POSITIVITY

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ABSTRACT

These notes provide a systematic account of certain aspects of the statistical structure of quantum theory. Here the all prevailing notion is that of a completely positive map and Stinespring's famous characterization thereof. I have also included a systematic treatment of "quantum dynamical semigroups", culminating in Lindblad's celebrated description of their generators.

CONTENTS

POSITIVITY

§1. *OPERATORS*

In what follows, H stands for a complex Hilbert space, the convention on the inner product $< , >$ being that it is conjugate linear in the first slot and linear in the second slot.

Denote by $B(H)$ the set of bounded linear operators on H.

- $\underline{L}_2(H)$ is the $*$-ideal in $B(H)$ consisting of the Hilbert-Schmidt operators.
- $\underline{L}_1(H)$ is the $*$-ideal in $B(H)$ consisting of the trace class operators.

Recall that $\underline{L}_2(H)$ is a Hilbert space while $\underline{L}_1(H)$ is a Banach space. Moreover, $\underline{L}_1(H) \subset \underline{L}_2(H)$ with

$$||A||_1 \geq ||A||_2 \geq ||A||.$$

[Note: By definition,

$$\begin{cases} ||A||_1 = \text{tr}(|A|) \\ ||A||_2 = (\text{tr}(|A|^2))^{1/2}. \end{cases}$$]

- $\underline{L}_\infty(H)$ is the norm closed $*$-ideal in $B(H)$ consisting of the compact operators.

<u>N.B.</u> We have

$$\underline{L}_1(H) \subset \underline{L}_2(H) \subset \underline{L}_\infty(H).$$

1.1 <u>LEMMA</u> Every closed ideal $I \subset B(H)$ is necessarily a $*$-ideal.

1.2 <u>REMARK</u> If H is infinite dimensional and separable, then there is only one proper norm closed ideal in $\mathcal{B}(H)$, viz. $\underline{L}_\infty(H)$. In general, for each infinite cardinal $\kappa \leq \dim H$, let I_κ be the set of all elements of $\mathcal{B}(H)$ whose range does not contain a norm closed subspace of dimension at least κ -- then I_κ is a proper norm closed ideal in $\mathcal{B}(H)$ and any such has this form for some κ.

[Note: Here, the term "dimension" is to be taken in the sense of Hilbert space theory, i.e., the cardinality of an orthonormal basis.]

Write

$$\begin{cases} \underline{L}_\infty(H)^* \text{ for the dual of } \underline{L}_\infty(H) \\ \underline{L}_1(H)^* \text{ for the dual of } \underline{L}_1(H). \end{cases}$$

1.3 <u>THEOREM</u> The arrow

$$\underline{L}_1(H) \to \underline{L}_\infty(H)^*$$

that sends T to λ_T $(T \in \underline{L}_1(H))$, where

$$\lambda_T(A) = \text{tr}(TA) \qquad (A \in \underline{L}_\infty(H)),$$

is an isometric isomorphism.

1.4 <u>THEOREM</u> The arrow

$$\mathcal{B}(H) \to \underline{L}_1(H)^*$$

that sends A to λ_A $(A \in \mathcal{B}(H))$, where

$$\lambda_A(T) = \mathrm{tr}(AT) \quad (T \in \underline{L}_1(H)),$$

is an isometric isomorphism.

Topologies:

- The strong operator topology on $B(H)$ is generated by the seminorms

$$||A||_x = ||Ax|| \quad (x \in H).$$

- The weak operator topology on $B(H)$ is generated by the seminorms

$$||A||_{x,y} = |\langle x,Ay\rangle| \quad (x,y \in H).$$

- The weak* operator topology on $B(H)$ is generated by the seminorms

$$||A||_T = |\mathrm{tr}(TA)| \quad (T \in \underline{L}_1(H)).$$

N.B. The weak* operator topology coincides with the weak* topology on $B(H)$ when $B(H)$ is viewed as the dual of $\underline{L}_1(H)$ via 1.4.

1.5 REMARK Each of these three topologies is weaker than the norm topology. None are metrizable unless H is finite dimensional, in which case they all agree with the norm topology.

Denote by $U(H)$ the set of all unitary operators on H -- then $U(H)$ is a group under operator multiplication.

1.6 LEMMA The (relative) strong, weak, and weak* operator topologies coincide on $U(H)$ and make $U(H)$ into a topological group.

1.7 <u>REMARK</u> The strong closure of $\mathcal{U}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ consists of all isometries. On the other hand, the weak closure of $\mathcal{U}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ is the entire closed unit ball of $\mathcal{B}(\mathcal{H})$.

1.8 <u>THEOREM</u> Suppose that $\mathcal{H}$ is infinite dimensional — then $\mathcal{U}(\mathcal{H})$ is contractible in the norm topology, as well as in the weak and strong operator topologies.

[Note: Needless to say, this is false if $\mathcal{H}$ is finite dimensional.]

Consider now the duals

$$\begin{cases} (\mathcal{B}(\mathcal{H}),\tau_s)^* & (\tau_s = \text{SOT}) \\ (\mathcal{B}(\mathcal{H}),\tau_w)^* & (\tau_w = \text{WOT}). \end{cases}$$

1.9 <u>LEMMA</u> We have

$$(\mathcal{B}(\mathcal{H}),\tau_s)^* = (\mathcal{B}(\mathcal{H}),\tau_w)^*.$$

In other words, the set of SOT-continuous linear functionals is identical with the set of WOT-continuous linear functionals. Furthermore, given such a λ, it can be shown that $\exists$

$$\begin{cases} x_1,\ldots,x_n \\ y_1,\ldots,y_n \end{cases} \in \mathcal{H}$$

for which

$$\lambda(A) = \sum_{k=1}^{n} \langle y_k, Ax_k \rangle \quad (A \in \mathcal{B}(H)).$$

1.10 <u>REMARK</u> Given $x,y \in H$, let

$$P_{x,y} = \langle y, - \rangle x.$$

Then $P_{x,y} \in \mathcal{B}(H)$ and

$$||P_{x,y}|| = ||x|| \, ||y||.$$

In fact, $P_{x,y} \in \underline{L}_1(H)$ and

$$\text{tr}(P_{x,y}) = \langle y, x \rangle.$$

Since any $T \in \underline{L}_1(H)$ of finite rank admits a representation

$$T = \sum_{k=1}^{n} P_{x_k, y_k}$$

for certain

$$\left[\begin{array}{l} x_1, \ldots, x_n \\ \\ y_1, \ldots, y_n \end{array}\right. \in H,$$

it follows that $\forall\, A \in \mathcal{B}(H)$,

$$\begin{aligned} \sum_{k=1}^{n} \langle y_k, Ax_k \rangle &= \sum_{k=1}^{n} \text{tr}(P_{Ax_k, y_k}) \\ &= \sum_{k=1}^{n} \text{tr}(AP_{x_k, y_k}) \\ &= \text{tr}(AT) \end{aligned}$$

$$= \lambda_T(A).$$

Therefore the SOT-dual (= the WOT-dual) can be identified with the set of finite rank operators equipped with the trace norm.

Let

$$\mathcal{B}(H)_+ = \{A \in \mathcal{B}(H) : A \geq 0\}.$$

Then a linear map $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ is said to be positive if

$$\Phi(\mathcal{B}(H)_+) \subset \mathcal{B}(H)_+.$$

[Note: Positive maps are necessarily bounded.]

N.B. Let

$$\mathcal{B}(H)_{SA} = \{A \in \mathcal{B}(H) : A^* = A\}.$$

Then

$$\Phi(\mathcal{B}(H)_{SA}) \subset \mathcal{B}(H)_{SA}$$

if Φ is positive, hence for all A in $\mathcal{B}(H)$,

$$\Phi(A^*) = \Phi(A)^*.$$

Given $A,B \in \mathcal{B}(H)$, put

$$A \circ B = \frac{1}{2}(AB + BA).$$

Then the operation

$$\circ:\mathcal{B}(H) \times \mathcal{B}(H) \to \mathcal{B}(H)$$

is called the Jordan product, a Jordan morphism being a linear map $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ such that $\forall$ A,B,

$$\Phi(A \circ B) = \Phi(A) \circ \Phi(B).$$

A linear map $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ is unital provided $\Phi(I) = I$.

- An order isomorphism is a unital linear bijection $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ such that

$$\Phi(\mathcal{B}(H)_+) = \mathcal{B}(H)_+.$$

- A Jordan isomorphism is a unital linear bijection $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ such that $\forall$ A,B,

$$\Phi(A \circ B) = \Phi(A) \circ \Phi(B).$$

1.11 LEMMA Suppose that $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ is a unital linear bijection -- then Φ is an order isomorphism iff Φ is a Jordan isomorphism.

Let $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ be a unital linear map.

- Φ is a *-morphism if $\Phi(A^*) = \Phi(A)^*$ and $\Phi(AB) = \Phi(A)\Phi(B)$
- Φ is a *-antimorphism if $\Phi(A^*) = \Phi(A)^*$ and $\Phi(AB) = \Phi(B)\Phi(A)$.

Assign to the terms *-isomorphism and *-antiisomorphism the obvious significance -- then both are Jordan isomorphisms.

1.12 THEOREM If $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ is a Jordan isomorphism, then Φ is either a *-isomorphism or a *-antiisomorphism.

1.13 LEMMA Every order isomorphism (or, equivalently, Jordan isomorphism (cf. 1.11)) is weak* continuous.

§2. STATES

The weak* continuous linear functionals on $\mathcal{B}(H)$ occupy a special position in the theory.

2.1 LEMMA A bounded linear functional $\lambda:\mathcal{B}(H) \to \underline{C}$ is weak* continuous iff there are sequences $\{x_k\}$, $\{y_k\}$ in H with

$$\begin{cases} \sum_k ||x_k||^2 < \infty \\ \sum_k ||y_k||^2 < \infty \end{cases}$$

such that

$$\lambda(A) = \sum_k \langle y_k, Ax_k \rangle \quad (A \in \mathcal{B}(H)).$$

[Note: The operator

$$T = \sum_k P_{x_k, y_k}$$

is trace class (the RHS being a trace norm convergent sum) and

$$\mathrm{tr}(T) = \sum_k \langle y_k, x_k \rangle .]$$

2.2 LEMMA A bounded linear functional $\lambda:\mathcal{B}(H) \to \underline{C}$ is weak* continuous iff there is a trace class operator T such that $\lambda = \lambda_T$.

Write $\mathcal{B}(H)_*$ for the norm closed subspace of $\mathcal{B}(H)^*$ consisting of the weak* continuous linear functionals on $\mathcal{B}(H)$ -- then $\mathcal{B}(H)_*$ is called the predual of $\mathcal{B}(H)$.

2.3 THEOREM $\mathcal{B}(H)_*$ is isometrically isomorphic to $\underline{L}_1(H):\lambda_T \leftrightarrow T$.

[Note: In view of 1.4,

$$(\mathcal{B}(H)_*)^* \approx \underline{L}_1(H)^* \approx \mathcal{B}(H).]$$

A linear functional $\lambda:\mathcal{B}(H) \to \underline{C}$ is said to be positive if

$$A \geq 0 \Rightarrow \lambda(A) \geq 0.$$

[Note: Positive linear functionals are necessarily bounded.]

2.4 LEMMA A positive linear functional $\lambda:\mathcal{B}(H) \to \underline{C}$ is weak* continuous iff there is an orthogonal sequence $\{x_k\}$ in H with

$$\sum_k ||x_k||^2 = ||\lambda||$$

such that

$$\lambda(A) = \sum_k \langle x_k, Ax_k \rangle \quad (A \in \mathcal{B}(H)).$$

2.5 LEMMA A positive linear functional $\lambda:\mathcal{B}(H) \to \underline{C}$ is weak* continuous iff there is a positive trace class operator T such that $\lambda = \lambda_T$.

A positive linear functional $\lambda:\mathcal{B}(H) \to \underline{C}$ is said to be completely additive if for every collection $\{P_i : i \in I\}$ of mutually orthogonal projections,

$$\lambda(\sum_{i \in I} P_i) = \sum_{i \in I} \lambda(P_i).$$

[Note: Let $F \subset 2^I$ be the set of finite subsets of I -- then $\sum_{i \in I} P_i$ is the strong limit of the

$$P_F = \sum_{i \in F} P_i \quad (F \in \mathcal{F})$$

while

$$\sum_{i \in I} \lambda(P_i) = \sup_{F \in \mathcal{F}} \sum_{i \in F} \lambda(P_i).]$$

2.6 THEOREM A positive linear functional $\lambda:\mathcal{B}(\mathcal{H}) \to \underline{C}$ is weak* continuous iff it is completely additive.

2.7 REMARK Suppose that $\lambda:\mathcal{B}(\mathcal{H}) \to \underline{C}$ is a positive linear functional with the following property: For any bounded increasing net $\{A_i : i \in I\}$ in $\mathcal{B}(\mathcal{H})_+$,

$$\lambda(\sup_{i \in I} A_i) = \lim_{i \in I} \lambda(A_i).$$

Then λ is completely additive, hence is weak* continuous.

A state on $\mathcal{B}(\mathcal{H})$ is a positive linear functional $\omega:\mathcal{B}(\mathcal{H}) \to \underline{C}$ such that $\omega(I) = 1$.

Let $S(\mathcal{B}(\mathcal{H}))$ be the state space of $\mathcal{B}(\mathcal{H})$ (meaning the set of states on $\mathcal{B}(\mathcal{H})$) -- then $S(\mathcal{B}(\mathcal{H}))$ is a convex set and its elements are continuous of norm 1.

[Note: The extreme points of $S(\mathcal{B}(\mathcal{H}))$ are called pure states.]

E.g.: Each unit vector $x \in \mathcal{H}$ gives rise to a state ω_x, viz.

$$\omega_x(A) = \langle x, Ax \rangle \quad (A \in \mathcal{B}(\mathcal{H})).$$

If $c \in \underline{C}$ and $|c| = 1$, then

$$\omega_{cx}(A) = \langle cx, Acx \rangle = \bar{c}c\langle x, Ax \rangle = \omega_x(A).$$

[Note: Let P_x be the orthogonal projection onto $\underline{C}x$ -- then

$$\omega_x(A) = tr(P_x A) \quad (A \in \mathcal{B}(\mathcal{H})).]$$

2.8 REMARK The ω_x $(||x|| = 1)$ are pure states but when $\mathcal{H}$ is infinite dimensional, there are many others.

2.9 LEMMA Let $\omega \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$ -- then

$$\omega(A^*) = \overline{\omega(A)}$$

and

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B).$$

Denote by $\mathcal{S}_{\underline{n}}(\mathcal{B}(\mathcal{H}))$ the subset of $\mathcal{S}(\mathcal{B}(\mathcal{H}))$ consisting of those ω which are weak* continuous or, equivalently, completely additive (cf. 2.6).

[Note: An element of $\mathcal{S}_{\underline{n}}(\mathcal{B}(\mathcal{H}))$ is termed normal.]

N.B. In the quantum mechanics literature, $\mathcal{S}_{\underline{n}}(\mathcal{B}(\mathcal{H}))$ is usually abbreviated to $\mathcal{S}(\mathcal{H})$, its elements then being referred to as states on $\mathcal{H}$.

A density operator is a positive trace class operator W with tr(W) = 1.

Let $\mathcal{W}(\mathcal{H})$ be the set of density operators -- then $\mathcal{W}(\mathcal{H})$ is a closed convex subset of $\underline{L}_1(\mathcal{H})$ and the arrow

$$\begin{cases} \mathcal{W}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}) \\ W \to \lambda_W \end{cases}$$

is bijective.

[Note: By definition,

$$\lambda_W(A) = \operatorname{tr}(WA) \quad (A \in \mathcal{B}(\mathcal{H})).$$

In particular:

$$\lambda_{P_x} = \omega_x \quad (||x|| = 1).]$$

2.10 EXAMPLE Take $\mathcal{H} = \underline{C}^2$ -- then relative to an orthonormal basis, the matrix representing a given $W \in \mathcal{W}(\mathcal{H})$ has the form

$$\frac{1}{2}\begin{bmatrix} 1 + z & x - \sqrt{-1}\, y \\ x + \sqrt{-1}\, y & 1 - z \end{bmatrix} \quad (x,y,z \in \underline{R}),$$

where $x^2 + y^2 + z^2 \le 1$. Therefore $\mathcal{W}(\mathcal{H})$ can be identified with the closed unit ball in $\underline{R}^3$, its boundary $x^2 + y^2 + z^2 = 1$ parameterizing the rank one orthogonal projections.

[Note: Let

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_y = \begin{bmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$\frac{1}{2}\begin{bmatrix} 1 + z & x - \sqrt{-1}\, y \\ x + \sqrt{-1}\, y & 1 - z \end{bmatrix}$$

$$= \frac{1}{2} (I_2 + x\sigma_x + y\sigma_y + z\sigma_z).]$$

2.11 <u>LEMMA</u> If $\{w_n\}$ is a sequence of nonnegative real numbers such that $\sum_n w_n = 1$ and if $\{W_n\}$ is a sequence of density operators, then $\{w_n W_n\}$ is summable w.r.t. the trace norm topology and its sum

$$W = \sum_n w_n W_n$$

is a density operator.

Let $W \in \mathcal{W}(H)$ -- then by a <u>decomposition</u> of W, we understand a collection $\{w_i : i \in I_W\}$ of positive real numbers subject to $\sum_{i \in I_W} w_i = 1$ and a collection $\{P_i : i \in I_W\}$ of rank one orthogonal projections such that

$$W = \sum_{i \in I_W} w_i P_i.$$

[Note: The index set I_W is at most countable.]

2.12 <u>LEMMA</u> Every $W \in \mathcal{W}(H)$ admits a decomposition.

<u>PROOF</u> Fix an orthonormal basis $\{e_i : i \in I\}$ -- then

$$1 = \mathrm{tr}(W)$$

$$= \sum_{i \in I} \langle e_i, We_i \rangle$$

$$= \sum_{i \in I} \langle \sqrt{W} e_i, \sqrt{W} e_i \rangle,$$

so

$$I_W = \{i \in I : ||\sqrt{W}\, e_i|| \neq 0\}$$

is at most countable. Given $i \in I_W$, put

$$w_i = ||\sqrt{W}\, e_i||^2$$

and

$$x_i = \frac{\sqrt{W}\, e_i}{\sqrt{w_i}} ,$$

from which

$$W = \sum_{i \in I_W} w_i P_{x_i} .$$

In fact, $\forall\, x \in H$,

$$\begin{aligned} \langle x, Wx \rangle &= \langle \sqrt{W}\, x, \sqrt{W}\, x \rangle \\ &= \sum_{i \in I} |\langle e_i, \sqrt{W}\, x \rangle|^2 \\ &= \sum_{i \in I_W} |\langle x, \sqrt{W}\, e_i \rangle|^2 . \end{aligned}$$

On the other hand,

$$\begin{aligned} &\langle x, \sum_{i \in I_W} w_i P_{x_i} x \rangle \\ &= \sum_{i \in I_W} w_i \langle x, P_{x_i} x \rangle \\ &= \sum_{i \in I_W} w_i \langle x, \langle x_i, x \rangle x_i \rangle \end{aligned}$$

$$= \sum_{i \in I_W} w_i |\langle x, x_i \rangle|^2$$

$$= \sum_{i \in I_W} w_i |\langle x, \frac{\sqrt{W}\, e_i}{\sqrt{w_i}} \rangle|^2$$

$$= \sum_{i \in I_W} |\langle x, \sqrt{W}\, e_i \rangle|^2.$$

[Note: In general, W will have many different decompositions (as can be seen already when $H = \underline{C}^2$) and it may very well happen that $P_{x_i} = P_{x_j}$ for distinct $i,j \in I_W$.]

2.13 LEMMA The extreme points of $W(H)$ are the rank one orthogonal projections.

PROOF Consider a $W \in W(H)$ such that $W^2 \neq W$ — then W has an eigenvalue $\lambda \in]0,1[$ corresponding to a rank one orthogonal projection E. This said, write

$$W = \lambda E + (1-\lambda) \left[\frac{W - \lambda E}{1 - \lambda} \right]$$

to see that W is not an extreme point of $W(H)$. Conversely, suppose that $W^2 = W$ and $W = \lambda W_1 + (1-\lambda) W_2$ for some $\lambda \in]0,1[$. Fix a unit vector $x_0 : W x_0 = x_0$ — then $\langle x, W_1 x \rangle = \langle x, W_2 x \rangle = 0$ if $x \perp x_0$. But $\langle x, W_1 x \rangle = ||\sqrt{W_1}\, x||^2$ and $\langle x, W_2 x \rangle = ||\sqrt{W_2}\, x||^2$, thus $\sqrt{W_1}\, x = \sqrt{W_2}\, x = 0$ if $x \perp x_0$, which implies that $W_1 x = W_2 x = 0$ if $x \perp x_0$. Since W_1 and W_2 are selfadjoint, $\exists\, c_1, c_2 \in \underline{C} : W_1 = c_1 W$, $W_2 = c_2 W$. And: $\text{tr}(W_1) = \text{tr}(W_2) = 1 \Rightarrow c_1 = c_2 = 1 \Rightarrow W_1 = W_2 = W$.

2.14 <u>REMARK</u> Let $W \in \mathcal{W}(H)$ -- then

$$\mathrm{tr}(W^2) \leq ||W||\mathrm{tr}(W) = ||W|| \leq \mathrm{tr}(W) = 1$$

and $W^2 = W$ iff $\mathrm{tr}(W^2) = 1$.

By $A_b(S(H))$ we shall understand the set of real valued bounded affine functions on $S(H)$ equipped with pointwise ordering and the supremum norm.

<u>N.B.</u> Each $A \in \mathcal{B}(H)_{SA}$ gives rise to an element $\hat{A} \in A_b(S(H))$, viz.

$$\hat{A}(\lambda_W) = \lambda_A(W) \qquad (W \in \mathcal{W}(H)).$$

[Note: $\hat{A}$ is real valued. In fact,

$$\overline{\lambda_A(W)} = \overline{\mathrm{tr}(AW)} = \mathrm{tr}(A^*W) = \mathrm{tr}(AW) = \lambda_A(W).]$$

2.15 <u>LEMMA</u> The arrow

$$\begin{cases} \mathcal{B}(H)_{SA} \longrightarrow A_b(S(H)) \\ \quad A \longrightarrow \hat{A} \end{cases}$$

is an order and norm preserving linear isomorphism.

<u>PROOF</u> Let $F \in A_b(S(H))$ -- then $F = C_1F_1 - C_2F_2$, where $C_1 \geq 0$, $C_2 \geq 0$ are constants and F_1, F_2 are elements of $A_b(S(H))$ whose range is contained in $[0,1]$. Accordingly, one might just as well assume outright that F is an affine mapping from $S(H)$ to $[0,1]$. But such an F can be uniquely extended to a positive linear functional $\Lambda_F: \underline{L}_1(H) \to \underline{C}$ (cf. infra). Since Λ_F is positive, it is bounded:

$\Lambda_F \in \underline{L}_1(H)^*$. So $\exists\, A \in B(H)$:

$$\Lambda_F = \lambda_A \quad \text{(cf. 1.4)}.$$

And A is necessarily selfadjoint:

$$\mathrm{tr}(AW) = \mathrm{tr}(A^*W) \ \forall\, W \in W(H)$$

$$\Rightarrow$$

$$A = A^* \quad \text{(cf. 7.4)}.$$

[Note: Here is a sketch of the extension procedure (a detailed rendition is given in §4). Thus put $\Lambda_F(0) = 0$ and for each positive trace class operator $T \neq 0$, set

$$\Lambda_F(T) = \mathrm{tr}(T)F\left(\frac{T}{\mathrm{tr}(T)}\right).$$

The fact that F is affine implies that

$$\Lambda_F(T_1 + T_2) = \Lambda_F(T_1) + \Lambda_F(T_2).$$

Next, extend Λ_F to the selfadjoint $T \in \underline{L}_1(H)$ via the prescription

$$\Lambda_F(T) = \Lambda_F(T^+) - \Lambda_F(T^-)$$

and then to arbitrary $T \in \underline{L}_1(H)$ by

$$\Lambda_F(T) = \Lambda_F(\mathrm{Re}\ T) + \sqrt{-1}\ \Lambda_F(\mathrm{Im}\ T).]$$

2.16 <u>THEOREM</u> There is a one-to-one correspondence between the order isomorphisms $B(H) \to B(H)$ and the affine bijections $S(H) \to S(H)$.

<u>PROOF</u> If $\Phi : B(H) \to B(H)$ is an order isomorphism, then Φ is weak* continuous (cf. 1.13) and the restriction of $\Phi^* : B(H)^* \to B(H)^*$ to $S(H)$ is an affine bijection.

In the other direction, suppose that $\zeta:S(H) \to S(H)$ is an affine bijection. Define

$$F_\zeta:A_b(S(H)) \to A_b(S(H))$$

by the rule $F_\zeta(f) = f \circ \zeta$ — then F_ζ can be regarded as an arrow

$$B(H)_{SA} \to B(H)_{SA} \quad \text{(cf. 2.15)}.$$

Extend F_ζ to all of $B(H)$ by sending A to

$$F_\zeta(\text{Re } A) + \sqrt{-1}\, F_\zeta(\text{Im } A)$$

to get an order isomorphism $\Phi_\zeta:B(H) \to B(H)$ such that $\Phi_\zeta^*:B(H)^* \to B(H)^*$ restricts to ζ:

$$\Phi_\zeta^* | S(H) = \zeta$$

or, spelled out, $\forall\ \omega \in S(H)$,

$$\omega(\Phi_\zeta(A)) = (\zeta\omega)(A) \quad (A \in B(H)).$$

Write $P(H)$ for the set of rank one orthogonal projections. Define a map

$$P(H) \times P(H) \to \underline{C}$$

by

$$(P_1,P_2) \to \text{tr}(P_1P_2).$$

2.17 <u>LEMMA</u> $\forall\ P_1,P_2 \in P(H)$,

$$0 \le \text{tr}(P_1P_2) \le 1.$$

<u>PROOF</u> Assume that $P_1 \ne P_2$ and choose unit vectors x_1,x_2:

$$\begin{cases} P_1 = P_{x_1} \\ P_2 = P_{x_2}. \end{cases}$$

Then

$$\begin{aligned} \text{tr}(P_1P_2) &= \text{tr}(P_{x_1}P_{x_2}) \\ &= \omega_{x_1}(P_{x_2}) \\ &= \langle x_1, P_{x_2}x_1\rangle \\ &= \langle x_1, \langle x_2, x_1\rangle x_2\rangle \\ &= \langle x_2, x_1\rangle\langle x_1, x_2\rangle \\ &= |\langle x_1, x_2\rangle|^2 \\ &\leq ||x_1||^2\ ||x_2||^2 = 1. \end{aligned}$$

Keeping to the preceding notation, let $A = P_1 - P_2$ and let P be the orthogonal projection onto the subspace spanned by x_1 and x_2 -- then

$$\begin{cases} A^*Ax_1 = (1 - \text{tr}(P_1P_2))x_1 \\ A^*Ax_2 = (1 - \text{tr}(P_1P_2))x_2 \end{cases}$$

=>

$$A^*A = (1 - \text{tr}(P_1P_2))P.$$

2.18 LEMMA $\forall \ P_1, P_2 \in \mathcal{P}(H)$,

$$\begin{cases} \|P_1 - P_2\| = (1 - \mathrm{tr}(P_1P_2))^{1/2} \\ \|P_1 - P_2\|_2 = \sqrt{2}\ (1 - \mathrm{tr}(P_1P_2))^{1/2} \\ \|P_1 - P_2\|_1 = 2(1 - \mathrm{tr}(P_1P_2))^{1/2}. \end{cases}$$

PROOF All assertions are trivial if $P_1 = P_2$, so assume that $P_1 \neq P_2$ -- then from the above,

$$\|A^*A\| = (1 - \mathrm{tr}(P_1P_2))\|P\|$$

or still,

$$\|A\|^2 = (1 - \mathrm{tr}(P_1P_2))$$

$$\Rightarrow$$

$$\|P_1 - P_2\| = (1 - \mathrm{tr}(P_1P_2))^{1/2}.$$

Next,

$$\begin{aligned} \|P_1 - P_2\|_2 &= (\mathrm{tr}(|A|^2))^{1/2} \\ &= (\mathrm{tr}(A^*A))^{1/2} \\ &= (\mathrm{tr}((1 - \mathrm{tr}(P_1P_2))P))^{1/2} \\ &= (1 - \mathrm{tr}(P_1P_2))^{1/2}\ (\mathrm{tr}(P))^{1/2} \\ &= \sqrt{2}\ (1 - \mathrm{tr}(P_1P_2))^{1/2}. \end{aligned}$$

Finally,

$$
\begin{aligned}
||P_1 - P_2||_1 &= \mathrm{tr}(|A|) \\
&= \mathrm{tr}((A^*A)^{1/2}) \\
&= \mathrm{tr}((1 - \mathrm{tr}(P_1P_2))^{1/2}P) \\
&= \mathrm{tr}(P)(1 - \mathrm{tr}(P_1P_2))^{1/2} \\
&= 2(1 - \mathrm{tr}(P_1P_2))^{1/2}.
\end{aligned}
$$

2.19 <u>REMARK</u> A <u>transition probability space</u> is a pair $(\mathcal{P},p)$, where $\mathcal{P}$ is a nonempty set and $p:\mathcal{P} \times \mathcal{P} \to [0,1]$ is a function such that

$$
\begin{cases}
p(\sigma,\tau) = p(\tau,\sigma) \\
p(\sigma,\tau) = 1 \iff \sigma = \tau.
\end{cases}
$$

E.g.: Let $p(\sigma,\tau) = \delta_{\sigma\tau}$. Or, what is germane, take $\mathcal{P} = \mathcal{P}(\mathcal{H})$ and let

$$p(P_1,P_2) = \mathrm{tr}(P_1P_2).$$

Put $\mathcal{H}^\times = \mathcal{H} - \{0\}$ and set

$$\underline{P}(\mathcal{H}) = \underline{C}^\times \backslash \mathcal{H}^\times.$$

Then $\mathcal{H}^\times$ carries the topology induced by the metric and we shall agree to equip $\underline{P}(\mathcal{H})$ with the quotient topology.

2.20 LEMMA $\underline{P}(H)$ is a Hausdorff space and the projection

$$H^{\times} \to \underline{P}(H)$$

is open and continuous.

[Note: $\underline{P}(H)$ is second countable if H is separable.]

2.21 REMARK It can be shown that $\underline{P}(H)$ is simply connected.

The transition topology on $P(H)$ is the initial topology determined by the functions $f_y : P(H) \to \underline{C}$ $(y \in H^{\times})$, where

$$f_y(P_x) = |\langle x,y\rangle|^2 \quad (||x|| = 1).$$

2.22 LEMMA Equip $P(H)$ with the transition topology -- then the canonical arrow $\underline{P}(H) \to P(H)$ is a homeomorphism.

[The canonical arrow is certainly bijective and continuous, thus one has only to prove that it is open, an elementary if tedious exercise.]

§3. *EFFECTS*

An <u>effect</u> is a positive operator E which is bounded above by the identity $I: 0 \leq E \leq I$.

Let $E(H)$ be the set of effects — then $E(H)$ is convex and partially ordered (but $E(H)$ is not a lattice unless $H = \underline{C}$). In addition, there is an arrow $\perp : E(H) \rightarrow E(H)$ that sends E to $E^{\perp} = I - E$. Obviously, $E^{\perp\perp} = E$ and $E \leq F \Rightarrow F^{\perp} \leq E^{\perp}$.

[Note: For use below,

$$E \in E(H) \Rightarrow 0 \leq E^2 \leq E \leq I$$

$$\Rightarrow E^2 \in E(H).]$$

Write $L(H)$ for the set of orthogonal projections — then $L(H)$ is a lattice which is contained in $E(H)$.

3.1 <u>LEMMA</u> $L(H)$ is the set of extreme points of $E(H)$.

<u>PROOF</u> Consider an $E \in E(H)$ such that $E^2 \neq E$ — then

$$I - (2E - E^2) = (I - E)^2 \in E(H)$$

$\Rightarrow$

$$2E - E^2 \in E(H).$$

But

$$E = \frac{1}{2} E^2 + \frac{1}{2} (2E - E^2),$$

thus E is not an extreme point of $E(H)$. Conversely, let $P \in L(H)$ and suppose that there are effects E_1, E_2 such that

$$P = \lambda E_1 + (1 - \lambda)E_2 \quad (0 < \lambda < 1).$$

If $Px = 0$, then

$$0 = \langle x, Px\rangle = \lambda\langle x, E_1 x\rangle + (1 - \lambda)\langle x, E_2 x\rangle$$

$$\geq \lambda\langle x, E_1 x\rangle \geq 0$$

$$\Rightarrow$$

$$0 = \langle x, E_1 x\rangle = \langle\sqrt{E_1}\, x, \sqrt{E_1}\, x\rangle$$

$$\Rightarrow \sqrt{E_1}\, x = 0 \Rightarrow E_1 x = 0.$$

Analogously,

$$I - P = \lambda(I - E_1) + (1 - \lambda)(I - E_2),$$

so

$$Px = x \Rightarrow E_1 x = x.$$

Therefore $E_1 = P = E_2$.

Given $E \in \mathcal{E}(\mathcal{H})$ and a unit vector x, let

$$\lambda(E, P_x) = \sup\{\lambda \in \underline{R} : \lambda P_x \leq E\}.$$

Then $\lambda(E, P_x)$ is called the strength of E along P_x.

[Note: Obviously,

$$0 \leq \lambda(E, P_x) \leq 1,$$

but, in general, the function $\lambda(\text{—}, P_x): \mathcal{E}(\mathcal{H}) \to [0,1]$ is not affine.]

3.2 <u>LEMMA</u> We have

$$\lambda(E,P_x)P_x \le E.$$

<u>PROOF</u> Choose a sequence $\lambda_n:\lambda_n P_x \le E$ and $\lambda_n \to \lambda(E,P_x)$ — then $\forall\, y \in \mathcal{H}$,

$$\lambda_n\langle y,P_x y\rangle \le \langle y,Ey\rangle$$

$$\Rightarrow$$

$$\lambda(E,P_x)\langle y,P_x y\rangle \le \langle y,Ey\rangle$$

$$\Rightarrow$$

$$\lambda(E,P_x)P_x \le E.$$

<u>N.B.</u>

$$E \wedge P_x = \lambda(E,P_x)P_x.$$

For by construction,

$$\begin{cases} \lambda(E,P_x)P_x \le P_x \\ \lambda(E,P_x)P_x \le E. \end{cases}$$

If now $F \le P_x$ $(F \in \mathcal{E}(\mathcal{H}))$, then $F = \lambda_F P_x$ $(\lambda_F \le 1)$ and $F \le E$ forces $\lambda_F \le \lambda(E,P_x)$.

3.3 <u>LEMMA</u> $\exists$ a unit vector $\bar{x}$ and a real number $\bar{\lambda}:\bar{\lambda}P_{\bar{x}} \le E$ and

$$\bar{\lambda}\langle x,P_{\bar{x}}x\rangle = \langle x,Ex\rangle.$$

<u>PROOF</u> If $\langle x,Ex\rangle = 0$, then we can take $\bar{x} = x$ and $\bar{\lambda} = 0$. On the other hand,

if $\langle x,Ex\rangle \neq 0$, then we can take

$$\bar{x} = \frac{Ex}{||Ex||} \quad \text{and} \quad \bar{\lambda} = \frac{||Ex||^2}{\langle x,Ex\rangle} .$$

3.4 <u>LEMMA</u> Let $E,F \in E(H)$ -- then $E \leq F$ iff

$$\lambda(E,P_x) \leq \lambda(F,P_x)$$

for all unit vectors x.

<u>PROOF</u> The direct implication is trivial:

$$\lambda(E,P_x)P_x \leq E \leq F$$

$$\Rightarrow \lambda(E,P_x) \leq \lambda(F,P_x).$$

To discuss the converse, choose $\bar{x}$ and $\bar{\lambda}$ per E (cf. 3.3) -- then

$$\bar{\lambda} \leq \lambda(E,P_{\bar{x}}) \leq \lambda(F,P_{\bar{x}})$$

$$\Rightarrow \bar{\lambda}P_{\bar{x}} \leq F$$

$\Rightarrow$

$$\langle x,Ex\rangle = \bar{\lambda}\langle x,P_{\bar{x}}x\rangle$$

$$= \langle x,\bar{\lambda}P_{\bar{x}}x\rangle$$

$$\leq \langle x,Fx\rangle.$$

Therefore $E \leq F$, x being arbitrary.

3.5 THEOREM $\forall\ E \in \mathcal{E}(H)$,

$$E = \vee\{\lambda(E,P_x)P_x : ||x|| = 1\}.$$

PROOF First, of course, $\forall\ x$

$$\lambda(E,P_x)P_x \leq E.$$

Next, if $F \in \mathcal{E}(H)$ and $\lambda(E,P_x)P_x \leq F$ for all x, then $\lambda(E,P_x) \leq \lambda(F,P_x)$ for all x, which implies that $E \leq F$ (cf. 3.4).

3.6 REMARK According to 3.1, $\mathcal{L}(H)$ is the set of extreme points of $\mathcal{E}(H)$. Here is another characterization: An effect E is an orthogonal projection iff $\forall\ x$, $\lambda(E,P_x) \in \{0,1\}$.

3.7 RAPPEL If $A \in \mathcal{B}(H)$ is selfadjoint, then

$$H = \text{Ker } A \oplus \overline{\text{Ran } A}.$$

In particular:

$$\text{Ran } A = H \Rightarrow A \text{ invertible.}$$

3.8 THEOREM Let $E \in \mathcal{E}(H)$ and suppose that $E^{1/2}$ is surjective -- then $\forall\ x$,

$$\lambda(E,P_x) = ||E^{-1/2}x||^{-2}.$$

PROOF To begin with, we claim that

$$||E^{-1/2}x||^{-2}P_x \leq E.$$

Thus put $\xi = E^{-1/2}x$ -- then $\forall\ y \in \mathcal{H}$,

$$<y,||E^{-1/2}x||^{-2}P_x y> = ||\xi||^{-2}<y,P_x y>$$

$$= ||\xi||^{-2}<y,<x,y>x>$$

$$= ||\xi||^{-2}<y,<E^{1/2}\xi,y>E^{1/2}\xi>$$

$$= ||\xi||^{-2}<y,E^{1/2}\xi><E^{1/2}\xi,y>$$

$$= ||\xi||^{-2}<E^{1/2}y,\xi><\xi,E^{1/2}y>$$

$$= ||\xi||^{-2}<E^{1/2}y,P_{\xi,\xi}E^{1/2}y> \quad \text{(cf. 1.10)}$$

$$\leq ||\xi||^{-2}||P_{\xi,\xi}||\ ||E^{1/2}y||^2$$

$$= ||\xi||^{-2}||\xi||^2<y,Ey>$$

$$= <y,Ey>.$$

Therefore

$$||E^{-1/2}x||^{-2}P_x \leq E,$$

as claimed. Matters are thus reduced to proving that

$$\lambda P_x \leq E \Rightarrow \lambda \leq ||E^{-1/2}x||^{-2}.$$

But $\forall\ y \in \mathcal{H}$,

$$\lambda|<x,y>|^2 \leq ||E^{1/2}y||^2$$

or still,

$$\lambda|\langle E^{1/2}\xi,y\rangle|^2 \le ||E^{1/2}y||^2$$

or still,

$$\lambda|\langle \xi,E^{1/2}y\rangle|^2 \le ||E^{1/2}y||^2.$$

Now take $y = E^{-1/2}\xi$:

$$\lambda|\langle \xi,\xi\rangle|^2 \le ||\xi||^2$$

$$\Rightarrow$$

$$\lambda||\xi||^4 \le ||\xi||^2$$

$$\Rightarrow$$

$$\lambda \le ||\xi||^{-2} = ||E^{-1/2}x||^{-2}.$$

[Note: If $E^{1/2}$ is surjective, then E is surjective, hence invertible. Proof: $\forall\ y \in H$, $E^{1/2}y = E(E^{-1/2}y)$. Consequently,

$$\frac{1}{||E^{-1/2}x||^2} = \frac{1}{\langle x,E^{-1}x\rangle} = \frac{1}{\mathrm{tr}(P_xE^{-1})} = \frac{1}{\omega_x(E^{-1})}\ .]$$

3.9 <u>REMARK</u> Fix $E \in E(H)$ -- then it can be shown that

$$\exists\ \lambda > 0:\lambda P_x \le E \iff x \in \mathrm{Ran}\ E^{1/2}.$$

So, in general

$$\lambda(E,P_x) = \begin{cases} ||E^{-1/2}x||^{-2} & \text{if } x \in \mathrm{Ran}\ E^{1/2} \\ \\ 0 & \text{if } x \notin \mathrm{Ran}\ E^{1/2}. \end{cases}$$

3.10 <u>EXAMPLE</u> Suppose given $\lambda_0 : 0 < \lambda_0 \leq 1$. Assume: $\forall\, x$, $\lambda(E,P_x) = \lambda_0$ -- then $E = \lambda_0 I$. To see this, observe that $E^{1/2}$ must be surjective (cf. 3.9), hence $\forall\, x$,

$$\langle x, E^{-1}x\rangle = \frac{1}{\lambda_0} = \frac{1}{\lambda_0}\langle x,x\rangle = \langle x, \frac{1}{\lambda_0} Ix\rangle$$

$$\Rightarrow E = \lambda_0 I.$$

3.11 <u>LEMMA</u> Let x be a unit vector and E an effect -- then

$$\lambda(E,P_x) \leq \langle x,Ex\rangle$$

and

$$\lambda(E,P_x) = \langle x,Ex\rangle \iff Ex = \lambda(E,P_x)x.$$

Recall that by 3.1, $\mathcal{L}(\mathcal{H})$ is the set of extreme points of $\mathcal{E}(\mathcal{H})$. In addition:

3.12 <u>LEMMA</u> Suppose that $\mathcal{H}$ is infinite dimensional -- then the weak closure of $\mathcal{L}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ is $\mathcal{E}(\mathcal{H})$.

§4. *AUTOMORPHISMS*

Let $\overline{\mathcal{U}(\mathcal{H})}$ denote the set of all antiunitary operators on $\mathcal{H}$. Put

$$\text{Aut } \mathcal{H} = \mathcal{U}(\mathcal{H}) \cup \overline{\mathcal{U}(\mathcal{H})}.$$

Then Aut $\mathcal{H}$ is a topological group in any of the three topologies figuring in 1.6 and $\mathcal{U}(\mathcal{H})$ is its identity component.

[Note: Recall that the product of two antiunitary operators is unitary while the product of a unitary operator and an antiunitary operator is antiunitary.]

<u>N.B.</u> The quotient

$$\Sigma(\mathcal{H}) = \{zI:|z| = 1\}\backslash\text{Aut } \mathcal{H}$$

is called the <u>symmetry group</u> of $\mathcal{H}$.

4.1 <u>LEMMA</u> If $U \in \mathcal{U}(\mathcal{H})$, then the map

$$A \to UAU^{-1} \quad (A \in \mathcal{B}(\mathcal{H}))$$

is a $*$-isomorphism, call it σ_U. On the other hand, if $\bar{U} \in \overline{\mathcal{U}(\mathcal{H})}$, then the map

$$A \to \bar{U}A^*\bar{U}^{-1}$$

is a $*$-antiisomorphism, call it $\sigma_{\bar{U}}$.

4.2 <u>EXAMPLE</u> Take $\mathcal{H} = \underline{C}$ and let K be the complex conjugation -- then $\Sigma(\mathcal{H}) = \{I,K\}$. Here $\mathcal{B}(\mathcal{H}) = \{T_z:z \in \underline{C}\}$, where $T_z w = zw$, thus $T_z^* = T_{\bar{z}}$ and T_z is unitary iff $|z| = 1$.

- $\sigma_{T_z} T_w = T_z T_w T_z^{-1}$

$$= T_z T_w T_{\bar{z}} = T_{zw\bar{z}} = T_w.$$

I.e.: σ_{T_z} is the identity map on $\mathcal{B}(\mathcal{H})$.

- $\sigma_{T_z K} T_w = T_z K T_w^* (T_z K)^{-1}$

$$= T_z K T_{\bar{w}} K T_{\bar{z}} = T_z T_w T_{\bar{z}} = T_w.$$

I.e.: $\sigma_{T_z K}$ is the identity map on $\mathcal{B}(\mathcal{H})$.

If $z \in \underset{\sim}{C}$ and $|z| = 1$, then

$$\begin{cases} \sigma_{zU} = \sigma_U & (U \in \mathcal{U}(\mathcal{H})) \\ \sigma_{z\bar{U}} = \sigma_{\bar{U}} & (\bar{U} \in \overline{\mathcal{U}(\mathcal{H})}). \end{cases}$$

E.g.: $\forall\ x \in \mathcal{H}$,

$$\sigma_{z\bar{U}} Ax = (z\bar{U})A^*(z\bar{U})^{-1}x$$

$$= z\bar{U}A^*\bar{U}^{-1}\bar{z}x$$

$$= z\bar{U}A^*z\bar{U}^{-1}x$$

$$= z\bar{U}zA^*\bar{U}^{-1}x$$

$$= z\bar{z}\bar{U}A^*\bar{U}^{-1}x$$

$$= \bar{U}A^*\bar{U}^{-1}x$$

$$= \sigma_{\bar{U}}Ax.$$

Given

$$\begin{cases} U \in \mathcal{U}(\mathcal{H}) \\ \bar{U} \in \overline{\mathcal{U}(\mathcal{H})}, \end{cases}$$

write

$$\left.\begin{matrix} [U] \\ [\bar{U}] \end{matrix}\right\} \in \Sigma(\mathcal{H})$$

for its equivalence class.

4.3 <u>LEMMA</u> If

$$\left.\begin{matrix} U \\ V \end{matrix}\right\} \in \mathcal{U}(\mathcal{H}) \text{ or } \left.\begin{matrix} \bar{U} \\ \bar{V} \end{matrix}\right\} \in \overline{\mathcal{U}(\mathcal{H})}$$

and if

$$\sigma_U = \sigma_V \text{ or } \sigma_{\bar{U}} = \sigma_{\bar{V}},$$

then

$$[U] = [V] \text{ or } [\bar{U}] = [\bar{V}].$$

<u>PROOF</u> It will be enough to deal with the unitary case and for this it can be assumed that $\mathcal{H} \neq \underline{C}$ (cf. 4.2). If $\sigma_U(A) = \sigma_V(A)$ for all $A \in \mathcal{B}(\mathcal{H})$, then in particular $\sigma_U(P) = \sigma_V(P)$ for all $P \in \mathcal{P}(\mathcal{H})$, hence $\forall$ unit vector $x \in \mathcal{H}$,

$$UP_xU^{-1} = VP_xV^{-1}$$

or still,

$$P_{Ux} = P_{Vx},$$

which implies that $Ux = z(x)Vx$ ($z(x) \in \underset{\sim}{C}$, $|z(x)| = 1$), the claim being that $z(x)$ is a constant independent of x. Take two unit vectors x_1, x_2.

<u>Case 1</u>: $\langle x_1, x_2\rangle \neq 0$ -- then

$$\begin{aligned}\langle x_1, x_2\rangle &= \langle Ux_1, Ux_2\rangle \\ &= \langle z(x_1)Vx_1, z(x_2)Vx_2\rangle \\ &= \overline{z(x_1)}z(x_2)\langle Vx_1, Vx_2\rangle \\ &= \overline{z(x_1)}z(x_2)\langle x_1, x_2\rangle\end{aligned}$$

$\Rightarrow$

$$\overline{z(x_1)}z(x_2) = 1 \Rightarrow z(x_1) = z(x_2).$$

<u>Case 2</u>: $\langle x_1, x_2\rangle = 0$ -- then

$$\langle x_1, \frac{x_1 + x_2}{||x_1 + x_2||}\rangle \neq 0$$

$\Rightarrow$

$$z(x_1) = z(\frac{x_1 + x_2}{||x_1 + x_2||}) \quad \text{(cf. Case 1)}$$

$\Rightarrow$

$$Ux_1 + Ux_2 = U(x_1 + x_2)$$

$$= ||x_1 + x_2||U(\frac{x_1 + x_2}{||x_1 + x_2||})$$

$$= ||x_1 + x_2||z(\frac{x_1 + x_2}{||x_1 + x_2||})V(\frac{x_1 + x_2}{||x_1 + x_2||})$$

$$= z(x_1)Vx_1 + z(x_1)Vx_2$$

$$= Ux_1 + z(x_1)Vx_2$$

=>

$$Ux_2 = z(x_1)Vx_2 \Rightarrow z(x_1) = z(x_2).$$

N.B. The proof shows that σ_U or $\sigma_{\bar{U}}$ is determined by its restriction to $\mathcal{P}(\mathcal{H})$.

4.4 LEMMA If $U \in \mathcal{U}(\mathcal{H})$ and $\bar{U} \in \overline{\mathcal{U}(\mathcal{H})}$, then $\sigma_U \neq \sigma_{\bar{U}}$ provided $\dim \mathcal{H} > 1$.

PROOF To get a contradiction, suppose that $\sigma_U = \sigma_{\bar{U}}$. Proceeding as in 4.3, $\forall$ unit vector $x \in \mathcal{H}$, $\exists$ $z(x) \in \underline{C}$ of absolute value 1 such that $Ux = z(x)\bar{U}x$. This said, consider a pair x_1, x_2 of orthogonal unit vectors — then

$$U(x_1 + x_2) = Ux_1 + Ux_2$$

$$= z(x_1)\bar{U}x_1 + z(x_2)\bar{U}x_2.$$

Meanwhile

$$U(x_1 + x_2) = ||x_1 + x_2||U(\frac{x_1 + x_2}{||x_1 + x_2||})$$

$$= ||x_1 + x_2||z(\frac{x_1 + x_2}{||x_1 + x_2||})\bar{U}(\frac{x_1 + x_2}{||x_1 + x_2||})$$

$$= z\left(\frac{x_1 + x_2}{||x_1 + x_2||}\right)\bar{U}x_1 + z\left(\frac{x_1 + x_2}{||x_1 + x_2||}\right)\bar{U}x_2.$$

But $\bar{U}x_1$ and $\bar{U}x_2$ are linearly independent. Therefore

$$z(x_1) = z\left(\frac{x_1 + x_2}{||x_1 + x_2||}\right) = z(x_2).$$

Now repeat the computation using instead

$$\frac{x_1 + \sqrt{-1}\, x_2}{||x_1 + \sqrt{-1}\, x_2||}$$

to conclude that

$$z(x_1) = z\left(\frac{x_1 + \sqrt{-1}\, x_2}{||x_1 + \sqrt{-1}\, x_2||}\right),\ z(x_2) = -\, z\left(\frac{x_1 + \sqrt{-1}\, x_2}{||x_1 + \sqrt{-1}\, x_2||}\right),$$

from which the sought for contradiction.

[Note: It is to be emphasized that 4.4 is false if $\dim H = 1$ (cf. 4.2).]

Write Aut $\mathcal{B}(H)$ for the group of Jordan isomorphisms.

4.5 <u>LEMMA</u> Every $\Phi \in \mathrm{Aut}\, \mathcal{B}(H)$ is an isometry: $\forall\, A \in \mathcal{B}(H)$,

$$||\Phi(A)|| = ||A|| \quad (A \in \mathcal{B}(H)).$$

So, in view of 1.12, every *-isomorphism and every *-antiisomorphism is an isometry.

[Note: For this, 1.12 can be obviated: One need only quote standard

C*-algebra generalities.]

4.6 <u>THEOREM</u> Every *-isomorphism $\Phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ can be implemented by a $U \in \mathcal{U}(\mathcal{H})$, i.e.,

$$\Phi(A) = UAU^{-1} \quad (A \in \mathcal{U}(\mathcal{H})).$$

<u>PROOF</u> Fix a unit vector $x \in \mathcal{H}$ and determine a unit vector $y \in \mathcal{H}$ by $P_y = \Phi^{-1}(P_x)$. Bearing in mind that y is cyclic for $\mathcal{B}(\mathcal{H})$, let

$$UAy = \Phi(A)x \quad (A \in \mathcal{B}(\mathcal{H})).$$

Then

$$\begin{aligned} ||Ay|| &= ||AP_y y|| = ||AP_y|| \\ &= ||\Phi(AP_y)|| = ||\Phi(A)\Phi(P_y)|| = ||\Phi(A)P_x|| \\ &= ||\Phi(A)P_x x|| = ||\Phi(A)x|| = ||UAy||. \end{aligned}$$

Therefore U is welldefined and isometric. Since the range of U is $\mathcal{B}(\mathcal{H})x$, it follows that U is unitary. Finally, for all $A,B \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} UAU^{-1}Bx &= UA\Phi^{-1}(B)y \\ &= \Phi(A\Phi^{-1}(B))x \\ &= \Phi(A)Bx \end{aligned}$$

$\Rightarrow$

$$UAU^{-1} = \Phi(A).$$

4.7 <u>THEOREM</u> Every *-antiisomorphism $\Phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ can be implemented by a

$\bar{U} \in \overline{U(H)}$, i.e.,

$$\Phi(A) = \bar{U}A^*\bar{U}^{-1} \quad (A \in B(H)).$$

PROOF Fix a conjugation K, thus $K \in \overline{U(H)}$ and $K^2 = I$. Define $\Psi: B(H) \to B(H)$ by

$$\Psi(A) = \Phi(KA^*K) \quad (A \in B(H)).$$

The map $A \to KA^*K$ is a *-antiisomorphism, so Ψ is a *-isomorphism. Using 4.6, choose $U \in U(H)$:

$$\Psi(A) = UAU^{-1} \quad (A \in B(H)).$$

Then

$$K(KA^*K)^*K = A$$

$$\Rightarrow$$

$$\begin{aligned}\Phi(A) &= \Phi(K(KA^*K)^*K) \\ &= \Psi(KA^*K) \\ &= UKA^*(UK)^{-1}.\end{aligned}$$

And

$$\bar{U} = UK \in \overline{U(H)}.$$

4.8 REMARK Neither U nor $\bar{U}$ is unique but rather is unique up to phase:

$$\begin{cases} \Phi \leftrightarrow [U] \in \Sigma(H) \\ \Phi \leftrightarrow [\bar{U}] \in \Sigma(H). \end{cases}$$

4.9 SCHOLIUM The canonical arrow

$$\Sigma(H) \to \text{Aut } B(H)$$

is surjective and is bijective if $\dim H > 1$.

Write $\text{Aut}\ \mathcal{W}(H)$ for the set of affine bijections $\zeta:\mathcal{W}(H) \to \mathcal{W}(H)$ -- then $\text{Aut}\ \mathcal{W}(H)$ is a group and there is a bijective arrow

$$\text{Aut}\ \mathcal{B}(H) \to \text{Aut}\ \mathcal{W}(H)$$

of restriction (cf. 2.16).

[Note: $\forall\ \zeta \in \text{Aut}\ \mathcal{W}(H)$,

$$\zeta(\mathcal{P}(H)) = \mathcal{P}(H).]$$

<u>N.B.</u> Fix $W \in \mathcal{W}(H)$.

• $\forall\ U \in \mathcal{U}(H)$,

$$\begin{aligned} \lambda_W(UAU^{-1}) &= \text{tr}(WUAU^{-1}) \\ &= \text{tr}(U^{-1}WUA) \\ &= \lambda_{U^{-1}WU}(A). \end{aligned}$$

• $\forall\ \bar{U} \in \overline{\mathcal{U}(H)}$,

$$\begin{aligned} \lambda_W(\bar{U}A^*\bar{U}^{-1}) &= \text{tr}(W\bar{U}A^*\bar{U}^{-1}) \\ &= \overline{\text{tr}(\bar{U}^{-1}W\bar{U}A^*)} \\ &= \overline{\lambda_{\bar{U}^{-1}W\bar{U}}(A^*)} \\ &= \lambda_{\bar{U}^{-1}W\bar{U}}(A). \end{aligned}$$

Write $\text{Aut}\ \mathcal{P}(H)$ for the set of those bijections $\rho:\mathcal{P}(H) \to \mathcal{P}(H)$ with the property that

$$\operatorname{tr}(\rho(P_1)\rho(P_2)) = \operatorname{tr}(P_1P_2)$$

$\forall\ P_1,P_2 \in \mathcal{P}(H)$.

[Note: Aut $\mathcal{P}(H)$ is a group.]

4.10 <u>LEMMA</u> Given $\zeta \in \text{Aut}\ \mathcal{W}(H)$, put $\rho = \zeta|\mathcal{P}(H)$ -- then $\rho \in \text{Aut}\ \mathcal{P}(H)$.

<u>PROOF</u> Let $\Phi_\zeta \in \text{Aut}\ \mathcal{B}(H)$ be the order isomorphism corresponding to ζ -- then either $\Phi_\zeta = \sigma_U$ ($\exists\ U \in \mathcal{U}(H)$) (cf. 4.6) or $\Phi_\zeta = \sigma_{\bar{U}}$ ($\exists\ \bar{U} \in \overline{\mathcal{U}(H)}$) (cf. 4.7), from which the assertion.

Consequently, there is an injective arrow

$$\text{Aut}\ \mathcal{W}(H) \rightarrow \text{Aut}\ \mathcal{P}(H)$$

of restriction.

4.11 <u>LEMMA</u> Every $\rho \in \text{Aut}\ \mathcal{P}(H)$ admits a unique extension to an element $\zeta \in \text{Aut}\ \mathcal{W}(H)$.

<u>PROOF</u> Let $W \in \mathcal{W}(H)$, consider a decomposition of W,

$$W = \sum_{i \in I_W} w_i P_i \quad \text{(cf. 2.12)},$$

and define $\zeta(W)$ by

$$\zeta(W) = \sum_{i \in I_W} w_i \rho(P_i).$$

To check that ζ is welldefined, suppose that

$$W = \sum_{j \in J_W} w_j P_j$$

is another decomposition of W. Given a unit vector $x \in H$, choose y:

$$\rho(P_y) = P_x \quad (||y|| = 1).$$

Then

$$\langle x, \Sigma w_i \rho(P_i)x - \Sigma w_j \rho(P_j)x\rangle$$

$$= \Sigma w_i \langle x, \rho(P_i)x\rangle - \Sigma w_j \langle x, \rho(P_j)x\rangle$$

$$= \Sigma w_i \mathrm{tr}(P_x \rho(P_i)) - \Sigma w_j \mathrm{tr}(P_x \rho(P_j))$$

$$= \Sigma w_i \mathrm{tr}(\rho(P_y)\rho(P_i)) - \Sigma w_j \mathrm{tr}(\rho(P_y)\rho(P_j))$$

$$= \Sigma w_i \mathrm{tr}(P_y P_i) - \Sigma w_j \mathrm{tr}(P_y P_j)$$

$$= \mathrm{tr}(P_y(\Sigma w_i P_i - \Sigma w_j P_j))$$

$$= \mathrm{tr}(P_y(W - W))$$

$$= 0.$$

Therefore

$$\sum_{i \in I_W} w_i \rho(P_i) = \sum_{j \in J_W} w_j \rho(P_j).$$

The verification that ζ is an affine bijection is straightforward.

The arrow

$$\mathrm{Aut}\, \mathcal{W}(H) \to \mathrm{Aut}\, \mathcal{P}(H)$$

of restriction is thus bijective and $\forall\ \rho \in \mathrm{Aut}\, \mathcal{P}(H)$, $\exists\ U \in \mathcal{U}(H)$ or $\exists\ \bar{U} \in \overline{\mathcal{U}(H)}$

such that

$$\rho = \sigma_U | P(H) \text{ or } \rho = \sigma_{\bar{U}} | P(H).$$

To recapitulate:

$$\text{Aut } B(H) \approx \text{Aut } W(H) \approx \text{Aut } P(H).$$

[Note: When $H = \underline{C}$, each of these three groups consists of $\{\sigma_I\}$ alone, whereas $\Sigma(H) = \{I,K\}$ (cf. 4.2).]

Write Aut $E(H)$ for the set of those bijections $\gamma: E(H) \rightarrow E(H)$ such that $\forall$ E,F $\in E(H)$,

$$E + F \in E(H) \Leftrightarrow \gamma(E) + \gamma(F) \in E(H)$$

and

$$E + F \in E(H) \Rightarrow \gamma(E + F) = \gamma(E) + \gamma(F).$$

Then Aut $E(H)$ is a group and there is an injective arrow

$$\text{Aut } B(H) \rightarrow \text{Aut } E(H)$$

of restriction which, as will be shown below, is actually bijective.

4.12 <u>LEMMA</u> Let $\gamma \in$ Aut $E(H)$ -- then $\forall$ E,F $\in E(H)$,

$$E \leq F \Leftrightarrow \gamma(E) \leq \gamma(F).$$

<u>PROOF</u> If $E \leq F$, then $F = (F - E) + E$, with $F - E \in E(H)$, hence

$$\gamma(F) = \gamma(F - E) + \gamma(E) \Rightarrow \gamma(E) \leq \gamma(F).$$

And conversely... .

4.13 <u>LEMMA</u> Let $\gamma \in$ Aut $E(H)$ -- then $\gamma(0) = 0$ and $\gamma(I) = I$.

<u>PROOF</u> First

$$0 = 0 + 0 \Rightarrow \gamma(0) = \gamma(0 + 0) = \gamma(0) + \gamma(0)$$

$$\Rightarrow \gamma(0) = 0.$$

Second

$$\begin{bmatrix} \gamma(I) \leq I \\ \gamma^{-1}(I) \leq I \end{bmatrix}$$

$\Rightarrow$

$$I = \gamma(\gamma^{-1}(I)) \leq \gamma(I) \leq I.$$

4.14 <u>LEMMA</u> Let $\gamma \in \text{Aut}\, \mathcal{E}(\mathcal{H})$ -- then $\forall\, E \in \mathcal{E}(\mathcal{H})$,

$$\gamma(E^{\perp}) = \gamma(E)^{\perp}.$$

<u>PROOF</u> In fact,

$$I = \gamma(I) = \gamma(E + E^{\perp}) = \gamma(E) + \gamma(E^{\perp})$$

$\Rightarrow$

$$\gamma(E^{\perp}) = I - \gamma(E) = \gamma(E)^{\perp}.$$

4.15 <u>LEMMA</u> Let $\gamma \in \text{Aut}\, \mathcal{E}(\mathcal{H})$ -- then $\forall\, E \in \mathcal{E}(\mathcal{H})$,

$$\gamma(rE) = r\gamma(E) \quad (0 \leq r \leq 1).$$

<u>PROOF</u> Assuming that $r \neq 0,1$, start with $r = \frac{1}{n}$ ($n \in \underline{N}, n > 1$) and write

$$E = \frac{1}{n} E + \cdots + \frac{1}{n} E \quad \text{(n terms)}.$$

Then

$$\gamma(E) = n\gamma(\frac{1}{n} E)$$

or still,

$$\gamma(\frac{1}{n} E) = \frac{1}{n} \gamma(E).$$

But this implies that

$$\gamma(rE) = r\gamma(E)$$

for all rational $r \in]0,1[$. To handle an irrational r, choose a sequence $\{r_n\}$ of rational $r_n \in]0,1[$:

$$r - \frac{1}{n} < r_n < r.$$

Then for $n >> 0$:

$$r_n E \le rE \le (r_n + \frac{1}{n})E$$

$$\Rightarrow$$

$$\gamma(r_n E) \le \gamma(rE) \le \gamma((r_n + \frac{1}{n})E)$$

$$\Rightarrow$$

$$r_n \gamma(E) \le \gamma(rE) \le (r_n + \frac{1}{n})\gamma(E),$$

so in the limit,

$$r\gamma(E) \le \gamma(rE) \le r\gamma(E).$$

4.16 <u>REMARK</u> $\forall\, E \in \mathcal{E}(\mathcal{H})$,

$$E = \bigvee_{P \in \mathcal{P}(\mathcal{H})} \lambda(E,P)P \quad \text{(cf. 3.5)},$$

thus $\forall\, \gamma \in \text{Aut}\, \mathcal{E}(\mathcal{H})$,

$$\gamma(E) = \bigvee_{P \in \mathcal{P}(\mathcal{H})} \gamma(\lambda(E,P)P)$$

$$= \bigvee_{P \in \mathcal{P}(\mathcal{H})} \lambda(E,P)\gamma(P).$$

4.17 LEMMA Let $\gamma \in \text{Aut}\, E(H)$ -- then γ is an affine bijection.

PROOF $\forall\, r \in [0,1]$ and $\forall\, E,F \in E(H)$,

$$\gamma(rE + (1 - r)F) = \gamma(rE) + \gamma((1 - r)F)$$

$$= r\gamma(E) + (1 - r)\gamma(F).$$

4.18 REMARK Aut $E(H)$ is strictly contained in the set of affine bijections $E(H) \to E(H)$, the point here being that an affine bijection ϕ need not send 0 to 0 but instead $\phi(0) = 0$ or $\phi(0) = I$.

We shall now extend a given $\gamma \in E(H)$ to an order isomorphism $\Phi_\gamma : B(H) \to B(H)$ with the property that

$$\Phi_\gamma | E(H) = \gamma.$$

Step 1: Extend γ to $B(H)_+$ by writing

$$\gamma_+(A) = ||A||\gamma\left(\frac{A}{||A||} \right)$$

if $A \neq 0$ and set $\gamma_+(0) = 0$. Note that

$$\gamma_+(rA) = r\gamma_+(A) \qquad (r \geq 0).$$

Furthermore, γ_+ is additive on $B(H)_+$:

$$\gamma_+(A + B) = ||A + B||\gamma\left(\frac{A + B}{||A + B||} \right)$$

$$= ||A + B||\gamma\left(\frac{A}{||A + B||} + \frac{B}{||A + B||} \right)$$

$$= ||A + B||\gamma\left(\frac{||A||}{||A + B||} \frac{A}{||A||} + \frac{||B||}{||A + B||} \frac{B}{||B||} \right)$$

$$= ||A + B|| \frac{||A||}{||A + B||} \gamma(\frac{A}{||A||}) + ||A + B|| \frac{||B||}{||A + B||} \gamma(\frac{B}{||B||})$$

$$= ||A||\gamma(\frac{A}{||A||}) + ||B||\gamma(\frac{B}{||B||})$$

$$= \gamma_+(A) + \gamma_+(B).$$

<u>Step 2</u>: Extend γ_+ to $\mathcal{B}(\mathcal{H})_{SA}$ by writing

$$\gamma_{SA}(A) = \gamma_+(A^+) - \gamma_+(A^-).$$

I.e.:

$$\gamma_{SA}(A) = \frac{1}{2} \gamma_+(|A| + A) - \frac{1}{2} \gamma_+(|A| - A).$$

Note that

$$\gamma_{SA}(rA) = r\gamma_{SA}(A) \quad (r \in \underline{R}).$$

Furthermore, γ_{SA} is additive on $\mathcal{B}(\mathcal{H})_{SA}$:

$$\gamma_{SA}(A + B) = \gamma_+((A + B)^+) - \gamma_+((A + B)^-)$$

$$= \frac{1}{2} \gamma_+(|A + B| + A + B) - \frac{1}{2} \gamma_+(|A + B| - A - B).$$

But

$$|A + B| + A + B$$

$$= |A + B| + |A| + |B| - (|A| - A) - (|B| - B)$$

=>

$$|A + B| + A + B + (|A| - A) + (|B| - B)$$

$$= |A + B| + |A| + |B|$$

=>

$$\gamma_+(|A + B| + A + B + (|A| - A) + (|B| - B))$$

$$= \gamma_+(|A + B| + A + B) + 2\gamma_+(A^-) + 2\gamma_+(B^-)$$

$$= \gamma_+(|A + B|) + \gamma_+(|A|) + \gamma_+(|B|)$$

=>

$$\gamma_+(|A + B| + A + B)$$

$$= \gamma_+(|A + B|) + \gamma_+(|A|) + \gamma_+(|B|)$$

$$- 2\gamma_+(A^-) - 2\gamma_+(B^-).$$

And likewise

$$\gamma_+(|A + B| - A - B)$$

$$= \gamma_+(|A + B|) + \gamma_+(|A|) + \gamma_+(|B|)$$

$$- 2\gamma_+(A^+) - 2\gamma_+(B^+).$$

Therefore

$$\frac{1}{2}\gamma_+(|A + B| + A + B) - \frac{1}{2}\gamma_+(|A + B| - A - B)$$

$$= \gamma_+(A^+) - \gamma_+(A^-) + \gamma_+(B^+) - \gamma_+(B^-)$$

$$= \gamma_{SA}(A) + \gamma_{SA}(B).$$

<u>Step 3</u>: Extend γ_{SA} to $\mathcal{B}(\mathcal{H})$ by writing

$$\Phi_\gamma(A) = \gamma_{SA}(\text{Re } A) + \sqrt{-1}\,\gamma_{SA}(\text{Im } A).$$

N.B. From the definitions,

$$\Phi_\gamma | E(H) = \gamma.$$

We then claim that $\Phi_\gamma \in \text{Aut } B(H)$. In any event, Φ_γ is unital ($\gamma(I) = I$ (cf. 4.13)) and positive (by construction). Moreover, Φ_γ is surjective (its range contains $E(H)$). As for injectivity, suppose that $\Phi_\gamma(A) = 0$, hence

$$\begin{cases} \gamma_{SA}(\text{Re } A) = 0 \\ \gamma_{SA}(\text{Im } A) = 0. \end{cases}$$

To conclude that $A = 0$, it suffices to show that $\text{Re } A = 0$ and $\text{Im } A = 0$. Let us check this for Re A, the argument for Im A being analogous. Thus write

$$\text{Re } A = (\text{Re } A)^+ - (\text{Re } A)^-.$$

Then

$$\gamma_{SA}(\text{Re } A) = 0 \Rightarrow \gamma_+((\text{Re } A)^+) = \gamma_+((\text{Re } A)^-).$$

If $(\text{Re } A)^+ = 0$ but $(\text{Re } A)^- \neq 0$, then

$$0 = \gamma_+((\text{Re } A)^-)$$

$$= ||(\text{Re } A)^-||\gamma\left(\frac{(\text{Re } A)^-}{||(\text{Re } A)^-||}\right)$$

$\Rightarrow$

$$\gamma\left(\frac{(\text{Re } A)^-}{||(\text{Re } A)^-||}\right) = 0 \Rightarrow (\text{Re } A)^- = 0,$$

a contradiction. Accordingly, we can assume that both $(\text{Re } A)^+ \neq 0$ and $(\text{Re } A)^- \neq 0$, so

$$||(\mathrm{Re}\ A)^+||\gamma\left(\frac{(\mathrm{Re}\ A)^+}{||(\mathrm{Re}\ A)^+||}\right) = ||(\mathrm{Re}\ A)^-||\gamma\left(\frac{(\mathrm{Re}\ A)^-}{||(\mathrm{Re}\ A)^-||}\right)$$

=>

$$\gamma\left(\frac{(\mathrm{Re}\ A)^+}{||(\mathrm{Re}\ A)^+|| + ||(\mathrm{Re}\ A)^-||}\right)$$

$$= \frac{||(\mathrm{Re}\ A)^+||}{||(\mathrm{Re}\ A)^+|| + ||(\mathrm{Re}\ A)^-||}\ \gamma\left(\frac{(\mathrm{Re}\ A)^+}{||(\mathrm{Re}\ A)^+||}\right)$$

$$= \frac{||(\mathrm{Re}\ A)^-||}{||(\mathrm{Re}\ A)^+|| + ||(\mathrm{Re}\ A)^-||}\ \gamma\left(\frac{(\mathrm{Re}\ A)^-}{||(\mathrm{Re}\ A)^-||}\right)$$

$$= \gamma\left(\frac{(\mathrm{Re}\ A)^-}{||(\mathrm{Re}\ A)^+|| + ||(\mathrm{Re}\ A)^-||}\right)$$

=>

$$(\mathrm{Re}\ A)^+ = (\mathrm{Re}\ A)^- \Rightarrow \mathrm{Re}\ A = 0.$$

Therefore the bottom line is that the arrow

$$\mathrm{Aut}\ \mathcal{B}(\mathcal{H}) \to \mathrm{Aut}\ \mathcal{E}(\mathcal{H})$$

of restriction is bijective.

4.19 <u>REMARK</u> Consider a bijection $\gamma:\mathcal{E}(\mathcal{H}) \to \mathcal{E}(\mathcal{H})$ such that $\forall\ E \in \mathcal{E}(\mathcal{H})$,

$$\gamma(E^\perp) = \gamma(E)^\perp$$

and $\forall\ E,F \in \mathcal{E}(\mathcal{H})$,

$$E \le F \Leftrightarrow \gamma(E) \le \gamma(F).$$

Suppose that $\dim \mathcal{H} > 1$ -- then it can be shown that $\gamma \in \text{Aut}\, \mathcal{E}(\mathcal{H})$.

[Note: The assertion is false if $\mathcal{H} = \underline{C}$. Indeed, on [0,1] the functional equation $f(1 - x) = 1 - f(x)$ admits infinitely many strictly increasing solutions that are not additive. E.g.: Given $c > 1$, consider

$$f_c(x) = \frac{x^c}{x^c + (1 - x)^c} .]$$

In terms of the inclusions

$$\begin{array}{ccccc} \mathcal{W}(\mathcal{H}) & \subset & \mathcal{E}(\mathcal{H}) & & \\ \cup & & \cup & \subset & \mathcal{B}(\mathcal{H}), \\ \mathcal{P}(\mathcal{H}) & \subset & \mathcal{L}(\mathcal{H}) & & \end{array}$$

it remains to consider $\text{Aut}\, \mathcal{L}(\mathcal{H})$, which we shall take to be the set of those bijections $\Lambda: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ such that $\forall\, P \in \mathcal{L}(\mathcal{H})$,

$$\Lambda(P^{\perp}) = \Lambda(P)^{\perp}$$

and $\forall\, P_1, P_2 \in \mathcal{L}(\mathcal{H})$,

$$P_1 \leq P_2 \Leftrightarrow \Lambda(P_1) \leq \Lambda(P_2).$$

Then $\text{Aut}\, \mathcal{L}(\mathcal{H})$ is a group and there is an injective arrow

$$\text{Aut}\, \mathcal{E}(\mathcal{H}) \rightarrow \text{Aut}\, \mathcal{L}(\mathcal{H})$$

of restriction which is trivially bijective if $\dim \mathcal{H} = 1$ but matters are not so simple if $\dim \mathcal{H} > 1$, a condition that will be assumed henceforth.

N.B. If we identify the lattice $\mathcal{L}(\mathcal{H})$ with the lattice of all closed linear subspaces M of $\mathcal{H}$, then $\text{Aut}\, \mathcal{L}(\mathcal{H})$ is the set of those bijective maps Λ that preserve orthogonality and order:

$$\begin{cases} \Lambda(M^{\perp}) = \Lambda(M)^{\perp} \\ M_1 \subset M_2 \iff \Lambda(M_1) \subset \Lambda(M_2). \end{cases}$$

4.20 <u>LEMMA</u> Let $\Lambda \in \text{Aut}\ L(H)$ — then $\Lambda(0) = 0$ and $\Lambda(I) = I$.

<u>PROOF</u> $\forall\ P \in L(H)$,

$$0 \leq P \Rightarrow \Lambda(0) \leq \Lambda(P).$$

But Λ is bijective, thus $\Lambda(0) = 0$. And then

$$\Lambda(I) = \Lambda(0^{\perp}) = \Lambda(0)^{\perp} = 0^{\perp} = I.$$

4.21 <u>LEMMA</u> Let $\Lambda \in \text{Aut}\ L(H)$. Let $P_1, P_2 \in L(H)$ and suppose that $P_1 \perp P_2$ — then $\Lambda(P_1) \perp \Lambda(P_2)$.

<u>PROOF</u> In fact,

$$\begin{aligned} P_1 \perp P_2 &\Rightarrow P_1^{\perp} \geq P_2 \\ &\Rightarrow \Lambda(P_1^{\perp}) = \Lambda(P_1)^{\perp} \geq \Lambda(P_2) \\ &\Rightarrow \Lambda(P_1) \perp \Lambda(P_2). \end{aligned}$$

<u>N.B.</u> $\forall\ \Lambda \in \text{Aut}\ L(H)$,

$$\Lambda(P(H)) = P(H).$$

Write $\text{Aut}_w\ P(H)$ for the set of those bijections $\rho_w : P(H) \to P(H)$ which preserve "transition probability zero", i.e.,

$$\mathrm{tr}(\rho_w(P_1)\rho_w(P_2)) = 0 \iff \mathrm{tr}(P_1P_2) = 0.$$

Then $\mathrm{Aut}_w\ \mathcal{P}(\mathcal{H})$ is a group containing $\mathrm{Aut}\ \mathcal{P}(\mathcal{H})$ as a subgroup.

4.22 <u>EXAMPLE</u> Take $\mathcal{H} = \underline{C}^2$ -- then in the notation of 2.10, the elements of $\mathcal{P}(\mathcal{H})$ are the matrices of the form

$$\frac{1}{2}(I_2 + x\sigma_x + y\sigma_y + z\sigma_z),$$

where $x^2 + y^2 + z^2 = 1$, so the bijections of $\mathcal{P}(\mathcal{H})$ correspond to the bijections

$$(x,y,z) \rightarrow (x',y',z')$$

of $\underline{S}^2$, the unit sphere in $\underline{R}^3$. Using variables $(1,\phi,\theta)$ $(0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi)$, define $f:\underline{S}^2 \rightarrow \underline{S}^2$ as follows:

$$f(1,\phi,\theta) = (1,\phi,\theta) \qquad (\phi \neq \frac{\pi}{2})$$

and

$$f(1,\frac{\pi}{2},\theta) = (1,\frac{\pi}{2},g(\theta)),$$

where

$$\begin{cases} g(\theta) = \dfrac{\theta^2}{\pi} & (0 \leq \theta \leq \pi) \\[2ex] g(\theta) = \dfrac{(\theta - \pi)^2}{\pi} + \pi & (\pi \leq \theta \leq 2\pi). \end{cases}$$

The function $\rho_f:\mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H})$ induced by f preserves transition probability zero, hence $\rho_f \in \mathrm{Aut}_w\ \mathcal{P}(\mathcal{H})$. Still, $f \notin \mathrm{Aut}\ \mathcal{P}(\mathcal{H})$.

[Note: In terms of cardinalities,

$$\#\text{Aut } \mathcal{P}(\underline{C}^2) = \underline{c}$$

and

$$\#\text{Aut}_w \ \mathcal{P}(\underline{C}^2) = 2^{\underline{c}}.]$$

4.23 <u>REMARK</u> The two dimensional situation is the anomaly: If $\dim \mathcal{H} > 2$, then it turns out that

$$\text{Aut } \mathcal{P}(\mathcal{H}) \approx \text{Aut}_w \ \mathcal{P}(\mathcal{H}) \quad \text{(cf. 5.18)}.$$

4.24 <u>LEMMA</u> Let $\rho_w \in \text{Aut}_w \ \mathcal{P}(\mathcal{H})$ -- then there is a unique $\Lambda(\rho_w) \in \text{Aut } \mathcal{L}(\mathcal{H})$ such that $\Lambda(\rho_w)(P) = \rho_w(P)$ for all $P \in \mathcal{P}(\mathcal{H})$.

<u>PROOF</u> Suppose, initially, that M is a nonzero linear subspace of $\mathcal{H}$ (M not necessarily closed). Let

$$\Lambda(\rho_w)(M) = \{x \in \rho_w(P_u)\mathcal{H}: u \in M, ||u|| = 1\},$$

put $\Lambda(\rho_w)(\{0\}) = \{0\}$, and observe that

$$\begin{cases} M = \Lambda(\rho_w)\Lambda(\rho_w^{-1})(M) \\ M = \Lambda(\rho_w^{-1})\Lambda(\rho_w)(M). \end{cases}$$

Next,

$$\begin{cases} u \in M, \ ||u|| = 1 \\ v \in M^\perp, \ ||v|| = 1 \end{cases}$$

=>

$$\mathrm{tr}(P_u P_v) = |\langle u,v\rangle|^2 = 0$$

=>

$$\mathrm{tr}(\rho_w(P_u)\rho_w(P_v)) = 0$$

=>

$$\Lambda(\rho_w)(M) \perp \Lambda(\rho_w)(M^\perp)$$

=>

$$\Lambda(\rho_w)(M^\perp) \subset \Lambda(\rho_w)(M)^\perp.$$

And

$$M^\perp = (\Lambda(\rho_w^{-1})\Lambda(\rho_w)(M))^\perp$$

$$\supset \Lambda(\rho_w^{-1})(\Lambda(\rho_w)(M)^\perp)$$

=>

$$\Lambda(\rho_w)(M^\perp) \supset \Lambda(\rho_w)(M)^\perp.$$

Therefore

$$\Lambda(\rho_w)(M^\perp) = \Lambda(\rho_w)(M)^\perp.$$

If now M is in addition closed, then $M^{\perp\perp} = M$, so

$$\Lambda(\rho_w)(M) = \Lambda(\rho_w)((M^\perp)^\perp)$$

$$= \Lambda(\rho_w)(M^\perp)^\perp,$$

which implies that $\Lambda(\rho_w)(M)$ is closed as well.

The arrow

$$\begin{cases} \mathrm{Aut}_W\, P(H) \to \mathrm{Aut}\; L(H) \\ \rho_W \to \Lambda(\rho_W) \end{cases}$$

respects composition and is bijective. In fact, injectivity is obvious while

$$\Lambda \in \mathrm{Aut}\; L(H) \Rightarrow \Lambda | P(H) \in \mathrm{Aut}_W\, P(H)$$

and this proves surjectivity.

§5. *GLEASON'S THEOREM*

The set $L(H)$ of orthogonal projections is a complete orthomodular lattice.

N.B. If $\{P_i : i \in I\}$ is a collection of orthogonal projections, then

$$\begin{cases} \bigwedge_{i\in I} P_i = \text{g.}\ell\text{.b. } \{P_i : i \in I\} \\ \bigvee_{i\in I} P_i = \ell\text{.u.b. } \{P_i : i \in I\}. \end{cases}$$

[Note: $\bigwedge_{i\in I}$ is the orthogonal projection with range

$$\bigcap_{i\in I} \text{Ran } P_i$$

and $\bigvee_{i\in I} P_i$ is the orthogonal projection with range the closure of the linear span of

$$\bigcup_{i\in I} \text{Ran } P_i.]$$

5.1 LEMMA If $\{P_i : i \in I\}$ is a collection of mutually orthogonal projections, then

$$\bigvee_{i\in I} P_i = \sum_{i\in I} P_i.$$

A charge on $L(H)$ is a function $\mu : L(H) \to [0,1]$ such that $\mu(I) = 1$ and $\forall\ P_1, P_2 \in L(H)$:

$$P_1 \perp P_2 \Rightarrow \mu(P_1 + P_2) = \mu(P_1) + \mu(P_2).$$

[Note: It then follows that $\mu(0) = 0$.]

5.2 <u>LEMMA</u> Let μ be a charge on $L(H)$ -- then μ is monotone, i.e.,

$$P_1 \leq P_2 \Rightarrow \mu(P_1) \leq \mu(P_2).$$

<u>PROOF</u> In fact,

$$P_2 = (P_2 - P_1) + P_1$$

$\Rightarrow$

$$\mu(P_2) = \mu(P_2 - P_1) + \mu(P_1)$$

$\Rightarrow$

$$\mu(P_2 - P_1) = \mu(P_2) - \mu(P_1)$$

$\Rightarrow$

$$\mu(P_1) \leq \mu(P_2).$$

5.3 <u>LEMMA</u> Let μ be a charge on $L(H)$. Assume: $P_1P_2 = P_2P_1$ -- then

$$\mu(P_1 \wedge P_2) + \mu(P_1 \vee P_2) = \mu(P_1) + \mu(P_2).$$

<u>PROOF</u> Since P_1P_2, $P_1(I - P_2)$, and $P_2(I - P_1)$ are mutually orthogonal projections, we have

$$\mu(P_1P_2) + \mu(P_1(I - P_2)) + \mu(P_2(I - P_1))$$

$$= \mu(P_1P_2 + P_1 - P_1P_2 + P_2 - P_2P_1)$$

$$= \mu(P_1 + P_2 - P_1P_2)$$

$$= \mu(P_1 \vee P_2).$$

But

$$\begin{cases} P_1 \perp P_2(I - P_1) \\ P_2 \perp P_1(I - P_2) \end{cases}$$

=>

$$\begin{cases} \mu(P_1) + \mu(P_2(I - P_1)) = \mu(P_1 + P_2 - P_1P_2) \\ \mu(P_2) + \mu(P_1(I - P_2)) = \mu(P_1 + P_2 - P_1P_2) \end{cases} = \mu(P_1 \vee P_2).$$

Therefore

$$\mu(P_1 \vee P_2)$$

$$= \mu(P_1P_2) + \mu(P_1(I - P_2)) + \mu(P_2(I - P_1))$$

$$= \mu(P_1 \wedge P_2) + \mu(P_1(I - P_2)) + \mu(P_2(I - P_1))$$

$$= \mu(P_1 \wedge P_2) + \mu(P_1 \vee P_2) - \mu(P_2) + \mu(P_1 \vee P_2) - \mu(P_1)$$

=>

$$\mu(P_1 \wedge P_2) + \mu(P_1 \vee P_2) = \mu(P_1) + \mu(P_2).$$

[Note: It is thus a corollary that

$$\mu(P_1 \vee P_2) \leq \mu(P_1) + \mu(P_2)$$

provided $P_1P_2 = P_2P_1$.]

5.4 <u>LEMMA</u> Let μ be a charge on $L(H)$. Suppose that $\{P_i : i \in I\}$ is a collection of mutually orthogonal projections -- then

$$I^+ = \{i \in I : \mu(P_i) > 0\}$$

is at most countable.

Let μ be a charge on $L(H)$ -- then an element $P \in L(H)$ is said to be <u>μ-null</u> if $\mu(P) = 0$. E.g.: 0 is μ-null. If the set of μ-null elements has a greatest member P_μ, then $I - P_\mu$ is called the <u>support of μ</u>, written spt μ.

5.5 <u>LEMMA</u> Suppose that spt μ exists -- then

$$\mu(P) = 0 \iff P \perp \text{spt}\,\mu.$$

5.6 <u>EXAMPLE</u> Fix a unit vector $x \in H$ and define $\mu_x : L(H) \to [0,1]$ by

$$\mu_x(P) = \text{tr}(P_x P) = \langle x, Px \rangle.$$

Then μ_x is a charge on $L(H)$ and

$$\mu_x(P) = 0 \iff P \perp P_x.$$

Therefore the set of μ_x-null elements has a greatest member, viz. $P_{\mu_x} = I - P_x$, hence spt $\mu_x = P_x$.

Let μ be a charge on $L(H)$ -- then μ is said to satisfy the <u>J-P condition</u> if

$$\left[\begin{array}{l} \mu(P_1) = 0 \\ \\ \mu(P_2) = 0 \end{array}\right. \quad \Rightarrow \mu(P_1 \vee P_2) = 0.$$

[Note: "J-P" stands for Jauch-Piron.]

5.7 <u>LEMMA</u> Suppose that spt μ exists -- then μ satisfies the J-P condition.

<u>PROOF</u> For

$$\left.\begin{array}{l} \mu(P_1) = 0 \\ \\ \mu(P_2) = 0 \end{array}\right] \Rightarrow P_1, P_2 \leq P_\mu .$$

On the other hand,

$$\left.\begin{array}{l} P_1 \leq P_1 \vee P_2 \\ \\ P_2 \leq P_1 \vee P_2 \end{array}\right] \Rightarrow P_1 \vee P_2 \leq P_\mu \Rightarrow \mu(P_1 \vee P_2) \leq \mu(P_\mu) = 0 \quad \text{(cf. 5.2)}.$$

• A charge μ on $L(H)$ is said to be <u>σ-additive</u> if for any sequence $\{P_n : n \in \underline{N}\}$ of mutually orthogonal projections,

$$\mu(\bigvee_{n\in\underline{N}} P_n) = \sum_{n\in\underline{N}} \mu(P_n).$$

[Note: According to 5.1,

$$\bigvee_{n\in\underline{N}} P_n = \sum_{n\in\underline{N}} P_n .]$$

• A charge μ on $L(H)$ is said to be <u>completely additive</u> if for any collection $\{P_i : i \in I\}$ of mutually orthogonal projections,

$$\mu(\bigvee_{i\in I} P_i) = \sum_{i\in I} \mu(P_i).$$

[Note: According to 5.1,

$$\bigvee_{i\in I} P_i = \sum_{i\in I} P_i.]$$

5.8 THEOREM Suppose that μ is σ-additive -- then spt μ exists iff μ is completely additive and satisfies the J-P condition.

PROOF Assume first that spt μ exists -- then in view of 5.7, we have only to show that μ is completely additive. Introduce I^+ as in 5.4 and put

$$P = \bigvee_{i\in I} P_i,\ P^+ = \bigvee_{i\in I^+} P_i,\ P_0 = \bigvee_{i\in I-I^+} P_i.$$

Then

$$P = P^+ \vee P_0,\ P^+P_0 = 0,$$

and

$$\mu(P) = \mu(P^+) + \mu(P_0) \quad \text{(cf. 5.3)}.$$

But

$$\begin{aligned} i \in I - I^+ &\Rightarrow P_i \leq P_\mu \\ &\Rightarrow P_0 \leq P_\mu \\ &\Rightarrow \mu(P_0) = 0. \end{aligned}$$

Since μ is σ-additive, it follows that

$$\begin{aligned} \mu(P) &= \mu(P^+) \\ &= \sum_{i\in I^+} \mu(P_i) = \sum_{i\in I} \mu(P_i). \end{aligned}$$

I.e.: μ is completely additive. Turning to the converse, we distinguish two cases.

<u>Case 1</u>: $\mu(P) > 0 \ \forall \ P \neq 0$. In this situation, it is clear that spt $\mu = I$.

<u>Case 2</u>: $\exists \ P \in L(H): P \neq 0$ and $\mu(P) = 0$. Zornify to get a maximal collection $\{P_i : i \in I\}$ of mutually orthogonal projections such that $\forall \ i$, $P_i \neq 0$ and $\mu(P_i) = 0$. Set

$$P_\mu = \bigvee_{i \in I} P_i.$$

Then by complete additivity, $\mu(P_\mu) = 0$ and the claim is that P_μ is the greatest μ-null element in $L(H)$. To see this, note that

$$\mu(P) = 0 \Rightarrow \mu(P \vee P_\mu) = 0 \quad \text{(J-P condition)}$$

$$\Rightarrow \mu(P \vee P_\mu - P_\mu) = 0.$$

However

$$(P \vee P_\mu - P_\mu) \perp P_\mu,$$

so by maximality,

$$P \vee P_\mu - P_\mu = 0.$$

Therefore

$$P \leq P \vee P_\mu = P_\mu,$$

as claimed.

5.9 <u>EXAMPLE</u> Fix $W \in \mathcal{W}(H)$ and put

$$\mu_W(P) = \mathrm{tr}(WP) \qquad (P \in L(H)).$$

Then μ_W is a σ-additive charge on $L(H)$. Let P_W be the orthogonal projection onto Ker W, so

$$\mu_W(P_W) = \mathrm{tr}(WP_W) = 0,$$

and P_W is the greatest μ_W-null element in $L(H)$ (hence $I - P_W = \mathrm{spt}\ \mu_W$). For suppose that $\mu_W(P) = 0$. Take an orthonormal basis $\{e_i : i \in I\}$ for Ran P and note that

$$0 = \mu_W(P) = \mathrm{tr}(WP)$$

$$= \sum_{i \in I} \langle e_i, WPe_i \rangle$$

$$= \sum_{i \in I} ||\sqrt{W}\, e_i||^2$$

$\Rightarrow$

$$\sqrt{W}\, e_i = 0 \Rightarrow We_i = 0$$

$\Rightarrow$

$$\mathrm{Ran}\ P \subset \mathrm{Ran}\ P_W \Rightarrow P \leq P_W.$$

Owing now to 5.8, μ_W is completely additive and satisfies the J-P condition.

[Note: The results embodied in 2.5 and 2.6 imply that, a priori, μ_W is a completely additive charge on $L(H)$.]

5.10 <u>THEOREM</u> Assume: $\dim H \neq 2$. Suppose that μ is σ-additive -- then μ is completely additive iff $\exists\ W \in W(H) : \mu = \mu_W$.

<u>N.B.</u> The point, of course, is the representation of a completely additive μ as

a μ_W.

[Note: The one dimensional case is trivial: $L(H) = \{0,I\}$ and

$$\begin{cases} \mu(0) = 0 \\ \mu(I) = 1 \end{cases} \Rightarrow \mu = \mu_I.$$

In general, the range of μ_W is $\{0,1\}$ only when $\dim H = 1$ (use a decomposition of W (cf. 2.12)).]

5.11 <u>REMARK</u> Gleason's theorem is 5.10 in the case when H is separable of dimension > 2 (σ-additivity and complete additivity are one and the same in the separable setting).

Let $\underline{S}(H) = \{x \in H: ||x|| = 1\}$ -- then a function $f:\underline{S}(H) \to \underline{R}$ is a <u>frame function</u> if $f(cx) = f(x)$ $\forall$ $c \in \underline{C}: |c| = 1$ and $\exists$ a constant $C(f)$ (the <u>weight</u> of f) such that $\forall$ orthonormal basis $\{e_i : i \in I\}$, the series $\sum_{i \in I} f(e_i)$ is absolutely convergent and has sum $C(f)$. E.g.: If $T \in \underline{L}_1(H)$ and is selfadjoint, then the function

$$f_T(x) = \langle x,Tx\rangle \quad (x \in \underline{S}(H))$$

is a frame function.

5.12 <u>LEMMA</u> Suppose that $\dim H > 2$ and let $f:\underline{S}(H) \to \underline{R}_{\geq 0}$ be a nonnegative frame function -- then $\exists$ a selfadjoint $T \in \underline{L}_1(H)$ such that $f = f_T$.

This is the technical crux of the matter but the proof is a bit involved so we'll postpone it for now (see the Appendix at the end of the §).

Returning to 5.10, let μ be a completely additive charge on $L(H)$. Define $f_\mu : \underline{S}(H) \to \underline{R}_{\geq 0}$ by

$$f_\mu(x) = \mu(P_x).$$

Then for any orthonormal basis $\{e_i : i \in I\}$, the complete additivity of μ gives

$$\begin{aligned}\sum_{i\in I} f_\mu(e_i) &= \sum_{i\in I} \mu(P_{e_i}) \\ &= \mu(\bigvee_{i\in I} P_{e_i}) \\ &= \mu(I) \\ &= 1.\end{aligned}$$

Therefore f_μ is a nonnegative frame function, hence by 5.12, $\exists\ W \in W(H)$:

$$f_\mu = f_W.$$

To prove that $\mu = \mu_W$, take a $P \in L(H)$ and choose an orthonormal basis $\{e_i : i \in I\}$, where $I = J \cup K$ $(J \cap K = \emptyset)$ and

$$\left[\begin{array}{l}\{e_j : j \in J\} \text{ an orthonormal basis for Ran } P \\ \{e_k : k \in K\} \text{ an orthonormal basis for } (\text{Ran } P)^\perp.\end{array}\right.$$

Then using once again the complete additivity of μ, we have

$$\begin{aligned}\mu(P) &= \mu(\bigvee_{j\in J} P_{e_j}) \\ &= \sum_{j\in J} \mu(P_{e_j})\end{aligned}$$

$$= \sum_{j \in J} f_\mu(e_j)$$

$$= \sum_{j \in J} f_W(e_j)$$

$$= \sum_{j \in J} \langle e_j, We_j \rangle$$

$$= \sum_{j \in J} \langle e_j, WPe_j \rangle$$

$$= \mathrm{tr}(WP)$$

$$= \mu_W(P).$$

So, modulo 5.12, the proof of 5.10 is complete.

5.13 EXAMPLE Gleason's theorem is false if $\mathcal{H} = \underline{C}^2$. Thus fix a set $\mathcal{R}$ of representatives for the antipodal equivalence relation on $\underline{S}^2$ and let

$$\begin{cases} \mu(0) = 0 \\ \mu(I) = 1 \end{cases}, \quad \mu(P) = \begin{cases} 1 & (P \in \mathcal{R}) \\ 0 & (P \notin \mathcal{R}). \end{cases}$$

Then $\nexists\, W : \mu = \mu_W$.

Additional insight into the structure of Aut $L(\mathcal{H})$ can be obtained by applying the machinery developed above.

5.14 LEMMA Let $\Lambda \in$ Aut $L(\mathcal{H})$. Suppose that $\{P_i : i \in I\}$ is a collection of orthogonal projections -- then

$$\begin{cases} \Lambda(\bigvee_{i\in I} P_i) = \bigvee_{i\in I} \Lambda(P_i) \\ \Lambda(\bigwedge_{i\in I} P_i) = \bigwedge_{i\in I} \Lambda(P_i). \end{cases}$$

5.15 <u>EXAMPLE</u> Fix $W \in \mathcal{W}(\mathcal{H})$, $\Lambda \in \text{Aut } \mathcal{L}(\mathcal{H})$, and put

$$\mu_{W,\Lambda}(P) = \text{tr}(W\Lambda(P)) \quad (P \in \mathcal{L}(\mathcal{H})).$$

Then $\mu_{W,\Lambda}$ is a completely additive charge on $\mathcal{L}(\mathcal{H})$. In fact, $\Lambda(I) = I$ (cf. 4.20), hence

$$\mu_{W,\Lambda}(I) = \text{tr}(W\Lambda(I)) = \text{tr}(W) = 1.$$

And if $\{P_i : i \in I\}$ is a collection of mutually orthogonal projections:

$$\begin{aligned} &\mu_{W,\Lambda}(\bigvee_{i\in I} P_i) \\ &\quad = \text{tr}(W\Lambda(\bigvee_{i\in I} P_i)) \\ &\quad = \text{tr}(W \bigvee_{i\in I} \Lambda(P_i)) \\ &\quad = \mu_W(\bigvee_{i\in I} \Lambda(P_i)) \\ &\quad = \sum_{i\in I} \mu_W(\Lambda(P_i)) \\ &\quad = \sum_{i\in I} \mu_{W,\Lambda}(P_i). \end{aligned}$$

5.16 <u>LEMMA</u> Assume: $\dim H > 2$ -- then $\forall\ \Lambda \in \text{Aut}\ L(H)$, there is a unique $\zeta(\Lambda) \in \text{Aut}\ W(H)$ such that

$$\zeta(\Lambda)(P) = \Lambda(P)$$

for all $P \in P(H)$.

<u>PROOF</u> In the notation of 5.15, consider $\mu_{W,\Lambda^{-1}}$ and determine $W' \in W(H)$ per 5.10: $\forall\ P \in L(H)$,

$$\mu_{W,\Lambda^{-1}}(P) = \mu_{W'}(P).$$

Definition:

$$\zeta(\Lambda)(W) = W'.$$

Then $\zeta(\Lambda) \in \text{Aut}\ W(H)$. To confirm that

$$\zeta(\Lambda)(P) = \Lambda(P)$$

for all $P \in P(H)$, it suffices to show that

$$\text{tr}(\zeta(\Lambda)(P_1)P_2) = \text{tr}(\Lambda(P_1)P_2)$$

for all $P_1,P_2 \in P(H)$ (cf. 7.4). But

$$\zeta(\Lambda)|P(H) \in \text{Aut}\ P(H) \quad \text{(cf. 4.10)}.$$

Therefore

$$\text{tr}(\zeta(\Lambda)(P_1)P_2)$$

$$= \text{tr}(\zeta(\Lambda)(P_1)\zeta(\Lambda)(\zeta(\Lambda)^{-1}(P_2)))$$

$$= \text{tr}(P_1\zeta(\Lambda)^{-1}(P_2))$$

$$= \text{tr}(P_1\zeta(\Lambda^{-1})(P_2))$$

$$= \mathrm{tr}(\zeta(\Lambda^{-1})(P_2)P_1)$$

$$= \mathrm{tr}(P_2\Lambda(P_1))$$

$$= \mathrm{tr}(\Lambda(P_1)P_2).$$

The arrow

$$\begin{cases} \mathrm{Aut}\ L(H) \rightarrow \mathrm{Aut}\ W(H) \\ \Lambda \rightarrow \zeta(\Lambda) \end{cases}$$

respects composition. Moreover, it is injective. For suppose that

$$\zeta(\Lambda)(W) = W \quad (W \in W(H)).$$

Then

$$\zeta(\Lambda)(P) = P \quad (P \in P(H))$$

or still,

$$\Lambda(P) = P \quad (P \in P(H)).$$

But this implies that Λ is the identity map. Thus fix $P_0 \in L(H)$ and write

$$P_0 = \bigvee_{i \in I} P_i,$$

where P_i ranges over the elements of $P(H)$ which are $\leq P_0$ — then

$$\Lambda(P_0) = \bigvee_{i \in I} \Lambda(P_i) \quad \text{(cf. 5.14)}$$

$$= \bigvee_{i \in I} P_i$$

$$= P_0.$$

Consider now the diagram

$$\begin{array}{ccc} \text{Aut } L(H) & \longrightarrow & \text{Aut } W(H) \\ \uparrow & & \downarrow \\ \text{Aut}_W \, P(H) & \longleftarrow & \text{Aut } P(H) \end{array} \qquad (\dim H > 2).$$

As it stands, the vertical arrows are isomorphisms and the horizontal arrows are one-to-one but it will be shown below that they too are isomorphisms.

5.17 <u>LEMMA</u> Consider the diagram of groups

$$\begin{array}{ccc} G_1 & \xrightarrow{\alpha} & G_2 \\ \delta \uparrow & & \downarrow \beta \\ G_4 & \xleftarrow[\gamma]{} & G_3. \end{array}$$

Assume: $\alpha,\beta,\gamma,\delta$ are injective and

$$\beta \circ \alpha \circ \delta \circ \gamma = \mathrm{id}_{G_3}.$$

Then $\alpha,\beta,\gamma,\delta$ are surjective.

<u>PROOF</u> The hypotheses imply that the three remaining compositions are the respective identity maps on G_4, G_1, G_2. E.g.: Given $g_4 \in G_4$,

$$\gamma \circ \beta \circ \alpha \circ \delta(g_4) = g_4'$$

$\Rightarrow$

$$\beta \circ \alpha \circ \delta(\gamma(\beta \circ \alpha \circ \delta(g_4))) = \beta \circ \alpha \circ \delta(g_4')$$

$\Rightarrow$

$$\beta \circ \alpha \circ \delta(g_4) = \beta \circ \alpha \circ \delta(g_4')$$

$$\Rightarrow g_4 = g_4'.$$

And this leads to the surjectivity. E.g.: Given $g_4 \in G_4$,

$$\gamma(g_3) = g_4$$

if $g_3 = \beta \circ \alpha \circ \delta(g_4)$.

Working with Aut $\mathcal{P}(H)$,

$$\begin{aligned} \rho \in \text{Aut } \mathcal{P}(H) &\Rightarrow \rho \in \text{Aut}_w \mathcal{P}(H) \\ &\Rightarrow \Lambda(\rho) \in \text{Aut } \mathcal{L}(H) \\ &\Rightarrow \zeta(\Lambda(\rho)) \in \text{Aut } \mathcal{W}(H). \end{aligned}$$

But the arrow

$$\text{Aut } \mathcal{W}(H) \to \text{Aut } \mathcal{P}(H)$$

is simply the arrow of restriction and $\forall\ P \in \mathcal{P}(H)$,

$$\begin{aligned} \zeta(\Lambda(\rho))(P) &= \Lambda(\rho)(P) \quad \text{(cf. 5.16)} \\ &= \rho(P) \quad \text{(cf. 4.24)}. \end{aligned}$$

Therefore 5.17 is applicable, hence all the arrows in the diagram

$$\begin{array}{ccc} \text{Aut } \mathcal{L}(H) & \longrightarrow & \text{Aut } \mathcal{W}(H) \\ \uparrow & & \downarrow \\ \text{Aut}_w \mathcal{P}(H) & \longleftarrow & \text{Aut } \mathcal{P}(H) \end{array} \qquad (\dim H > 2)$$

are isomorphisms.

5.18 REMARK In particular:

$$\dim \mathcal{H} > 2 \Rightarrow \text{Aut } \mathcal{P}(\mathcal{H}) \approx \text{Aut}_w \mathcal{P}(\mathcal{H}) \quad \text{(cf. 4.23)}.$$

[Note: Suppose that $\dim \mathcal{H} = 2$ and consider the diagram

$$\text{Aut } \mathcal{W}(\mathcal{H}) \to \text{Aut } \mathcal{P}(\mathcal{H}) \to \text{Aut}_w \mathcal{P}(\mathcal{H}) \to \text{Aut } \mathcal{L}(\mathcal{H}).$$

Then the first and third arrows are isomorphisms while the second arrow is injective but not surjective (cf. 4.22).]

N.B. Maintaining the assumption that $\dim \mathcal{H} > 2$, $\forall\ \Lambda \in \text{Aut } \mathcal{L}(\mathcal{H})$, $\exists\ U \in \mathcal{U}(\mathcal{H})$ or $\exists\ \bar{U} \in \overline{\mathcal{U}(\mathcal{H})}$ such that

$$\Lambda = \sigma_U | \mathcal{L}(\mathcal{H}) \text{ or } \Lambda = \sigma_{\bar{U}} | \mathcal{L}(\mathcal{H}).$$

In fact,

$$\Lambda | \mathcal{P}(\mathcal{H}) \in \text{Aut } \mathcal{P}(\mathcal{H})$$

and $\exists\ U \in \mathcal{U}(\mathcal{H})$ or $\exists\ \bar{U} \in \overline{\mathcal{U}(\mathcal{H})}$ such that

$$\Lambda | \mathcal{P}(\mathcal{H}) = \sigma_U | \mathcal{P}(\mathcal{H}) \text{ or } \Lambda | \mathcal{P}(\mathcal{H}) = \sigma_{\bar{U}} | \mathcal{P}(\mathcal{H}).$$

On the other hand, given any $P \in \mathcal{P}(\mathcal{H})$, we can write

$$P = \bigvee_{i \in I} P_i,$$

where P_i ranges over the elements of $\mathcal{P}(\mathcal{H})$ which are $\leq P$. Therefore

$$\Lambda(P) = \bigvee_{i \in I} \Lambda(P_i) \quad \text{(cf. 5.14)}$$

and

$$\bigvee_{i\in I} \Lambda(P_i) = \begin{cases} \bigvee_{i\in I} \sigma_U(P_i) = \sigma_U(\bigvee_{i\in I} P_i) = \sigma_U(P) \\ \\ \bigvee_{i\in I} \sigma_{\bar{U}}(P_i) = \sigma_{\bar{U}}(\bigvee_{i\in I} P_i) = \sigma_{\bar{U}}(P). \end{cases}$$

APPENDIX

ASSERTION G: Suppose that $\dim \mathcal{H} > 2$ and let $f:\underline{S}(\mathcal{H}) \to \underline{R}_{\geq 0}$ be a nonnegative frame function -- then $\exists$ a selfadjoint $T \in \underline{L}_1(\mathcal{H})$ such that $f = f_T$ (cf. 5.12).

I do not intend to give a complete proof of this result. Nevertheless, it is instructive to isolate the essentials behind the argument.

N.B. In what follows, we shall allow our Hilbert spaces to be either real or complex.

[Note: In the real case, a frame function f is "even", i.e., $f(x) = f(-x)$ $(||x|| = 1)$.]

Frame functions of the form f_T, where $T \in \underline{L}_1(\mathcal{H})$ is selfadjoint, are termed admissible.

[Note: Recall that

$$f_T(x) = \langle x,Tx\rangle \quad (||x|| = 1).]$$

The technical key to the whole business is to first prove Assertion G in the special case when $\mathcal{H} = \underline{R}^3$ and $\underline{S}(\mathcal{H}) = \underline{S}^2$. In other words: If $f:\underline{S}^2 \to \underline{R}_{\geq 0}$ is even and $\exists$ C(f) such that for any orthonormal basis $\{e_1,e_2,e_3\}$, $f(e_1) + f(e_2) + f(e_3) = C(f)$, then f is admissible. Proof: Omitted... .

Granted this, the proof of Assertion G hinges on two preliminary steps.

[Note: Every frame function f can be extended to all of H by the prescription

$$F(x) = \begin{cases} ||x||^2 f(\frac{x}{||x||}) & (x \neq 0) \\ 0 & (x = 0).] \end{cases}$$

If H is a complex Hilbert space, then a closed set $S_0 \subset H$ is a <u>real subspace</u> if (i) $a,b \in \underline{R}$ & $u,v \in S_0 \Rightarrow au + bv \in S_0$ and (ii) $< , > |S_0 \times S_0 \subset \underline{R}$.

<u>Step 1</u>: Suppose that H is complex of dimension 2 and let $f:\underline{S}(H) \to \underline{R}_{\geq 0}$ be a nonnegative frame function. Assume: The restriction of f to the unit sphere of every real subspace of H is admissible — then f is admissible.

[Let M = sup f, thus $0 \leq M \leq C(f)$ and f actually takes on the value M. To see this, choose unit vectors x_n:

$$\begin{cases} f(x_n) \to M \\ x_n \to x_0. \end{cases}$$

Set

$$y_n = \frac{<x_n,x_0>}{|<x_n,x_0>|} x_n.$$

Then $||y_n|| = 1$, $\lim y_n = x_0$, and the real linear span $S_0(n)$ of x_0 and y_n is a real subspace, hence $\exists$ a symmetric $T_n : S_0(n) \to S_0(n)$ such that

$$f(x) = \langle x, T_n x\rangle \quad (x \in S_0(n), ||x|| = 1).$$

Write

$$|f(x_0) - M| \le |f(x_0) - f(y_n)| + |f(y_n) - M|.$$

• $|f(x_0) - f(y_n)|$

$$= |\langle x_0, T_n x_0\rangle - \langle y_n, T_n y_n\rangle|$$

$$= |\langle x_0 - y_n, T_n(x_0 + y_n)\rangle|$$

$$\le ||x_0 - y_n|| \; ||x_0 + y_n|| \; ||T_n||$$

$$\le 2||T_n|| \; ||x_0 - y_n||$$

$$\le 2C(f)||x_0 - y_n||.$$

• $|f(y_n) - M|$

$$= |f(\frac{\langle x_n, x_0\rangle}{|\langle x_n, x_0\rangle|} x_n) - M|$$

$$= | \frac{|\langle x_n, x_0\rangle|}{|\langle x_n, x_0\rangle|} f(x_n) - M|$$

$$= |f(x_n) - M|.$$

So

$$f(x_0) = M.$$

Let y_0 be any unit vector orthogonal to x_0 and let S_0 be the real linear span of x_0 and y_0. Note that

$$\begin{cases} F(x_0) = M \\ F(y_0) = C(f) - f(x_0) = C(f) - M. \end{cases}$$

Since the quadratic form corresponding to f in S_0 attains its supremum on the unit sphere at x_0, it follows that

$$F(ax_0 + by_0) = a^2M + b^2(C(f) - M) \qquad (a,b \in \underline{R}).$$

If $a,b \in \underline{C}$, $ab \neq 0$, then

$$F(ax_0 + by_0) = F(\frac{a}{|a|}(|a|x_0 + \frac{b|a|}{a} y_0))$$

$$= F(|a|x_0 + |b|y_0'),$$

where

$$y_0' = \frac{|a|}{a} \frac{b}{|b|} y_0$$

is a unit vector orthogonal to x_0. Therefore

$$F(ax_0 + by_0) = |a|^2M + |b|^2(C(f) - M).$$

The same relation is valid if either a or b vanishes. Consequently, $F(x) = \langle x,Tx \rangle$, T the diagonal matrix

$$\begin{bmatrix} M & 0 \\ 0 & C(f) - M \end{bmatrix}$$

w.r.t. the orthonormal basis $\{x_0,y_0\}$. But this means that f is admissible.]

<u>Step 2</u>: Suppose that H is complex of dimension >2 and let $f:\underline{S}(H) \to \underline{R}_{\geq 0}$ be a nonnegative frame function. Assume: The restriction of f to the unit sphere of every two dimensional subspace of H is admissible — then f is admissible.

[Written out, the assumption is that on each two dimensional subspace $K \subset H$, $\exists$ a selfadjoint $T_K:K \to K$ such that

$$f(x) = \langle x,T_K x\rangle \quad (x \in K, \ ||x|| = 1).$$

If $x,y \in H$ are linearly independent, then x,y span a two dimensional subspace K and we put

$$T(x,y) = \langle x,T_K y\rangle.$$

If $x,y \in H$ are linearly dependent, say $x = cy$ ($c \in \underline{C}$), take for K any two dimensional subspace containing x, let

$$T(x,y) = \langle x,T_K y\rangle,$$

and note that

$$\langle x,T_K y\rangle = \langle x,T_K cy\rangle = c\langle x,T_K x\rangle = cF(x),$$

which is independent of the choice of K. Therefore

$$T:H \times H \to \underline{C}$$

is welldefined. And by construction,

$$\begin{cases} T(x,x) \geq 0 \\ T(x,y) = \overline{T(y,x)} \\ T(x,cy) = cT(x,y) \quad (c \in \underline{C}). \end{cases}$$

We then claim that

$$T(x,y_1) + T(x,y_2) = T(x,y_1 + y_2).$$

In fact,

$$\text{Re } T(x,y) = \frac{1}{4} (F(x + y) - F(x - y)).$$

So

$$\text{Re } T(x,y_1) + \text{Re } T(x,y_2)$$

$$= \frac{1}{4} (F(x + y_1) - F(x - y_1) + F(x + y_2) - F(x - y_2)).$$

But

$$2F(x) + 2F(y) = F(x + y) + F(x - y)$$

$$\Rightarrow$$

$$F(x + y_1) + F(x + y_2) - F(x - y_1) - F(x - y_2)$$

$$= \frac{1}{2} (F(x + y_1 + x + y_2) + F(x + y_1 - x - y_2)$$

$$- F(x - y_1 + x - y_2) - F(x - y_1 - x + y_2))$$

$$= \frac{1}{2} (F(2x + y_1 + y_2) + F(y_1 - y_2)$$

$$- F(2x - y_1 - y_2) - F(- y_1 + y_2))$$

$$= \frac{1}{2} (F(2x + y_1 + y_2) - F(2x - y_1 - y_2))$$

$$\Rightarrow$$

$$\text{Re } T(x,y_1) + \text{Re } T(x,y_2)$$

$$= \frac{1}{2} \frac{1}{4} (F(2x + y_1 + y_2) - F(2x - y_1 - y_2))$$

$$= \frac{1}{2} \text{Re } T(2x,y_1 + y_2)$$

$$= \text{Re } T(x,y_1 + y_2).$$

Next

$$T(x, -\sqrt{-1}\, y) = \operatorname{Re} T(x, -\sqrt{-1}\, y) + \sqrt{-1} \operatorname{Im} T(x, -\sqrt{-1}\, y)$$

or still,

$$T(x, -\sqrt{-1}\, y) = -\sqrt{-1}\, T(x,y) = -\sqrt{-1}\, (\operatorname{Re} T(x,y) + \sqrt{-1} \operatorname{Im} T(x,y))$$

=>

$$\operatorname{Im} T(x,y) = \operatorname{Re} T(x, -\sqrt{-1}\, y)$$

=>

$$\operatorname{Im} T(x,y_1) + \operatorname{Im} T(x,y_2)$$

$$= \operatorname{Re} T(x, -\sqrt{-1}\, y_1) + \operatorname{Re} T(x, -\sqrt{-1}\, y_2)$$

$$= \operatorname{Re} T(x, -\sqrt{-1}\, (y_1 + y_2))$$

$$= \operatorname{Im} T(x, y_1 + y_2).$$

This settles the claim. The final detail is the boundedness of T:

$$|T(x,y)| \leq |\operatorname{Re} T(x,y)| + |\operatorname{Im} T(x,y)|$$

$$= |\operatorname{Re} T(x,y)| + |\operatorname{Re} T(x, -\sqrt{-1}\, y)|$$

$$\leq \frac{1}{4} (F(x + y) + F(x - y) + F(x - \sqrt{-1}\, y) + F(x + \sqrt{-1}\, y))$$

$$\leq \frac{C(f)}{4} (||x + y||^2 + ||x - y||^2 + ||x - \sqrt{-1}\, y||^2 + ||x + \sqrt{-1}\, y||^2)$$

$$\leq C(f)\ (||x||^2 + ||y||^2)$$

=>

$$|T(x,y)| \leq C(f)\ (||x|| = 1,\ ||y|| = 1).$$

Thus there exists a unique $T \geq 0$: $\forall\, x,y \in \mathcal{H}$,

$$T(x,y) = \langle x,Ty \rangle.$$

But this means that f is admissible.]

We are now in a position to prove Assertion G.

Let $\mathcal{K} \subset \mathcal{H}$ be any two dimensional subspace and let $\mathcal{S}_0 \subset \mathcal{K}$ be a real subspace. Take $\mathcal{S}_0$ two dimensional (matters are trivial if $\mathcal{S}_0$ is one dimensional) and fix a unit vector $x_0 \perp \mathcal{S}_0$. In the real linear subspace spanned by x_0 and $\mathcal{S}_0$, the restriction of f to "$\underline{S}^2$" is admissible. Therefore the restriction of f to the unit sphere of $\mathcal{S}_0$ is admissible. So Step 1 implies that $f|\underline{S}(\mathcal{K})$ is admissible and then Step 2 implies that f itself is admissible.

<u>RAPPEL</u> A cardinal κ is <u>measurable</u> if κ is uncountable and $\exists$ a positive measure $m \neq 0$ on the power set of κ such that $m(\{x\}) = 0\ \forall\, x \in \kappa$.

<u>LEMMA</u> Suppose that $\dim \mathcal{H}$ is uncountable -- then every σ-additive charge μ is completely additive iff $\dim \mathcal{H}$ is not measurable.

<u>PROOF</u> Fix a σ-additive charge μ and assume that $\dim \mathcal{H}$ is not measurable. Fix a collection $\{P_i : i \in I\}$ of mutually orthogonal projections. For each $i \in I$, choose an orthonormal basis in Ran P_i and complete their union to an orthonormal basis $\{e_k : k \in K\}$ for $\mathcal{H}$. Define a positive measure $m \neq 0$ on the power set of K by

$$m(L \subset K) = \mu\Big(\bigvee_{\ell \in L} P_{e_\ell}\Big) \quad (L \neq \emptyset)$$

and set $m(\emptyset) = 0$ -- then m is finite, hence

$$D = \{k \in K : m(\{k\}) > 0\}$$

is at most countable. Because K and K-D have the same cardinality, the restriction of m to the power set of K-D must vanish identically. Let

$$K' = \{k \in K: \exists\ i \in I \text{ st } e_k \in \text{Ran } P_i\}$$

and

$$I' = \{i \in I: \text{Ran } P_i \cap \text{Ran} \bigvee_{k \in D} P_{e_k} \neq 0\}.$$

Then we have

$$\begin{aligned}
\mu(\bigvee_{i \in I} P_i) &= \mu(\bigvee_{k \in K'} P_{e_k}) \\
&= m(K') \\
&= m((K' \cap D) \cup (K' \cap (K-D))) \\
&= m(K' \cap D) + m(K' \cap (K-D)) \\
&= m(K' \cap D) \\
&= \mu(\bigvee_{k \in K' \cap D} P_{e_k}) \\
&= \mu(\bigvee_{i \in I'} P_i) \\
&= \sum_{i \in I'} \mu(P_i) \\
&= \sum_{i \in I} \mu(P_i).
\end{aligned}$$

Therefore μ is completely additive. To treat the converse, fix an orthonormal basis $\{e_k : k \in K\}$ for H and suppose that $m \neq 0$ is a probability measure on 2^K which vanishes at every point of K. Define $\mu: L(H) \to [0,1]$ by

$$\mu(P) = \int_K ||P_{e_k}||^2 dm(k).$$

Then

$$\mu(I) = \int_K ||e_k||^2 dm(k)$$

$$= \int_K 1 dm(k) = 1.$$

Moreover, in view of the monotone convergence theorem, μ is σ-additive, thus by hypothesis is completely additive. Accordingly,

$$1 = \mu(I) = \mu(\bigvee_{k \in K} P_{e_k})$$

$$= \sum_{k \in K} \mu(P_{e_k})$$

$$= 0,$$

a contradiction. So dim H is not measurable.

REMARK Under the preceding assumptions, every σ-additive charge μ on $L(H)$ extends to an element of $S_{\underline{n}}(B(H))$ $(\equiv S(H))$ (cf. 5.10), i.e., extends to a normal state. If, however, μ is merely a charge on $L(H)$, then without any conditions of a set-theoretic nature, it can be shown that μ extends to an element of $S(B(H))$, i.e., extends to a state.

§6. *STATISTICAL MODELS*

An effect algebra is a system $(E,0,1,\oplus)$, where $0,1$ are distinct elements of E and $\oplus$ is a partial binary operation on E that satisfies the following conditions.

E_0: $\forall\, a \in E$, $a \oplus 0$ is defined and equals a.

E_1: If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

E_2: If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

E_3:: $\forall\, a \in E$, there is a unique $a' \in E$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$.

E_4: If $a \oplus 1$ is defined, then $a = 0$.

The elements of E are called effects.

6.1 EXAMPLE Let $(\Omega,\mathcal{A})$ be a measurable space (meaning that Ω is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of Ω). Given $A,B \in \mathcal{A}$, define $A \oplus B = A \cup B$ if $A \cap B = \emptyset$ -- then $(\mathcal{A},\emptyset,\Omega,\oplus)$ is an effect algebra.

6.2 LEMMA The condition E_0 is redundant, i.e., is implied by $E_1 - E_4$.

PROOF Use E_3 to get $1':1 \oplus 1' = 1$ or still, $1' \oplus 1 = 1$ (cf. E_1), hence $1' = 0$ (cf. E_4). Now write

$$1 = 1 \oplus 1'$$

$$= 1 \oplus 0$$

$$= (a \oplus a') \oplus 0 \quad (\text{cf. } E_3)$$

$$= (a' \oplus a) \oplus 0 \quad (\text{cf. } E_1)$$

$$= a' \oplus (a \oplus 0) \quad (\text{cf. } E_2).$$

But

$$a' \oplus a'' = 1 \Rightarrow a'' = a$$

$$\Rightarrow a \oplus 0 = a.$$

6.3 EXAMPLE The unit interval $[0,1]$ is an effect algebra under the partial binary operation $a \oplus b = a + b$ whenever $a + b \leq 1$ (here $a' = 1 - a$).

6.4 EXAMPLE Given a measurable space $(\Omega,\mathcal{A})$, let $\mathcal{E}(\Omega,\mathcal{A})$ be the set of all Borel measurable functions $f:\Omega \to [0,1]$ — then $\mathcal{E}(\Omega,\mathcal{A})$ can be viewed as an effect algebra in the obvious way.

6.5 EXAMPLE If $\mathcal{H}$ is a complex Hilbert space, then $\mathcal{E}(\mathcal{H})$ is an effect algebra: $E \oplus F = E + F$ provided $E + F \leq I$ (here, $E' = E^{\perp} = I - E$).

6.6 LEMMA Let $(E,0,1,\oplus)$ be an effect algebra. Assume: $a \oplus b = a \oplus c$ — then $b = c$.

6.7 LEMMA Let $(E,0,1,\oplus)$ be an effect algebra. Assume: $a \oplus b = 0$ — then $a = b = 0$.

Given an effect algebra $(E,0,1,\oplus)$, write

$$a \leq b \text{ if } \exists\, c \in E : a \oplus c = b.$$

6.8 LEMMA The binary relation $\leq$ is a partial ordering on E and $0 \leq a \leq 1$ for all $a \in E$.

[Note: The fact that $\leq$ is reflexive, i.e., $a \leq a$, follows from E_0: $a \oplus 0 = a$.]

N.B. In 6.3, 6.4, 6.5, $\leq$ is the usual partial ordering.

6.9 REMARK In general, $(E,\leq)$ is not a lattice.

6.10 LEMMA Let $(E,0,1,\oplus)$ be an effect algebra — then $a \oplus b$ is defined iff $a \leq b'$.

6.11 LEMMA Let $(E,0,1,\oplus)$ be an effect algebra — then

$$a \leq b \Rightarrow b' \leq a'.$$

An element $a \in E$ is sharp if $a \wedge a'$ exists and equals 0. Denote the set of sharp elements in E by E_S — then $0,1 \in E_S$ and in 6.3, $E_S = \{0,1\}$, in 6.4, $E_S = \{\chi_A : A \in \mathcal{A}\}$, in 6.5, $E_S = \mathcal{L}(\mathcal{H})$.

The operation $\oplus$ can be extended to any finite set of elements by recursion: $a_1,\ldots,a_n$ are summable if $a_1 \oplus \cdots \oplus a_{n-1}$ exists and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exists, in which case we put

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n.$$

A subset $D \subset E$ is summable if every finite subset of D is summable. We then define the sum $\oplus D$ as the supremum of all partial finite sums (assuming that the supremum exists in E). An effect algebra is a <u>σ-effect algebra</u> if every countable summable subset has a sum.

6.12 <u>LEMMA</u> An effect algebra $(E,0,1,\oplus)$ is a σ-effect algebra iff for every increasing sequence $a_1 \leq a_2 \leq \ldots$, the supremum $\vee_n a_n$ exists.

<u>N.B.</u> The effect algebras in 6.3, 6.4, and 6.5 are σ-effect algebras.

Let E,F be effect algebras -- then a map $\phi:E \rightarrow F$ is <u>additive</u> if

$$a \oplus b \text{ defined} \Rightarrow \phi(a) \oplus \phi(b) \text{ defined}$$

and

$$\phi(a \oplus b) = \phi(a) \oplus \phi(b).$$

If $\phi:E \rightarrow F$ is additive and if $\phi(1) = 1$, then ϕ is a <u>morphism</u>. A morphism ϕ is an <u>isomorphism</u> if ϕ is bijective and ϕ^{-1} is a morphism.

6.13 <u>LEMMA</u> Suppose that $\phi:E \rightarrow F$ is a morphism -- then $\phi(0) = 0$.

<u>PROOF</u> In fact,

$$\begin{aligned} 1 = \phi(1) &= \phi(1 \oplus 0) \\ &= \phi(1) \oplus \phi(0) \\ &= 1 \oplus \phi(0) \\ &= \phi(0) \oplus 1 \end{aligned}$$

$$\Rightarrow$$

$$\phi(0) = 0 \quad (\text{cf. } E_4).$$

6.14 LEMMA Suppose that $\phi:E \to F$ is a morphism -- then $\forall\ a \in E$, $\phi(a') = \phi(a)'$.

PROOF In fact,

$$1 = \phi(1) = \phi(a \oplus a')$$

$$= \phi(a) \oplus \phi(a')$$

$$\Rightarrow$$

$$\phi(a') = \phi(a)' \quad (\text{cf. } E_3).$$

A morphism $\phi:E \to F$ between σ-effect algebras is a σ-morphism if for every increasing sequence $a_1 \leq a_2 \leq \ldots$, we have

$$\phi(\vee_n a_n) = \vee_n \phi(a_n).$$

6.15 EXAMPLE Let $(\Omega,\mathcal{A})$ be a measurable space, $\mathcal{H}$ a complex Hilbert space -- then a semispectral measure on $(\Omega,\mathcal{A})$ is a σ-morphism $\mathcal{A} \to \mathcal{E}(\mathcal{H})$ (cf. §7).

A state on an effect algebra E is a morphism $s:E \to [0,1]$, thus $s(0) = 0$, $s(1) = 1$, and $s(a \oplus b) = s(a) + s(b)$. Write $\mathcal{S}(E)$ for the set of states on E -- then $\mathcal{S}(E)$ is a convex set.

6.16 EXAMPLE Let X be a nonempty set -- then $([0,1]^X,0,1,\oplus)$ is an effect algebra, where $0(x) = 0$, $1(x) = 1$, $(f \oplus g)(x) = f(x) + g(x)$ $(x \in X)$. Denoting by $B(X)$ the Banach space of bounded real valued functions on X, a map $s:[0,1]^X \to [0,1]$ is a state iff $\exists$ a (necessarily unique) positive linear functional

$\lambda:B(X) \to \underline{R}$ with $\lambda(1) = 1$ such that $\lambda|[0,1]^X = s$. In particular: $S([0,1])$ is a singleton, viz. the identity map on $[0,1]$.

6.17 <u>EXAMPLE</u> Let $[0,1]^2 = [0,1] \times [0,1]$ -- then $[0,1]^2$ is an effect algebra, where $(a_1,b_1) \oplus (a_2,b_2)$ is defined iff $a_1 \oplus a_2$ and $b_1 \oplus b_2$ are defined in $[0,1]$, in which case

$$(a_1,b_1) \oplus (a_2,b_2) = (a_1 \oplus a_2, b_1 \oplus b_2).$$

Here

$$\left[\begin{array}{l} S([0,1]^2) = \{s_\alpha : 0 \le \alpha \le 1\} \\ \quad s_\alpha(a,b) = \alpha a + (1 - \alpha)b. \end{array}\right.$$

A <u>σ-state</u> on a σ-effect algebra E is a σ-morphism $s:E \to [0,1]$, hence $s(\bigoplus_n a_n) = \bigoplus_n s(a_n)$. Write $S_\sigma(E)$ for the set of σ-states on E -- then $S_\sigma(E)$ is a σ-convex set.

[Note: Obviously, $S_\sigma(E) \subset S(E)$.]

6.18 <u>EXAMPLE</u> Let (Ω,A) be a measurable space and let $M_1^+(\Omega,A)$ be the convex set of probability measures on (Ω,A). Given $\mu \in M_1^+(\Omega,A)$, put

$$\mu(f) = \int_\Omega f d\mu \quad (f \in E(\Omega,A)).$$

Then $\mu \in S_\sigma(E(\Omega,A))$ (monotone convergence theorem). Moreover, every $s \in S_\sigma(E(\Omega,A))$ is of this form.

6.19 <u>REMARK</u> The extreme points of $M_1^+(\Omega,\mathcal{A})$ are the probability measures μ such that $\forall\ A \in \mathcal{A}$, either $\mu(A) = 0$ or $\mu(A) = 1$ (observe that $0 < \mu(A) < 1$

=>

$$\mu = \mu(A)\left[\frac{\mu(- \cap A)}{\mu(A)}\right] + (1 - \mu(A))\left[\frac{\mu(- \cap (\Omega - A))}{\mu(\Omega - A)}\right]).$$

E.g.: The Dirac measures $\delta_\omega (\omega \in \Omega)$ are extreme points of $M_1^+(\Omega,\mathcal{A})$ (however distinct ω might give rise to the same δ_ω). In general, there are others. Thus take Ω uncountable and let $\mathcal{A}$ be the set of all subsets of Ω that are countable or have a countable complement. Let $\mu(A) = 0$ (A countable), $\mu(A) = 1$ (A uncountable) -- then μ is an extreme point of $M_1^+(\Omega,\mathcal{A})$ but $\mu \neq \delta_\omega$ $(\forall\ \omega \in \Omega)$. On the other hand, if Ω is Polish (i.e., if Ω is a complete separable metric space) and if $\mathcal{A} = \text{Bor}\ \Omega$ (the σ-algebra of Borel subsets of Ω), then the extreme points of $M_1^+(\Omega,\mathcal{A})$ are the Dirac measures, so in this situation there is no pathology.

6.20 <u>EXAMPLE</u> Let $\mathcal{H}$ be a separable complex Hilbert space -- then

$$\mathcal{S}_\sigma(\mathcal{E}(\mathcal{H})) \approx \mathcal{W}(\mathcal{H}).$$

[Employing a step-by-step procedure, one can extend a given $s \in \mathcal{S}_\sigma(\mathcal{E}(\mathcal{H}))$ to an element $\lambda_W \in \mathcal{S}(\mathcal{H})$. And:

$$s(E) = \lambda_W(E) = \text{tr}(WE) \quad (E \in \mathcal{E}(\mathcal{H})).]$$

Fix a Polish space M and let E be a σ-effect algebra — then an <u>observable</u>

on E is a σ-morphism $X:\text{Bor } M \to E$. Spelled out,

$$X(\emptyset) = 0, \ X(M) = 1$$

and

$$S_i \cap S_j = \emptyset (i \neq j) \Rightarrow X(\bigcup_n S_n) = \bigoplus_n X(S_n).$$

We shall call X sharp if $\forall\, S \in \text{Bor } M$, $X(S)$ is sharp.

Notation: $\mathcal{O}_M(E)$ is the set of observables on E.

6.21 EXAMPLE Fix $a \in E$ and take $M = \underline{R}$ -- then the prescription $X_a:\text{Bor } \underline{R} \to E$, where

$$X_a(S) = \begin{cases} 1 \text{ if } \{0,1\} \cap S = \{0,1\} \\ a \text{ if } \{0,1\} \cap S = \{1\} \\ a' \text{ if } \{0,1\} \cap S = \{0\} \\ 0 \text{ if } \{0,1\} \cap S = \emptyset, \end{cases}$$

defines an observable on E, the indicator of a.

6.22 REMARK Every $\mu \in M_1^+(M, \text{Bor } M)$ can be regarded as an observable on $E = [0,1]$.

Suppose given a convex set $\mathcal{S}$ whose elements are called states and a set $\mathcal{O}$ whose elements are called observables.

M_1: To each $\Lambda \in \mathcal{S}$ and to each $X \in \mathcal{O}$, there is attached a probability measure m_Λ^X on Bor M.

M2: $\forall\ \Lambda_1, \Lambda_2 \in \mathcal{S}$ and $0 < w < 1$,

$$m^X_{w\Lambda_1 + (1 - w)\Lambda_2} = w m^X_{\Lambda_1} + (1 - w) m^X_{\Lambda_2} \quad \forall\ X \in \mathcal{O}.$$

M3: $\forall\ X_1 \in \mathcal{O}$ and $\forall$ Borel function $\Phi : M \to M$, $\exists\ X_2 \in \mathcal{O} : \forall\ \Lambda \in \mathcal{S}$,

$$m_\Lambda^{X_2} = m_\Lambda^{X_1} \circ \Phi^{-1}.$$

A pair $(\mathcal{S}, \mathcal{O})$ subject to $M_1 - M_3$ is called a statistical model based on M.

6.23 THEOREM Let E be a σ-effect algebra -- then the pair $(\mathcal{S}_\sigma(E), \mathcal{O}_M(E))$ is a statistical model based on M.

PROOF Given $s \in \mathcal{S}_\sigma(E)$, $X \in \mathcal{O}_M(E)$, put

$$m_s^X = s \circ X \in M_1^+ (M, \text{Bor } M).$$

[Note: $\forall$ Borel function $\Phi : M \to M$, the composition $X \circ \Phi^{-1}$ is again an observable on E.]

We shall now specialize to when $E = \mathcal{E}(\Omega, A)$.

6.24 LEMMA Suppose that

$$X : \text{Bor } M \to \mathcal{E}(\Omega, A)$$

is a sharp observable -- then

$$S \cap T = \emptyset \Rightarrow X(S)X(T) = 0.$$

PROOF Let

$$\begin{cases} X(S) = \chi_A \\ & (A,B \in A). \\ X(T) = \chi_B \end{cases}$$

Then

$$\chi_A + \chi_B \leq 1 \Rightarrow A \cap B = \emptyset$$

$$\Rightarrow \chi_A \chi_B = 0.$$

If $f:\Omega \to M$ is Borel measurable and if

$$X_f(S) = \chi_{f^{-1}(S)} \quad (S \in \text{Bor } M),$$

then

$$X_f:\text{Bor } M \to E(\Omega,A)$$

is a sharp observable.

6.25 LEMMA Suppose that

$$X:\text{Bor } M \to E(\Omega,A)$$

is a sharp observable — then $\exists$ a Borel measurable function $f:\Omega \to M$ such that $X = X_f$.

PROOF To illustrate the ideas, consider the case when $M = \underline{R}$ and, for convenience, work with the elements of A rather than their characteristic functions — then

$$\bigcup_{x \in \underline{R}} X(\{x\}) = \Omega \quad \text{(see below)}$$

and since the union on the LHS is disjoint, we can define $f:\Omega \rightarrow \underline{R}$ by stipulating that $f(\omega)$ is to be the $x \in \underline{R}$ such that $\omega \in X(\{x\})$. Therefore

$$f^{-1}(S) = X(S) \quad (S \in \text{Bor } \underline{R}).$$

Write

$$\underline{R} = \bigcup_{k=-\infty}^{\infty} [k,k+1[.$$

Then

$$\Omega = \bigcup_{k=-\infty}^{\infty} X([k,k+1[).$$

Fix $\omega \in \Omega$ and choose $k:\omega \in X([k,k+1[)$. Let $I = [k,k+1]$:

$$X(I) = X([k,k+1[) \cup X(\{k+1\})$$

$$\Rightarrow \omega \in X(I).$$

Put $I_1 = I$ and define $I_n = [a_n, b_n]$ recursively:

$$\omega \in X(I_n) = X([a_n, \frac{a_n+b_n}{2}]) \cup X([\frac{a_n+b_n}{2}, b_n])$$

$$\Rightarrow \omega \in X(I_{n+1}),$$

where

$$I_{n+1} = [a_n, \frac{a_n+b_n}{2}] \text{ or } [\frac{a_n+b_n}{2}, b_n].$$

By the nested interval principle,

$$\bigcap_{n=1}^{\infty} I_n = \{x\} \quad (\exists\, x \in \underline{R}).$$

But

$$X\left(\bigcap_{n=1}^{\infty} I_n\right) = \bigcap_{n=1}^{\infty} X(I_n)$$

$$\Rightarrow \omega \in X(\{x\}).$$

And this proves that

$$\bigcup_{x \in \underline{R}} X(\{x\}) = \Omega.$$

[Note: The general case is analogous. In this connection, recall that a metric space (X,d) is complete iff for every descending sequence $C_1 \supset C_2 \supset \ldots$ of nonempty closed sets such that $\text{diam } C_n \to 0$, the intersection $\bigcap_n C_n$ is not empty.]

If $\phi: E(M, \text{Bor } M) \to E(\Omega, A)$ is a σ-morphism, then the restriction $\phi|\text{Bor } M$ is an observable. There is also a converse. For suppose that $X: \text{Bor } M \to E(\Omega, A)$ is an observable. Write $X(\omega, S) = X(S)(\omega)$ — then

$$\mu_{X,\omega} = X(\omega, \text{—}) \in M_1^+(M, \text{Bor } M)$$

and the prescription

$$\bar{X}F(\omega) = \int_M F(m) d\mu_{X,\omega}(m) \qquad (F \in E(M, \text{Bor } M))$$

extends X to a σ-morphism

$$\bar{X}: E(M, \text{Bor } M) \to E(\Omega, A).$$

Indeed, $\forall$ $S \in \text{Bor } M$,

$$\bar{X}\chi_S(\omega) = \int_M \chi_S(m) d\mu_{X,\omega}(m)$$

$$= \mu_{X,\omega}(S)$$

$$= X(\omega,S)$$

$$= X(S)(\omega).$$

So, in summary, there is a one-to-one correspondence between observables $X:\text{Bor } M \rightarrow E(\Omega,A)$ and σ-morphisms $\phi:E(M,\text{Bor } M) \rightarrow E(\Omega,A)$.

6.26 EXAMPLE Since

$$S_\sigma(E(\Omega,A)) = M_1^+(\Omega,A) \quad \text{(cf. 6.18)},$$

it follows from 6.23 that the pair

$$(M_1^+(\Omega,A),\mathcal{O}_M(E(\Omega,A)))$$

is a statistical model based on M:

$$m_\mu^X = \mu \circ X \quad (X:\text{Bor } M \rightarrow E(\Omega,A)).$$

[Note: If

$$\begin{cases} X:\text{Bor } M \rightarrow E(\Omega,A) \\ Y:\text{Bor } M \rightarrow E(\Omega,A) \end{cases}$$

are observables and if

$$m_\mu^X = m_\mu^Y \ \forall \ \mu \in M_1^+(\Omega,A),$$

then X = Y. Proof: $\forall$ S $\in$ Bor M and $\forall$ $\omega \in \Omega$,

$$X(S)(\omega) = \delta_\omega(X(S)) = \delta_\omega(Y(S)) = Y(S)(\omega).]$$

6.27 <u>REMARK</u> The arrow

$$\begin{cases} M_1^+(\Omega,A) \to M_1^+(M,\text{Bor } M) \\ \mu \longrightarrow m_\mu^X \end{cases}$$

is an affine map. In addition, it is weakly continuous and

$$m_{\delta_\omega}^X(S) = X(S)(\omega)$$

is a measurable function of ω. Conversely, any affine map $M_1^+(\Omega,A) \to M_1^+(M,\text{Bor } M)$ with these two properties is of this form for a unique observable $X:\text{Bor } M \to E(\Omega,A)$.

§7. *SEMISPECTRAL MEASURES*

Let $(\Omega,\mathcal{A})$ be a measurable space, $\mathcal{H}$ a complex Hilbert space -- then a semispectral measure on $(\Omega,\mathcal{A})$ is a σ-morphism $\mathcal{A} \to \mathcal{E}(\mathcal{H})$ (cf. 6.15). In other words, a semispectral measure on $(\Omega,\mathcal{A})$ is a function $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ such that

$$E(\emptyset) = 0, \ E(\Omega) = I$$

and

$$A_i \cap A_j = \emptyset \ (i \neq j) \Rightarrow E(A_1 \cup A_2 \cup \ldots) = E(A_1) + E(A_2) + \ldots,$$

where the convergence in the summation on the right is in the strong operator topology. A spectral measure on $(\Omega,\mathcal{A})$ is a semispectral measure E such that $\forall\ A \in \mathcal{A}$, $E(A)$ is sharp, i.e., $E(A) \in \mathcal{L}(\mathcal{H})$.

[Note: Suppose that $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ is a function such that

$$E(\emptyset) = 0, \ E(\Omega) = I.$$

Then E is a semispectral measure iff $\forall\ x \in \mathcal{H}$, the map $A \to \langle x,E(A)x\rangle$ is countably additive. So, for instance, given any $\mu \in M_1^+(\Omega,\mathcal{A})$, the prescription

$$E(A) = \mu(A)I \quad (A \in \mathcal{A})$$

defines a semispectral measure.]

7.1 REMARK Write $M(\Omega,\mathcal{A};\mathcal{H})$ for the set of semispectral measures on $(\Omega,\mathcal{A})$ -- then $M(\Omega,\mathcal{A};\mathcal{H})$ is a convex set and every spectral measure is an extreme point but, in general, there will be others (cf. 7.8).

7.2 LEMMA Let $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ be a semispectral measure -- then E is a spectral measure iff $\forall\ A,B \in \mathcal{A}$,

$$E(A \cap B) = E(A)E(B).$$

Let $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ be a semispectral measure -- then $\forall\ W \in \mathcal{W}(\mathcal{H})$, the composition

$$\lambda_W \circ E \in M_1^+(\Omega,\mathcal{A}).$$

And the arrow

$$\begin{cases} \mathcal{W}(\mathcal{H}) \to M_1^+(\Omega,\mathcal{A}) \\ W \to \lambda_W \circ E \end{cases}$$

is an affine map.

7.3 <u>THEOREM</u> If $\Phi:\mathcal{W}(\mathcal{H}) \to M_1^+(\Omega,\mathcal{A})$ is an affine map, then there is a unique semispectral measure $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ such that $\Phi(W) = \lambda_W \circ E$.

[The point is this: Every affine map $f:\mathcal{W}(\mathcal{H}) \to [0,1]$ can be represented by a unique $E \in \mathcal{E}(\mathcal{H})$, i.e., $f(W) = \lambda_W(E) = \text{tr}(WE)$ $(W \in \mathcal{W}(\mathcal{H}))$.]

7.4 <u>RAPPEL</u> Let $A \in \mathcal{B}(\mathcal{H})$. Suppose that $\text{tr}(AW) = 0\ \forall\ W \in \mathcal{W}(\mathcal{H})$ -- then $A = 0$.

[Let $x \in \underline{S}(\mathcal{H})$ and take $W = P_x$:

$$\text{tr}(AP_x) = \langle x,Ax\rangle = 0\ldots\ .]$$

7.5 <u>LEMMA</u> If

$$\begin{cases} E:\mathcal{A} \to \mathcal{E}(\mathcal{H}) \\ F:\mathcal{A} \to \mathcal{E}(\mathcal{H}) \end{cases}$$

are semispectral measures and if

$$\lambda_W \circ E = \lambda_W \circ F \ \forall \ W \in \mathcal{W}(\mathcal{H}),$$

then E = F.

7.6 <u>REMARK</u> Suppose that $\mathcal{H}$ is separable — then $S_\sigma(\mathcal{E}(\mathcal{H})) \approx \mathcal{W}(\mathcal{H})$ (cf. 6.20). Now fix a Polish space M — then the observables X:Bor M $\rightarrow \mathcal{E}(\mathcal{H})$ are precisely the semispectral measures on (M,Bor M):

$$\mathcal{O}_M(\mathcal{E}(\mathcal{H})) = \mathcal{M}(M,\text{Bor } M;\mathcal{H}).$$

And the pair

$$(\mathcal{W}(\mathcal{H}),\mathcal{M}(M,\text{Bor } M;\mathcal{H}))$$

is a statistical model based on M (cf. 6.23).

[Note: Consider the case when $M = \underline{R}$ — then in this situation there is a one-to-one correspondence between the set of selfadjoint operators on $\mathcal{H}$ and the set of spectral measures on ($\underline{R}$,Bor $\underline{R}$). Here, of course, it is a question of potentially unbounded operators A:Dom A $\rightarrow \mathcal{H}$ and we shall denote by E^A the spectral measure Bor $\underline{R} \rightarrow \mathcal{L}(\mathcal{H})$ attached to A (thus for f:$\underline{R} \rightarrow \underline{R}$ Borel, the spectral measure attached to f(A) is the assignment $S \rightarrow E^A(f^{-1}(S))$).]

If $\Omega = \{\omega_1,\omega_2,\ldots\}$ is finite or countable and $\mathcal{A}$ is the set of all subsets of Ω, i.e., $\mathcal{A} = 2^\Omega$, then a semispectral measure $E:\mathcal{A} \rightarrow \mathcal{E}(\mathcal{H})$ is completely determined by the $E(\omega_i) \equiv E(\{\omega_i\})$. So, $\forall$ subset $A \subset \Omega$,

$$E(A) = \sum_{\omega_i \in A} E(\omega_i).$$

In particular:

$$\sum_{\omega_i \in \Omega} E(\omega_i) = I.$$

N.B. Suppose given effects E_k $(k = 1,\ldots,n)$:

$$\sum_{k=1}^{n} E_k = I.$$

Take $\Omega = \{1,\ldots,n\}$, $A = 2^{\Omega}$, and put

$$E(A) = \sum_{k \in A} E_k.$$

Then the arrow

$$E:2^{\Omega} \to E(H)$$

is a semispectral measure.

[Note: One can identify $M_1^+(\Omega,A)$ with the simplex

$$\{(\lambda_1,\ldots,\lambda_n):\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\}.]$$

7.7 EXAMPLE Take $H = \underline{C}^2$ and let

$$E_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E_3 = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad E_4 = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then E_1,E_2,E_3,E_4 are effects and

$$E_1 + E_2 + E_3 + E_4 = \begin{bmatrix} 1 & 0 \\ & \\ 0 & 1 \end{bmatrix}.$$

Therefore this data generates a semispectral measure.

7.8 EXAMPLE Take $\mathcal{H} = \underline{C}^2$ and let P_1,P_2,P_3 be orthogonal projections onto three vectors with angles between them $\frac{2\pi}{3}$ — then

$$\frac{2}{3} P_1 + \frac{2}{3} P_2 + \frac{2}{3} P_3 = I.$$

The associated semispectral measure is an extreme point of

$$\mathcal{M}(\Omega,2^\Omega;\underline{C}^2) \qquad (\Omega = \{1,2,3\})$$

but it is not a spectral measure.

If $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ is a semispectral measure, then $\forall\ x,y \in \mathcal{H}$, the arrow

$$A \to \langle x,E(A)y\rangle$$

is a complex measure $\mu_{x,y}$ on Ω whose total variation is at most $||x||\ ||y||$. Given a bounded measurable function $f:\Omega \to \underline{C}$, put

$$B_f(x,y) = \int_\Omega f(\omega)d\mu_{x,y}(\omega).$$

Then

$$B_f:\mathcal{H} \times \mathcal{H} \to \underline{C}$$

is a sesquilinear form. Since

$$|B_f(x,y)| \leq ||f||_\infty \, ||x|| \, ||y||,$$

it follows that $\exists$ a bounded linear operator $T_f \in \mathcal{B}(H)$ such that

$$B_f(x,y) = \langle x, T_f y \rangle$$

for all $x,y \in H$ and $||T_f|| \leq ||f||_\infty$. In suggestive notation, one writes

$$T_f = \int_\Omega f(\omega) dE(\omega).$$

Example: $\forall\, A \in \mathcal{A}$, $T_{\chi_A} = E(A)$.

7.9 <u>LEMMA</u> We have

$$\begin{cases} T_{f+g} = T_f + T_g \\ T_{cf} = cT_f \\ T_{\bar{f}} = T_f^*. \end{cases}$$

7.10 <u>RAPPEL</u> If S is a subset of $\mathcal{B}(H)$, then its <u>commutant</u> S' is

$$\{T \in \mathcal{B}(H) : TS = ST \ \forall\, S \in S\}$$

and its <u>bicommutant</u> S'' is $(S')'$. ETC:

$$\begin{cases} S \subset S'' \\ S' = S''' = \ldots \\ S'' = S'''' = \ldots . \end{cases}$$

[Note: S is commutative iff $S \subset S'$. And

$$\mathcal{S} \subset \mathcal{S}' \Rightarrow \mathcal{S}'' \subset \mathcal{S}' = \mathcal{S}''' = (\mathcal{S}'')'.$$

Therefore $\mathcal{S}''$ is commutative.]

Put

$$\mathcal{S}^* = \{S^*: S \in \mathcal{S}\}$$

and call $\mathcal{S}$ <u>selfadjoint</u> if $\mathcal{S} = \mathcal{S}^*$. If $\mathcal{S}$ is selfadjoint, then $\mathcal{S}'$ and $\mathcal{S}''$ are $*$-subalgebras of $\mathcal{B}(\mathcal{H})$.

[Note: A $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is called a <u>W*-algebra</u> if it coincides with its bicommutant.]

The set

$$\{E\} = \{E(A): A \in \mathcal{A}\}$$

is selfadjoint, hence $\{E\}'$ and $\{E\}''$ are $*$-subalgebras of $\mathcal{B}(\mathcal{H})$.

7.11 <u>LEMMA</u> $\forall$ f,

$$T_f \in \{E\}''.$$

A semispectral measure $E: \mathcal{A} \to \mathcal{E}(\mathcal{H})$ is said to be <u>commutative</u> if $\forall$ $A, B \in \mathcal{A}$,

$$E(A)E(B) = E(B)E(A).$$

[Note: Every spectral measure is commutative (cf. 7.2).]

7.12 <u>LEMMA</u> Suppose that E is commutative -- then E'' is commutative, hence $\forall$ f,g,

$$T_f T_g = T_g T_f.$$

7.13 LEMMA Suppose that E is commutative -- then $\forall$ f, T_f is normal:

$$T_f T_f^* = T_f^* T_f.$$

[Note: Recall that $T_f^* = T_{\bar{f}}$.]

N.B. If E is a spectral measure, then $\forall$ f,g,

$$T_f T_g = T_{fg}.$$

So, in this situation, the arrow $f \to T_f$ is a norm decreasing $*$-homomorphism from the commutative Banach algebra $B(\Omega)$ of bounded complex valued measurable functions on Ω into an algebra of normal operators on $\mathcal{H}$.

7.14 REMARK Let A be selfadjoint (unbounded in general). Given $\lambda \in \underline{R}$, put $E_\lambda^A = E^A(]-\infty,\lambda])$ -- then $\forall\ x \in \mathcal{H}$, $F_x(\lambda) = \langle x, E_\lambda^A x\rangle$ is an increasing right continuous function on $\underline{R}$ and

$$\begin{aligned}\mu_{x,x}(]a,b]) &= \langle x, E^A(]a,b])x\rangle \\ &= F_x(b) - F_x(a),\end{aligned}$$

thus $\mu_{x,x}$ is the Stieltjes measure induced by F_x (and F_x is the cumulative distribution function of $\mu_{x,x}$). Here

$$\text{Dom } A = \{x \in \mathcal{H} : \int_{\underline{R}} \lambda^2 d\langle x, E_\lambda^A x\rangle < \infty\}$$

and $\forall\ x \in$ Dom A,

$$\begin{cases} \langle x,Ax\rangle = \int_{\underline{R}} \lambda d\langle x,E_\lambda^A x\rangle \\ ||Ax||^2 = \int_{\underline{R}} \lambda^2 d\langle x,E_\lambda^A x\rangle. \end{cases}$$

[Note: If $x \neq y$, then the function $\lambda \rightarrow \langle x,E_\lambda^A y\rangle$ is of bounded variation (as can be seen by polarization) and $\mu_{x,y}$ is the associated Stieltjes measure.]

7.15 RAPPEL Assume that H is separable and let $\{A_i : i \in I\}$ be a commutative set of bounded selfadjoint operators -- then $\exists\ A \in B(H)_{SA}$ and Borel functions $f_i:\underline{R} \rightarrow \underline{R}$ such that $\forall\ i$,

$$A_i = f_i(A) = \int_{\underline{R}} f_i dE^A.$$

7.16 EXAMPLE Assume that H is separable and let $E:\text{Bor}\ \underline{R} \rightarrow E(H)$ be a commutative semispectral measure -- then $\exists\ A \in B(H)_{SA}$ and Borel functions $w_S:\underline{R} \rightarrow \underline{R}$ such that $\forall\ S$,

$$E(S) = \int_{\underline{R}} w_S dE^A.$$

We turn now to a method of constructing semispectral measures from spectral measures.

Suppose given two measurable spaces (Ω_1,A_1) and (Ω_2,A_2) -- then a probability kernel is a map

$$K:\Omega_1 \times A_2 \rightarrow [0,1]$$

such that $K(\text{—},A_2)$ is measurable for every $A_2 \in A_2$ and $K(\omega_1,\text{—}) \in M_1^+(\Omega_2,A_2)$ for every $\omega_1 \in \Omega_1$. E.g.: If $f:\Omega_1 \to \Omega_2$ is measurable, then

$$K(\omega_1,A_2) = \chi_{A_2}(f(\omega_1))$$

is a probability kernel.

7.17 EXAMPLE Consider the setup of 7.16 -- then the integration can be taken over the spectrum $\sigma(A)$ of A. Moreover, one can always arrange matters in such a way that

$$0 \leq w_S(\lambda) \leq 1 \quad (S \in \text{Bor } \underline{R}, \lambda \in \sigma(A))$$

and $\forall\ \lambda \in \sigma(A)$, $S \to w_S(\lambda)$ is a probability measure on $\underline{R}$. So if

$$\begin{cases} \Omega_1 = \sigma(A),\ \mathcal{A}_1 = \text{Bor } \sigma(A) \\ \Omega_2 = \underline{R},\ \mathcal{A}_2 = \text{Bor } \underline{R}, \end{cases}$$

then $K(\lambda,S) = w_S(\lambda)$ is a probability kernel.

Let $E_1:\mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ be a spectral measure. Define $E_2:\mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$ by the rule

$$E_2(A_2) = \int_{\Omega_1} K(\omega_1,A_2)dE_1(\omega_1)$$

or still,

$$E_2(A_2) = T_{K(\text{---},A_2)}.$$

7.18 LEMMA E_2 is a commutative semispectral measure.

PROOF To check that $E_2(A_2) \in \mathcal{E}(H)$, note that

$$
\begin{aligned}
0 &\le \langle x, E_2(A_2)x\rangle \\
&= \int_{\Omega_1} K(\omega_1, A_2)\, d\mu_{x,x}(\omega_1) \\
&\le \int_{\Omega_1} 1\, d\mu_{x,x}(\omega_1) \\
&= \langle x, T_1 x\rangle \\
&= \langle x, T_{\chi_{\Omega_1}} x\rangle \\
&= \langle x, E_1(\Omega_1)x\rangle \\
&= \langle x, x\rangle .
\end{aligned}
$$

7.19 EXAMPLE Let $\Lambda = [\lambda_{mn}]$ $(m = 1,\ldots,M,\ n = 1,\ldots,N)$ be an M-by-N matrix which is stochastic in the sense that $\lambda_{mn} \ge 0$ and $\forall\, n$, $\sum_m \lambda_{mn} = 1$. Put

$$
\left[\begin{array}{l}
\Omega_1 = \{1,\ldots,N\} \quad (A_1 = 2^{\Omega_1}) \\
\Omega_2 = \{1,\ldots,M\} \quad (A_2 = 2^{\Omega_2}).
\end{array}\right.
$$

Fix a spectral measure $n \to E_1(n)$ and define a probability kernel K by

$$
K(n, A_2) = \sum_{m \in A_2} \lambda_{mn} .
$$

Then the induced semispectral measure $m \rightarrow E_2(m)$ is given by

$$E_2(m) = \sum_n \lambda_{mn} E_1(n).$$

7.20 <u>RAPPEL</u> Take $\mathcal{H} = L^2(\underline{R})$ and let

$$(Qf)(\lambda) = \lambda f(\lambda),$$

where

$$\text{Dom } Q = \{f: \int_{\underline{R}} \lambda^2 |f(\lambda)|^2 d\lambda < \infty\}.$$

Then Q is selfadjoint (but unbounded) and the associated spectral measure

$$E^Q: \text{Bor } \underline{R} \rightarrow \mathcal{L}(L^2(\underline{R}))$$

is the prescription

$$E^Q(S)f = \chi_S f.$$

[Note: Q is the <u>position operator</u>. If $U_F: L^2(\underline{R}) \rightarrow L^2(\underline{R})$ is the unitary operator provided by the Plancherel theorem, then $P = U_F^{-1} Q U_F$ is the <u>momentum operator</u>. Explicated,

$$(Pf)(\lambda) = -\sqrt{-1}\, f'(\lambda),$$

where

$$\text{Dom } P = \{f: \int_{\underline{R}} |f'(\lambda)|^2 d\lambda < \infty\},$$

the derivative being in the sense of distributions. So, $\forall\, x \in L^2(\underline{R})$ $(||x|| = 1)$,

$$\omega_x(E^P(S)) = \text{tr}(P_x E^P(S))$$

$$= \langle x, E^P(S)x\rangle$$

$$= \langle x, U_F^{-1}\chi_S U_F x\rangle$$

$$= \langle U_F x, \chi_S U_F x\rangle$$

$$= \int_S |U_F x(\lambda)|^2 d\lambda.]$$

Fix $\rho \geq 0: \rho(\lambda) = \rho(-\lambda)$ and $\int_{\underline{R}} \rho(\lambda)d\lambda = 1$.

7.21 <u>EXAMPLE</u> The assignment

$$S \to \int_{\underline{R}} (\rho*\chi_S)(\lambda)dE^Q(\lambda)$$

defines a semispectral measure Q_ρ, called an <u>approximate position operator</u>, thus $\forall\ \phi,\psi \in L^2(\underline{R})$,

$$\langle\phi, Q_\rho(S)\psi\rangle = \int_{\underline{R}} (\rho*\chi_S)(\lambda)\overline{\phi(\lambda)}\psi(\lambda)d\lambda.$$

Put

$$\Omega_1 = \Omega_2 = \underline{R},\ A_1 = A_2 = \text{Bor } \underline{R}$$

and define

$$K_\rho: \underline{R} \times \text{Bor } \underline{R} \to [0,1]$$

by

$$K_\rho(\lambda, S) = (\rho*\chi_S)(\lambda).$$

Then K_ρ is a probability kernel, so the formation of Q_ρ is simply an application of the foregoing generalities.

[Note: Q itself is formally recovered by taking $\rho = \delta$ $(\delta * \chi_S = \chi_S)$.]

N.B. Approximate momentum operators can be defined analogously.

7.22 EXAMPLE Fix a real valued unit vector $\xi \in L^2(\underline{R})$:

$$\xi \in \text{Dom } Q \cap \text{Dom } P$$

and

$$\left[\begin{array}{l} <\xi,Q\xi> = 0 \\ <\xi,P\xi> = 0 \end{array}\right. , \ \xi \text{ even.}$$

Given $\alpha, \beta \in \underline{R}$, put

$$\xi_{\alpha\beta}(\lambda) = e^{\sqrt{-1}\ \beta\lambda}\xi(\lambda - \alpha) \qquad (\lambda \in \underline{R}).$$

Then

$$\left[\begin{array}{l} <\xi_{\alpha\beta},Q\xi_{\alpha\beta}> = \alpha \\ <\xi_{\alpha\beta},P\xi_{\alpha\beta}> = \beta. \end{array}\right.$$

Given

$$W \in \mathcal{W}(L^2(\underline{R})),$$

write

$$W(\alpha,\beta) = \frac{1}{2\pi} <\xi_{\alpha\beta},W\xi_{\alpha\beta}>.$$

Then

$$\int_{\underline{R}^2} W(\alpha,\beta)d\alpha d\beta = 1.$$

To check this, consider a decomposition of W:

$$W = \sum_{i \in I_W} w_i P_i \quad \text{(cf. 2.12)}$$

=>

$$\int_{\underline{R}^2} W(\alpha,\beta)\,d\alpha d\beta$$

$$= \sum_{i \in I_W} w_i \frac{1}{2\pi} \int_{\underline{R}^2} \langle \xi_{\alpha\beta}, P_i \xi_{\alpha\beta} \rangle d\alpha d\beta.$$

And

$$\frac{1}{2\pi} \int_{\underline{R}^2} \langle \xi_{\alpha\beta}, P_i \xi_{\alpha\beta} \rangle d\alpha d\beta$$

$$= \frac{1}{2\pi} \int_{\underline{R}^2} \langle \xi_{\alpha\beta}, \langle x_i, \xi_{\alpha\beta} \rangle x_i \rangle d\alpha d\beta$$

$$= \frac{1}{2\pi} \int_{\underline{R}^2} |\langle \xi_{\alpha\beta}, x_i \rangle|^2 d\alpha d\beta$$

$$= \int_{\underline{R}^2} \left|\frac{1}{\sqrt{2\pi}} \langle \xi_{\alpha\beta}, x_i \rangle\right|^2 d\alpha d\beta$$

$$= \int_{\underline{R}^2} |x_i(\lambda) \bar{\xi}(\lambda - \alpha)|^2 d\lambda d\alpha \quad \text{(Plancherel)}$$

$$= \int_{\underline{R}^2} |x_i(\lambda)|^2 \; |\xi(\alpha)|^2 d\lambda d\alpha$$

$$= ||x_i||^2 \; ||\xi||^2$$

$$= 1.$$

Therefore

$$\int_{\underline{R}^2} W(\alpha,\beta)\,d\alpha d\beta = \sum_{i \in I_W} w_i = 1.$$

For each $S \in \text{Bor } \underline{R}^2$, the arrow

$$W \to \int_S W(\alpha,\beta)\,d\alpha d\beta$$

is an affine map $\mathcal{W}(L^2(\underline{R})) \to [0,1]$, hence there exists a unique $E(S) \in \mathcal{E}(L^2(\underline{R}))$ such that

$$\int_S W(\alpha,\beta)\,d\alpha d\beta = \text{tr}(WE(S)) \qquad \text{(cf. 7.2)}.$$

The definitions imply that

$$E:\text{Bor } \underline{R}^2 \to \mathcal{E}(L^2(\underline{R}))$$

is a semispectral measure and $\forall\ S \in \text{Bor } \underline{R}$,

$$\begin{cases} E(S \times \underline{R}) = \int_{\underline{R}} (|\xi|^2 * \chi_S)(\lambda)\,dE^Q(\lambda) \\[2ex] E(\underline{R} \times S) = \int_{\underline{R}} (|\hat{\xi}|^2 * \chi_S)(\lambda)\,dE^P(\lambda) \end{cases}$$

or still, in terms of the marginals,

$$\begin{cases} E(\text{—} \times \underline{R}) = Q_{|\xi|^2} \\[2ex] E(\underline{R} \times \text{—}) = P_{|\hat{\xi}|^2} \,. \end{cases}$$

[Note: E is not commutative.]

Fix a Polish space M, take $\mathcal{H}$ separable, and let $E:\text{Bor } M \to \mathcal{E}(\mathcal{H})$ be a commutative semispectral measure:

$$E(S)E(T) = E(T)E(S) \quad \forall\ S,T \in \text{Bor } M.$$

Put

$$\begin{cases} A = E(S) \\ B = E(T) \end{cases}$$

and let

$$\begin{cases} E^A \\ E^B \end{cases} : \text{Bor } \underline{R} \to L(H)$$

be the spectral measures determined by $\begin{cases} A \\ B \end{cases}$.

N.B. Since $AB = BA$, for all $U,V \in \text{Bor } \underline{R}$,

$$E^A(U)E^B(V) = E^B(V)E^A(U).$$

Abbreviate $M_1^+(M,\text{Bor } M)$ to $M_1^+(M)$ and let

$$A_M \subset 2^{M_1^+(M)}$$

be the σ-algebra generated by the sets of the form

$$\{\mu:\mu(S_1)<c_1,\ldots,\mu(S_n)<c_n\},$$

where the $S_i \in \text{Bor } M$ and the $c_i \in \underline{R}$, thus $(M_1^+(M),A_M)$ is a measurable space. Given $S \in \text{Bor } M$, define

$$\pi_S:M_1^+(M) \to \underline{R}$$

by

$$\pi_S(\mu) = \mu(S).$$

Then π_S is Borel measurable. Therefore

$$\pi_S \in \mathcal{E}(M_1^+(M), \mathcal{A}_M) \qquad \text{(cf. 6.4)}$$

and

$$\pi_S^{-1}: \text{Bor } \underline{R} \to \mathcal{A}_M.$$

7.23 LEMMA $\exists$ a spectral measure $X: \mathcal{A}_M \to \mathcal{L}(\mathcal{H})$ such that $\forall$ $U \in$ Bor $\underline{R}$,

$$E^A(U) = X(\pi_S^{-1}(U)).$$

[Note: It can be shown that X is necessarily unique.]

7.24 THEOREM (Holevo) We have

$$A = E(S) = \int_{M_1^+(M)} \pi_S(\mu) dX(\mu).$$

PROOF Let $x \in \mathcal{H}: ||x|| = 1$ and define $\mu_x: \mathcal{L}(\mathcal{H}) \to [0,1]$ as in 5.6:

$$\mu_x(P) = \langle x, Px \rangle.$$

Then

$$\mu_x \circ X \in M_1^+(M_1^+(M), \mathcal{A}_M).$$

And

$$\langle x, (\int_{M_1^+(M)} \pi_S(\mu) dX(\mu)) x \rangle$$

$$= \int_{M_1^+(M)} \pi_S(\mu) d(\mu_x \circ X)(\mu)$$

$$= \int_{\underline{R}} \lambda d(\mu_x \circ X \circ \pi_S^{-1})(\lambda)$$

$$= \int_{\underline{R}} \lambda d(\mu_x \circ E^A)(\lambda)$$

$$= \int_{\underline{R}} \lambda d\langle x, E_\lambda^A x\rangle$$

$$= \langle x, Ax\rangle = \langle x, E(S)x\rangle.$$

[Note: If

$$\begin{cases} \Omega_1 = M_1^+(M),\ \mathcal{A}_1 = \mathcal{A}_M \\ \Omega_2 = M,\ \mathcal{A}_2 = \text{Bor } M, \end{cases}$$

then $K(\mu,S) = \pi_S(\mu)$ is a probability kernel.]

7.25 <u>EXAMPLE</u> Take $M = \underline{R}$ and let the data be as in 7.17, thus

$$E(S) = \int_{\sigma(A)} K(\lambda,S)dE^A(\lambda) \quad (S \in \text{Bor } \underline{R}).$$

Denote by κ the function $\sigma(A) \to M_1^+(\underline{R})$ that sends λ to $K(\lambda,\text{—})$ -- then κ is measurable and

$$X = E^A \circ \kappa^{-1}: \mathcal{A}_{\underline{R}} \to L(H)$$

is a spectral measure such that

$$\int_{M_1^+(\underline{R})} \pi_S(\mu)dX(\mu) = \int_{M_1^+(\underline{R})} \pi_S(\mu)d(E^A \circ \kappa^{-1})(\mu)$$

$$= \int_{\sigma(A)} (\pi_S \circ \kappa)(\lambda)dE^A(\lambda)$$

$$= \int_{\sigma(A)} K(\lambda,S)dE^A(\lambda)$$

$$= E(S).$$

[Note: This capital "A" is not the same as the "A" figuring in the statement of 7.24.]

Semispectral measures can show up unexpectedly. As a "for instance", consider <u>von Neumann's inequality</u>: If $T \in \mathcal{B}(\mathcal{H})$ is a contraction (meaning that $||T|| \leq 1$) and if

$$p(z) = \sum_{k=0}^{n} a_k z^k, \ p(T) = \sum_{k=0}^{n} a_k T^k,$$

then

$$||p(T)|| \leq \sup_{|z|=1} |p(z)|.$$

To prove this, one can assume outright that $||T|| < 1$ (since $\lim_{r \uparrow 1} rT = T$). Let $\Delta = \{z:|z| = 1\}$ -- then the series

$$C(z) = I + \sum_{k=1}^{\infty} (z^k (T^*)^k + \bar{z}^k T^k)$$

is uniformly convergent on Δ, so $\forall\ x,y \in \mathcal{H}$,

$$\int_\Delta z^n \langle x,C(z)y\rangle dz = \langle x,T^n y\rangle \quad (n = 0,1,\ldots).$$

Given $S \in \text{Bor}\ \Delta$, the integral

$$\int_S \langle x,C(z)y\rangle dz$$

makes sense and

$$0 \leq \int_S \langle x,C(z)x\rangle dz$$

$$\leq (1 + 2 \sum_{k=1}^{\infty} ||T||^k)||x||^2.$$

In addition, $\forall$ x,y $\in$ $\mathcal{H}$, the function

$$\begin{cases} \text{Bor } \Delta \to \underline{C} \\ S \to \int_S \langle x, C(z)y\rangle dz \end{cases}$$

is countably additive, so $\exists$ a semispectral measure E:Bor $\Delta \to \mathcal{E}(\mathcal{H})$ such that

$$\langle x, E(S)y\rangle = \int_S \langle x, C(z)y\rangle dz.$$

And $\forall$ n $\geq$ 0,

$$\int_\Delta z^n dE(z) = T^n$$

$$\Rightarrow$$

$$\begin{aligned} ||p(T)|| &= ||\sum_{k=0}^{n} a_k T^k|| \\ &= ||\int_\Delta (\sum_{k=0}^{n} a_k z^k) dE(z)|| \\ &= ||\int_\Delta p(z) dE(z)|| \\ &\leq ||p(z)||_\infty = \sup_{|z|=1} |p(z)|. \end{aligned}$$

7.26 <u>EXAMPLE</u> Suppose that U $\in$ $\mathcal{U}(\mathcal{H})$ and consider the trigonometric polynomial $\sum_{k=-n}^{n} a_k z^k$ on Δ -- then

$$\begin{aligned} ||\sum_{k=-n}^{n} a_k U^k|| &= ||U^n \sum_{k=-n}^{n} a_k U^k|| \\ &= ||\sum_{k=-n}^{n} a_k U^{k+n}|| \end{aligned}$$

$$\leq \sup_{|z|=1} \left| \sum_{k=-n}^{n} a_k z^{k+n} \right|$$

$$= \sup_{|z|=1} \left| \sum_{k=-n}^{n} a_k z^{k} \right| .$$

§8. *THE KOLMOGOROV CONSTRUCTION*

Let Ω be a nonempty set -- then a kernel on Ω is a map $K:\Omega \times \Omega \to \underline{C}$.

Definition: A kernel K on Ω is positive definite if $\forall\ n \in \underline{N}$ and for all

$$\left[\begin{array}{l} \omega_1,\ldots,\omega_n \in \Omega \\ \\ c_1,\ldots,c_n \in \underline{C}, \end{array}\right.$$

we have

$$\sum_{i,j=1}^{n} \bar{c}_i c_j K(\omega_i,\omega_j) \geq 0.$$

8.1 EXAMPLE Take $\Omega = H$, a complex Hilbert space -- then $K(x,y) = \langle x,y\rangle$ is a positive definite kernel on H.

Let $A = [a_{ij}]$ be an n-by-n matrix ($a_{ij} \in \underline{C}$) -- then A is said to be positive definite if for every sequence $c_1,\ldots,c_n$ of n complex numbers,

$$\sum_{i,j=1}^{n} \bar{c}_i c_j a_{ij} \geq 0.$$

[Note: A positive definite n-by-n matrix determines a positive definite kernel on $\{1,\ldots,n\}$ (and vice-versa).]

8.2 REMARK If K is a positive definite kernel on Ω, then the matrix $[K(\omega_i,\omega_j)]$ is positive definite, hence in particular

$$\overline{K(\omega,\omega')} = K(\omega',\omega).$$

8.3 LEMMA If $A = [a_{ij}]$ and $B = [b_{ij}]$ are positive definite, then so is $C = [a_{ij}b_{ij}]$ (the entrywise product of A and B).

PROOF Let

$$x_{ij} = c_i\bar{c}_j b_{ji}.$$

Then $X = [x_{ij}]$ is positive definite:

$$\begin{aligned}\sum_{i,j=1}^{n} \bar{z}_i z_j x_{ij} &= \sum_{i,j=1}^{n} \bar{z}_i z_j c_i \bar{c}_j b_{ji} \\ &= \sum_{i,j=1}^{n} \overline{(\bar{z}_j c_j)}(\bar{z}_i c_i) b_{ji} \\ &= \sum_{i,j=1}^{n} \overline{(\bar{z}_i c_i)}(\bar{z}_j c_j) b_{ij} \\ &\geq 0.\end{aligned}$$

Therefore $\operatorname{tr}(AX) \geq 0$, i.e.,

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}x_{ji} &= \sum_{i,j=1}^{n} a_{ij}c_j\bar{c}_i b_{ij} \\ &= \sum_{i,j=1}^{n} \bar{c}_i c_j a_{ij} b_{ij} \\ &\geq 0.\end{aligned}$$

Denote by $K(\Omega)$ the set whose elements are the positive definite kernels on Ω -- then 8.3 implies that $K(\Omega)$ is closed under pointwise multiplication.

8.4 <u>LEMMA</u> If $A = [a_{ij}]$ is positive definite, then so is $[E(A)_{ij}]$, where

$$E(A)_{ij} = e^{a_{ij}}.$$

Corollary: $K \in K(\Omega) \Rightarrow e^K \in K(\Omega)$.

8.5 <u>THEOREM</u> (The Kolmogorov Construction) Let K be a positive definite kernel on Ω -- then $\exists$ a complex Hilbert space H_K and a map $\Lambda:\Omega \to H_K$ such that

$$K(\omega,\omega') = \langle\Lambda(\omega),\Lambda(\omega')\rangle$$

and the set $\{\Lambda(\omega):\omega \in \Omega\}$ is total in H_K.

<u>PROOF</u> Consider the vector space $\underline{C}^{(\Omega)}$ of all complex valued functions $f:\Omega \to \underline{C}$ such that $f(\omega) = 0$ except for at most a finite set of ω. Put

$$\langle f,f'\rangle = \sum_{\omega,\omega'} \overline{f(\omega)}f'(\omega')K(\omega,\omega').$$

Then the pair $(\underline{C}^{(\Omega)},\langle\ ,\ \rangle)$ is a complex, potentially non Hausdorff, pre-Hilbert space. To get a genuine pre-Hilbert space, divide out by $N = \{f:\langle f,f\rangle = 0\}$ (which thanks to the Schwarz inequality, is linear) and then take for H_K the completion of $\underline{C}^{(\Omega)}/N$. As for Λ, simply observe that

$$K(\omega,\omega') = \langle\delta_\omega,\delta_{\omega'}\rangle.$$

[Note: If H_K' is another complex Hilbert space and if $\Lambda':\Omega \to H_K'$ is another map satisfying the preceding conditions, then there is an isometric isomorphism $T:H_K \to H_K'$ such that $T\Lambda(\omega) = \Lambda'(\omega)\ \forall\ \omega \in \Omega$.]

8.6 <u>EXAMPLE</u> Take $\Omega = H$, a complex Hilbert space, and let

$$K(x,y) = e^{\langle x,y\rangle} \quad (x,y \in H).$$

Then K is a positive definite kernel on H and $H_K = \mathrm{BO}(H)$, the bosonic Fock space over H.

[Note: Here $\Lambda:H \to \mathrm{BO}(H)$ is the map $x \to \underline{\exp}\ x$:

$$e^{\langle x,y\rangle} = \langle \underline{\exp}\ x, \underline{\exp}\ y\rangle.]$$

8.7 <u>EXAMPLE</u> Let $H_1,\ldots,H_n$ be complex Hilbert spaces with respective inner products $\langle\ ,\ \rangle_1,\ldots,\langle\ ,\ \rangle_n$. Put

$$K(x,y) = \prod_{k=1}^{n} \langle x_k,y_k\rangle_k,$$

where

$$\begin{bmatrix} x = (x_1,\ldots,x_n) \\ y = (y_1,\ldots,y_n). \end{bmatrix}$$

Then K is a positive definite kernel on $H_1 \times \cdots \times H_n$ and

$$H_K = H_1 \otimes \cdots \otimes H_n,$$

the Hilbert space tensor product of $H_1,\ldots,H_n$.

8.8 REMARK Write HILB for the category whose objects are the complex Hilbert spaces and whose morphisms are the bounded linear operators — then the functor

$$\otimes:\underline{\text{HILB}} \times \underline{\text{HILB}} \rightarrow \underline{\text{HILB}}$$

and the object $\underline{C}$ together with the canonical natural isomorphisms serve to equip HILB with the structure of a symmetric monoidal category. This data does not directly reflect the presence of an inner product but it appears indirectly since HILB is also a $*$-category, i.e., $\forall\ H_1, H_2 \in \text{Ob}\ \underline{\text{HILB}}$, $\exists$ a map

$$*:\text{Mor}(H_1,H_2) \equiv B(H_1,H_2)$$

that sends a morphism $A:H_1 \rightarrow H_2$ to its adjoint $A^*:H_2 \rightarrow H_1$ subject to

$$I^* = I,\ (AB)^* = B^*A^*,\ A^{**} = A.$$

For the applications, it will be necessary to extend the definition of kernel, replacing the target $\underline{C}$ ($\approx B(\underline{C})$) by $B(H)$ (H a complex Hilbert space), thus now $K:\Omega \times \Omega \rightarrow B(H)$.

Definition: A kernel K on Ω is positive definite if $\forall\ n \in \underline{N}$ and for all

$$\begin{bmatrix} \omega_1,\ldots,\omega_n \in \Omega \\ x_1,\ldots,x_n \in H, \end{bmatrix}$$

we have

$$\sum_{i,j=1}^{n} \langle x_i, K(\omega_i,\omega_j)x_j\rangle \geq 0.$$

N.B. The condition on K amounts to requiring that $\forall$ n, the operator matrix

$$\begin{bmatrix} K(\omega_1,\omega_1) & K(\omega_1,\omega_2) & \cdots & K(\omega_1,\omega_n) \\ K(\omega_2,\omega_1) & K(\omega_2,\omega_2) & \cdots & K(\omega_2,\omega_n) \\ \vdots & \vdots & & \vdots \\ K(\omega_n,\omega_1) & K(\omega_n,\omega_2) & \cdots & K(\omega_n,\omega_n) \end{bmatrix}$$

defines an element of $\mathcal{B}(\overset{n}{\oplus} \mathcal{H})_+$.

[Note: $\forall\ x \in \mathcal{H}$, the matrix

$$\begin{bmatrix} \langle x,K(\omega,\omega)x\rangle & \langle x,K(\omega,\omega')x\rangle \\ \langle x,K(\omega',\omega)x\rangle & \langle x,K(\omega',\omega')x\rangle \end{bmatrix}$$

is positive definite. Therefore

$$\langle x,K(\omega,\omega')x\rangle = \overline{\langle x,K(\omega',\omega)x\rangle}$$

$\Rightarrow$

$$\langle K(\omega,\omega')^*x,x\rangle = \langle K(\omega',\omega)x,x\rangle$$

$\Rightarrow$

$$K(\omega,\omega')^* = K(\omega',\omega).]$$

8.9 <u>EXAMPLE</u> Let $\mathcal{H},\mathcal{K}$ be complex Hilbert spaces and suppose that $A:\Omega \to \mathcal{B}(\mathcal{H},\mathcal{K})$ is a map. Put

$$K(\omega,\omega') = A(\omega)^*A(\omega').$$

Then K is a positive definite kernel. In fact,

$$\sum_{i,j=1}^{n} \langle x_i,K(\omega_i,\omega_j)x_j\rangle = \sum_{i,j=1}^{n} \langle x_i,A(\omega_i)^*A(\omega_j)x_j\rangle$$

$$= \sum_{i,j=1}^{n} \langle A(\omega_i)x_i, A(\omega_j)x_j\rangle$$

$$= ||\sum_{k=1}^{n} A(\omega_k)x_k||^2 \geq 0.$$

There is no difficulty in extending 8.5 to the present setting.

8.10 <u>THEOREM</u> Let K be a positive definite kernel on Ω — then $\exists$ a complex Hilbert space $\mathcal{H}_K$ and a mapping $\rho:\Omega \times \mathcal{H} \rightarrow \mathcal{H}_K$ linear in the second variable such that

$$\langle x, K(\omega,\omega')x'\rangle = \langle \rho(\omega,x), \rho(\omega',x')\rangle$$

and $\rho(\Omega,\mathcal{H})$ is total in $\mathcal{H}_K$.

<u>PROOF</u> Replace $\underline{C}^{(\Omega)}$ by $\mathcal{H}^{(\Omega)}$ and write

$$\langle f,f'\rangle = \sum_{\omega,\omega'} \langle f(\omega), K(\omega,\omega')f(\omega')\rangle.$$

[Note: If $\mathcal{H}_K'$ is another complex Hilbert space and if $\rho':\Omega \times \mathcal{H} \rightarrow \mathcal{H}_K'$ is another map satisfying the preceding conditions, then there is an isometric isomorphism $T:\mathcal{H}_K \rightarrow \mathcal{H}_K'$ such that $T \circ \rho = \rho'$.]

<u>N.B.</u> Take $\mathcal{H} = \underline{C}$ ($\approx \mathcal{B}(\underline{C})$) and let $\Lambda(\omega) = \rho(\omega,1)$ — then

$$K(\omega,\omega') = \langle 1, K(\omega,\omega')1\rangle$$

$$= \langle \rho(\omega,1), \rho(\omega',1)\rangle$$

$$= \langle \Lambda(\omega), \Lambda(\omega')\rangle.$$

8.11 <u>REMARK</u> Given $\omega \in \Omega$, let $A(\omega) = \rho(\omega,\text{—})$ -- then $A(\omega) \in \mathcal{B}(\mathcal{H},\mathcal{H}_K)$ and

$$K(\omega,\omega') = A(\omega)^*A(\omega').$$

Let $K(\Omega;\mathcal{H})$ stand for the set of positive definite kernels $K:\Omega \times \Omega \to \mathcal{B}(\mathcal{H})$. Given $K, K' \in K(\Omega;\mathcal{H})$, write $K \geq K'$ if $K - K' \in K(\Omega;\mathcal{H})$.

8.12 <u>EXAMPLE</u> Given $K \in K(\Omega;\mathcal{H})$ and $E \in \mathcal{E}(\mathcal{H}_K)$, put

$$K_E(\omega,\omega') = A(\omega)^*EA(\omega').$$

Then $K_E \in K(\Omega;\mathcal{H})$ and $K \geq K_E$.

[We have

$$\begin{aligned}\sum_{i,j=1}^{n} \langle x_i,K_E(\omega_i,\omega_j)x_j\rangle &= \sum_{i,j=1}^{n} \langle x_i,A(\omega_i)^*EA(\omega_j)x_j\rangle \\ &= \sum_{i,j=1}^{n} \langle A(\omega_i)x_i,EA(\omega_j)x_j\rangle \\ &= \sum_{i,j=1}^{n} \langle\sqrt{E}\, A(\omega_i)x_i,\sqrt{E}\, A(\omega_j)x_j\rangle \\ &= \Big|\Big|\sum_{k=1}^{n} \sqrt{E}\, A(\omega_k)x_k\Big|\Big|^2 \geq 0.\end{aligned}$$

And

$$\begin{aligned}&K(\omega,\omega') - K_E(\omega,\omega') \\ &\quad = A(\omega)^*IA(\omega') - A(\omega)^*EA(\omega') \\ &\quad = A(\omega)^*(I - E)A(\omega') \\ &\quad = K_{I - E}(\omega,\omega').]\end{aligned}$$

8.13 <u>LEMMA</u> Let $K, K' \in K(\Omega;H)$ and define

$$\begin{cases} A(\omega) \in \mathcal{B}(H, H_K) \\ & (\omega \in \Omega) \\ A'(\omega) \in \mathcal{B}(H, H_{K'}) \end{cases}$$

as in 8.11 -- then $K \geq K'$ iff $\exists$ a contraction $T: H_K \rightarrow H_{K'}$ such that

$$A'(\omega) = TA(\omega) \quad (\omega \in \Omega).$$

[Note: The totality of $\rho(\Omega, H)$ implies that T is unique. And

$$\begin{aligned} K'(\omega,\omega') &= (A'(\omega))^*A'(\omega') \\ &= (TA(\omega))^*TA(\omega') \\ &= A(\omega)^*T^*TA(\omega'), \end{aligned}$$

where $T^*T \in \mathcal{B}(H_K)$ is a positive contraction (cf. 8.12).]

There is another version of 8.10 which is based on the following observation.

8.14 <u>LEMMA</u> A map $K: \Omega \times \Omega \rightarrow \mathcal{B}(H)$ is a positive definite kernel iff for any finitely supported collection $\{A_\omega \in \mathcal{B}(H): \omega \in \Omega\}$,

$$\sum_{\omega,\omega'} A_\omega^* K(\omega,\omega') A_{\omega'} \geq 0.$$

I.e.: The LHS is in $\mathcal{B}(H)_+$.

[Note: Finitely supported means, of course, that

$$\#\{\omega \in \Omega: A_\omega \neq 0\} < \infty.]$$

To make use of this, some additional structure will be required.

So let E be a right $\mathcal{B}(H)$-module -- then a $\mathcal{B}(H)$-valued pre-inner product on E is a function $< , >: E \times E \to \mathcal{B}(H)$ such that $\forall$ a,b,c $\in$ E, $\forall$ A $\in \mathcal{B}(H)$, $\forall \lambda \in \underline{C}$:

(i) $\langle a,b + c\rangle = \langle a,b\rangle + \langle a,c\rangle$;

(ii) $\langle a,\lambda b\rangle = \lambda\langle a,b\rangle$;

(iii) $\langle a,bA\rangle = \langle a,b\rangle A$;

(iv) $\langle a,b\rangle^* = \langle b,a\rangle$;

(v) $\langle a,a\rangle \geq 0$ ($\Rightarrow \langle a,a\rangle \in \mathcal{B}(H)_+$).

If

$$\langle a,a\rangle = 0 \Rightarrow a = 0,$$

then $< , >$ is called a $\mathcal{B}(H)$-valued inner product.

[Note: $< , >$ is "conjugate linear" in the first variable:

$$\langle aA,b\rangle = A^*\langle a,b\rangle.]$$

A pre-Hilbert $\mathcal{B}(H)$-module is a right $\mathcal{B}(H)$-module E equipped with a $\mathcal{B}(H)$-valued pre-inner product $< , >$.

8.15 LEMMA Suppose that E is a pre-Hilbert $\mathcal{B}(H)$-module -- then $\forall$ a,b $\in$ E,

$$\langle a,b\rangle^*\langle a,b\rangle \leq ||\langle a,a\rangle||\langle b,b\rangle.$$

PROOF Take $||\langle a,a\rangle|| = 1$ and let A $\in \mathcal{B}(H)$:

$$\begin{aligned} 0 &\leq \langle aA - b,aA - b\rangle \\ &= A^*\langle a,a\rangle A - \langle b,a\rangle A - A^*\langle a,b\rangle + \langle b,b\rangle \\ &\leq A^*A - \langle b,a\rangle A - A^*\langle a,b\rangle + \langle b,b\rangle. \end{aligned}$$

Now take $A = \langle a,b \rangle$ to get

$$0 \le \langle a,b\rangle^*\langle a,b\rangle - \langle b,a\rangle \langle a,b\rangle - \langle a,b\rangle^*\langle a,b\rangle + \langle b,b\rangle$$

or still,

$$\langle a,b\rangle^*\langle a,b\rangle \le \langle b,b\rangle.$$

Put

$$||a|| = ||\langle a,a\rangle||^{1/2} \quad (a \in E).$$

Then 8.15 implies that $||.||$ is a seminorm on E:

$$\begin{cases} ||a + b|| \le ||a|| + ||b|| \\ ||\lambda a|| \le |\lambda| \; ||a||. \end{cases}$$

Moreover, $||.||$ is a norm if the pre-inner product is actually an inner product.

Let

$$N_E = \{a \in E: ||a|| = 0\}.$$

Then N_E is a submodule of E and the pre-inner product and seminorm drop to an inner product and norm on the quotient module E/N_E.

E is said to be a <u>Hilbert $\mathcal{B}(H)$-module</u> if the seminorm is a norm and E is complete.

[Note: Identify $\underline{C}$ and $\mathcal{B}(\underline{C})$ -- then the Hilbert $\underline{C}$-modules are the complex Hilbert spaces.]

8.16 <u>LEMMA</u> The completion of E/N_E is a Hilbert $\mathcal{B}(H)$-module.

8.17 <u>EXAMPLE</u> View $\mathcal{B}(H)$ itself as a right $\mathcal{B}(H)$-module and put

$$\langle A,B\rangle = A^*B \quad (A,B \in \mathcal{B}(H)).$$

Then $\mathcal{B}(H)$ is a Hilbert $\mathcal{B}(H)$-module.

8.18 THEOREM Let K be a positive definite kernel on Ω -- then $\exists$ a Hilbert $\mathcal{B}(H)$-module E_K and a mapping $\Lambda:\Omega \rightarrow E_K$ such that

$$K(\omega,\omega') = \langle\Lambda(\omega),\Lambda(\omega')\rangle.$$

PROOF The set of all finitely supported collections $\{A_\omega \in \mathcal{B}(H):\omega \in \Omega\}$ is a right $\mathcal{B}(H)$-module and the prescription

$$\langle\{A_\omega\},\{A'_\omega\}\rangle = \sum_{\omega,\omega'} A^*_\omega K(\omega,\omega')A'_\omega$$

equips it with the structure of a pre-Hilbert $\mathcal{B}(H)$-module (cf. 8.14), thus by definition,

$$K(\omega,\omega') = \langle\{\delta_{\omega\zeta}I:\zeta \in \Omega\},\{\delta_{\omega'\zeta'}I:\zeta' \in \Omega\}\rangle.$$

The rest is clear: Mod out by the elements of seminorm zero and then complete (cf. 8.16).

[Note: There is also an assertion of uniqueness.]

8.19 REMARK As has been seen earlier, $\underline{C}$-valued positive definite kernels can be multiplied pointwise but while this operation makes sense for $\mathcal{B}(H)$-valued positive definite kernels, the pointwise product of two positive definite kernels need not be positive definite. The escape from this difficulty is simple: Replace pointwise multiplication by pointwise composition of mappings. So suppose that H,H' are complex Hilbert spaces and let $K:\Omega \times \Omega \rightarrow \mathcal{B}(\mathcal{B}(H),\mathcal{B}(H'))$ be a map -- then the following conditions are equivalent.

- For all

$$\omega_1,\ldots,\omega_n \in \Omega$$

and all

$$\begin{cases} A_1,\ldots,A_n \in \mathcal{B}(H) \\ A_1',\ldots,A_n' \in \mathcal{B}(H'), \end{cases}$$

$$\sum_{i,j=1}^{n} A_i'^{*}K(\omega_i,\omega_j)(A_i^{*}A_j)A_j' \geq 0.$$

- The map

$$\underline{k}:(\mathcal{B}(H) \times \Omega) \times (\mathcal{B}(H) \times \Omega) \to \mathcal{B}(H')$$

that sends

$$((A,\omega),(A',\omega'))$$

to

$$K(\omega,\omega')(A^{*}A')$$

is a positive definite kernel.

Under these circumstances, we shall again refer to K as a <u>positive definite kernel</u> (8.18 is applicable via $\underline{k}$, hence

$$K(\omega,\omega')(A^{*}) = K(\omega',\omega)(A)^{*}).$$

If now H,H',H'' are complex Hilbert spaces and if

$$\begin{cases} K:\Omega \times \Omega \to \mathcal{B}(\mathcal{B}(H),\mathcal{B}(H')) \\ L:\Omega \times \Omega \to \mathcal{B}(\mathcal{B}(H'),\mathcal{B}(H'')) \end{cases}$$

are positive definite kernels, then

$$L \circ K: \Omega \times \Omega \to \mathcal{B}(\mathcal{B}(H), \mathcal{B}(H''))$$

is a positive definite kernel. Here

$$(L \circ K)(\omega, \omega') = L(\omega, \omega') \circ K(\omega, \omega').$$

Let E be a Hilbert module -- then E is a Banach space. Write $\mathcal{B}(E)$ for the set of bounded linear operators $T: E \to E$ and let $\mathcal{B}^*(E)$ denote the subset of $\mathcal{B}(E)$ consisting of those $T \in \mathcal{B}(E)$ for which there is a $T^* \in \mathcal{B}(E)$ such that $\langle T^*a, b\rangle = \langle a, Tb\rangle \ \forall \ a, b \in E$. In other words: $\mathcal{B}^*(E)$ is the set of bounded linear operators on E possessing an adjoint w.r.t. the $\mathcal{B}(H)$-valued inner product on E.

[Note: The adjoint T^* of a $T \in \mathcal{B}^*(E)$ is unique, belongs to $\mathcal{B}^*(E)$, and $T^{**} = T$. Therefore $\mathcal{B}^*(E)$ is a unital $*$-algebra with involution $T \to T^*$. More is true: $\forall \ T \in \mathcal{B}^*(E)$, $||T^*T|| = ||T||^2$. Since $\mathcal{B}^*(E)$ is a closed subalgebra of $\mathcal{B}(E)$, it follows that $\mathcal{B}^*(E)$ is a unital C*-algebra.]

N.B. Every $T \in \mathcal{B}^*(E)$ is $\mathcal{B}(H)$-linear: $T(aA) = T(a)A$. In general, however, an arbitrary bounded $\mathcal{B}(H)$-linear map $E \to E$ need not have an adjoint.

[Note: Another point is that as elements of $\mathcal{B}(H)$,

$$\langle T(a), T(a)\rangle \leq ||T||^2 \langle a, a\rangle \quad (a \in E).]$$

8.20 EXAMPLE Given $a, b \in E$, define $\Theta_{a,b}: E \to E$ by

$$\Theta_{a,b}(c) = a\langle b, c\rangle \quad (c \in E).$$

Then $\Theta_{a,b} \in \mathcal{B}^*(E)$, where

$$\Theta^*_{a,b} = \Theta_{b,a}.$$

[Note: An element $P \in \mathcal{B}^*(E)$ is a <u>projection</u> if $P = P^* = P^2$. E.g.: $\Theta_{a,a}$ is a projection provided $<a,a> = I$.]

Let A be a unital $*$-algebra with unit e. Suppose that $\Phi:A \to \mathcal{B}(H)$ is a linear map (it is not assumed that Φ sends e to I) -- then Φ gives rise to a kernel $K_\Phi:A \times A \to \mathcal{B}(H)$, viz.

$$K_\Phi(\xi,\eta) = \Phi(\xi^*\eta).$$

8.21 <u>EXAMPLE</u> Suppose that $\Phi:A \to \underline{C}$ is linear and <u>positive</u>, i.e., $\Phi(\xi^*\xi) \geq 0$ $\forall\ \xi \in A$ -- then K_Φ is positive definite:

$$\sum_{i,j=1}^{n} \bar{c}_i c_j K_\Phi(\xi_i,\xi_j)$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \Phi(\xi_i^*\xi_j)$$

$$= \sum_{i,j=1}^{n} \Phi((c_i\xi_i)^* c_j\xi_j)$$

$$= \Phi((\sum_{i=1}^{n} c_i\xi_i)^*(\sum_{j=1}^{n} c_j\xi_j))$$

$$\geq 0.$$

Let E be a Hilbert $\mathcal{B}(H)$-module -- then a unital $*$-representation of A on E is a unital $*$-homomorphism $\pi:A \to \mathcal{B}^*(E)$. Given $a \in E$, define $\Phi_a:A \to \mathcal{B}(H)$ by

$$\Phi_a(\xi) = <a,\pi(\xi)a> \quad (\xi \in A).$$

8.22 <u>LEMMA</u> Put $K_a = K_{\Phi_a}$ — then $K_a: A \times A \to \mathcal{B}(H)$ is positive definite.

<u>PROOF</u> Bearing in mind 8.14, suppose that $\{A_\xi \in \mathcal{B}(H) : \xi \in A\}$ is finitely supported:

$$\sum_{\xi,\xi'} A_\xi^* K_a(\xi,\xi') A_{\xi'}$$

$$= \sum_{\xi,\xi'} A_\xi^* \langle a, \pi(\xi^*\xi') \rangle A_{\xi'}$$

$$= \sum_{\xi,\xi'} A_\xi^* \langle a, \pi(\xi)^* \pi(\xi') a \rangle A_{\xi'}$$

$$= \sum_{\xi,\xi'} A_\xi^* \langle \pi(\xi) a, \pi(\xi') a \rangle A_{\xi'}$$

$$= \sum_{\xi,\xi'} \langle (\pi(\xi) a) A_\xi, (\pi(\xi') a) A_{\xi'} \rangle$$

$$= \sum_{\xi,\xi'} \langle \pi(\xi)(a A_\xi), \pi(\xi')(a A_{\xi'}) \rangle$$

$$= \langle \sum_\eta \pi(\eta)(a A_\eta), \sum_\eta \pi(\eta)(a A_\eta) \rangle$$

$$\geq 0.$$

8.23 <u>RAPPEL</u> Let A be a unital $*$-algebra — then an element $\xi \in A$ is <u>unitary</u> if ξ^{-1} exists and equals ξ^*. One calls A a <u>U*-algebra</u> if A is the linear span of its unitary elements. E.g.: Every unital Banach $*$-algebra is a U*-algebra (hence, in particular, every unital C*-algebra is a U*-algebra).

[Note: Recall that a Banach $*$-algebra is a Banach algebra A equipped with an isometric involution: $||\xi|| = ||\xi^*||$ $(\xi \in A)$ $(||e|| = 1)$.]

N.B. For a class of examples of U*-algebras which are not unital Banach *-algebras, take any group G (discrete topology) and consider $\underline{C}^{(G)}$ under convolution equipped with the involution $f \to f^*$, $f^*(\sigma) = \overline{f(\sigma^{-1})}$ -- then the δ_σ are unitary and their linear span is all of $\underline{C}^{(G)}$.

8.24 THEOREM Let A be a U*-algebra and suppose that $\Phi: A \to B(H)$ is a linear map. Assume: K_Φ is positive definite -- then $\exists$ a Hilbert $B(H)$-module E, a unital *-representation π of A on E, and an element $a \in E$ such that

$$\Phi(\xi) = \langle a, \pi(\xi) a\rangle \quad (\xi \in A).$$

Moreover, the linear span of $\pi(A)(aB(H))$ is dense in E.

PROOF Consider the algebraic tensor product $A \otimes B(H)$, viewed as a right $B(H)$-module in the obvious way. Define

$$\langle\ ,\ \rangle : A \otimes B(H) \times A \otimes B(H) \to B(H)$$

by

$$\langle \sum_{i=1}^{n} \xi_i \otimes A_i, \sum_{j=1}^{m} \eta_j \otimes B_j\rangle$$

$$= \sum_{i,j} A_i^* \Phi(\xi_i^* \eta_j) B_j.$$

Then $A \otimes B(H)$ is a pre-Hilbert $B(H)$-module. On the other hand, $A \otimes B(H)$ is also a left A-module and

$$N \equiv N_{A \otimes B(H)}$$

is an A-submodule of $A \otimes B(H)$. To see this, let $\xi \in A$ be unitary -- then

$$<\xi(\sum_{i=1}^{n} \xi_i \otimes A_i), \xi(\sum_{i=1}^{n} \xi_i \otimes A_i)>$$

$$= <\sum_{i=1}^{n} \xi\xi_i \otimes A_i, \sum_{i=1}^{n} \xi\xi_i \otimes A_i>$$

$$= \sum_{i,j} A_i^* \Phi((\xi\xi_i)^*\xi\xi_j)A_j$$

$$= \sum_{i,j} A_i^* \Phi(\xi_i^*\xi^*\xi\xi_j)A_j$$

$$= \sum_{i,j} A_i^* \Phi(\xi_i^*\xi_j)A_j$$

$$= <\sum_{i=1}^{n} \xi_i \otimes A_i, \sum_{i=1}^{n} \xi_i \otimes A_i>$$

=>

$$\xi N \subset N.$$

But A is spanned by its unitary elements, thus

$$AN \subset N.$$

Proceeding, set

$$E_0 = A \otimes B(H)/N$$

and given $\xi \in A$, write

$$\pi_0(\xi)(X + N) = \xi X + N \quad (X \in A \otimes B(H)).$$

Then $\pi_0(\xi^*) = \pi_0(\xi)^*$ and there is no difficulty in showing that π_0 extends to a unital $*$-representation π of A on the completion E of E_0 ($\pi_0(\xi)$ is an isometry if ξ is unitary). Finally, viewing

$$a = e \otimes I + N$$

as an element of E, we have

$$\begin{aligned}
\langle a,\pi(\xi)a\rangle &= \langle e \otimes I, \xi \otimes I\rangle \\
&= I^*\Phi(e^*\xi)I \\
&= \Phi(e\xi) \\
&= \Phi(\xi)
\end{aligned}$$

and the linear span of

$$\begin{aligned}
\pi(A)(aB(H)) &= \{\pi(\xi)(aA)\} \\
&= \{\xi \otimes A + N\}
\end{aligned}$$

is E_0 which is dense in E.

8.25 <u>REMARK</u> Suppose that $\Phi(e) = I$ — then from the above, $\langle a,a\rangle = I$. Therefore $P_a \equiv \Theta_{a,a} \in B^*(E)$ is a projection (cf. 8.20) and the arrow

$$A \rightarrow \Theta_{aA,a}$$

ia a $*$-isomorphism of $B(H)$ onto the closed $*$-subalgebra $P_a B^*(E) P_a$ of $B^*(E)$. Furthermore, $\forall\ b \in E$,

$$\begin{aligned}
&P_a\pi(\xi)P_a(b) \\
&\quad = P_a\pi(\xi)(a\langle a,b\rangle) \\
&\quad = P_a(\pi(\xi)(a)\langle a,b\rangle) \\
&\quad = P_a(\pi(\xi)a)\langle a,b\rangle \\
&\quad = a\langle a,\pi(\xi)a\rangle\langle a,b\rangle
\end{aligned}$$

$$= a\Phi(\xi)\langle a,b\rangle$$

$$= \Theta_{a\Phi(\xi),a}(b).$$

I.e.:

$$\Phi(\xi) \rightarrow \Theta_{a\Phi(\xi),a} = P_a\pi(\xi)P_a \quad (\xi \in \mathcal{A}).$$

So, in summary: If K_Φ is positive definite and if $\Phi(e) = I$, then $\exists$ a C*-algebra $\mathcal{B}$ containing $\mathcal{B}(\mathcal{H})$, a projection $P \in \mathcal{B}$ such that $\mathcal{B}(\mathcal{H}) = P\mathcal{B}P$, and a $*$-homomorphism $\pi:\mathcal{A} \rightarrow \mathcal{B}$ with $\Phi(\xi) = P\pi(\xi)P \ \forall \ \xi \in \mathcal{A}$.

8.26 <u>EXAMPLE</u> (The GNS Construction) Let $\mathcal{A}$ be a U*-algebra and suppose that $\Phi:\mathcal{A} \rightarrow \underline{C}$ is a positive linear functional (cf. 8.21) -- then we can write

$$\Phi(\xi) = \langle a,\pi(\xi)a\rangle \quad (\xi \in \mathcal{A}) \quad (\text{cf. } 8.24).$$

Here E is a Hilbert $\mathcal{B}(\underline{C})$-module or still, E is a complex Hilbert space, and a is π-cyclic: $\pi(\mathcal{A})a$ is dense in E.

§9. *-SEMIGROUPS

Let H,K be complex Hilbert spaces and let $R \in B(H,K)$ -- then an element $B \in B(K)$ is called an R-dilation of an element $A \in B(H)$ if

$$A = R^*BR.$$

One writes $A = \Delta_R B$ and calls A an R-compression of B.

[Note: If $R^*R = I$, then R is an isometry and H can be viewed as a closed linear subspace of K. With this understanding, R^* is the orthogonal projection P_H of K onto H and $\forall\ x \in H$, $Ax = P_H Bx$. To reinforce this convention, it is customary to write $A = pr_H B$ and to call A a projection of B and B a dilation of A.]

9.1 EXAMPLE If $T \in B(H)$ is the projection of $U \in U(K)$, then T is a contraction. Proof: $\forall\ x \in H$,

$$||Tx|| = ||P_H Ux|| \le ||Ux|| = ||x||$$

$$\Rightarrow ||T|| \le 1.$$

Conversely, let $T \in B(H)$ be a contraction -- then $\exists\ K \supset H$ and $U \in U(K)$ such that $T = pr_H U$.

[Put $K = H \oplus H$ and let

$$U = \begin{bmatrix} T & D_T \\ -D_{T^*} & T^* \end{bmatrix} \in B(K),$$

where

$$\begin{cases} D_T = (I - TT^*)^{1/2} & (0 \leq I - TT^* \leq I) \\ D_{T^*} = (I - T^*T)^{1/2} & (0 \leq I - T^*T \leq I). \end{cases}$$

Identify H with the first factor of K -- then

$$Tx = P_H Ux \quad (x \leftrightarrow (x,0)),$$

P_H the orthogonal projection of K onto H, and the claim is that $U \in U(K)$. From the definitions,

$$U^*U = \begin{bmatrix} T^* & -D_{T^*} \\ D_T & T \end{bmatrix} \begin{bmatrix} T & D_T \\ -D_{T^*} & T^* \end{bmatrix}$$

and

$$UU^* = \begin{bmatrix} T & D_T \\ -D_{T^*} & T^* \end{bmatrix} \begin{bmatrix} T^* & -D_{T^*} \\ D_T & T \end{bmatrix}.$$

or still,

$$U^*U = \begin{bmatrix} T^*T + D_{T^*}^2 & T^*D_T - D_{T^*}T^* \\ D_TT - TD_{T^*} & D_T^2 + TT^* \end{bmatrix}$$

and

$$UU^* = \begin{bmatrix} TT^* + D_T^2 & -TD_{T^*} + D_T T \\ -D_{T^*}T^* + T^*D_T & D_{T^*}^2 + T^*T \end{bmatrix}$$

or still,

$$U^*U = \begin{bmatrix} I & T^*D_T - D_{T^*}T^* \\ D_T T - TD_{T^*} & I \end{bmatrix}$$

and

$$UU^* = \begin{bmatrix} I & -TD_{T^*} + D_T T \\ -D_{T^*}T^* + T^*D_T & I \end{bmatrix}.$$

But the off diagonal entries vanish. To see this, one has only to show that

$$D_T T = TD_{T^*} \ (\Rightarrow T^*D_T = D_{T^*}T^*).$$

In any event,

$$D_T^2 T = TD_{T^*}^2$$

$$\Rightarrow$$

$$D_T^{2n} T = TD_{T^*}^{2n}$$

$$\Rightarrow$$

$$p(D_T^2)T = Tp(D_{T^*}^2),$$

p an arbitrary polynomial. Since

$$\begin{cases} D_T = (D_T^2)^{1/2} \\ D_{T^*} = (D_{T^*}^2)^{1/2} \end{cases}$$

and since $\exists$ a sequence of polynomials p_n such that $\forall$ $A \in \mathcal{B}(\mathcal{H})_+$,

$$p_n(A) \to A^{1/2} \quad \text{(SOT)},$$

it indeed follows that

$$D_T T = T D_{T^*}.]$$

[Note: These considerations imply that every element of $\mathcal{B}(\mathcal{H})$ has a normal dilation.]

9.2 <u>EXAMPLE</u> Every effect $E \in \mathcal{E}(\mathcal{H})$ can be dilated to a projection.

[Take $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ and work with

$$\begin{bmatrix} E & (E - E^2)^{1/2} \\ (E - E^2)^{1/2} & E \end{bmatrix}.]$$

A <u>*-semigroup</u> is a semigroup Γ with unit e equipped with an involution $*:\Gamma \to \Gamma$, thus

$$\xi^{**} = \xi, \ (\xi\eta)^* = \eta^*\xi^*, \ e^* = e.$$

E.g.: Every group G is a *-semigroup ($\sigma^* = \sigma^{-1}$).

[Note: Every unital *-algebra $\mathcal{A}$ is a *-semigroup (per multiplication).]

<u>N.B.</u> We have

$$e^* = ee^* = e^{**}e^* = (ee^*)^* = e^{**} = e.$$

A <u>representation</u> of a $*$-semigroup Γ on a complex Hilbert space H is a homomorphism $\pi:\Gamma \to \mathcal{B}(H)$, π being termed a <u>$*$-representation</u> provided $\pi(\xi^*) = \pi(\xi)^* \ \forall \ \xi \in \Gamma$.

9.3 <u>REMARK</u> Suppose that $\pi:\Gamma \to \mathcal{B}(H)$ is a $*$-representation of Γ -- then $\pi(e)$ is an orthogonal projection and π is said to be <u>unital</u> if $\pi(e) = I$. In general,

$$H = \pi(e)H \oplus (I - \pi(e))H.$$

Therefore π is the orthogonal direct sum of a unital $*$-representation and a null $*$-representation.

[Note: A representation of a group G is a unital $*$-representation iff it is a unitary representation.]

9.4 <u>EXAMPLE</u> Let (Ω,A) be a measurable space -- then A is a $*$-semigroup:

$$A_1 \cdot A_2 = A_1 \cap A_2, \ \Omega \cdot A = A, \ A^* = A.$$

So, if $\pi:A \to \mathcal{B}(H)$ is a $*$-representation, then

$$\begin{cases} \pi(A)^2 = \pi(A^2) = \pi(A) \\ \pi(A)^* = \pi(A^*) = \pi(A). \end{cases}$$

Thus the $\pi(A)$ are orthogonal projections.

Let Γ be a $*$-semigroup. Suppose that $\Phi:\Gamma \to \mathcal{B}(H)$ is a function and $\pi:\Gamma \to \mathcal{B}(K)$ is a $*$-representation -- then we write $\Phi = \Delta_R\pi$ if $\forall \ \xi \in \Gamma$, $\Phi(\xi) = \Delta_R\pi(\xi)$ and we write $\Phi = \mathrm{pr}_H\pi$ if $\forall \ \xi \in \Gamma$, $\Phi(\xi) = \mathrm{pr}_H\pi(\xi)$.

[Note: Call π <u>minimal</u> if $\pi(\Gamma)R\mathcal{H}$ is total in $\mathcal{K}$. A minimal π is necessarily unital.]

9.5 LEMMA Suppose that $\Phi:\Gamma \to \mathcal{B}(\mathcal{H})$ is a function. Let

$$\begin{cases} \pi_1:\Gamma \to \mathcal{B}(\mathcal{K}_1) \\ \pi_2:\Gamma \to \mathcal{B}(\mathcal{K}_2) \end{cases}$$

be *-representations of Γ for which $\exists\ R_1 \in \mathcal{B}(\mathcal{H},\mathcal{K}_1)$ & $R_2 \in \mathcal{B}(\mathcal{H},\mathcal{K}_2)$:

$$\Delta_{R_1}\pi_1 = \Phi = \Delta_{R_2}\pi_2.$$

Assume: π_1 and π_2 are minimal -- then there is an isometric isomorphism $T:\mathcal{K}_1 \to \mathcal{K}_2$ such that $TR_1 = R_2$ and $T\pi_1 = \pi_2 T$.

PROOF Extend the arrow

$$\sum_i \pi_1(\xi_i)R_1x_i \to \sum_i \pi_2(\xi_i)R_2x_i$$

in the obvious way.

[Note: We have

$$\begin{aligned} &<\pi_1(\xi)R_1x,\pi_1(\xi)R_1x> \\ &\quad = <R_1^*\pi_1(\xi)^*\pi_1(\xi)R_1x,x> \\ &\quad = <R_1^*\pi_1(\xi^*)\pi_1(\xi)R_1x,x> \\ &\quad = <R_1^*\pi_1(\xi^*\xi)R_1x,x> \\ &\quad = <R_2^*\pi_2(\xi^*\xi)R_2x,x> \end{aligned}$$

$$= \langle\pi_2(\xi)R_2x,\pi_2(\xi)R_2x\rangle.]$$

9.6 LEMMA Suppose that $\Phi:\Gamma \to \mathcal{B}(\mathcal{H})$ is a function. Let

$$\begin{bmatrix} \pi_1:\Gamma \to \mathcal{B}(\mathcal{K}_1) \\ \pi_2:\Gamma \to \mathcal{B}(\mathcal{K}_2) \end{bmatrix}$$

be $*$-representations of Γ for which

$$\mathrm{pr}_{\mathcal{H}}\pi_1 = \Phi = \mathrm{pr}_{\mathcal{H}}\pi_2.$$

Assume:

$$\begin{bmatrix} \pi_1(\Gamma)\mathcal{H} \text{ is total in } \mathcal{K}_1 \\ \pi_2(\Gamma)\mathcal{H} \text{ is total in } \mathcal{K}_2. \end{bmatrix}$$

Then there is an isometric isomorphism $T:\mathcal{K}_1 \to \mathcal{K}_2$ such that $Tx = x \ \forall \ x \in \mathcal{H}$ and $T\pi_1 = \pi_2 T$.

[This is a special case of 9.5. Bear in mind that

$$\begin{bmatrix} R_1:\mathcal{H} \to \mathcal{K}_1 \\ R_2:\mathcal{H} \to \mathcal{K}_2 \end{bmatrix}$$

are isometric embeddings and $\forall \ x \in \mathcal{H}$

$$\text{"}R_1x = x\text{"}$$

$$\text{"}R_2x = x\text{".}]$$

Let Γ be a $*$-semigroup -- then $F(\Gamma,\mathcal{H})$ will stand for the complex linear space consisting of all functions $f:\Gamma \to H$ such that $f(\xi) = 0$ except for at most a finite number of ξ.

<u>N.B.</u> There is a map $\Gamma \times H \to F(\Gamma,H)$, viz. $(\xi,x) \to f_{\xi,x}$, where

$$f_{\xi,x}(\eta) = \begin{cases} x \text{ if } \eta = \xi \\ 0 \text{ if } \eta \neq \xi. \end{cases}$$

Suppose that $\Phi:\Gamma \to B(H)$ is a function -- then Φ gives rise to a kernel $K_\Phi:\Gamma \times \Gamma \to B(H)$:

$$K_\Phi(\xi,\eta) = \Phi(\xi^*\eta).$$

And the condition that K_Φ be positive definite is that $\forall\ f \in F(\Gamma,H)$,

$$\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\eta)f(\eta)\rangle \geq 0.$$

9.7 <u>LEMMA</u> If K_Φ is positive definite, then $\forall\ \xi,\eta \in \Gamma$,

$$\Phi(\xi^*\eta)^* = \Phi(\eta^*\xi).$$

9.8 <u>EXAMPLE</u> Take $\Gamma = \underline{Z}$ -- then a function $\Phi:\underline{Z} \to B(H)$ is simply a collection $\{T_n:n \in \underline{Z}\}$, where $\forall\ n$, $T_n \in B(H)$, so K_Φ is positive definite iff for every finite sequence

$$x_{-n},\ldots,x_{-1},x_0,x_1,\ldots,x_n \in H,$$

we have

$$\sum_{k,\ell=-n}^{n} \langle x_k,T_{\ell-k}x_\ell\rangle \geq 0.$$

Impose now the condition $T_0 = I$ and $\forall\ n$, $T_{-n} = T_n^*$ -- then the assumption that K_Φ

is positive definite forces the T_n to be contractions. This can be seen as follows. Fix $x_0 \in H$, $n_0 \in \underline{Z}$, and define $f_0 \in F(\Gamma,H)$ by

$$f_0(0) = x_0,\ f_0(n_0) = -\ T_{-n_0}x_0,$$

letting $f_0(n) = 0$ otherwise:

$$\sum_{n,m} \langle f_0(n), T_{m-n}f_0(m)\rangle \geq 0.$$

I.e.:

$$\langle x_0, \sum_m T_m f_0(m)\rangle + \langle f_0(n_0), \sum_m T_{m-n_0}f_0(m)\rangle$$

$$= \langle x_0,x_0\rangle + \langle x_0, T_{n_0}(-\ T_{-n_0}x_0)\rangle$$

$$+ \langle -\ T_{-n_0}x_0, T_{-n_0}x_0\rangle + \langle -\ T_{-n_0}x_0, I(-\ T_{-n_0}x_0)\rangle$$

$$= \langle x_0,x_0\rangle - \langle T^*_{n_0}x_0, T_{-n_0}x_0\rangle$$

$$= \langle x_0,x_0\rangle - \langle T_{-n_0}x_0, T_{-n_0}x_0\rangle \geq 0.$$

Therefore the T_n are contractions, as claimed. On the other hand, let us start with a contraction T and put

$$\begin{cases} T_n = T^n & (n = 0,1,\ldots) \\ T_{-n} = (T^*)^n & (n = 1,2,\ldots). \end{cases}$$

Then $T_0 = I$ and $\forall\ n,\ T_{-n} = T^*_n$. In addition, K_Φ is positive definite. Thus let

$$T(r,\theta) = \sum_{-\infty}^{\infty} r^{|m|}e^{\sqrt{-1}\ m\theta}T_m\ \ (0 \leq r < 1, 0 \leq \theta \leq 2\pi).$$

Since $||T_m|| \leq 1$, the series is convergent in norm. Set $z = re^{\sqrt{-1}\,\theta}$ -- then

$$T(r,\theta) =$$

$$(\frac{1}{2} I + \sum_1^\infty z^m T^m) + (\frac{1}{2} I + \sum_1^\infty \bar{z}^m (T^*)^m$$

$$= \mathrm{Re}(I + 2 \sum_1^\infty z^m T^m)$$

$$= \mathrm{Re}(I + zT)(I - zT)^{-1}$$

=>

$$\langle x, T(r,\theta)x\rangle$$

$$= ||y||^2 - |z|^2||Ty||^2 \geq 0 \quad (y = (I - zT)^{-1}x).$$

Fix a finite sequence

$$x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n \in H$$

and in the above, take

$$x = \sum_{k=-n}^{n} e^{-\sqrt{-1}\,k\theta} x_k.$$

Then

$$\sum_{k,\ell=-n}^{n} e^{\sqrt{-1}\,(k-\ell)\theta} \langle x_k, T(r,\theta)x_\ell\rangle \geq 0$$

or still,

$$\sum_{k,\ell=-n}^{n} \sum_{-\infty}^{\infty} e^{\sqrt{-1}\,(k-\ell+m)\theta} r^{|m|} \langle x_k, T_m x_\ell\rangle \geq 0$$

or still,

$$\sum_{-\infty}^{\infty} e^{\sqrt{-1}\,\theta m} \sum_{k,\ell=-n}^{n} r^{|m+\ell-k|} \langle x_k, T_{m+\ell-k} x_\ell\rangle \geq 0$$

=>

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{-\infty}^{\infty} e^{\sqrt{-1}\,\theta m}(\ldots)d\theta \geq 0$$

=>

$$\sum_{k,\ell=-n}^{n} r^{|\ell-k|}\langle x_k, T_{\ell-k}x_\ell\rangle \geq 0$$

=>

$$\sum_{k,\ell=-n}^{n} \langle x_k, T_{\ell-k}x\ \rangle$$

$$= \lim_{r\uparrow 1}\sum_{k,\ell=-n}^{n} r^{|\ell-k|}\langle x_k, T_{\ell-k}x_k\rangle$$

$$\geq 0.$$

9.9 LEMMA If $\Phi = \Delta_R\pi$ for some $*$-representation π, then K_Φ is positive definite.

PROOF In fact, $\forall\ f \in F(\Gamma,H)$,

$$\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\eta)f(\eta)\rangle$$

$$= \sum_{\xi,\eta} \langle f(\xi),R^*\pi(\xi^*\eta)Rf(\eta)\rangle$$

$$= \sum_{\xi,\eta} \langle \pi(\xi)Rf(\xi),\pi(\eta)Rf(\eta)\rangle$$

$$= \left|\left|\sum_{\mu} \pi(\mu)Rf(\mu)\right|\right|^2 \geq 0.$$

Suppose that $\Phi:\Gamma \to \mathcal{B}(H)$ is a function — then Φ is said to satisfy the <u>boundedness condition</u> if $\exists$ a map $\beta:\Gamma \to \underline{R}_{\geq 0}$ such that $\forall\ \mu \in \Gamma$ and $\forall\ f \in F(\Gamma,H)$,

$$\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\mu^*\mu\eta)f(\eta)\rangle$$

$$\leq \beta(\mu) \sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\eta)f(\eta)\rangle.$$

9.10 <u>LEMMA</u> If $\Phi = \Delta_R\pi$ for some $*$-representation π, then Φ satisfies the boundedness condition.

<u>PROOF</u> In fact, $\forall\ \mu \in \Gamma$ and $\forall\ f \in F(\Gamma,H)$,

$$\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\mu^*\mu\eta)f(\eta)\rangle$$

$$= \sum_{\xi,\eta} \langle f(\xi),R^*\pi(\xi^*\mu^*\mu\eta)Rf(\eta)\rangle$$

$$= \sum_{\xi,\eta} \langle \pi(\mu)\pi(\xi)Rf(\xi),\pi(\mu)\pi(\eta)Rf(\eta)\rangle$$

$$= ||\pi(\mu)\sum_{\nu} \pi(\nu)Rf(\nu)||^2$$

$$\leq ||\pi(\mu)||^2 \sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\eta)f(\eta)\rangle.$$

9.11 <u>THEOREM</u> (Sz.-Nagy) Suppose that $\Phi:\Gamma \to \mathcal{B}(H)$ is a function. Assume: K_Φ is positive definite and Φ satisfies the boundedness condition — then $\exists$ a complex Hilbert space K, a minimal $*$-representation π of Γ on K, and a linear map

$$\left[\begin{array}{l} \mathcal{F}(\Gamma,\mathcal{H}) \to \mathcal{K} \\ \quad f \to \hat{f} \end{array}\right.$$

onto a dense linear subspace of $\mathcal{K}$ such that $\Phi = \Delta_R \pi$ and $\forall\ f,g \in \mathcal{F}(\Gamma,\mathcal{H})$ & $\forall\ \mu \in \Gamma$,

$$\langle \hat{f},\pi(\mu)\hat{g}\rangle = \sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\mu\eta)g(\eta)\rangle.$$

9.12 <u>EXAMPLE</u> If G is a group, then the boundedness condition is automatic and specialization of 9.11 leads to the following classical assertion. Given a positive definite function $\chi:G \to \underline{C}$ with $\chi(e) = 1$, put $K_\chi(\sigma,\tau) = \chi(\sigma^{-1}\tau)$ $(\sigma,\tau \in G)$ -- then the kernel K_χ is positive definite, hence $\exists$ a complex Hilbert space $\mathcal{H}_\chi$, a homomorphism $U_\chi:G \to \mathcal{U}(\mathcal{H}_\chi)$, and a cyclic unit vector $x_\chi \in \mathcal{H}_\chi$ such that $\forall\ \sigma \in G$:

$$\chi(\sigma) = \langle x_\chi,U_\chi(\sigma)x_\chi\rangle.$$

9.13 <u>EXAMPLE</u> Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction -- then $\exists\ \mathcal{K} \supset \mathcal{H}$ and $U \in \mathcal{U}(\mathcal{K})$ such that $T^n = pr_{\mathcal{H}}U^n$ or still, $T^n = P_{\mathcal{H}}(U^n|\mathcal{H})$ $(n = 0,1,\ldots)$, $P_{\mathcal{H}}:\mathcal{K} \to \mathcal{H}$ the orthogonal projection (cf. 9.1 and 9.8). Furthermore, $\{U^n\mathcal{H}:n \in \underline{Z}\}$ is total in $\mathcal{K}$.

[Note: Let

$$E^U:\text{Bor}[0,2\pi] \to \mathcal{L}(\mathcal{K})$$

be the spectral measure attached to U:

$$U = \int_0^{2\pi} e^{\sqrt{-1}\,\theta} dE^U(\theta)$$

$\Rightarrow$

$$U^n = \int_0^{2\pi} e^{\sqrt{-1}\,n\theta} dE^U(\theta).$$

Then $E = P_{H} \circ E^{U}|H$ is a semispectral measure and

$$T^{n} = \int_{0}^{2\pi} e^{\sqrt{-1}\, n\theta} dE(\theta),$$

a conclusion that should be compared with that arrived at during the course of deriving von Neumann's inequality in §7.]

While the proof of 9.11 is "canonical", it is on the lengthy side so we shall break the argument up into a series of steps.

<u>Step 1</u>: Given $f \in F(\Gamma,H)$, define a function $\hat{f}:\Gamma \to H$ by

$$\hat{f}(\xi) = \sum_{\eta} \Phi(\xi^{*}\eta)f(\eta).$$

Then

$$\hat{F}(\Gamma,H) = \{\hat{f}:f \in F(\Gamma,H)\}$$

is a complex linear space. Put

$$\langle\hat{f},\hat{g}\rangle = \sum_{\xi} \langle\hat{f}(\xi),g(\xi)\rangle$$

and consider the RHS:

$$\begin{aligned}
\sum_{\xi} \langle\hat{f}(\xi),g(\xi)\rangle &= \sum_{\xi} \langle\sum_{\eta} \Phi(\xi^{*}\eta)f(\eta),g(\xi)\rangle \\
&= \sum_{\xi}\sum_{\eta} \langle f(\eta),\Phi(\xi^{*}\eta)^{*}g(\xi)\rangle \\
&= \sum_{\xi}\sum_{\eta} \langle f(\eta),\Phi(\eta^{*}\xi)g(\xi)\rangle \quad \text{(cf. 9.7)} \\
&= \sum_{\eta} \langle f(\eta),\sum_{\xi} \Phi(\eta^{*}\xi)g(\xi)\rangle \\
&= \sum_{\eta} \langle f(\eta),\hat{g}(\eta)\rangle.
\end{aligned}$$

Then

$$\begin{cases} \hat{f} = \hat{f}_1 \\ \hat{g} = \hat{g}_1 \end{cases}$$

=>

$$\begin{aligned} <\hat{f},\hat{g}> &= \sum_{\xi} <\hat{f}(\xi),g(\xi)> \\ &= \sum_{\xi} <\hat{f}_1(\xi),g(\xi)> \\ &= \sum_{\eta} <f_1(\eta),\hat{g}(\eta)> \\ &= \sum_{\eta} <f_1(\eta),\hat{g}_1(\eta)> \\ &= \sum_{\xi} <\hat{f}_1(\xi),g_1(\xi)> \\ &= <\hat{f}_1,\hat{g}_1>. \end{aligned}$$

Therefore

$$< \, , \, >: \hat{F}(\Gamma,H) \times \hat{F}(\Gamma,H) \to \underline{C}$$

iw welldefined.

<u>Step 2</u>: Since K_Φ is positive definite,

$$\begin{aligned} <\hat{f},\hat{f}> &= \sum_{\xi} <\hat{f}(\xi),f(\xi)> \\ &= \sum_{\xi}\sum_{\eta} <f(\eta),\Phi(\eta^*\xi)f(\xi)> \end{aligned}$$

$$= \sum_{\xi,\eta} \langle f(\xi), \Phi(\xi^*\eta) f(\eta) \rangle$$

$$\geq 0.$$

And this implies that

$$|\langle \hat{f}, \hat{g} \rangle|^2 \leq \langle \hat{f}, \hat{f} \rangle \langle \hat{g}, \hat{g} \rangle.$$

Step 3: We claim that

$$\langle f, f \rangle = 0 \Rightarrow f = 0.$$

For then $\langle \hat{f}, \hat{g} \rangle = 0 \ \forall \ g$, i.e.,

$$\sum_{\xi} \langle \hat{f}(\xi), g(\xi) \rangle = 0 \ \forall \ g.$$

In this relation, take $g = g_{\eta,y}$:

$$0 = \sum_{\xi} \langle \hat{f}(\xi), g_{\eta,y}(\xi) \rangle$$

$$= \langle \hat{f}(\eta), y \rangle$$

$$\Rightarrow \hat{f} = 0,$$

η and y being arbitrary. Therefore $\langle \ , \ \rangle$ is an inner product and $\hat{F}(\Gamma, H)$ is a pre-Hilbert space.

Step 4: Given $\hat{f}$ and $\mu \in \Gamma$, write

$$F_{\mu}(\xi) = \hat{f}(\mu^*\xi)$$

$$\equiv \sum_{\eta} \Phi(\xi^*\mu\eta) f(\eta) \quad (\xi \in \Gamma).$$

Let $y_j = f(\eta_j)$ $(j = 1,\ldots,m)$ be the nonzero values of f, thus

$$F_\mu(\xi) = \sum_{j=1}^{m} \Phi(\xi^*\mu\eta_j)y_j.$$

Define $\tilde{F}_\mu$ as follows:

$$\tilde{F}_\mu(\nu) = \sum_{\mu\eta_j=\nu} y_j \text{ if } \mu\eta_\ell = \nu \ (\exists\ \ell)$$

and

$$\tilde{F}_\mu(\nu) = 0 \text{ if } \mu\eta_j \neq \nu \ (j = 1,\ldots,m).$$

Then

$$\tilde{F}_\mu \in F(\Gamma,H)$$

and $\forall\ \xi \in \Gamma$,

$$F_\mu(\xi) = \sum_{k=1}^{n} \Phi(\xi^*\nu_k)\tilde{F}_\mu(\nu_k),$$

where $\{\nu_1,\ldots,\nu_n\} = \{\mu\eta_1,\ldots,\mu\eta_m\}$ and $\nu_k \neq \nu_\ell$ for $k \neq \ell$. Consequently,

$$F_\mu \in F(\Gamma,H).$$

Step 5: Define $\pi\langle\mu\rangle$ by the rule

$$(\pi\langle\mu\rangle\hat{f})(\xi) = \hat{f}(\mu^*\xi).$$

Then

$$\pi\langle\mu\rangle:\hat{F}(\Gamma,H) \to \hat{F}(\Gamma,H)$$

is linear and

$$\pi\langle\mu\nu\rangle = \pi\langle\mu\rangle\pi\langle\nu\rangle.$$

In addition,

$$\langle\pi\langle\mu\rangle\hat{f},\hat{f}\rangle = \sum_{\xi} \langle\hat{f}(\mu^*\xi),f(\xi)\rangle$$

$$= \sum_{\xi}\sum_{\eta} \langle\Phi(\xi^*\mu\eta)f(\eta),f(\xi)\rangle$$

$$= \sum_{\xi} \sum_{\eta} \langle f(\eta), \Phi(\xi^*\mu\eta)^* f(\xi)\rangle$$

$$= \sum_{\xi} \sum_{\eta} \langle f(\eta), \Phi(\eta^*\mu^*\xi) f(\xi)\rangle$$

$$= \sum_{\eta} \langle f(\eta), \sum_{\xi} \Phi((\mu\eta)^*\xi) f(\xi)\rangle$$

$$= \sum_{\eta} \langle f(\eta), \hat{f}(\mu\eta)\rangle$$

$$= \langle \hat{f}, \pi\langle\mu^*\rangle\hat{f}\rangle.$$

So, by polarization,

$$\langle \pi\langle\mu\rangle\hat{f}, \hat{g}\rangle = \langle \hat{f}, \pi\langle\mu^*\rangle\hat{g}\rangle.$$

In particular:

$$\langle \pi\langle\mu\rangle\hat{f}, \pi\langle\mu\rangle\hat{f}\rangle$$

$$= \langle \hat{f}, \pi\langle\mu^*\rangle\pi\langle\mu\rangle\hat{f}\rangle$$

$$= \langle \hat{f}, \pi\langle\mu^*\mu\rangle\hat{f}\rangle.$$

Step 6: We have

$$\langle \hat{f}, \pi\langle\mu^*\mu\rangle\hat{f}\rangle$$

$$= \sum_{\eta} \langle f(\eta), \hat{f}((\mu^*\mu)^*\eta)\rangle$$

$$= \sum_{\eta} \langle f(\eta), \hat{f}(\mu^*\mu\eta)\rangle$$

$$= \sum_{\xi,\eta} \langle f(\xi), \Phi(\xi^*\mu^*\mu\eta) f(\eta)\rangle,$$

which, by the boundedness condition, is

$$\leq \beta(\mu) \sum_{\xi,\eta} <f(\xi),\Phi(\xi^*\eta)f(\eta)>$$

$$= \beta(\mu)<\hat{f},\hat{f}>.$$

Therefore

$$<\pi<\mu>\hat{f},\pi<\mu>\hat{f}> \leq \beta(\mu)<\hat{f},\hat{f}>.$$

<u>Step 7</u>: Let K be the completion of $\hat{F}(\Gamma,H)$ per the inner product $<\ ,\ >$ — then $\pi<\mu>$ admits a unique extension to an element $\pi(\mu) \in \mathcal{B}(K)$ and

$$\pi:\Gamma \rightarrow \mathcal{B}(K)$$

is a *-representation of Γ. Moreover, by construction the arrow

$$\begin{cases} F(\Gamma,H) \rightarrow K \\ \quad f \rightarrow \hat{f} \end{cases}$$

has the stated properties.

<u>Step 8</u>: Define $R:H \rightarrow K$ by

$$Rx = \hat{f}_{e,x}.$$

Then $R \in \mathcal{B}(H,K)$:

$$||Rx||^2 = <\hat{f}_{e,x},\hat{f}_{e,x}>$$

$$= <x,\Phi(e^*e)x>$$

$$= <x,\Phi(e)x>.$$

And

$$<x,R^*\pi(\mu)Rx>$$

$$= <Rx,\pi(\mu)Rx>$$

$$= <\hat{f}_{e,x}, \pi(\mu)\hat{f}_{e,x}>$$

$$= \sum_{\xi}\sum_{\eta} <f_{e,x}(\eta), \Phi(\eta^*\mu\xi) f_{e,x}(\xi)>$$

$$= <x, \Phi(\mu)x>$$

$\Rightarrow$

$$\Phi = \Delta_R\pi.$$

<u>Step 9</u>: Let $f \in F(\Gamma, H)$ -- then

$$\hat{f}(\xi) = \sum_{\eta} \Phi(\xi^*\eta) f(\eta)$$

$$= \sum_{\eta}\sum_{\mu} \Phi((\eta^*\xi)^*\mu) f_{e,f(\eta)}(\mu)$$

$$= \sum_{\eta} \hat{f}_{e,f(\eta)}(\eta^*\xi)$$

$$= \sum_{\eta} (\pi(\eta)\hat{f}_{e,f(\eta)})(\xi)$$

$$= \sum_{\eta} (\pi(\eta) Rf(\eta))(\xi).$$

<u>Step 10</u>: π is minimal, i.e., $\pi(\Gamma)RH$ is total in K. This is because each $\hat{f} \in \hat{F}(\Gamma, H)$ is a finite linear combination of elements of $\pi(\Gamma)RH$ and $\hat{F}(\Gamma, H)$ is dense in K.

The proof of 9.11 is now complete.

9.14 <u>REMARK</u> Suppose that $\Phi(e) = I$ -- then $\Phi = pr_H\pi$. Proof: π minimal $\Rightarrow$

π unital

$$\Rightarrow$$

$$R^*R = R^*\pi(e)R = \Phi(e) = I.$$

[Note: Here is a corollary: $\forall\ \xi \in \Gamma$,

$$\Phi(\xi)^*\Phi(\xi) \leq \Phi(\xi^*\xi).$$

Indeed, $\forall\ x \in H$,

$$\begin{aligned} <x,\Phi(\xi)^*\Phi(\xi)x> &= ||\Phi(\xi)x||^2 \\ &\leq ||\pi(\xi)x||^2 \\ &= <x,\pi(\xi^*\xi)x> \\ &= <x,\Phi(\xi^*\xi)x>. \end{aligned}$$

So

$$\xi = \xi^* \Rightarrow \Phi(\xi)^2 \leq \Phi(\xi^2).]$$

In certain situations, the assumption that K_Φ is positive definite forces the boundedness condition.

9.15 EXAMPLE Suppose that A is a unital Banach $*$-algebra thought of as a $*$-semigroup w.r.t. multiplication. Let $\Phi:A \rightarrow B(H)$ be a linear map and assume that K_Φ is positive definite. Given $f \in F(A,H)$, define $\omega_f:A \rightarrow \underline{C}$ by

$$\omega_f(\mu) = \sum_{\xi,\eta} <f(\xi),\Phi(\xi^*\mu\eta)f(\eta)>.$$

Then ω_f is a positive linear functional:

$$\omega_f(\mu^*\mu) = \sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\mu^*\mu\eta)f(\eta)\rangle$$

$$= \langle\hat{f},\pi\langle\mu^*\mu\rangle\hat{f}\rangle$$

$$= \langle\pi\langle\mu\rangle\hat{f},\pi\langle\mu\rangle\hat{f}\rangle$$

$$\geq 0.$$

So, by standard generalities,

$$|\omega_f(\mu)| \leq \omega_f(e)(r(\mu^*\mu))^{1/2},$$

$r(.)$ the spectral radius. That the boundedness condition is satisfied is thus manifest.

9.16 <u>LEMMA</u> Fix a $\star$-semigroup Γ and let $\Phi:\Gamma \rightarrow \mathcal{B}(\mathcal{H})$ be a function. Assume: K_Φ is positive definite -- then the following are equivalent.

1. $\exists$ a map $\beta:\Gamma \rightarrow \underline{\mathrm{R}}_{\geq 0}$ such that $\forall\ \mu \in \Gamma$ and $\forall\ f \in \mathcal{F}(\Gamma,\mathcal{H})$,

$$\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\mu^*\mu\eta)f(\eta)\rangle$$

$$\leq \beta(\mu) \sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\eta)f(\eta)\rangle.$$

2. $\exists$ a map $\beta:\Gamma \rightarrow \underline{\mathrm{R}}_{\geq 0}$ such that $\forall\ \mu \in \Gamma$ and $\forall\ x \in \mathcal{H}$,

$$\langle x,\Phi(\xi^*\mu^*\mu\xi)x\rangle \leq \beta(\mu)\langle x,\Phi(\xi^*\xi)x\rangle.$$

3. $\exists$ a positive constant K and a submultiplicative map $\alpha:\Gamma \rightarrow \underline{\mathrm{R}}_{\geq 0}$ such that $\forall\ \mu \in \Gamma$,

$$||\Phi(\mu)|| \leq K\alpha(\mu).$$

4. $\exists$ a map $\rho:\Gamma \to \underline{R}_{\geq 0}$ such that $\forall\ \mu \in \Gamma$ and $\forall\ f \in F(\Gamma,H)$,

$$\liminf_{k \to \infty} \Big(\sum_{\xi,\eta} \langle f(\xi), \Phi(\xi^*(\mu^*\mu)^{2k}\eta) f(\eta)\rangle \Big)^{2^{-k}} \leq \rho(\mu).$$

<u>PROOF</u>

- 1 => 2: This is obvious (same β).

- 2 => 3: Suppose that $||x|| = 1$, $||y|| = 1$ — then

$$|\langle y,\Phi(\mu)x\rangle|^2 \leq \langle y,\Phi(e)y\rangle\ \langle x,\Phi(\mu^*\mu)x\rangle$$

$$\leq \beta(\mu)\langle y,\Phi(e)y\rangle\ \langle x,\Phi(e)x\rangle$$

$$\leq \beta(\mu)||\Phi(e)||^2$$

=>

$$||\Phi(\mu)|| \leq \beta(\mu)^{1/2}||\Phi(e)||.$$

Therefore we can take $\alpha = \beta^{1/2}$, $K = ||\Phi(e)||$. Choosing β to be minimal gives $\alpha(\mu\nu) \leq \alpha(\mu)\alpha(\nu)$, the asserted submultiplicativity of α.

- 3 => 4: Note that

$$||\Phi(\xi^*(\mu^*\mu)^{2k}\eta)||$$

$$\leq K\alpha(\xi^*(\mu^*\mu)^{2k}\eta)$$

$$\leq K\alpha(\xi^*)\alpha((\mu^*\mu)^{2k})\alpha(\eta)$$

$$\leq K\alpha(\xi^*)\alpha(\eta)\alpha(\mu^*\mu)^{2k}$$

=>

$$\beta(\mu) \leq \alpha(\mu^*\mu).$$

• 4 => 1: First

$$|\langle\pi\langle\mu^*\mu\rangle\hat{f},\hat{f}\rangle|^2 \leq ||\pi\langle\mu^*\mu\rangle\hat{f}||^2\ ||\hat{f}||^2.$$

I.e.:

$$|\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\mu^*\mu\eta)f(\eta)\rangle|^2$$

$$\leq (\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*(\mu^*\mu)^2\eta)f(\eta)\rangle)$$

$$\times (\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\eta^*)f(\eta)\rangle).$$

Thus, by iteration,

$$\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\mu^*\mu\eta)f(\eta)\rangle$$

$$\leq (\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*(\mu^*\mu)^{2k}\eta)f(\eta)\rangle)^{2^{-k}}$$

$$\times (\sum_{\xi,\eta} \langle f(\xi),\Phi(\xi^*\eta^*)f(\eta)\rangle)^{1-2^{-k}}.$$

And so forth.

9.17 <u>THEOREM</u> Suppose that $\Phi:\Gamma \to \mathcal{B}(\mathcal{H})$ is a function. Assume: K_Φ is positive definite and $\exists\ M > 0$:

$$||\Phi(\mu)|| \leq M\ \forall\ \mu \in \Gamma.$$

Then Φ satisfies the boundedness condition, hence 9.11 is applicable.

[Thanks to 9.16 (3 => 1), this is immediate.]

9.18 <u>REMARK</u> It can happen that K_Φ is positive definite, yet Φ fails to satisfy the boundedness condition. For example, let Γ be the $*$-semigroup of all complex polynomials on the real line (the $*$-operation being complex conjugation). Given $\xi \in \Gamma$, put

$$\Phi(\xi) = \int_0^\infty \xi(t)e^{-t}dt$$

to get a function $\Phi:\Gamma \rightarrow \underline{C}$ ($\approx \mathcal{B}(\underline{C})$) -- then it is clear that K_Φ is positive definite. Still, Φ does not satisfy the boundedness condition. Thus write ξ_0 for the polynomial $t \rightarrow t$ and note that

$$\Phi(\xi_0^n) = n!.$$

If now Φ did satisfy the boundedness condition, then it would satisfy 3 in 9.16 for a submultiplicative α, hence

$$\begin{aligned} n! &= |\Phi(\xi_0^n)| \\ &\leq K\alpha(\xi_0^n) \\ &\leq K\alpha(\xi_0)^n \end{aligned}$$

=>

$$\alpha(\xi_0) \geq (\frac{n!}{K})^{1/n},$$

an impossibility.

[Note: Sans Stirling,

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \rightarrow 0 \Rightarrow \frac{1}{(n!)^{1/n}} \rightarrow 0.]$$

9.19 LEMMA Suppose that K is a complex Hilbert space containing H as a closed subspace. Let $A \in B(H)$, $T \in B(K)$ -- then $T|H = A$ iff

$$A = pr_H T \text{ and } A^*A = pr_H T^*T$$

or still, iff

$$(A^*)^i A^j = pr_H (T^*)^i T^j \quad (i,j = 0,1,\ldots,).$$

PROOF It need only be shown that the conditions

$$A = pr_H T \text{ and } A^*A = pr_H T^*T$$

imply $T|H = A$. To this end, let $x \in H$ -- then

$$\begin{aligned} ||Ax||^2 &= \langle x, A^*Ax\rangle \\ &= \langle x, P_H T^*Tx\rangle \\ &= \langle P_H x, T^*Tx\rangle \\ &= \langle x, T^*Tx\rangle \\ &= \langle Tx, Tx\rangle \\ &= ||Tx||^2. \end{aligned}$$

But at the same time,

$$||Ax|| = ||P_H Tx||.$$

Therefore

$$||Tx|| = ||P_H Tx||$$

$$\Rightarrow$$

$$Tx = P_H Tx.$$

I.e.:

$$Tx = Ax \Rightarrow T|H = A.$$

Let $A \in B(H)$ -- then A is said to be subnormal if $\exists$ a complex Hilbert space K containing H as a closed subspace and a normal $T \in B(K)$ such that $T|H = A$.

[Note: In general, every $A \in B(H)$ can be represented as the projection of a normal $T \in B(K)$ in some extension space K: $A = \mathrm{pr}_H T$ (cf. 9.1).]

9.20 EXAMPLE Every isometry A is subnormal. For A is a contraction, hence $\exists\ K \supset H$ and $U \in U(K)$ such that $A = \mathrm{pr}_H U$ (cf. 9.1). But $\forall\ x \in H$,

$$||x|| = ||Ax|| = ||P_H Ux|| \leq ||Ux|| = ||x||$$

$$\Rightarrow P_H Ux = Ux$$

$$\Rightarrow U|H = A.$$

9.21 LEMMA Suppose that $A \in B(H)$ is subnormal -- then $\forall\ x_0, x_1, \ldots, x_n \in H$,

$$\sum_{i,j=0}^{n} \langle A^i x_j, A^j x_i \rangle \geq 0.$$

PROOF The LHS equals

$$\sum_{i,j=0}^{n} \langle T^i x_j, T^j x_i \rangle$$

or still,

$$\sum_{i,j=0}^{n} \langle x_j, (T^i)^* T^j x_i \rangle$$

or still,

$$\sum_{i,j=0}^{n} \langle x_j, T^j (T^i)^* x_i \rangle$$

or still,

$$\sum_{i,j=0}^{n} \langle (T^j)^* x_j, (T^i)^* x_i \rangle$$

$$= \left|\left| \sum_{k=0}^{n} (T^k)^* x_k \right|\right|^2$$

$$\geq 0.$$

9.22 THEOREM Let $A \in \mathcal{B}(H)$ and suppose that $\forall\ x_0, x_1, \ldots, x_n \in H$,

$$\sum_{i,j=0}^{n} \langle A^i x_j, A^j x_i \rangle \geq 0.$$

Then A is subnormal.

PROOF Take for Γ the set of all pairs ξ of nonnegative integers (i,j), with

$$\begin{cases} (i,j)(i',j') = (i + i', j + j') \\ (i,j)^* = (j,i)\ ,\ e = (0,0). \end{cases}$$

Then Γ is a $*$-semigroup. Define $\Phi : \Gamma \rightarrow \mathcal{B}(H)$ by

$$\Phi(\xi) = \Phi(i,j) = (A^*)^i A^j.$$

Claim: K_Φ is positive definite. Proof:

$$\sum_{\xi,\xi'} \langle f(\xi), \Phi(\xi^* \xi') f(\xi') \rangle$$

$$= \sum_{(i,j)} \sum_{(i',j')} \langle f(i,j), \Phi(j + i', i + j') f(i',j') \rangle$$

$$= \sum_{(i,j)} \sum_{(i',j')} \langle f(i,j), (A^*)^{j+i'} A^{i+j'} f(i',j') \rangle$$

$$= \sum_{(i,j)} \sum_{(i',j')} \langle A^{j+i'} f(i,j), A^{i+j'} f(i',j') \rangle$$

$$= \sum_{i} \sum_{i'} \langle A^{i'} x_i, A^{i} x_{i'} \rangle$$

$$\geq 0.$$

Here

$$\begin{cases} x_i = \sum_j A^j f(i,j) \\ x_{i'} = \sum_{j'} A^{j'} f(i',j'). \end{cases}$$

As for the boundedness condition, we have

$$||\Phi(i,j)|| \leq ||A||^{i+j}$$

and

$$\alpha(i,j) = ||A||^{i+j}$$

is (sub)multiplicative:

$$\alpha((i,j)(i',j')) = \alpha(i + i', j + j')$$

$$= ||A||^{i+i'+j+j'}$$

$$= \alpha(i,j)\alpha(i',j').$$

Therefore one can quote 3 => 1 in 9.16. Now apply 9.11 to get $\pi:\Phi = pr_{H}\pi$

(cf. 9.14 ($\Phi(e) = \Phi(0,0) = I$)). Since the semigroup operation is commutative, the $\pi(\xi)$ are normal. Let $\xi_0 = (0,1)$ -- then a given $\xi = (i,j)$ can be written as

$$\xi = (1,0)^i(0,1)^j$$

$$= (\xi_0^*)^i \xi_0^j,$$

so

$$\pi(\xi) = (T^*)^i T^j,$$

where $T = \pi(\xi_0)$. Accordingly,

$$(A^*)^i A^j = \Phi(i,j) = pr_{\mathcal{H}}\pi(i,j) = pr_{\mathcal{H}}(T^*)^i T^j \quad (i,j = 0,1,\ldots)$$

$$\Rightarrow$$

$$T|\mathcal{H} = A \quad \text{(cf. 9.19).}$$

9.23 <u>REMARK</u> Retaining the notation of 9.16,

$$\alpha(i,j) = ||A||^{i+j}$$

$$\Rightarrow$$

$$\beta(i,j) = ||A||^{2(i+j)} \Rightarrow \beta(1,0) = ||A||^2.$$

Consequently,

$$\sum_{i,j=0}^{n} \langle A^{i+1}x_j, A^{j+1}x_i \rangle$$

$$\leq ||A||^2 \sum_{i,j=0}^{n} \langle A^i x_j, A^j x_i \rangle.$$

APPENDIX

Our definition of $*$-semigroup incorporates the condition that Γ be unital, an assumption that simplifies the theory. Here, we shall retain the involution $*:\Gamma \to \Gamma$ but drop the existence of the unit e: Γ will thus denote a nonunital $*$-semigroup.

What can be said about 9.11? For the most part, no use was made of the unit which, in fact, makes an appearance only near the end, viz. in Step 8 (construction of R). Still, 9.11 does go through in the nonunital case provided we append an extra hypothesis (see below).

Going back to the beginning for the moment, the first point is that 9.5 and 9.6 remain valid (the proofs, however, are a little more complicated). Next, the boundedness condition does not change but 9.16 needs a slight revision: Replace 3 by

3'. $\exists$ a function $K:\Gamma \to \underline{\mathbb{R}}_{\geq 0}$ and a submultiplicative map $\alpha:\Gamma \to \underline{\mathbb{R}}_{\geq 0}$ such that $\forall\ \mu \in \Gamma$,

$$||\Phi(\xi^*\mu^*\mu\xi)|| \leq K(\xi)\alpha(\mu).$$

Turning to the nonunital version of 9.11, introduce the following condition:

$\underline{\text{C}}$: $\exists$ a net $\{e_i : i \in I\}$ in Γ such that $\forall\ \xi \in \Gamma$,

$$\begin{cases} \Phi(\xi e_i) \to \Phi(\xi) \\ \Phi(e_i^*\xi) \to \Phi(\xi) \end{cases}$$

weakly and

$$\gamma = \sup_{i \in I} ||\Phi(e_i^* e_i)|| < \infty.$$

<u>RAPPEL</u> Let $\mathcal{H}$ be a complex Hilbert space. Suppose that $\{x_i : i \in I\}$ is a bounded weak Cauchy net in $\mathcal{H}$ -- then there is a unique $x \in \mathcal{H}$ such that $x_i \rightarrow x$ weakly.

<u>RAPPEL</u> Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces. Suppose that $\{T_i : i \in I\}$ is a weak Cauchy net in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\forall \ x \in \mathcal{H}$,

$$\sup_{i \in I} ||T_i x|| < \infty.$$

Then $\exists \ T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $T_i \rightarrow T$ weakly.

[$\forall \ x \in \mathcal{H}$, $\{T_i x : i \in I\}$ is a bounded weak Cauchy net in $\mathcal{K}$, hence is weakly convergent to a unique element of $\mathcal{K}$, call it Tx -- then $T:\mathcal{H} \rightarrow \mathcal{K}$ is linear. But, by the uniform boundedness principle,

$$\sup_{i \in I} ||T_i|| = M < \infty.$$

Therefore $\forall \ x \in \mathcal{H}$ & $\forall \ y \in \mathcal{K}$,

$$|\langle y, Tx \rangle| = \lim_{i \in I} |\langle y, T_i x \rangle|$$

$$\leq M||y|| \ ||x||.$$

So T is bounded and by construction, $T_i \rightarrow T$ weakly.]

[Note: In general, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is not weakly complete. E.g.: Take $\mathcal{H} = \mathcal{K}$ infinite dimensional and $T:\mathcal{H} \rightarrow \mathcal{H}$ linear and unbounded. Given a finite dimensional subspace $F \subset \mathcal{H}$, let $T_F = T \circ P_F$, P_F the orthogonal projection of $\mathcal{H}$ onto F -- then $\{T_F\}$ is a weak Cauchy net in $\mathcal{B}(\mathcal{H})$ which does not converge weakly in $\mathcal{B}(\mathcal{H})$.]

Given $\xi \in \Gamma$, define $R_\xi : \mathcal{H} \to \mathcal{K}$ by

$$(R_\xi x)(\mu) = \hat{f}_{\xi,x}(\mu) \quad (= \Phi(\mu^*\xi)x).$$

<u>LEMMA</u> $\forall\ \xi \in \Gamma,\ R_\xi \in \mathcal{B}(\mathcal{H},\mathcal{K})$.

<u>PROOF</u> Suppose that $x_n \to 0$ in $\mathcal{H}$ -- then

$$\begin{aligned} ||R_\xi x_n||^2 &= <\hat{f}_{\xi,x_n},\hat{f}_{\xi,x_n}> \\ &= <x_n,\Phi(\xi^*\xi)x_n> \\ &\leq ||x_n||^2\ ||\Phi(\xi^*\xi)|| \\ &\to 0. \end{aligned}$$

<u>LEMMA</u> $\forall\ \xi,\eta \in \Gamma$, we have

$$R_\xi^* R_\eta = \Phi(\xi^*\eta).$$

<u>PROOF</u> $\forall\ x,y \in \mathcal{H}$,

$$\begin{aligned} <x,R_\xi^* R_\eta y> &= <R_\xi x,R_\eta y> \\ &= <\hat{f}_{\xi,x},\hat{f}_{\eta,y}> \\ &= <x,\Phi(\xi^*\eta)y>. \end{aligned}$$

Continuing the discussion, granted C (maintaining, of course, the assumption that K_Φ is positive definite and satisfies the boundedness condition), put

$R_i = R_{e_i}$ $(i \in I)$ — then

$$<R_ix,R_ix> = <\hat{f}_{e_{i,x}},\hat{f}_{e_{i,x}}>$$

$$= <x,\Phi(e_i^*e_i)x>$$

$$\leq \gamma||x||^2.$$

Thus $\exists$ M > 0:

$$\sup_{i\in I} ||R_i|| = M < \infty \text{ (uniform boundedness principle).}$$

On the other hand, the definitions imply that $\{R_i : i \in I\}$ is a weak Cauchy net, hence $\exists\ R \in \mathcal{B}(H,K)$:

$$R = \lim_{i\in I} R_i \quad \text{(WOT).}$$

With this preparation, we are finally in a position to deal with Step 8 in the proof of 9.11, viz. the assertion that $\Phi = \Delta_R\pi$, i.e., $\forall\ \mu \in \Gamma$,

$$\Phi(\mu) = R^*\pi(\mu)R.$$

First

$$\pi(\mu)R = \pi(\mu)\lim_{i\in I} R_i \quad \text{(WOT)}$$

$$= \lim_{i\in I} \pi(\mu)R_i \quad \text{(WOT).}$$

But

$$\pi(\mu)R_i = \pi(\mu)R_{e_i}.$$

And

$$(\pi(\mu)R_{e_i}x)(\xi) = (\pi(\mu)\hat{f}_{e_{i,x}})(\xi)$$

$$= \Phi(\xi^*\mu e_i)x$$

$$= (R_{\mu e_i}x)(\xi).$$

Therefore

$$\pi(\mu)R_i = R_{\mu e_i}$$

$$\Rightarrow$$

$$\pi(\mu)R = \lim_{i\in I} R_{\mu e_i} \quad \text{(WOT)}.$$

<u>LEMMA</u> We have

$$R_\mu = \lim_{i\in I} R_{\mu e_i} \quad \text{(WOT)}.$$

<u>PROOF</u> $\forall\ x \in H$ and $\forall\ f \in F(\Gamma,H)$,

$$<R_{\mu e_i}x,\hat{f}> = <\hat{f}_{\mu e_i,x},\hat{f}>$$

$$= \sum_\xi <\sum_\eta \Phi(\xi^*\eta)f_{\mu e_i,x}(\eta),f(\xi)>$$

$$= \sum_\xi <\Phi(\xi^*\mu e_i)x,f(\xi)>$$

$$\to \sum_\xi <\Phi(\xi^*\mu)x,f(\xi)>$$

$$= \sum_\xi <\hat{f}_{\mu,x}(\xi),f(\xi)>$$

$$= <\hat{f}_{\mu,x},\hat{f}>$$

$$= <R_\mu x,\hat{f}>.$$

So, in conclusion,

$$\pi(\mu)R = \lim_{i\in I} R_{\mu e_i} \quad \text{(WOT)}$$

$$= R_\mu.$$

But then

$$R^*\pi(\mu)R = R^*R_\mu$$

$$= (\lim_{i\in I} R_i^*)R_\mu \quad \text{(WOT)}$$

$$= \lim_{i\in I} (R_i^* R_\mu) \quad \text{(WOT)}$$

$$= \lim_{i\in I} (R^*_{e_i} R_\mu) \quad \text{(WOT)}$$

$$= \lim_{i\in I} \Phi(e_i^*\mu) \quad \text{(WOT)}$$

$$= \Phi(\mu),$$

as desired.

<u>REMARK</u> π is minimal (cf. Step 10 of the proof of 9.11).

[Note that

$$(Rx)(\xi) = \Phi(\xi^*)x$$

=>

$$(\pi(\mu)Rx)(\xi) = \Phi(\xi^*\mu)x,$$

which leads to the formula in Step 9.]

§10. *DILATION THEOREMS*

In this §, we shall consider some important applications of 9.11.

10.1 RAPPEL If $\{T_i : i \in I\}$ is an increasing net of positive operators converging weakly to an operator T, then $T_i \to T$ strongly and

$$T = \text{l.u.b. } \{T_i : i \in I\}.$$

Suppose that $(\Omega, \mathcal{A})$ is a measurable space and $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ is a semispectral measure.

10.2 THEOREM (Naimark) There exists a complex Hilbert space $\bar{\mathcal{H}}$ containing $\mathcal{H}$ as a closed subspace and a spectral measure $\bar{E}:\mathcal{A} \to \mathcal{L}(\bar{\mathcal{H}})$ such that $\forall\ A \in \mathcal{A}$,

$$E(A) = P_{\mathcal{H}}\bar{E}(A)\,|\mathcal{H},$$

$P_{\mathcal{H}}:\bar{\mathcal{H}} \to \mathcal{H}$ the orthogonal projection of $\bar{\mathcal{H}}$ onto $\mathcal{H}$. Furthermore, the requirement that

$$\{\bar{E}(A)\mathcal{H} : A \in \mathcal{A}\}$$

be total in $\bar{\mathcal{H}}$ determines $\bar{\mathcal{H}}$ and $\bar{E}$ up to isometric isomorphism (cf. 9.6).

PROOF Recalling that $\mathcal{A}$ is a *-semigroup (cf. 9.4), view $E:\mathcal{A} \to \mathcal{E}(\mathcal{H})$ $(\subset \mathcal{B}(\mathcal{H}))$ as the "Φ" of 9.11. There are then two points.

1. The kernel

$$\begin{cases} \mathcal{A} \times \mathcal{A} \to \mathcal{B}(\mathcal{H}) \\ (A,B) \to E(A \cap B) \end{cases}$$

is positive definite: For all

$$\left[\begin{array}{l} A_1,\ldots,A_n \in \mathcal{A} \\ \\ x_1,\ldots,x_n \in \mathcal{H}, \end{array}\right.$$

we have

$$\sum_{i,j=1}^{n} \langle x_i, E(A_i \cap A_j)x_j\rangle \geq 0.$$

To establish this, put

$$\mu = \sum_{i=1}^{n} \mu_{x_i,x_i}.$$

Then by polarization and the Schwarz inequality,

$$\mu_{x_i,x_j} \ll \mu \quad (i,j = 1,\ldots,n).$$

Denote the corresponding Radon-Nikodym derivative by f_{ij} and let $\lambda_1,\ldots,\lambda_n \in \underline{C}$: $\forall\ A \in \mathcal{A}$,

$$\begin{aligned} 0 &\leq \langle \sum_i \lambda_i x_i, E(A)(\sum_j \lambda_j x_j)\rangle \\ &= \sum_{i,j} \bar{\lambda}_i \lambda_j \langle x_i, E(A)x_j\rangle \\ &= \sum_{i,j} \bar{\lambda}_i \lambda_j \mu_{x_i,x_j}(A) \\ &= \sum_{i,j} \bar{\lambda}_i \lambda_j \int_A f_{ij}\, d\mu. \end{aligned}$$

Therefore

$$\sum_{i,j=1}^{n} \bar{\lambda}_i \lambda_j f_{ij}(\omega) \geq 0 \quad [\mu \text{ a.e.}].$$

On the other hand,

$$\left|\sum_{i=1}^{n} \lambda_i \chi_{A_i}\right|^2 = \sum_{i,j=1}^{n} \bar{\lambda}_i \lambda_j \chi_{A_i \cap A_j} \geq 0.$$

Now apply 8.3 to conclude that

$$\sum_{i,j=1}^{n} \chi_{A_i \cap A_j} f_{ij}(\omega) \geq 0 \quad [\mu \text{ a.e.}],$$

which upon integration over Ω gives

$$0 \leq \sum_{i,j=1}^{n} \int_{\Omega} \chi_{A_i \cap A_j} f_{ij} d\mu$$

$$= \sum_{i,j=1}^{n} \int_{A_i \cap A_j} f_{ij} d\mu$$

$$= \sum_{i,j=1}^{n} \langle x_i, E(A_i \cap A_j) x_j \rangle.$$

2. E satisfies the boundedness condition:

$$\sum_{i,j=1}^{n} \langle x_i, E(A_i \cap A \cap A_j) x_j \rangle$$

$$\leq \beta(A) \sum_{i,j=1}^{n} \langle x_i, E(A_i \cap A_j) x_j \rangle.$$

Indeed, the assignment

$$A \to \sum_{i,j=1}^{n} \langle x_i, E(A_i \cap A \cap A_j) x_j \rangle \ (\geq 0)$$

is a measure whose value at Ω is

$$\sum_{i,j=1}^{n} <x_i, E(A_i \cap A_j)x_j>.$$

The boundedness condition is thus satisfied by taking $\beta(A) = 1$ $(A \in \mathcal{A})$.

Thanks to 1 and 2, 9.11 is applicable. Denote the "K" there by $\bar{\mathcal{H}}$ and the "π" there by $\bar{E}$:

$$\begin{cases} \bar{E}:\mathcal{A} \to \mathcal{B}(\bar{\mathcal{H}}) \\ E = pr_{\mathcal{H}}\bar{E} \quad (\text{cf. } 9.12\ (E(\Omega) = I)) \end{cases}$$

with

$$\{\bar{E}(A)\mathcal{H}:A \in \mathcal{A}\}$$

total in $\bar{\mathcal{H}}$. The fact that $\bar{E}$ is a unital *-representation of $\mathcal{A}$ on $\bar{\mathcal{H}}$ implies that

$$\begin{cases} \bar{E}(A)^2 = \bar{E}(A) \\ \bar{E}(A)^* = \bar{E}(A) \end{cases}, \quad \bar{E}(\Omega) = I.$$

I.e.: $\bar{E}:\mathcal{A} \to \mathcal{L}(\bar{\mathcal{H}})$. It remains to prove that $\bar{E}$ is a spectral measure. To begin with, $\bar{E}$ is finitely additive. For let $A_1,A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = \emptyset$ -- then $\forall$ $A,B \in \mathcal{A}$ and $\forall$ $x,y \in \mathcal{H}$,

$$<\bar{E}(B)y, (\bar{E}(A_1) + \bar{E}(A_2))\bar{E}(A)x>$$

$$= <\bar{E}(A)(\bar{E}(A_1) + \bar{E}(A_2))\bar{E}(B)y,x>$$

$$= <\bar{E}(A \cap A_1 \cap B)y + \bar{E}(A \cap A_2 \cap B)y,P_{\mathcal{H}}x>$$

$$= <P_{\mathcal{H}}\bar{E}(A \cap A_1 \cap B)y + P_{\mathcal{H}}\bar{E}(A \cap A_2 \cap B)y,x>$$

$$= \langle E(A \cap A_1 \cap B)y + E(A \cap A_2 \cap B)y, x\rangle$$

$$= \langle E(A \cap (A_1 \cup A_2) \cap B)y, x\rangle$$

$$= \langle P_{\mathcal{H}}\bar{E}(A \cap (A_1 \cup A_2) \cap B)y, x\rangle$$

$$= \langle \bar{E}(A)\bar{E}(A_1 \cup A_2)\bar{E}(B)y, P_{\mathcal{H}}x\rangle$$

$$= \langle \bar{E}(B)y, \bar{E}(A_1 \cup A_2)\bar{E}(A)x\rangle$$

$\Rightarrow$

$$\bar{E}(A_1 \cup A_2) = \bar{E}(A_1) + \bar{E}(A_2).$$

Finally, if $A_1 \subset A_2 \subset \ldots$, then

$$\bar{E}(A_1) \leq \bar{E}(A_2) \leq \ldots$$

and by the same procedure, we find that

$$\lim \bar{E}(A_n) = \bar{E}\Big(\bigcup_{n=1}^{\infty} A_n\Big)$$

weakly, hence strongly (cf. 10.1).

In the proof, minimality was used to force the countable additivity of $\bar{E}$. However, minimality may not have a direct interpretation, while the construction of a larger "$\bar{\mathcal{H}}$" and "$\bar{E}$" does.

10.3 <u>EXAMPLE</u> Take Ω Polish, let $\mathcal{A}$ = Bor Ω, and fix a Borel measure μ. Suppose given a collection of unit vectors e_ω $(\omega \in \Omega)$ such that the function

$\omega \to ||e_\omega||$ is continuous with

$$I = \int_\Omega P_{e_\omega} d\mu(\omega) \quad \text{(SOT)}.$$

So $\forall\ x \in \mathcal{H}$,

$$x = \int_\Omega \langle e_\omega, x\rangle e_\omega d\mu(\omega)$$

and $\forall\ y, x \in \mathcal{H}$,

$$\langle y,x\rangle = \langle y, \int_\Omega \langle e_\omega,x\rangle e_\omega d\mu(\omega)\rangle$$

$$= \int_\Omega \langle y,e_\omega\rangle\langle e_\omega,x\rangle d\mu(\omega).$$

Put

$$E(A) = \int_A P_{e_\omega} d\mu(\omega) \quad (A \in \mathcal{A}).$$

Then E is a semispectral measure. Let $\bar{\mathcal{H}} = L^2(\Omega,\mu)$ and define a spectral measure $\bar{E}$ by

$$\bar{E}(A)f = \chi_A f \quad (A \in \mathcal{A}).$$

Identify $\mathcal{H}$ with a closed subspace of $\bar{\mathcal{H}}$ via the isometry $x \to f_x$, where

$$f_x(\omega) = \langle e_\omega,x\rangle \quad (||x||^2 = \int_\Omega ||\langle e_\omega,x\rangle||^2 d\mu(\omega)).$$

Let

$$\kappa(\omega,\omega') = \langle e_\omega, e_{\omega'}\rangle.$$

Then

$$\int_\Omega \kappa(\omega,\omega') f_x(\omega') d\mu(\omega')$$

$$= \int_\Omega \langle e_\omega,e_{\omega'}\rangle\langle e_{\omega'},x\rangle d\mu(\omega')$$

$$= \langle e_\omega,x\rangle$$

$$= f_x(\omega).$$

On the other hand, $\forall\ f \in H^{\perp}$,

$$\int_{\Omega} \kappa(\omega,\omega')f(\omega')d\mu(\omega')$$

$$= \int_{\Omega} \langle e_{\omega}, e_{\omega'}\rangle f(\omega')d\mu(\omega')$$

$$= \int_{\Omega} \overline{\langle e_{\omega'}, e_{\omega}\rangle} f(\omega')d\mu(\omega')$$

$$= \int_{\Omega} \overline{f_{e_{\omega}}(\omega')} f(\omega')d\mu(\omega')$$

$$= 0.$$

Therefore the orthogonal projection P_{H} of $\bar{H}$ onto H is an integral operator with kernel κ. Finally,

$$E = P_{H}\bar{E}|H.$$

In fact,

$$P_{H}\bar{E}f_{x}(\omega)$$

$$= \int_{\Omega} \kappa(\omega,\omega')\chi_{A}(\omega')f_{x}(\omega')d\mu(\omega')$$

$$= \int_{A} \langle e_{\omega}, e_{\omega'}\rangle\langle e_{\omega'}, x\rangle d\mu(\omega').$$

And, by comparison,

$$f_{E(A)x}(\omega) = \langle e_{\omega}, E(A)x\rangle$$

$$= \langle e_{\omega}, \int_{A} \langle e_{\omega'}, x\rangle e_{\omega'} d\mu(\omega')\rangle$$

$$= \int_{A} \langle e_{\omega}, e_{\omega'}\rangle\langle e_{\omega'}, x\rangle d\mu(\omega').$$

10.4 <u>LEMMA</u> Let Γ be a *-semigroup, $\Phi:\Gamma \to \mathcal{B}(H)$ a function. Assume: K_{Φ} is positive definite — then for all $x \in H$ and for all

$$\begin{bmatrix} \xi_1,\ldots,\xi_n \in \Gamma \\ c_1,\ldots,c_n \in \underline{C}, \end{bmatrix}$$

we have

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \langle x, \Phi(\xi_i^* \xi_j) x \rangle \geq 0.$$

10.5 <u>EXAMPLE</u> Let A be a U^*-algebra and suppose that $\Phi: A \rightarrow B(H)$ is a linear map. Assume: K_Φ is positive definite — then Φ satisfies the boundedness condition. To see this, let $\mu_1,\ldots,\mu_n \in A$ be unitary — then $\forall\ x \in H$ and $\forall\ \xi \in \Gamma$,

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \langle x, \Phi(\xi^* \mu_i^* \mu_j \xi) x \rangle$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \langle x, \Phi((\mu_i \xi)^* \mu_j \xi) x \rangle$$

$$\geq 0.$$

Therefore the matrix

$$[\langle x, \Phi((\mu_i \xi)^* \mu_j \xi) x \rangle]$$

is positive definite, hence its operator norm is $\leq$ its trace which equals

$$\sum_{i=1}^{n} \langle x, \Phi((\mu_i \xi)^* \mu_i \xi) x \rangle$$

$$= \sum_{i=1}^{n} \langle x, \Phi(\xi^* \mu_i^* \mu_i \xi) x \rangle$$

$$= \sum_{i=1}^{n} \langle x, \Phi(\xi^*\xi)x\rangle \quad (\mu_i^*\mu_i = e)$$

$$= n\langle x, \Phi(\xi^*\xi)x\rangle$$

=>

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \langle x, \Phi(\xi^*\mu_i^*\mu_j\xi)x\rangle$$

$$\leq (n \sum_{i=1}^{n} |c_i|^2)\langle x, \Phi(\xi^*\xi)x\rangle.$$

Since any $\mu \in \Gamma$ can be written as a finite linear combination of unitary elements, it follows from 2 in 9.16 that Φ satisfies the boundedness condition.

[Note: Specialized to the case when A is a unital Banach *-algebra, we have thus recovered 9.15 by a different argument.]

10.6 REMARK Here is another proof. Using 8.24, write

$$\Phi(\mu) = \langle a, \pi(\mu)a\rangle \quad (\mu \in A).$$

Then

$$||\Phi(\mu)|| = ||\langle a, \pi(\mu)a\rangle||$$

$$\leq ||a|| \; ||\pi(\mu)a|| \quad \text{(cf. 8.12)}$$

$$\leq ||a||^2 \; ||\pi(\mu)||.$$

In 3 of 9.16, take $K = ||a||^2$ and set $\alpha(\mu) = ||\pi(\mu)||$ to conclude again that Φ satisfies the boundedness condition.

10.7 THEOREM Let A be a U*-algebra and suppose that $\Phi: A \rightarrow B(H)$ is a linear map. Assume: K_Φ is positive definite -- then $\Phi = \Delta_R \pi$, where π is a minimal $*$-representation of A on some complex Hilbert space K.

<u>PROOF</u> In view of 10.5, this is implied by 9.11 modulo one detail: Linearity of π. But, due to the minimality of π, $\pi(A)RH$ is total in K and $\forall\ x,y \in H$,

$$
\begin{aligned}
&<\pi(\eta)Ry,(\pi(\xi_1) + \pi(\xi_2))\pi(\xi)Rx> \\
&\quad = <Ry,(\pi(\eta^*\xi_1\xi) + \pi(\eta^*\xi_2\xi))Rx> \\
&\quad = <y,R^*\pi(\eta^*\xi_1\xi)Rx + R^*\pi(\eta^*\xi_2\xi)Rx> \\
&\quad = <y,\Phi(\eta^*\xi_1\xi)x + \Phi(\eta^*\xi_2\xi)x> \\
&\quad = <y,\Phi(\eta^*\xi_1\xi + \eta^*\xi_2\xi)x> \\
&\quad = <y,\Phi(\eta^*(\xi_1 + \xi_2)\xi)x> \\
&\quad = <y,R^*\pi(\eta^*(\xi_1 + \xi_2)\xi)Rx> \\
&\quad = <\pi(\eta)Ry,\pi(\xi_1 + \xi_2)\pi(\xi)Rx>
\end{aligned}
$$

$\Rightarrow$

$$\pi(\xi_1 + \xi_2) = \pi(\xi_1) + \pi(\xi_2).$$

Ditto:

$$\pi(c\xi) = c\pi(\xi) \qquad (c \in \underline{C}).$$

10.8 <u>REMARK</u> In 10.7, take for A a unital Banach $*$-algebra -- then $\Phi = \Delta_R \pi$ $(= R^*\pi R)$ is continuous:

$$\Phi \in \mathcal{B}(A, \mathcal{B}(H)).$$

Proof: A *-representation of a Banach *-algebra on a Hilbert space is necessarily continuous (in fact, contractive).

[Note:

$$||\Phi|| = ||R||^2 = ||R^*R||.$$

For

$$\Phi(\xi) = R^*\pi(\xi)R$$

=>

$$\begin{aligned} ||\Phi(\xi)|| &\leq ||R^*|| \ ||\pi(\xi)|| \ ||R|| \\ &\leq ||R||^2 \ ||\pi|| \ ||\xi|| \\ &\leq ||R||^2 \ ||\xi|| \quad (||\pi|| \leq 1) \end{aligned}$$

=>

$$||\Phi|| \leq ||R||^2.$$

On the other hand,

$$\Phi(e) = R^*R \quad (\pi(e) = I)$$

=>

$$\begin{aligned} ||R^*R|| &= ||\Phi(e)|| \\ &\leq ||\Phi|| \ ||e|| \\ &\leq ||\Phi|| \quad (||e|| = 1).] \end{aligned}$$

10.9 <u>RAPPEL</u> An <u>approximate unit</u> in a Banach *-algebra A is a norm bounded net $\{e_i : i \in I\}$ such that $\forall \ \xi \in A$,

$$\begin{cases} \lim_{i \in I} ||e_i\xi - \xi|| = 0 \\ \lim_{i \in I} ||\xi e_i - \xi|| = 0. \end{cases}$$

10.10 LEMMA If A is a Banach *-algebra and if A admits an approximate unit, then every positive linear functional on A is continuous.

[Note: If $\omega: A \to \underline{C}$ is a positive linear functional, then

$$|\omega(\xi^* \eta \xi)| \le r(\eta)\omega(\xi^*\xi) \qquad (\eta = \eta^*),$$

$r(.)$ the spectral radius.]

This lemma leads to an "automatic continuity" result. Thus let $\Phi: A \to B(H)$ be a linear map. Assume: K_Φ is positive definite — then

$$\Phi \in B(A, B(H)).$$

Proof: $\forall\, x \in H$, the linear functional

$$\omega_x(\xi) = \langle x, \Phi(\xi)x\rangle$$

is positive, hence by 10.10 is continuous. Polarization then implies that $\forall\, x,y \in H$,

$$\omega_{x,y}(\xi) = \langle x, \Phi(\xi)y\rangle$$

is continuous, thus $\exists\, M_{x,y} > 0$:

$$|\omega_{x,y}(\xi)| \le M_{x,y}||\xi||.$$

But by the uniform boundedness principle,

$$||\Phi|| = \sup_{||\xi|| \le 1} ||\Phi(\xi)|| < \infty$$

$\iff \forall\, x,y \in H$,

$$\sup_{||\xi|| \le 1} |\langle x, \Phi(\xi)y\rangle| < \infty.$$

[Note: In the case when A is unital, this approach provides another route to 10.8.]

10.11 <u>THEOREM</u> Let A be a Banach *-algebra with an approximate unit and suppose that $\Phi:A \to \mathcal{B}(H)$ is a linear map. Assume: K_Φ is positive definite -- then $\Phi = \Delta_R\pi$, where π is a minimal *-representation of A on some complex Hilbert space K.

<u>PROOF</u> It is a question of applying the considerations in the Appendix to §9. Since Φ is continuous, $\forall\ \xi \in A$,

$$\left[\begin{array}{l} \Phi(\xi e_i) \to \Phi(\xi) \\ \\ \Phi(e_i^*\xi) \to \Phi(\xi) \end{array}\right. \qquad \text{in } \mathcal{B}(H) \text{ (norm convergence).}$$

On the other hand, if

$$\sup_{i\in I} ||e_i|| \leq C,$$

then

$$\begin{aligned} ||\Phi(e_i^*e_i)|| &\leq ||\Phi||\ ||e_i^*e_i|| \\ &\leq ||\Phi||\ ||e_i^*||\ ||e_i|| \\ &\leq ||\Phi||\ ||e_i||^2 \\ &\leq ||\Phi||C^2 \end{aligned}$$

=>

$$\gamma = \sup_{i\in I} ||\Phi(e_i^*e_i)|| < \infty.$$

I.e.: Condition C is satisfied. There remains the verification of the boundedness condition. Thus fix $x \in H$ -- then

$$\omega_x(\xi^*\mu^*\mu\xi)$$

$$= \langle x,\Phi(\xi^*\mu^*\mu\xi)x\rangle$$

$$\leq r(\mu^*\mu)\omega_x(\xi^*\xi)$$

$$= r(\mu^*\mu)\langle x,\Phi(\xi^*\xi)x\rangle.$$

So condition 2 of 9.16 is met if we let $\beta(\mu) = r(\mu^*\mu)$.

10.12 EXAMPLE (The GNS Construction) Specialize 10.11 to the case when Φ is a positive linear functional -- then K_Φ is positive definite (cf. 8.21) and $\exists\ a \in \mathcal{K}$:

$$\Phi(\xi) = \langle a,\pi(\xi)a\rangle \quad \text{(cf. 8.26)}.$$

For let $a = R(1)$ $(R:\underline{C} \to \mathcal{K})$:

$$\langle R(1),\pi(\xi)R(1)\rangle$$

$$= \langle 1,(R^*\pi(\xi)R)(1)\rangle$$

$$= \langle 1,\Phi(\xi)1\rangle$$

$$= \Phi(\xi).$$

[Note: π is minimal, hence $\pi(A)R\underline{C}$ is total in $\mathcal{K}$. But $\pi(A)R\underline{C} = \pi(A)R(1)$, which is linear, so $R(1)$ is π-cyclic.]

10.13 THEOREM (Stinespring) Let A be a C*-algebra and suppose that $\Phi:A \to \mathcal{B}(\mathcal{H})$ is a linear map. Assume: K_Φ is positive definite -- then $\Phi = \Delta_R\pi$, where π is a minimal *-representation of A on some complex Hilbert space $\mathcal{K}$.

[This is a special case of 10.11 (every C*-algebra admits an approximate unit).]

Let A be a U*-algebra and suppose that $\Phi:A \to \mathcal{B}(\mathcal{H})$ is a linear map. Assume: K_Φ

is positive definite and $\Phi(e) = I$ -- then 10.7 is in force and $RR^* \in \mathcal{B}(K)$ is an orthogonal projection.

10.14 LEMMA We have

$$\{\xi \in A : \Phi(\xi^*\xi) = \Phi(\xi)^*\Phi(\xi)\}$$

$$= \{\xi \in A : \Phi(\eta\xi) = \Phi(\eta)\Phi(\xi) \ \forall \ \eta \in A\}.$$

PROOF If $\Phi(\eta\xi) = \Phi(\eta)\Phi(\xi) \ \forall \ \eta \in A$, take $\eta = \xi^*$ to get $\Phi(\xi^*\xi) = \Phi(\xi^*)\Phi(\xi) = \Phi(\xi)^*\Phi(\xi)$ (cf. 9.7). To go the other way, assume that $\Phi(\xi^*\xi) = \Phi(\xi)^*\Phi(\xi)$, thus

$$R^*\pi(\xi^*\xi)R = (R^*\pi(\xi)R)^* \ (R^*\pi(\xi)R)$$

or still,

$$R^*\pi(\xi)^*\pi(\xi)R = R^*\pi(\xi)^*RR^*\pi(\xi)R.$$

Let

$$T = (I - RR^*)\pi(\xi)RR^*.$$

Then

$$T^*T = RR^*\pi(\xi)^*(I - RR^*)(I - RR^*)\pi(\xi)RR^*$$

$$= RR^*\pi(\xi)^*(I - RR^*)\pi(\xi)RR^*$$

$$= RR^*\pi(\xi)^*\pi(\xi)RR^* - RR^*\pi(\xi)^*RR^*\pi(\xi)RR^*$$

$$= R(R^*\pi(\xi)^*\pi(\xi)R - R^*\pi(\xi)^*RR^*\pi(\xi)R)R^*$$

$$= 0$$

$\Rightarrow$

$$T = 0$$

$\Rightarrow$

$$\pi(\xi)RR^* = RR^*\pi(\xi)RR^*.$$

So, $\forall\ \eta \in A$,

$$\begin{aligned} R\Phi(\eta\xi)R^* &= RR^*\pi(\eta\xi)RR^* \\ &= RR^*\pi(\eta)\pi(\xi)RR^* \\ &= RR^*\pi(\eta)RR^*\pi(\xi)RR^* \\ &= R\Phi(\eta)\Phi(\xi)R^* \end{aligned}$$

=>

$$R^*R\Phi(\eta\xi)R^*R = R^*R\Phi(\eta)\Phi(\xi)R^*R$$

=>

$$\Phi(\eta\xi) = \Phi(\eta)\Phi(\xi) \quad (R^*R = I).$$

[Note: If

$$\begin{cases} \Phi(\xi_1^*\xi_1) = \Phi(\xi_1)^*\Phi(\xi_1) \\ \Phi(\xi_2^*\xi_2) = \Phi(\xi_2)^*\Phi(\xi_2), \end{cases}$$

then $\forall\ \eta \in A$,

$$\begin{aligned} \Phi(\eta\xi_1\xi_2) &= \Phi(\eta\xi_1)\Phi(\xi_2) \\ &= \Phi(\eta)\Phi(\xi_1)\Phi(\xi_2) \\ &= \Phi(\eta)\Phi(\xi_1\xi_2). \end{aligned}$$

Therefore

$$\{\xi \in A : \Phi(\xi^*\xi) = \Phi(\xi)^*\Phi(\xi)\}$$

is a unital subalgebra of A on which Φ is multiplicative.]

10.15 <u>LEMMA</u> If $\Phi: A \to \mathcal{B}(H)$ sends the unitary elements of A to the unitary elements of A to the unitary elements of $B(H)$, then Φ is a *-homomorphism.

<u>PROOF</u> Given a unitary $\xi \in A$, we have

$$\Phi(\xi)^*\Phi(\xi) = I = \Phi(e) = \Phi(\xi^*\xi).$$

So, $\forall\ \eta \in A$,

$$\Phi(\eta\xi) = \Phi(\eta)\Phi(\xi) \quad \text{(cf. 10.14)}.$$

Therefore Φ is multiplicative on A. But Φ is also a *-map (cf. 9.7), hence Φ is a *-homomorphism.

Given a *-algebra A, define $\gamma:A \to [0,\infty]$ by

$$\gamma(\xi) = \sup_{\pi} ||\pi(\xi)||,$$

where π ranges over the *-representations of A on a complex Hilbert space — then A is said to be a <u>GN-algebra</u> if $\gamma(\xi)$ is finite for all $\xi \in A$.

10.16 <u>EXAMPLE</u> Every U*-algebra A is a GN-algebra. For if $\xi \in A$ is unitary, then

$$\begin{aligned} ||\pi(\xi)||^2 &= ||\pi(\xi)^*\pi(\xi)|| \\ &= ||\pi(\xi^*\xi)|| \\ &= ||\pi(e)|| \le 1. \end{aligned}$$

10.17 <u>EXAMPLE</u> Every Banach *-algebra A is a GN-algebra. In fact, $\forall\ \xi \in A$,

$$\gamma(\xi) \le r(\xi^*\xi)^{1/2} \le ||\xi^*\xi||^{1/2}.$$

10.18 <u>EXAMPLE</u> The *-algebra A of all complex polynomials $p:\underline{R} \to \underline{C}$ is not a

GN-algebra.

[Note: The multiplication is pointwise and the involution is complex conjugation.]

Suppose that A is a GN-algebra -- then $\gamma:A \rightarrow \underline{R}_{\geq 0}$ is a submultiplicative seminorm and

$$\gamma(\xi^*\xi) = \gamma(\xi)^2 \quad (\xi \in A).$$

Let

$$A_R = \bigcap_{\pi} \text{Ker } \pi \quad (\equiv \{\xi:\gamma(\xi) = 0\}).$$

Then A_R is a *-ideal in A and γ induces a C*-norm on the quotient A/A_R. Denote the completion of A/A_R by $C^*(A)$, the <u>enveloping C*-algebra</u> of A, and write p_A for the canonical *-homomorphism $A \rightarrow C^*(A)$.

10.19 <u>EXAMPLE</u> It can happen that $A = A_R$. Thus let A be the set of all complex $\underline{N}$-by-$\underline{N}$ matrices that have only finitely many nonzero entries in each row and each column -- then A has no nonzero *-representations on a complex Hilbert space.

[Note: The multiplication is matrix multiplication and the involution is conjugate transpose.]

10.20 <u>LEMMA</u> Suppose that A is a GN-algebra. If B is a C*-algebra and $\Phi:A \rightarrow B$ is a *-homomorphism, then there is a unique *-homomorphism $\bar{\Phi}:C^*(A) \rightarrow B$ such that $\Phi = \bar{\Phi} \circ p_A$.

10.21 <u>LEMMA</u> Suppose that A and A' are GN-algebras and $\phi:A \rightarrow A'$ is a

$*$-homomorphism -- then there is a unique $*$-homomorphism $C^*(\phi):C^*(A) \to C^*(A')$ rendering the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ p_A\ } & C^*(A) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle C^*(\phi)} \\ A' & \xrightarrow[\ p_{A'}\ ]{} & C^*(A') \end{array}$$

commutative.

Suppose still that A is a GN-algebra and let $\Phi:A \to \mathcal{B}(H)$ be a linear map. Assume: Φ is <u>γ-bounded</u>, i.e., $\exists$ M > 0 such that

$$||\Phi(\xi)|| \leq M\gamma(\xi) \qquad (\xi \in A).$$

Then Φ vanishes on A_R, hence drops to A/A_R, and from there extends by continuity to a linear map $\bar{\Phi}:C^*(A) \to \mathcal{B}(H)$ with the property that $\Phi = \bar{\Phi} \circ p_A$.

10.22 <u>LEMMA</u> K_Φ is positive definite iff $K_{\bar{\Phi}}$ is positive definite.

§11. *COMPLETELY POSITIVE MAPS*

Let A be a $*$-algebra.

- Write A_+ for the set of all finite sums of the form $\sum_{i=1}^{n} \xi_i^* \xi_i$ and call the elements of A_+ <u>positive</u>.

- Write A_{SA} for the set of all $\xi \in A$ with the property that $\xi^* = \xi$ and call the elements of A_{SA} <u>selfadjoint</u>.

11.1 <u>LEMMA</u> The linear span of A_+ is A^2.

<u>PROOF</u> In fact,

$$\xi^*\eta = \frac{1}{4} \sum_{k=0}^{3} (\sqrt{-1})^{-k} (\xi + (\sqrt{-1})^k \eta)^* (\xi + (\sqrt{-1})^k \eta).$$

11.2 <u>LEMMA</u> The linear span of A_{SA} is A.

<u>PROOF</u> In fact,

$$\xi = (\xi + \xi^*)/2 + \sqrt{-1}\ (\xi - \xi^*)/2\sqrt{-1}.$$

11.3 <u>LEMMA</u> We have

$$A_+ - A_- = A_{SA} \cap A^2.$$

<u>PROOF</u> The definitions imply that the LHS is contained in the RHS. On the other hand,

$$\sum_{i=1}^{n} \xi_i \eta_i \in A_{SA} \cap A^2$$

=>

$$\sum_{i=1}^{n} \xi_i \eta_i = (\sum_{i=1}^{n} \xi_i \eta_i)^*$$

$$= \frac{1}{2} \sum_{i=1}^{n} (\xi_i \eta_i + \eta_i^* \xi_i^*)$$

$$= \frac{1}{2} \sum_{i=1}^{n} ((\xi_i^* + \eta_i)^*(\xi_i^* + \eta_i) - \xi_i \xi_i^* - \eta_i^* \eta_i)$$

$$\in A_+ - A_-.$$

[Note: Therefore

$$A_+ - A_- = A_{SA} \iff A = A^2.]$$

Let A,B be $*$-algebras -- then a linear map $\Phi:A \to B$ is said to be <u>positive</u> if

$$\Phi(A_+) \subset B_+.$$

11.4 <u>LEMMA</u> Suppose that $\Phi:A \to B$ is a positive map -- then $\forall\ \xi,\eta \in A$,

$$\Phi(\xi^*\eta)^* = \Phi(\eta^*\xi).$$

<u>PROOF</u> $\forall\ c \in \underline{C}$,

$$0 \le \Phi((c\xi + \eta)^*(c\xi + \eta))$$

$$= |c|^2\Phi(\xi^*\xi) + \bar{c}\Phi(\xi^*\eta) + c\Phi(\eta^*\xi) + \Phi(\eta^*\eta).$$

Take $c = 1$ and $c = \sqrt{-1}$ to conclude that

$$\left[\begin{array}{l} \Phi(\xi^*\eta) + \Phi(\eta^*\xi) = \Phi(\xi^*\eta)^* + \Phi(\eta^*\xi)^* \\ \\ \Phi(\xi^*\eta) - \Phi(\eta^*\xi) = \Phi(\eta^*\xi)^* - \Phi(\xi^*\eta)^* \end{array}\right. \quad \in B^2$$

and then add these equations.

[Note: Therefore $A = A^2$

$$\Rightarrow$$

$$\Phi(\xi)^* = \Phi(\xi^*) \quad (\xi \in A).$$

In particular, this is the case when A is unital.]

Write $M_n(A)$ for the algebra of n-by-n matrices with entries from A made into a $*$-algebra by the specification

$$[\xi_{ij}]^* = [\xi_{ji}^*].$$

And ditto for $\mathcal{B}$.

[Note: If $M_n(A)$ is identified with $A \otimes M_n(\underline{C})$, then

$$(\xi \otimes A)^* = \xi^* \otimes A^*,$$

A^* the conjugate transpose.]

11.5 <u>EXAMPLE</u> Let

$$[A_{ij}] \in M_n(\mathcal{B}(H)).$$

Then

$$[A_{ij}] \in M_n(\mathcal{B}(H))_+$$

iff $\forall A_1,\ldots,A_n \in \mathcal{B}(H)$,

$$\sum_{i,j=1}^{n} A_i^* A_{ij} A_j \geq 0.$$

If now $\Phi: A \to \mathcal{B}$ is a linear map, then $\forall$ n, Φ gives rise to a linear map

$\Phi_n : M_n(A) \to M_n(B)$, viz.

$$\Phi_n([\xi_{ij}]) = [\Phi(\xi_{ij})].$$

Definition: Φ is <u>n-positive</u> if Φ_n is positive and Φ is <u>completely positive</u> if Φ_n is n-positive $\forall$ n.

[Note: Φ n-positive => Φ m-positive $(m \leq n)$.]

<u>N.B.</u> It is false that "positive" => "completely positive". E.g.: The arrow of transposition

$$\begin{bmatrix} M_2(\underline{C}) \to M_2(\underline{C}) \\ A \to A^T \end{bmatrix}$$

is positive but not 2-positive.

11.6 <u>EXAMPLE</u> A $*$-homomorphism $\Phi : A \to B$ is positive:

$$\Phi(\sum_{i=1}^{n} \xi_i^* \xi_i) = \sum_{i=1}^{n} \Phi(\xi_i^* \xi_i)$$

$$= \sum_{i=1}^{n} \Phi(\xi_i)^* \Phi(\xi_i) \in B_+.$$

Since Φ_n is also a $*$-homomorphism, it follows that Φ is completely positive.

11.7 <u>EXAMPLE</u> Suppose that H and K are complex Hilbert spaces and $\pi : A \to B(K)$ is a $*$-representation -- then $\forall$ $R \in B(H,K)$, $\Phi = \Delta_R \pi$ is completely positive.

11.8 <u>REMARK</u> An n-positive map $\Phi : M_n(\underline{C}) \to M_n(\underline{C})$ is necessarily completely positive. On the other hand, one can construct examples of k-positive maps

$\Phi:M_n(\underline{C}) \to M_n(\underline{C})$ which are not (k+1)-positive ($k = 1,\ldots,n-1$ $(n > 1)$).

Every n-tuple $\xi_1,\ldots,\xi_n$ of elements of A determines a positive element $\underline{\xi}$ of $M_n(A)$, viz.:

$$\underline{\xi} = \Xi^*\Xi = [\xi_i^*\xi_j],$$

where Ξ is the element of $M_n(A)$ whose k^{th} row is $\xi_1,\ldots,\xi_n$ and whose other entries are 0 (any k between 1 and n).

11.9 <u>LEMMA</u> Let $\underline{\xi} \in M_n(A)$ be positive -- then $\underline{\xi}$ is a finite sum of positive elements of the form $[\xi_i^*\xi_j]$.

<u>PROOF</u> By definition, $\underline{\xi}$ is a finite sum of elements of the form $\Xi^*\Xi$ ($\Xi \in M_n(A)$). Decompose Ξ as $\Xi_1 + \cdots + \Xi_n$, Ξ_k the k^{th} row of Ξ and 0 otherwise -- then

$$\Xi^*\Xi = \Xi_1^*\Xi_1 + \cdots + \Xi_n^*\Xi_n$$

and each term on the right is of the form $[\xi_i^*\xi_j]$.

11.10 <u>THEOREM</u> Suppose that $\Phi:A \to \mathcal{B}(H)$ is a linear map -- then Φ is completely positive iff K_Φ is positive definite.

<u>PROOF</u> If Φ is completely positive and if $\xi_1,\ldots,\xi_n \in A$, then

$$[\xi_i^*\xi_j] \in M_n(A)_+$$

$$\Rightarrow [\Phi(\xi_i^*\xi_j)] \in M_n(\mathcal{B}(H))_+ \approx \mathcal{B}(\overset{n}{\oplus} H)_+.$$

Therefore K_Φ is positive definite. Conversely, let $\underline{\xi} \in M_n(A)_+$ -- then in view of 11.9, to prove that $\Phi_n(\underline{\xi}) \in B(\overset{n}{\oplus} H)_+$, one can assume that $\underline{\xi}$ has the form $[\xi_i^*\xi_j]$, in which case matters are immediate.

Consequently, the requirement that a linear map $\Phi:A \rightarrow B(H)$ be completely positive is: $\forall\ n \in \underline{N}$ and for all

$$\begin{bmatrix} \xi_1,\ldots,\xi_n \in A \\ x_1,\ldots,x_n \in H, \end{bmatrix}$$

we have

$$\sum_{i,j=1}^{n} \langle x_i, \Phi(\xi_i^*\xi_j)x_j\rangle \geq 0.$$

Here is a variant.

11.11 <u>LEMMA</u> Suppose that $\Phi:A \rightarrow B(H)$ is a linear map. Assume: $\Phi(\xi)^* = \Phi(\xi^*)$ $(\xi \in A)$ -- then Φ is completely positive iff

$$\forall \begin{bmatrix} \xi_1,\ldots,\xi_n \\ & \in A \\ \eta_1,\ldots,\eta_n \end{bmatrix}$$

and $\forall\ x \in H$, we have

$$\sum_{i,j=1}^{n} \langle x, \Phi(\eta_i^*)\Phi(\xi_i^*\xi_j)\Phi(\eta_j)x\rangle \geq 0.$$

<u>PROOF</u> Let x run over a set $\{x_i \in H: i \in I\}$ such that $\{\Phi(A)x_i: i \in I\}$ is total

in $\mathcal{H}$.

11.12 RAPPEL Suppose that $\omega:\mathcal{B}(\mathcal{H}) \to \underline{C}$ is a bounded linear functional such that $||\omega|| = \omega(I) = 1$ — then ω is positive, hence is a state on $\mathcal{B}(\mathcal{H})$.

11.13 LEMMA Let $\Phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital linear map. Assume: Φ is bounded and $||\Phi|| \leq 1$ — then Φ is positive.

PROOF Given $x \in \underline{S}(\mathcal{H})$, define $\omega_x:\mathcal{B}(\mathcal{H}) \to \underline{C}$ by $\omega_x(A) = \langle x,\Phi(A)x\rangle$ — then ω_x is a linear functional of norm ≤ 1. But $\omega_x(I) = 1$, so

$$\omega_x \in S(\mathcal{B}(\mathcal{H})) \quad \text{(cf. 11.12)}.$$

Therefore

$$A \in \mathcal{B}(\mathcal{H})_+ \Rightarrow \omega_x(A) \geq 0.$$

Since this is true for all $x \in \underline{S}(\mathcal{H})$, it follows that

$$A \in \mathcal{B}(\mathcal{H})_+ \Rightarrow \Phi(A) \in \mathcal{B}(\mathcal{H})_+.$$

11.14 EXAMPLE Let $\Phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital linear map. Assume: Φ is bounded of norm ≤ 1, idempotent (i.e., $\Phi^2 = \Phi$) and

$$\Phi(XYZ) = X\Phi(Y)Z \quad (X,Z \in \text{Ran } \Phi \ \& \ Y \in \mathcal{B}(\mathcal{H})).$$

Then Φ is completely positive. To see this, note that Φ is at least positive (cf. 11.13), thus $\forall\ A \in \mathcal{B}(\mathcal{H})$, $\Phi(A)^* = \Phi(A^*)$ (cf. 11.4). Now let

$$\left[\begin{array}{l} A_1,\ldots,A_n \\ \\ B_1,\ldots,B_n \end{array}\right. \in \mathcal{B}(\mathcal{H}).$$

Then

$$\sum_{i,j=1}^{n} \Phi(B_i)^* A_i^* A_j \Phi(B_j)$$

$$= (\sum_{k=1}^{n} A_k \Phi(B_k))^* (\sum_{k=1}^{n} A_k \Phi(B_k)) \geq 0$$

=>

$$\sum_{i,j=1}^{n} \langle x, \Phi(B_i^*) \Phi(A_i^* A_j) \Phi(B_j) x \rangle$$

$$= \sum_{i,j=1}^{n} \langle x, \Phi(B_i)^* \Phi(A_i^* A_j) \Phi(B_j) x \rangle$$

$$= \sum_{i,j=1}^{n} \langle x, \Phi(\Phi(B_i)^* A_i^* A_j \Phi(B_j)) x \rangle$$

$$= \langle x, \Phi(\sum_{i,j=1}^{n} \Phi(B_i)^* A_i^* A_j \Phi(B_j)) x \rangle$$

$$\geq 0.$$

Therefore Φ is completely positive (cf. 11.11).

Let X be a LCH space, $C_\infty(X)$ the algebra of complex valued continuous functions on X that vanish at infinity. Equip $C_\infty(X)$ with the sup norm and let the involution be complex conjugation.

N.B. The dual $C_\infty(X)^*$ of $C_\infty(X)$ can be identified with M(X), the space of complex Radon measures on X:

$$\mu \to I_\mu, \; I_\mu(f) = \int_X f d\mu.$$

Here

$$||\mu|| = |\mu|(X),$$

$|\mu|$ the total variation of μ.

11.15 <u>THEOREM</u> If $\Phi:C_\infty(X) \to \mathcal{B}(\mathcal{H})$ is positive, then Φ is completely positive.

<u>PROOF</u> It suffices to prove that K_Φ is a positive definite kernel (cf. 11.10). So let

$$\begin{bmatrix} f_1,\ldots,f_n \in C_\infty(X) \\ x_1,\ldots,x_n \in \mathcal{H}. \end{bmatrix}$$

Then the claim is that

$$\sum_{i,j=1}^{n} \langle x_i, \Phi(\bar{f}_i f_j) x_j \rangle \geq 0.$$

To begin with, $\forall\ x \in \mathcal{H}$, the assignment

$$f \to \langle x, \Phi(f)x \rangle \quad (f \in C_\infty(X))$$

is a positive linear functional, hence is given by integration w.r.t. a Radon measure μ_x:

$$\langle x, \Phi(f)x \rangle = \int_X f d\mu_x.$$

Therefore

$$\sum_{i=1}^{n} \langle x_i, \Phi(f) x_i \rangle = \int_X f d\mu,$$

where

$$\mu = \sum_{i=1}^{n} \mu_{x_i}.$$

Because $\Phi(\bar{f}f) \in \mathcal{B}(\mathcal{H})_+$, we have

$$|<x_i,\Phi(\bar{f}f)x_j>|$$

$$\leq ||\Phi(\bar{f}f)^{1/2}x_i||\ ||\Phi(\bar{f}f)^{1/2}x_j||$$

$$= <x_i,\Phi(\bar{f}f)x_i>^{1/2}<x_j,\Phi(\bar{f}f)x_j>^{1/2}.$$

So, thanks to the Radon-Nikodym theorem, $\exists\ \phi_{ij} \in L^1(X,\mu)$ such that $\forall\ f \in C_\infty(X)$,

$$<x_i,\Phi(f)x_j> = \int_X f\phi_{ij}d\mu.$$

Next, for all $\lambda_1,\ldots,\lambda_n \in \underline{C}$:

$$\sum_{i,j=1}^{n} \bar{\lambda}_i\lambda_j<x_i,\Phi(\bar{f}f)x_j>$$

$$= <\sum_{i=1}^{n} \lambda_i x_i,\Phi(\bar{f}f)\sum_{j=1}^{n} \lambda_j x_j>$$

$$\geq 0.$$

Consequently,

$$\sum_{i,j=1}^{n} \bar{\lambda}_i\lambda_j\phi_{ij}(x) \geq 0 \quad [\mu\ \text{a.e.}].$$

But the matrix $[\bar{f}_i f_j]$ is positive definite at every $x \in X$, thus on the basis of 8.3,

$$\sum_{i,j=1}^{n} \bar{f}_i f_j\phi_{ij} \geq 0 \quad [\mu\ \text{a.e.}].$$

And this implies that

$$\sum_{i,j=1}^{n} <x_i,\Phi(\bar{f}_i f_j)x_j>$$

$$= \sum_{i,j=1}^{n} \int_X \bar{f}_i f_j \phi_{ij} d\mu$$

$$= \int_X \left(\sum_{i,j=1}^{n} \bar{f}_i f_j \phi_{ij} \right) d\mu$$

$$\geq 0.$$

11.16 <u>REMARK</u> If A is an arbitrary commutative C*-algebra, then $\exists$ a LCH space X and an isomorphism $A \to C_\infty(X)$. Therefore 11.15 can be restated: Every positive $\Phi: A \to \mathcal{B}(H)$ is necessarily completely positive.

11.17 <u>LEMMA</u> Let A be a U*-algebra and suppose that $\Phi: A \to \mathcal{B}(H)$ is a positive map -- then Φ is γ-bounded (hence lifts to a positive map $\bar{\Phi}: C^*(A) \to \mathcal{B}(H)$).

<u>PROOF</u> We claim that $\forall \xi \in A$,

$$||\Phi(\xi)|| \leq 2||\Phi(e)||\gamma(\xi).$$

Indeed

$$||\Phi(\xi)|| \leq 2 \sup_{x \in \underline{S}(H)} |\langle x, \Phi(\xi)x\rangle|.$$

Now fix $x \in \underline{S}(H)$ -- then

$$\omega_x(\xi) = \langle x, \Phi(\xi)x\rangle$$

is a positive linear functional, thus $\exists$ a unital *-representation of A on some complex Hilbert space K and an element $a \in K$ such that

$$\omega_x(\xi) = \langle a, \pi(\xi)a\rangle \quad \text{(cf. 8.26)}.$$

So

$$||a||^2 = \langle a, \pi(e)a\rangle$$

$$= \langle x, \Phi(e)x\rangle$$

$$\leq ||\Phi(e)||$$

=>

$$|<x,\Phi(\xi)x>| \leq ||a||^2||\pi(\xi)||$$

$$\leq ||\Phi(e)||\gamma(\xi).$$

11.18 LEMMA Let A be a Banach *-algebra with an approximate unit and suppose that $\Phi:A \to \mathcal{B}(H)$ is a positive map -- then Φ is γ-bounded (hence lifts to a positive map $\bar{\Phi}:C^*(A) \to \mathcal{B}(H)$).

PROOF As in 11.17, it suffices to estimate

$$|<x,\Phi(\xi)x>| \quad (\xi \in A, x \in \underset{\sim}{S}(H)).$$

To this end, note first that the discussion following 10.10 implies that

$$\Phi \in \mathcal{B}(A,\mathcal{B}(H)).$$

But $\forall$ $i \in I$,

$$|<x,\Phi(e_i\xi)x>|^2 \leq <x,\Phi(e_i^*e_i)x><x,\Phi(\xi^*\xi)x>.$$

Therefore

$$|<x,\Phi(\xi)x>|^2 \leq M<x,\Phi(\xi^*\xi)x>,$$

where

$$M = ||\Phi|| \sup_{i\in I} ||e_i^*e_i||.$$

So, if

$$\omega_x(\xi) = <x,\Phi(\xi)x>,$$

then

$$||\omega_x||_H \equiv \sup\{|\omega_x(\xi)|^2:\omega_x(\xi^*\xi) \leq 1\}$$

$$\leq M,$$

a bound which is independent of $x \in \underline{S}(H)$. Using 10.12, write

$$\omega_x(\xi) = \langle a, \pi(\xi) a \rangle.$$

Bearing in mind that $\pi(A)a$ is dense in K,

$$\begin{aligned}
||a||^2 &= \sup_{y \in \underline{S}(K)} |\langle a, y \rangle|^2 \\
&= \sup_{\pi(\xi) a \in \underline{S}(K)} |\langle a, \pi(\xi) a \rangle|^2 \\
&= \sup_{\omega_x(\xi^* \xi) \leq 1} |\omega_x(\xi)|^2 \\
&= ||\omega_x||_H \\
&\leq M
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
|\langle x, \Phi(\xi) x \rangle| &\leq ||a||^2 ||\pi(\xi)|| \\
&\leq M\gamma(\xi).
\end{aligned}$$

11.19 THEOREM Let A be a commutative U*-algebra or a commutative Banach *-algebra with an approximate unit and suppose that $\Phi: A \to B(H)$ is a positive map -- then Φ is completely positive.

PROOF It is clear that

$$A \text{ commutative} \Rightarrow C^*(A) \text{ commutative}.$$

But $\bar{\Phi}: C^*(A) \to B(H)$ is positive, hence is completely positive (cf. 11.16), so

$K_{\bar{\Phi}}$ is positive definite (cf. 11.10). Therefore K_Φ is positive definite (cf. 10.22), hence Φ is completely positive (cf. 11.10).

Let X be a LCH space -- then $\forall$ p $\in$ X, there is an arrow

$$ev_p : M_n(C_\infty(X)) \to M_n(\underline{C})$$

of evaluation.

11.20 <u>LEMMA</u> Let $F = [f_{ij}] \in M_n(C_\infty(X))$. Assume: $\forall$ p $\in$ X,

$$ev_p(F) \in M_n(\underline{C})_+.$$

Then

$$F \in M_n(C_\infty(X))_+.$$

11.21 <u>THEOREM</u> Let A be a *-algebra and suppose that $\Phi: A \to C_\infty(X)$ is positive -- then Φ is completely positive.

<u>PROOF</u> Let $\underline{\xi} \in M_n(A)_+$. To establish that

$$\Phi_n(\underline{\xi}) \in M_n(C_\infty(X))_+,$$

we can and will assume that $\underline{\xi} = [\xi_i^* \xi_j]$ (cf. 11.9) -- then $\forall$ p $\in$ X, the matrix

$$\Phi_n(\underline{\xi})(p) \in M_n(\underline{C})$$

is positive definite. Proof:

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \Phi_n(\underline{\xi})(p)_{ij}$$

$$= \sum_{i,j=1}^{n} \bar{c}_i c_j \Phi(\xi_i^* \xi_j)(p)$$

$$= \Phi(\sum_{i,j=1}^{n} \bar{c}_i c_j \xi_i^* \xi_j)(p)$$

$$= \Phi((\sum_{k=1}^{n} c_k \xi_k)^*(\sum_{k=1}^{n} c_k \xi_k))(p)$$

$$\geq 0.$$

Therefore

$$\Phi_n(\underline{\xi}) \in M_n(C_\infty(X))_+ \quad \text{(cf. 11.20)}.$$

[Note: One can, of course, replace $C_\infty(X)$ by an arbitrary commutative C*-algebra (cf. 11.16).]

11.22 REMARK It can be shown that if A and B are C*-algebras, then there is a positive but not 2-positive map $\Phi:A \to B$ unless A or B is commutative.

11.23 EXAMPLE A linear functional $\Phi:A \to \underline{C}$ is positive if $\forall\ \xi \in A$, $\Phi(\xi^*\xi) \geq 0$. Claim: Positive linear functionals are completely positive. To see this, in 11.21 take X to be a singleton -- then $C_\infty(X) = C(X) = \underline{C}$ ... .

[Note: One can also proceed from first principles: Use 8.21 and quote 11.10 (A was unital in the discussion preceding 8.21 but this plays no role in the argument).]

11.24 LEMMA Let A be a unital *-algebra and suppose that $\Phi:A \to B(H)$ is 2-positive -- then $\forall\ \xi \in A$,

$$\Phi(\xi)^*\Phi(\xi) \leq ||\Phi(e)||\Phi(\xi^*\xi).$$

PROOF $\forall\ \xi \in A$, we have

$$\Phi_2\left(\begin{bmatrix} 1 & \xi \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & \xi \\ 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \Phi(e) & \Phi(\xi) \\ \Phi(\xi)^* & \Phi(\xi^*\xi) \end{bmatrix} \in M_2(\mathcal{B}(\mathcal{H}))_+.$$

So, $\forall$ x,y $\in \mathcal{H}$,

$$\langle x,\Phi(e)x\rangle + \langle x,\Phi(\xi)y\rangle + \langle y,\Phi(\xi)^*x\rangle + \langle y,\Phi(\xi^*\xi)y\rangle$$

$$\geq 0.$$

If $\Phi(e) = 0$ but $\Phi(\xi) \neq 0$ ($\exists$ ξ), choose y subject to $\Phi(\xi)y \neq 0$ and then take x to be a large multiple of $- \Phi(\xi)y$, from which an obvious contradiction. Therefore

$$\Phi \neq 0 \Rightarrow \Phi(e) \neq 0.$$

Now let

$$x = - ||\Phi(e)||^{-1}\Phi(\xi)y,$$

multiply the above inequality by $||\Phi(e)||$, and then vary y over $\mathcal{H}$ to get

$$||\Phi(e)||^{-1}\Phi(\xi)^*\Phi(e)\Phi(\xi)$$

$$- 2\Phi(\xi)^*\Phi(\xi) + ||\Phi(e)||\Phi(\xi^*\xi) \geq 0.$$

It remains only to note that

$$\Phi(\xi)^*(I - ||\Phi(e)||^{-1}\Phi(e))\Phi(\xi) \in \mathcal{B}(\mathcal{H})_+.$$

11.25 <u>LEMMA</u> Let A be a unital Banach *-algebra and suppose that $\Phi:A \rightarrow \mathcal{B}(\mathcal{H})$

is positive. Assume: $\xi \in A$ is normal, i.e., $\xi\xi^* = \xi^*\xi$ -- then

$$\Phi(\xi)^*\Phi(\xi) \leq ||\Phi(e)||\Phi(\xi^*\xi).$$

PROOF Let $A(\xi)$ be the closed *-subalgebra of A generated by e and ξ -- then $A(\xi)$ is a unital commutative Banach *-algebra, hence the restriction of Φ to $A(\xi)$ is completely positive (cf. 11.19), in particular is 2-positive, so 11.24 is applicable.

11.26 RAPPEL The closed unit ball in $B(H)$ is the closed convex hull of $U(H)$.

[Note: In fact, every $A \in B(H)$ with $||A|| < 1$ is a convex combination of unitary operators.]

11.27 THEOREM Let H and K be complex Hilbert spaces and suppose that $\Phi:B(H) \to B(K)$ is a positive map -- then Φ is bounded and $||\Phi|| = ||\Phi(I)||$.

PROOF That Φ is bounded is standard. This said, let $U \in U(H)$:

$$\begin{aligned} ||\Phi(U)||^2 &= ||\Phi(U)^*\Phi(U)|| \\ &\leq ||\Phi(I)||\ ||\Phi(U^*U)|| \quad \text{(cf. 11.25)} \\ &= ||\Phi(I)||^2. \end{aligned}$$

The continuity of Φ and 11.26 then imply that $||\Phi|| = ||\Phi(I)||$.

In other words: "Positive maps $\Phi:B(H) \to B(K)$ attain their norm at the identity operator".

11.28 EXAMPLE Is it true that if $||\Phi|| = ||\Phi(I)||$ and $\Phi(I) \geq 0$, then Φ is positive? The answer is "no". E.g.: Consider the linear map $\Phi : M_2(\underline{C}) \to M_2(\underline{C})$ defined by

$$\Phi\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}.$$

§12. *OPERATOR SYSTEMS*

Let $\mathcal{H}$ be a complex Hilbert space -- then an <u>operator system</u> is a linear subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ such that $I \in \mathcal{S}$ and $\mathcal{S} = \mathcal{S}^*$.

12.1 <u>REMARK</u> In view of the Gelfand-Naimark theorem, every unital C*-algebra A "is" an operator system.

Given an operator system $\mathcal{S}$, put

$$\begin{cases} \mathcal{S}_+ = \mathcal{S} \cap \mathcal{B}(\mathcal{H})_+ \\ \mathcal{S}_{SA} = \mathcal{S} \cap \mathcal{B}(\mathcal{H})_{SA}. \end{cases}$$

Let $A \in \mathcal{S}_{SA}$ -- then

$$A = A^+ - A^-$$

but it need not be true that $A^+, A^- \in \mathcal{S}$.

12.2 <u>EXAMPLE</u> Take $\mathcal{H} = \underline{C}^3$ and identify $\mathcal{B}(\mathcal{H})$ with $M_3(\underline{C})$. Let

$$\mathcal{S} = \{A : a_{11} = a_{22} = a_{33}\}.$$

Then $\mathcal{S}$ is an operator system and

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathcal{S}.$$

But

$$A^+ = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A^- = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

do not belong to S.

Nevertheless, it is still possible to express $A \in S_{SA}$ as the difference of two elements in S_+:

$$A = \frac{1}{2}(||A||I + A) - \frac{1}{2}(||A||I - A).$$

Since the real and imaginary parts of each element of S also belong to S, it follows that S is the linear span of S_+.

12.3 <u>LEMMA</u> $\forall\, A_1, A_2 \in S_+$,

$$||A_1 - A_2|| \leq \max\{||A_1||, ||A_2||\}.$$

The notion of positive linear functional on S is clear and the standard facts obtain.

12.4 <u>LEMMA</u> If $\omega: S \to \underline{C}$ is a positive linear functional, then ω is bounded and $||\omega|| = \omega(I)$.

12.5 <u>LEMMA</u> If $\omega: S \to \underline{C}$ is a bounded linear functional and if $||\omega|| = \omega(I)$, then ω is positive.

12.6 <u>THEOREM</u> (Krein) Every positive linear functional $\omega:S \to \underline{C}$ can be extended to a positive linear functional $\bar{\omega}:B(H) \to \underline{C}$.

<u>PROOF</u> Thanks to 12.4, ω is bounded and $||\omega|| = \omega(I)$. Now use the Hahn-Banach theorem and extend ω to a bounded linear functional $\bar{\omega}:B(H) \to \underline{C}$ with $||\bar{\omega}|| = ||\omega||$. That $\bar{\omega}$ is positive is implied by 12.5.

If

$$\begin{cases} S \subset B(H) \\ T \subset B(K) \end{cases}$$

are operator systems, then a linear map $\Phi:S \to T$ is said to be <u>positive</u> if

$$\Phi(S_+) \subset T_+.$$

<u>N.B.</u> Φ is <u>$*$-linear</u>:

$$\Phi(A)^* = \Phi(A^*) \qquad (A \in S).$$

12.7 <u>EXAMPLE</u> In general, a positive map $\Phi:S \to B(K)$ can not be extended to a positive map $\bar{\Phi}:B(H) \to B(K)$. To illustrate this, take $H = \underline{C}^4$, let

$$X = \mathrm{diag}(1,\sqrt{-1},-1,-\sqrt{-1}),$$

and denote by $S \subset M_4(\underline{C})$ the operator system spanned by I,X,X^* — then

$$aI + bX + cX^*$$

is selfadjoint iff

$$\begin{cases} c = \bar{b} \\ a = \bar{a} \end{cases}$$

and is positive iff in addition

$$a \geq 2 \max (|\mathrm{Re}\, b|, |\mathrm{Im}\, b|).$$

Consider now the linear map $\Phi: S \rightarrow M_2(\underset{\sim}{C})$ that sends

$$aI + bX + cX^* \text{ to } \begin{bmatrix} a & \sqrt{2}\, b \\ \sqrt{2}\, c & a \end{bmatrix}.$$

Then Φ is positive and

$$\begin{cases} ||\Phi(I)|| = 1 \\ ||\Phi(X)|| = \sqrt{2}. \end{cases}$$

Since $||X|| = 1$, this implies that

$$||\Phi|| \geq \sqrt{2} > ||\Phi(I)||$$

But if there were a positive extension $\bar{\Phi}: M_4(\underset{\sim}{C}) \rightarrow M_2(\underset{\sim}{C})$, then

$$||\bar{\Phi}|| = ||\bar{\Phi}(I)|| \quad \text{(cf. 11.27)},$$

an impossibility.

12.8 <u>LEMMA</u> Suppose that $\Phi: S \rightarrow T$ is positive -- then Φ is bounded and

$$||\Phi|| \leq 2\, ||\Phi(I)||.$$

<u>PROOF</u> There are three steps.

1. If $A \in S_+$, then

$$0 \leq A \leq ||A||I \Rightarrow 0 \leq \Phi(A) \leq ||A||\Phi(I)$$

$\Rightarrow$

$$||\Phi(A)|| \leq ||A|| \ ||\Phi(I)||.$$

2. If $A \in S_{SA}$, then

$$||\Phi(A)|| = ||\Phi(\frac{1}{2}(||A||I + A) - \frac{1}{2}(||A||I - A))||$$

$$\leq \frac{1}{2}\max\{||\Phi(||A||I + A)||, ||\Phi(||A||I - A)||\} \quad \text{(cf. 12.3)}$$

$$\leq ||A|| \ ||\Phi(I)||.$$

3. If $A \in S$, then

$$A = \text{Re } A + \sqrt{-1} \text{ Im } A$$

$\Rightarrow$

$$||\Phi(A)|| \leq ||\text{Re } A|| + ||\text{Im } A||$$

$$\leq 2||A|| \ ||\Phi(I)||.$$

12.9 <u>REMARK</u> The constant 2 in 12.8 is sharp. However, under certain circumstances, it can be reduced. For instance, if X is a compact Hausdorff space and if $\Phi:C(X) \to \mathcal{B}(H)$ is positive, then Φ is completely positive (cf. 11.15) and $||\Phi|| = ||\Phi(I)||$ (cf. 10.8).

12.10 <u>LEMMA</u> Suppose that $\Phi:S \to \mathcal{B}(K)$ is positive -- then Φ extends to a positive map on the norm closure of S.

Given an operator system S, it makes sense to form $M_n(S)$, an operator system per

$$M_n(\mathcal{B}(\mathcal{H})) \approx \mathcal{B}(\overset{n}{\oplus} \mathcal{H})$$

and we shall put

$$\begin{cases} M_n(\mathcal{S})_+ = M_n(\mathcal{S}) \cap M_n(\mathcal{B}(\mathcal{H}))_+ \\ M_n(\mathcal{S})_{SA} = M_n(\mathcal{S}) \cap M_n(\mathcal{B}(\mathcal{H}))_{SA}. \end{cases}$$

If $\mathcal{S},\mathcal{T}$ are operator systems, then a linear map $\Phi:\mathcal{S} \rightarrow \mathcal{T}$ induces linear maps

$$\Phi_n:M_n(\mathcal{S}) \rightarrow M_n(\mathcal{T}) \quad (n = 1,2,\ldots)$$

and Φ is termed <u>completely positive</u> if $\forall$ n, Φ_n is positive.

12.11 <u>EXAMPLE</u> Given

$$A = [a_{ij}], \ B = [b_{ij}] \in M_n(\underline{C}),$$

their <u>Schur product</u> $A*B$ is $[a_{ij}b_{ij}]$ (the entrywise product of A and B (cf. 8.3)). Now fix A and define a linear map $S_A:M_n(\underline{C}) \rightarrow M_n(\underline{C})$ by $S_A(B) = A*B$ -- then the following are equivalent:

1. A is positive definite; 2. S_A is positive; 3. S_A is completely positive.

12.12 <u>LEMMA</u> Fix an operator system $\mathcal{S}$ and let X be a compact Hausdorff space. Suppose that $\Phi:C(X) \rightarrow \mathcal{S}$ is positive -- then Φ is completely positive (cf. 11.15).

12.13 <u>LEMMA</u> Fix an operator system $\mathcal{S}$ and let X be a compact Hausdorff space. Suppose that $\Phi:\mathcal{S} \rightarrow C(X)$ is positive -- then Φ is completely positive (cf. 11.21).

[Note: Specialized to the case when X is a point, the conclusion is that every positive linear functional $\omega:S \to \underline{C}$ is completely positive.]

N.B. We have

$$||\Phi|| = ||\Phi_1|| \leq ||\Phi_2|| \leq \ldots \leq ||\Phi_n|| \leq \ldots .$$

Set

$$||\Phi||_{cb} = \sup \{||\Phi_n||:n \in \underline{N}\}.$$

Then Φ is said to be completely bounded if $||\Phi||_{cb} < \infty$ and completely contractive if $||\Phi||_{cb} \leq 1$.

12.14 EXAMPLE The arrow of transposition

$$\begin{cases} M_n(\underline{C}) \to M_n(\underline{C}) \\ A \to A^T \end{cases}$$

is completely bounded and, in fact, $||T||_{cb} = n$.

Suppose that $\Phi:S \to T$ is completely positive -- then Φ is completely bounded.

Proof:

$$||\Phi_n|| \leq 2||\Phi_n(I)|| \quad \text{(cf. 12.8)}$$

$$= 2||\Phi(I)||$$

=>

$$||\Phi||_{cb} \leq 2||\Phi(I)||.$$

With more work this can be sharpened to:

$$||\Phi||_{cb} = ||\Phi|| = ||\Phi(I)||.$$

<u>Matrix Trick</u> $\forall\ A \in \mathcal{B}(H)$,

$$||A|| \leq 1 \iff \begin{bmatrix} I & A \\ A^* & I \end{bmatrix} \in M_2(\mathcal{B}(H))_+.$$

12.15 <u>THEOREM</u> If S,T are operator systems and if $\Phi:S \to T$ is completely positive, then Φ is completely bounded with

$$||\Phi||_{cb} = ||\Phi|| = ||\Phi(I)||.$$

<u>PROOF</u> The assertion is trivial if $\Phi(I) = 0$ (for then $\Phi = 0$ (cf. 12.8)), so assume that $\Phi(I) \neq 0$. Noting that

$$\begin{cases} M_2(M_n(S)) \approx M_{2n}(S) \\ M_2(M_n(T)) \approx M_{2n}(T), \end{cases}$$

take a contraction $A \in M_n(S)$:

$$||A|| \leq 1 \Rightarrow \begin{bmatrix} I & A \\ A^* & I \end{bmatrix} \geq 0.$$

Then

$$\Phi_{2n}\left(\begin{bmatrix} I & A \\ A^* & I \end{bmatrix}\right)$$

$$= \begin{bmatrix} \Phi_n(I) & \Phi_n(A) \\ \Phi_n(A)^* & \Phi_n(I) \end{bmatrix} \geq 0.$$

But

$$0 \leq \Phi_n(I) \leq ||\Phi_n(I)||I$$

$$= ||\Phi(I)||I.$$

Therefore

$$\begin{bmatrix} ||\Phi(I)||I & \Phi_n(A) \\ \Phi_n(A)^* & ||\Phi(I)||I \end{bmatrix}$$

$$= \begin{bmatrix} ||\Phi(I)||I - \Phi_n(I) & 0 \\ 0 & ||\Phi(I)||I - \Phi_n(I) \end{bmatrix}$$

$$+ \begin{bmatrix} \Phi_n(I) & \Phi_n(A) \\ \Phi_n(A)^* & \Phi_n(I) \end{bmatrix}$$

≥ 0

$$\Rightarrow \begin{bmatrix} I & \Phi_n(A)/||\Phi(I)|| \\ \Phi_n(A)^*/||\Phi(I)|| & I \end{bmatrix} \geq 0$$

$\Rightarrow$

$$||\Phi_n(A)/||\Phi(I)|| \ || \leq 1$$

$\Rightarrow$

$$||\Phi_n(A)|| \leq ||\Phi(I)||$$

$\Rightarrow$

$$||\Phi_n|| \leq ||\Phi(I)||$$

$\Rightarrow$

$$||\Phi||_{cb} \leq ||\Phi(I)||.$$

On the other hand, trivially

$$||\Phi(I)|| \leq ||\Phi|| \leq ||\Phi||_{cb}.$$

[Note: Just what space "I" operates on is to be inferred from context.]

12.16 RAPPEL Let $A \in \mathcal{B}(\mathcal{H})$ -- then $-I \leq A \leq I$ iff $\forall\ t \in \underline{R}$,

$$||A - \sqrt{-1}\ tI|| \leq (1 + t^2)^{1/2}.$$

12.17 LEMMA If S,T are operator systems and if $\Phi:S \rightarrow T$ is a unital linear map, then Φ is completely positive iff Φ is completely contractive.

PROOF One direction is immediate:

$$\Phi(I) = I \Rightarrow ||\Phi(I)|| = 1$$

$$\Rightarrow ||\Phi||_{cb} = 1 \quad \text{(cf. 12.15)}.$$

As for the converse, $\forall\ A \in S: -I \leq A \leq I$, we have

$$\begin{aligned} ||\Phi(A) - \sqrt{-1}\, tI|| &= ||\Phi(A) - \sqrt{-1}\, t\Phi(I)|| \\ &= ||\Phi(A - \sqrt{-1}\, tI)|| \\ &\leq ||\Phi||\ ||A - \sqrt{-1}\, tI|| \\ &\leq ||A - \sqrt{-1}\, tI|| \\ &\leq (1 + t^2)^{1/2} \end{aligned}$$

$\Rightarrow$

$$-I \leq \Phi(A) \leq I.$$

Given $E \in S: 0 \leq E \leq I$, let $A = 2E - I$ — then $-I \leq A \leq I$, so

$$-I \leq \Phi(A) \leq I$$

or still,

$$0 \leq \Phi(E) \leq I.$$

This proves that Φ is positive. The same argument also works for Φ_n. Therefore Φ is completely positive.

Given an operator system S, write $L(S, M_n(\underline{C}))$ for the vector space of linear maps from S to $M_n(\underline{C})$ and write $L(M_n(S), \underline{C})$ for the vector space of linear maps from $M_n(S)$ to $\underline{C}$.

- There is an arrow

$$L(S, M_n(\underline{C})) \to L(M_n(S), \underline{C}),$$

viz. $\Phi \to \Lambda_\Phi$, where

$$\Lambda_\Phi([A_{ij}]) = \frac{1}{n} \sum_{i,j=1}^{n} \langle e_i, \Phi(A_{ij}) e_j \rangle.$$

• There is an arrow

$$L(M_n(S),\underline{C}) \to L(S,M_n(\underline{C})),$$

viz. $\Lambda \to \Phi_\Lambda$, where

$$\Phi_\Lambda(A)_{ij} = n\Lambda(E_{ij} \otimes A).$$

[Note: Here, $e_1,\ldots,e_n$ is the canonical basis for $\underline{C}^n$ and E_{ij}, $1 \le i, j \le n$, are the matrix units in $M_n(\underline{C})$, thus $E_{ij} \otimes A$ is the element of $M_n(S)$ whose ij^{th} entry is A and whose other entries are 0.]

N.B. We have

$$\begin{cases} \Phi \to \Lambda_\Phi \to \Phi_{\Lambda_\Phi} = \Phi \\ \Lambda \to \Phi_\Lambda \to \Lambda_{\Phi_\Lambda} = \Lambda. \end{cases}$$

12.18 LEMMA Let $\Phi \in L(S,M_n(\underline{C}))$. Assume: Φ_n is positive -- then Λ_Φ is positive.

PROOF Let

$$X \in \underline{C}^{n^2} = \underline{C} \oplus \cdots \oplus \underline{C}$$

be the vector

$$e_1 \oplus \cdots \oplus e_n.$$

Then

$$\Lambda_\Phi([A_{ij}]) = \frac{1}{n} \langle X,\Phi_n([A_{ij}])X\rangle.$$

12.19 LEMMA Let $\Phi \in L(S,M_n(\underline{C}))$. Assume: Λ_Φ is positive -- then Φ is completely positive.

PROOF Put $\omega = \Lambda_\Phi$ and use 12.6 to extend ω to a positive linear functional $\bar{\omega}:M_n(\mathcal{B}(H)) \to \underline{C}$ — then the map

$$\Phi_{\bar{\omega}}:\mathcal{B}(H) \to M_n(\underline{C})$$

extends Φ, thus matters are reduced to proving that $\Psi \equiv \Phi_{\bar{\omega}}$ is completely positive or still, that Ψ is m-positive ($m = 1,2,\ldots$). So let $A \in M_m(\mathcal{B}(H))_+$, say

$$A = [A_i^* A_j] \quad \text{(cf. 11.9)}.$$

Since $\Psi_m([A_i^*A_j])$ operates on $\underline{C}^{mn}$, to check positivity, it suffices to work with

$$x = x_1 \oplus \cdots \oplus x_m \in \underline{C}^{mn},\ x_i \in \underline{C}^n,\ x_i = \sum_{p=1}^{n} c_{ip}e_p.$$

This said, write

$$\begin{aligned}
&<x,\Psi_m([A_i^*A_j])x> \\
&= \sum_{i,j=1}^{m} <x_i,\Psi(A_i^*A_j)x_j> \\
&= \sum_{i,j=1}^{m}\ \sum_{p,q=1}^{n} \bar{c}_{ip}c_{jq}<e_p,\Psi(A_i^*A_j)e_q> \\
&= \sum_{i,j=1}^{m}\ \sum_{p,q=1}^{n} \bar{c}_{ip}c_{jq}n\bar{\omega}(E_{pq} \otimes A_i^*A_j).
\end{aligned}$$

Put

$$C_i = \begin{bmatrix} c_{i1} & c_{i2} & \cdots & c_{in} \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (1 \le i \le m).$$

Then

$$C_i^* C_j = \sum_{p,q=1}^{n} \bar{c}_{ip} c_{jq} E_{pq}.$$

Therefore

$$\begin{aligned} &\langle x, \Psi_m([A_i^* A_j])x\rangle \\ &= n \sum_{i,j=1}^{m} \bar{\omega}(C_i^* C_j \otimes A_i^* A_j) \\ &= n\bar{\omega}((\sum_{k=1}^{m} C_k \otimes A_k)^*(\sum_{k=1}^{m} C_k \otimes A_k)) \\ &\ge 0. \end{aligned}$$

12.20 <u>REMARK</u> Suppose that $\Phi:S \to M_n(\underline{C})$ is completely positive — then Φ_n is positive, hence Λ_Φ is positive (cf. 12.18), and the proof of 12.19 produces a completely positive extension $\Psi:B(H) \to M_n(\underline{C})$ of Φ.

[Note: We have

$$||\Psi|| = ||\Psi(I)|| = ||\Phi(I)|| = ||\Phi|| \quad \text{(cf. 12.15)}.$$

Let X be a normed linear space and let K be a complex Hilbert space. In

$\mathcal{B}(X,\mathcal{B}(K))^*$ (the dual of $\mathcal{B}(X,\mathcal{B}(K))$), write $\mathcal{B}(X,\mathcal{B}(K))_*$ for the closed linear span of the linear functionals $\Lambda_{x,\lambda}$, where $x \in X$, $\lambda \in \mathcal{B}(K)_*$ (cf. 2.3), and

$$\Lambda_{x,\lambda}(A) = \lambda(Ax) \quad (A \in \mathcal{B}(X,\mathcal{B}(K))).$$

Then $\mathcal{B}(X,\mathcal{B}(K))$ is isometrically isomorphic to the dual of $\mathcal{B}(X,\mathcal{B}(K))_*$, hence can be equipped with the weak* topology. Taking into account Alaoglu's theorem, it follows that a bounded net $\{A_i : i \in I\}$ of bounded linear maps $A_i : X \to \mathcal{B}(K)$ has a convergent subnet.

<u>N.B.</u> A bounded net $\{A_i : i \in I\}$ in $\mathcal{B}(X,\mathcal{B}(K))$ converges to A weak* iff $\forall\ T \in \underline{L}_1(K)$,

$$\operatorname{tr}((A_i x)T) \to \operatorname{tr}((Ax)T) \quad (x \in X)$$

or still, $\forall\ u,v \in K$,

$$\operatorname{tr}((A_i x)P_{u,v}) \to \operatorname{tr}((Ax)P_{u,v}) \quad \text{(cf. 1.10)},$$

i.e.,

$$\operatorname{tr}(P_{(A_i x)u,v}) \to \operatorname{tr}(P_{(Ax)u,v}),$$

i.e.,

$$\langle v, (A_i x)u\rangle \to \langle v, (Ax)u\rangle.$$

Let $X = \mathcal{B}(H)$ and equip $\mathcal{B}(\mathcal{B}(H),\mathcal{B}(K))$ with the weak* topology. Denote by

$$CP_r(\mathcal{B}(H),\mathcal{B}(K))$$

the set of completely positive maps $\mathcal{B}(H) \to \mathcal{B}(K)$ of norm $\le r$.

12.21 <u>LEMMA</u> $CP_r(\mathcal{B}(H),\mathcal{B}(K))$ is compact in the (relative) weak* topology.

12.22 <u>THEOREM</u> (Arveson) Let $S \subset \mathcal{B}(H)$ be an operator system, K a complex

Hilbert space. Suppose that $\Phi:S \rightarrow \mathcal{B}(K)$ is completely positive -- then Φ admits a completely positive extension $\Psi:\mathcal{B}(H) \rightarrow \mathcal{B}(K)$.

<u>PROOF</u> Given a finite dimensional subspace $F \subset K$, put

$$\Phi_F(A) = P_F\Phi(A)|F \quad (A \in S),$$

where P_F is the orthogonal projection per F -- then $\Phi_F:S \rightarrow \mathcal{B}(F)$ is completely positive. But $\mathcal{B}(F) \approx M_n(\underline{C})$ ($n = \text{rank } P_F$), hence $\exists$ a completely positive extension $\Psi_F:\mathcal{B}(H) \rightarrow \mathcal{B}(F)$ of Φ_F (cf. 12.20). Pass now from Ψ_F to $\overline{\Psi}_F = \Psi_F P_F$, so

$$\overline{\Psi}_F:\mathcal{B}(H) \rightarrow \mathcal{B}(K)$$

and $\overline{\Psi}_F$ is again completely positive. Order the F by inclusion to get a net $\{\overline{\Psi}_F\}$ in $CP_{||\Phi||}(\mathcal{B}(H),\mathcal{B}(K))$:

$$\begin{aligned} ||\overline{\Psi}_F|| &= ||\overline{\Psi}_F(I)|| \quad \text{(cf. 12.15)} \\ &= ||\Psi_F(I)P_F|| \\ &\leq ||\Psi_F(I)|| \\ &= ||\Phi_F(I)|| \\ &\leq ||\Phi(I)|| \\ &= ||\Phi|| \quad \text{(cf. 12.15)}. \end{aligned}$$

Using 12.21, choose a subnet of $\{\overline{\Psi}_F\}$ that converges to some element $\Psi \in CP_{||\Phi||}(\mathcal{B}(H),\mathcal{B}(K))$ -- then the claim is that

$$\Psi|S = \Phi.$$

To see this, fix $A \in S$, take $u,v \in K$, let $F_{u,v} = \text{span}\{u,v\}$, and consider any $F \supset F_{u,v}$:

$$\begin{aligned}\langle v,\Phi(A)u\rangle &= \langle v,P_F\Phi(A)u\rangle \\ &= \langle v,\Phi_F(A)u\rangle \\ &= \langle v,\Psi_F(A)u\rangle \\ &= \langle v,\Psi_F(A)P_F u\rangle \\ &= \langle v,\bar{\Psi}_F(A)u\rangle.\end{aligned}$$

Let $I \subset B(H)$ be an operator system -- then I is said to be injective if given operator systems $T \subset T' \subset B(K)$, every completely positive $\Phi:T \to I$ has a completely positive extension $\Phi':T' \to I$.

12.23 EXAMPLE $B(H)$ is injective. For, according to 12.22, a completely positive $\Phi:T \to B(H)$ admits a completely positive extension $\Psi:B(K) \to B(H)$, thus one can take $\Phi' = \Psi|T'$.

12.24 REMARK Every unital C*-algebra A can be regarded as an operator system (cf. 12.1), so it makes sense to ask whether A is injective or not.

12.25 LEMMA I is injective iff $\exists$ a completely positive projection $\Pi:B(H) \to I$ of $B(H)$ onto I.

PROOF Suppose first that I is injective. In the above, take $K = H$ and let

$$\begin{cases} \mathcal{T} = \mathcal{I} \\ \mathcal{T}' = \mathcal{B}(\mathcal{H}). \end{cases}$$

Then the identity map $\mathcal{I} \to \mathcal{I}$ has a completely positive extension $\Pi:\mathcal{B}(\mathcal{H}) \to \mathcal{I}$ which is obviously idempotent. Turning to the converse, assume that $\mathcal{I}$ has the stated property, consider operator systems $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{B}(\mathcal{K})$, and let $\Phi:\mathcal{T} \to \mathcal{I}$ be completely positive. Postcompose Φ with the inclusion $\iota:\mathcal{I} \to \mathcal{B}(\mathcal{H})$ to get a completely positive map $\iota \circ \Phi:\mathcal{T} \to \mathcal{B}(\mathcal{H})$ -- then 12.22 provides us with a completely positive extension $\Psi:\mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ and $\Phi' = \Pi \circ (\Psi|\mathcal{T}'):\mathcal{T}' \to \mathcal{I}$ is a completely positive extension of $\Phi:\forall\ B \in \mathcal{T}$,

$$\Phi'(B) = \Pi(\Psi(B)) = \Pi(\iota \circ \Phi(B)) = \Phi(B).$$

[Note: Schematically,

$$\begin{array}{ccc} \mathcal{T} \subset \mathcal{T}' & \subset & \mathcal{B}(\mathcal{K}) \\ \Phi \downarrow & & \downarrow \Psi \\ \mathcal{I} & \underset{\Pi}{\overset{\iota}{\rightleftarrows}} & \mathcal{B}(\mathcal{H}) \end{array} \quad .]$$

Take $\mathcal{I}$ injective and Π per 12.25.

<u>N.B.</u> Π is unital $\Rightarrow ||\Pi|| = ||\Pi(I)|| = 1$ (cf. 12.15).

12.26 <u>LEMMA</u> $\forall$ A,B $\in \mathcal{B}(\mathcal{H})$,

$$\Pi(\Pi(A)B) = \Pi(\Pi(A)\Pi(B)) = \Pi(A\Pi(B)).$$

<u>PROOF</u> By linearity, it can be assumed that A and B are selfadjoint -- then

$$T = \begin{bmatrix} 0 & \Pi(A) \\ \Pi(A) & B \end{bmatrix} \text{ and } \Pi_2(T) = \begin{bmatrix} 0 & \Pi(A) \\ \Pi(A) & \Pi(B) \end{bmatrix}$$

are selfadjoint elements of $M_2(\mathcal{B}(H))$, hence

$$\Pi_2(T)^2 \le \Pi_2(T^2) \quad \text{(cf. 11.24)}.$$

Therefore

$$\begin{bmatrix} \Pi(A)^2 & \Pi(A)\Pi(B) \\ \Pi(B)\Pi(A) & \Pi(A)^2 + \Pi(B)^2 \end{bmatrix}$$

$$\le \begin{bmatrix} \Pi(\Pi(A)^2) & \Pi(\Pi(A)B) \\ \Pi(B\Pi(A)) & \Pi(\Pi(A)^2 + B^2) \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} \Pi(\Pi(A)^2) & \Pi(\Pi(A)\Pi(B)) \\ \Pi(\Pi(B)\Pi(A)) & \Pi(\Pi(A)^2 + \Pi(B)^2) \end{bmatrix}$$

$$\le \begin{bmatrix} \Pi(\Pi(A)^2) & \Pi(\Pi(A)B) \\ \Pi(B\Pi(A)) & \Pi(\Pi(A)^2 + B^2) \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 0 & \Pi(\Pi(A)B) - \Pi(\Pi(A)\Pi(B)) \\ \Pi(B\Pi(A)) - \Pi(\Pi(B)\Pi(A)) & \text{——} \end{bmatrix}$$

$$\ge 0.$$

But

$$(\Pi(\Pi(A)B) - \Pi(\Pi(A)\Pi(B))^*$$

$$= \Pi((\Pi(A)B)^*) - \Pi((\Pi(A)\Pi(B))^*)$$

$$= \Pi(B^*\Pi(A^*)) - \Pi(\Pi(B^*)\Pi(A^*))$$

$$= \Pi(B\Pi(A)) - \Pi(\Pi(B)\Pi(A)).$$

Thus we have

$$\begin{bmatrix} 0 & X \\ X^* & \text{——} \end{bmatrix} \geq 0,$$

where

$$X = \Pi(\Pi(A)B) - \Pi(\Pi(A)\Pi(B)).$$

Fix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix} \in M_2(\mathcal{B}(H))_{SA}$$

with the property that

$$\begin{bmatrix} 0 & X \\ X^* & \text{——} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix}^2 .$$

Then

$$a_{11} = a_{12} = 0 \Rightarrow X = 0.$$

I.e.:

$$\Pi(\Pi(A)B) = \Pi(\Pi(A)\Pi(B)).$$

Finally

$$\Pi(A\Pi(B))^* = \Pi(\Pi(B^*)A^*)$$

$$= \Pi(\Pi(B)A)$$

$$= \Pi(\Pi(B)\Pi(A))$$

=>

$$\Pi(A\Pi(B)) = \Pi(\Pi(B)\Pi(A))^*$$

$$= \Pi(\Pi(A)\Pi(B)).$$

12.27 <u>THEOREM</u> Suppose that $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$ is an injective operator system -- then the assignment

$$\begin{cases} \mathcal{I} \times \mathcal{I} \to \mathcal{I} \\ (A,B) \to A \underset{\Pi}{\circ} B = \Pi(AB) \end{cases}$$

defines a multiplication on $\mathcal{I}$. Furthermore, $\mathcal{I}$ together with this multiplication and its given *-operation and norm is a C*-algebra with multiplicative identity I.

<u>PROOF</u> The multiplication is associative (cf. 12.26):

$$\begin{cases} A \underset{\Pi}{\circ} (B \underset{\Pi}{\circ} C) = \Pi(A\Pi(BC)) = \Pi(\Pi(A)\Pi(BC)) = \Pi(\Pi(A)BC) = \Pi(ABC) \\ (A \underset{\Pi}{\circ} B) \underset{\Pi}{\circ} C = \Pi(\Pi(AB)C) = \Pi(\Pi(AB)\Pi(C)) = \Pi(AB\Pi(C)) = \Pi(ABC). \end{cases}$$

Next, $\forall \, A \in \mathcal{I}$,

$$\begin{cases} I \underset{\Pi}{\circ} A = \Pi(IA) = \Pi(A) = A \\ A \underset{\Pi}{\circ} I = \Pi(AI) = \Pi(A) = A \end{cases}$$

and $\forall$ A,B $\in$ I,

$$(A \underset{\Pi}{\circ} B)^* = \Pi(AB)^* = \Pi(B^*A^*) = B^* \underset{\Pi}{\circ} A^*.$$

Therefore I is a unital *-algebra. But the continuity of Π implies that I is norm complete. And $\forall$ A,B $\in$ S,

$$\begin{aligned} ||A \underset{\Pi}{\circ} B|| &= ||\Pi(AB)|| \\ &\leq ||\Pi||\ ||AB|| \leq ||AB|| \leq ||A||\ ||B||. \end{aligned}$$

Therefore I is a complex Banach algebra. It remains to verify the C*-condition, viz.

$$||A^* \underset{\Pi}{\circ} A|| = ||A||^2.$$

On the one hand,

$$\begin{aligned} ||A^* \underset{\Pi}{\circ} A|| &= ||\Pi(A^*A)|| \\ &\leq ||\Pi||\ ||A^*A|| \\ &= ||A^*A|| = ||A||^2, \end{aligned}$$

while on the other,

$$\begin{aligned} \Pi(A^*A) &\geq \Pi(A)^*\Pi(A) \quad \text{(cf. 11.24)} \\ &= A^*A \end{aligned}$$

=>

$$\begin{aligned} ||A^* \underset{\Pi}{\circ} A|| &= ||\Pi(A^*A)|| \\ &\geq ||A^*A|| = ||A||^2. \end{aligned}$$

Let A_Π stand for I supplied with the C*-algebra structure set out in 12.27 --

then $\forall$ n, the arrow

$$\Pi_n : M_n(\mathcal{B}(\mathcal{H})) \to M_n(\mathcal{B}(\mathcal{H}))$$

determines a C*-algebra structure on the range $\mathcal{A}_{\Pi_n}$.

12.28 <u>LEMMA</u> There is a canonical $*$-isomorphism

$$M_n(\mathcal{A}_\Pi) \approx \mathcal{A}_{\Pi_n}.$$

<u>PROOF</u> In fact, $\forall$ A,B $\in M_n(\mathcal{A}_\Pi)$, their product is the matrix

$$[\sum_k A_{ik} \underset{\Pi}{\circ} B_{kj}] = [\sum_k \Pi(A_{ik}B_{kj})]$$

$$= \Pi_n([\sum_k A_{ik}B_{kj}])$$

$$= \Pi_n(AB).$$

[Note: A $*$-isomorphism of C*-algebras is necessarily isometric, thus one can identify $M_n(\mathcal{A}_\Pi)$ and $\mathcal{A}_{\Pi_n}$. Accordingly, $M_n(\mathcal{A}_\Pi)$ has the relative $\mathcal{B}(\overset{n}{\oplus} \mathcal{H})$ norm and

$$M_n(\mathcal{I}) \cap \mathcal{B}(\overset{n}{\oplus} \mathcal{H})_+ = M_n(\mathcal{A}_\Pi)_+.]$$

<u>N.B.</u> In general, $\mathcal{A}_\Pi$ is not a C*-subalgebra of $\mathcal{B}(\mathcal{H})$.

[Note: It is easy to prove, however, that $\mathcal{A}_\Pi$ is monotonically complete in the sense that every bounded increasing net $A_i \in (\mathcal{A}_\Pi)_{SA}$ has a least upper bound.]

12.29 <u>REMARK</u> The multiplication in $\mathcal{I}$ is independent of the choice of Π in the following sense. Suppose that

$$\begin{cases} \Pi_1 : \mathcal{B}(\mathcal{H}) \to \mathcal{I} \\ \Pi_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{I} \end{cases}$$

are completely positive projections of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{I}$ -- then the diagram

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & A_{\Pi_1} \\ \Vert & & \\ \mathcal{I} & \longrightarrow & A_{\Pi_2} \end{array}$$

has a unital filler $\Theta : A_{\Pi_1} \to A_{\Pi_2}$ which is a $*$-isomorphism.

Let $A, A' \in \mathcal{I}$ and let $B \in \mathcal{B}(\mathcal{H})$ -- then (cf. 12.26)

$$\begin{cases} \Pi(A\Pi(B)) = \Pi(\Pi(A)B) = \Pi(AB) \\ \Pi(\Pi(B)A') = \Pi(B\Pi(A')) = \Pi(BA'). \end{cases}$$

12.30 <u>LEMMA</u> Suppose that $\mathcal{I}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ -- then

$$\Pi(ABA') = A\Pi(B)A'.$$

<u>PROOF</u> Since $\mathcal{I}$ is closed under multiplication,

$$A\Pi(B)A' \in \mathcal{I}$$

$\Rightarrow$

$$\Pi(A\Pi(B)A') = A\Pi(B)A'.$$

But

$$\Pi(A\Pi(B)A') = \Pi(A\Pi(\Pi(B)A'))$$

$$= \Pi(A\Pi(BA'))$$

$$= \Pi(ABA').$$

[Note: Therefore $\Pi:\mathcal{B}(\mathcal{H}) \to \mathcal{I}$ is an $\mathcal{I}$-bimodule map. In particular, $\mathcal{I}$ might be a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$ in which case Π is called a <u>completely positive conditional expectation</u>.]

Let $\mathcal{A}$ be a unital *-subalgebra of $\mathcal{B}(\mathcal{H})$ -- then $\mathcal{A}$ is said to be <u>generated by projections</u> if the linear span of the projections in $\mathcal{A}$ is norm dense in $\mathcal{A}$. E.g.: W*-algebras have this property.

12.31 <u>LEMMA</u> Suppose that $\mathcal{A}$ is generated by projections. Assume: $\exists$ a unital idempotent $\Phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ of norm 1 whose range is $\mathcal{A}$ -- then $\forall\ \xi,\eta \in \mathcal{A}$ & $\forall\ A \in \mathcal{B}(\mathcal{H})$,

$$\Phi(\xi A\eta) = \xi\Phi(A)\eta.$$

<u>PROOF</u> To begin with, Φ is positive (cf. 11.13), thus $\forall\ A \in \mathcal{B}(\mathcal{H})$, $\Phi(A)^* = \Phi(A^*)$ (cf. 11.4). This said, it suffices to make the verification when ξ,η are projections in $\mathcal{A}$ and then one has only to show that $\Phi(\xi A) = \xi\Phi(A)$.

Indeed,

$$\Phi(A\eta)^* = \Phi((A\eta)^*)$$

$$= \Phi(\eta^*A^*)$$

$$= \eta^*\Phi(A^*)$$

$$= \eta^*\Phi(A)^*$$

=>

$$\Phi(A\eta) = (\eta^*\Phi(A)^*)^* = \Phi(A)\eta.$$

Let $\xi' = I - \xi$ -- then $\forall\ A, A' \in \mathcal{B}(H)$,

$$
\begin{aligned}
&||\xi A + \xi' A'||^2 \\
&\quad = ||(\xi A + \xi' A')^*(\xi A + \xi' A')|| \\
&\quad = ||A^*\xi A + A'^*\xi' A'|| \\
&\quad \leq ||A^*\xi A|| + ||A'^*\xi' A'|| \\
&\quad = ||A^*\xi^*\xi A|| + ||A'^*\xi'^*\xi' A'|| \\
&\quad = ||(\xi A)^*\xi A|| + ||(\xi' A')^*\xi' A'|| \\
&\quad = ||\xi A||^2 + ||\xi' A'||^2.
\end{aligned}
$$

So, $\forall\ t \in \underline{R}_{>0}$,

$$
\begin{aligned}
&(1 + t)^2||\xi'\Phi(\xi A)||^2 \\
&\quad = ||\xi'\Phi(\xi A) + t\xi'\Phi(\xi A)||^2 \\
&\quad = ||\xi'\Phi(\xi A) + t\Phi(\xi'\Phi(\xi A))||^2 \\
&\quad = ||\xi'\Phi(\xi A + t\xi'\Phi(\xi A))||^2 \\
&\quad \leq ||\xi A + t\xi'\Phi(\xi A)||^2 \\
&\quad \leq ||\xi A||^2 + ||t\xi'\Phi(\xi A)||^2 \\
&\quad = ||\xi A||^2 + t^2||\xi'\Phi(\xi A)||^2
\end{aligned}
$$

=>

$$(2t + 1)||\xi'\Phi(\xi A)||^2 \leq ||\xi A||^2$$

=>

$$||\xi'\Phi(\xi A)||^2 = 0$$

$\Rightarrow$

$$\xi'\Phi(\xi A) = 0$$

$\Rightarrow$

$$\Phi(\xi A) = \xi\Phi(\xi A).$$

Replacing ξ by ξ' and repeating the argument leads to

$$\xi\Phi((I - \xi)A) = 0$$

or still,

$$\xi\Phi(A) = \xi\Phi(\xi A).$$

Therefore

$$\Phi(\xi A) = \xi\Phi(A),$$

as desired.

[Note: Consequently, Φ is completely positive (cf. 11.14), hence $\mathcal{A}$ is injective (cf. 12.25).]

12.32 <u>EXAMPLE</u> Take $\mathcal{A} = \underline{C}I$ and let Φ be as in 12.31 -- then $\forall$ $A \in \mathcal{B}(\mathcal{H})$,

$$\Phi(A) = \omega(A)I \quad (\omega(A) \in \underline{C})$$

and $\omega:\mathcal{B}(\mathcal{H}) \to \underline{C}$ is a state on $\mathcal{B}(\mathcal{H})$ $(I = \Phi(I) = \omega(I)I \Rightarrow \omega(I) = 1)$. Conversely, if $\omega:\mathcal{B}(\mathcal{H}) \to \underline{C}$ is a state on $\mathcal{B}(\mathcal{H})$ and if we let

$$\Phi(A) = \omega(A)I,$$

then Φ is unital $(\omega(I) = 1)$, idempotent $(\Phi(\Phi(A)) = \Phi(\omega(A)I) = \omega(A)\Phi(I) = \omega(A)I = \Phi(A))$, and $||\Phi|| = ||\omega|| = 1$.

APPENDIX

The following result can sometimes be used to reduce a nonunital situation to a unital situation.

<u>LEMMA</u> Let $S \subset B(H)$ be an operator system, K a complex Hilbert space. Suppose that $\Phi:S \to B(K)$ is completely positive -- then $\exists$ a unital completely positive $\Psi:S \to B(K)$ such that

$$\Phi(A) = \Phi(I)^{1/2}\Psi(A)\Phi(I)^{1/2} (A \in S).$$

<u>PROOF</u> Let $T = \Phi(I) \in B(K)_+$ and let P_T be its support projection -- then P_T is the minimal $P \in L(K)$ such that $TP = T$ and the sequence

$$T^{1/2}(T + I/n)^{-1/2} \in B(K)_+$$

is increasing and converges to P_T in the strong operator topology. Fix $x \in \underline{S}(H)$ and define $\Psi^n:S \to B(K)$ by

$$\Psi^n(A) = (T + I/n)^{-1/2}\Phi(A)(T + I/n)^{-1/2} + \langle x,Ax\rangle (I - P_T).$$

Then Ψ^n is completely positive and we claim that $\forall$ $A \in S$, the strong limit of $\{\Psi^n(A)\}$ exists. To establish this, it can be assumed that $0 \leq A \leq I$, hence $0 \leq \Phi(A) \leq \Phi(I) = T$, so $\exists$ $C_A \in B(K)$:

$$\Phi(A)^{1/2} = \begin{cases} C_A T^{1/2} \\ \\ T^{1/2} C_A^*. \end{cases}$$

Therefore

$$\left\{\begin{array}{l} \Phi(A)^{1/2}(T + I/n)^{-1/2} = C_A T^{1/2}(T + I/n)^{-1/2} \to C_A P_T \\ \\ (T + I/n)^{-1/2}\Phi(A)^{1/2} = (T + I/n)^{-1/2}T^{1/2}C_A^* \to P_T C^*_A \end{array}\right. \quad (\mathrm{SOT}).$$

But

$$\left\{\begin{array}{l} ||\Phi(A)^{1/2}(T + I/n)^{-1/2}|| \\ \\ ||(T + I/n)^{-1/2}\Phi(A)^{1/2}|| \end{array}\right. \leq ||C_A||.$$

Since multiplication is jointly continuous on bounded sets in the strong operator topology, it follows that

$$(T + I/n)^{-1/2}\Phi(A)(T + I/n)^{-1/2} \to P_T C_A^* C_A P_T$$

strongly, from which the claim. Now define $\Psi: S \to B(K)$ by

$$\Psi(A) = \lim \Psi^n(A).$$

Then Ψ is completely positive. Moreover, Ψ is unital:

$$\begin{aligned} \Psi(I) &= \lim \Psi^n(I) \\ &= \lim((T + I/n)^{-1/2}\Phi(I)(T + I/n)^{-1/2} + \langle x,x\rangle\ (I - P_T)) \\ &= \lim((T + I/n)^{-1/2}T^{1/2}T^{1/2}(T + I/n)^{-1/2} + (I - P_T)) \\ &= P_T P_T + I - P_T \\ &= I. \end{aligned}$$

And lastly,

$$(T + I/n)^{1/2}\Psi^n(A)(T + I/n)^{1/2}$$

$$= \Phi(A) + \langle x,Ax\rangle \; (T + I/n)(I - P_T)$$

=>

$$T^{1/2}\Psi(A)T^{1/2} = \Phi(A) + \langle x,Ax\rangle \; T(I - P_T)$$

$$= \Phi(A) + \langle x,Ax\rangle \; (T - TP_T)$$

$$= \Phi(A).$$

§13. RADON-NIKODYM THEORY

Let Γ be a $*$-semigroup.

Notation: Given functions $\Phi,\Phi':\Gamma \to \mathcal{B}(\mathcal{H})$ whose associated kernels $K_\Phi, K_{\Phi'}$ are positive definite, write $\Phi \geq \Phi'$ if $K_\Phi - K_{\Phi'}$ is positive definite.

Suppose that

$$\begin{cases} \Phi_1:\Gamma \to \mathcal{B}(\mathcal{H}_1) \\ \Phi_2:\Gamma \to \mathcal{B}(\mathcal{H}_2) \end{cases}$$

are representable, i.e.,

$$\begin{cases} \Phi_1 = \Delta_{R_1}\pi_1 \quad (R_1 \in \mathcal{B}(\mathcal{H}_1,\mathcal{K}_1)) \\ \Phi_2 = \Delta_{R_2}\pi_2 \quad (R_2 \in \mathcal{B}(\mathcal{H}_2,\mathcal{K}_2)), \end{cases}$$

where

$$\begin{cases} \pi_1:\Gamma \to \mathcal{B}(\mathcal{K}_1) \\ \pi_2:\Gamma \to \mathcal{B}(\mathcal{K}_2) \end{cases}$$

are minimal $*$-representations of Γ on

$$\begin{cases} \mathcal{K}_1 \\ \mathcal{K}_2, \end{cases}$$

respectively.

[Note: K_{Φ_1}, K_{Φ_2} are necessarily positive definite (cf. 9.9) and both Φ_1,Φ_2 satisfy the boundedness condition (cf. 9.10).]

13.1 <u>LEMMA</u> Let $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Assume: $\exists\ \tilde{X} \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that

$$\tilde{X}R_1 = R_2X \text{ and } \tilde{X}\pi_1(\xi) = \pi_2(\xi)\tilde{X} \quad (\xi \in \Gamma).$$

Then

$$||\tilde{X}||^2\Phi_1 \geq X^*\Phi_2X.$$

<u>PROOF</u> Given

$$\left[\begin{array}{l} \xi_1, \ldots, \xi_n \in \Gamma \\ x_1, \ldots, x_n \in \mathcal{H}_1, \end{array}\right.$$

we have

$$\sum_{i,j=1}^{n} \langle x_i, X^*\Phi_2(\xi_i^*\xi_j)Xx_j\rangle$$

$$= \sum_{i,j=1}^{n} \langle x_i, X^*R_2^*\pi_2(\xi_i^*\xi_j)R_2Xx_j\rangle$$

$$= \sum_{i,j=1}^{n} \langle R_2Xx_i, \pi_2(\xi_i)^*\pi_2(\xi_j)R_2Xx_j\rangle$$

$$= \sum_{i,j=1}^{n} \langle \pi_2(\xi_i)R_2Xx_i, \pi_2(\xi_j)R_2Xx_j\rangle$$

$$= ||\sum_{k=1}^{n} \pi_2(\xi_k)R_2Xx_k||^2$$

$$= ||\sum_{k=1}^{n} \pi_2(\xi_k)\tilde{X}R_1x_k||^2$$

$$= ||\sum_{k=1}^{n} \tilde{X}\pi_1(\xi_k)R_1x_k||^2$$

$$= ||\tilde{X}(\sum_{k=1}^{n} \pi_1(\xi_k)R_1x_k)||^2$$

$$\leq ||\tilde{X}||^2 \ ||\sum_{k=1}^{n} \pi_1(\xi_k)R_1x_k||^2$$

$$= ||\tilde{X}||^2 \sum_{i,j=1}^{n} \langle x_i, \Phi_1(\xi_i^*\xi_j)x_j\rangle.$$

Therefore

$$||\tilde{X}||^2\Phi_1 \geq X^*\Phi_2X.$$

13.2 <u>RAPPEL</u> Let H be a complex Hilbert space. Suppose that $A \subset B(H)$ is a $*$-subalgebra and has a trivial null space ($\xi x = 0 \ \forall \ \xi \in A \Rightarrow x = 0$) -- then the closure $\bar{A}$ of A in the strong operator topology contains the identity operator I.

13.3 <u>LEMMA</u> Let $X \in B(H_1,H_2)$. Assume:

$$r^2\Phi_1 \geq X^*\Phi_2X \quad (\exists \ r > 0).$$

Then there is a unique $\tilde{X} \in B(K_1,K_2)$ such that

$$\tilde{X}R_1 = R_2X \text{ and } \tilde{X}\pi_1(\xi) = \pi_2(\xi)\tilde{X} \quad (\xi \in \Gamma).$$

<u>PROOF</u> Proceeding as in 13.1,

$$||\sum_{k=1}^{n} \pi_2(\xi_k)R_2Xx_k||^2$$

$$= \sum_{i,j=1}^{n} \langle x_i, X^*\Phi_2(\xi_i^*\xi_j)Xx_j\rangle$$

$$\leq r^2 \sum_{i,j=1}^{n} \langle x_i, \Phi_1(\xi_i^*\xi_j)x_j\rangle$$

$$= r^2 \|\sum_{k=1}^{n} \pi_1(\xi_k)R_1x_k\|^2,$$

so there is a unique $\tilde{X} \in \mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)$ such that $\forall\ \xi \in \Gamma$,

$$\tilde{X}\pi_1(\xi)R_1 = \pi_2(\xi)R_2X.$$

And then $\forall\ \xi,\eta \in \Gamma$ & $\forall\ x \in \mathcal{H}_1$,

$$\begin{aligned}\tilde{X}\pi_1(\xi)\pi_1(\eta)R_1x &= \tilde{X}\pi_1(\xi\eta)R_1x \\ &= \pi_2(\xi\eta)R_2Xx \\ &= \pi_2(\xi)\pi_2(\eta)R_2Xx \\ &= \pi_2(\xi)\tilde{X}\pi_1(\eta)R_1x.\end{aligned}$$

Since $\pi_1(\Gamma)R_1\mathcal{H}_1$ is total in $\mathcal{K}_1$, it follows that $\tilde{X}$ intertwines π_1 and π_2. It remains to prove that $\tilde{X}R_1 = R_2X$. For this purpose, consider the $*$-subalgebra $\mathcal{W} \subset \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ of all diagonal matrices

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},$$

where

$$W_i = \sum_k c_k \pi_i(\xi_k) \quad (i = 1,2).$$

Because of minimality, $\mathcal{W}$ has a trivial nullspace. Accordingly, thanks to 13.2, its closure $\bar{\mathcal{W}}$ in the strong operator topology contains the identity operator

$$I = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}.$$

But $\forall\ x \in \mathcal{H}_1$,

$$\tilde{X}W_1R_1x = \tilde{X}\sum_k c_k\pi_1(\xi_k)R_1x$$

$$= \sum_k c_k\tilde{X}\pi_1(\xi_k)R_1x$$

$$= \sum_k c_k\pi_2(\xi_k)R_2Xx$$

$$= (\sum_k c_k\pi_2(\xi_k))R_2Xx$$

$$= W_2R_2Xx.$$

Now let W approach I to conclude that $\tilde{X}R_1 = R_2X$.

<u>N.B.</u> $\forall\ x \in \mathcal{H}_1$,

$$||\pi_2(\xi)R_2Xx||^2 \le r^2||\pi_1(\xi)R_1x||^2$$

=>

$$||\tilde{X}\pi_1(\xi)R_1x||^2 = ||\pi_2(\xi)R_2Xx||^2$$

$$\leq r^2||\pi_1(\xi)R_1x||^2$$

=>

$$||\tilde{X}|| \leq r.$$

13.4 <u>THEOREM</u> Suppose that $X \in \mathcal{B}(H_1,H_2)$ intertwines Φ_1 and Φ_2:

$$X\Phi_1(\xi) = \Phi_2(\xi)X \quad (\xi \in \Gamma).$$

Then there is a unique $\tilde{X} \in \mathcal{B}(K_1,K_2)$ such that

$$\tilde{X}R_1 = R_2X \text{ and } \tilde{X}\pi_1(\xi) = \pi_2(\xi)\tilde{X} \quad (\xi \in \Gamma).$$

Moreover,

$$||\tilde{X}|| \leq ||X||.$$

<u>PROOF</u> First, for all

$$\begin{cases} \xi_1,\ldots,\xi_n \in \Gamma \\ x_1,\ldots,x_n \in H_1, \end{cases}$$

$$\sum_{i,j=1}^{n} \langle x_i, X^*\Phi_2(\xi_i^*\xi_j)Xx_j\rangle$$

$$= \sum_{i,j=1}^{n} \langle x_i, X^*X\Phi_1(\xi_i^*\xi_j)x_j\rangle.$$

Next, $\forall\ \xi \in \Gamma$,

$$X\Phi_1(\xi^*) = \Phi_2(\xi^*)X$$

=>

$$(X\Phi_1(\xi^*))^* = (\Phi_2(\xi^*)X)^*$$

$\Rightarrow$

$$\Phi_1(\xi^*)^*X^* = X^*\Phi_2(\xi^*)^*$$

$\Rightarrow$

$$\Phi_1(\xi)X^* = X^*\Phi_2(\xi) \quad \text{(cf. 9.7)}$$

$\Rightarrow$

$$\begin{aligned} X^*X\Phi_1(\xi) &= X^*\Phi_2(\xi)X \\ &= \Phi_1(\xi)X^*X. \end{aligned}$$

Consequently, in $\overset{n}{\oplus} \mathcal{H}_1$, the operator

$$\Delta = \mathrm{diag}(X^*X)$$

commutes with

$$\Xi = [\Phi_1(\xi_i^*\xi_j)].$$

But

$$\Delta, \Xi \in \mathcal{B}(\overset{n}{\oplus} \mathcal{H})_+$$

$\Rightarrow$

$$\Delta\Xi \in \mathcal{B}(\overset{n}{\oplus} \mathcal{H})_+.$$

So, $\forall\ \underline{x} \in \overset{n}{\oplus} \mathcal{H}_1$,

$$\begin{aligned} 0 &\le \langle \underline{x}, \Delta\Xi\underline{x} \rangle \\ &= \langle \underline{x}, \Delta\Xi^{1/2}\Xi^{1/2}\underline{x} \rangle \\ &= \langle \Xi^{1/2}\underline{x}, \Delta\Xi^{1/2}\underline{x} \rangle \\ &\le \|\Delta\|\ \langle \Xi^{1/2}\underline{x}, \Xi^{1/2}\underline{x} \rangle \end{aligned}$$

$$\le ||X^*X|| \langle \underline{x}, \Xi\underline{x} \rangle$$

$$\le ||X||^2 \langle x, \Xi\underline{x} \rangle.$$

Therefore

$$\sum_{i,j=1}^{n} \langle x_i, X^*\Phi_2(\xi_i^*\xi_j)Xx_j \rangle$$

$$\le ||X||^2 \sum_{i,j=1}^{n} \langle x_i, \Phi_1(\xi_i^*\xi_j)x_j \rangle.$$

The existence (and uniqueness) of $\tilde{X}$ is thus guaranteed by 13.3 (and $r = ||X||$ $\Rightarrow ||\tilde{X}|| \le ||X||$).

13.5 REMARK This result implies that there is a "lifting map" $X \to \tilde{X}$ from the set of operators that intertwine Φ_1 and Φ_2 to the set of operators that intertwine π_1 and π_2.

- If X is a contraction, then so is $\tilde{X}$.
- If X is unitary, then so is $\tilde{X}$.

[Note: Suppose that $X:H_1 \to H_2$ intertwines Φ_1 and Φ_2 -- then $X^*:H_2 \to H_1$ intertwines Φ_2 and Φ_1 and its lift is $\tilde{X}^*:K_2 \to K_1$.]

Let A be a U*-algebra -- then 10.7 is applicable.

[Note: If $\Phi:A \to B(H)$ is completely positive (or, equivalently, K_Φ is positive definite (cf. 11.10)), then $\Phi = \Delta_R\pi$ is "the canonical representation" of Φ.]

Denote by $CP(A,\mathcal{B}(H))$ the set of completely positive maps $A \to \mathcal{B}(H)$ and for $\Phi,\Phi' \in CP(A,\mathcal{B}(H))$, write $\Phi \geq \Phi'$ if $\Phi - \Phi' \in CP(A,\mathcal{B}(H))$.

13.6 LEMMA Suppose that $\Phi \geq \Phi'$ — then there is a unique contraction $T \in \mathcal{B}(K,K')$ such that

$$TR = R' \text{ and } T\pi(\xi) = \pi'(\xi)T \quad (\xi \in A).$$

PROOF In 13.3, take

$$\begin{bmatrix} \Phi_1 = \Phi \ (\Rightarrow \Phi = \Delta_R\pi, \ R:H \to K) \\ \\ \Phi_2 = \Phi' \ (\Rightarrow \Phi' = \Delta_{R'}\pi', \ R':H \to K'), \end{bmatrix}$$

$r = 1$, and $X = I$ to produce $\tilde{X}:K \to K'$ of norm ≤ 1 such that

$$\tilde{X}R = R' \text{ and } \tilde{X}\pi(\xi) = \pi'(\xi)\tilde{X} \quad (\xi \in A).$$

Now change the lettering and write T in place of $\tilde{X}$.

[Note: Using the notation of §8, put $A(\xi) = \pi(\xi)R$ — then

$$\begin{aligned} K_\Phi(\xi,\eta) &= \Phi(\xi^*\eta) \\ &= R^*\pi(\xi^*\eta)R \\ &= R^*\pi(\xi)^*\pi(\eta)R \\ &= A(\xi)^*A(\eta) \quad \text{(cf. 8.11)}. \end{aligned}$$

According to 8.13, the assumption that $\Phi \geq \Phi'$ (which is equivalent to the assumption that $K_\Phi \geq K_{\Phi'}$) entails the existence of a unique contraction $T:K \to K'$ such that

$$A'(\xi) = TA(\xi)$$

or still,

$$\pi'(\xi)R' = T\pi(\xi)R.$$

This "T" is the same as the "T" figuring in 13.6. In fact, $R' = TR$ (take $\xi = e$). As for the relation $T\pi = \pi' T$, it suffices to verify it on $\pi(A)RH$. But

$$\left[\begin{array}{l} T\pi(\xi)\pi(\eta)Rx = T\pi(\xi\eta)Rx = \pi'(\xi\eta)R'x \\ \\ \pi'(\xi)T\pi(\eta)Rx = \pi'(\xi)\pi'(\eta)R'x = \pi'(\xi\eta)R'x. \end{array}\right]$$

13.7 <u>LEMMA</u> We have

$$T^*T \in \pi(A)' \quad \text{(cf. 7.10)}.$$

Put

$$D_\Phi(\Phi') = T^*T.$$

Then $D_\Phi(\Phi')$ is called the <u>Radon-Nikodym derivative</u> of Φ' w.r.t. Φ.

<u>N.B.</u> $\forall\ \xi \in A$,

$$\begin{aligned} \Phi'(\xi) &= R'^*\pi'(\xi)R' \\ &= R^*T^*\pi'(\xi)TR \\ &= R^*\pi(\xi)T^*TR \\ &= R^*T^*T\pi(\xi)R \quad \text{(cf. 13.7)} \\ &= R^*D_\Phi(\Phi')\pi(\xi)R \\ &= R^*D_\Phi(\Phi')^{1/2}\pi(\xi)D_\Phi(\Phi')^{1/2}R. \end{aligned}$$

[Note:

$$T\pi(\xi^*) = \pi'(\xi^*)T$$

$\Rightarrow$

$$\pi(\xi^*)^*T^* = T^*\pi'(\xi^*)^*$$

or still,

$$\pi(\xi)T^* = T^*\pi'(\xi).]$$

13.8 EXAMPLE Suppose that $\Phi:A \to \underline{C}$ is a positive linear functional -- then Φ is completely positive (cf. 11.23) and the GNS construction enables one to write

$$\Phi(\xi) = \langle a,\pi(\xi)a\rangle \quad (\xi \in A) \qquad \text{(cf. 8.26 and 10.12)}.$$

So

$$\Phi' \leq \Phi \Rightarrow \Phi'(\xi) = \langle a,D_{\Phi}(\Phi')\pi(\xi)a\rangle \qquad (\xi \in A).$$

13.9 EXAMPLE Let X be a compact Hausdorff space. Suppose that μ is a Radon measure on X -- then the assignment

$$f \to \int_X f d\mu \quad (f \in C(X))$$

defines a positive linear functional Φ_μ on C(X). As such, Φ_μ is completely positive (cf. 11.23). To explicate 10.7 in this situation, let π_μ be the canonical representation of C(X) on $K_\mu = L^2(X,\mu)$, thus

$$\pi_\mu(f) = M_f,$$

where M_f is the operator of multiplication by f. Here $H_\mu = \underline{C}$ and $R_\mu:H_\mu \to K_\mu$ sends a complex number z to the constant function on X with value z. It is then immediate that

$$\Phi_\mu(f) = R_\mu^*\pi_\mu(f)R_\mu$$

and $\pi_\mu(C(X))R_\mu H_\mu$ is total in K_μ. Assume now that ν is another Radon measure on X with

$$\int_X f d\mu \geq \int_X f d\nu \quad (f \in C(X))$$

or still, $\Phi_\mu \geq \Phi_\nu$ — then

$$D_{\Phi_\mu}(\Phi_\nu) \equiv D_\mu(\nu) \in \pi_\mu(C(X))' \quad \text{(cf. 13.7).}$$

I.e.:

$$D_\mu(\nu) \in \{M_\phi : \phi \in L^\infty(X,\mu)\},$$

M_ϕ the multiplication operator per ϕ. Therefore, in suggestive notation,

$$D_\mu(\nu) = d\nu/d\mu \in L^\infty(X,\mu),$$

which is in accordance with the facts since the Radon-Nikodym derivative is only determined [μ a.e.] (note that $||D_\mu(\nu)||_\infty \leq 1$). Indeed,

$$\Phi_\nu(f) = \int_X f d\nu$$

$$= \int_X f(d\nu/d\mu)d\mu.$$

Given $\Phi \in CP(A, \mathcal{B}(H))$, put

$$[0,\Phi] = \{\Phi' \in CP(A,\mathcal{B}(H)) : \Phi \geq \Phi'\}.$$

Then $[0,\Phi]$ is a convex set.

[Note: $\forall\ \Phi',\Psi' \in [0,\Phi]$ and $\forall\ t \in [0,1]$,

$$t\Phi' + (1 - t)\Psi' \in [0,\Phi]$$

and

$$D_\Phi(t\Phi' + (1 - t)\Psi') = tD_\Phi(\Phi') + (1 - t)D_\Phi(\Psi').]$$

Let $A \in \pi(A)'$ and define a linear map $\Phi_A : A \rightarrow \mathcal{B}(H)$ by

$$\Phi_A(\xi) = R^*A\pi(\xi)R \quad (\xi \in A).$$

13.10 <u>LEMMA</u> If $\Phi_A = 0$, then $A = 0$.

<u>PROOF</u> Bearing in mind that $\pi(A)RH$ is total in K, $\forall\ x,y \in H$, we have

$$\begin{aligned} &<\pi(\xi)Rx, A\pi(\eta)Ry> \\ &= <x, R^*\pi(\xi)^*A\pi(\eta)Ry> \\ &= <x, R^*\pi(\xi^*)A\pi(\eta)Ry> \\ &= <x, R^*A\pi(\xi^*\eta)Ry> \\ &= <x, \Phi_A(\xi^*\eta)y> \\ &= 0. \end{aligned}$$

13.11 <u>LEMMA</u> If $A \in \pi(A)' \cap E(K)$, then $\Phi_A \in CP(A, B(H))$ and, in fact, $\Phi_A \in [0,\Phi]$ (cf. 8.12).

[Note: Restricting A to $\pi(A)' \cap E(K)$ rather than just $E(K)$ does not conflict with 8.12. The point is this: Every completely positive Φ gives rise to a positive definite kernel $K_\Phi : A \times A \to B(H)$ but there may be positive definite kernels $K : A \times A \to B(H)$ that do not come from a completely positive Φ. In particular: If $E \in E(K)$ is arbitrary, then

$$K_E(\xi,\eta) = R^*\pi(\xi)^*E\pi(\eta)R$$

is a positive definite kernel and $K_\Phi \geq K_E$ but in general, $K_E \neq K_{\Phi'}$ for some $\Phi' \in [0,\Phi]$.]

<u>N.B.</u> There are two other elementary points.

- If $A_1, A_2 \in \pi(A)'$ and if $0 \leq A_1 \leq A_2 \leq I$, then $0 \leq \Phi_{A_1} \leq \Phi_{A_2} \leq \Phi$.

• If $A_1, A_2 \in \pi(A)'$ and if $0 \le \Phi_{A_1} \le \Phi_{A_2} \le \Phi$, then $0 \le A_1 \le A_2 \le I$.

13.12 THEOREM The arrow

$$\pi(A)' \cap E(K) \to [0,\Phi]$$

that sends A to Φ_A is bijective.

PROOF To establish surjectivity, take a $\Phi' \in [0,\Phi]$ and let $A = D_\Phi(\Phi')$ -- then, as observed above, $\forall\ \xi \in A$,

$$\Phi'(\xi) = R^* D_\Phi(\Phi')\pi(\xi)R,$$

so

$$\Phi' = \Phi_{D_\Phi(\Phi')}.$$

13.13 REMARK Turned around, the arrow

$$[0,\Phi] \to \pi(A)' \cap E(K)$$

that sends Φ' to $D_\Phi(\Phi')$ is bijective.

13.14 EXAMPLE Take $A = B(H)$ $(\dim H = n)$ but, in a change of notation, let K play the role of "H" in the foregoing theory and take it finite dimensional $(\dim K = m)$. Define

$$\Phi_\tau \in CP(B(H), B(K))$$

by

$$\Phi_\tau(A) = \frac{tr(A)}{n} I_K \quad (A \in B(H)).$$

Given $x \in H, y \in K$, let

$$P_{x,y}: K \to H$$

be the operator

$$P_{x,y} = \langle y, \text{---} \rangle x \quad \text{(cf. 1.10)}.$$

Fix an orthonormal basis

$$\begin{cases} e_1, \ldots, e_n \text{ for } H \\ f_1, \ldots, f_m \text{ for } K. \end{cases}$$

Then

$$\Phi_\tau(A) = \sum_{i=1}^{n} \sum_{k=1}^{m} V_{ik}^* A V_{ik} \quad (A \in B(H)),$$

where

$$V_{ik} = \frac{1}{\sqrt{n}} P_{e_i, f_k}.$$

In fact,

$$\sum_{i=1}^{n} \sum_{k=1}^{m} V_{ik}^* A V_{ik} y$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{m} V_{ik}^* A \langle f_k, y \rangle e_i$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{m} \langle f_k, y \rangle V_{ik}^* A e_i$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \langle f_k, y \rangle \langle e_i, A e_i \rangle f_k$$

$$= \frac{1}{n} \sum_{i=1}^{n} \langle e_i, Ae_i \rangle \sum_{k=1}^{m} \langle f_k, y \rangle f_k$$

$$= \frac{tr(A)}{n} I_{K} y$$

$$= \Phi_\tau(A).$$

Let R_τ be the map

$$K \to H \otimes K \otimes H$$

that sends $y \in K$ to

$$\sum_{i=1}^{n} \sum_{k=1}^{m} V_{ik} y \otimes f_k \otimes e_i$$

and let

$$\pi : B(H) \to B(H \otimes K \otimes H)$$

be the $*$-representation of $B(H)$ on $H \otimes K \otimes H$ that sends $A \in B(H)$ to

$$A \otimes I_{K \otimes H}.$$

Then the definitions imply that $\Phi_\tau = \Delta_{R_\tau} \pi$ is the canonical representation of Φ_τ.

Suppose now that $\Phi \in CP(B(H), B(K))$. Claim: $\exists\, C_\Phi > 0$ such that

$$C_\Phi \Phi_\tau \geq \Phi.$$

To see this, note first that

$$\frac{1}{n} \sum_{i,j=1}^{n} P_{e_i, e_j} \otimes P_{e_i, e_j} \in B(H \otimes H)$$

is the orthogonal projection of $H \otimes H$ onto the subspace determined by

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i \in \underline{S}\,(H \otimes H).$$

Since Φ is completely positive, the operator

$$M_\Phi = (\Phi \otimes I_H)\left(\frac{1}{n}\sum_{i,j=1}^{n} P_{e_i,e_j} \otimes P_{e_i,e_j}\right)$$

$$= \frac{1}{n}\sum_{i,j=1}^{n} \Phi(P_{e_i,e_j}) \otimes P_{e_i,e_j}$$

$$\in B(K \otimes H)$$

is positive. In addition,

$$<f_k \otimes e_i, M_\Phi(f_\ell \otimes e_j)>$$

$$= \frac{1}{n} <f_k \otimes e_i, \sum_{\alpha,\beta=1}^{n} \Phi(P_{e_\alpha,e_\beta}) \otimes P_{e_\alpha,e_\beta}(f_\ell \otimes e_j)>$$

$$= \frac{1}{n}\sum_{\alpha,\beta=1}^{n} <f_k \otimes e_i, \Phi(P_{e_\alpha,e_\beta})f_\ell \otimes P_{e_\alpha,e_\beta}e_j>$$

$$= \frac{1}{n}\sum_{\alpha,\beta=1}^{n} <f_k \otimes e_i, \Phi(P_{e_\alpha,e_\beta})f_\ell \otimes <e_\beta,e_j>e_\alpha>$$

$$= \frac{1}{n}\sum_{\alpha=1}^{n} <f_k \otimes e_i, \Phi(P_{e_\alpha,e_j})f_\ell \otimes e_\alpha>$$

$$= \frac{1}{n}\sum_{\alpha=1}^{n} <f_k, \Phi(P_{e_\alpha,e_j})f_\ell><e_i,e_\alpha>$$

$$= \frac{1}{n} <f_k, \Phi(P_{e_i,e_j})f_\ell>.$$

So, $\forall\ A \in B(H)$ and $\forall\ y \in K$,

$$R_\tau^*(A \otimes M_\Phi)R_\tau y$$

$$= \frac{1}{n}\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{m} <e_i,Ae_j><f_k \otimes e_i, M_\Phi(f_\ell \otimes e_j)>P_{f_k,f_\ell}y$$

$$= \frac{1}{n^2} \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{m} \langle e_i, Ae_j\rangle \langle f_k, \Phi(P_{e_i,e_j}) f_\ell\rangle \langle f_\ell, y\rangle f_k.$$

Write

$$A = \sum_{i,j=1}^{n} \langle e_i, Ae_j\rangle P_{e_i,e_j}.$$

Then

$$y = \sum_{\ell=1}^{m} \langle f_\ell, y\rangle f_\ell$$

=>

$$\Phi(A)y = \sum_{\ell=1}^{m} \langle f_\ell, y\rangle \Phi(A) f_\ell$$

$$= \sum_{\ell=1}^{m} \langle f_\ell, y\rangle \sum_{k=1}^{m} \langle f_k, \Phi(A) f_\ell\rangle f_k$$

$$= \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{m} \langle e_i, Ae_j\rangle \langle f_k, \Phi(P_{e_i,e_j}) f_\ell\rangle \langle f_\ell, y\rangle f_k.$$

Therefore

$$R_\tau^*(A \otimes M_\Phi) R_\tau y = \frac{1}{n^2} \Phi(A) y$$

or still,

$$R_\tau^*\left(A \otimes \frac{M_\Phi}{||M_\Phi||}\right) R_\tau y = \frac{1}{n^2 ||M_\Phi||} \Phi(A) y.$$

Since

$$I_H \otimes \frac{M_\Phi}{||M_\Phi||} \in \pi(\mathcal{B}(H))' \cap \mathcal{E}(K \otimes H),$$

it follows that

$$C_\Phi \Phi_\tau \geq \Phi$$

if

$$C_\Phi = n^2||M_\Phi||.$$

[Note: Set

$$\Phi' = \frac{1}{C_\Phi}\Phi.$$

Then

$$D_\Phi(\Phi') = I_H \otimes \frac{M_\Phi}{||M_\Phi||}.]$$

13.15 RAPPEL A nontrivial *-representation $\pi:A \rightarrow B(K)$ is topologically irreducible iff $\pi(A)' \cap L(K) = \{0,I\}$ or still, iff $\pi(A)' = \underline{C}I$.

Let $\Phi \in CP(A,B(H))$ be nonzero -- then Φ is said to be pure if $[0,\Phi] = \{t\Phi:0 \leq t \leq 1\}$.

13.16 LEMMA The pure elements of $CP(A,B(H))$ are those Φ for which π is topologically irreducible.

PROOF Suppose that Φ is pure -- then

$$\pi(A)' \cap E(K) = \{D_\Phi(t\Phi):0 \leq t \leq 1\}$$

$$= \{tI:0 \leq t \leq 1\}$$

=>

$$\pi(A)' \cap L(K) = \{0,I\}.$$

Therefore π is topologically irreducible. Conversely,

$$\pi(A)' = \underline{C}I$$

$\Rightarrow$

$$\pi(A)' \cap E(K) = \{tI : 0 \leq t \leq 1\}$$

$\Rightarrow$

$$[0,\Phi] = \{t\Phi : 0 \leq t \leq 1\}.$$

13.17 <u>REMARK</u> Let $V: V \to H$ be an isometry ($\Rightarrow V^*V = I$) -- then

$$\Phi \text{ pure} \Rightarrow V^*\Phi V \text{ pure}.$$

In fact, the canonical decomposition of $V^*\Phi V$ is

$$V^*R^*\pi RV = (RV)^*\pi RV.$$

[Note: Since π is topologically irreducible, it is automatic that $\pi(A)RVV$ is total in K.]

13.18 <u>EXAMPLE</u> Let X be a compact Hausdorff space -- then the pure elements of $CP(C(X),B(H))$ are the functions $\Phi: C(X) \to H$ of the form

$$\Phi(f) = f(x)A \quad (x \in X),$$

where $A \in B(H)_+$ has rank 1.

13.19 <u>REMARK</u> Consider the setup of 13.14 -- then Φ_τ is not pure unless $H = K = \underline{C}$. Indeed, the commutant of

$$\pi(B(H)) = B(H) \otimes \underline{C}I_{K \otimes H}$$

in

$$B(H \otimes K \otimes H) = B(H) \otimes B(K \otimes H)$$

is

$$\underset{\sim}{C}I_{H} \otimes B(K \otimes H).$$

13.20 <u>LEMMA</u> Fix $\Phi \neq 0$ in $CP(A,B(H))$ — then the extreme points of the convex set $[0,\Phi]$ are the Φ_P $(P \in \pi(A)' \cap L(K))$ (cf. 3.1).

We shall now return to where we started. So let Γ be a $*$-semigroup and suppose that $\Phi:\Gamma \to B(H)$ is a function which is representable, i.e., $\Phi = \Delta_R\pi$ $(R \in B(H,K)$ and $\pi:\Gamma \to B(K)$ a minimal $*$-representation of Γ on $K)$ — then there is an arrow $X \to \tilde{X}$ from $\Phi(\Gamma)'$ to $\pi(\Gamma)'$ subject to $\tilde{X}R = RX$ (cf. 13.4).

[Note: By construction,

$$\tilde{X}(\sum_i \pi(\xi_i)Rx_i) = \sum_i \pi(\xi_i)RXx_i \quad (\xi_i \in \Gamma, x_i \in H).]$$

<u>N.B.</u> $\tilde{X} \in \{RR^*\}'$. In fact,

$$(X^*)^{\sim} = (\tilde{X})^* \quad \text{(cf. 13.5)}.$$

Therefore

$$\begin{aligned}\tilde{X}RR^* &= RXR^* \\ &= R(RX^*)^* \\ &= R(\tilde{X^*}R)^* \\ &= RR^*\tilde{X}.\end{aligned}$$

Thus the range of the arrow $X \to \tilde{X}$ is actually $\pi(\Gamma)' \cap \{RR^*\}'$. As such, it is a $*$-homomorphism. Furthermore, $\tilde{X} = 0$ iff $RX = 0$.

13.21 <u>LEMMA</u> The arrow $X \to \tilde{X}$ is surjective.

<u>PROOF</u> Write $R = (RR^*)^{1/2}W$, $W:H \to K$ a partial isometry and

$$WW^*(RR^*)^{1/2} = (RR^*)^{1/2}WW^* = (RR^*)^{1/2}.$$

Given $T \in \pi(\Gamma)' \cap \{RR^*\}'$, put

$$X = W^*TW \in \mathcal{B}(H).$$

Then

$$\begin{aligned} RX &= (RR^*)^{1/2}WX \\ &= (RR^*)^{1/2}WW^*TW \\ &= (RR^*)^{1/2}TW \\ &= T(RR^*)^{1/2}W \\ &= TR. \end{aligned}$$

And $\forall\ \xi \in \Gamma$,

$$\begin{aligned} X\Phi(\xi) &= XR^*\pi(\xi)R \\ &= W^*TWR^*\pi(\xi)R \\ &= W^*TWW^*(RR^*)^{1/2}\pi(\xi)R \\ &= W^*T(RR^*)^{1/2}\pi(\xi)R \\ &= W^*(RR^*)^{1/2}T\pi(\xi)R \\ &= R^*\pi(\xi)TR \\ &= R^*\pi(\xi)T(RR^*)^{1/2}W \\ &= R^*\pi(\xi)(RR^*)^{1/2}TW \\ &= R^*\pi(\xi)(RR^*)^{1/2}WW^*TW \end{aligned}$$

$$= R^*\pi(\xi)RX$$

$$= \Phi(\xi)X.$$

Therefore $\tilde{X} = T$.

13.22 <u>LEMMA</u> Suppose that Ker $R = \{0\}$ — then the arrow $X \rightarrow \tilde{X}$ is a $*$-isomorphism:

$$\Phi(\Gamma)' \approx \pi(\Gamma)' \cap \{RR^*\}'.$$

<u>PROOF</u> In view of 13.21, the claim is that $\tilde{X} = 0 \Rightarrow X = 0$. To get a contradiction, say $X \neq 0$. Choose $x \in \mathcal{H}: Xx \neq 0$. But

$$\tilde{X} = 0 \Rightarrow \tilde{X}R = 0 \Rightarrow RX = 0 \Rightarrow RXx = 0 \Rightarrow Xx = 0\ldots .$$

13.23 <u>REMARK</u> If $\Phi(e) = I$, then $R^*R = I$ (cf. 9.14). Therefore Ker $R = \{0\}$ and 13.22 is applicable.

The assumption that R has a trivial nullspace has certain consequences. E.g.: π topologically irreducible $\Rightarrow \pi(\Gamma)' = \underline{C}I \Rightarrow \Phi$ topologically irreducible.

[Note: $\forall\ \xi \in \Gamma$, $\Phi(\xi)^* = \Phi(\xi^*)$ (cf. 9.7). Therefore $\Phi(\Gamma) \subset \mathcal{B}(\mathcal{H})_{SA}$ and the statement that Φ is topologically irreducible means that the only closed subspaces of $\mathcal{H}$ which are invariant under $\Phi(\Gamma)$ are $\{0\}$ and $\mathcal{H}$.]

13.24 <u>EXAMPLE</u> Take $\Gamma = A$, a U^*-algebra. Let $\Phi \in CP(A,\mathcal{B}(\mathcal{H}))$ and assume that Φ is pure and unital — then π is topologically irreducible (cf. 13.16), hence Φ is topologically irreducible.

Write $CPU(A,\mathcal{B}(H))$ for the subset of $CP(A,\mathcal{B}(H))$ consisting of the completely positive unital maps $\Phi:A \to \mathcal{B}(H)$ — then it is clear that $CPU(A,\mathcal{B}(H))$ is convex.

13.25 EXAMPLE If $\Phi \in CP(A,\mathcal{B}(H))$ is pure and unital, then Φ is an extreme point of $CPU(A,\mathcal{B}(H))$. Thus consider a convex decomposition $\Phi = \lambda_1\Phi_1 + \lambda_2\Phi_2$ $(0 < \lambda_1,\lambda_2$ & $\lambda_1 + \lambda_2 = 1)$, where $\Phi_1,\Phi_2 \in CPU(A,\mathcal{B}(H))$ -- then

$$\begin{cases} 0 \leq \lambda_1\Phi_1 \leq \Phi \\ 0 \leq \lambda_2\Phi_2 \leq \Phi \end{cases} \Rightarrow \begin{cases} \lambda_1\Phi_1 = t_1\Phi \\ \lambda_2\Phi_2 = t_2\Phi. \end{cases}$$

But

$$\begin{cases} \Phi_1(I) = \Phi(I) = I \\ \Phi_2(I) = \Phi(I) = I \end{cases} \Rightarrow \begin{cases} \lambda_1 = t_1 \\ \lambda_2 = t_2 \end{cases} \Rightarrow \begin{cases} \Phi_1 = \Phi \\ \Phi_2 = \Phi. \end{cases}$$

Therefore Φ is an extreme point of $CPU(A,\mathcal{B}(H))$.

13.26 LEMMA Suppose that $\Phi \in CPU(A,\mathcal{B}(H))$ is an extreme point of $CPU(A,\mathcal{B}(H))$ -- then the map

$$\begin{cases} \pi(A)' \to \mathcal{B}(H) \\ A \longrightarrow R^*AR \end{cases}$$

is injective.

PROOF Assume that $R^*AR = 0$ and without loss of generality, take A selfadjoint. Fix $s,t \in \underline{R}_{>0}$:

$$\frac{1}{4} I \leq sA + tI \leq \frac{3}{4} I$$

and put

$$X = sA + tI.$$

Then $R^*R = I$, so

$$\frac{1}{4} I \le tI \le \frac{3}{4} I,$$

thus $0 < t < 1$. Now form

$$\begin{cases} \Phi_X = R^*X\pi R \\ \Phi_{I-X} = R^*(I-X)\pi R. \end{cases}$$

Obviously,

$$\Phi = \Phi_X + \Phi_{I-X}$$

or still,

$$\Phi = t(t^{-1}\Phi_X) + (1-t)((1-t)^{-1}\Phi_{I-X}).$$

But

$$\begin{cases} \Phi_X(e) = tI \\ \Phi_{I-X}(e) = (1-t)I. \end{cases}$$

Therefore

$$\Phi = t^{-1}\Phi_X = (1-t)^{-1}\Phi_{I-X}.$$

Consequently,

$$R^*X\pi R = tR^*\pi R = R^*(tI)\pi R$$

$$\Rightarrow X = tI \quad \text{(cf. 13.10)}$$

$$\Rightarrow tI = sA + tI \Rightarrow sA = 0 \Rightarrow A = 0.$$

13.27 <u>REMARK</u> In 13.26, take H finite dimensional -- then it is a corollary that $\pi(A)'$ is finite dimensional.

As it happens, the converse to 13.26 holds as well.

13.28 LEMMA If $\Phi \in \text{CPU}(A,\mathcal{B}(H))$ and if the map

$$\begin{cases} \pi(A)' \to \mathcal{B}(H) \\ A \longrightarrow R^*AR \end{cases}$$

is injective, then Φ is an extreme point of $\text{CPU}(A,\mathcal{B}(H))$.

PROOF Consider a convex decomposition $\Phi = \lambda_1\Phi_1 + \lambda_2\Phi_2$ $(0 < \lambda_1,\lambda_2$ & $\lambda_1 + \lambda_2 = 1)$ -- then $\lambda_1\Phi_1 \in [0,\Phi]$, so $\exists\ A \in \pi(A)':\lambda_1\Phi_1 = R^*A\pi R$ (cf. 13.12), hence

$$\lambda_1 I = R^*(\lambda_1 I)R = R^*AR$$

$$\Rightarrow$$

$$\lambda_1 I = A.$$

Therefore

$$\lambda_1\Phi_1 = R^*(\lambda_1 I)\pi R = \lambda_1\Phi$$

$$\Rightarrow \Phi_1 = \Phi \Rightarrow \Phi_2 = \Phi.$$

13.29 RAPPEL Let $\pi:A \to \mathcal{B}(H)$ be a $*$-representation -- then $\pi(A)H$ is total iff the only $x \in H$ with the property that $\pi(\xi)x = 0\ \forall\ \xi \in A$ is $x = 0$.

[Note: Therefore

$$\pi \text{ unital} \Rightarrow \pi(A)H \text{ total.}]$$

13.30 <u>EXAMPLE</u> Suppose that $\pi:A \to B(H)$ is a unital $*$-representation -- then $\pi \in \text{CPU}(A,B(H))$ (cf. 11.6) and in view of 13.29, its canonical decomposition is "$\pi = \pi$" ($R = I$), thus π is an extreme point of $\text{CPU}(A,B(H))$ (cf. 13.28).

[Note: This shows that in general an extreme point of $\text{CPU}(A,B(H))$ need not be pure (π is pure iff π is topologically irreducible (cf. 13.16)).]

§14. *MATRIX STATES*

Given operator systems

$$\begin{cases} S \subset B(H) \\ T \subset B(K), \end{cases}$$

denote by $CP(S,T)$ the set of completely positive maps $S \to T$ and by $CPU(S,T)$ the set of completely positive unital maps $S \to T$.

14.1 <u>EXAMPLE</u> Let $\Phi: M_n(\underline{C}) \to M_m(\underline{C})$ be a linear map — then Φ is completely positive iff it has the form

$$\Phi(A) = \sum_{i=1}^{nm} V_i^* A V_i \qquad (A \in M_n(\underline{C})),$$

where the $V_i \in M_{n,m}(\underline{C})$, so Φ is unital iff $\sum_{i=1}^{nm} V_i^* V_i = I$.

[Note: $CPU(M_n(\underline{C}), M_m(\underline{C}))$ is a convex set. What are its extreme points? Answer: Those Φ which admit an expansion

$$\Phi(A) = \sum_{i=1}^{\ell} V_i^* A V_i \qquad (A \in M_n(\underline{C})),$$

subject to $\ell \leq nm$, $\sum_{i=1}^{\ell} V_i^* V_i = I$, and $\{V_i^* V_j : 1 \leq i,j \leq \ell\}$ linearly independent ($\Rightarrow \ell^2 \leq m^2 \Rightarrow \ell \leq m$).]

If $\Phi, \Phi' \in CP(S,T)$, write $\Phi \geq \Phi'$ provided $\Phi - \Phi' \in CP(S,T)$.

14.2 <u>LEMMA</u> If $\Phi,\Phi' \in CPU(S,T)$, then $\Phi \geq \Phi'$ iff $\Phi = \Phi'$.

<u>PROOF</u> Assume that $\Phi - \Phi'$ is completely positive -- then

$$||\Phi - \Phi'|| = ||\Phi(I) - \Phi'(I)|| \quad \text{(cf. 12.15)}$$
$$= 0.$$

Let $\Phi \in CP(S,T)$ be nonzero -- then Φ is said to be <u>pure</u> if $\Phi \geq \Phi' \Rightarrow \Phi' = t\Phi (0 \leq t \leq 1)$.

14.3 <u>LEMMA</u> Suppose that $\Phi \in CPU(S,\mathcal{B}(K))$ is pure -- then $\Phi(S)' = \underline{C}I$.

<u>PROOF</u> The range of Φ is a $*$-closed subspace of $\mathcal{B}(K)$, hence its commutant $\Phi(S)'$ is a $*$-subalgebra of $\mathcal{B}(K)$. But

$$\Phi(S)' = \Phi(S)''' = (\Phi(S)')'',$$

so $\Phi(S)'$ is a W^*-algebra, thus the linear span of the projections in $\Phi(S)'$ is norm dense in $\Phi(S)'$. This said, fix a nonzero $P \in \Phi(S)'$ -- then

$$\Phi = P\Phi P + (I - P)\Phi(I - P)$$
$$\Rightarrow$$
$$\Phi \geq P\Phi P \Rightarrow P\Phi P = t\Phi \quad (0 \leq t \leq 1)$$
$$\Rightarrow P = tI \Rightarrow t \neq 0$$
$$\Rightarrow P = I.$$

Therefore $\Phi(S)' = \underline{C}I$.

[Note: It follows that

$$\Phi(S) \subset \mathcal{B}(K)_{SA}$$

is topologically irreducible (cf. 13.24).]

Let $S \subset B(H)$ be an operator system -- then a state on S is a positive linear functional $\omega: S \to \underline{C}$ such that $\omega(I) = 1$ (cf. 12.4 and 12.5), thus $CPU(S,\underline{C})$ is the set of states on S (recall that positive linear functionals are completely positive (cf. 12.13)).

N.B. View $CPU(S,\underline{C})$ as a subset of S^* (the dual of S) and equip S^* with the weak* topology -- then $CPU(S,\underline{C})$ is bounded and weak* closed, hence is weak* compact (Alaoglu).

[Note: $CPU(S,\underline{C})$ is not empty (cf. 14.5 infra) and convex. So, thanks to the Krein-Milman theorem, $CPU(S,\underline{C})$ is the weak* closed convex hull of its extreme points.]

14.4 RAPPEL If X_0 is a linear subspace of a normed linear space X and if $\rho_0: X_0 \to \underline{C}$ is a bounded linear functional, then there is a bounded linear functional $\rho: X \to \underline{C}$ such that $||\rho|| = ||\rho_0||$ and $\rho|X_0 = \rho_0$.

14.5 LEMMA If $A \in S_{SA}$ and if $\lambda \in \sigma(A)$ (the spectrum of A), then $\exists$ a state $\omega \in CPU(S,\underline{C})$ such that $\omega(A) = \lambda$.

PROOF For all complex numbers a and b,

$$|\lambda a + b| \leq ||aA + bI||.$$

Therefore the prescription

$$\omega_0(aA + b) = \lambda a + b$$

defines a bounded linear functional ω_0 on the linear subspace $\{aA + bI\}$ of S with $||\omega_0|| = \omega_0(I) = 1$. Now apply 14.4 to get a bounded linear functional ω on S such that $||\omega|| = ||\omega_0||$ and

$$\begin{cases} \omega(A) = \omega_0(A) = \lambda \\ \omega(I) = \omega_0(I) = 1. \end{cases}$$

Then ω is positive (cf. 12.5), thus $\omega \in \mathrm{CPU}(S,\underline{C})$.

14.6 <u>THEOREM</u> Let $A \in S$. Suppose that $\omega(A) = 0 \ \forall$ state ω -- then $A = 0$.

<u>PROOF</u> If A is selfadjoint, then $\sigma(A) = \{0\}$ (cf. 14.5). But either $||A||$ or $-||A|| \in \sigma(A)$, so $||A|| = 0 \Rightarrow A = 0$. If A is not selfadjoint, write

$$A = \mathrm{Re}\, A + \sqrt{-1}\, \mathrm{Im}\, A \quad (\mathrm{Re}\, A, \mathrm{Im}\, A \in S).$$

Then $\forall\ \omega$,

$$\omega(A) = 0$$

$$\Rightarrow$$

$$0 = \omega(\mathrm{Re}\, A) + \sqrt{-1}\, \omega(\mathrm{Im}\, A)$$

$$\Rightarrow$$

$$\begin{cases} \omega(\mathrm{Re}\, A) = 0 \\ \omega(\mathrm{Im}\, A) = 0 \end{cases} \qquad (\omega(S_{SA}) \subset \underline{R})$$

$$\Rightarrow$$

$$\begin{cases} \mathrm{Re}\, A = 0 \\ \mathrm{Im}\, A = 0 \end{cases} \Rightarrow A = 0.$$

14.7 LEMMA Let $\omega \in \text{CPU}(S,\underline{C})$ -- then ω is pure iff ω is an extreme point of $\text{CPU}(S,\underline{C})$.

PROOF That

"pure => extreme"

can be gleaned from 13.25 (matters go through with no change if A is replaced by S). As for the converse, viz.

"extreme => pure",

suppose that $0 \leq \omega' \leq \omega$ ($\omega' \in \text{CP}(S,\underline{C})$), thus $0 \leq \omega'(I) \leq \omega(I) = 1$ and

$$||\omega'|| = \omega'(I) \quad \text{(cf. 12.4)},$$

so $\omega'(I) = 0 \Rightarrow \omega' = 0$, while $\omega'(I) = 1 \Rightarrow \omega = \omega'$ (cf. 14.2). On the other hand, if $0 < \omega'(I) < 1$, then we can write

$$\omega = \lambda_1\omega_1 + \lambda_2\omega_2,$$

where

$$\begin{bmatrix} \lambda_1 = \omega'(I),\ \lambda_2 = 1 - \omega'(I) \\ \omega_1 = \frac{1}{\lambda_1}\omega',\ \omega_2 = \frac{1}{\lambda_2}(\omega - \omega'). \end{bmatrix}$$

Since $\omega_1,\omega_2 \in \text{CPU}(S,\underline{C})$ and since by assumption, ω is an extreme point of $\text{CPU}(S,\underline{C})$, it follows that $\omega_1 = \omega$ or still, $\omega' = \lambda_1\omega$, as desired.

[Note: Therefore every state is a weak* limit of convex combinations of pure states.]

14.8 THEOREM Let $A \in S$. Suppose that $\omega(A) = 0$ $\forall$ pure state ω -- then $A = 0$ (cf. 14.6).

14.9 <u>THEOREM</u> Every pure state ω on S can be extended to a pure state $\bar{\omega}$ on $B(H)$.

<u>PROOF</u> Let $S_\omega(B(H))$ be the subset of $S(B(H))$ consisting of those states that extend ω -- then $S_\omega(B(H))$ is not empty (cf. 12.6). Moreover, $S_\omega(B(H))$ is a weak* closed convex subset of $S(B(H))$, thus is weak* compact, so by the Krein-Milman theorem is the weak* closed convex hull of its extreme points. Fix one such extreme point $\bar{\omega}$. Claim: $\bar{\omega}$ is a pure state on $B(H)$. Bearing in mind 14.7, consider a convex decomposition $\bar{\omega} = \lambda_1\bar{\omega}_1 + \lambda_2\bar{\omega}_2$ $(0 < \lambda_1,\lambda_2$ & $\lambda_1 + \lambda_2 = 1)$, where $\bar{\omega}_1,\bar{\omega}_2 \in S(B(H))$, and restrict this data to S:

$$\omega = \bar{\omega}|S = \lambda_1\bar{\omega}_1|S + \lambda_2\bar{\omega}_2|S$$

$$\Rightarrow$$

$$\bar{\omega}_1|S = \bar{\omega}_2|S = \omega$$

$$\Rightarrow$$

$$\bar{\omega}_1,\bar{\omega}_2 \in S_\omega(B(H))$$

$$\Rightarrow$$

$$\bar{\omega}_1 = \bar{\omega}_2 = \bar{\omega}.$$

A <u>matrix state</u> on S is an element $\Phi \in \mathrm{CPU}(S,M_n(\underline{C}))$.

[Note: According to 12.15,

$$||\Phi|| = ||\Phi(I)|| = ||I|| = 1.]$$

14.10 <u>REMARK</u> $\mathrm{CPU}(S,M_n(C))$ is not empty. Proof: Take any state $\omega:S \to \underline{C}$

and consider

$$\Phi_\omega(A) = \omega(A)I_n \quad (A \in S).$$

14.11 LEMMA Suppose that $\Phi:S \to \underline{C}$ is a linear map — then Φ is a matrix state iff Φ_n is positive.

View $CPU(S,M_n(\underline{C}))$ as a subset of $B(S,M_n(\underline{C}))$ and equip $B(S,M_n(\underline{C}))$ with the weak* topology $(M_n(\underline{C}) = B(\underline{C}^n))$ (cf. §12) -- then $CPU(S,M_n(\underline{C}))$ is weak* compact, hence, being convex, is the weak* closed convex hull of its extreme points (Krein-Milman).

[Note: It can happen that $CPU(S,M_n(\underline{C}))$ has no pure elements at all (recalling 13.16, take $S = C(X)$ (X a compact Hausdorff space) and consider $CPU(C(X),M_n(\underline{C})(n > 1))$. This is not a contradiction since an extreme point of $CPU(S,M_n(\underline{C}))$ need not be pure (the proof in 14.7 that "extreme => pure" breaks down if $n > 1$).]

14.12 RAPPEL If $\Phi \in CP(S,M_n(\underline{C}))$, then $\exists$ a $\Psi \in CPU(S,M_n(\underline{C}))$ such that

$$\Phi(A) = \Phi(I)^{1/2}\Psi(A)\Phi(I)^{1/2} \quad (A \in S).$$

[Note: For the details, see the Appendix to §12.]

14.13 THEOREM Suppose that $\Phi:S \to M_n(\underline{C})$ is a pure matrix state -- then Φ can be extended to a pure matrix state $\bar{\Phi}:B(H) \to M_n(\underline{C})$.

<u>PROOF</u> Let $S_\Phi(\mathcal{B}(\mathcal{H}))$ be the set of completely positive maps $\mathcal{B}(\mathcal{H}) \to M_n(\underline{C})$ that extend Φ — then $S_\Phi(\mathcal{B}(\mathcal{H}))$ is not empty (cf. 12.22), convex, and its elements are unital. Fix an extreme point $\bar{\Phi} \in S_\Phi(\mathcal{B}(\mathcal{H}))$ (such exist ...). Claim: $\bar{\Phi}$ is pure. To see this, we first argue as in 4.9 and deduce that $\bar{\Phi}$ is an extreme point of $\mathrm{CPU}(\mathcal{B}(\mathcal{H}), M_n(\underline{C}))$. Owing now to 13.26, the map

$$\begin{cases} \bar{\pi}(\mathcal{B}(\mathcal{H}))' \to M_n(\underline{C}) \quad (= \mathcal{B}(\underline{C}^n)) \\ \\ A \longrightarrow \bar{R}^*A\bar{R} \end{cases}$$

is injective (here, $\bar{\Phi} = \bar{R}^*\bar{\pi}\bar{R}$ is the canonical decomposition of $\bar{\Phi}$). To conclude that $\bar{\Phi}$ is pure, one has only to establish that $\bar{\pi}$ is topologically irreducible (cf. 13.16), which is equivalent to showing that $\dim \bar{\pi}(\mathcal{B}(\mathcal{H}))' = 1$ or still, that

$$\dim\{\bar{R}^*A\bar{R}: A \in \bar{\pi}(\mathcal{B}(\mathcal{H}))'\} = 1.$$

So let $P \in \bar{\pi}(\mathcal{B}(\mathcal{H}))'$ be a projection -- then $\bar{R}^*P\bar{R}$ is a scalar multiple of the identity. In fact, $\bar{\Phi}_P \in [0,\bar{\Phi}]$ (cf. 13.11), where

$$\bar{\Phi}_P = \bar{R}^*P\bar{\pi}\bar{R}.$$

Write (cf. 14.12)

$$\bar{\Phi}_P = \bar{\Phi}_P(I)^{1/2}\bar{\Psi}_P\bar{\Phi}_P(I)^{1/2} \qquad (\bar{\Psi}_P(I) = I),$$

which, upon restriction to S, gives:

$$\Phi \geq \bar{\Phi}_P(I)^{1/2}(\bar{\Psi}_P|S)\bar{\Phi}_P(I)^{1/2}$$

$$\Rightarrow$$

$$\bar{\Phi}_P(I)^{1/2}(\bar{\Psi}_P|S)\bar{\Phi}_P(I)^{1/2} = t_P\Phi \qquad (0 \leq t_P \leq 1)$$

=>

$$\bar{\Phi}_P(I) = t_P I$$

=>

$$\bar{R}^* P \bar{R} = t_P I.$$

14.14 <u>REMARK</u> This argument is completely general: Any pure $\Phi \in \text{CPU}(S, \mathcal{B}(K))$ admits a pure extension $\Phi \in \text{CPU}(\mathcal{B}(H), \mathcal{B}(K))$.

14.15 <u>EXAMPLE</u> Let $S \subset M_n(\underline{C})$ be an operator system and suppose that $\Phi \in \text{CPU}(S, M_m(\underline{C}))$ is pure -- then $m \leq n$ and $\exists$ an isometry $V: \underline{C}^m \to \underline{C}^n$ such that

$$\Phi(A) = V^* A V \quad (A \in S).$$

Thus choose $\bar{\Phi}: M_n(\underline{C}) \to M_m(\underline{C})$ per 14.13: $\bar{\Phi} = \bar{R}^* \bar{\pi} \bar{R}$. But, being topologically irreducible, $\bar{\pi}$ is unitarily equivalent to the identity representation:

$$\bar{\pi}(A) = U^* A U \quad (A \in M_n(\underline{C})).$$

Therefore

$$\bar{\Phi}(A) = \bar{R}^* U^* A U \bar{R} \quad (A \in M_n(\underline{C})).$$

And $V = U\bar{R}$ is an isometry from $\underline{C}^m \to \underline{C}^n$ ($\Rightarrow m \leq n$).

Let $\Phi \in \text{CPU}(M_n(\underline{C}), M_n(C))$ -- then, in the notation of 14.1,

$$\Phi(A) = \sum_i V_i^* A V_i \quad (A \in M_n(\underline{C})),$$

where

$$\sum_i V_i^* V_i = I.$$

Write

$$\mathrm{FIX}_{\Phi} = \{A:\Phi(A) = A\}$$

and let

$$V = \{V_i, V_i^*\}.$$

Then

$$V' \subset \mathrm{FIX}_{\Phi}.$$

Note too that

$$A \in \mathrm{FIX}_{\Phi} \Rightarrow A^* \in \mathrm{FIX}_{\Phi}.$$

14.16 <u>REMARK</u> FIX_{Φ} is an operator system. However, in general, FIX_{Φ} is not an algebra. E.g.: Define $\Phi:M_3(\underline{C}) \to M_3(\underline{C})$ by

$$\Phi\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & \frac{a_{11} + a_{22}}{2} \end{bmatrix}.$$

Then

$$\mathrm{FIX}_{\Phi} = \{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{a + b}{2} \end{bmatrix} : a,b \in \underline{C}\}$$

contains no nontrivial subalgebras.

[Note: One choice for the V_i is

$$V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix}.$$

Here $\mathcal{V}' = \underline{C}I$.]

14.17 LEMMA The following conditions are equivalent: (i) $\text{FIX}_\Phi = \mathcal{V}'$; (ii) FIX_Φ is a unital $*$-algebra; (iii) $A \in \text{FIX}_\Phi \Rightarrow A^*A \in \text{FIX}_\Phi$.

PROOF It is clear that (i) => (ii) and (ii) => (iii). So assume (iii) is in force and let $A \in \text{FIX}_\Phi$:

$$\begin{aligned} 0 &\le \sum_i [A,V_i][A,V_i]^* \\ &= AA^* + \Phi(AA^*) - \Phi(A)A^* - A\Phi(A^*) \\ &= AA^* + AA^* - AA^* - AA^* \\ &= 0 \end{aligned}$$

$$\Rightarrow$$

$$[A,V_i] = 0 \ (\forall\ i).$$

Analogously,

$$[A,V_i^*] = 0 \ (\forall\ i).$$

Therefore $A \in \mathcal{V}'$, which implies that FIX_Φ is contained in $\mathcal{V}'$, hence is equal to $\mathcal{V}'$.

14.18 EXAMPLE Let $\mathcal{S} \subset M_n(\underline{C})$ be an irreducible operator system. Suppose that $\Phi \in CPU(M_n(\underline{C}), M_n(\underline{C}))$ has the property that $FIX_\Phi \supset \mathcal{S}$ -- then it can be shown that

$$FIX_\Phi = M_n(\underline{C}),$$

thus, as a corollary, the identity map on $\mathcal{S}$ is pure.

APPENDIX

The following result is a "pure analog" of the lemma in the Appendix to §12.

LEMMA Suppose that $\Phi \in CP(\mathcal{S}, M_n(\underline{C}))$ is pure -- then $\exists\ m \leq n$, a pure element $\Psi \in CPU(\mathcal{S}, M_m(\underline{C}))$, and a linear map $\gamma : \underline{C}^n \to \underline{C}^m$ such that

$$\Phi(A) = \gamma^* \Psi(A) \gamma \qquad (A \in \mathcal{S}).$$

PROOF Put $T = \Phi(I)$. Assuming that $T \neq 0$, let m be the dimension of the range of T (thus $m \leq n$) and write $\underline{C}^n = \text{Ker } T \oplus \text{Ran } T$. Denote the orthogonal projection of $\underline{C}^n$ onto Ran T by P -- then $P \in \Phi(\mathcal{S})'$ (exercise), so

$$\begin{aligned} \Phi &= P\Phi P + (I-P)\Phi(I-P) \\ &\geq P\Phi P. \end{aligned}$$

Since $P\Phi P$ is completely positive and Φ is pure, $\exists\ t \in [0,1]$:

$$P\Phi P = t\Phi$$

$$\Rightarrow$$

$$P\Phi(I)P = t\Phi(I) \Rightarrow t = 1$$

$$\Rightarrow$$

$$P\Phi P = \Phi.$$

With respect to the decomposition $\underline{C}^n = \text{Ker } T \oplus \text{ Ran } T$, $T = 0 \oplus R$, where $R:\text{Ran } T \to \text{Ran } T$ is selfadjoint and has strictly positive eigenvalues. Set

$$\begin{cases} T_+ = 0 \oplus R^{1/2} \\ T_- = 0 \oplus R^{-1/2}. \end{cases}$$

Then

$$PT_+ = T_+P,\ PT_- = T_-P$$

and

$$T_-TT_- = P.$$

Fix an isometry $W:\underline{C}^m \to \underline{C}^n$ for which $\text{Ran } W = \text{Ran } T$ (hence $P = WW^*$, $I = W^*W$). Define $\Psi \in CP(S, M_m(\underline{C}))$ by

$$\Psi = W^*T_-\Phi T_-W.$$

Then

$$\Psi(I) = W^*T_-\Phi(I)T_-W$$

$$= W^*T_-TT_-W$$

$$= W^*PW$$

$$= W^*WW^*W$$

$$= II = I$$

$\Rightarrow$

$$\Psi \in CPU(S, M_m(\underline{C})).$$

We claim next that Ψ is pure. To check this, suppose that $\Psi \geq \Psi'$, thus

$$W\Psi W^* \geq W\Psi' W^*.$$

But

$$W\Psi W^* = WW^*T_-\Phi T_-WW^*$$

$$= PT_-\Phi T_-P$$

$$= T_-P\Phi PT_-$$

$$= T_-\Phi T_-$$

$\Rightarrow$

$$\Phi \geq T_+W\Psi' W^*T_+$$

$\Rightarrow$

$$T_+W\Psi' W^*T_+ = t\Phi \quad (0 \leq t \leq 1)$$

$\Rightarrow$

$$W\Psi' W^* = tT_-\Phi T_-$$

$\Rightarrow$

$$\Psi' = I\Psi' I$$

$$= W^*W\Psi' W^*W$$

$$= W^*(tT_-\Phi T_-)W$$

$$= t(W^*T_-\Phi T_-W)$$

$$= t\Psi.$$

Therefore Ψ is pure. Finally, let $\gamma = W^*T_+$:

$$\gamma^*\Psi\gamma = T_+W\Psi W^*T_+$$

$$= T_+WW^*T_-\Phi T_-WW^*T_+$$

$$= T_+PT_-\Phi T_-PT_+$$

$$= T_+T_-P\Phi PT_-T_+$$

$$= P\Phi P$$

$$= \Phi.$$

§15. *COMPLETELY BOUNDED MAPS*

Let H be a complex Hilbert space -- then an <u>operator space</u> is a linear subspace $S \subset B(H)$. In particular: Every operator system is an operator space.

15.1 <u>EXAMPLE</u> If H_1 and H_2 are complex Hilbert spaces, then $B(H_1,H_2)$ is an operator space.

[The arrow

$$A \to \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$$

is an isometric embedding of $B(H_1,H_2)$ into $B(H_1 \oplus H_2)$.]

15.2 <u>REMARK</u> Every Banach space A "is" an operator space. To see this, equip its dual with the weak* topology and let X be the closed unit ball -- then X is a compact Hausdorff space and there is an isometric embedding $A \to C(X)$ $(\subset B(\ell_2(X)))$.

The notion of "completely bounded linear map" between operator systems carries over without change to operator spaces.

So let S and T be operator spaces and denote by $CB(S,T)$ the set of completely bounded $\Phi:S \to T$ -- then

$$||\Phi|| \leq ||\Phi||_{cb} \Rightarrow CB(S,T) \subset B(S,T).$$

And, when equipped with the cb-norm, $CB(S,T)$ becomes a normed linear space.

15.3 <u>LEMMA</u> Suppose that T is complete -- then $CB(S,T)$ is complete.

15.4 <u>THEOREM</u> (Wittstock) Let $S \subset B(H)$ be an operator space, K a complex Hilbert space. Suppose that $\Phi:S \to B(K)$ is completely bounded -- then Φ admits a completely bounded extension $\Psi:B(H) \to B(K)$ with $||\Phi||_{cb} = ||\Psi||_{cb}$.

This result is in the Hahn-Banach mode (cf. 12.22 for its completely positive analog) and various proofs are known, one of which will be detailed below.

15.5 <u>THEOREM</u> Let $S \subset B(H)$ be an operator space, K a complex Hilbert space. Suppose that $\Phi:S \to B(K)$ is completely bounded -- then $\exists$ a complex Hilbert space X, a unital $*$-representation $\pi:B(H) \to B(X)$, and operators $R_1,R_2 \in B(K,X)$ such that

$$\Phi(A) = R_1^*\pi(A)R_2 \quad (A \in S).$$

Moreover,

$$||R_1|| \ ||R_2|| \leq ||\Phi||_{cb}.$$

[Note: Therefore

$$||R_1|| \ ||R_2|| = ||\Phi||_{cb}.$$

In fact, $||\pi||_{cb} \leq 1$, so

$$||\Phi||_{cb} \leq ||R_1|| \ ||R_2||.]$$

Let us grant 15.5 for the moment and put

$$\Psi(A) = R_1^*\pi(A)R_2 \quad (A \in B(H)).$$

Then Ψ extends Φ and $||\Phi||_{cb} = ||\Psi||_{cb}$, from which 15.4.

We shall now turn to the proof of 15.5, which requires some preparation.

Let E be a vector space over $\underline{R}$ -- then a function $p:E \to \underline{R}$ is <u>sublinear</u> if $\forall$ a,b $\in$ E, $p(a+b) \leq p(a) + p(b)$, and $\forall$ a $\in$ E, $\forall$ $t \geq 0$, $p(ta) = tp(a)$.

15.6 <u>RAPPEL</u> If $p:E \to \underline{R}$ is sublinear, then $\exists$ a linear $f:E \to \underline{R}$ such that $\forall$ a $\in$ E, $f(a) \leq p(a)$.

Let E_+ be a cone in E -- then a function $q:E_+ \to \underline{R}$ is <u>superlinear</u> if $\forall$ a,b $\in E_+$, $q(a) + q(b) \leq q(a+b)$ and $\forall$ a $\in E_+$, $\forall$ $t \geq 0$, $q(ta) = q(a)$.

<u>INTERPOLATION PRINCIPLE</u> Let $q:E_+ \to \underline{R}$ be superlinear and let $p:E \to \underline{R}$ be sublinear. Assume: $\forall$ a $\in E_+$, $q(a) \leq p(a)$ -- then $\exists$ a linear $f:E \to \underline{R}$ such that

$$\begin{cases} q(a) \leq f(a) & (a \in E_+) \\ f(a) \leq p(a) & (a \in E). \end{cases}$$

[Put

$$r(a) = \inf\{p(a+b) - q(b):b \in E_+\}.$$

Then r is sublinear, $-p(-a) \leq r(a) \leq p(a)$ for all a $\in$ E, and $r(-b) \leq -q(b)$ for all b $\in E_+$. Consider any linear $f:E \to \underline{R}$ such that $f(a) \leq r(a)$ $(a \in E)$ (cf. 15.6).]

Given an element

$$\tau = \sum_k A_k \otimes y_k$$

in the algebraic tensor product $S \otimes K$ and an element $T \in \mathcal{B}(K,H)$, write

$$T\tau = \sum_k A_k T y_k,$$

thus $T\tau \in H$.

Given finite sequences

$$\begin{cases} \tau_i : i = 1,\ldots,n \in S \otimes K \\ y_i : i = 1,\ldots,n \in K, \end{cases}$$

write $\{\tau_i\} \leq \{y_i\}$ if $\forall\ T \in \mathcal{B}(K,H)$,

$$\sum_i ||T\tau_i||^2 \leq \sum_i ||Ty_i||^2.$$

15.7 <u>LEMMA</u> We have: $\{\tau_i\} \leq \{y_i\}$ iff $\exists$ an element $[A_{ij}] \in M_n(S)$ of norm ≤ 1 such that $\forall\ i = 1,\ldots,n$,

$$\tau_i = \sum_j A_{ij} \otimes y_j.$$

<u>N.B.</u> The map $\Phi:S \rightarrow \mathcal{B}(K)$ induces an arrow $\hat{\Phi}:S \otimes K \rightarrow K$, viz.

$$\hat{\Phi}(\tau) = \hat{\Phi}(\sum_k A_k \otimes y_k)$$

$$= \sum_k \Phi(A_k)y_k.$$

It will be convenient to divide up the proof of 15.5 into steps.

<u>Step 1</u>: It follows from 15.7 that

$$\{\tau_i\} \leq \{y_i\} \Rightarrow \sum_i ||\hat{\Phi}(\tau_i)||^2$$

$$\leq ||\Phi||_{cb}^2 \sum_i ||y_i||^2.$$

Step 2: Let E be the set of all functions $\phi:\mathcal{B}(K,H) \to \underline{R}$ such that for some finite sequence $y_1,\ldots,y_n$ in K and all $T \in \mathcal{B}(K,H)$,

$$|\phi(T)| \leq \sum_i ||Ty_i||^2.$$

Then E is a vector space over $\underline{R}$ and the subset E_+ of positive functions is a cone in E.

Step 3: Define $p:E \to \underline{R}$ by

$$p(\phi) = \inf\{||\Phi||_{cb}^2 \sum_i ||y_i||^2\},$$

where the inf runs over all finite sequences $y_1,\ldots,y_n$ in K such that $\phi(T) \leq \sum_i ||Ty_i||^2$ for all $T \in \mathcal{B}(K,H)$ — then p is sublinear.

Step 4: Define $q:E_+ \to \underline{R}$ by

$$q(\phi) = \sup\{\sum_i ||\hat{\Phi}(\tau_i)||^2\},$$

where the sup runs over all finite sequences $\tau_1,\ldots,\tau_n$ in $S \otimes K$ such that $\sum_i ||T\tau_i||^2 \leq \phi(T)$ for all $T \in \mathcal{B}(K,H)$ — then q is superlinear.

Step 5: Since

$$q(\phi) \leq p(\phi) \quad (\phi \in E_+),$$

the Interpolation Principle implies that $\exists$ a linear $f:E \to \underline{R}$ such that

$$\begin{cases} q(\phi) \leq f(\phi) & (\phi \in E_+) \\ f(\phi) \leq p(\phi) & (\phi \in E). \end{cases}$$

<u>Step 6</u>: Extend f by linearity to a function on the complexification $E + \sqrt{-1}\,E$ and denote it still by f. Write $\mathcal{F}$ for the set consisting of all functions $F:\mathcal{B}(\mathcal{K},\mathcal{H}) \to \mathcal{H}$ such that the map $T \to ||F(T)||^2$ lies in E. Noting that the function $T \to \langle F_1(T), F_2(T)\rangle$ lies in $E + \sqrt{-1}\,E$, put

$$\langle F_1, F_2\rangle = f(\langle F_1(\cdot), F_2(\cdot)\rangle) \qquad (F_1, F_2 \in \mathcal{F}).$$

This prescription defines a pre-inner product on $\mathcal{F}$, hence, in the usual way, leads to a Hilbert space $\mathcal{X}$.

<u>Step 7</u>: Given an element $y \in \mathcal{K}$, define $F_y:\mathcal{B}(\mathcal{K},\mathcal{H}) \to \mathcal{H}$ by $F_y(T) = Ty$ -- then $F_y \in \mathcal{F}$, thus

$$\langle F_y, F_y\rangle \le ||\Phi||_{cb}^2\, ||y||^2,$$

so there is a linear operator $V_1:\mathcal{K} \to \mathcal{X}$ with $||V_1|| \le ||\Phi||_{cb}$ such that $V_1 y$ is the equivalence class of F_y in $\mathcal{X}$.

<u>Step 8</u>: Up to equivalence classes, define a unital *-representation $\pi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{X})$ by

$$\pi(A)F(T) = AF(T).$$

<u>Step 9</u>: $\forall\, A_i \in \mathcal{S}$, $\forall\, y_i \in \mathcal{K}$,

$$||\sum_i \Phi(A_i)y_i||^2 \le f(\phi),$$

where

$$\phi(T) = ||\sum_i A_i F_{y_i}(T)||^2.$$

Consequently,

$$||\sum_i \Phi(A_i)y_i|| \le ||\sum_i \pi(A_i)V_1 y_i||$$

and the recipe

$$V_2\left(\sum_i \pi(A_i)V_1y_i\right) = \sum_i \Phi(A_i)y_i$$

extends to a linear operator $V_2 : X \to K$ of norm ≤ 1.

Step 10: Obviously, $||V_1|| \, ||V_2|| \leq ||\Phi||_{cb}$. And, by construction, $\forall$ A $\in S$,

$$V_2\pi(A)V_1y = \Phi(A)y \quad (y \in K).$$

To complete the proof of 15.5, one has only to change the notation: Let

$$\begin{cases} R_1 = V_2^* \\ R_2 = V_1. \end{cases}$$

15.8 REMARK Unlike the completely positive case, there are no known conditions that force the data to be unique up to unitary equivalence.

15.9 THEOREM Let S be an operator system. Suppose that $\Phi : S \to B(K)$ is completely bounded -- then $\exists$ completely positive maps $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ from S to $B(K)$ such that

$$\Phi = \Phi_1 - \Phi_2 + \sqrt{-1}\,(\Phi_3 - \Phi_4).$$

PROOF Put (cf. 15.5)

$$\begin{cases} \Phi_1 = 4^{-1}(R_2 + R_1)^*\pi(R_2 + R_1) \\ \Phi_2 = 4^{-1}(R_2 - R_1)^*\pi(R_2 - R_1) \end{cases}$$

and

$$\begin{cases} \Phi_3 = 4^{-1}(R_2 + \sqrt{-1}\, R_1)^*\pi(R_2 + \sqrt{-1}\, R_1) \\ \Phi_4 = 4^{-1}(R_2 - \sqrt{-1}\, R_1)^*\pi(R_2 - \sqrt{-1}\, R_1). \end{cases}$$

Let V be a complex vector space -- then $M_n(V)$ is an $M_n(\underline{C})$ bimodule w.r.t. matrix multiplication.

Notation: Given $A \in M_n(V)$, $B \in M_m(V)$, write

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in M_{n+m}(V).$$

Assume now that for each $n \in \underline{N}$, $||\cdot||_n$ is a norm on $M_n(V)$ -- then V is said to be a <u>matricial normed space</u> if

- $\forall\, A \in M_n(V)$ & $\forall\, B \in M_m(V)$:

$$\left|\left| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right|\right|_{n+m} = \max\{||A||_n, ||B||_m\}.$$

- $\forall\, \alpha,\beta \in M_n(\underline{C})$ & $\forall\, A \in M_n(V)$:

$$||\alpha A\beta||_n \le ||\alpha||\ ||A||_n\ ||\beta||.$$

If V,V' are matricial normed spaces and if $\Phi:V \to V'$ is a linear operator, then $\forall$ n, there is an induced arrow $\Phi_n:M_n(V) \to M_n(V')$ and Φ is said to be <u>completely bounded</u> if

$$||\Phi||_{cb} = \sup\{||\Phi_n||:n \in \underline{N}\}$$

is finite. In addition, Φ is a complete isometry provided Φ is invertible and

$$||\Phi||_{cb} = ||\Phi^{-1}||_{cb} = 1.$$

15.10 EXAMPLE Every operator space $S \subset B(H)$ is a matricial normed space. [The norm on $M_n(S)$ is, of course, the norm which it inherits as a subspace of $M_n(B(H)) = B(\overset{n}{\oplus} H)$.]

15.11 THEOREM (Ruan) If E is a matricial normed space, then $\exists$ a complex Hilbert space H, an operator space $S \subset B(H)$, and a complete isometry $\Phi: E \to S$.

[Note: I am going to omit the proof of this result but, in essence, what it says is that a matricial normed space can be regarded as an "abstract operator space".]

Let S and T be operator spaces -- then $\forall$ n, there is a linear identification

$$M_n(CB(S,T)) \approx CB(S,M_n(T))$$

and the cb-norm on the RHS can be transferred to a norm $||\cdot||_n$ on the LHS.

15.12 LEMMA CB(S,T) is a matricial normed space.

§16. *OPERATIONS AND CHANNELS*

Let H be a complex Hilbert space -- then the weak* topology on $B(H)$ is the initial topology determined by the elements of $B(H)_*$, i.e., is the smallest topology for which each $\lambda \in B(H)_*$ is continuous. Accordingly, a function $f:X \to B(H)$ from a topological space X to $B(H)$ equipped with the weak* topology is continuous iff $\forall\ \lambda \in B(H)_*$, the composition $\lambda \circ f:X \to \underline{C}$ is continuous.

16.1 RAPPEL Let $\{A_i:i \in I\}$ be a bounded increasing net in $B(H)_+$ and let A be its supremum -- then

$$A_i \to A \text{ in the strong operator topology}$$

and

$$A_i \to A \text{ in the weak* operator topology.}$$

16.2 LEMMA Let H and K be complex Hilbert spaces. Suppose that $\Phi:B(H) \to B(K)$ is positive -- then Φ is weak* continuous iff for every bounded increasing net $\{A_i:i \in I\}$ in $B(H)_+$,

$$\Phi(\sup_{i\in I} A_i) = \sup_{i\in I} \Phi(A_i).$$

PROOF Assume first that Φ is weak* continuous:

$$A_i \to \sup_{i\in I} A_i \text{ (weak*)} \quad \text{(cf. 16.1)}$$

$$\Rightarrow$$

$$\Phi(A_i) \to \Phi(\sup_{i\in I} A_i) \text{ (weak*).}$$

On the other hand, $\{\Phi(A_i):i \in I\}$ is a bounded increasing net in $\mathcal{B}(K)_+$, hence

$$\Phi(A_i) \to \sup_{i\in I} \Phi(A_i) \text{ (weak*)} \quad \text{(cf. 16.1)}.$$

Therefore

$$\Phi(\sup_{i\in I} A_i) = \sup_{i\in I} \Phi(A_i).$$

Conversely, if Φ has the stated property, then $\forall$ positive $\lambda \in \mathcal{B}(K)_*$, $\lambda \circ \Phi$ is weak* continuous (cf. 2.7). But an arbitrary element of $\mathcal{B}(K)_*$ can be written as a linear combination of four positive elements. So, $\forall\ \lambda \in \mathcal{B}(K)_*$, $\lambda \circ \Phi$ is weak* continuous, thus Φ is weak* continuous.

16.3 EXAMPLE Suppose that $\phi:\underline{L}_1(H) \to \underline{L}_1(H)$ is linear and positive ($T \geq 0 \Rightarrow \phi(T) \geq 0$) -- then ϕ is bounded: $\exists\ C_\phi > 0$ such that

$$||\phi(T)||_1 \leq C_\phi ||T||_1 \quad (T \in \underline{L}_1(H)).$$

Let $\Phi = \phi^*$, thus

$$\Phi:\underline{L}_1(H)^* \to \underline{L}_1(H)^*$$

or still,

$$\Phi:\mathcal{B}(H) \to \mathcal{B}(H) \quad \text{(cf. 1.4)}.$$

Explicated: $\forall\ A \in \mathcal{B}(H)\ \&\ \forall\ T \in \underline{L}_1(H)$,

$$\text{tr}(\Phi(A)T) = \text{tr}(A\phi(T)).$$

Then

$$\Phi(\mathcal{B}(H)_+) \subset \mathcal{B}(H)_+.$$

Moreover, Φ is weak* continuous. For, in the notation of 16.1,

$$\operatorname{tr}(\Phi(A_i)T) = \operatorname{tr}(A_i\phi(T))$$

$$\to \operatorname{tr}((\sup_{i\in I} A_i)\phi(T)) = \operatorname{tr}(\Phi(\sup_{i\in I} A_i)T).$$

I.e.:

$$\Phi(A_i) \to \Phi(\sup_{i\in I} A_i) \quad (\text{weak*}).$$

Meanwhile

$$\Phi(A_i) \to \sup_{i\in I} \Phi(A_i) \quad (\text{weak*}).$$

Therefore

$$\Phi(\sup_{i\in I} A_i) = \sup_{i\in I} \Phi(A_i).$$

And this implies that Φ is weak* continuous (cf. 16.2).

Assume henceforth that H is separable and let K be another complex Hilbert space, which we shall also assume is separable. Let $\pi:\mathcal{B}(H) \to \mathcal{B}(K)$ be a unital $*$-homomorphism, so $||\pi(A)|| \leq ||A||$ $(A \in \mathcal{B}(H))$.

16.4 <u>RAPPEL</u> There is an orthogonal decomposition

$$K = K_0 \oplus \bigoplus_{k\in\kappa} K_k,$$

where K_0 and K_k are π-invariant.

• The restriction π_0 of π to K_0 annihilates the elements of $\underline{L}_\infty(H)$, thus π_0 is actually a representation of the quotient $\mathcal{B}(H)/\underline{L}_\infty(H)$.

• The restriction of π_k of π to K_k is unitarily equivalent to the standard representation of $\mathcal{B}(H)$ on H, so $\forall$ k, $\exists$ a unitary operator $U_k:H \to K_k$ such that

$\forall\, A \in \mathcal{B}(H)$,

$$A = U_k^* \pi(A) U_k.$$

[Note: The index set κ is at most countable.]

Suppose that $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ is completely positive and let $\Phi = \Delta_R \pi$ be its canonical decomposition.

[Note: Recall that $\pi:\mathcal{B}(H) \to \mathcal{B}(K)$ is a unital $*$-homomorphism and K is necessarily separable (even finite dimensional if this is the case of H).]

Denote by P_0 the orthogonal projection of K onto K_0 and by P_k the orthogonal projection of K onto K_k. Put

$$\begin{cases} V_0 = P_0 R \quad (H \to K_0) \\ V_k = U_k^* P_k R \quad (H \to K_k \to H). \end{cases}$$

If κ is finite, matters are straightforward: $\forall\, A \in \mathcal{B}(H)$,

$$\begin{aligned} \Phi(A) &= R^* \pi(A) R \\ &= R^*(P_0 \pi(A) P_0 + \sum_{k\in\kappa} P_k \pi(A) P_k) R \\ &= R^* P_0 \pi(A) P_0 R + \sum_{k\in\kappa} R^* P_k U_k A U_k^* P_k R \\ &= V_0^* \pi_0(A) V_0 + \sum_{k\in\kappa} V_k^* A V_k. \end{aligned}$$

<u>N.B.</u> Take H finite dimensional and recover 14.1 ($n = m$ & $\pi_0 \equiv 0$).

The situation when κ is countable, say $\kappa = \{1,2,\ldots\}$, is more complicated because there are issues of convergence.

16.5 LEMMA $\forall A \in \mathcal{B}(\mathcal{H})$, the series

$$\sum_{k=1}^{\infty} V_k^* A V_k$$

is weak* convergent.

PROOF It suffices to establish this for an effect $E \in \mathcal{E}(\mathcal{H})$, thus $0 \le E \le I$. Put

$$E_n = \sum_{k \le n} V_k^* E V_k.$$

Then

$$E_n \in \mathcal{B}(\mathcal{H})_+ \text{ and } E_n \le E_{n+1}.$$

In addition, $\forall x \in \mathcal{H}$,

$$\begin{aligned} \langle x, V_k^* E V_k x\rangle &= \langle V_k x, E V_k x\rangle \\ &\le \langle V_k x, V_k x\rangle \\ &= \langle x, V_k^* V_k x\rangle \end{aligned}$$

$\Rightarrow$

$$V_k^* E V_k \le V_k^* V_k$$

$\Rightarrow$

$$\begin{aligned} E_n &\le \sum_{k \le n} V_k^* V_k \\ &\le \sum_{k \le n} R^* P_k U_k U_k^* P_k R \\ &= \sum_{k \le n} R^* P_k R \\ &= R^* (\sum_{k \le n} P_k) R \\ &\le R^* R. \end{aligned}$$

Therefore

$$\lim_{n \to \infty} E_n \text{ (weak*)}$$

exists (cf. 16.1).

[Note: The conclusion of 16.5 is order independent, i.e., if $\kappa = \{k_1, k_2, \ldots\}$ is another enumeration of κ, then

$$\sum_{i \le n} V_{k_i}^* A V_{k_i}$$

converges weak* to

$$\sum_{k=1}^{\infty} V_k^* A V_k.]$$

Let

$$P_{(n)} = P_0 + \sum_{k \le n} P_k.$$

Then

$$P_{(n)} \to I \text{ (weak*)}$$

and $\forall\, A \in \mathcal{B}(\mathcal{H})$,

$$R^* P_{(n)} \pi(A) P_{(n)} R = V_0^* \pi_0(A) V_0 + \sum_{k=1}^{n} V_k^* A V_k.$$

By the above, the RHS converges weak* to

$$V_0^* \pi_0(A) V_0 + \sum_{k=1}^{\infty} V_k^* A V_k$$

and we claim that the LHS converges weak* to

$$R^* \pi(A) R \ (= \Phi(A)).$$

16.6 <u>LEMMA</u> Let $R \in \mathcal{B}(H,K)$ -- then

$$T \in \underline{L}_1(H) \Rightarrow RTR^* \in \underline{L}_1(K)$$

and $\forall\ X \in \mathcal{B}(K)$,

$$\mathrm{tr}(R^*XRT) = \mathrm{tr}(XRTR^*).$$

From the definitions, $P_{(n)}$ commutes with $\pi(A)$, so

$$P_{(n)}\pi(A)P_{(n)} = P_{(n)}\pi(A) \rightarrow \pi(A) \quad (\text{weak*}).$$

In 16.6, take

$$X = P_{(n)}\pi(A)P_{(n)}.$$

Then $\forall\ T \in \underline{L}_1(H)$,

$$\mathrm{tr}(R^*P_{(n)}\pi(A)P_{(n)}RT)$$

$$= \mathrm{tr}(P_{(n)}\pi(A)P_{(n)}RTR^*)$$

$$\rightarrow \mathrm{tr}(\pi(A)RTR^*).$$

But

$$\mathrm{tr}(\pi(A)RTR^*) = \mathrm{tr}(R^*\pi(A)RT) \quad (\text{cf. } 16.6).$$

And this settles the claim, $T \in \underline{L}_1(H)$ being arbitrary.

To recapitulate:

16.7 <u>THEOREM</u> Suppose that $\Phi:\mathcal{B}(H) \rightarrow \mathcal{B}(H)$ is completely positive -- then $\exists$ $V_k \in \mathcal{B}(H)$ $(k \in \kappa)$ such that

$$\Phi(A) = V_0^*\pi_0(A)V_0 + \sum_{k\in\kappa} V_k^*AV_k \quad (A \in \mathcal{B}(H)).$$

[Note: The series converges in the weak* operator topology (and in the strong operator topology).]

We want now to impose an assumption on Φ that, among other things, will serve to eliminate "π_0" from 16.7.

Assumption: Φ is weak* continuous -- then 16.2 is in force, hence

$$\Phi(A_i) \to \Phi(\sup_{i \in I} A_i) \ (= \sup_{i \in I} \Phi(A_i))$$

both weak* and strongly.

16.8 LEMMA Let $\{A_i : i \in I\}$ be a bounded increasing net in $\mathcal{B}(\mathcal{H})_+$ and let A be its supremum -- then $\forall$ X,Y $\in \mathcal{B}(\mathcal{H})$,

$$\Phi(X^*A_iY) \to \Phi(X^*AY)$$

in the strong operator topology.

PROOF Write

$$\begin{aligned} &\Phi(X^*A_iY) \\ &= \frac{1}{4}\Phi((X + Y)^*A_i(X + Y)) \\ &\quad - \frac{1}{4}\Phi((X - Y)^*A_i(X - Y)) \\ &\quad - \frac{\sqrt{-1}}{4}\Phi((X + \sqrt{-1}\,Y)^*A_i(X + \sqrt{-1}\,Y)) \\ &\quad + \frac{\sqrt{-1}}{4}\Phi((X - \sqrt{-1}\,Y)^*A_i(X - \sqrt{-1}\,Y)). \end{aligned}$$

16.9 LEMMA $\pi:\mathcal{B}(H) \to \mathcal{B}(K)$ is weak* continuous.

PROOF Use the notation of the proof of 9.11, replacing μ by A_i, ξ by X, and η by Y to get

$$<\pi(A_i)\hat{f},\hat{g}>$$

$$= \sum_X \sum_Y <\Phi(X^*A_iY)f(Y),g(X)>.$$

Bearing in mind that $\sum_X \sum_Y$ is actually a finite sum, it then follows from 16.8 that

$$\sum_X \sum_Y <\Phi(X^*A_iY)f(Y),g(X)>$$

converges to

$$\sum_X \sum_Y <\Phi(X^*AY)f(Y),g(X)>.$$

I.e.:

$$<\pi(A_i)\hat{f},\hat{g}>$$

converges to

$$<\pi(A)\hat{f},\hat{g}>.$$

Therefore $\pi(A_i)$ converges to $\pi(A)$ weakly (recall that $\hat{F}(\mathcal{B}(H),H)$ is dense in K). But $\{\pi(A_i):i \in I\}$ is an increasing net of positive operators, hence $\pi(A_i) \to \pi(A)$ strongly (cf. 10.1) and

$$\pi(A) = \sup_{i \in I} \pi(A_i).$$

Consequently, π is weak* continuous (cf. 16.2).

[Note: The net $\{\pi(A_i):i \in I\}$ is norm bounded:

$$||\pi(A_i)|| \le ||A_i|| \le ||A||.$$

This said, there is a generality to the effect that if $\{T_\alpha\}$ is a norm bounded net in $\mathcal{B}(\mathcal{K})$ with the property that

$$\langle T_\alpha y, y'\rangle \to \langle Ty, y'\rangle \quad (y, y' \in \mathcal{Y}),$$

where $\mathcal{Y} \subset \mathcal{K}$ is dense (or just total), then $T_\alpha \to T$ weakly.]

Armed with this result, it is then easy to see that $\mathcal{K}_0$ consists of the zero vector alone. In fact, this is automatic if $\mathcal{H}$ is finite dimensional, so take $\mathcal{H}$ infinite dimensional and, to get a contradiction, assume that $\exists\, y_0 \in \mathcal{K}_0 : ||y_0|| = 1$. Put

$$\omega_0(A) = \langle y_0, \pi(A) y_0\rangle \quad (A \in \mathcal{B}(\mathcal{H})).$$

Since π is unital, $\omega_0(I) = 1$. Moreover, ω_0 is weak* continuous. Proof:

$$\omega_0(A) = \mathrm{tr}(P_{y_0} \pi(A)) \quad (A \in \mathcal{B}(\mathcal{H})).$$

Therefore

$$I = \bigvee_{P \in \mathcal{P}(\mathcal{H})} P$$

=>

$$\omega_0(I) = \sum_{P \in \mathcal{P}(\mathcal{H})} \omega_0(P) = 0 \quad (\text{cf. } 2.6 \text{ and } 5.1).$$

16.10 <u>THEOREM</u> (Kraus) Suppose that $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is completely positive and weak* continuous -- then $\exists\, V_k \in \mathcal{B}(\mathcal{H})$ $(k \in \kappa)$ such that

$$\Phi(A) = \sum_{k \in \kappa} V_k^* A V_k \quad (A \in \mathcal{B}(\mathcal{H})).$$

[Note: The series converges in the weak* operator topology (and in the strong operator topology).]

16.11 <u>REMARK</u> For each $T \in \underline{L}_1(H)$, the series

$$\sum_{k \in \kappa} V_k T V_k^*$$

is trace norm convergent. For if $m > n$ and T is positive, then

$$|| \sum_{k \leq m} V_k T V_k^* - \sum_{k \leq n} V_k T V_k^* ||_1$$

$$= || \sum_{k=n+1}^{m} V_k T V_k^* ||_1$$

$$= \mathrm{tr}(\sum_{k=n+1}^{m} V_k T V_k^*)$$

$$= \mathrm{tr}(\sum_{k=n+1}^{m} V_k^* V_k T)$$

$$= \mathrm{tr}((\sum_{k \leq m} V_k^* V_k) T) - \mathrm{tr}((\sum_{k \leq n} V_k^* V_k) T).$$

But $\sum_{k \leq n} V_k^* V_k$ converges weak* (cf. 16.5). Therefore the sequence $\sum_{k \leq n} V_k T V_k^*$ is Cauchy in the trace norm, thus

$$\sum_{k \in \kappa} V_k T V_k^*$$

makes sense (it is order independent). The extension of these considerations to an arbitrary $T \in \underline{L}_1(H)$ is immediate. So, if $\phi:\underline{L}_1(H) \to \underline{L}_1(H)$ is defined by the rule

$$\phi(T) = \sum_{k \in \kappa} V_k T V_k^*,$$

then ϕ is linear and positive. Furthermore, its dual $\phi^*:\underline{L}_1(H)^* \to \underline{L}_1(H)^*$ can be identified with Φ (cf. 16.3).

An operation is a completely positive weak* continuous map $\Phi:\mathcal{B}(H) \rightarrow \mathcal{B}(H)$ subject to the constraint $\Phi(I) \leq I$.

[Note: The weak* continuity is automatic if H is finite dimensional.]

N.B. An operation is contractive:

$$||\Phi|| = ||\Phi(I)|| \leq 1 \quad \text{(cf. 11.27)}.$$

The effect corresponding to an operation Φ is, by definition, $\Phi(I)$.

16.12 EXAMPLE If $E \in \mathcal{E}(H)$, then the assignment $A \rightarrow E^{1/2}AE^{1/2}$ is an operation and the associated effect is $E^{1/2}E^{1/2} = E$.

16.13 LEMMA Let V_k $(k = 1,2,\ldots)$ be a sequence in $\mathcal{B}(H)$. Assume: $\forall$ finite subset $F \subset \underline{N}$,

$$\sum_{k \in F} V_k^* V_k \leq I.$$

Then the prescription

$$\Phi(A) = \sum_{k=1}^{\infty} V_k^* A V_k \quad (A \in \mathcal{B}(H))$$

defines an operation.

16.14 EXAMPLE Suppose that the V_k are positive -- then the Lüders operation associated with the V_k is the prescription

$$A \rightarrow \sum_{k=1}^{\infty} V_k^{1/2} A V_k^{1/2} \quad (A \in \mathcal{B}(H)).$$

A channel is a unital operation $\Phi:\Phi(I) = I$.

16.15 EXAMPLE If $U \in \mathcal{U}(\mathcal{H})$, then the assignment $A \to U^*AU$ is a channel.

[Note: More generally, if $U_1,\ldots,U_n \in \mathcal{U}(\mathcal{H})$ and if $\lambda_1,\ldots,\lambda_n \in [0,1]$ with $\sum_{k=1}^{n} \lambda_k = 1$, then the assignment

$$A \to \sum_{k=1}^{n} \lambda_k U_k^* A U_k$$

ia a channel.]

16.16 EXAMPLE Fix a positive trace class operator $T \neq 0$ — then the assignment

$$A \to \frac{\text{tr}(AT)}{\text{tr}(T)} I$$

ia a channel.

16.17 EXAMPLE Take $\mathcal{H} = \underline{C}^n$ and let $\mathcal{U}(n) = \mathcal{U}(\underline{C}^n)$ — then the assignment

$$A \to \int_{\mathcal{U}(n)} U^*AU dU$$

ia a channel, call it Φ_{av}.

[Note: Here, dU is normalized Haar measure on $\mathcal{U}(n)$. Since $\Phi_{av}(A)U_0 = U_0\Phi_{av}(A)$ for all $U_0 \in \mathcal{U}(n)$, $\exists$ a linear functional $\lambda:\mathcal{B}(\mathcal{H}) \to \underline{C}$ such that

$$\Phi_{av}(A) = \lambda(A) I.$$

But

$$\text{tr}(A) = \text{tr}(\Phi_{av}(A))$$

$$= \lambda(A)\mathrm{tr}(I)$$

$$= \lambda(A)n$$

=>

$$\Phi_{av}(A) = \frac{\mathrm{tr}(A)}{n} I.]$$

Let $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ be an operation -- then by 16.10,

$$\Phi(A) = \sum_{k\in\kappa} V_k^* A V_k,$$

where

$$\sum_{k\in\kappa} V_k^* V_k \le I,$$

a condition which carries with it some additional structure.

16.18 <u>RAPPEL</u> Let K be a complex Hilbert space -- then $W \in \mathcal{B}(K)$ is a <u>coisometry</u> provided $WW^* = I$.

16.19 <u>LEMMA</u> Suppose that $T \in \mathcal{B}(H)$ is a contraction -- then $\exists$ a complex Hilbert space K_T containing H as a closed subspace and a coisometry $W \in \mathcal{B}(K_T)$ such that H is W-invariant and $W|H = T$.

16.20 <u>THEOREM</u> $\exists$ a complex Hilbert space K containing H as a closed subspace and coisometries $W_k \in \mathcal{B}(K)$ $(k \in \kappa)$ such that $W_k W_\ell^* = 0$ $(k \ne \ell)$ with $W_k H \subset H$ and $W_k|H = V_k$ $\forall$ k.

<u>PROOF</u> Let $H_\kappa = \overset{\kappa}{\oplus} H$ and define $T \in \mathcal{B}(H_\kappa)$ by

$$T(x_1,x_2,\ldots) = (V_1x_1,V_2x_2,\ldots).$$

Then T is a contraction, so 16.19 secures $K_T \supset H_\kappa$ and a coisometry $W \in B(K_T)$ such that H_κ is W-invariant and $W|H_\kappa = T$. Write $K_T = H_\kappa \oplus H_\kappa^\perp$ -- then w.r.t. this decomposition of K_T, the matrix of W is

$$\begin{bmatrix} V_1 & 0 & 0 & \ldots & X_1 \\ V_2 & 0 & 0 & \ldots & X_2 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & Y \end{bmatrix}.$$

Here $X_k:H_\kappa^\perp \to H$ and $Y:H_\kappa^\perp \to H_\kappa^\perp$. Because $WW^* = I$, we have

$$V_kV_\ell^* + X_kX_\ell^* = \delta_{k\ell}I.$$

In addition,

$$\begin{cases} X_kY^* = 0 \\ YY^* = I. \end{cases}$$

If $Y^*Y = I$, then all $X_k = 0$ and the V_k are coisometries with orthogonal initial spaces, thus matters are trivial. Otherwise, $Y^*Y \neq I$ and there exists a collection $Z_k \in B(H_\kappa^\perp)$ of coisometries which have orthogonal initial spaces. Let $K = H \oplus H_\kappa^\perp$ and define $W_k \in B(K)$ by

$$W_k = \begin{bmatrix} V_k & X_k \\ & \\ 0 & Z_kY \end{bmatrix}.$$

Then the W_k are coisometries with the stated properties.

Since W_k is a coisometry, $W_k^*W_k$ is an orthogonal projection. On the other hand,

$$W_k^*W_kW_\ell^*W_\ell = 0 \quad (k \neq \ell).$$

Therefore

$$\sum_{k\in\kappa} W_k^*W_k \in L(K).$$

Define now a map $\Theta:\mathcal{B}(K) \to \mathcal{B}(K)$ by stipulating that

$$\Theta(B) = \sum_{k\in\kappa} W_k^*BW_k \quad (B \in \mathcal{B}(K)).$$

16.21 <u>LEMMA</u> Θ is a $*$-endomorphism.

Given $x,y \in H$ and $A \in \mathcal{B}(H)$, we have

$$\begin{aligned}
<x,P_H\Theta(AP_H)y> &= <P_Hx, \sum_{k\in\kappa} W_k^*AP_HW_ky> \\
&= \sum_{k\in\kappa} <W_kx,AW_ky> \\
&= \sum_{k\in\kappa} <V_kx,AV_ky> \\
&= <x, \sum_{k\in\kappa} V_k^*AV_ky> \\
&= <x,\Phi(A)y>
\end{aligned}$$

$\Rightarrow$

$$\Phi(A) = P_H\Theta(AP_H)|H.$$

Next

$$\langle x, P_{H}\Theta^{2}(AP_{H})y\rangle = \langle x, P_{H}\Theta(\Theta(AP_{H}))y\rangle$$

$$= \langle P_{H}x, \sum_{\ell\in\kappa} W_{\ell}^{*}\Theta(AP_{H})W_{\ell}y\rangle$$

$$= \sum_{\ell\in\kappa} \langle W_{\ell}x, \Theta(AP_{H})W_{\ell}y\rangle$$

$$= \sum_{\ell\in\kappa} \langle V_{\ell}x, \Theta(AP_{H})V_{\ell}y\rangle$$

$$= \sum_{\ell\in\kappa} \langle V_{\ell}x, \sum_{k\in\kappa} W_{k}^{*}AP_{H}W_{k}V_{\ell}y\rangle$$

$$= \sum_{k\in\kappa}\sum_{\ell\in\kappa} \langle W_{k}V_{\ell}x, AP_{H}W_{k}V_{\ell}y\rangle$$

$$= \sum_{k\in\kappa}\sum_{\ell\in\kappa} \langle V_{k}V_{\ell}x, AV_{k}V_{\ell}y\rangle$$

$$= \langle x, \sum_{k\in\kappa}\sum_{\ell\in\kappa} V_{\ell}^{*}V_{k}^{*}AV_{k}V_{\ell}y\rangle.$$

On the other hand,

$$\langle x, \Phi^{2}(A)y\rangle = \langle x, \Phi(\Phi(A))y\rangle$$

$$= \langle x, \Phi(\sum_{k\in\kappa} V_{k}^{*}AV_{k})y\rangle$$

$$= \langle x, \sum_{k\in\kappa} \Phi(V_{k}^{*}AV_{k})y\rangle$$

$$= \langle x, \sum_{k\in\kappa}\sum_{\ell\in\kappa} V_{\ell}^{*}V_{k}^{*}AV_{k}V_{\ell}y\rangle.$$

Therefore

$$\Phi^{2}(A) = P_{H}\Theta^{2}(AP_{H})|H.$$

16.22 <u>THEOREM</u> Suppose that $\Phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is an operation -- then $\exists$ a complex Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and a *-endomorphism $\Theta:\mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ such that $\forall$ $n \in \underline{N}$,

$$\Phi^n(A) = P_{\mathcal{H}}\Theta^n(AP_{\mathcal{H}})|\mathcal{H} \quad (A \in \mathcal{B}(\mathcal{H})).$$

[This follows from the foregoing by iteration.]

Although I shall omit the details, it should be mentioned that it is possible to rework 16.20 so as to ensure that

$$\sum_{k\in\kappa} V_k^* V_k = I \Rightarrow \sum_{k\in\kappa} W_k^* W_k = I.$$

Consequently, if Φ is a channel, then Θ is unital and

$$I = \Phi(I) = P_{\mathcal{H}}\Theta(P_{\mathcal{H}})|\mathcal{H}.$$

§17. *SEMIGROUP THEORY*

This is a vast subject, a small portion of which will be reviewed below.

Let X be a complex Banach space -- then initially, a (one parameter) semigroup on X is just a collection of bounded linear operators $T_t:X \rightarrow X$ parameterized by $t \geq 0$ such that

$$\begin{cases} T_0 = I \\ T_s T_t = T_{s+t} \quad (s,t \geq 0). \end{cases}$$

17.1 <u>EXAMPLE</u> The "trivial semigroup" is the prescription $T_t = I$ $(t \geq 0)$.

<u>N.B.</u> Typically, X is a complex Hilbert space $\mathcal{H}$ or a unital C*-algebra $\mathcal{A}$ (e.g., $\mathcal{B}(\mathcal{H})$).

17.2 <u>EXAMPLE</u> Take $X = \underline{C}$ -- then the prescription $T_t z = e^{-t} z$ $(t \geq 0)$ defines a semigroup and $T_t 1 \neq 1$ $\forall$ $t > 0$.

17.3 <u>EXAMPLE</u> Take $X = \underline{C}^2$ -- then the prescription

$$T_t \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{z_1 + z_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \frac{z_1 - z_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (t \geq 0)$$

defines a semigroup and

$$T_t \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} \quad \forall\ t > 0.$$

Starting with $\{T_t : t \geq 0\}$, one then proceeds to impose various continuity conditions.

First, consider the following topologies on $\mathcal{B}(X)$.

The norm topology: A net T_i $(i \in I)$ in $\mathcal{B}(X)$ converges to T iff $||T_i - T|| \to 0$.

The strong topology: A net T_i $(i \in I)$ in $\mathcal{B}(X)$ converges to T iff $\forall\ x \in X$, $||T_i x - Tx|| \to 0$.

The weak topology: A net T_i $(i \in I)$ in $\mathcal{B}(X)$ converges to T iff $\forall\ \lambda \in X^*$ & $\forall\ x \in X$, $\lambda(T_i x - Tx) \to 0$.

Returning to our semigroup $\{T_t : t \geq 0\}$, one says that it is norm continuous, strongly continuous, or weakly continuous according to whether the map

$$\begin{cases} \underline{R}_{\geq 0} \to \mathcal{B}(X) \\ \quad t \to T_t \end{cases}$$

is continuous when $\mathcal{B}(X)$ is endowed with the norm topology, strong topology, or weak topology, respectively.

17.4 LEMMA A semigroup $\{T_t : t \geq 0\}$ is strongly continuous iff it is weakly continuous.

Because of this, it suffices to consider strongly continuous semigroups only (norm continuous semigroups being a special case).

17.5 EXAMPLE Take $X = C[0,1]$ (sup norm). Put $T_0 = I$ and for $t > 0$, write

$$(T_t f)(x) = x^t f(x) - x^t \log x \, f(0) \quad (0 < x \leq 1),$$

letting $(T_t f)(0) = 0$ -- then $\{T_t : t \geq 0\}$ is strongly continuous on $]0,\infty[$ but is not strongly continuous on $[0,\infty[$. In fact,

$$\lim_{t \to 0} ||T_t|| = \infty.$$

17.6 LEMMA Suppose that $\{T_t : t \geq 0\}$ is strongly continuous -- then $\exists$ constants $a \in \underline{R}$ and $M \geq 1$ such that

$$||T_t|| \leq Me^{at}$$

for all $t \geq 0$.

PROOF $\forall \; x \in X$, the function

$$t \to T_t x \quad (0 \leq t \leq 1)$$

is in $C([0,1],X)$, hence

$$\sup_{0 \leq t \leq 1} ||T_t x|| < \infty.$$

Therefore, by the uniform boundedness principle, $\exists \; M \geq 1$:

$$||T_t|| \leq M \; (0 \leq t \leq 1) \;\; (T_0 = T \Rightarrow M \geq 1).$$

If now $a = \log M$ and if for a given $t > 0$, n is the least integer $\geq t$, then

$$||T_t|| = ||(T_{t/n})^n|| \le M^n \le M^{t+1} = Me^{at}.$$

17.7 <u>EXAMPLE</u> The function $t \to ||T_t||$ need not be bounded. E.g.: Take $X = \underline{C}^2$ and let

$$T_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Then

$$\lim_{t \to \infty} ||T_t|| = \infty.$$

17.8 <u>EXAMPLE</u> The function $t \to ||T_t||$ need not be continuous. E.g.: Take $X = L^2[0,1]$ and let

$$(T_t f)(s) = \begin{cases} f(s + t) & (0 \le s + t \le 1) \\ 0 & (s + t > 1). \end{cases}$$

Then

$$\begin{cases} ||T_t|| = 1 & (0 \le t < 1) \\ T_t = 0 & (t > 1). \end{cases}$$

17.9 <u>LEMMA</u> A semigroup $\{T_t : t \ge 0\}$ is strongly continuous iff $\forall\, x \in X$,

$$\lim_{t \to \infty} T_t x = x.$$

<u>PROOF</u> The necessity is obvious. To establish the sufficiency, fix $t_0 > 0$ and

let $x \in X$ — then

$$\lim_{h \downarrow 0} ||T_{t_0+h}x - T_{t_0}x||$$

$$= \lim_{h \downarrow 0} ||T_{t_0}(T_h x - x)||$$

$$\leq ||T_{t_0}|| \lim_{h \downarrow 0} ||T_h x - x|| = 0,$$

from which continuity on the right. As for continuity on the left, take $h < 0$ and write

$$||T_{t_0+h}x - T_{t_0}x||$$

$$= ||T_{t_0+h}(x - T_{-h}x)||$$

$$\leq ||T_{t_0+h}|| \, ||T_{-h}x - x||$$

$$\leq C||T_{-h}x - x||,$$

where

$$||T_t|| \leq C \ (0 \leq t \leq t_0) \ \ (\text{cf. } 17.6).$$

17.10 <u>LEMMA</u> Suppose that $\{T_t : t \geq 0\}$ is strongly continuous — then the map

$$\begin{cases} [0,\infty[\times X \to X \\ \quad (t,x) \to T_t x \end{cases}$$

is jointly continuous.

PROOF Take $s \geq t$ and note that

$$||T_s y - T_t x|| \leq ||T_s(y - x)|| + ||(T_s - T_t)x||$$

$$\leq ||T_s|| \, ||y - x|| + ||T_t|| \, ||T_{s-t}x - x||.$$

17.11 RAPPEL Assuming that the semigroup $\{T_t : t \geq 0\}$ is strongly continuous, let Dom L be the set of all $x \in X$ for which

$$\lim_{t \to 0} \frac{T_t x - x}{t}$$

exists and define L on Dom L by the equality

$$Lx = \lim_{t \to 0} \frac{T_t x - x}{t}.$$

Then Dom L is a dense linear subspace of X and L is a closed linear operator (unbounded in general). In addition,

$$x \in \text{Dom } L \Rightarrow T_t x \in \text{Dom } L$$

and

$$\frac{d}{dt} T_t x = LT_t x = T_t Lx.$$

Finally, $\forall\, x \in X$ & $\forall\, t > 0$,

$$\int_0^t T_s x \, ds \in \text{Dom } L$$

and

$$T_t x - x = L\left(\int_0^t T_s x \, ds\right)$$

$$= \int_0^t T_s Lx \, ds \text{ if } x \in \text{Dom } L.$$

[Note: L is called the generator of the semigroup $\{T_t : t \geq 0\}$.]

17.12 REMARK As a complement to 17.4, let w-lim stand for limit in the weak topology on X -- then it can be shown that the set of X such that

$$\underset{t \to 0}{\text{w-lim}} \frac{T_t x - x}{t}$$

exists coincides with the set of $x \in X$ such that

$$\lim_{t \to 0} \frac{T_t x - x}{t}$$

exists and the linear operators defined thereby are identical. In brief:

"weak generator" = "generator".

17.13 LEMMA If $\{T'_t : t \geq 0\}$ and $\{T''_t : t \geq 0\}$ are two strongly continuous semigroups with the same generator L, then $T'_t = T''_t \ \forall \ t \geq 0$.

PROOF Fix $t > 0$, let $x \in \text{Dom } L$, and define $f:[0,t] \to X$ by

$$f(s) = T'_{t-s} T''_s x \quad (0 \leq s \leq t).$$

Then

$$\begin{cases} f(0) = T'_t x \\ f(t) = T''_t x. \end{cases}$$

On the other hand,

$$\begin{aligned} \frac{d}{ds} f(s) &= - T'_{t-s} L T''_s x + T'_{t-s} L T''_s x \\ &= 0. \end{aligned}$$

Therefore f(s) is constant on [0,t], so $T_t' = T_t''$ on Dom L. But Dom L is dense in X, thus $T_t' = T_t''$.

Given $L \in \mathcal{B}(X)$, put

$$e^{tL} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n.$$

Then the series on the RHS is norm convergent and the assignment $t \to e^{tL}$ defines a norm continuous semigroup with L as its generator.

17.14 <u>EXAMPLE</u> Let $X = \underset{\sim}{C}^2$ -- then

$$L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow e^{tL} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow e^{tL} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow e^{tL} = \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}.$$

17.15 <u>LEMMA</u> Every norm continuous semigroup $\{T_t : t \geq 0\}$ is of the form

$$T_t = e^{tL} \quad (t \geq 0)$$

for some bounded operator $L \in \mathcal{B}(X)$.

[Note: We have

$$\lim_{t \to 0} \left|\left|\frac{T_t - I}{t} - L\right|\right| = 0.]$$

17.16 <u>EXAMPLE</u> Let $X = \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ a complex Hilbert space). Fix $H \in \mathcal{B}(\mathcal{H})_{SA}$ and put

$$T_t A = e^{\sqrt{-1}\ tH} A e^{-\sqrt{-1}\ tH} \quad (t \geq 0).$$

Then $\{T_t : t \geq 0\}$ is norm continuous and its generator L is given by

$$LA = \sqrt{-1}\ [H,A].$$

17.17 <u>REMARK</u> There exist strongly continuous semigroups $\{T_t : t \geq 0\}$ such that the series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L^n x$$

converges only for $t = 0$ or $x = 0$.

17.18 <u>THEOREM</u> Suppose that $\{T_t : t \geq 0\}$ is strongly continuous with generator L such that

$$||T_t|| \leq Me^{at}$$

for some $a \in \underline{R}$ and $M \geq 1$ (cf. 17.6). Let $L' \in \mathcal{B}(X)$ and specify that

$$\text{Dom}(L + L') = \text{Dom}\ L.$$

Then $L + L'$ is the generator of a strongly continuous semigroup $\{T'_t : t \geq 0\}$ for which

$$||T'_t|| \leq M \exp(a + M||L'||)t \quad (t \geq 0).$$

And in norm,

$$T'_t x = \lim_{n \to \infty} (T_{t/n} e^{(t/n)L'})^n x \quad (x \in X).$$

[Note: The "bounded perturbation" L + L' is necessarily closed. If L' is not bounded, then L + L' need not be a generator (e.g. take L unbounded and consider L + (- L)...).]

N.B. There are integral equations

$$\begin{cases} T'_t x = T_t x + \int_0^t T_{t-s} L' T'_s x ds \\ T'_t x = T_t x + \int_0^t T'_s L' T_{t-s} x ds \end{cases} \quad (x \in X).$$

17.19 LEMMA Maintain the notation of 17.18 -- then $\exists$ C > 0 such that

$$||T_t - T'_t|| \le tC \quad (0 \le t \le 1).$$

PROOF In fact, $\forall$ x $\in$ X,

$$||T_t x - T'_t x|| = ||\int_0^t T_{t-s} L' T'_s x ds||$$

$$\le \int_0^t ||T_{t-s} L' T'_s x|| ds$$

$$\le t \sup_{0 \le u \le 1} ||T_u|| \sup_{0 \le u' \le 1} ||T'_{u'}|| \, ||L'|| \, ||x||.$$

[Note: One can construct examples of strongly continuous semigroups

$$\begin{cases} \{T^1_t : t \ge 0\} \\ \{T^2_t : t \ge 0\} \end{cases}$$

such that

$$||T_t^1 - T_t^2|| \le Ct \quad (0 \le t \le 1),$$

yet

$$\text{Dom } L^1 \cap \text{Dom } L^2 = \{0\},$$

thus L^1 and L^2 do not differ by an element of $\mathcal{B}(X)$.]

17.20 <u>EXAMPLE</u> Let $X = \underline{c}_0$ (the elements of X are all sequences $\{c_n\}$ $(c_n \in \underline{C})$ such that $\lim_{n \to \infty} c_n = 0$, equipped with the sup norm). Define a strongly continuous semigroup $\{T_t : t \ge 0\}$ on X by

$$T_t\{c_n\} = \{e^{\sqrt{-1}\, nt} c_n\}.$$

Then

$$L\{c_n\} = \{\sqrt{-1}\, nc_n\},$$

where

$$\text{Dom } L = \{\{c_n\} \in \underline{c}_0 : \{\sqrt{-1}\, nc_n\} \in \underline{c}_0\}.$$

Perturb L by $L_n' \in \mathcal{B}(X)$:

$$L_n'\{c_n\} = \{0,\ldots,0,nc_n,0,\ldots\}.$$

Then $L + L_n'$ is the generator of a strongly continuous semigroup $\{T_{n,t}' : t \ge 0\}$ (cf. 17.18) and $\forall\ x = \{c_n\} \in \text{Dom } L$, we have

$$||(L + L_n')x - Lx|| = ||L_n'x|| = n|c_n| \to 0 \ \ (n \to \infty).$$

I.e.: $L + L_n'$ converges pointwise to L on Dom L. Still,

$$||T_{n,t}'|| \ge e^{nt},$$

so $\forall$ $t > 0$, $\exists$ $x \in X:\{T'_{n,t}x\}$ does not converge (uniform boundedness principle).

If $A \in \mathcal{B}(X)$, then its dual $A^* \in \mathcal{B}(X^*)$:

$$(A^*\lambda)(x) = \lambda(Ax) \quad (x \in X, \lambda \in X^*).$$

And

$$||A^*|| = ||A||.$$

17.21 <u>RAPPEL</u> Let A be a densely defined linear operator on X -- then its dual A^* has for its domain the set of all $\lambda \in X^*$ with the property that the linear functional $x \to \lambda(Ax)$ is norm continuous on Dom A, $A^*\lambda \in X^*$ being, by definition, the extension of this linear functional to X.

- If A is densely defined, then A^* is weak* closed (hence is norm closed).
- If A is closed and densely defined, then A^* is weak* densely defined (as well as weak* closed).

17.22 <u>REMARK</u> If X is reflexive, then the weak and weak* topologies on X^* coincide, so if A is closed and densely defined, then Dom A^* is a norm dense linear subspace of X^*.

<u>N.B.</u> In addition to the norm topology, the strong topology, and the weak topology, $\mathcal{B}(X^*)$ also carries the <u>weak* topology</u>: A net Λ_i $(i \in I)$ in $\mathcal{B}(X^*)$ converges to Λ iff $\forall$ $x \in X$ & $\forall$ $\lambda \in X^*$, $\langle x, \Lambda_i\lambda - \Lambda\lambda\rangle \to 0$.

[Note: Technically, this is the "point weak* topology" but for simplicity, we shall omit the adjective "point" from the terminology.]

Suppose that $\{T_t:t \geq 0\}$ is a strongly continuous semigroup on X with generator L -- then $\{T_t^*:t \geq 0\}$ is a semigroup on X*.

[Note: If $\{T_t:t \geq 0\}$ is norm continuous, then so is $\{T_t^*:t \geq 0\}$:

$$||T_{t_1}^* - T_{t_2}^*|| = ||T_{t_1} - T_{t_2}||.]$$

17.23 <u>LEMMA</u> $\{T_t^*:t \geq 0\}$ is weak* continuous.

<u>PROOF</u> One has only to observe that

$$|\langle x,(T_{t_1}^* - T_{t_2}^*)\lambda\rangle| = |\langle (T_{t_1} - T_{t_2})x,\lambda\rangle|$$

$$\leq ||T_{t_1}x - T_{t_2}x||\ ||\lambda||.$$

If X is reflexive, then it follows from 17.23 that the semigroup $\{T_t^*:t \geq 0\}$ is weakly continuous, hence is strongly continuous (cf. 17.4), but this fails to be true in general.

17.24 <u>EXAMPLE</u> Take $X = C_\infty(\underline{R})$ and define T_t by

$$(T_t f)(x) = f(x + t).$$

Then $T_t^*\delta_x = \delta_{x+t}$ and

$$||T_t^*\delta_x - \delta_x|| = 2 \quad (t \neq 0).$$

Therefore the semigroup $\{T_t^*:t \geq 0\}$ is not strongly continuous (cf. 17.9).

Since L is closed and densely defined, L^* is weak* closed and weak* densely defined.

17.25 <u>LEMMA</u> Dom L^* is a T_t^*-invariant linear subspace of X^* and for all $\lambda \in$ Dom L^*,

$$L^*T_t^*\lambda = T_t^*L^*\lambda.$$

17.26 <u>LEMMA</u> $\forall\ x \in X$ & $\forall\ \lambda \in$ Dom L^*,

$$\langle x, T_t^*\lambda - \lambda\rangle = \int_0^t \langle x, T_s^*L^*\lambda\rangle ds \quad (t > 0).$$

<u>PROOF</u> We have

$$\langle x, T_t^*\lambda - \lambda\rangle = \langle T_t x - x, \lambda\rangle$$

$$= \langle L(\int_0^t T_s x ds), \lambda\rangle \quad \text{(cf. 17.11)}$$

$$= \langle \int_0^t T_s x ds, L^*\lambda\rangle$$

$$= \int_0^t \langle T_s x, L^*\lambda\rangle ds$$

$$= \int_0^t \langle x, T_s^*L^*\lambda\rangle ds.$$

[Note: Analogously, $\forall\ x \in$ Dom L and $\forall\ \lambda \in X^*$,

$$\langle Lx, \int_0^t T_s^*\lambda ds\rangle = \langle L(\int_0^t T_s x ds), \lambda\rangle$$

$$= \langle T_t x - x, \lambda\rangle$$

$$= \langle x, T_t^*\lambda - \lambda\rangle$$

$\Rightarrow$

$$\int_0^t T_s^*\lambda ds \in \text{Dom } L^*.]$$

Let Dom L_w^* be the set of all $\lambda \in X^*$ for which

$$\lim_{t \to 0} \frac{T_t^*\lambda - \lambda}{t}$$

exists (weak*) and define L_w^* on Dom L_w^* by the equality

$$L_w^*\lambda = \lim_{t \to 0} \frac{T_t^*\lambda - \lambda}{t}.$$

Then L_w^* is called the <u>weak* generator</u> of the semigroup $\{T_t^*:t \geq 0\}$.

17.27 <u>THEOREM</u> The dual L^* equals the weak* generator L_w^*.

<u>PROOF</u> We shall begin by showing that $L^* \subset L_w^*$. So fix $\lambda \in$ Dom L^* — then $\forall\ x \in X$,

$$\lim_{t \to 0} \frac{1}{t} \langle x, T_t^*\lambda - \lambda\rangle = \lim_{t \to 0} \frac{1}{t} \int_0^t \langle x, T_s^*L^*\lambda\rangle ds$$

$$= \langle x, L^*\lambda\rangle.$$

Therefore $\lambda \in$ Dom L_w^* and $L_w^*\lambda = L\lambda$. Conversely, for any $\lambda \in$ Dom L_w^* and for any $x \in$ Dom L,

$$\langle x, L_w^*\lambda\rangle = \lim_{t \to 0} \frac{1}{t} \langle x, T_t^*\lambda - \lambda\rangle$$

$$= \lim_{t \to 0} \frac{1}{t} \langle T_t x - x, \lambda\rangle$$

$$= \langle Lx, \lambda\rangle.$$

Therefore $\lambda \in \text{Dom } L^*$ and $L^*\lambda = L_w^*\lambda$, i.e., $L_w^* \subset L^*$.

17.28 <u>REMARK</u> In the reflexive case, $\{T_t^*:t \geq 0\}$ is strongly continuous and its generator is L^*.

17.29 <u>EXAMPLE</u> Contrary to what obtains in the strongly continuous situation (cf. 17.13), the weak* generator of $\{T_t^*:t \geq 0\}$ need not determine $\{T_t^*:t \geq 0\}$ uniquely within the class of all weak* continuous semigroups on X^*. E.g., consider the setup in 17.24, thus $X^* = M(\underline{R})$. With Lebesgue measure as the reference, each $\mu \in M(\underline{R})$ admits a decomposition $\mu = \mu_{ac} + \mu_s$ into an absolutely continuous part and a singular part, so in obvious notation,

$$M(\underline{R}) = M(\underline{R})_{ac} \oplus M(\underline{R})_s,$$

where both $M(\underline{R})_{ac}$ and $M(\underline{R})_s$ are closed in $M(\underline{R})$ and invariant under T_t^*. Given $\alpha > 0$, define a weak* continuous semigroup Λ_t^α by

$$\Lambda_t^\alpha \mu = T_t^* \mu_{ac} + T_{\alpha t}^* \mu_s.$$

Then $t \rightarrow \Lambda_t^\alpha \mu$ is strongly continuous iff $\mu = \mu_{ac}$. Therefore the maximal subspace of $M(\underline{R})$ on which each Λ_t^α is strongly continuous is the same for each α, i.e., is $M(\underline{R})_{ac}$, and on this space, the action does not depend on α, hence all the Λ_t^α have the same weak* generator (see 17.30 infra).

[Note: The theory developed below implies that $\{T_t:t \geq 0\}$ is the only strongly continuous semigroup on X whose dual has weak* generator L^*.]

Let

$$X^0 = \{\lambda \in X^*: \lim_{t \to 0} ||T_t^*\lambda - \lambda|| = 0\}.$$

Then X^0 is T_t^*-invariant and we put

$$T_t^0 = T_t^*|X^0.$$

N.B. X^0 is a norm closed linear subspace of X^*. Proof: Suppose that $\lambda_n \to \lambda$ $(\lambda_n \in X^0)$ -- then

$$\begin{aligned} ||T_t^*\lambda - \lambda|| &= ||T_t^*\lambda - T_t^*\lambda_n + T_t^*\lambda_n - \lambda_n + \lambda_n - \lambda|| \\ &\le ||T_t^*(\lambda - \lambda_n)|| + ||T_t^*\lambda_n - \lambda_n|| + ||\lambda_n - \lambda|| \\ &\le ||T_t^*|| \, ||\lambda_n - \lambda|| + ||T_t^*\lambda_n - \lambda_n|| + ||\lambda_n - \lambda|| \\ &= ||T_t|| \, ||\lambda_n - \lambda|| + ||T_t^*\lambda_n - \lambda_n|| + ||\lambda_n - \lambda||. \end{aligned}$$

Choose $C > 0$:

$$||T_t|| \le C \quad (0 \le t \le 1) \quad (\text{cf. } 17.6).$$

Given $\varepsilon > 0$, choose N:

$$||\lambda_N - \lambda|| < \frac{\varepsilon}{2(C+1)}$$

and choose $t(\varepsilon,N) \le 1$:

$$0 \le t < t(\varepsilon,N) \Rightarrow ||T_t^*\lambda_N - \lambda_N|| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} ||T_t^*\lambda - \lambda|| &< C\,\frac{\varepsilon}{2(C+1)} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2(C+1)} \\ &= \varepsilon \quad (0 \le t < t(\varepsilon,N)). \end{aligned}$$

17.30 LEMMA Dom L^* is contained in X^O.

PROOF Let $\lambda \in$ Dom L^* -- then $\forall\, x \in X$,

$$|<x,T_t^*\lambda - \lambda>|$$

$$= |\int_0^t <x,T_s^*L^*\lambda>ds| \qquad \text{(cf. 17.26)}$$

$$\leq \int_0^t |<T_s x,L^*\lambda>|ds$$

$$\leq \int_0^t ||T_s x||\ ||L^*\lambda||ds$$

$$\leq t(\sup_{0\leq s\leq t} ||T_s||)||x||\ ||L^*\lambda||$$

=>

$$||T_t^*\lambda - \lambda|| \leq t(\sup_{0\leq s\leq t} ||T_s||)||L^*\lambda||.$$

Therefore

$$\text{Dom } L^* \subset X^O.$$

Accordingly, the norm closure $\overline{\text{Dom } L^*}$ is contained in X^O but even more is true.

17.31 LEMMA We have

$$\overline{\text{Dom } L^*} = X^O.$$

PROOF Let $\lambda \in X^O$ — then $\forall\, x \in X$,

$$|<x,\frac{1}{t}\int_0^t T_s^*\lambda ds - \lambda>|$$

$$\leq \sup_{0\leq s\leq t} ||T_s^*\lambda - \lambda||\ ||x||$$

=>

$$||\frac{1}{t}\int_0^t T_s^*\lambda ds - \lambda|| \leq \sup_{0\leq s\leq t} ||T_s^*\lambda - \lambda|| \to 0 \text{ as } t \downarrow 0.$$

But

$$\frac{1}{t}\int_0^t T_s^*\lambda ds \in \text{Dom } L^* \quad \text{(cf. 17.26)}.$$

Therefore $\lambda \in \overline{\text{Dom L}}$.

Because X^O is a norm closed linear subspace of X^*, it is a Banach space and from the definitions, $\{T_t^O : t \geq 0\}$ is a strongly continuous semigroup on X^O (cf. 17.9, bearing in mind that $\forall$ t, $||T_t^*|| = ||T_t||$).

17.32 REMARK If X is reflexive, then $X^O = X^*$ and $\{T_t^* : t \geq 0\}$ is strongly continuous, a point that has been noted earlier.

Let L^O be the generator of the semigroup $\{T_t^O : t \geq 0\}$.

17.33 LEMMA Dom L^O is weak* dense in X^*.

PROOF In fact, Dom L^* is weak* dense in X^*, hence the same is true of X^O (cf. 17.30). But Dom L^O is norm dense in X^O.

To relate L^O to L^*, introduce the part of L^* in X^O, viz. the operator L^*/X^O

whose domain is $\{\lambda \in \text{Dom } L^*: L^*\lambda \in X^O\}$ with $L^*/X^O\lambda = L^*\lambda$.

17.34 <u>LEMMA</u> L^O is the part of L^* in X^O and L^* is the weak* closure of L^O.

17.35 <u>EXAMPLE</u> Take $X = L^1(\underline{R})$ and define T_t by

$$(T_t f)(x) = f(x + t).$$

Then $X^* = L^\infty(\underline{R})$,

$$(T_t^*\phi)(x) = \phi(x - t),$$

and $X^O = BC_u(\underline{R})$, the bounded uniformly continuous functions on $\underline{R}$. Here the generator L is differentiation and Dom L is the set of all $f \in L^1(\underline{R})$ which are absolutely continuous subject to $f' \in L^1(\underline{R})$. However, Dom $L^O \neq$ Dom L^* (consider the function $x \to |\sin x|$).

17.36 <u>LEMMA</u> Let $\lambda \in X^*$ -- then the following conditions are equivalent:

(i) $\lambda \in \text{Dom } L^*$;

(ii) $\limsup_{t \downarrow 0} t^{-1}||T_t^*\lambda - \lambda|| < \infty$;

(iii) $\liminf_{t \downarrow 0} t^{-1}||T_t^*\lambda - \lambda|| < \infty$.

<u>PROOF</u> That (i) => (ii) is contained in the proof of 17.30 and (ii) => (iii) is trivial, so assume (iii). Choose a sequence $t_n \downarrow 0$ and a constant $C > 0$:

$$\frac{1}{t_n} ||T_{t_n}^* \lambda - \lambda|| \le C \ \forall \ n.$$

Then the linear functional $x \to \lambda(Lx)$ is norm continuous on Dom L. Proof:

$$|\lambda(Lx)| = |\lim_{n \to \infty} \frac{1}{t_n} \langle T_{t_n} x - x, \lambda\rangle|$$

$$= |\lim_{n \to \infty} \frac{1}{t_n} \langle x, T^*_{t_n} \lambda - \lambda\rangle|$$

$$\leq C||x||.$$

Therefore $\lambda \in$ Dom L*.

If $x \in$ Dom L, then

$$T_t x - x = \int_0^t T_s Lx ds \quad \text{(cf. 17.11)}$$

=>

$$||T_t x - x|| \leq t(\sup_{0 \leq s \leq t} ||T_s||)||Lx||$$

=>

$$\limsup_{t \downarrow 0} t^{-1}||T_t x - x|| < \infty.$$

Nevertheless, in general, this can not be reversed.

17.37 EXAMPLE Take $X = C_\infty(\underline{R})$ and define T_t by

$$(T_t f)(x) = f(x + t) \quad \text{(cf. 17.24)}.$$

If $f \in C_\infty(\underline{R})$ is absolutely continuous with derivative $f' \in L^\infty(\underline{R})$ but $f' \notin C_\infty(\underline{R})$, then

$$\limsup_{t \downarrow 0} t^{-1}||T_t f - f|| \leq ||f'||_\infty.$$

Still, $f \notin$ Dom L.

17.38 RAPPEL Suppose that $L:\text{Dom } L \to X$ is closed and densely defined -- then the resolvent set $\rho(L)$ of L is the set of all complex numbers z such that $zI - L:\text{Dom } L \to X$ is bijective -- then

$$R(z:L) \equiv (zI - L)^{-1}$$

is a bounded linear operator on X (closed graph theorem).

[Note: The spectrum $\sigma(L)$ of L is the complement in $\underline{C}$ of $\rho(L)$.]

Suppose that $\{T_t:t \geq 0\}$ is strongly continuous with generator L such that

$$||T_t|| \leq Me^{at}$$

for some $a \in \underline{R}$ and $M \geq 1$ (cf. 17.6).

17.39 LEMMA The spectrum of L is contained in $\{z:\text{Re } z \leq a\}$, hence the resolvent set of L contains $\{z:\text{Re } z > a\}$.

17.40 LEMMA If $\text{Re } z > a$, then

$$(zI - L)^{-1}x = \int_0^\infty e^{-zt}T_t x dt \quad (x \in X).$$

[Note: Therefore

$$||(zI - L)^{-1}|| \leq M(\text{Re } z - a)^{-1}.]$$

17.41 THEOREM (Post-Widder Inversion Formula) $\forall\ x \in X$,

$$T_t x = \lim_{n \to \infty} (\frac{n}{t} R(\frac{n}{t}:L))^n x = \lim_{n \to \infty} (I - \frac{t}{n} L)^{-n} x$$

uniformly on compacta in t.

Let $\mathcal{H}$ be a complex Hilbert space -- then in what follows, it will be a question of a semigroup $\{T_t : t \geq 0\}$ on the Banach space $\mathcal{B}(\mathcal{H})$ which is weak* continuous.

<u>N.B.</u> Recall that $\mathcal{B}(\mathcal{H})$ can be identified with the dual of $\underline{L}_1(\mathcal{H})$ (cf. 1.4). So, to say that $\{T_t : t \geq 0\}$ is weak* continuous amounts to saying that the function

$$\begin{cases} \underline{R}_{\geq 0} \to \mathcal{B}(\mathcal{B}(\mathcal{H})) \\ t \to T_t \end{cases}$$

is continuous when

$$\mathcal{B}(\mathcal{B}(\mathcal{H})) \equiv \mathcal{B}(\underline{L}_1(\mathcal{H})^*)$$

is endowed with the weak* topology, thus $\forall\ A \in \mathcal{B}(\mathcal{H})\ \&\ \forall\ T \in \underline{L}_1(\mathcal{H})$,

$$\lim_{t \to t_0} \operatorname{tr}((T_t A)T) = \operatorname{tr}((T_{t_0} A)T).$$

In particular: $\forall\ x,y \in \mathcal{H}$,

$$\lim_{t \to t_0} \operatorname{tr}((T_t A)P_{x,y}) = \operatorname{tr}((T_{t_0} A)P_{x,y})$$

or still,

$$\lim_{t \to t_0} \operatorname{tr}(P_{(T_t A)x,y}) = \operatorname{tr}(P_{(T_{t_0} A)x,y})$$

or still,

$$\lim_{t \to t_0} \langle y, (T_t A)x\rangle = \langle y, (T_{t_0} A)x\rangle.$$

17.41 <u>LEMMA</u> Suppose that $\{T_t : t \geq 0\}$ is weak* continuous -- then $\exists$ constants

$a \in \underline{R}$ and $M \geq 1$ such that

$$||T_t|| \leq Me^{at}$$

for all $t \geq 0$.

<u>PROOF</u> We first claim that $||T_t||$ is bounded in some neighborhood of the origin: $\exists\ \delta > 0$ and $M(\delta) \geq 1$:

$$||T_t|| \leq M(\delta) \qquad (0 \leq t \leq \delta).$$

Assume not, thus there would exist a sequence $t_n > 0 : t_n \to 0$ and $||T_{t_n}|| \geq n$. So, by the uniform boundedness principle, $\exists\ A \neq 0$ in $\mathcal{B}(H)$ such that $\{||T_{t_n} A||\}$ is unbounded. Put $A_n = T_{t_n} A$ and let $x,y \in H$ -- then

$$\langle y, A_n x\rangle \to \langle y, Ax\rangle.$$

Therefore the sequence $\{A_n x\}$ is weakly convergent to Ax, hence $\exists\ C_x > 0 : \forall\ n$,

$$||A_n x|| \leq C_x,$$

which, by another application of the uniform boundedness principle, implies that $\exists\ C > 0 : \forall\ n$,

$$||A_n|| \leq C,$$

a contradiction. Proceeding, given $t > 0$, write $t = k\delta + \tau$ $(0 \leq \tau < \delta, k \in \underline{Z}_{\geq 0})$:

$$\begin{aligned} ||T_t|| &= ||T_{k\delta+\tau}|| \\ &= ||T_{k\delta} \circ T_\tau|| \\ &\leq ||T_\delta||^k\ ||T_\tau|| \end{aligned}$$

$$\leq M(\delta)^{k+1}$$

$$\leq M(\delta)M(\delta)^{t/\delta}$$

$$= M(\delta)e^{at} \quad (a = \delta^{-1} \log M(\delta)).$$

17.42 LEMMA Suppose that $\{T_t : t \geq 0\}$ is weak* continuous. Assume: $\forall\ A \in \mathcal{B}(H)$ and $\forall\ x \in H$,

$$\lim_{t \to 0} ||(T_t A)(x) - Ax|| = 0.$$

Then $\forall\ A \in \mathcal{B}(H)$ and $\forall\ x \in H$,

$$\lim_{t \to t_0} ||(T_t A)(x) - (T_{t_0} A)(x)|| = 0.$$

[One can argue as in the proof of 17.9 (continuity on the left at $t_0 > 0$ being secured by an application of 17.41).]

[Note: The condition

$$\lim_{t \to t_0} ||(T_t A)(x) - (T_{t_0} A)(x)|| = 0$$

is weaker than strong convergence which would read

$$\lim_{t \to t_0} ||T_t A - T_{t_0} A|| = 0.]$$

17.43 EXAMPLE Suppose that $\{T_t : t \geq 0\}$ is weak* continuous. Assume: $\forall\ t, T_t : \mathcal{B}(H) \to \mathcal{B}(H)$ is 2-positive and $||T_t I|| \leq 1$ — then $\forall\ A \in \mathcal{B}(H)$ and $\forall\ x \in H$,

$$\lim_{t \to 0} ||(T_t A)(x) - Ax|| = 0.$$

To see this, start by writing

$$||(T_tA)(x) - Ax||^2$$

$$= <(T_tA)(x) - Ax,(T_tA)(x) - Ax>$$

$$= <x,(T_tA)^*(T_tA)(x)>$$

$$- 2\mathrm{Re}\ <Ax,(T_tA)(x)> + <Ax,Ax>.$$

But

$$(T_tA)^*(T_tA) \leq ||T_tI||T_t(A^*A) \quad \text{(cf. 11.24)}$$

$$\leq T_t(A^*A) \quad (||T_tI|| \leq 1).$$

Therefore

$$||(T_tA)(x) - Ax||^2$$

$$\leq <x,T_t(A^*A)(x)>$$

$$- 2\mathrm{Re}\ <Ax,(T_tA)(x)> + <Ax,Ax>$$

$$\xrightarrow[(t \to 0)]{} <x,A^*Ax> - 2<Ax,Ax> + <Ax,Ax>$$

$$= 0.$$

Maintaining the assumption that $\{T_t:t \geq 0\}$ is weak* continuous, suppose further that $\forall\ t \geq 0$, the map $T_t:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is positive and normal† — then $\{T_t:t \geq 0\}$ gives rise to a predual semigroup $\{(T_t)_*:t \geq 0\}$, i.e., a semigroup on $\mathcal{B}(\mathcal{H})_*$ or still,

†Let $\Phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a positive linear map — then Φ is <u>normal</u> iff Φ is weak* continuous.

on $\underline{L}_1(H)$ (cf. 2.3). Spelled out, $\forall$ $A \in \mathcal{B}(H)$ & $\forall$ $T \in \underline{L}_1(H)$,

$$\mathrm{tr}((T_t A)T) = \mathrm{tr}(A(T_t)_* T).$$

17.44 <u>LEMMA</u> The predual semigroup $\{(T_t)_*: t \geq 0\}$ is strongly continuous.

<u>PROOF</u> It suffices to show that $\{(T_t)_*: t \geq 0\}$ is weakly continuous (cf. 17:4): $\forall$ $\lambda \in \underline{L}_1(H)^*$ & $\forall$ $T \in \underline{L}_1(H)$,

$$\lim_{t \to t_0} \lambda((T_t)_* T - (T_{t_0})_* T) \to 0.$$

But $\exists$ a unique $A \in \mathcal{B}(H)$ such that $\lambda = \lambda_A$ (cf. 1.4), so

$$\begin{aligned}
&\lim_{t \to t_0} \lambda((T_t)_* T - (T_{t_0})_* T) \\
&= \lim_{t \to t_0} \mathrm{tr}(A((T_t)_* T - (T_{t_0})_* T)) \\
&= \lim_{t \to t_0} \mathrm{tr}((T_t A)T - (T_{t_0} A)T) \\
&= 0.
\end{aligned}$$

The dual $\{((T_t)_*)^*: t \geq 0\}$ is precisely $\{T_t: t \geq 0\}$ (cf. 16.3). Accordingly, if L is the generator of $\{T_t: t \geq 0\}$, then L is weak* closed and weak* densely defined. Its domain consists of those $A \in \mathcal{B}(H)$ such that

$$\lim_{t \to 0} \frac{T_t A - A}{t}$$

exists (weak*). And, owing to 17.31,

$$\overline{\mathrm{Dom}\ L} = \mathcal{B}(H)^{\mathrm{O}},$$

Dom L itself being characterized by 17.36.

[Note: In obvious notation, $(L_*)^* = L$.]

17.45 RAPPEL If $A_i \to A$ (weak*), then $A_i^* \to A^*$ (weak*).

[$\forall\ T \in \underline{L}_1(H)$,

$$\operatorname{tr}(TA_i^*) = \operatorname{tr}(T^{**}A_i^*) = \operatorname{tr}((A_iT^*)^*) = \overline{\operatorname{tr}(A_iT^*)}$$

$$\to \overline{\operatorname{tr}(AT^*)} = \overline{\overline{\operatorname{tr}((AT^*)^*)}} = \operatorname{tr}(TA^*).]$$

Since T_t is positive, $\forall\ A \in \mathcal{B}(H)$,

$$T_tA^* = (T_tA)^*.$$

17.46 LEMMA Let $A \in \operatorname{Dom} L$ — then $A^* \in \operatorname{Dom} L$ and $LA^* = (LA)^*$.

PROOF By hypothesis,

$$\frac{T_tA - A}{t} \to LA \text{ (weak*)}.$$

Therefore

$$\frac{(T_tA - A)^*}{t} = \frac{T_tA^* - A^*}{t}$$

$$\to (LA)^* \text{ (weak*)}.$$

So, $A^* \in \operatorname{Dom} L$ and $LA^* = (LA)^*$.

[Note: If $\forall\ t \geq 0$, $T_tI = I$, then $I \in \operatorname{Dom} L$ and Dom L is an operator system.]

17.47 RAPPEL The subset of $\mathcal{B}(\mathcal{B}(H))$ whose elements are the $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$

which are weak* continuous is a norm closed linear subspace of $\mathcal{B}(\mathcal{B}(H))$.

If $\{T_t : t \geq 0\}$ is norm continuous, then a fortiori, $\{T_t : t \geq 0\}$ is weak* continuous. Supposing still that the $T_t : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ are positive, the relations

$$\begin{cases} \lim\limits_{t \rightarrow 0} \left\|\dfrac{T_t - I}{t} - L\right\| = 0 \\[2ex] \lim\limits_{n \rightarrow \infty} \left\|T_t - \sum\limits_{k=0}^{n} \dfrac{t^k}{k!} L^k\right\| = 0 \end{cases}$$

imply that the T_t are normal iff L is weak* continuous (cf. 17.47).

<u>N.B.</u> The semigroup $\{T_t : t \geq 0\}$ is <u>unital</u> if $T_t I = I \ \forall \ t \geq 0$.

17.48 <u>EXAMPLE</u> Fix $H \in \mathcal{B}(H)_{SA}$ and let

$$T_t A = \frac{1}{\sqrt{2\pi t}} \int_{\underline{R}} e^{-x^2/2t} \, e^{\sqrt{-1}\, xH} A \, e^{-\sqrt{-1}\, xH} ds \quad (A \in \mathcal{B}(H)).$$

Then $\{T_t : t \geq 0\}$ is norm continuous and unital. Moreover, $\forall \ t \geq 0$, the map $T_t : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ is positive and normal. Finally, the generator L is given by

$$\begin{aligned} LA &= -\frac{1}{2} [H,[H,A]] \\ &= -\frac{1}{2} (H^2 A + AH^2 - 2HAH). \end{aligned}$$

[Note: From the formula, it is obvious that L is weak* continuous.]

17.49 <u>EXAMPLE</u> If $U \in \mathcal{U}(\mathcal{H})$ and $\lambda > 0$, then the prescription

$$T_t A = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} U^{*n} A U^n \quad (A \in \mathcal{B}(\mathcal{H}))$$

defines a norm continuous unital semigroup $\{T_t : t \geq 0\}$ whose generator L is given by

$$LA = \lambda(U^*AU - A)$$

or still,

$$LA = -\frac{1}{2}(V^*VA + AV^*V - 2V^*AV)$$

if $V = \sqrt{\lambda}\, U$. And the T_t are positive and normal.

17.50 <u>REMARK</u> Take $\mathcal{H}$ separable and suppose that the semigroup $\{T_t : t \geq 0\}$ is norm continuous with the T_t completely positive and normal -- then in the terminology of §16, the T_t are operations if $T_t I \leq I \ \forall\, t$ and the T_t are channels if $T_t I = I \ \forall\, t$.

§18. *GENERATORS*

Let X be a complex Banach space -- then a semigroup $\{T_t : t \geq 0\}$ on X is said to be contractive if $\forall\ t$, T_t is a contraction (i.e., $||T_t|| \leq 1$).

18.1 EXAMPLE Take $X = L^2(\underline{R}^n)$ and define T_t by

$$(T_t f)(x) = (4\pi t)^{-n/2} \int_{\underline{R}^n} e^{-|x-y|^2/4t} f(x)dx.$$

Then the semigroup $\{T_t : t \geq 0\}$ is strongly continuous and contractive. Its generator L is the laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

where

$$\text{Dom } \Delta = \{f : \Delta f \in L^2(\underline{R}^n)\}.$$

Here Δf is understood in the sense of distributions, hence Dom Δ is the Sobolev space $W^{2,2}(\underline{R}^n)$.

18.2 THEOREM (Hille-Yosida) Suppose that $L : \text{Dom } L \to X$ is closed and densely defined -- then L is the generator of a strongly continuous contractive semigroup iff

$$\rho(L) \supset \underline{R}_{>0} \text{ and } ||(rI - L)^{-1}|| \leq \frac{1}{r} \ (r > 0).$$

[Note:

$$||T_t|| \leq 1 \Rightarrow a = 0.$$

Therefore the conditions are certainly necessary (cf. 17.39 and 17.40). That they are also sufficient is one of the pillars of the classical theory.]

Suppose that $L:\text{Dom } L \to X$ is a densely defined linear operator -- then L is said to be <u>dissipative</u> if $\forall\ x \in \text{Dom } L\ \&\ \forall\ r \in \underline{R}_{>0}$,

$$||(rI - L)x|| \geq r||x||.$$

[Note: Therefore $rI - L$ is injective and

$$||(rI - L)^{-1}x|| \leq \frac{1}{r}\ ||x||$$

for all x in the range of $rI - L$, or still, in $(rI - L)\text{Dom } L$.]

18.3 <u>LEMMA</u> If L is dissipative, then L admits closure.

<u>PROOF</u> Consider a sequence $\{x_n : x_n \in \text{Dom } L\}$:

$$\begin{cases} x_n \to 0 \\ Lx_n \to y. \end{cases}$$

Claim: $y = 0$. To see this, note that $\forall\ x \in \text{Dom } L$,

$$||r(rI - L)x_n + (rI - L)x|| \geq r||rx_n + x||$$

$$\Rightarrow$$

$$||-ry + (rI - L)x|| \geq r||x|| \quad (n \to \infty)$$

$$\Rightarrow$$

$$||-y + x - \frac{1}{r}Lx|| \geq ||x||$$

$$\Rightarrow$$

$$||-y + x|| \geq ||x|| \geq (r \to \infty).$$

But Dom L is dense in X, hence $y = 0$.

<u>N.B.</u> The closure $\bar{L}$ is again dissipative.

18.4 <u>LEMMA</u> If L is closed and dissipative, then $rI - L$ is surjective for some $r > 0$ iff $rI - L$ is surjective for all $r > 0$.

<u>PROOF</u> Let $O \subset \underset{\sim}{R}_{>0}$ be the set of r such that $rI - L$ is surjective. Take O nonempty -- then

$$\rho(L) \text{ open} \Rightarrow O \text{ open.}$$

But O is also closed ($\Rightarrow O = \underset{\sim}{R}_{>0}$). Thus let $r_n \in O : r_n \to r > 0$. Given $y \in X$, $\exists$ $x_n \in \text{Dom } L$:

$$r_n x_n - Lx_n = y.$$

And

$$||x_n|| \leq r_n^{-1}||(r_n I - L)x_n||$$

$$= r_n^{-1}||y|| \leq C \ (\exists\ C > 0).$$

Now write

$$r_m||x_n - x_m|| \leq ||r_m(x_n - x_m) - L(x_n - x_m)||$$

$$= ||r_m x_n - r_m x_m - Lx_n + Lx_m||$$

$$= ||r_m x_n - y - Lx_m - Lx_n + Lx_m||$$

$$= ||r_m x_n - y - Lx_n||$$

$$= ||r_m x_n - r_n x_n||$$

$$= |r_m - r_n| \, ||x_n||$$

$$\leq C|r_m - r_n|$$

to see that $\{x_n\}$ is Cauchy, so $x_n \to x$

=>

$$Lx_n = r_n x_n - y \to rx - y.$$

Because L is closed,

$$x \in \text{Dom } L \text{ and } Lx = rx - y.$$

Therefore

$$rx - Lx = (rI - L)x = y$$

=>

$$\text{Ran}(rI - L) = X$$

=> $r \in O$ => O closed.

18.5 <u>LEMMA</u> If L is dissipative and if $rI - L$ is surjective for some $r > 0$, then L is closed.

<u>PROOF</u> In fact,

$$(rI - L)^{-1} \in \mathcal{B}(X),$$

hence $rI - L$ is closed, which implies that L is closed.

Given $x \in X$, its <u>duality set</u> is defined by

$$\mathcal{D}(x) = \{\lambda_x \in X^*: \lambda_x(x) = ||x||^2 = ||\lambda_x||^2\}.$$

[Note: $\mathcal{D}(x)$ is not empty (Hahn-Banach).]

E.g.: If $X = H$, a complex Hilbert space, then $\mathcal{D}(x) = \{x\}$.

18.6 LEMMA Suppose that $L:\text{Dom } L \to X$ is a densely defined linear operator -- then L is dissipative iff $\forall\ x \in \text{Dom } L$, $\exists\ \lambda_x \in \mathcal{D}(x)$ such that

$$\text{Re } \lambda_x(Lx) \leq 0.$$

PROOF Assume first that the stated inequality is in force. Take $x \in \text{Dom } L$: $||x|| = 1$ and choose $\lambda_x \in \mathcal{D}(x)$ accordingly ($\Rightarrow \lambda_x(x) = ||x||^2 = ||\lambda_x||^2 = 1$) -- then $\forall\ r > 0$,

$$\begin{aligned} ||(rI - L)x|| &= ||\lambda_x||\ ||(rI - L)x|| \\ &\geq |\lambda_x((rI - L)x)| \\ &\geq \text{Re } \lambda_x((rI - L)x) \\ &= \text{Re}(r\lambda_x(x) - \lambda_x(Lx)) \\ &\geq r, \end{aligned}$$

which is the implication in one direction. Proceeding to the converse, fix $x \in \text{Dom } L$: $||x|| = 1$ and assume that

$$||(rI - L)x|| \geq r$$

for all $r > 0$. Choose

$$\lambda_r \in \mathcal{D}((rI - L)x)$$

and put

$$\eta_r = \frac{\lambda_r}{||\lambda_r||}.$$

Then

$$r \leq ||(rI - L)x||$$

$$= |\eta_r((rI - L)x)|$$

$$= r\mathrm{Re}\ \eta_r(x) - \mathrm{Re}\ \eta_r(Lx)$$

$$\leq \min\{r - \mathrm{Re}\ \eta_r(Lx),\ r\mathrm{Re}\ \eta_r(x) + ||Lx||\}$$

=>

$$\mathrm{Re}\ \eta_r(Lx) \leq 0 \text{ and } 1 - \frac{1}{r}||Lx|| \leq \mathrm{Re}\ \eta_r(x).$$

Let η be a weak* accumulation point of the net $\{\eta_r\}$ $(r \to \infty)$. So:

$$\mathrm{Re}\ \eta(Lx) \leq 0 \text{ and } 1 \leq \mathrm{Re}\ \eta(x),$$

with, of course, $||\eta|| \leq 1$, thus

$$1 \leq \mathrm{Re}\ \eta(x) \leq |\eta(x)| \leq ||x|| = 1$$

=>

$$\eta \in \mathcal{D}(x).$$

Put $\lambda_x = \eta$ to complete the proof.

18.7 <u>RAPPEL</u> Given $x \in X$, an element $\lambda \in X^*$ is called a <u>tangent functional</u> at x if $\lambda(x) = ||\lambda||\ ||x||$.

Write $\mathfrak{T}(x)$ for the set of tangent functionals at x — then $\mathcal{D}(x) \subset \mathfrak{T}(x)$.

[Note: The containment $\mathcal{D}(x) \subset \mathfrak{T}(x)$ is proper if $x \neq 0$. For if $\lambda \in \mathcal{D}(x)$, then $\frac{\lambda}{2} \in \mathfrak{T}(x)$ but $\frac{\lambda}{2} \notin \mathcal{D}(x)$.]

18.8 <u>LEMMA</u> Suppose that L is dissipative — then $\forall\ x \in \mathrm{Dom}\ L\ \&\ \forall\ \lambda \in \mathfrak{T}(x)$,

$$\mathrm{Re}\ \lambda(Lx) \leq 0.$$

<u>PROOF</u> Fix $x \in \text{Dom } L$ and let $\lambda \in \mathfrak{T}(x)$. Put

$$\eta = \frac{\lambda}{||\lambda||}.$$

Then $\forall\ r > 0$,

$$||(I + rL)(x)|| \geq \text{Re } \eta((I + rL)(x))$$

$$= ||x|| + r\text{Re } \eta(Lx).$$

Therefore

$$\text{Re } \eta(Lx) \leq \limsup_{r \downarrow 0} \frac{1}{r}\ (||(I + rL)(x)|| - ||x||).$$

Consider now any $y \in \text{Dom } L$:

$$||(I + rL)(x)|| \leq ||x + ry|| + r||y - Lx||$$

$$\leq ||(I - rL)(x + ry)|| + r||y - Lx||$$

$$\leq ||x|| + 2r||y - Lx|| + r^2||Ly||.$$

Therefore

$$\text{Re } \eta(Lx) \leq \limsup_{r \downarrow 0} \frac{1}{r}\ (2r||y - Lx|| + r^2||Ly||)$$

$$= 2||y - Lx||.$$

But Dom L is dense in X, hence

$$\text{Re } \eta(Lx) = 0.$$

18.9 <u>LEMMA</u> Suppose that $\{T_t : t \geq 0\}$ is strongly continuous and contractive -- then its generator L is dissipative.

<u>PROOF</u> Take any $x \in \text{Dom } L$ and let $\lambda \in \mathfrak{T}(x)$ -- then

$$\lambda(Lx) = \lim_{t \to 0} \frac{1}{t} \lambda(T_t x - x)$$

$$= \lim_{t \to 0} \frac{1}{t} (\lambda(T_t x) - ||\lambda|| \, ||x||).$$

But

$$|\lambda(T_t x)| \leq ||\lambda|| \, ||T_t x||$$

$$\leq ||\lambda|| \, ||T_t|| \, ||x||$$

$$\leq ||\lambda|| \, ||x||.$$

Therefore

$$\text{Re } \lambda(Lx) \leq 0$$

and one may quote 18.6.

18.10 EXAMPLE Let $A \in \mathcal{B}(X)$ -- then $A - ||A||I$ is dissipative. For

$$||\exp(t(A - ||A||I))||$$

$$= ||\exp(tA)\exp(- t||A||I)||$$

$$\leq ||\exp(tA)|| \, ||\exp(- t||A||I)||$$

$$\leq e^{t||A||} \, e^{-t||A||}$$

$$= 1.$$

18.11 THEOREM (Lumer-Phillips) Suppose that $L:\text{Dom } L \to X$ is a densely defined linear operator -- then L is the generator of a strongly continuous contractive semigroup $\{T_t : t \geq 0\}$ iff L is dissipative and for some $r > 0$, $\text{Ran}(rI - L) = X$.

<u>PROOF</u> The necessity follows from 18.9 and the fact that $\rho(L) \supset \underline{R}_{>0}$. As for the sufficiency, 18.5 implies that L is closed. But L closed and dissipative forces the surjectivity of $rI - L$ for all $r > 0$ (cf. 18.4), thus $\rho(L) \supset \underline{R}_{>0}$ and

$$||(rI - L)^{-1}|| \leq \frac{1}{r} \ (r > 0).$$

One can therefore apply 18.2.

18.12 <u>EXAMPLE</u> If $X = H$, a complex Hilbert space, then a densely defined L is the generator of a strongly continuous contractive semigroup $\{T_t : t \geq 0\}$ iff $\forall\, x \in \text{Dom } L$,

$$\text{Re}\langle x, Lx\rangle \leq 0$$

and for all $r > 0$,

$$\text{Ran}(rI - L) = H.$$

Suppose that $L:\text{Dom } L \rightarrow X$ is a densely defined linear operator -- then we shall call L a <u>generator</u> if L is dissipative and for all $r > 0$,

$$\text{Ran}(rI - L) = X.$$

18.13 <u>EXAMPLE</u> If L and L^* are dissipative, then $\bar{L}$ is a generator.

[In view of 18.3, L admits closure, hence $\bar{L}$ makes sense, and, as mentioned there, $\bar{L}$ is dissipative, so matters reduce to proving that $\forall\, r > 0$,

$$\text{Ran}(rI - \bar{L}) = X$$

or still, $\forall\, r > 0$,

$$\overline{\text{Ran}(rI - L)} = X.$$

To get a contradiction, assume that $\exists\, r > 0$:

$$\overline{\mathrm{Ran}(rI - L)} \neq X.$$

Then, by the Hahn-Banach theorem, $\exists\, \lambda \neq 0$ in X^* such that

$$<(rI - L)x,\lambda> = 0 \quad (x \in \mathrm{Dom}\ L).$$

Therefore $\lambda \in \mathrm{Dom}\ L^*$ and

$$<x,(rI - L^*)\lambda> = 0 \quad (x \in \mathrm{Dom}\ L).$$

But Dom L is dense in X, hence $(rI - L^*)\lambda = 0$, which violates the injectivity of $(rI - L^*)$ (by hypothesis, L^* is dissipative).]

18.14 <u>THEOREM</u> Let L be a generator and suppose given a linear operator A with

$$\mathrm{Dom}\ A \supset \mathrm{Dom}\ L$$

subject to

$$||Ax|| \leq a||x|| + b||Lx|| \quad (x \in \mathrm{Dom}\ L)$$

for some $a \geq 0$ and $0 < b < 1$. Assume: Either A or L + A is dissipative -- then L + A is a generator.

[Note: The domain of L + A is Dom L.]

It will be convenient to proceed via a series of lemmas.

18.15 <u>LEMMA</u> If A is dissipative, then $L + \alpha A$ is dissipative for all $\alpha \geq 0$.

<u>PROOF</u> According to 18.8, $\forall\, x \in \mathrm{Dom}\ L$ & $\forall\, \lambda \in \mathfrak{C}(x)$,

$$\mathrm{Re}\ \lambda(Lx) \leq 0.$$

But, being dissipative, the same holds for A, thus $\forall\, x \in \mathrm{Dom}\ L$ & $\forall\, \lambda \in \mathfrak{C}(x)$,

$$\mathrm{Re}\ \lambda((L + \alpha A)(x)) \leq 0.$$

Therefore $L + \alpha A$ is dissipative (cf. 18.6).

18.16 LEMMA If $L + A$ is dissipative, then $L + \alpha A$ is dissipative for all $\alpha \geq 0$.

PROOF In fact, $\forall \lambda \in \mathfrak{C}(x)$ $(x \in \text{Dom } L)$,

$$\begin{aligned} &\text{Re } \lambda((L + \alpha A)(x)) \\ &\quad = \alpha \text{Re } \lambda((L + A)(x)) + (1 - \alpha)\text{Re } \lambda(Lx) \\ &\quad \leq 0. \end{aligned}$$

Therefore $L + \alpha A$ is dissipative (cf. 18.6).

18.17 LEMMA If $0 \leq \alpha_1 < \frac{1}{2b}$, then $L + \alpha_1 A$ is a generator.

PROOF To begin with, $\forall x \in X$ & $\forall r > 0$,

$$\begin{aligned} &||rA(I - rL)^{-1}x|| \\ &\leq a||r(I - rL)^{-1}x|| + b||rL(I - rL)^{-1}x||. \end{aligned}$$

But

$$\begin{aligned} &||rL(I - rL)^{-1}x|| \\ &= ||- rL(I - rL)^{-1}x|| \\ &= ||((I - rL) - I)(I - rL)^{-1}x|| \\ &= ||(I - (I - rL)^{-1}x||. \end{aligned}$$

So, bearing in mind that

$$||(I - rL)^{-1}|| \leq 1,$$

it follows that

$$||rA(I - rL)^{-1}x|| \le (ar + 2b)||x||.$$

Now choose r_0 such that $\alpha_1(ar + 2b) < 1$ for $0 \le r < r_0$ and put

$$A_r = \alpha_1 rA(I - rL)^{-1}.$$

Then

$$||A_r|| \le \alpha_1(ar + 2b) < 1$$

$\Rightarrow$

$$(I - A_r)^{-1} \in \mathcal{B}(X).$$

Therefore

$$\begin{aligned} &\mathrm{Ran}(I - r(L + \alpha_1 A)) \\ &= \mathrm{Ran}((I - A_r)(I - rL)) \\ &= \mathrm{Ran}(I - A_r) \\ &= \mathrm{Dom}(I - A_r)^{-1} = X. \end{aligned}$$

Since $L + \alpha_1 A$ is dissipative, 18.5 implies that $L + \alpha_1 A$ is closed, thus by 18.4,

$$rI - (L + \alpha_1 A)$$

is surjective for all $r > 0$. Consequently, $L + \alpha_1 A$ is a generator.

18.18 <u>LEMMA</u> If $0 \le \alpha_2 < \frac{1}{4b}$, then $L + \alpha_1 A + \alpha_2 A$ is a generator.

<u>PROOF</u> $\forall\ x \in \mathrm{Dom}\ L$,

$$\begin{aligned} ||Ax|| &\le a||x|| + b||Lx|| \\ &\le a||x|| + b||Lx - \alpha_1 Ax + \alpha_1 Ax|| \\ &\le a||x|| + b||(L + \alpha_1 A)x|| + b\alpha_1||Ax|| \end{aligned}$$

$$\leq a||x|| + b||(L + \alpha_1 A)x|| + \frac{1}{2}||Ax||$$

=>

$$||Ax|| \leq 2a||x|| + 2b||(L + \alpha_1 A)x||.$$

From here, one may argue as in 18.17 to conclude that $L + \alpha_1 A + \alpha_2 A$ is a generator.

Iteration then implies that

$$L + (\sum_{k=1}^{n} \alpha_k)A \quad (0 \leq \alpha_k < \frac{1}{2^k b})$$

is a generator. But

$$\frac{1}{2b} + \frac{1}{4b} + \cdots + \frac{1}{2^n b} = \frac{1 - 2^{-n}}{b}.$$

Therefore $L + \alpha A$ is a generator for all

$$0 \leq \alpha < \frac{1 - 2^{-n}}{b}.$$

Since $b < 1$,

$$n >> 0 \Rightarrow \frac{1 - 2^{-n}}{b} > 1.$$

Accordingly, $L + A$ is a generator, the contention of 18.14.

18.19 <u>REMARK</u> It is an interesting point of detail that

$$18.14 \Rightarrow 17.18.$$

18.20 <u>EXAMPLE</u> The assumption that $b < 1$ in 18.14 cannot, in general, be replaced by $b = 1$. E.g.: Let H be a complex Hilbert space and let A be selfadjoint but unbounded -- then $\sqrt{-1}\, A$ and $-\sqrt{-1}\, A$ generate strongly continuous contractive semigroups. On the other hand,

$$||-\sqrt{-1}\, A|| = ||\sqrt{-1}\, A||,$$

so the conditions of 18.14 are met with $a = 0$, $b = 1$. However, $\sqrt{-1}\, A + (-\sqrt{-1}\, A)$ is the zero operator on Dom A, thus is not closed, thus is not a generator.

18.21 REMARK If A^* is densely defined, then $b = 1$ is permissible in 18.14 provided the conclusion is modified to read: $\overline{L + A}$ is a generator.

If A is bounded and dissipative, then 18.14 implies that $L + A$ is a generator. Proof: $\forall\, x \in \text{Dom } L$,

$$\begin{aligned} ||Ax|| &\leq ||A||\ ||x|| \\ &\leq ||A||\ ||x|| + b||Lx|| \quad (0 < b < 1). \end{aligned}$$

18.22 EXAMPLE Let $X = \mathcal{B}(H)$ (H a complex Hilbert space). Consider a contractive semigroup $\{T_t : t \geq 0\}$. Assume that $\{T_t : t \geq 0\}$ is weak* continuous and, in addition, that $\forall\, t \geq 0$, the map $T_t : \mathcal{B}(H) \to \mathcal{B}(H)$ is positive and normal -- then the predual semigroup $\{(T_t)_* : t \geq 0\}$ is strongly continuous (cf. 17.44) and contractive $(||(T_t)_*|| = ||((T_t)_*)^*|| = ||T_t|| \leq 1)$. Suppose now that $A \in \mathcal{B}(H)$ is the dual of an element T of $\underline{L}_1(H)$: $A = T^*$ -- then $T - ||T||I$ is dissipative (cf. 18.10), thus

$$L_* + T - ||T||I$$

is a generator, so $L_* + T$ is the generator of a strongly continuous semigroup (cf. 17.18). Therefore

$$(L_* + T)^* = L + A$$

is the weak* generator of a weak* continuous semigroup on $\mathcal{B}(H)$.

[Note: The growth bound on the semigroup per $L_* + T$ is $\exp(||T||t)$, hence the growth bound on the semigroup per $L + A$ is $\exp(||A||t)$.]

The formulation of 18.11 simplifies if L is bounded.

18.23 <u>THEOREM</u> Let $L \in \mathcal{B}(X)$ -- then the norm continuous semigroup $\{e^{tL}:t \geq 0\}$ is contractive iff L is dissipative.

Of course, we need only deal with the sufficiency, the crunch being 18.27 infra.

18.24 <u>RAPPEL</u> Let $L \in \mathcal{B}(X)$ -- then the frontier $\partial\sigma(L)$ of the spectrum $\sigma(L)$ is contained in the approximate point spectrum of L.

[Note: This means that given $z \in \partial\sigma(L)$, $\exists$ a sequence $x_n \in X:||x_n|| = 1$ and

$$\lim_{n \to \infty} ||Lx_n - zx_n|| = 0.]$$

18.25 <u>LEMMA</u> Suppose that $L \in \mathcal{B}(X)$ is dissipative -- then

$$\partial\sigma(L) \subset \{z:\text{Re } z \leq 0\}.$$

<u>PROOF</u> Fix $z \in \partial\sigma(L)$ and choose the x_n as above. Put $\lambda_n = \lambda_{x_n}$ (cf. 18.6), thus

$$\text{Re } \lambda_n(Lx_n) \leq 0.$$

But

$$\lambda_n(Lx_n - zx_n) \to 0 \quad (n \to \infty)$$

=>

$$\lambda_n(Lx_n) - z\lambda_n(x_n) \to 0 \quad (n \to \infty)$$

=>

$$\lambda_n(Lx_n) - z \to 0 \quad (n \to \infty)$$

=>

$$\text{Re } \lambda_n(Lx_n) \to \text{Re } z \quad (n \to \infty).$$

Therefore $\text{Re } z \leq 0$.

[Note: Recall that $\lambda_n(x_n) = ||x_n||^2 = ||\lambda_n||^2 = 1$, so

$$|\lambda_n(Lx_n - zx_n)| \leq ||\lambda_n|| \, ||Lx_n - zx_n||$$

$$= ||Lx_n - zx_n|| \to 0 \quad (n \to \infty).]$$

18.26 <u>RAPPEL</u> Let $L \in \mathcal{B}(X)$ -- then the set of $z \in \sigma(L)$ such that z is not an eigenvalue and $\text{Ran}(zI - L)$ is closed but not all of X is an open subset of $\underline{C}$.

18.27 <u>LEMMA</u> Suppose that $L \in \mathcal{B}(X)$ is dissipative -- then $\rho(L) \supset \underline{R}_{>0}$.

<u>PROOF</u> $\forall\ r > 0$, $rI - L$ is injective, hence r is not an eigenvalue. On the other hand, $rI - L$ is bounded, thus closed, so $(rI - L)^{-1}$ is closed. But

$$(rI - L)^{-1}: \text{Ran}(rI - L) \to X$$

is bounded. Therefore its domain $\text{Ran}(rI - L)$ is closed. And, in fact, Ran(rI - L) = X (otherwise, use of 18.26 would lead to a contradiction of 18.24).]

18.28 <u>RAPPEL</u> Let a,b be two points in a Banach space E -- then the function

$$t \to ||a + tb|| \quad (t \in \underline{R})$$

has a derivative on the right and a derivative on the left at every $t_0 \in \underline{R}$.

In 18.28, take $E = \mathcal{B}(X)$, $a = I$, $b = L$, $t_0 = 0$ -- then

$$\lim_{t \downarrow 0} \frac{||I + tL|| - 1}{t}$$

exists.

Notation: Given a unit vector $x \in X$, fix $\lambda_x \in \mathcal{D}(x)$ $(\Rightarrow \lambda_x(x) = ||x||^2 = ||\lambda_x||^2 = 1)$ and put

$$\Theta(L) = \sup_{x:||x|| = 1} \text{Re } \lambda_x(Lx).$$

18.29 <u>LEMMA</u> $\forall\, L \in \mathcal{B}(X)$,

$$\Theta(L) \leq \lim_{t \downarrow 0} \frac{||I + tL|| - 1}{t}.$$

<u>PROOF</u> We have

$$|\lambda_x(x + tLx)| = |1 + t\text{Re } \lambda_x(Lx) + \sqrt{-1}\; t \text{ Im } \lambda_x(Lx)|$$

$$= ((1 + t\text{Re } \lambda_x(Lx))^2 + t^2(\text{Im } \lambda_x(Lx))^2)^{1/2}$$

$$= 1 + t\text{Re } \lambda_x(Lx) + o(t).$$

Therefore

$$\text{Re } \lambda_x(Lx) = \lim_{t \downarrow 0} \frac{|\lambda_x(x + tLx)| - 1}{t}.$$

But

$$|\lambda_x(x + tLx)| \leq ||\lambda_x|| \; ||(I + tL)x||$$

$$\leq ||I + tL||.$$

Therefore

$$\text{Re } \lambda_x(Lx) \leq \lim_{t \downarrow 0} \frac{||I + tL|| - 1}{t}$$

=>

$$\Theta(L) \leq \lim_{t \downarrow 0} \frac{||I + tL|| - 1}{t}.$$

18.30 <u>LEMMA</u> Let $L \in \mathcal{B}(X)$ and suppose that

$$\lim_{t \downarrow 0} \frac{||I + tL|| - 1}{t} \leq 0.$$

Then L is dissipative, hence $\{e^{tL}:t \geq 0\}$ is a norm continuous contractive semigroup (cf. 18.23).

<u>PROOF</u> For $\Theta(L) \leq 0$, thus one may cite 18.6.

[Note: To establish that L is dissipative, it suffices to work with unit vectors (see the proof of 18.6).]

Now specialize and take $X = \mathcal{B}(H)$ (H a complex Hilbert space).

18.31 <u>THEOREM</u> Let $L \in \mathcal{B}(\mathcal{B}(H))$ and assume that L is *-linear — then e^{tL}

is positive for all positive t iff

$$LI + U^*(LI)U \geq (LU^*)U + U^*(LU)$$

for all $U \in \mathcal{U}(\mathcal{H})$.

The initial step is to reduce matters to when $LI = 0$. Thus let $K = -LI/2$ and define

$$L' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

by

$$L'A = KA + AK.$$

Then $\forall\, t \in \underline{R}$,

$$e^{tL'}(A) = e^{tK}Ae^{tK}$$

and $\{e^{tL'} : t \in \underline{R}\}$ is a norm continuous group of positive maps on $\mathcal{B}(\mathcal{H})$. But in norm, $\forall\, t \geq 0$

$$e^{t(L+L')}(A) = \lim_{n \to \infty} (e^{(t/n)L}e^{(t/n)L'})^n A \quad \text{(cf. 17.18)}.$$

Therefore e^{tL} is positive for all positive t iff $e^{t(L+L')}$ is positive for all positive t. To complete the reduction, it remains only to note that

$$(L + L')I = LI - LI/2 - LI/2 = 0.$$

Obviously,

$$LI = 0 \Rightarrow e^{tL}(I) = I.$$

So, if e^{tL} is positive, then

$$||e^{tL}|| = 1 \quad \text{(cf. 11.27)}.$$

Now let $U \in \mathcal{U}(\mathcal{H})$:

$$\langle x, e^{tL}(U^*)e^{tL}(U)x\rangle$$

$$= \langle e^{tL}(U)x, e^{tL}(U)x \rangle$$

$$= ||e^{tL}(U)x||^2$$

$$\leq ||e^{tL}(U)||^2 ||x||^2$$

$$\leq (||e^{tL}|| \; ||U||)^2 ||x||^2$$

$$\leq \langle x,x \rangle$$

$\Rightarrow$

$$e^{tL}(U^*)e^{tL}(U) \leq I.$$

Differentiation at $t = 0$ then gives

$$0 \geq (LU^*)U + U^*(LU).$$

[Note: At this point, replace "L" by $L + L'$ to get

$$0 \geq ((L + L')U^*)U + U^*((L + L')U)$$

or still,

$$0 \geq (LU^*)U + U^*(LU)$$

$$- (LI/2)U^*U + U^*(- LI/2)U + U^*(- LI/2)U + U^*U(- LI/2)$$

or still,

$$LI + U^*(LI)U \geq (LU^*)U + U^*(LU).]$$

Conversely, since e^{tL} is unital, to prove that e^{tL} is positive, it suffices to prove that $||e^{tL}|| \leq 1$ (cf. 11.13) and for this, we shall apply 18.30. To begin with,

$$||I + tL|| = \sup_{U \in \mathcal{U}(H)} ||U + tLU|| \quad \text{(cf. 11.26)}.$$

But

$$||U + tLU||^2 = ||(U + tLU)^*(U + tLU)||$$

$$= ||I + t((LU^*)U + U^*(LU)) + t^2(LU)^*LU||$$

which, for $0 \le t \le \frac{1}{2||L||}$, is

$$\le 1 + t^2||L||^2 \quad \text{(see below)}.$$

So

$$||I + tL|| \le (1 + t^2||L||^2)^{1/2}$$

$$\Rightarrow \quad \lim_{t \downarrow 0} \frac{||I + tL|| - 1}{t} \le \lim_{t \downarrow 0} \frac{(1 + t^2||L||^2)^{1/2} - 1}{t} = 0.$$

Therefore $||e^{tL}|| \le 1$ (cf. 18.30).

18.32 <u>RAPPEL</u> Let $T \in \mathcal{B}(\mathcal{H})_{SA}$ — then

$$||T|| = \sup_{x \in \underline{S}(\mathcal{H})} |\langle x, Tx\rangle|.$$

The assumption is that

$$0 \ge (LU^*)U + U^*(LU)$$

and, from the definitions,

$$||(LU^*)U + U^*(LU)|| \le 2||L||.$$

This said, given $x \in \underline{S}(\mathcal{H})$, consider

$$\langle x, (I + t((LU^*)U + U^*(LU)) + t^2(LU)^*LU)x\rangle$$

$$= 1 + t\langle x, ((LU^*)U + U^*(LU))x\rangle + t^2\langle x, (LU)^*LUx\rangle.$$

Then

$$0 \le t \le \frac{1}{2||L||}$$

$$\Rightarrow$$

$$1 + t\langle x, ((LU^*)U + U^*(LU))x\rangle \ge 0.$$

Thus, with this restriction on t, we have

$$||I + t((LU^*)U + U^*(LU)) + t^2(LU)^*LU||$$

$$= \sup_{x\in \underline{S}(H)} |\langle x, (I + t((LU^*)U + U^*(LU)) + t^2(LU)^*LU)x\rangle|$$

$$= \sup_{x\in \underline{S}(H)} \langle x, (I + t((LU^*)U + U^*(LU)) + t^2(LU)^*LU)x\rangle$$

$$= \sup_{x\in \underline{S}(H)} 1 + t\langle x, ((LU^*)U + U^*(LU))x\rangle + t^2\langle x, (LU)^*LUx\rangle$$

$$\le \sup_{x\in \underline{S}(H)} 1 + t^2\langle x, (LU)^*LUx\rangle$$

$$\le 1 + t^2||L||^2.$$

§19. *DISSIPATIONS*

Fix a complex Hilbert space $\mathcal{H}$. Let $L \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ and assume that L is $*$-linear. Consider the following conditions.

1. e^{tL} is positive $(t \geq 0)$.

2. $(rI - L)^{-1}$ is positive $(r > ||L||)$.

3. $B^*(LA)B \geq 0$ if $AB = 0$, where $A \in \mathcal{B}(\mathcal{H})_+$, $B \in \mathcal{B}(\mathcal{H})$.

4. $\forall\ A \in \mathcal{B}(\mathcal{H})_{SA}$,

$$LA^2 + A(LI)A \geq (LA)A + A(LA).$$

19.1 <u>THEOREM</u> We have

$$4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 4.$$

The proof is spelled out in the lines below.

19.2 <u>LEMMA</u> Assume 4, let $A \in \mathcal{B}(\mathcal{H})_+$, let $\omega \in S(\mathcal{B}(\mathcal{H}))$, and suppose that $\omega(A) = 0$ — then $\omega(LA) \geq 0$.

<u>PROOF</u> $\forall\ B \in \mathcal{B}(\mathcal{H})$,

$$\omega(\sqrt{A}B) = 0 = \omega(B\sqrt{A}) \quad (\text{cf. } 2.9).$$

But

$$LA + \sqrt{A}(LI)\sqrt{A} \geq (L\sqrt{A})\sqrt{A} + \sqrt{A}(L\sqrt{A})$$

$\Rightarrow$

$$\omega(LA) + \omega(\sqrt{A}(LI)\sqrt{A}) \geq \omega((L\sqrt{A})\sqrt{A}) + \omega(\sqrt{A}(L\sqrt{A}))$$

$\Rightarrow$

$$\omega(LA) \geq 0.$$

19.3 <u>RAPPEL</u> Let $T \in \mathcal{B}(H)_{SA}$ — then $T \in \mathcal{B}(H)_+$ iff $\forall\ \omega \in S(\mathcal{B}(H))$, $\omega(T) \geq 0$.

<u>4 => 3</u>: $\forall\ \omega \in S(\mathcal{B}(H))$,

$$\omega(B^*AB) = 0.$$

On the other hand, $\omega(B^*\_\_B)$ is at least a positive linear functional, thus on the basis of 19.2,

$$\omega(B^*(LA)B) \geq 0.$$

But

$$B^*(LA)B \in \mathcal{B}(H)_{SA},$$

so

$$B^*(LA)B \geq 0 \quad \text{(cf. 19.3)}.$$

<u>3 => 2</u>: It will be enough to show that if $A \in \mathcal{B}(H)_{SA}$ and if $(rI - L)A \in \mathcal{B}(H)_+$, then $A \in \mathcal{B}(H)_+$. To this end, write $A = A^+ - A^-$. Since $A^+A^- = 0$, we have $A^-(LA^+)A^- \geq 0$. In addition,

$$\begin{aligned} 0 &\leq A^-(I - L/r)(A)A^- \\ &= A^-AA^- - A^-(L/r)(A)A^- \\ &= A^-(A^+ - A^-)A^- - A^-(L/r)(A^+ - A^-)A^- \\ &= -(A^-)^3 - A^-(L/r)(A^+)A^- + A^-(L/r)(A^-)A^- \end{aligned}$$

$\Rightarrow$

$$(A^-)^3 \le (A^-)^3 + A^-(L/r)(A^+)A^-$$

$$\le A^-(L/r)(A^-)A^-.$$

If $A^- \neq 0$, then

$$||A^-||^3 = ||(A^-)^3||$$

$$\le ||A^-(L/r)(A^-)A^-||$$

$$\le ||L/r|| \; ||A^-||^3$$

$$< ||A^-||^3,$$

a contradiction. Therefore $A^- = 0$, so $A = A^+ \in \mathcal{B}(\mathcal{H})_+$.

[Note: The map

$$(rI - L)^{-1}:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

is $*$-linear, hence respects $\mathcal{B}(\mathcal{H})_{SA}$.]

2 => 1: This can be seen by writing

$$e^{tL} = \lim_{n \to \infty} (I - \frac{t}{n} L)^{-n} \quad \text{(cf. 17.41)}.$$

19.4 LEMMA Assume 1 and further that $LI = 0$ — then $\forall A \in \mathcal{B}(\mathcal{H})_{SA}$,

$$LA^2 \ge (LA)A + A(LA).$$

PROOF Obviously,

$$LI = 0 \Rightarrow e^{tL}(I) = I.$$

So, if e^{tL} is positive, then $||e^{tL}|| = 1$ (cf. 11.27), hence

$$e^{tL}(A^2) \ge e^{tL}(A)^2 \quad \text{(cf. 11.25)}.$$

Differentiation at $t = 0$ gives

$$LA^2 \geq (LA)A + A(LA).$$

1 => 4: As in the proof of 18.31, introduce L' and then pass to $L + L'$. Since $(L + L')I = 0$, from the lemma we have

$$(L + L')A^2 \geq ((L + L')A)A + A((L + L')A)$$

or still,

$$LA^2 + (- LI/2)A^2 + A^2(- LI/2)$$

$$\geq (LA)A + (- LI/2)A^2 + A(- LI/2)A$$

$$+ A(LA) + A(- LI/2)A + A^2(- LI/2)$$

or still,

$$LA^2 + A(LI)A \geq (LA)A + A(LA).$$

19.5 REMARK Replace 4 by 4': $\forall A \in \mathcal{B}(H)_{SA}$,

$$LA^2 \geq (LA)A + A(LA).$$

Then the proof that 4 => 3 goes through under 4', thus 4' implies that e^{tL} is positive $(t \geq 0)$.

19.6 LEMMA Suppose that $LI \leq 0$ — then LI is dissipative.

PROOF From spectral theory (cf. infra),

$$||(rI - LI)^{-1}|| \leq \frac{1}{r} \ (r > 0).$$

So, $\forall x \in H$,

$$||(rI - LI)x|| = ||(rI - LI)(rI - LI)^{-1}y||$$

$$= ||y||.$$

But

$$||(rI - LI)^{-1}y|| \le ||(rI - LI)^{-1}|| \; ||y||$$

$$\le \frac{||y||}{r}$$

=>

$$||y|| = ||(rI - LI)x||$$

$$\ge r||(rI - LI)^{-1}y||$$

$$= r||x||.$$

[Note: Write

$$(rI - LI)^{-1} = \int_{\sigma(LI)} \frac{1}{r - \lambda} dE_\lambda^{LI}.$$

Then

$$LI \le 0 \Rightarrow \sigma(LI) \subset \underline{R}_{\le 0}$$

=>

$$||(rI - LI)^{-1}|| \le \sup_{\lambda \in \sigma(LI)} \frac{1}{r - \lambda} \le \frac{1}{r}.]$$

19.7 <u>LEMMA</u> Suppose that $LI \le 0$ -- then $\forall \, t \ge 0$,

$$||e^{tL}(I)|| \le 1.$$

<u>PROOF</u> The assignment $t \to e^{tL(I)}$ is a norm continuous semigroup on H (not $B(H)$...). Since its generator LI is dissipative (cf. 19.6), we have

$$||e^{tL(I)}|| \le 1 \quad \text{(cf. 18.23)}.$$

But

$$e^{tL}(I) = e^{tL(I)}.$$

19.8 <u>THEOREM</u> e^{tL} is a positive contraction for all $t \geq 0$ iff

$$LA^2 \geq (LA)A + A(LA)$$

for all $A \in \mathcal{B}(\mathcal{H})_{SA}$.

<u>PROOF</u> If the e^{tL} are positive contractions, then

$$e^{tL}(I)e^{tL}(I) \leq I$$

$$\Rightarrow$$

$$(LI)I + I(LI) \leq 0$$

$$\Rightarrow LI \leq 0.$$

So, $\forall\, x \in \mathcal{H}$ & $\forall\, A \in \mathcal{B}(\mathcal{H})_{SA}$,

$$\langle x, A(LI)Ax\rangle = \langle Ax, (LI)Ax\rangle \leq 0$$

$$\Rightarrow$$

$$A(LI)A \leq 0.$$

$$\Rightarrow$$

$$LA^2 \geq LA^2 + A(LI)A$$

$$\geq (LA)A + A(LA) \quad \text{(cf. 1 => 4 in 19.1).}$$

Conversely, the relation

$$LA^2 \geq (LA)A + A(LA) \qquad (A \in \mathcal{B}(\mathcal{H})_{SA})$$

implies that the e^{tL} are positive (cf. 19.5) and, on general grounds (cf. 11.27),

$$||e^{tL}|| = ||e^{tL}(I)||.$$

Now take A = I to get

$$LI \geq 2LI \Rightarrow LI \leq 0.$$

Therefore

$$||e^{tL}(I)|| \leq 1 \quad \text{(cf. 19.7)}.$$

Let A be a $*$-algebra.

- A <u>$*$-dissipation</u> is a $*$-linear map $\delta: A \to A$ such that

$$\delta(\xi^*\xi) \geq \delta(\xi^*)\xi + \xi^*\delta(\xi) \quad (\xi \in A).$$

- A <u>$*$-derivation</u> is a $*$-linear map $\delta: A \to A$ such that

$$\delta(\xi\eta) = \delta(\xi)\eta + \xi\delta(\eta) \quad (\xi,\eta \in A).$$

[Note: Recall that δ is $*$-linear if δ is linear and $\delta(\xi)^* = \delta(\xi^*)$ (thus $\xi = \xi^* \Rightarrow \delta(\xi)^* = \delta(\xi)$).]

<u>N.B.</u> While these definitions are the point of departure for a "general theory", we shall deal only with the case where $A = B(H)$, H a complex Hilbert space.

19.9 <u>EXAMPLE</u> If $\delta: B(H) \to B(H)$ is a $*$-dissipation, then

$$\delta(I) = \delta(II) \geq 2\delta(I) \Rightarrow \delta(I) \leq 0.$$

19.10 <u>EXAMPLE</u> If $\delta: B(H) \to B(H)$ is a $*$-derivation, then

$$\delta(I) = \delta(II) = 2\delta(I) \Rightarrow \delta(I) = 0.$$

19.11 <u>THEOREM</u> Suppose that $\delta: B(H) \to B(H)$ is a $*$-dissipation — then δ is dissipative.

PROOF Fix $A \in \mathcal{B}(H)$ and let $\lambda \in \mathfrak{T}(A^*A):||\lambda|| = 1$ ($\Rightarrow \lambda(A^*A) = ||A^*A|| = ||A||^2$) -- then $\lambda(I) = 1$ (hence λ is positive (cf. 11.12)). Proof: Write $\lambda(I) = a + \sqrt{-1}\, b$ and note that

$$a^2 + b^2 = |\lambda(I)|^2 \leq 1,$$

$$(a - 2)^2 + b^2 = |\lambda(I - 2A^*A/||A||^2)|^2 \leq 1,$$

from which $a = 1$ and $b = 0$. This settled, define $\lambda_A \in \mathcal{B}(H)^*$ by

$$\lambda_A(B) = \lambda(A^*B).$$

Then $\forall\ B:||B|| = 1$, we have

$$\begin{aligned} |\lambda_A(B)|^2 &= |\lambda(A^*B)|^2 \\ &\leq \lambda(A^*A)\lambda(B^*B) \quad \text{(cf. 2.9)} \\ &\leq ||A||^2\, ||B||^2 \\ &\leq ||A||^2 \end{aligned}$$

$\Rightarrow$

$$||\lambda_A|| \leq ||A||.$$

On the other hand,

$$||A||^2 = \lambda(A^*A) = \lambda_A(A) \leq ||\lambda_A||\ ||A||$$

$\Rightarrow$

$$||A|| \leq ||\lambda_A||.$$

Therefore

$$\lambda_A(A) = ||A||^2 = ||\lambda_A||\ ||A||$$

$\Rightarrow$

$$\lambda_A \in \mathfrak{T}(A).$$

Next

$$2\mathrm{Re}\ \lambda_A(\delta(A)) = \lambda_A(\delta(A)) + \overline{\lambda_A(\delta(A))}$$

$$= \lambda(A^*\delta(A)) + \overline{\lambda(A^*\delta(A))}$$

$$= \lambda(A^*\delta(A)) + \lambda((A^*\delta(A))^*) \quad \text{(cf. 2.9)}$$

$$= \lambda(A^*\delta(A)) + \lambda(\delta(A)^*A)$$

$$= \lambda(\delta(A^*)A) + \lambda(A^*\delta(A))$$

$$\leq \lambda(\delta(A^*A))$$

$$= - \lambda(\delta(||A||^2 I - A^*A)) + ||A||^2\lambda(\delta(I)).$$

But

$$\delta(I) \leq 0 \Rightarrow \lambda(\delta(I)) \leq 0.$$

And

$$- \lambda(\delta(||A||^2 I - A^*A))$$

$$= - \lambda(\delta(B^2)) \quad (B = (||A||^2 I - A^*A)^{1/2})$$

$$= \lambda(- \delta(B^2))$$

$$\leq \lambda(- \delta(B)B - B\delta(B))$$

$$= - \lambda(\delta(B)B) - \lambda(B\delta(B))$$

$$= 0,$$

since, e.g.,

$$|\lambda(\delta(B)B)|^2$$

$$\leq \lambda(\delta(B)^2)\lambda(B^2) \quad \text{(cf. 2.9)}$$

$$= \lambda(\delta(B)^2)(||A||^2\lambda(I) - \lambda(A^*A))$$

$$= \lambda(\delta(B)^2)(0)$$

$$= 0.$$

Therefore δ is dissipative (see below).

[Note: $\forall\, A \in \mathcal{B}(H)$,

$$\text{Re } \lambda_A(\delta(A)) \leq 0.$$

So

$$||\lambda_A||\ ||A|| = \text{Re } \lambda_A(A)$$

$$\leq \text{Re } \lambda_A((I - r\delta)(A)) \quad (r > 0)$$

$$\leq ||\lambda_A||\ ||(I - r\delta)(A)||$$

$\Rightarrow$

$$||A|| \leq ||(I - r\delta)(A)||.$$

I.e.: δ is dissipative.]

19.12 LEMMA Suppose that $\delta:\mathcal{B}(H) \to \mathcal{B}(H)$ is a *-dissipation -- then δ is bounded.

PROOF In fact, δ is dissipative (cf. 19.11), hence admits closure (cf. 18.3) or still, δ is closed, thus is bounded (closed graph theorem).

[Note: In particular, *-derivations are bounded.]

19.13 REMARK If $\delta:\mathcal{B}(H) \to \mathcal{B}(H)$ is a *-dissipation, then δ is the generator of a norm continuous contractive semigroup $\{e^{t\delta}:t \geq 0\}$ (cf. 18.23).

19.14 THEOREM Suppose that $\delta:\mathcal{B}(H) \to \mathcal{B}(H)$ is a bounded *-linear map -- then δ is a *-dissipation iff $\forall\ A \in \mathcal{B}(H)$,

$$e^{t\delta}(A^*A) \geq e^{t\delta}(A^*)e^{t\delta}(A) \quad (t \geq 0).$$

PROOF Differentiation at $t = 0$ gives

$$\delta(A^*A) \geq \delta(A^*)A + A^*\delta(A).$$

Conversely, if δ is a *-dissipation, then $e^{t\delta}$ is positive (cf. 19.5). Fix $A \in \mathcal{B}(H)$ and let

$$f(t) = e^{t\delta}(A^*A) - e^{t\delta}(A^*)e^{t\delta}(A) \quad (t \geq 0).$$

Then

$$f'(t) = \delta e^{t\delta}(A^*A) - (\delta e^{t\delta}(A^*))e^{t\delta}(A) - e^{t\delta}(A^*)(\delta e^{t\delta}(A))$$

=>

$$f(t) - e^{t\delta}f(0) = \int_0^t \frac{d}{ds}\,(e^{(t-s)\delta}f(s))ds$$

$$= -\int_0^t e^{(t-s)\delta}\delta f(s)ds + \int_0^t e^{(t-s)\delta}\,\frac{d}{ds}\,f(s)ds$$

$$= \int_0^t e^{(t-s)\delta}(\delta(e^{s\delta}(A^*)e^{s\delta}(A))$$

$$- (\delta e^{s\delta}(A^*)e^{s\delta}(A) - e^{s\delta}(A^*)(\delta e^{s\delta}(A)))ds$$

$$\geq 0$$

=>

$$f(t) \geq e^{\delta t}f(0) = 0$$

=>

$$e^{t\delta}(A^*A) \geq e^{t\delta}(A^*)e^{t\delta}(A) \quad (t \geq 0).$$

We shall now narrow the focus of 19.1.

Let $L \in \mathcal{B}(\mathcal{B}(H))$ and take L *-linear -- then L is said to be <u>conditionally completely positive</u> if $\forall\, n \in \underline{N}$ and for all

$$\begin{cases} A_1,\ldots,A_n \in \mathcal{B}(H) \\ x_1,\ldots,x_n \in H, \end{cases}$$

we have

$$\sum_{i,j=1}^{n} \langle x_i, L(A_i^* A_j) x_j \rangle \geq 0$$

whenever $A_1 x_1 + \cdots + A_n x_n = 0$.

19.15 <u>LEMMA</u> L is conditionally completely positive iff $\forall\, n \in \underline{N}$ and for all

$$\begin{cases} A_1,\ldots,A_n \in \mathcal{B}(H) \\ B_1,\ldots,B_n \in \mathcal{B}(H), \end{cases}$$

we have

$$\sum_{i,j=1}^{n} B_i^* L(A_i^* A_j) B_j \geq 0$$

whenever $A_1 B_1 + \cdots + A_n B_n = 0$.

Let L_n be the extension of L to $M_n(\mathcal{B}(H))$.

19.16 <u>LEMMA</u> L is conditionally completely positive iff $\forall\, n \in \underline{N}$ and for all finite collections

$$\underline{A}_k, \underline{B}_k \in M_n(\mathcal{B}(H)),$$

we have

$$\sum_{k,\ell} \underline{B}_k^* L_n(\underline{A}_k^* \underline{A}_\ell) \underline{B}_\ell \geq 0$$

whenever $\sum_k \underline{A}_k \underline{B}_k = 0$.

[Note: The verification hinges on 11.5 and 11.9.]

<u>N.B.</u> $\forall\, n \in \underline{N}$,

$$(e^{tL})_n = e^{tL_n}.$$

19.17 <u>THEOREM</u> Let $L \in \mathcal{B}(\mathcal{B}(H))$ and assume that L is $*$-linear — then e^{tL} is completely positive for all positive t iff L is conditionally completely positive.

<u>PROOF</u> Suppose first that the e^{tL} are completely positive. Fix $n \in \underline{N}$ and let

$$\left[\begin{array}{l} A_1, \ldots, A_n \in \mathcal{B}(H) \\ \\ B_1, \ldots, B_n \in \mathcal{B}(H) \end{array}\right.$$

subject to $A_1 B_1 + \cdots + A_n B_n = 0$ — then

$$\sum_{i,j=1}^{n} B_i^* \left(\frac{e^{tL}(A_i^* A_j) - A_i^* A_j}{t}\right) B_j$$

$$= \frac{1}{t} \sum_{i,j=1}^{n} B_i^* e^{tL}(A_i^* A_j) B_j \geq 0$$

=>

$$\sum_{i,j=1}^{n} B_i^* L(A_i^* A_j) B_j \geq 0 \quad (t \to 0).$$

Conversely, thanks to 19.16,

$$\underline{AB} = 0 \Rightarrow \underline{B}^* L_n(\underline{A}^* \underline{A}) \underline{B} \geq 0.$$

Therefore condition 3 of 19.1 is in force (details below), so e^{tL_n} is positive. As this is true $\forall$ n, it follows that e^{tL} is completely positive.

[Note: If $\underline{A} \in M_n(\mathcal{B}(H))_+$, then $\underline{A} = \sqrt{\underline{A}}\,\sqrt{\underline{A}}$ and $\underline{AB} = 0 \Leftrightarrow \sqrt{\underline{A}}\,\underline{B} = 0$ (recall that $\sqrt{\underline{A}}$ is the strong limit of a sequence of polynomials in $\underline{A}$).]

19.18 <u>THEOREM</u> Let $L \in \mathcal{B}(\mathcal{B}(H))$ and assume that L is *-linear -- then e^{tL} is completely positive and contractive for all positive t iff $\forall\, n \in \underline{N}$ and for all

$$A_1, \ldots, A_n \in \mathcal{B}(H),$$

we have

$$[L(A_i^* A_j)] \geq [L(A_i^*) A_j + A_i^* L(A_j)].$$

Here is an initial preliminary to the proof.

19.19 <u>LEMMA</u> Let $\Phi: \mathcal{B}(H) \to \mathcal{B}(H)$ be a bounded *-linear map -- then Φ is completely positive iff $\forall\, n \in \underline{N}$ and for all

$$A_1, \ldots, A_n \in \mathcal{B}(H),$$

we have

$$||\Phi|| [\Phi(A_i^* A_j)] \geq [\Phi(A_i^*) \Phi(A_j)].$$

PROOF That the condition is necessary is a consequence of 11.24 and 12.15 ($||\Phi|| = ||\Phi_n||$). That the condition is sufficient is immediate: $\forall\ x_1,\ldots,x_n \in H$,

$$||\Phi|| \sum_{i,j=1}^{n} \langle x_i, \Phi(A_i^* A_j) x_j \rangle \geq ||\sum_{k=1}^{n} \Phi(A_k) x_k||^2 \geq 0.$$

So, if $\{e^{tL}: t \geq 0\}$ is a semigroup of completely positive contractions, then

$$[e^{tL}(A_i^* A_j)] \geq ||e^{tL}||\ [e^{tL}(A_i^* A_j)] \geq [e^{tL}(A_i^*) e^{tL}(A_j)]$$

$$\Rightarrow$$

$$[L(A_i^* A_j)] \geq [L(A_i^*) A_j + A_i^* L(A_j)].$$

To reverse this, some additional considerations will be required.

N.B. In its simplest form, the condition on L implies that the e^{tL} are positive contractions (cf. 19.8).

19.20 LEMMA Let $\omega \in S(M_n(B(H)))$. Assume:

$$\omega([A_i^* A_j]) = 0.$$

Then

$$\omega([L(A_i^* A_j)]) \geq 0.$$

PROOF Define $\underline{A}_0 \in M_n(B(H))$ by

$$\begin{bmatrix} A_1 & \cdots & A_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

Then $\omega(\underline{A}_0^*\underline{A}_0) = 0$, thus $\forall\ \underline{X}_0 \in M_n(\mathcal{B}(H))$,

$$\begin{cases} \omega(\underline{X}_0\underline{A}_0) = 0 \\ \omega(\underline{A}_0^*\underline{X}_0) = 0 \end{cases} \qquad \text{(cf. 2.9).}$$

In particular: $\forall\ X_1,\ldots,X_n \in \mathcal{B}(H)$,

$$\begin{cases} \omega([X_iA_j]) = 0 \\ \omega([A_i^*X_j]) = 0. \end{cases}$$

Therefore

$$\begin{aligned} &\omega([L(A_i^*A_j)]) \\ &\quad \geq \omega([L(A_i^*)A_j]) + \omega([A_i^*L(A_j)]) \\ &\quad = 0. \end{aligned}$$

19.21 <u>LEMMA</u> L is conditionally completely positive.

<u>PROOF</u> Fix $n \in \underline{N}$ and let

$$\begin{cases} A_1,\ldots,A_n \in \mathcal{B}(H) \\ x_1,\ldots,x_n \in H \end{cases}$$

subject to $A_1x_1 + \cdots + A_nx_n = 0$ — then the claim is that

$$\sum_{i,j=1}^{n} \langle x_i, L(A_i^*A_j)x_j\rangle \geq 0.$$

Let

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \overset{n}{\oplus} H$$

and let $\omega_{\underline{x}}$ denote the associated positive linear functional:

$$\omega_{\underline{x}}(\underline{A}) = \langle \underline{x}, \underline{A}x \rangle \qquad (\underline{A} \in M_n(\mathcal{B}(H))).$$

Then

$$\omega_{\underline{x}}([A_i^* A_j]) = \sum_{i=1}^{n} \langle x_i, \sum_{j=1}^{n} A_i^* A_j x_j \rangle$$

$$= 0.$$

Therefore

$$\omega_{\underline{x}}([L(A_i^* A_j)]) \geq 0 \qquad \text{(cf. 19.20)}.$$

I.e.:

$$\sum_{i,j=1}^{n} \langle x_i, L(A_i^* A_j) x_j \rangle \geq 0.$$

Consequently, the e^{tL} are completely positive (cf. 19.17).

19.22 <u>THEOREM</u> Let $L \in \mathcal{B}(\mathcal{B}(H))$ and assume that L is *-linear -- then e^{tL} is completely positive and unital iff $LI = 0$ and $\forall\, n \in \underline{N}$ and for all

$$\begin{cases} A_1, \ldots, A_n \in \mathcal{B}(H) \\ B_1, \ldots, B_n \in \mathcal{B}(H), \end{cases}$$

we have

$$\sum_{i,j=1}^{n} B_i^* (L(A_i^* A_j) - L(A_i^*) A_j - A_i^* L(A_j)) B_j \geq 0.$$

PROOF Suppose first that e^{tL} is completely positive and unital. The relation $LI = 0$ being obvious, consider the asserted inequality. Put $A_0 = I$, $B_0 = -\sum_{i=1}^{n} A_i B_i$ -- then $\sum_{i=0}^{n} A_i B_i = 0$. Since L is conditionally completely positive (cf. 19.17),

$$\sum_{i,j=0}^{n} B_i^* L(A_i^* A_j) B_j \geq 0 \quad \text{(cf. 19.15)}$$

or still,

$$\left(-\sum_{i=1}^{n} A_i B_i\right)^* LI \left(-\sum_{i=1}^{n} A_i B_i\right)$$

$$+ \left(-\sum_{i=1}^{n} A_i B_i\right)^* \sum_{j=1}^{n} L(A_j) B_j$$

$$+ \sum_{i=1}^{n} B_i^* L(A_i^*) \left(-\sum_{j=1}^{n} A_j B_j\right) + \sum_{i,j=1}^{n} B_i^* L(A_i^* A_j) B_j \geq 0.$$

But $LI = 0$, hence

$$\sum_{i,j=1}^{n} B_i^* (L(A_i^* A_j) - L(A_i^*) A_j - A_i^* L(A_j)) B_j \geq 0.$$

As for the converse, $LI = 0 \Rightarrow e^{tL}$ unital. Furthermore, if the inequality obtains, then the matrix

$$[L(A_i^* A_j) - L(A_i^*) A_j - A_i^* L(A_j)]$$

lies in $M_n(\mathcal{B}(\mathcal{H}))_+$ (cf. 11.5), so e^{tL} is completely positive (cf. 19.18).

Let $\Phi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be completely positive. Fix $K \in \mathcal{B}(\mathcal{H})$ and define $L: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ by

$$LA = \Phi(A) + K^* A + AK \quad (A \in \mathcal{B}(\mathcal{H})).$$

Then it is clear that L is $*$-linear and conditionally completely positive. The converse is also true and is of pivotal importance for the theory.

19.23 <u>THEOREM</u> Suppose that $L \in \mathcal{B}(\mathcal{B}(H))$ is $*$-linear and conditionally completely positive -- then $\exists$ a completely positive map $\Phi:\mathcal{B}(H) \to \mathcal{B}(H)$ and an operator $K \in \mathcal{B}(H)$ such that

$$LA = \Phi(A) + K^*A + AK \quad (A \in \mathcal{B}(H)).$$

<u>PROOF</u> Fix $x_0 \in \underline{S}(H)$, define K by

$$\begin{aligned}<y,Kx> &= <x_0,LP_{x_0,y}x> \\ &\quad - \frac{1}{2} <x_0,LP_{x_0,x_0}x_0><y,x>,\end{aligned}$$

and then define Φ by

$$\Phi(A) = LA - K^*A - AK \quad (A \in \mathcal{B}(H)).$$

To check that Φ is completely positive, let

$$\begin{cases} A_1,\ldots,A_n \in \mathcal{B}(H) \\ x_1,\ldots,x_n \in H, \end{cases}$$

and consider

$$\sum_{i,j=1}^{n} <x_i,\Phi(A_i^*A_j)x_j>$$

or still,

$$\sum_{i,j=1}^{n} <x_i,L(A_i^*A_j)x_j>$$

$$- \sum_{i,j=1}^{n} <x_i,K^*A_i^*A_jx_j> - \sum_{i,j=1}^{n} <x_i,A_i^*A_jKx_j>,$$

which we claim is the same as

$$\sum_{i,j=0}^{n} \langle x_i, L(A_i^*A_j)x_j\rangle$$

if

$$A_0 = -\sum_{k=1}^{n} A_k P_{x_k,x_0}.$$

Granted this,

$$A_0x_0 + A_1x_1 + \cdots + A_nx_n$$

$$= -A_1P_{x_1,x_0}x_0 - \cdots - A_nP_{x_n,x_0}x_0$$

$$+ A_1x_1 + \cdots + A_nx_n$$

$$= -A_1x_1 - \cdots - A_nx_n + A_1x_1 + \cdots + A_nx_n$$

$$= 0$$

$\Rightarrow$

$$\sum_{i,j=0}^{n} \langle x_i, L(A_i^*A_j)x_j\rangle \geq 0,$$

L being conditionally completely positive. Therefore Φ is completely positive. As for the claim, in the expression

$$\sum_{i,j=0}^{n} \langle x_i, L(A_i^*A_j)x_j\rangle$$

isolate

$$\langle x_0, L(A_0^*A_0)x_0\rangle$$

and

$$\left[\begin{array}{l} \sum_{j=1}^{n} \langle x_0, L(A_0^* A_j) x_j \rangle \\ \\ \sum_{i=1}^{n} \langle x_i, L(A_i^* A_0) x_0 \rangle . \end{array}\right.$$

We have

$$\langle x_0, L(A_0^* A_0) x_0 \rangle$$

$$= \langle x_0, L((\sum_{i=1}^{n} A_i P_{x_i, x_0})^* (\sum_{j=1}^{n} A_j P_{x_j, x_0})) x_0 \rangle$$

$$= \sum_{i,j=1}^{n} \langle x_0, L(P_{x_0, x_i} A_i^* A_j P_{x_j, x_0}) x_0 \rangle .$$

But

$$P_{x_0, x_i} A_i^* A_j P_{x_j, x_0} y$$

$$= P_{x_0, x_i} A_i^* A_j \langle x_0, y \rangle x_j$$

$$= P_{x_0, x_i} \langle x_0, y \rangle A_i^* A_j x_j$$

$$= \langle x_0, y \rangle \langle x_i, A_i^* A_j x_j \rangle x_0$$

$$= \langle A_i x_i, A_j x_j \rangle P_{x_0, x_0} y .$$

So

$$\langle x_0, L(A_0^* A_0) x_0 \rangle$$

$$= \langle x_0, L P_{x_0, x_0} x_0 \rangle \sum_{i,j=1}^{n} \langle A_i x_i, A_j x_j \rangle .$$

Next

$$\sum_{j=1}^{n} \langle x_0, L(A_0^*A_j)x_j\rangle$$

$$= -\sum_{i,j=1}^{n} \langle x_0, L(P_{x_0,x_i} A_i^*A_j)x_j\rangle$$

$$= -\sum_{i,j=1}^{n} \langle L(P_{x_0,x_i} A_i^*A_j)^* x_0, x_j\rangle$$

$$= -\sum_{i,j=1}^{n} \langle L(A_j^*A_i P_{x_i,x_0})x_0, x_j\rangle$$

$$= -\sum_{i,j=1}^{n} \langle L(P_{A_j^*A_i x_i, x_0})x_0, x_j\rangle$$

$$= -\sum_{i,j=1}^{n} \langle x_0, L(P_{A_j^*A_i x_i, x_0})^* x_j\rangle$$

$$= -\sum_{i,j=1}^{n} \langle x_0, L(P_{x_0, A_j^*A_i x_i})x_j\rangle.$$

Analogously

$$\sum_{i=1}^{n} \langle x_i, L(A_i^*A_0)x_0\rangle$$

$$= -\sum_{i,j=1}^{n} \langle L(P_{x_0, A_i^*A_j x_j})x_i, x_0\rangle.$$

It remains to deal with

$$-\sum_{i,j=1}^{n} \langle x_i, K^*A_i^*A_j x_j\rangle - \sum_{i,j=1}^{n} \langle x_i, A_i^*A_j K x_j\rangle$$

or still,

$$-\sum_{i,j=1}^{n} \langle Kx_i, A_i^*A_jx_j\rangle - \sum_{i,j=1}^{n} \langle A_j^*A_ix_i, Kx_j\rangle$$

or still,

$$-\sum_{i,j=1}^{n} \overline{\langle A_i^*A_jx_j, Kx_i\rangle} - \sum_{i,j=1}^{n} \langle A_j^*A_ix_i, Kx_j\rangle$$

or still,

$$-\sum_{i,j=1}^{n} \overline{\langle x_0, L(P_{x_0, A_i^*A_jx_j})x_i\rangle} - \sum_{i,j=1}^{n} \langle x_0, L(P_{x_0, A_j^*A_ix_i})x_j\rangle$$

$$+ \frac{1}{2} \overline{\langle x_0, LP_{x_0,x_0}x_0\rangle} \sum_{i,j=1}^{n} \overline{\langle A_i^*A_jx_j, x_i\rangle}$$

$$+ \frac{1}{2} \langle x_0, LP_{x_0,x_0}x_0\rangle \sum_{i,j=1}^{n} \langle A_j^*A_ix_i, x_j\rangle.$$

Since L is $*$-linear,

$$LP_{x_0,x_0} \in \mathcal{B}(H)_{SA},$$

hence

$$\langle x_0, LP_{x_0,x_0}x_0\rangle$$

is real. And

$$\sum_{i,j=1}^{n} \overline{\langle A_i^*A_jx_j, x_i\rangle}$$

$$= \sum_{i,j=1}^{n} \langle x_i, A_i^*A_jx_j\rangle$$

$$= \sum_{i,j=1}^{n} \langle A_ix_i, A_jx_j\rangle$$

while

$$\sum_{i,j=1}^{n} \langle A_j^* A_i x_i, x_j \rangle$$

$$= \sum_{i,j=1}^{n} \langle A_i x_i, A_j x_j \rangle.$$

We have thus accounted for

$$\langle x_0, L(A_0^* A_0) x_0 \rangle.$$

What's left is obvious.

Put

$$H = \frac{\sqrt{-1}}{2} (K - K^*).$$

Then

$$H \in \mathcal{B}(\mathcal{H})_{SA}.$$

19.24 LEMMA Suppose that $LI = 0$ -- then

$$LA = \Phi(A) - \frac{1}{2} (\Phi(I)A + A\Phi(I)) + \sqrt{-1}\, [H,A] \quad (A \in \mathcal{B}(\mathcal{H})).$$

PROOF In fact, $\forall\, A \in \mathcal{B}(\mathcal{H})$,

$$- \frac{1}{2} (\Phi(I)A + A\Phi(I))$$

$$- \frac{1}{2} ((K - K^*)A - A(K - K^*))$$

$$= \frac{1}{2} ((K^* + K)A + A(K^* + K))$$

$$- \frac{1}{2} ((K - K^*)A - A(K - K^*))$$

$$= \frac{1}{2}\,(2K^*A + 2AK)$$

$$= K^*A + AK.$$

19.25 <u>REMARK</u> Take $\mathcal{H}$ separable, consider the representation of L per 19.24, and impose the additional condition that L is weak* continuous — then Φ is weak* continuous, hence in view of 16.10,

$$\Phi(A) = \sum_{k \in K} V_k^* A V_k \quad (A \in \mathcal{B}(\mathcal{H})).$$

Accordingly, $\forall\, A \in \mathcal{B}(\mathcal{H})$,

$$LA = -\frac{1}{2} \sum_{k \in K} (V_k^* V_k A + A V_k^* V_k - 2V_k^* A V_k) + \sqrt{-1}\,[H,A].$$

[Note: This representation of L is, of course, far from unique.]

§20. *DERIVATIONS*

Fix a complex Hilbert space $\mathcal{H}$. Let $H \in \mathcal{B}(\mathcal{H})_{SA}$ and given $t \in \underline{R}$, put

$$T_t A = e^{\sqrt{-1}\, tH} A e^{-\sqrt{-1}\, tH} \quad (A \in \mathcal{B}(\mathcal{H})).$$

Then the assignment $t \rightarrow T_t$ is a norm continuous one parameter group of *-automorphisms of $\mathcal{B}(\mathcal{H})$ and its generator L is given by

$$LA = \sqrt{-1}\ [H,A] \quad (\text{cf. } 17.16).$$

20.1 <u>THEOREM</u> Suppose that $\{T_t : t \in \underline{R}\}$ is a norm continuous one parameter group of *-automorphisms of $\mathcal{B}(\mathcal{H})$ -- then $\exists\ H \in \mathcal{B}(\mathcal{H})_{SA}$ such that $\forall\ t \in \underline{R}$,

$$T_t A = e^{\sqrt{-1}\, tH} A e^{-\sqrt{-1}\, tH} \quad (A \in \mathcal{B}(\mathcal{H})).$$

The proof depends on two preliminary lemmas.

20.2 <u>LEMMA</u> Let $\delta \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ -- then $e^{t\delta} (t \in \underline{R})$ is a one parameter group of *-automorphisms of $\mathcal{B}(\mathcal{H})$ iff δ is a *-derivation.

<u>PROOF</u> The derivative at $t = 0$ of the function

$$t \rightarrow e^{t\delta}(AB^*) = (e^{t\delta}A)(e^{t\delta}B)^*$$

is

$$\delta(AB^*) = (\delta A)B^* + A(\delta B)^*,$$

thus δ is a *-derivation. To go the other way, assume first that $0 \leq s \leq t$ and consider the function

$$s \rightarrow e^{(t-s)\delta}((e^{s\delta}A)(e^{s\delta}B)^*).$$

Since δ is a *-derivation, the derivative of this function vanishes identically on $[0,t]$, hence

$$e^{t\delta}(AB^*) = (e^{t\delta}A)(e^{t\delta}B)^*.$$

I.e.: $e^{t\delta}$ is a *-homomorphism. An analogous argument shows that the inverse $e^{-t\delta}$ has the same property, from which the assertion.

N.B. A *-derivation $\delta:\mathcal{B}(H) \to \mathcal{B}(H)$ is necessarily bounded (cf. 19.12).

20.3 LEMMA Suppose that $\delta:\mathcal{B}(H) \to \mathcal{B}(H)$ is a *-derivation — then $\exists\, H \in \mathcal{B}(H)_{SA}$ such that $\forall\, A \in \mathcal{B}(H)$,

$$\delta(A) = \sqrt{-1}\,[H,A].$$

PROOF Fix $y \in \underline{S}(H)$ and define $\tilde{H} \in \mathcal{B}(H)$ by

$$\tilde{H}x = \delta(P_{x,y})(y) \qquad (P_{x,y} = \langle y, — \rangle x).$$

Given $A \in \mathcal{B}(H)$, we have

$$(\tilde{H}A - A\tilde{H})(x) = \delta(P_{Ax,y})(y) - A(\delta(P_{x,y})(y))$$

$$= \delta(AP_{x,y})(y) - (A(\delta P_{x,y}))(y)$$

$$= ((\delta A)P_{x,y})(y) + (A(\delta P_{x,y}))(y) - (A(\delta P_{x,y}))(y)$$

$$= ((\delta A)P_{x,y})(y)$$

$$= \delta A(P_{x,y}(y))$$

$$= \delta A(\langle y,y\rangle x)$$

$$= \delta A(x).$$

Therefore

$$\delta A = \tilde{H}A - A\tilde{H}.$$

It remains to be shown that $\tilde{H}$ can be modified to $\sqrt{-1}\,H$, where $H \in \mathcal{B}(\mathcal{H})_{SA}$. To this end, note that

$$\tilde{H}A^* - A^*\tilde{H} = \delta(A^*)$$

$$= \delta(A)^*$$

$$= A^*\tilde{H}^* - \tilde{H}^*A^*$$

or still,

$$\tilde{H}A - A\tilde{H} = A\tilde{H}^* - \tilde{H}^*A.$$

Accordingly,

$$\left(\frac{\tilde{H} - \tilde{H}^*}{2}\right)A - A\left(\frac{\tilde{H} - \tilde{H}^*}{2}\right)$$

$$= \frac{1}{2}\,(\tilde{H}A - A\tilde{H}) + \frac{1}{2}\,(A\tilde{H}^* - \tilde{H}^*A)$$

$$= \frac{1}{2}\,(\tilde{H}A - A\tilde{H}) + \frac{1}{2}\,(\tilde{H}A - A\tilde{H})$$

$$= \delta A,$$

so we can take

$$H = \frac{1}{\sqrt{-1}}\,\left(\frac{\tilde{H} - \tilde{H}^*}{2}\right).$$

Turning to the proof of the theorem, let δ be the generator of $\{T_t : t \in \underline{R}\}$ -- then the first lemma implies that δ is a $*$-derivation and the second lemma implies that

$$\delta = \sqrt{-1}\,[H, —].$$

Finally

$$T_tA = e^{t\delta}A = e^{\sqrt{-1}\, tH}Ae^{-\sqrt{-1}\, tH} \quad \text{(cf. 17.16).}$$

The assumption that $\{T_t : t \in \underline{R}\}$ is norm continuous can be substantially weakened but then matters become more technical in execution.

20.4 <u>THEOREM</u> Suppose that $\{T_t : t \in \underline{R}\}$ is a weak* continuous one parameter group of $*$-automorphisms of $\mathcal{B}(H)$ -- then

$$T_tA = U_tAU_t^* \quad (A \in \mathcal{B}(H)),$$

where $t \to U_t$ is a strongly continuous group of unitary operators on H.

<u>N.B.</u> Stone's theorem says that $\exists$ a selfadjoint operator H (in general unbounded) such that $\forall$ t,

$$U_t = e^{\sqrt{-1}\, H}.$$

20.5 <u>REMARK</u> It can be shown that if $\{T_t : t \in \underline{R}\}$ is a strongly continuous one parameter group of $*$-automorphisms of $\mathcal{B}(H)$, then $\{T_t : t \in \underline{R}\}$ is necessarily norm continuous.

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