: The constants belong to W_δ^k iff $\delta < -\frac{n}{2}$.

[Since

$$\partial^\alpha \sigma^c = O(r^{c-|\alpha|}),$$

it suffices to take $k = 0$. But

$$\sigma^{2c}\sigma^{2\delta}r^{n-1} = O(r^{2c + 2\delta + n-1})$$

$$= O(r^{-(-(2c + 2\delta + n-1))}).$$

And

$$-(2c + 2\delta + n-1) > 1 \iff c < -(\delta + \tfrac{n}{2}).]$$

<u>FACT</u> Multiplication $f \to f\sigma^c$ defines a continuous map $W_\delta^k \to W_{\delta-c}^k$.

<u>LEMMA</u> The operator

$$\partial_i : W_\delta^k \to W_{\delta+1}^{k-1} \quad (k \geq 1)$$

is a bounded linear transformation.

<u>Heuristics</u> One reason for introducing the W_δ^k is that they are better suited for the study of certain elliptic differential operators. Take, e.g., the laplacian Δ corresponding to η (the usual flat metric on $\underline{R}^n$). As a densely defined operator on $L^2(\underline{R}^n)$, its maximal domain is the set of $f \in L^2(\underline{R}^n)$ such that $\Delta f \in L^2(\underline{R}^n)$ in the sense of distributions, i.e., is the ordinary Sobolev space $H^2(\underline{R}^n)$ (and there, Δ is selfadjoint). Viewed as a map $\Delta : H^2(\underline{R}^n) \to L^2(\underline{R}^n)$, the kernel of Δ is trivial:

$$\Delta f = 0 \implies 0 = -\int_{\underline{R}^n} f \Delta f \, d^n x$$

$$= \int_{\underline{R}^n} |\mathrm{grad}\ f|^2 \, d^n x$$

$\Rightarrow$

$$f = 0.$$

On the other hand, the range of Δ is not closed. For if it were, then $\exists\, C > 0$: $\forall\, f \in H^2(\underline{R}^n)$,

$$||f||_{H^2(\underline{R}^n)} \leq C||\Delta f||_{L^2(\underline{R}^n)}.$$

But such a relation cannot be true. To see this, let

$$(S_R f)(x) = f(Rx).$$

Then

$$\Delta S_R f = R^2 S_R \Delta f.$$

Therefore

$$\begin{aligned} ||f||_{L^2(\underline{R}^n)} &= R^{-n/2}||S_{1/R}f||_{L^2(\underline{R}^n)} \\ &\leq R^{-n/2}||S_{1/R}f||_{H^2(\underline{R}^n)} \\ &\leq CR^{-n/2}||\Delta S_{1/R}f||_{L^2(\underline{R}^n)} \\ &= CR^{-2}||\Delta f||_{L^2(\underline{R}^n)}, \end{aligned}$$

an impossibility.

Put

$$L_\delta^2 = W_\delta^0.$$

Then the L_δ^2 are, by definition, the weighted L^2-spaces ($L_0^2 = L^2(\underline{R}^n)$).

<u>FACT</u> Suppose that $f \in L^2_\delta$ has the property that $\Delta f \in L^2_{\delta+2}$ in the sense of distributions -- then $f \in W^2_\delta$.

Observation: The dual of L^2_δ is $L^2_{-\delta}$. Indeed,

$$u \in L^2_\delta,\ v \in L^2_{-\delta}$$

$$\Rightarrow$$

$$\int_{\underline{R}^n} |uv| d^n x = \int_{\underline{R}^n} |u|\sigma^\delta \cdot |v|\sigma^{-\delta} d^n x$$

$$\leq \left(\int_{\underline{R}^n} |u|^2 \sigma^{2\delta} d^n x\right)^{1/2} \left(\int_{\underline{R}^n} |v|^2 \sigma^{-2\delta} d^n x\right)$$

$$< \infty.$$

Remark: The dual of W^k_δ contains $L^2_{-\delta}$. However, to completely explicate it, one has to introduce a weighted Sobolev space $W^{-k}_{-\delta}$, which is a certain subset of the space of tempered distributions on $\underline{R}^n$ and, by construction, is the dual of W^k_δ.

If $k \geq k'$, $\delta \geq \delta'$, then

$$W^k_\delta \subset W^{k'}_{\delta'}.$$

<u>RELLICH LEMMA</u> Suppose that $k > k'$, $\delta > \delta'$ -- then the injection

$$W^k_\delta \to W^{k'}_{\delta'}$$

is compact.

[Note: In other words, if $\{f_n\} \subset W^k_\delta$ is a bounded sequence, then there is a subsequence $\{f_{n_k}\}$ which converges in $W^{k'}_{\delta'}$.]

Remark: The injection

$$W_\delta^k \to W_\delta^{k-1} \quad (k \geq 1)$$

is continuous but not compact.

<u>EMBEDDING LEMMA I</u> We have

$$W_\delta^k \subset C_{\delta'}^{k'}$$

if

$$\begin{cases} k' < k - \frac{n}{2} \\ \delta' < \delta + \frac{n}{2}. \end{cases}$$

Application: Fix $k > \frac{n}{2}$ — then $\forall\ f \in W_\delta^k$,

$$\sigma^c|f| = o(1)$$

provided $c < \delta + \frac{n}{2}$.

[Take $k' = 0$, choose $\delta' : c < \delta' < \delta + \frac{n}{2}$, and write

$$\begin{aligned} \sigma^c|f| &= \sigma^{c-\delta'}\sigma^{\delta'}|f| \\ &= \sigma^{c-\delta'}O(1) \\ &= o(1)O(1) = o(1).] \end{aligned}$$

[Note: If $0 < \delta + \frac{n}{2}$, then

$$|f| = o(1).]$$

EMBEDDING LEMMA II We have

$$C^{k'}_{\delta'} \subset W^k_\delta$$

if

$$\begin{cases} k' \geq k \\ \delta' > \delta + \frac{n}{2}. \end{cases}$$

Example: If f is C^2 and if

$$f = O(\frac{1}{r}),\ \partial_i f = O(\frac{1}{r^2}),\ \partial_i\partial_j f = O(\frac{1}{r^3}),$$

then

$$f \in C^2_1 \subset W^2_\delta \quad (\delta < 1 - \frac{n}{2}).$$

POINCARÉ INEQUALITY Suppose that $\delta > -\frac{n}{2}$ — then $\exists\ C > 0$ such that $\forall\ f \in W^1_\delta$,

$$\int_{\underline{R}^n} |f|^2 \sigma^{2\delta} d^n x \leq C \int_{\underline{R}^n} |\text{grad } f|^2 \sigma^{2(\delta+1)} d^n x.$$

[Note: Take $\delta = -1$ to get

$$\int_{\underline{R}^n} |f|^2 \sigma^{-2} d^n x \leq C \int_{\underline{R}^n} |\text{grad } f|^2 d^n x.]$$

PRODUCT LEMMA Pointwise multiplication induces a continuous bilinear map

$$W^{k_1}_{\delta_1} \times W^{k_2}_{\delta_2} \to W^k_\delta$$

if

$$k_1, k_2 \geq k,\ k < k_1 + k_2 - \frac{n}{2},\ \delta < \delta_1 + \delta_2 + \frac{n}{2}.$$

Application: Suppose that $k > \frac{n}{2}$, $\delta > -\frac{n}{2}$ -- then W_δ^k is closed under the formation of products.

The theory outlined above admits an obvious extension to the case of functions $f:\underline{R}^n \to \underline{R}^m$ but it is customary to abbreviate and use the symbol W_δ^k in this situation as well.

[Note: To say that a tensor $T \in \mathcal{D}_q^p(\underline{R}^n)$ is in W_δ^k simply means that its components $T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$ are in W_δ^k.]

Notation: Let $I:\underline{R}^n \to \underline{R}^n$ be the identity map and write $W_\delta^k(I)$ for the set of functions $f:\underline{R}^n \to \underline{R}^n$ such that $f - I \in W_\delta^k$.

[Note: The arrow $W_\delta^k(I) \to W_\delta^k$ that sends f to f - I is bijective, thus $W_\delta^k(I)$ can be topologized by demanding that it be a homeomorphism.]

Denote now by

$$D_{\delta-1}^{k+1} \quad (k > \frac{n}{2},\ \delta > -\frac{n}{2})$$

the set of diffeomorphisms

$$\phi:\underline{R}^n \to \underline{R}^n$$

such that

$$\phi - I \in W_{\delta-1}^{k+1}$$

and equip it with the topology inherited from $W_{\delta-1}^{k+1}(I)$.

[Note: Given $\phi \in D_{\delta-1}^{k+1}$, write $\phi = I + F$, where $F \in W_{\delta-1}^{k+1}$. Fix $\varepsilon > 0$:

$$(\delta - 1) + \frac{n}{2} > -1 + \varepsilon > -1.$$

Then

$$W^{k+1}_{\delta-1} \subset C^{1}_{-1+\varepsilon}$$

$$\Rightarrow$$

$$F = O(r^{1-\varepsilon}).$$

Therefore the derivative $D\phi$ of ϕ (viewed as an $n \times n$ matrix of partial derivatives) is the identity matrix plus a matrix whose entries are in C^{0}_{ε}, hence are $O(\frac{1}{r^{\varepsilon}})$.]

<u>THEOREM</u> $D^{k+1}_{\delta-1}$ is closed under composition and inversion, thus is a group (in fact, a topological group). Moreover, $D^{k+1}_{\delta-1}$ operates continuously to the right on $W^{k'}_{\delta'}$ ($k' \leq k + 1$, $\delta' \in \underline{R}$) by pullback:

$$\begin{cases} W^{k'}_{\delta'} \times D^{k+1}_{\delta-1} \to W^{k'}_{\delta'} \\ \quad (f,\phi) \qquad \to \phi^{*}f \ (= f \circ \phi). \end{cases}$$

Terminology: A diffeomorphism $\phi:\underline{R}^{n} \to \underline{R}^{n}$ is called an <u>asymptotic symmetry</u> of class (k,δ) if $\phi \in D^{k+1}_{\delta-1}$.

<u>LEMMA</u> Let $\phi \in D^{k+1}_{\delta-1}$. Suppose that $T \in \mathcal{D}^{0}_{q}(\underline{R}^{n})$ $(q > 0)$ is in $W^{k'}_{\delta'}$ ($k' \leq k$, $\delta' \in \underline{R}$) -- then the same is true of $\phi^{*}T$.

[Take $q = 2$ and write

$$(\phi^{*}T)_{ij} = \sum_{a,b=1}^{n} \frac{\partial(x^{a}\circ\phi)}{\partial x^{i}} \frac{\partial(x^{b}\circ\phi)}{\partial x^{j}} T(\partial_{a},\partial_{b})\circ \phi$$

$$= \sum_{a,b=1}^{n} (\delta^a{}_i + F^a{}_i)(\delta^b{}_j + F^b{}_j) T_{ab} \circ \phi.$$

Here

$$\left[\begin{array}{l} F^a{}_i, F^b{}_j \in W^k_\delta \quad (\Rightarrow F^a{}_i F^b{}_j \in W^k{}_\delta) \\ \\ T_{ab} \circ \phi \in W^{k'}_{\delta'}. \end{array}\right.$$

But $k > \frac{n}{2}$, $\delta > -\frac{n}{2}$, and $k' \leq k$, which implies that the product of an element in W^k_δ with an element of $W^{k'}_{\delta'}$ is again in $W^{k'}_{\delta'}$.]

The definition of W^k_δ can be extended in the obvious way to "sufficiently regular" open subsets of $\underline{R}^n$, e.g., to

<u>exterior domains</u>:

$$\underline{E}_R = \{x: |x| > R\}$$

or

<u>annular domains</u>:

$$\underline{A}_R = \{x: R < |x| < 2R\}.$$

Suppose that $f \in W^k_\delta(\underline{E}_R)$ $(R \geq 1)$ — then for elementary reasons,

$$||f||_{W^k_\delta(\underline{A}_R)} \doteq R^{\delta + \frac{n}{2}} ||S_R f||_{H^k(\underline{A}_1)},$$

the implicit positive constants being independent of R,f, and where, as before $(S_R f)(x) = f(Rx)$:

<u>LEMMA</u> If $k > \frac{n}{2}$, then

$$\sup_{\underline{A}_R} \sigma^{\delta+\frac{n}{2}} |f| \leq K||f||_{W_\delta^k(\underline{A}_R)} \quad (R \geq 1).$$

[Applying the usual Sobolev inequality to $S_R f$ on $\underline{A}_1$, for $x \in \underline{A}_R$ we have

$$\begin{aligned}
\sigma^{\delta+\frac{n}{2}}(x)|f(x)| &\leq \sqrt{5}\, R^{\delta+\frac{n}{2}} |f(x)| \\
&\leq \sqrt{5}\, R^{\delta+\frac{n}{2}} \sup_{\underline{A}_R} |f(x)| \\
&= \sqrt{5}\, R^{\delta+\frac{n}{2}} \sup_{\underline{A}_1} |f(Rx)| \\
&\leq \sqrt{5}\, C(\underline{A}_1) R^{\delta+\frac{n}{2}} ||S_R f||_{H^k(\underline{A}_1)} \\
&\leq \sqrt{5}\, C(\underline{A}_1) C ||f||_{W_\delta^k(\underline{A}_R)} .]
\end{aligned}$$

[Note: $K > 0$ is independent of R, f.]

Let $f \in W_\delta^k$ and take $k > \frac{n}{2}$ -- then the estimate

$$\sigma^c |f| = o(1) \quad (c < \delta + \frac{n}{2})$$

can be sharpened to

$$\sigma^{\delta+\frac{n}{2}} |f| = o(1).$$

To see this, just note that

$$||f||_{W^k_\delta} < \infty \Rightarrow ||f||_{W^k_\delta(\underline{A}_R)} = o(1)$$

and then quote the lemma.

<u>Section 68</u>: <u>Asymptotically Euclidean Riemannian Structures</u> As in the previous section, it will be assumed that $n \geq 3$.

Definition: Let q be a riemannian structure on $\underline{R}^n$ -- then q is said to be <u>asymptotically euclidean</u> of class (k,δ) $(k > \frac{n}{2},\ \delta > -\frac{n}{2})$ if

$$q - \eta \in W_\delta^k.$$

[Note: Here η is the usual flat metric on $\underline{R}^n$.]

<u>SUBLEMMA</u> Let $\phi \in D_{\delta-1}^{k+1}$ -- then

$$\phi_* \eta - \eta \in W_\delta^k,$$

hence $\phi_* \eta$ is asymptotically euclidean of class (k,δ).

[We have

$$(\phi_* \eta)_{ij} = \sum_{a,b=1}^{n} \frac{\partial(x^a \circ \phi^{-1})}{\partial x^i} \frac{\partial(x^b \circ \phi^{-1})}{\partial x^j} \eta(\partial_a, \partial_b) \circ \phi^{-1}$$

$$= \sum_{a,b=1}^{n} \frac{\partial(x^a \circ \phi^{-1})}{\partial x^i} \frac{\partial(x^b \circ \phi^{-1})}{\partial x^j} \eta_{ab}$$

$$= \sum_{a=1}^{n} \frac{\partial(x^a \circ \phi^{-1})}{\partial x^i} \frac{\partial(x^a \circ \phi^{-1})}{\partial x^j}.$$

But

$$\phi^{-1} \in D_{\delta-1}^{k+1}$$

$$\Rightarrow$$

$$\phi^{-1} - I \in W_{\delta-1}^{k+1}$$

=>

$$x^a \circ \phi^{-1} - x^a \in W^{k+1}_{\delta-1}$$

=>

$$\frac{\partial(x^a \circ \phi^{-1})}{\partial x^b} = \delta^a{}_b + F^a{}_b,$$

where $F^a{}_b \in W^k_\delta$. Therefore

$$(\phi_* \eta)_{ij} = \sum_{a=1}^{n} (\delta^a{}_i + F^a{}_i)(\delta^a{}_j + F^a{}_j)$$

$$= \sum_{a=1}^{n} \delta^a{}_i \delta^b{}_j + F^i{}_j + F^j{}_i + \sum_{a=1}^{n} F^a{}_i F^a{}_j$$

$$= \eta_{ij} + F^i{}_j + F^j{}_i + \sum_{a=1}^{n} F^a{}_i F^a{}_j$$

=>

$$(\phi_* \eta)_{ij} - \eta_{ij} = F^i{}_j + F^j{}_i + \sum_{a=1}^{n} F^a{}_i F^a{}_j.$$

And the RHS of this equation is in W^k_δ.]

[Note: Recall that W^k_δ is closed under the formation of products $(k > \frac{n}{2}, \ \delta > -\frac{n}{2})$.]

<u>LEMMA</u> Suppose that q is asymptotically euclidean of class (k,δ) -- then $\forall \ \phi \in D^{k+1}_{\delta-1}$, $\phi_* q$ is asymptotically euclidean of class (k,δ).

[Bearing in mind that $q - \eta \in \mathcal{D}^0_2(\underline{R}^n)$ $(\Rightarrow \phi_*(q-\eta) \in W^k_\delta)$, one has only to write

$$\phi_* q - \eta = \phi_* q - \phi_* \eta + \phi_* \eta - \eta$$

$$= \phi_*(q-\eta) + \phi_*\eta - \eta.]$$

From this point on, it will be assumed that $n = 3$. Therefore the threshold values for (k,δ) are

$$\begin{cases} k > \frac{3}{2} \\ \delta > -\frac{3}{2}. \end{cases}$$

Obviously,

$$C_1^k \subset W_\delta^k \quad (-\frac{3}{2} < \delta < -\frac{1}{2}).$$

In particular:

$$C_1^k \subset W_{-1}^k.$$

On the other hand,

$$W_\delta^k \subset C_{\delta'}^{k-2} \quad (\delta' < \delta + \frac{3}{2}).$$

N.B.

$$f \in W_\delta^2 \Rightarrow r^{\delta + \frac{3}{2}}|f| = o(1).$$

So

$$f \in W_{-1}^2 \Rightarrow r^{\frac{1}{2}}|f| = o(1).$$

LEMMA Suppose that q is asymptotically euclidean of class (k,δ) -- then

$$q^{ij} - \eta_{ij} \in L_\delta^2.$$

The proof of this hinges on some preliminary considerations.

To begin with, we claim that

$$\det q - 1 \in W^k_\delta .$$

Thus write

$$\det q = \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix}$$

$$= q_{11} \begin{vmatrix} q_{22} & q_{23} \\ q_{32} & q_{33} \end{vmatrix} - q_{21} \begin{vmatrix} q_{12} & q_{13} \\ q_{32} & q_{33} \end{vmatrix} + q_{31} \begin{vmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{vmatrix} .$$

$$\bullet\ q_{11}(q_{22}q_{33} - q_{23}q_{32})$$

$$= (q_{11} - 1 + 1)((q_{22} - 1 + 1)(q_{33} - 1 + 1) - q_{23}q_{32})$$

$$= (q_{11} - 1 + 1)((q_{22} - 1)(q_{33} - 1)$$

$$+ (q_{22} - 1) + (q_{33} - 1) - q_{23}q_{32} + 1)$$

$$= (q_{11} - 1)(q_{22} - 1)(q_{33} - 1)$$

$$+ (q_{11} - 1)(q_{22} - 1) + (q_{11} - 1)(q_{33} - 1)$$

$$- (q_{11} - 1)(q_{23}q_{32}) + (q_{11} - 1)$$

$$+ (q_{22} - 1)(q_{33} - 1) + (q_{22} - 1) + (q_{33} - 1) - q_{23}q_{32}$$

$$+ 1.$$

- $- q_{21}(q_{12}q_{33} - q_{13}q_{32})$

$$= - q_{21}(q_{12}(q_{33} - 1 + 1) - q_{13}q_{32})$$

$$= - q_{21}(q_{12}(q_{33} - 1) + q_{12} - q_{13}q_{32}).$$

- $q_{31}(q_{12}q_{23} - q_{13}q_{22})$

$$= q_{31}(q_{12}q_{23} - q_{13}(q_{22} - 1 + 1))$$

$$= q_{31}(q_{12}q_{23} - q_{13}(q_{22} - 1) - q_{13}).$$

Now move the +1 to the other side and use the fact that W_δ^k is an algebra.

Since $\det q > 0$ and since $\det q - 1 \in W_\delta^k$, hence is $O(\frac{1}{r^\varepsilon})$ for some $\varepsilon > 0$, it follows that $\exists\ C_1 > 0,\ C_2 > 0$:

$$C_1 \leq \det q \leq C_2.$$

Observation:

$$(\det q)q^{ij} = \text{cof } q_{ij}$$

$$\Rightarrow$$

$$(\det q)q^{ij} - \eta_{ij} \in W_\delta^k.$$

With this preparation, the verification that

$$\int_{\underline{R}^3} \sigma^{2\delta} |q^{ij} - \eta_{ij}|^2 \, d^3x < \infty$$

is straightforward.

<u>$i \neq j$</u>: We have

$$\int_{\underline{R}^3} \sigma^{2\delta} |q^{ij}|^2 \, d^3x$$

$$= \int_{\underline{R}^3} \frac{1}{(\det q)^2} \sigma^{2\delta} |(\det q) q^{ij}|^2 \, d^3x$$

$$\leq \frac{1}{C_1^2} \int_{\underline{R}^3} \sigma^{2\delta} |(\det q) q^{ij}|^2 \, d^3x < \infty.$$

<u>$i = j$</u>: We have

$$\int_{\underline{R}^3} \sigma^{2\delta} |q^{ii} - 1|^2 \, d^3x$$

$$= \int_{\underline{R}^3} \frac{1}{(\det q)^2} \sigma^{2\delta} |(\det q)(q^{ii} - 1)|^2 \, d^3x$$

$$\leq \frac{1}{C_1^2} \int_{\underline{R}^3} \sigma^{2\delta} |(\det q)(q^{ii} - 1)|^2 \, d^3x.$$

But

$$(\det q)(q^{ii} - 1)$$

$$= (\det q) q^{ii} - 1 + (1 - \det q).$$

And both

$$\begin{cases} (\det q) q^{ii} - 1 \\ 1 - \det q \end{cases}$$

are in L_δ^2, hence so is their sum.

Notation: $Q_{AE}(k,\delta)$ is the set of asymtotically euclidean riemannian structures on $\underline{R}^3$ of class (k,δ) $(k \geq 2, \delta \geq -1)$.

[Note: Accordingly, $\forall$ (k,δ), $W^k_\delta \subset W^2_{-1}$ and

$$Q_{AE}(k,\delta) \subset Q_{AE}(2,-1).]$$

Remark: If q is asymptotically flat, then q is asymptotically euclidean of class (2,-1), i.e.,

$$Q_\infty \subset Q_{AE}(2,-1).$$

Let $q \in Q_{AE}(2,-1)$ — then q is said to satisfy the <u>integrability condition</u> if

$$\int_{\underline{R}^3} |S(q)| d^3x < \infty.$$

[Note: In view of the relation

$$\sqrt{C_1} \leq \sqrt{q} \leq \sqrt{C_2},$$

it is clear that q satisfies the integrability condition iff

$$\int_{\underline{R}^3} |S(q)| \sqrt{q}\, d^3x < \infty.]$$

To recast the integrability condition, write

$$S(q) = q^{j\ell}(\Gamma^i{}_{\ell j,i} - \Gamma^i{}_{ij,\ell} + \Gamma^a{}_{\ell j}\Gamma^i{}_{ia} - \Gamma^a{}_{ij}\Gamma^i{}_{\ell a}).$$

Since the q^{ab} are O(1) and the $q_{ab,c} \in W^1_0 \subset W^0_0 = L^2(\underline{R}^3)$, any product of the form

$$O(1) q_{ij,k} q_{i'j',k'}$$

is integrable. Therefore

$$q^{j\ell}(\Gamma^a{}_{\ell j}\Gamma^i{}_{ia} - \Gamma^a{}_{ij}\Gamma^i{}_{\ell a}) \in L^1(\underline{R}^3).$$

Next

$$q^{j\ell}(\Gamma^{i}{}_{\ell j,i} - \Gamma^{i}{}_{ij,\ell})$$

$$= \frac{1}{2} q^{j\ell}[(\partial_i q^{ik})(q_{k\ell,j} + q_{kj,\ell} - q_{\ell j,k})]$$

$$- \frac{1}{2} q^{j\ell}[(\partial_\ell q^{ik})(q_{ki,j} + q_{kj,i} - q_{ij,k})]$$

$$+ q^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell}).$$

The first and second terms are integrable. Indeed

$$\begin{cases} \partial_i q^{ik} = - q^{iu}q_{uv,i}q^{vk} \\ \partial_\ell q^{ik} = - q^{iu}q_{uv,\ell}q^{vk}, \end{cases}$$

so the preceding reasoning is applicable. The integrability of $S(q)$ is thus equivalent to the integrability of

$$q^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell}).$$

Now write

$$q^{ij}q^{k\ell}(q_{ik,j,\ell} - q_{ij,k,\ell})$$

$$= q_{i\ell,i,\ell} - q_{ii,\ell,\ell}$$

$$+ ((q^{ij} - \eta_{ij})q^{k\ell} + \eta_{ij}(q^{k\ell} - \eta_{k\ell}))(q_{ik,j,\ell} - q_{ij,k,\ell}).$$

We have

$$\left.\begin{array}{l} q_{ik,j,\ell} \\ q_{ij,k,\ell} \end{array}\right\} \in W_1^0 \equiv L_1^2.$$

On the other hand, thanks to the lemma above,

$$\left.\begin{array}{l} q^{ij} - \eta_{ij} \\ \\ q^{k\ell} - \eta_{k\ell} \end{array}\right] \in L^2_{-1} \ (\delta = -1).$$

But the product of an element in L^2_1 with an element in L^2_{-1} is integrable. And multiplying such a product by a term which is $O(1)$ does not affect integrability. Therefore

$$((q^{ij} - \eta_{ij})q^{k\ell} + \eta_{ij}(q^{k\ell} - \eta_{k\ell}))(q_{ik,j,\ell} - q_{ij,k,\ell})$$

is integrable.

Let

$$X = X^{\ell} \frac{\partial}{\partial x^{\ell}},$$

where

$$X^{\ell} = q_{i\ell,i} - q_{ii,\ell}.$$

Scholium:

$$S(q) \in L^1(\underline{R}^3) \iff \text{div } X \in L^1(\underline{R}^3).$$

Consequently, if q satisfies the integrability condition, then

$$\int_{\underline{R}^3} (\text{div } X) d^3x = \lim_{R \to \infty} \int_{\underline{D}^3(R)} (\text{div } X) d^3x$$

$$= \lim_{R \to \infty} \int_{\underline{S}^2(R)} (X \cdot \underline{n}) \omega^2_R.$$

I.e.:

$$\int_{\underline{S}^2(\infty)} (q_{i\ell,i} - q_{ii,\ell})\Omega_\infty^\ell$$

exists.

Remark: In the literature, it is sometimes asserted that S(q) is integrable iff

$$\int_{\underline{S}^2(\infty)} (q_{i\ell,i} - q_{ii,\ell})\Omega_\infty^\ell$$

exists, a statement which is patently false. The point, of course, is that an improper integral is not necessarily a Lebesgue integral, hence the mere existence of

$$\lim_{R \to \infty} \int_{\underline{S}^2(R)} (X\cdot\underline{n})\omega_R^2$$

$$= \lim_{R \to \infty} \int_{\underline{D}^3(R)} (\mathrm{div}\ X)d^3x$$

does not imply that div $X \in L^1(\underline{R}^3)$.

Let $Q_{AE}^*(k,\delta)$ stand for the subset of $Q_{AE}(k,\delta)$ consisting of those q which satisfy the integrability condition.

<u>LEMMA</u> Suppose that $q \in Q_{AE}^*(k,\delta)$ — then $\forall\ \phi \in D_{\delta-1}^{k+1}$, $\phi_* q \in Q_{AE}^*(k,\delta)$.

[Earlier considerations imply that $\phi_* q \in Q_{AE}(k,\delta)$. This said, let $\psi = \phi^{-1}$, thus $\psi^* = \phi_*$ and

$$S(\phi_* q) = S(\psi^* q) = S(q) \circ \psi.$$

Fix $C > 0$: $\det(D\psi) \geq C > 0$. Put $y = \psi(x)$ -- then $d^3y = \det(D\psi)d^3x$ and

$$\int_{\underline{R}^3} |S(q)(\psi(x))| d^3x$$

$$= \int_{\underline{R}^3} |S(q)(y)| \frac{1}{\det(D\psi)} d^3y$$

$$\leq \frac{1}{C} \int_{\underline{R}^3} |S(q)(y)| d^3y < \infty.]$$

Definition: The <u>energy</u> is the function

$$P^0 : Q^*_{AE}(k,\delta) \to \underline{R}$$

given by the prescription

$$P^0(q) = \int_{\underline{S}^2(\infty)} (q_{i\ell,i} - q_{ii,\ell})\Omega^{\ell}_{\infty}.$$

<u>N.B.</u> If the partial derivatives of q_{ij} are $O(\frac{1}{r^{2+\varepsilon}})$, then $P^0(q) = 0$.

Example: Let

$$q = u^4\eta,$$

where $u \in C^{\infty}_{>0}(\underline{R}^3)$ and for $r >> 0$,

$$u = 1 + \frac{A}{r} + \mu \quad (\mu = O^{\infty}(\frac{1}{r^2})).$$

Then q is asymptotically flat (hence $q \in Q_{AE}(2, -1)$) and

$$P^0(q) = 32\pi A.$$

To begin with:

1. $\partial_i(u^4\eta_{i1}) = \partial_1(u^4)$.

2. $\partial_i(u^4\eta_{i2}) = \partial_2(u^4)$.

3. $\partial_i(u^4\eta_{i3}) = \partial_3(u^4)$.

4. $-\ \partial_1(u^4\eta_{ii}) = -\ \partial_1(3u^4)$.

5. $-\ \partial_2(u^4\eta_{ii}) = -\ \partial_2(3u^4)$.

6. $-\ \partial_3(u^4\eta_{ii}) = -\ \partial_3(3u^4)$.

=>

$$\left|\begin{array}{l} 1 - 4 = -\ 2\partial_1 u^4 \\ 2 - 5 = -\ 2\partial_2 u^4. \\ 3 - 6 = -\ 2\partial_3 u^4. \end{array}\right.$$

But

$$(1 + \frac{A}{r} + \mu)^4$$

$$= (1 + \frac{A}{r})^4 + 4(1 + \frac{A}{r})^3\mu$$

$$+ 6(1 + \frac{A}{r})^2\mu^2 + 4(1 + \frac{A}{r})\mu^3 + \mu^4.$$

Therefore the only term that is relevant is

$$(1 + \frac{A}{r})^4 = 1 + \frac{4A}{r} + \frac{6A^2}{r^2} + \frac{4A^3}{r^3} + \frac{A^4}{r^4}\ .$$

However, of the terms on the RHS, only

$$\frac{4A}{r}$$

can contribute and we have

$$\partial_1(\tfrac{1}{r}) = -\frac{x^1}{r^3},\ \partial_2(\tfrac{1}{r}) = -\frac{x^2}{r^3},\ \partial_3(\tfrac{1}{r}) = -\frac{x^3}{r^3}.$$

Taking $R >> 0$, matters thus reduce to

$$8AR^2 \int_0^{2\pi} \int_0^{\pi} (\cos\theta \sin\phi(\frac{R\cos\theta\sin\phi}{R^3})$$

$$+ \sin\theta\sin\phi(\frac{R\sin\theta\sin\phi}{R^3}) + \cos\phi(\frac{R\cos\phi}{R^3}))\sin\phi\, d\phi\, d\theta$$

$$= 8A\cdot 4\pi = 32\pi A.$$

[Note: This is a legal computation. It does not depend on whether $S(q) \in L^1(\underline{R}^3)$ or, equivalently, whether div $X \in L^1(\underline{R}^3)$. In the case at hand,

$$q_{i\ell,i,\ell} - q_{ii,\ell,\ell}$$

$$= -2[\partial_1\partial_1 u^4 + \partial_2\partial_2 u^4 + \partial_3\partial_3 u^4]$$

and since $\mu = O^{\infty}(\frac{1}{r^2})$, it can be set equal to zero. The potential trouble then lies with the divergence of

$$(\frac{x^1}{r^3}, \frac{x^2}{r^3}, \frac{x^3}{r^3}),$$

there being no actual difficulty in that

$$\operatorname{div} \frac{\vec{r}}{r^3} = 0.$$

So, in this situation, $S(q) \in L^1(\underline{R}^3)$.

Exercise: Suppose that for $r >> 0$,

$$u = 1 + \frac{A}{r} + \sum_{i=1}^{3} B_i \frac{x^i}{r^3} + \mu \left(\mu = O^\infty\left(\frac{1}{r^3}\right)\right).$$

Then

$$B_i = \frac{3}{64\pi} \int_{\underline{S}^2(\infty)} x^i (q_{i\ell,i} - q_{ii,\ell}) \Omega_\infty^\ell.$$

Given $q \in Q_{AE}^*(k,\delta)$, let $\mathcal{O}_q$ be its orbit under the left action of $D_{\delta-1}^{k+1}$ by pushforward:

$$\mathcal{O}_q = \{\phi_* q : \phi \in D_{\delta-1}^{k+1}\}.$$

The lemma implies that P^0 is finite on $\mathcal{O}_q$. However, much more is true: P^0 is constant on $\mathcal{O}_q$.

<u>THEOREM</u> Let $q \in Q_{AE}^*(k,\delta)$ -- then $\forall\ \phi \in D_{\delta-1}^{k+1}$,

$$P^0(\phi_* q) = P^0(q).$$

The most difficult case is when $k = 2$, $\delta = -1$, so we'll concentrate on it.

<u>Estimation Principle</u> Fix $R_0 \geq 1$. Suppose that $f \in W_0^1(\underline{E}_{R_0})$ -- then

$$\int_{\underline{S}^2(R)} |f| d\Omega = o(R^{\frac{1}{2}}).$$

[Start with the fact that

$$||f||_{W_0^1(\underline{A}_R)} \doteq R^{\frac{3}{2}} ||S_R f||_{H^1(\underline{A}_1)} \quad (R \geq R_0).$$

Next, in view of the trace theorem from ordinary Sobolev theory (viz. that restriction to a compact hypersurface entails the loss of one half of a derivative),

$$S_R f \in H^1(\underline{A}_1) \Rightarrow S_R f|\underline{S}^2 \in H^{\frac{1}{2}}(\underline{S}^2),$$

with

$$||S_R f||_{H^{\frac{1}{2}}(\underline{S}^2)} \leq C||S_R f||_{H^1(\underline{A}_1)}.$$

On general grounds,

$$H^s(\underline{S}^2) \subset L^q(\underline{S}^2)$$

if

$$\frac{1}{q} = \frac{1}{2} - \frac{s}{2}.$$

In particular:

$$H^{\frac{1}{2}}(\underline{S}^2) \subset L^4(\underline{S}^2).$$

Therefore

$$\int_{\underline{S}^2(R)} |f| d\Omega = R^2 \int_{\underline{S}^2} |S_R f| d\Omega$$

$$= R^2 ||S_R f||_{L^1(\underline{S}^2)}$$

$$\leq CR^2 ||S_R f||_{L^4(\underline{S}^2)}$$

$$\leq CR^2 ||S_R f||_{H^{\frac{1}{2}}(\underline{S}^2)}$$

$$\leq CR^2 ||S_R f||_{H^1(\underline{A}_1)}$$

$$\leq CR^{\frac{1}{2}} ||f||_{W_0^1(A_R)}$$

$$\leq R^{\frac{1}{2}} o(1)$$

=>

$$\int_{\underline{S}^2(R)} |f| d\Omega = o(R^{\frac{1}{2}}).]$$

[Note: As usual in estimates of this type, C is a positive constant that can vary from line to line.]

Application: If $f \in W_0^1$ and $F \in W_{-1}^2$, then

$$\int_{\underline{S}^2(R)} |f| \cdot |F| d\Omega = o(1).$$

[Recall that

$$F \in W_{-1}^2 \Rightarrow F = o(R^{-\frac{1}{2}}),$$

so

$$R^{\frac{1}{2}} |F| \leq C.$$

But then

$$\int_{\underset{=}{S}^2(R)} |f|\cdot|F|d\Omega$$

$$= \int_{\underset{=}{S}^2(R)} R^{-\frac{1}{2}}|f|\cdot R^{\frac{1}{2}}|F|d\Omega$$

$$\leq CR^{-\frac{1}{2}} \int_{\underset{=}{S}^2(R)} |f|d\Omega$$

$$= CR^{-\frac{1}{2}} o(R^{\frac{1}{2}})$$

$$= o(1).]$$

Passing to the proof of the theorem, we shall begin with the special situation when $q = \eta$, the objective being to show that $P^0(\phi_*\eta) = 0$.

Let $y^a = x^a \circ \phi^{-1}$ — then

$$(\phi_*\eta)_{i\ell,i} - (\phi_*\eta)_{ii,\ell}$$

$$= \partial_i(y^a_{,i}y^a_{,\ell}) - \partial_\ell(y^a_{,i}y^a_{,i})$$

$$= y^a_{,i,i}y^a_{,\ell} + y^a_{,i}y^a_{,\ell,i} - y^a_{,i,\ell}y^a_{,i} - y^a_{,i}y^a_{,i,\ell}$$

$$= y^a_{,i,i}y^a_{,\ell} - y^a_{,i,\ell}y^a_{,i}$$

$$= \partial^2_{ii}y^a\partial_\ell y^a - \partial^2_{i\ell}y^a\partial_i y^a$$

$$= \partial^2_{ii}y^a(\partial_\ell y^a - \delta^a{}_\ell + \delta^a{}_\ell)$$

$$- \partial^2_{i\ell}y^a(\partial_i y^a - \delta^a{}_i + \delta^a{}_i)$$

$$= \partial^2_{ii}y^\ell - \partial^2_{i\ell}y^i$$

$$+ \partial^2_{ii}y^a(\partial_\ell y^a - \delta^a{}_\ell) - \partial^2_{i\ell}y^a(\partial_i y^a - \delta^a{}_i).$$

• $y^a - x^a \in W^3_{-2}$

$$\Rightarrow \quad y^a{}_{,b} - \delta^a{}_b \in W^2_{-1}$$

$$\Rightarrow \quad y^a{}_{,b,c} \in W^1_0.$$

Because of this, each of the terms

$$\left[\begin{array}{l} \partial^2_{ii}y^a(\partial_\ell y^a - \delta^a{}_\ell) \\ \\ \partial^2_{i\ell}y^a(\partial_i y^a - \delta^a{}_i) \end{array}\right.$$

has the form $f \cdot F$, where $f \in W^1_0$ and $F \in W^2_{-1}$, so their integrals over $\underline{S}^2(R)$ will not contribute when $R \to \infty$. We are thus left with

$$\partial^2_{ii}y^\ell - \partial^2_{i\ell}y^i.$$

Rappel: For any $X \in \mathcal{D}^1(\underline{R}^3)$,

$$\iota_X(dx^1 \wedge dx^2 \wedge dx^3)\,|\,\underline{S}^2(R) = \langle X, \underline{n} \rangle\, \omega_R^2.$$

Therefore

$$\begin{cases} \iota_{\frac{\partial}{\partial x^1}} (dx^1 \wedge dx^2 \wedge dx^3)\,|\,\underline{S}^2(R) = dx^2 \wedge dx^3\,|\,\underline{S}^2(R) = \frac{x^1}{R}\, \omega_R^2 \\ \iota_{\frac{\partial}{\partial x^2}} (dx^1 \wedge dx^2 \wedge dx^3)\,|\,\underline{S}^2(R) = -\, dx^1 \wedge dx^3\,|\,\underline{S}^2(R) = \frac{x^2}{R}\, \omega_R^2 \\ \iota_{\frac{\partial}{\partial x^3}} (dx^1 \wedge dx^2 \wedge dx^3)\,|\,\underline{S}^2(R) = dx^1 \wedge dx^2\,|\,\underline{S}^2(R) = \frac{x^3}{R}\, \omega_R^2. \end{cases}$$

But

$$\begin{cases} *dx^1 = dx^2 \wedge dx^3 \\ *dx^2 = -\, dx^1 \wedge dx^3 \\ *dx^3 = dx^1 \wedge dx^2. \end{cases}$$

Accordingly, in a mild abuse of notation,

$$\int_{\underline{D}^3(R)} \operatorname{div} X = \int_{\underline{S}^2(R)} (X^1 * dx^1 + X^2 * dx^2 + X^3 * dx^3).$$

The relation $P^0(\phi_* \eta) = 0$ then follows upon observing that

$$(\partial_{ii}^2 Y^\ell - \partial_{i\ell}^2 Y^i) * dx^\ell = d(\epsilon_{ik\ell}\, \partial_i Y^\ell dx^k).$$

<u>Details</u> To illustrate the procedure, note that the coefficient of $dx^1 \wedge dx^2$ on the LHS is

$$\partial_i(\partial_i y^3 - \partial_3 y^i)$$

or still,

$$\partial_1\partial_1 y^3 - \partial_1\partial_3 y^1 + \partial_2\partial_2 y^3 - \partial_2\partial_3 y^2 + \partial_3\partial_3 y^3 - \partial_3\partial_3 y^3$$

$$= \partial_1\partial_1 y^3 - \partial_1\partial_3 y^1 + \partial_2\partial_2 y^3 - \partial_2\partial_3 y^2.$$

As for the RHS, write

$$d(\varepsilon_{ik\ell}\partial_i y^\ell dx^k) = \varepsilon_{ik\ell}\partial_j\partial_i y^\ell dx^j \wedge dx^k$$

$$= \varepsilon_{ik1}\partial_j\partial_i y^1 dx^j \wedge dx^k$$

$$+ \varepsilon_{ik2}\partial_j\partial_i y^2 dx^j \wedge dx^k + \varepsilon_{ik3}\partial_j\partial_i y^3 dx^j \wedge dx^k$$

$$= \varepsilon_{i21}\partial_j\partial_i y^1 dx^j \wedge dx^2 + \varepsilon_{i31}\partial_j\partial_i y^1 dx^j \wedge dx^3$$

$$+ \varepsilon_{i12}\partial_j\partial_i y^2 dx^j \wedge dx^1 + \varepsilon_{i32}\partial_j\partial_i y^2 dx^j \wedge dx^3$$

$$+ \varepsilon_{i13}\partial_j\partial_i y^3 dx^j \wedge dx^1 + \varepsilon_{i23}\partial_j\partial_i y^3 dx^j \wedge dx^2.$$

Then the coefficient of $dx^1 \wedge dx^2$ is

$$\varepsilon_{i21}\partial_1\partial_i y^1 - \varepsilon_{i12}\partial_2\partial_i y^2 - \varepsilon_{i13}\partial_2\partial_i y^3 + \varepsilon_{i23}\partial_1\partial_i y^3$$

or still,

$$\varepsilon_{321}\partial_1\partial_3 y^1 - \varepsilon_{312}\partial_2\partial_3 y^2 - \varepsilon_{213}\partial_2\partial_2 y^3 + \varepsilon_{123}\partial_1\partial_1 y^3$$

or still,

$$-\partial_1\partial_3 y^1 - \partial_2\partial_3 y^2 + \partial_2\partial_2 y^3 + \partial_1\partial_1 y^3,$$

as desired.

Passing now to the general case, let again $\psi = \phi^{-1}$ -- then

$$(\psi^*q)_{i\ell,i} - (\psi^*q)_{ii,\ell}$$

$$= \partial_i(\partial_i y^a \partial_\ell y^b q_{ab} \circ \psi) - \partial_\ell(\partial_i y^a \partial_i y^b q_{ab} \circ \psi)$$

$$= \partial^2_{ii} y^a \partial_\ell y^b q_{ab} \circ \psi + \partial_i y^a \partial^2_{i\ell} y^b q_{ab} \circ \psi + \partial_i y^a \partial_\ell y^b \partial_i (q_{ab} \circ \psi)$$

$$- \partial^2_{\ell i} y^a \partial_i y^b q_{ab} \circ \psi - \partial_i y^a \partial^2_{\ell i} y^b q_{ab} \circ \psi - \partial_i y^a \partial_i y^b \partial_\ell (q_{ab} \circ \psi)$$

$$= \partial^2_{ii} y^a \partial_\ell y^b q_{ab} \circ \psi + \partial_i y^a \partial_\ell y^b \partial_i (q_{ab} \circ \psi)$$

$$- \partial^2_{\ell i} y^a \partial_i y^b q_{ab} \circ \psi - \partial_i y^a \partial_i y^b \partial_\ell (q_{ab} \circ \psi).$$

To discuss

$$\partial^2_{ii} y^a \partial_\ell y^b q_{ab} \circ \psi,$$

write

$$\partial_\ell y^b = \partial_\ell y^b - \delta^b{}_\ell + \delta^b{}_\ell.$$

Then

$$\partial^2_{ii}y^a\partial_\ell y^b q_{ab} \circ \psi$$

$$= \partial^2_{ii}y^a(\partial_\ell y^b - \delta^b{}_\ell)q_{ab} \circ \psi + \partial^2_{ii}y^a\delta^b{}_\ell q_{ab} \circ \psi$$

$$= \partial^2_{ii}y^a(\partial_\ell y^b - \delta^b{}_\ell)q_{ab} \circ \psi + \partial^2_{ii}y^a q_{a\ell} \circ \psi.$$

Since $q \in Q_{AE}(2, -1)$, we have

$$q_{ab} = \eta_{ab} + F_{ab},$$

where $F_{ab} \in W^2_{-1}$, hence

$$q_{ab} \circ \psi = \eta_{ab} + F_{ab} \circ \psi.$$

And still, $F_{ab} \circ \psi \in W^2_{-1}$. Therefore

$$\partial^2_{ii}y^a(\partial_\ell y^b - \delta^b{}_\ell)q_{ab} \circ \psi$$

$$= \partial^2_{ii}y^a(\partial_\ell y^b - \delta^b{}_\ell)\eta_{ab}$$

$$+ \partial^2_{ii}y^a(\partial_\ell y^b - \delta^b{}_\ell)F_{ab} \circ \psi.$$

Recalling that W^2_{-1} is closed under the formation of products, the upshot is that the integral of

$$\partial^2_{ii}y^a(\partial_\ell y^b - \delta^b{}_\ell)q_{ab} \circ \psi$$

over $\underline{S}^2(R)$ is $o(1)$. There remains

$$\partial^2_{ii}y^a q_{a\ell} \circ \psi$$

or, equivalently,

$$\partial^2_{ii}y^a(q_{a\ell} \circ \psi - \eta_{a\ell}) + \partial^2_{ii}y^a(q_{a\ell} \circ \psi).$$

But

$$\partial^2_{ii}y^a(q_{a\ell} \circ \psi - \eta_{a\ell})$$

is ignorable, leaving

$$\partial^2_{ii}y^\ell.$$

[Note: Analogously,

$$- \partial^2_{\ell i}y^a \partial_i y^b q_{ab} \circ \psi$$

provides the contribution

$$- \partial^2_{i\ell}y^i.]$$

To discuss

$$\partial_i y^a \partial_\ell y^b \partial_i (q_{ab} \circ \psi),$$

write

$$\begin{cases} \partial_i y^a = \partial_i y^a - \delta^a{}_i + \delta^a{}_i \\ \partial_\ell y^b = \partial_\ell y^b - \delta^b{}_\ell + \delta^b{}_\ell. \end{cases}$$

Then

$$\partial_i y^a \partial_\ell y^b \partial_i (q_{ab} \circ \psi)$$

$$= \partial_i y^a \partial_\ell y^b \partial_i (q_{ab} \circ \psi - \eta_{ab} + \eta_{ab})$$

$$= \partial_i y^a \partial_\ell y^b \partial_i (q_{ab} \circ \psi - \eta_{ab})$$

$$= (\partial_i y^a - \delta^a{}_i)(\partial_\ell y^b - \delta^b{}_\ell)\partial_i (q_{ab} \circ \psi - \eta_{ab})$$

$$+ (\partial_i y^a - \delta^a{}_i)\delta^b{}_\ell \partial_i (q_{ab} \circ \psi - \eta_{ab}) + \delta^a{}_i (\partial_\ell y^b - \delta^b{}_\ell)\partial_i (q_{ab} \circ \psi - \eta_{ab})$$

$$+ \delta^a{}_i \delta^b{}_\ell \partial_i (q_{ab} \circ \psi - \eta_{ab}).$$

The terms on the first and second line can, for the usual reasons, be set aside. In this connection, bear in mind that

$$q_{ab} \circ \psi - \eta_{ab} \in W^2_{-1}$$

$$\Rightarrow$$

$$\partial_i (q_{ab} \circ \psi - \eta_{ab}) \in W^1_0.$$

It remains to deal with

$$\delta^a{}_i \delta^b{}_\ell \partial_i (q_{ab} \circ \psi - \eta_{ab})$$

$$= \partial_i (q_{i\ell} \circ \psi - \eta_{i\ell})$$

$$= \partial_i (q_{i\ell} \circ \psi)$$

or still,

$$\partial_i (q_{i\ell} \circ (\psi - I + I))$$

$$= \partial_i(q_{i\ell} \circ (\psi - I)) + \partial_i q_{i\ell}.$$

But

$$\psi - I \in W^3_{-2}$$

$$\Rightarrow$$

$$D\psi - [I] \in W^2_{-1}.$$

So, by the chain rule,

$$\partial_i(q_{i\ell} \circ (\psi - I))$$

is a sum of terms of the form $f \cdot F$ ($f \in W^1_0$, $F \in W^2_{-1}$), thus is ignorable. All that is left, then, is

$$\partial_i q_{i\ell} (\equiv q_{i\ell,i}).$$

[Note: Analogously,

$$- \partial_i y^a \partial_i y^b \partial_\ell (q_{ab} \circ \psi)$$

provides the contribution

$$- \partial_\ell q_{ii} (\equiv - q_{ii,\ell}).]$$

Summary:

$$(\phi_* q)_{i\ell,i} - (\phi_* q)_{ii,\ell}$$

can be written in the form

$$\Phi_\ell + q_{i\ell,i} - q_{ii,\ell} + \partial^2_{ii} y^\ell - \partial^2_{i\ell} y^i,$$

where

$$\int_{\underline{S}^2(R)} |\Phi_\ell| d\Omega = o(1).$$

It was shown above that

$$(\partial^2_{ii}y^\ell - \partial^2_{i\ell}y^i)*dx^\ell = d(\varepsilon_{ik\ell}\partial_i y^\ell dx^k).$$

Therefore

$$\int_{\underline{S}^2(\infty)} ((\phi_* q)_{i\ell,i} - (\phi_* q)_{ii,\ell})\Omega^\ell_\infty = \int_{\underline{S}^2(\infty)} (q_{i\ell,i} - q_{ii,\ell})\Omega^\ell_\infty.$$

I.e.:

$$P^0(\phi_* q) = P^0(q),$$

the contention of the theorem.

Remark: The invariance of the energy definitely depends on the assumption that the diffeomorphism ϕ is an element of $D^{k+1}_{\delta-1}$. To see this, fix constants $C \geq 0$, $\alpha > 0$ and let

$$f(t) = t + Ct^{1-\alpha} \quad (t >> 0).$$

Working in a neighborhood of infinity, put $\rho = f^{-1}(r)$ ($\Rightarrow r = f(\rho)$) and take $y^a = \frac{\rho}{r} x^a$ ($\Rightarrow x^a = y^a(1 + \frac{C}{\rho^\alpha})$) — then it can be shown that

$$P^0(\phi_* \eta) = 16\pi \times \begin{cases} \infty & (\alpha < \frac{1}{2}) \\ C^2/8 & (\alpha = \frac{1}{2}) \\ 0 & (\alpha > \frac{1}{2}). \end{cases}$$

Section 69: Laplacians Continuing to work in $\underline{R}^3$, in this section we shall formulate a few background results from elliptic theory and illustrate their use by deriving some consequences which will play a role later on.

Criterion Assume:

- $\Phi \in C^\infty(I)$, where $I \subset \underline{R}$ is an open interval (possibly infinite).
- $f \in W_\delta^k$ $(k > \frac{3}{2}, \delta > -\frac{3}{2})$ with $[\inf f, \sup f] \subset I$.

Let $0 \leq k' \leq k$, $\delta' \in \underline{R}$ -- then

$$f' \in W_{\delta'}^{k'} \Rightarrow \Phi(f) f' \in W_{\delta'}^{k'}.$$

Rappel: Suppose that q is asymptotically euclidean of class (k,δ) — then

$$q^{ij} - \eta_{ij} \in L_\delta^2.$$

More is true:

$$q^{ij} - \eta_{ij} \in W_\delta^k.$$

[Note: It was shown in the last section that

$$(\det q) q^{ij} - \eta_{ij} \in W_\delta^k.]$$

To see this, take $\Phi(x) = \frac{1}{x+1}$ $(x > -1)$. Since $\det q - 1 \in W_\delta^k$ and since

$$[\inf(\det q - 1), \sup(\det q - 1)] \subset]-1,\infty[,$$

it makes sense to form

$$\Phi(\det q - 1) = \frac{1}{\det q}.$$

Accordingly, $\forall f \in W_\delta^k$,

$$\frac{1}{\det q} \cdot f \in W_\delta^k.$$

$\underline{i \neq j}$: $(\det q)q^{ij} \in W_\delta^k$

=>

$$q^{ij} = \frac{1}{\det q} \cdot ((\det q)q^{ij}) \in W_\delta^k.$$

$\underline{i = j}$: $(\det q)(q^{ii} - 1) \in W_\delta^k$

=>

$$q^{ii} - 1 = \frac{1}{\det q} \cdot (\det q)(q^{ii} - 1) \in W_\delta^k.$$

Consider the laplacian Δ corresponding to η -- then it is clear that

$$\Delta : W_\delta^k \to W_{\delta+2}^{k-2}.$$

Now let $q \in Q_{AE}(k,\delta)$ and consider

$$\Delta_q = \frac{1}{\sqrt{q}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} (\sqrt{q}\, q^{ij} \frac{\partial}{\partial x^j}).$$

Then it is still the case that

$$\Delta_q : W_\delta^k \to W_{\delta+2}^{k-2}.$$

<u>Details</u> First

$$f \in W_\delta^k \Rightarrow \frac{\partial f}{\partial x^j} \in W_{\delta+1}^{k-1}.$$

And

$$q^{ij} \frac{\partial f}{\partial x^j} = (q^{ij} - \eta_{ij} + \eta_{ij}) \frac{\partial f}{\partial x^j}$$

$$= (q^{ij} - \eta_{ij}) \frac{\partial f}{\partial x^j} + \eta_{ij} \frac{\partial f}{\partial x^j}$$

is also in $W^{k-1}_{\delta+1}$:

$$\left[\begin{array}{l} k-1 < k - \frac{3}{2} + k-1 \\ \\ \delta+1 < \delta + \frac{3}{2} + \delta+1. \end{array}\right.$$

Next take $\Phi(x) = \sqrt{1+x}$ $(x > -1)$, hence

$$\Phi(\det q - 1) = \sqrt{\det q}$$

$$\Rightarrow$$

$$\sqrt{q}\, q^{ij} \frac{\partial f}{\partial x^j} \in W^{k-1}_{\delta+1}$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x^i} (\sqrt{q}\, q^{ij} \frac{\partial f}{\partial x^j}) \in W^{k-2}_{\delta+2}.$$

Finally choose $\Phi(x) = \frac{1}{\sqrt{1+x}}$ $(x > -1)$ to get

$$\frac{1}{\sqrt{q}} \frac{\partial}{\partial x^i} (\sqrt{q}\, q^{ij} \frac{\partial f}{\partial x^j}) \in W^{k-2}_{\delta+2}.$$

N.B. By the same argument, Δ_q induces a map

$$W^k_{\delta'} \to W^{k-2}_{\delta'+2} \quad (\delta' \in \underline{R}).$$

THEOREM Let $q \in Q_{AE}(k,\delta)$. Suppose that $-\frac{3}{2} < \delta' < -\frac{1}{2}$ — then

$$\Delta_q:W^k_{\delta'} \to W^{k-2}_{\delta'+2}$$

is an isomorphism.

[Note: Take $\delta' = -1$ to get that

$$\Delta_q:W^k_{-1} \to W^{k-2}_1$$

is an isomorphism, in particular that

$$\Delta_q:W^2_{-1} \to W^0_1$$

is an isomorphism provided $q \in Q_{AE}(2, -1)$.]

Let E and F be Hilbert spaces — then a bounded linear transformation $T:E \to F$ is said to be Fredholm if Ker T is finite dimensional, Ran T is closed, and Co Ker T = F/Ran T is finite dimensional.

[Note: T is semi-Fredholm if Ker T is finite dimensional and Ran T is closed.]

If $T:E \to F$ is Fredholm, then its index is

$$\text{ind } T = \dim \text{Ker } T - \dim \text{Co Ker } T.$$

Example: The operator

$$\Delta:W^2_{-3/2} \to W^0_{1/2}$$

has a trivial kernel (a bounded harmonic function is a constant and the constants do not belong to $W^2_{-3/2}$). Still, its range is not closed, so Δ is not Fredholm.

Let $V \in W^{k-2}_{\delta+2}$ — then V determines an arrow

$$W^k_{\delta'} \to W^{k-2}_{\delta'+2},$$

viz. $f \to Vf$.

Convention: Henceforth, it will be assumed that V is, in addition, C^∞.

THEOREM Let $q \in Q_{AE}(k,\delta)$. Suppose that $-\frac{3}{2} < \delta' < -\frac{1}{2}$ — then

$$\Delta_q - V: W^k_{\delta'} \to W^{k-2}_{\delta'+2}$$

is Fredholm with index 0 and is an isomorphism if $V \geq 0$.

There are various results that go under the name "maximum principle". Here are two, tailored to our specific situation.

Strong Maximum Principle If f is a nonnegative C^∞ function such that

$$\Delta_q f - Vf \leq 0$$

and if $f(x_0) = 0$ at some $x_0 \in \underline{R}^3$, then f vanishes identically.

[Note: There is no sign restriction on V.]

Weak Maximum Principle If $f = o(1)$ is a C^∞ function such that

$$\Delta_q f - Vf \geq 0$$

and if $V \geq 0$, then $f \leq 0$.

SUBLEMMA If $q \in Q_{AE}(k,\delta)$, then $S(q) \in W^{k-2}_{\delta+2}$.

[For, as was shown above,

$$q^{ij} - \eta_{ij} \in W^k_\delta.$$

And the product of two elements in $W^{k-1}_{\delta+1}$ lies in $W^{k-2}_{\delta+2}$:

$$\left[\begin{array}{l} k - 2 < k - 1 + k - 1 - \frac{n}{2} \\ \\ \delta + 2 < \delta + 1 + \delta + 1 + \frac{n}{2}.] \end{array}\right.$$

Definition: Let $q \in Q_{AE}(k,\delta)$ — then the operator

$$\Delta_q - \frac{1}{8} S(q)$$

is called the conformal laplacian attached to q.

Conformal Replacement Principle Let $q \in Q_{AE}(k,\delta)$ $(-1 \leq \delta < -\frac{1}{2})$. Assume: $S(q) \geq 0$ — then $\exists$ $\chi \in C^{\infty}_{>0}(\underline{R}^3)$ subject to $\chi - 1 \in W^k_\delta$ such that

$$S(\chi^4 q) = 0.$$

[Viewing χ as the unknown, put $q' = \chi^4 q$. The rule for the change of scalar curvature under a conformal transformation then gives:

$$\chi^5 S(q') = - 8\Delta_q\chi + S(q)\chi$$

or still,

$$\chi^5 S(q') = - 8(\Delta_q\chi - \frac{1}{8} S(q)\chi).$$

Since $S(q) \geq 0$ and belongs to $W^{k-2}_{\delta+2}$, the conformal laplacian

$$\Delta_q - \frac{1}{8} S(q) : W^k_\delta \to W^{k-2}_{\delta+2}$$

is an isomorphism, thus there exists a unique $\bar{\chi} \in W^k_\delta$ such that

$$(\Delta_q - \frac{1}{8} S(q))\bar{\chi} = \frac{1}{8} S(q).$$

Define χ by $\chi - 1 = \bar{\chi}$ — then

$$\Delta_q \chi - \frac{1}{8} S(q)\chi$$

$$= \Delta_q \bar{\chi} - \frac{1}{8} S(q)(\bar{\chi} + 1)$$

$$= \frac{1}{8} S(q) - \frac{1}{8} S(q)$$

$$= 0.$$

Elliptic regularity implies that χ is C^∞, so it remains to show that $\chi > 0$. To this end, let $0 \le a \le 1$ and determine $\bar{\chi}_a \in W_\delta^k$ via

$$(\Delta_q - \frac{1}{8} S(q))\bar{\chi}_a = \frac{a}{8} S(q).$$

Put $\chi_a = 1 + \bar{\chi}_a$ and let $I = \{a:\chi_a > 0\}$ — then I is not empty ($\chi_0 = 1$). On the other hand,

$$\{f \in C_\varepsilon^0 : f > -1\}$$

is open in C_ε^0 and the map $a \rightarrow \chi_a \in C_\varepsilon^0$ is continuous ($\varepsilon > 0$ & $\varepsilon < \frac{1}{2} \le \delta + \frac{3}{2}$). Therefore I is open. But I is also closed. For $a_0 \in \bar{I} \Rightarrow \chi_{a_0} \ge 0$. However, $\chi_{a_0} \rightarrow 1$ at infinity, so, thanks to the strong maximum principle, $\chi_{a_0} > 0$. I.e.: $a_0 \in I$. Consequently, $I = [0,1]$, hence $\chi_1 = \chi > 0$.]

Remark: $\exists$ $C > 0$ such that

$$C \le \chi \le 1.$$

• $\underline{C \le \chi}$: Choose $R >> 0: \chi \ge \frac{1}{2}$ in $\underline{R}^3 - \underline{D}^3(R)$. As for the restriction $\chi|\underline{D}^3(R)$, it is positive, thus by compactness, $\exists\ c > 0$:

$$\chi \in \underline{D}^3(R) \Rightarrow \chi(x) \ge c.$$

Take $C = \min(\frac{1}{2}, c)$.

• $\underline{\chi \le 1}$: Since $\bar{\chi} = o(1)$ and since

$$(\Delta_q - \frac{1}{8} S(q))\bar{\chi} = \frac{1}{8} S(q) \ge 0,$$

the weak maximum principle implies that $\bar{\chi} \le 0$ or, equivalently, that $\chi \le 1$.

<u>LEMMA</u> Let $q \in Q_{AE}(k,\delta)$. Suppose that $\chi \in C^{\infty}_{>0}(\underline{R}^3)$ subject to $\chi - 1 \in W^k_\delta$ -- then

$$\chi^4 q \in Q_{AE}(k,\delta).$$

[Write

$$\chi^4 q - \eta = \chi^4(q-\eta) + (\chi^4 - 1)\eta.$$

Let $I = \underline{R}$ and $\Phi(x) = (1+x)^4$ -- then $\Phi(\chi-1) = \chi^4$, so, in view of the criterion,

$$\chi^4(q-\eta) \in W^k_\delta.$$

And

$$\Phi(x) - 1 = \Psi(x)x \quad (\Psi \in C^{\infty}(\underline{R}))$$

$$\Rightarrow$$

$$\Phi(\chi-1) - 1 = \Psi(\chi-1)(\chi-1) \in W^k_\delta.$$

I.e.: $\chi^4 - 1 \in W^k_\delta$.]

In particular: Replacing q by $q' = \chi^4 q$ in the conformal replacement principle does not take one outside of $Q_{AE}(k,\delta)$ $(\chi - 1 \in W^k_\delta,\ -1 \le \delta < -\frac{1}{2})$.

<u>LEMMA</u> Suppose that $q \in Q^*_{AE}(k,\delta)$ with $S(q) \geq 0$ — then $q' \in Q^*_{AE}(k,\delta)$ and

$$P^0(q') = P^0(q) - 8 \lim_{R \to \infty} \int_{\underline{D}^3(R)} \Delta_q \chi \operatorname{vol}_q.$$

Trivially, $q' \in Q^*_{AE}(k,\delta)$ $(S(q') = 0)$. This said, to fix the ideas let $k = 2$, $\delta = -1$.

Rappel: If $f \in W^1_0$ and $F \in W^2_{-1}$, then

$$\int_{\underline{S}^2(R)} |f| \cdot |F| d\Omega = o(1).$$

We have

$$q'_{i\ell,i} - q'_{ii,\ell} = (\partial_i \chi^4) q_{i\ell} - (\partial_\ell \chi^4) q_{ii}$$

$$+ \chi^4 (q_{i\ell,i} - q_{ii,\ell}).$$

- $\chi^4 (q_{i\ell,i} - q_{ii,\ell})$

$$= (\chi^4 - 1)(q_{i\ell,i} - q_{ii,\ell}) + q_{i\ell,i} - q_{ii,\ell}.$$

Since $\chi^4 - 1 \in W^2_{-1}$ and

$$\left.\begin{array}{l} q_{i\ell} - \eta_{i\ell} \\ \\ q_{ii} - \eta_{ii} \end{array}\right] \in W^2_{-1}$$

$\Rightarrow$

$$\left.\begin{array}{l} q_{i\ell,i} \\ \\ q_{ii,\ell} \end{array}\right] \in W^1_0,$$

it follows that

$$(\chi^4 -1)(q_{i\ell,i} - q_{ii,\ell})$$

can be ignored.

- $(\partial_i\chi^4)q_{i\ell} - (\partial_\ell\chi^4)q_{ii}$

$$= 4\chi^3(\partial_i\chi)q_{i\ell} - 4\chi^3(\partial_\ell\chi)q_{ii}$$

$$= 4\chi^3(\partial_i\chi)(q_{i\ell} - \eta_{i\ell}) - 4\chi^3(\partial_\ell\chi)(q_{ii} - \eta_{ii})$$

$$+ 4\chi^3(\partial_i\chi)\eta_{i\ell} - 4\chi^3(\partial_\ell\chi)\eta_{ii}.$$

The products

$$\begin{cases} 4\chi^3(q_{i\ell} - \eta_{i\ell}) \\ 4\chi^3(q_{ii} - \eta_{ii}) \end{cases}$$

are in W^2_{-1}. On the other hand,

$$\partial_i\chi, \partial_\ell\chi \in W^1_0.$$

Therefore

$$4\chi^3(\partial_i\chi)(q_{i\ell} - \eta_{i\ell}) - 4\chi^3(\partial_\ell\chi)(q_{ii} - \eta_{ii})$$

will not contribute. Write

$$4\chi^3(\partial_i\chi)\eta_{i\ell} - 4\chi^3(\partial_\ell\chi)\eta_{ii}$$

$$= 4(\chi^3 -1)(\partial_i\chi)\eta_{i\ell} - 4(\chi^3 -1)(\partial_\ell\chi)\eta_{ii}$$

$$+ 4((\partial_i\chi)\eta_{i\ell} - (\partial_\ell\chi)\eta_{ii}).$$

Then

$$4(\chi^3 -1)(\partial_i\chi)\eta_{i\ell} - 4(\chi^3 -1)(\partial_\ell\chi)\eta_{ii}$$

will not contribute, leaving

$$4((\partial_i\chi)\eta_{i\ell} - (\partial_\ell\chi)\eta_{ii})$$

or still,

$$4(\partial_\ell\chi - 3(\partial_\ell\chi))$$

$$= - 8\partial_\ell\chi.$$

Summary:

$$\int_{\underline{S}^2(R)} (q'_{i\ell,i} - q'_{ii,\ell})\Omega_R^\ell$$

$$= \int_{\underline{S}^2(R)} (q_{i\ell,i} - q_{ii,\ell})\Omega_R^\ell$$

$$- 8 \int_{\underline{S}^2(R)} (\partial_\ell\chi)\Omega_R^\ell + o(1).$$

<u>SUBLEMMA</u> $\sqrt{q} -1 \in W_{-1}^2$.

[Let $I = \{x:x > -1\}$ and write

$$\sqrt{1+x} -1 = \psi(x)x \quad (\psi \in C^\infty(I)).$$

Bearing in mind that $\det q -1 \in W_{-1}^2$, the criterion formulated at the beginning then implies that

$$\sqrt{q} - 1 = \sqrt{1 + (\det q - 1)} - 1$$

$$= \Psi(\det q - 1)(\det q - 1)$$

is in W^2_{-1}.]

Consequently,

$$- 8 \int_{\underline{S}^2(R)} (\partial_\ell \chi)\Omega^\ell_R$$

$$= - 8 \int_{\underline{S}^2(R)} (\sqrt{q}\ \partial_\ell \chi)\Omega^\ell_R + 8 \int_{\underline{S}^2(R)} ((\sqrt{q} - 1)\partial_\ell \chi)\Omega^\ell_R$$

$$= - 8 \int_{\underline{S}^2(R)} (\sqrt{q}\ \partial_\ell \chi)\Omega^\ell_R + o(1).$$

But

$$(\partial_k \chi) q^{\ell k} = \partial_k \chi (q^{\ell k} - \eta_{\ell k} + \eta_{\ell k})$$

$$= \partial_k \chi (q^{\ell k} - \eta_{\ell k}) + (\partial_k \chi)\eta_{\ell k}$$

$$= \partial_k \chi (q^{\ell k} - \eta_{\ell k}) + \partial_\ell \chi.$$

And

$$q^{\ell k} - \eta_{\ell k} \in W^2_{-1}$$

$$\Rightarrow$$

$$\sqrt{q}\ (q^{\ell k} - \eta_{\ell k}) \in W^2_{-1}.$$

Therefore

$$- 8 \int_{\underline{S}^2(R)} (\sqrt{q}\ \partial_\ell \chi)\Omega^\ell_R$$

$$= -8 \int_{\underline{S}^2(R)} (\sqrt{q}\, q^{\ell k} \frac{\partial \chi}{\partial x^k}) \Omega_R^\ell + o(1)$$

$$= -8 \int_{\underline{D}^3(R)} \frac{\partial}{\partial x^\ell} (\sqrt{q}\, q^{\ell k} \frac{\partial \chi}{\partial x^k}) d^3x + o(1)$$

$$= -8 \int_{\underline{D}^3(R)} \frac{1}{\sqrt{q}} \frac{\partial}{\partial x^\ell} (\sqrt{q}\, q^{\ell k} \frac{\partial \chi}{\partial x^k}) \sqrt{q}\, d^3x + o(1)$$

$$= -8 \int_{\underline{D}^3(R)} \Delta_q \chi \, \mathrm{vol}_q + o(1).$$

Now let $R \to \infty$ to get:

$$P^0(q') = P^0(q) - 8 \lim_{R \to \infty} \int_{\underline{D}^3(R)} \Delta_q \chi \, \mathrm{vol}_q.$$

[Note: It is not claimed that $\Delta_q \chi$ is integrable.]

<u>Energy Reduction Principle</u> This is the assertion that

$$P^0(q') \le P^0(q).$$

In fact,

$$S(q) \ge 0 \ \& \ \chi > 0$$

$\Rightarrow$

$$\Delta_q \chi = \frac{1}{8} S(q) \chi \ge 0$$

$\Rightarrow$

$$\int_{\underline{D}^3(R)} \Delta_q \chi \, \mathrm{vol}_q \ge 0.$$

One can then quote the lemma.

Section 70: Positive Energy Retain the assumptions and notation of the preceding section.

THEOREM Let $q \in Q^*_{AE}(4,\delta)$ $(-1 \leq \delta < -\frac{1}{2})$. Assume: $S(q) \geq 0$ — then $P^0(q) \geq 0$.

While we are not yet in a position to establish this result, in view of the energy reduction principle, to prove that $P^0(q) \geq 0$, it suffices to prove that $P^0(q') \geq 0$.

This said, replace q' by q (so now $S(q) = 0$). Fix a one parameter family of C^∞ cutoff functions ψ_θ $(\theta > 0)$ satisfying the following conditions.

1. $0 \leq \psi_\theta \leq 1$.
2. $\psi_\theta(x)$ depends only on $|x|$ and is a decreasing function of $|x|$.
3. $\psi_\theta(x) = 1$ if $|x| \leq \theta$.
4. $\psi_\theta(x) = 0$ if $|x| \geq 2\theta$.
5. $\exists\ C > 0 : \forall\ \theta$,

$$\theta|\psi_\theta'| + \theta^2|\psi_\theta''| \leq C.$$

Put

$$q_\theta = \psi_\theta q + (1 - \psi_\theta)\eta.$$

Then it is clear that

$$S(q_\theta) = \begin{cases} 0 & (r \leq \theta) \\ & \\ 0 & (r \geq 2\theta). \end{cases}$$

[Note: From the definitions,

$$S(q_\theta) = O(|x|^{-\frac{5}{2}})$$

for $\theta \leq |x| \leq 2\theta$ uniformly in $\theta >> 0$.]

LEMMA We have

$$[\int_{\underline{R}^3} |S(q_\theta)|^{3/2} \sqrt{q_\theta}\, d^3x]^{2/3} = O(\theta^{-\frac{1}{2}}).$$

[Note: The implied constant on the right is independent of θ.]

There is no guarantee that $S(q_\theta)$ is nonnegative, hence the conformal replacement principle is not applicable a priori. Still, as will be shown below, for all $\theta >> 0$, $\exists\ \chi_\theta \in C_{>0}^\infty(\underline{R}^3)$ subject to $\chi_\theta - 1 \in W_\delta^4$ $(-1 \leq \delta < -\frac{1}{2})$ such that

$$S(\chi_\theta^4 q_\theta) = 0.$$

Rappel: The conformal laplacian

$$\Delta_{q_\theta} - \frac{1}{8} S(q_\theta) : W_\delta^4 \to W_{\delta+2}^2 \quad (-1 \leq \delta < -\frac{1}{2})$$

is Fredholm with index 0.

So, to conclude that

$$\Delta_{q_\theta} - \frac{1}{8} S(q_\theta)$$

is an isomorphism, one has only to show that

$$\Delta_{q_\theta} - \frac{1}{8} S(q_\theta)$$

is injective $(\theta >> 0)$.

<u>N.B.</u> Granted this, the existence of χ_θ is then immediate (argue as in the conformal replacement principle).

There are a couple of technicalities that have to be taken care of first.

<u>Integration by Parts</u> Let $q \in Q_{AE}(k,\delta)$ $(\delta \geq -1)$. Suppose that $u, v \in W_\delta^k$ — then

$$\int_{\underline{R}^3} q(\text{grad}_q\, u, \text{grad}_q\, v)\sqrt{q}\, d^3x = -\int_{\underline{R}^3} u(\Delta_q v)\sqrt{q}\, d^3x.$$

Notation: Put

$$\nabla_q f = \text{grad}_q\, f \text{ and } |\nabla_q f|^2 = q(\nabla_q f, \nabla_q f).$$

<u>Sobolev Inequality</u> Let $q \in Q_{AE}(k,\delta)$ $(\delta \geq -1)$. Suppose that $f \in W_\delta^k$ — then

$$\left[\int_{\underline{R}^3} |f|^6 \sqrt{q}\, d^3x\right]^{1/3} \leq C_q \int_{\underline{R}^3} |\nabla_q f|^2 \sqrt{q}\, d^3x.$$

[Note: The positive constant C_q is independent of f and the C_{q_θ} are uniform in $\theta: C_{q_\theta} < C_0$ $(\forall\ \theta >> 0)$.]

Turning now to the injectivity of

$$\Delta_{q_\theta} - \frac{1}{8} S(q_\theta),$$

fix $\theta_0 : \theta > \theta_0 \Rightarrow$

$$\frac{1}{8}\left[\int_{\underline{R}^3} |S(q_\theta)|^{3/2} \sqrt{q_\theta}\, d^3x\right]^{2/3} < \frac{1}{C_0}.$$

Let $f \in W_\delta^4$ $(-1 \leq \delta < -\frac{1}{2})$:

$$\Delta_{q_\theta} f - \frac{1}{8} S(q_\theta) f = 0 \quad (\theta > \theta_0)$$

and, to derive a contradiction, assume that $f \neq 0$ — then

$$f\Delta_q f - \frac{1}{8} S(q_\theta) f^2 = 0$$

=>

$$0 = \int_{\underline{R}^3} (f\Delta_{q_\theta} f - \frac{1}{8} S(q_\theta) f^2) \sqrt{q_\theta}\, d^3x$$

=>

$$0 = \int_{\underline{R}^3} (|\nabla_{q_\theta} f|^2 + \frac{1}{8} S(q_\theta) f^2) \sqrt{q_\theta}\, d^3x$$

=>

$$\int_{\underline{R}^3} |\nabla_{q_\theta} f|^2 \sqrt{q_\theta}\, d^3x$$

$$= \frac{1}{8} |\int_{\underline{R}^3} S(q_\theta) f^2 \sqrt{q_\theta}\, d^3x|.$$

But for $\theta > \theta_0$,

$$\frac{1}{8} |\int_{\underline{R}^3} S(q_\theta) f^2 \sqrt{q_\theta}\, d^3x|$$

$$\leq \frac{1}{8} [\int_{\underline{R}^3} |S(q_\theta)|^{3/2} \sqrt{q_\theta}\, d^3x]^{2/3} [\int_{\underline{R}^3} |f|^6 \sqrt{q_\theta}\, d^3x]^{1/3}$$

$$< \frac{1}{C_0} [\int_{\underline{R}^3} |f|^6 \sqrt{q_\theta}\, d^3x]^{1/3}$$

$$\leq \int_{\underline{R}^3} |\nabla_{q_\theta} f|^2 \sqrt{q_\theta}\, d^3x.$$

Therefore $1 < 1$ ... .

Accordingly, if $\theta > \theta_0$, then $\exists\ \chi_\theta \in C^\infty_{>0}(\underline{R}^3)$ subject to $\chi_\theta - 1 \in W^4_\delta$ $(-1 \leq \delta < -\frac{1}{2})$ such that

$$S(q'_\theta) = 0,$$

where $q'_\theta = \chi_\theta^4 q_\theta$.

LEMMA We have

$$\lim_{\theta \to \infty} P^0(q'_\theta) = P^0(q).$$

Take $\theta > \theta_0$ — then in a certain exterior domain $\underline{E}_{R_\theta}$, $q'_\theta = u_\theta^4 \eta$ $\quad (u_\theta = \chi_\theta | \underline{E}_{R_\theta})$ and there, $S(u_\theta^4 \eta) = 0$, thus

$$0 = u_\theta^5 S(u_\theta^4 \eta) = - 8\Delta u_\theta + S(\eta) u_\theta$$

$$= - 8\Delta u_\theta,$$

i.e.,

$$\Delta u_\theta = 0.$$

This means that u_θ is harmonic. But $u_\theta \to 1$ at infinity, so there is an expansion

$$u_\theta(x) = 1 + \frac{A_\theta}{r} + \mu_\theta(x) \quad (\mu_\theta = O^\infty(\frac{1}{r^2})).$$

And

$$P^0(q'_\theta) = 32\pi A_\theta.$$

N.B. Since $P(q'_\theta) \to P(q)$ $(\theta \to \infty)$, matters have been reduced to proving that $A_\theta \geq 0$.

LEMMA If $A_\theta < 0$, then there exists a riemannian structure q''_θ on $\underline{R}^3$ with the following properties:

1. $S(q''_\theta) \geq 0$.

2. $\exists\ x: S(q''_\theta)|_x > 0$.

3. $\exists\ R: q''_\theta | \underline{E}_R = \eta | \underline{E}_R$.

But this is impossible. Thus let M be a compact connected C^∞ manifold of dimension ≥ 3 -- then there are three possibilities.

(A) $\exists$ a riemannian structure g on $M: S(g) \geq 0$ and $S(g) \not\equiv 0$.

(B) $\exists$ a riemannian structure g on $M: S(g) \equiv 0$ and $M \notin A$.

(C) $\nexists$ a riemannian structure g on $M: S(g) \geq 0$.

Example: $\forall\ n \geq 3$,

$$\begin{cases} \underline{S}^n \in A \\ \underline{T}^n \in B \\ \underline{T}^n \# \underline{T}^n \in C. \end{cases}$$

In particular: $\underline{T}^3$ does not admit a riemannian structure $g: S(g) \geq 0$ and $S(g)|_x > 0$ at some x.

Now take a cube centered at the origin which strictly contains $\underline{D}^3(R)$ and identify opposite sides to get a torus -- then q''_θ induces a riemannian structure g on this torus: $S(g) \geq 0$ and $S(g)|_x > 0$ at some x, a contradiction.

The proof of the lemma depends on an elementary preliminary fact.

<u>SUBLEMMA</u> Suppose that $u = \frac{1}{r} + v$ is harmonic in

$$\underline{A} = \{x : 1 < |x| < 6\}.$$

Then $\exists\ \delta > 0 : |v| < \delta$ implies $\exists\ H \in C^{\infty}_{>0}(\underline{A})$ with the following properties:

1. $\Delta H \geq 0$ and $\Delta H \not\equiv 0$.
2. $H = u$ near $|x| = 1$.
3. H = constant near $|x| = 6$.

[Assume first that $v = 0$ and construct a function $f \in C^{\infty}_{>0}(]1,6[)$ subject to:

1. $f(x) = \frac{1}{x} \quad (1 < x \leq 2)$.
2. $f(x) = \text{constant} \quad (5 \leq x < 6)$.
3. $f''(x) + \frac{2}{x} f'(x) > 0 \quad (2 < x < 5)$.

For the particulars, see below. Since $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$ on radial functions, one can let $H(x) = f(|x|)$. To treat the general case, fix a rotationally invariant C^{∞} cutoff function $\psi : \underline{R}^3 \rightarrow [0,1]$ such that

$$\begin{cases} \psi(x) = 1 \text{ if } |x| \leq 3 \\ \psi(x) = 0 \text{ if } |x| \geq 4 \end{cases}$$

and consider

$$H(x) = f(|x|) + \psi(x)v(x).$$

Elliptic theory can then be used to secure δ (for, by hypothesis, v is harmonic).

[Note: Denote by K the constant figuring in the definition of f — then it is clear that $H = K$ near $|x| = 6$, so the constant of property 3 is independent of v.]

<u>Details</u> Observe first that

$$f''(x) + \frac{2}{x} f'(x) = \frac{1}{x^2} \frac{d}{dx} (x^2 f'(x)).$$

Motivated by this, for ε small and positive, let

$$x^2 f'(x) = \phi_\varepsilon(x) \quad (1 < x < 6).$$

Here $\phi_\varepsilon(x) = -1$ $(1 < x \leq 2)$, then climbs from -1 to $-\varepsilon$ between 2 and $2 + \varepsilon$, then slowly strictly increases hitting 0 at 5 (usual $e^{-1/x}$ stuff), and finally $\phi_\varepsilon(x) = 0$ $(5 \leq x < 6)$. Take $f(x) = \frac{1}{x}$ $(1 < x \leq 2)$ and if $x > 2$,

$$f(x) = \int_2^x \frac{1}{t^2} \phi_\varepsilon(t)dt + \frac{1}{2}.$$

Obviously, $f(5)$ is positive provided ε is close enough to zero. And

$$f''(x) + \frac{2}{x} f'(x) > 0 \quad (2 < x < 5).$$

The function u_θ is harmonic in a certain exterior domain $\underline{E}_{R_\theta}$. Choose a positive integer k such that $R_\theta < 6^k$ and consider

$$v(x) = \frac{6^k}{A_\theta} \mu_\theta(6^k x) \quad (1 < |x| < 6).$$

Then for $k >> 0$, $|v| < \delta$ and the function

$$\chi(x) = 1 + \frac{A_\theta}{6^k} H \left(\frac{x}{6^k}\right) \quad (6^k < |x| < 6^{k+1})$$

is positive.

[Note: Therefore

$$L = 1 + \frac{A_\theta}{6^k} K$$

is positive.]

Put

$$q''_\theta = \begin{cases} q'_\theta & (|x| \le 6^k) \\ \chi^4\eta & (6^k < |x| < 6^{k+1}) \\ L^4\eta & (|x| \ge 6^{k+1}). \end{cases}$$

This makes sense:

- On $\underline{S}^2(6^k)$, $q'_\theta = u_\theta^4\eta$.

But near $\underline{S}^2(6^k)$,

$$\chi(x) = 1 + \frac{A_\theta}{6^k} H(\frac{x}{6^k})$$

$$= 1 + \frac{A_\theta}{6^k} [\frac{6^k}{|x|} + \frac{6^k}{A_\theta} \mu_\theta(x)]$$

$$= 1 + \frac{A_\theta}{r} + \mu_\theta(x)$$

$$= u_\theta(x).$$

And near $\underline{S}^2(6^{k+1})$,

$$\chi(x) = 1 + \frac{A_\theta}{6^k} H(\frac{x}{6^k})$$

$$= 1 + \frac{A_\theta}{6^k} K$$

$$= L.$$

After rescaling, we might just as well take $L = 1$. So, to complete the proof, one merely has to explicate $S(q''_\theta)$ and this is only an issue if $6^k < |x| < 6^{k+1}$. But when x is thus restricted,

$$\chi^5 S(\chi^4 \eta) = -8\Delta\chi + S(\eta)\chi$$

$$= -8\Delta\chi.$$

And

$$-8\Delta\chi\Big|_x = \frac{1}{6^{3k}}(-8A_\theta)(\Delta H\Big|_{\frac{x}{6^k}})$$

$$\geq 0.$$

Here, of course, $A_\theta < 0 \Rightarrow -8A_\theta > 0$. Moreover,

$$\exists\ x : \Delta H\Big|_{\frac{x}{6^k}} > 0.$$

Consequently, $S(q''_\theta)$ has properties 1 and 2.

Remark: It can be shown that if $S(q) \geq 0$ and $P^0(q) = 0$, then $(\underline{R}^3, q)$ is isometric to $(\underline{R}^3, \eta)$.

Example: There is one special set of circumstances where one can immediately assert that $P^0(q) \geq 0$. To this end, work in all of $\underline{R}^3$ and take $q = u^4\eta$ $(u \in C^\infty_{>0}(\underline{R}^3))$. Assume: $\Delta u \leq 0$ and in a certain exterior domain $\underline{E}_R$, u is harmonic with

$$u(x) = 1 + \frac{A}{r} + \mu(x) \quad (\mu = O^\infty(\frac{1}{r^2})).$$

Then

$$A \geq 0.$$

In fact, $1 - u = o(1)$ and $\Delta(1-u) \geq 0$, thus the weak maximum principle implies that $1 - u \leq 0$ or still, $1 \leq u$. Therefore

$$1 + \frac{A}{r} + O(\frac{1}{r^2}) \geq 1$$

$\Rightarrow$

$$\frac{A}{r} + O(\frac{1}{r^2}) \geq 0$$

$\Rightarrow$

$$A + O(\frac{1}{r}) \geq 0$$

$\Rightarrow$

$$A \geq 0.$$

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