RECONSTRUCTION THEORY

Garth Warner

Department of Mathematics

University of Washington

ABSTRACT

Suppose that G is a compact group. Denote by Rep G the category whose objects are the continuous finite dimensional unitary representations of G and whose morphisms are the intertwining operators -- then Rep G is a monoidal $\star$-category with certain properties $P_1, P_2, \ldots$. Conversely, if $\underline{C}$ is a monoidal $\star$-category possessing properties $P_1, P_2, \ldots$, can one find a compact group G, unique up to isomorphism, such that Rep G "is" $\underline{C}$? The central conclusion of reconstruction theory is that the answer is affirmative.

CONTENTS

§1. *MONOIDAL CATEGORIES*

Given categories $\underline{C}$,$\underline{D}$, their <u>product</u> is the category $\underline{C} \times \underline{D}$ defined by

$$\text{Ob}(\underline{C} \times \underline{D}) = \text{Ob } \underline{C} \times \text{Ob } \underline{D}$$

$$\text{Mor}((X,Y),(X',Y')) = \text{Mor}(X,X') \times \text{Mor}(Y,Y')$$

$$\text{id}_{X \times Y} = \text{id}_X \times \text{id}_Y,$$

with composition

$$(f',g') \circ (f,g) = (f' \circ f, g' \circ g).$$

Now take $\underline{C} = \underline{D}$ -- then a <u>monoidal category</u> is a category $\underline{C}$ equipped with a functor $\otimes:\underline{C} \times \underline{C} \to \underline{C}$ (the <u>multiplication</u>) and an object $e \in \text{Ob } \underline{C}$ (the <u>unit</u>), together with natural isomorphisms R, L, and A, where

$$\begin{cases} R_X:X \otimes e \to X \\ L_X:e \otimes X \to X \end{cases}$$

and

$$A_{X,Y,Z}:X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,$$

subject to the following assumptions.

(MC_1) The diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes (Z \otimes W)) & \xrightarrow{A} & (X \otimes Y) \otimes (Z \otimes W) & \xrightarrow{A} & ((X \otimes Y) \otimes Z) \otimes W \\ \text{id} \otimes A \downarrow & & & & \uparrow A \otimes \text{id} \\ X \otimes ((Y \otimes Z) \otimes W) & & \xrightarrow[A]{\hspace{6em}} & & (X \otimes (Y \otimes Z)) \otimes W \end{array}$$

commutes.

(MC_2) The diagram

$$\begin{array}{ccc} X \otimes (e \otimes Y) & \xrightarrow{A} & (X \otimes e) \otimes Y \\ {\scriptstyle id \otimes L}\downarrow & & \downarrow{\scriptstyle R \otimes id} \\ X \otimes Y & = & X \otimes Y \end{array}$$

commutes.

[Note: The "coherency" principle then asserts that "all" diagrams built up from instances of R, L, A (or their inverses), and id by repeated application of $\otimes$ necessarily commute. In particular, the diagrams

$$\begin{array}{ccc} e \otimes (X \otimes Y) & \xrightarrow{A} & (e \otimes X) \otimes Y \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L \otimes id} \\ X \otimes Y & = & X \otimes Y \end{array} \qquad \begin{array}{ccc} X \otimes (Y \otimes e) & \xrightarrow{A} & (X \otimes Y) \otimes e \\ {\scriptstyle id \otimes R}\downarrow & & \downarrow{\scriptstyle R} \\ X \otimes Y & = & X \otimes Y \end{array}$$

commute and $L_e = R_e : e \otimes e \to e$.]

N.B. Technically, the categories

$$\left[\begin{array}{l} \underline{C} \times (\underline{C} \times \underline{C}) \\ \\ (\underline{C} \times \underline{C}) \times \underline{C} \end{array}\right.$$

are not the same so it doesn't quite make sense to say that the functors

$$\underline{C} \times (\underline{C} \times \underline{C}) \to \underline{C} \left[\begin{array}{l} (X,(Y,Z)) \to X \otimes (Y \otimes Z) \\ \\ (f,(g,h)) \to f \otimes (g \otimes h) \end{array}\right.$$

$$(\underline{C} \times \underline{C}) \times \underline{C} \to \underline{C} \quad \begin{cases} ((X,Y),Z) \to (X \otimes Y) \otimes Z \\ ((f,g),h) \to (f \otimes g) \otimes h \end{cases}$$

are naturally isomorphic. However, there is an obvious isomorphism

$$\underline{C} \times (\underline{C} \times \underline{C}) \xrightarrow{\iota} (\underline{C} \times \underline{C}) \times \underline{C}$$

and the assumption is that $A:F \to G \circ \iota$ is a natural isomorphism, where

$$\begin{array}{ccc} \underline{C} \times (\underline{C} \times \underline{C}) & \xrightarrow{F} & \underline{C} \\ \iota \downarrow & & \\ (\underline{C} \times \underline{C}) \times \underline{C} & \xrightarrow[G]{} & \underline{C}. \end{array}$$

Accordingly,

$$\forall \ (X,(Y,Z)) \in \text{Ob } \underline{C} \times (\underline{C} \times \underline{C})$$

and

$$\forall \ (f,(g,h)) \in \text{Mor } \underline{C} \times (\underline{C} \times \underline{C}),$$

the square

$$\begin{array}{ccc} X \otimes (Y \otimes Z) & \xrightarrow{A_{X,Y,Z}} & (X \otimes Y) \otimes Z \\ f \otimes (g \otimes h) \downarrow & & \downarrow (f \otimes g) \otimes h \\ X' \otimes (Y' \otimes Z') & \xrightarrow[A_{X',Y',Z'}]{} & (X' \otimes Y') \otimes Z' \end{array}$$

commutes.

<u>Interchange Principle</u> If

$$\begin{cases} f \in \text{Mor}(X,X') \\ g \in \text{Mor}(Y,Y'), \end{cases}$$

then

$$(f \otimes \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \otimes g) = f \otimes g = (\mathrm{id}_{X'} \otimes g) \circ (f \otimes \mathrm{id}_Y).$$

[Note: Since $\otimes:\underline{C} \times \underline{C} \to \underline{C}$ is a functor, in general

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g').]$$

1.1 EXAMPLE Given a field $\underline{k}$, let $\underline{\mathrm{VEC}}_{\underline{k}}$ be the category whose objects are the vector spaces over $\underline{k}$ and whose morphisms are the linear transformations -- then $\underline{\mathrm{VEC}}_{\underline{k}}$ is monoidal: Take $X \otimes Y$ to be the algebraic tensor product and let e be $\underline{k}$.

[Note: If

$$\begin{cases} f:X \to X' \\ g:Y \to Y', \end{cases}$$

then

$$\otimes (f,g) = f \otimes g:X \otimes Y \to X' \otimes Y'$$

sends $x \otimes y$ to $f(x) \otimes g(y)$.]

Let H and K be complex Hilbert spaces -- then their algebraic tensor product $H \otimes K$ can be equipped with an inner product given on elementary tensors by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle$$

and its completion $H \underline{\otimes} K$ is a complex Hilbert space.

N.B. If

$$\begin{cases} A \in B(H_1, H_2) \\ B \in B(K_1, K_2), \end{cases}$$

then

$$A \otimes B : H_1 \otimes K_1 \to H_2 \otimes K_2.$$

extends by continuity to a bounded linear operator

$$A \underline{\otimes} B : H_1 \underline{\otimes} K_1 \to H_2 \underline{\otimes} K_2.$$

Denote by HILB the category whose objects are the complex Hilbert spaces and whose morphisms are the bounded linear operators.

1.2 EXAMPLE HILB is a monoidal category.

PROOF Define a functor

$$\otimes : \underline{\text{HILB}} \times \underline{\text{HILB}} \to \underline{\text{HILB}}$$

by

$$\otimes(H, K) = H \underline{\otimes} K$$

and

$$\otimes(H_1 \xrightarrow{A} H_2, K_1 \xrightarrow{B} K_2) = A \underline{\otimes} B$$

and let e be $\mathbb{C}$.

1.3 REMARK Both $\underline{\text{Vec}}_{\underline{k}}$ and HILB admit a second monoidal structure: Take for the multiplication the direct sum $\oplus$ and take for the unit the zero object $\{0\}$.

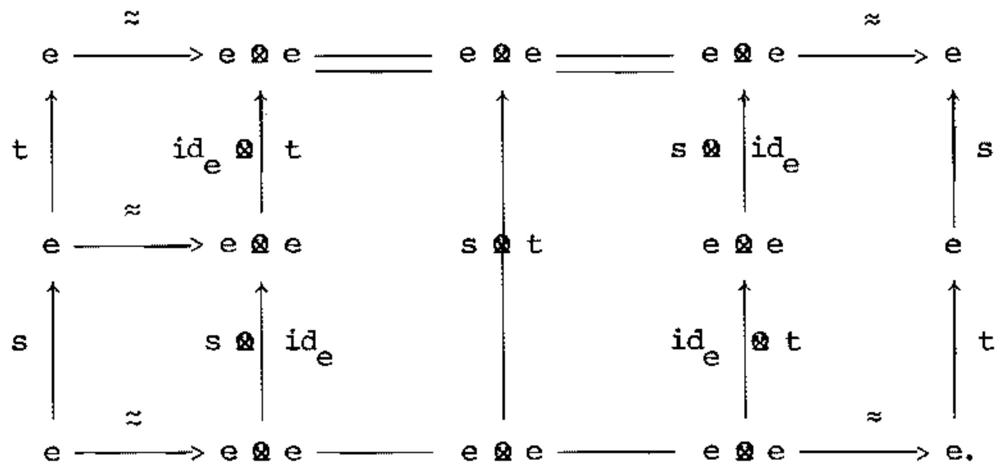
Put

$$\underline{M}(\underline{C}) = \text{Mor}(e, e).$$

Then $\underline{M}(\underline{C})$ is a monoid with categorical composition as monoid multiplication.

1.4 LEMMA The monoid $\underline{M}(\underline{C})$ is commutative.

PROOF Take $s,t \in \underline{M}(\underline{C})$ and consider the commutative diagram



Then

$$R_e^{-1} \circ (s \circ t) \circ R_e = R_e^{-1} \circ (t \circ s) \circ R_e$$

$$\Rightarrow$$

$$s \circ t = t \circ s.$$

Given $f \in \mathrm{Mor}(X,Y)$ and $s \in \underline{M}(\underline{C})$, define $s \cdot f$ to be the composition

$$X \xrightarrow{L^{-1}} e \otimes X \xrightarrow{s \otimes f} e \otimes Y \xrightarrow{L} Y.$$

1.5 LEMMA We have

$$\left[\begin{array}{lcl} id_e \cdot f & = & f \\ s \cdot (t \cdot f) & = & (s \circ t) \cdot f \\ (t \cdot g) \circ (s \cdot f) & = & (t \circ s) \cdot (g \circ f) \\ (s \cdot f) \otimes (t \cdot g) & = & (s \circ t) \cdot (f \otimes g). \end{array}\right.$$

A monoidal category $\underline{C}$ is said to be <u>strict</u> if R, L, and A are identities. So, if $\underline{C}$ is strict, then

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$$

and

$$\begin{cases} X \otimes e = X \\ e \otimes X = X. \end{cases}$$

[Note: While monoidal, neither $\underline{\text{VEC}}_{\underline{k}}$ nor $\underline{\text{HILB}}$ is strict monoidal.]

<u>N.B.</u> Take $\underline{C}$ strict and consider $\underline{M}(\underline{C})$ -- then $\forall\ f,g \in \underline{M}(\underline{C})$,

$$f \otimes g = f \circ g = g \circ f = g \otimes f.$$

1.6 <u>EXAMPLE</u> Let $\underline{S}$ be the category whose objects are the nonnegative integers and whose morphisms are specified by the rule

$$\text{Mor}(n,m) = \begin{cases} \emptyset \text{ if } n \neq m \\ S_n \text{ if } n = m, \end{cases}$$

composition in Mor(n,n) being group multiplication in S_n. Define

$$\otimes : \underline{S} \times \underline{S} \to \underline{S}$$

on objects by

$$\otimes(n,m) = n + m$$

and on morphisms by

$$\otimes(n \xrightarrow{\sigma} n,\ m \xrightarrow{\tau} m) = \rho_{n,m}(\sigma,\tau),$$

where

$$\rho_{n,m}:\mathcal{S}_n \times \mathcal{S}_m \to \mathcal{S}_{n+m}$$

is the canonical map, i.e.,

$$\rho_{n,m}(\sigma,\tau) = \begin{bmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & n+m \\ \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) & n+\tau(1) & n+\tau(2) & \cdots & n+\tau(m) \end{bmatrix},$$

and let e = 0 -- then with these choices, $\mathcal{S}$ is a strict monoidal category.

[Note: $\mathcal{S}$ is equivalent to the category whose objects are the finite sets and whose morphisms are the bijective maps.]

1.7 EXAMPLE Let $\underline{\text{MAT}}_{\underline{k}}$ be the category whose objects are the positive integers and whose morphisms are specified by the rule

$$\text{Mor}(n,m) = M_{n,m}(\underline{k}),$$

the n-by-m matrices with coefficients in $\underline{k}$. Here $\text{id}_n:n \to n$ is the unit diagonal n-by-n matrix and composition

$$\circ:\text{Mor}(n,m) \times \text{Mor}(m,p) \to \text{Mor}(n,p)$$

is

$$B \circ A = AB,$$

the product on the right being ordinary multiplication of matrices. Define

$$\otimes:\underline{\text{MAT}}_{\underline{k}} \times \underline{\text{MAT}}_{\underline{k}} \to \underline{\text{MAT}}_{\underline{k}}$$

on objects by

$$\otimes(n,m) = nm$$

and on morphisms by

$$\otimes(n \xrightarrow{A} m,\ p \xrightarrow{B} q)$$

$$= \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} \in \text{Mor}(np,mq)$$

and let $e = 1$ -- then with these choices, $\underline{\text{MAT}}_{\underline{k}}$ is a strict monoidal category.

[Note: Write $\underline{\text{FDVEC}}_{\underline{k}}$ for the full subcategory of $\underline{\text{VEC}}_{\underline{k}}$ whose objects are finite dimensional -- then there is an equivalence $\underline{\text{MAT}}_{\underline{k}} \to \underline{\text{FDVEC}}_{\underline{k}}$. Thus assign to each object n the vector space $\underline{k}^n$ and to each morphism $n \xrightarrow{A} m$ the linear map from $\underline{k}^n$ to $\underline{k}^m$ that sends $(x_1,\ldots,x_n) \in \underline{k}^n$ to $(y_1,\ldots,y_m) \in \underline{k}^m$, where y_i is the i^{th} entry of the 1-by-m matrix $[x_1,\ldots,x_n]A$.]

1.8 EXAMPLE Given a C*-algebra A, let End A be the category whose objects are the unital *-homomorphisms $\Phi:A \to A$ and whose arrows $\Phi \to \Psi$ are the intertwiners, i.e.,

$$\text{Mor}(\Phi,\Psi) = \{T \in A : T\Phi(A) = \Psi(A)T \ \forall\ A \in A\}.$$

Here, the composition of arrows, when defined, is given by the product in A and $1_A \in \text{Mor}(\Phi,\Phi)$ is 1_Φ. Define

$$\otimes:\text{End } A \times \text{End } A \to \text{End } A$$

on objects by

$$\otimes(\Phi,\Phi') = \Phi \circ \Phi'$$

and on morphisms by

$$\otimes(\Phi \xrightarrow{T} \Psi,\ \Phi' \xrightarrow{T'} \Psi')$$

$$= T\Phi(T')(= \Psi(T')T) \in \mathrm{Mor}(\Phi \circ \Phi',\ \Psi \circ \Psi')$$

and let $e = \mathrm{id}_A$ -- then with these choices, End A is a strict monoidal category.

[Note: $\forall$ A $\in$ A, we have

$$T\Phi(T')(\Phi \circ \Phi')(A)$$

$$= T\Phi(T')\Phi(\Phi'(A))$$

$$= T\Phi(T'\Phi'(A))$$

$$= \Psi(T'\Phi'(A))T$$

$$= \Psi(\Psi'(A)T')T$$

$$= \Psi(\Psi'(A))\Psi(T')T$$

$$= (\Psi \circ \Psi')(A)T\Phi(T').]$$

1.9 EXAMPLE Given a category $\underline{C}$, let $[\underline{C},\underline{C}]$ be the metacategory whose objects are the functors $F:\underline{C} \to \underline{C}$ and whose morphisms are the natural transformations Ξ from F to G. Define

$$\otimes:[\underline{C},\underline{C}] \to [\underline{C},\underline{C}]$$

on objects by

$$\otimes(F,F') = F \circ F'$$

and on morphisms by

$$\otimes(F \xrightarrow{\Xi} G,\ F' \xrightarrow{\Xi'} G')$$

$$= \Xi \otimes \Xi',$$

where

$$(\Xi \otimes \Xi')_X$$

$$= \Xi_{G'X} \circ F\Xi'_X \ (= G\Xi'_X \circ \Xi_{F'X}),$$

and let $e = \mathrm{id}_{\underline{C}}$ (the identity functor) -- then with these choices, $[\underline{C},\underline{C}]$ is a strict monoidal category.

[Note: If

$$\left[\begin{array}{l} \Xi \in \mathrm{Nat}(F,G) \\ \\ \Xi' \in \mathrm{Nat}(F',G'), \end{array}\right.$$

then

$$\left[\begin{array}{l} \forall\ X,Y \in \mathrm{Ob}\ \underline{C} \\ \\ \forall\ X',Y' \in \mathrm{Ob}\ \underline{C} \end{array}\right. \quad \text{and} \quad \left[\begin{array}{l} \forall\ f \in \mathrm{Mor}(X,Y) \\ \\ \forall\ f' \in \mathrm{Mor}(X',Y'), \end{array}\right.$$

there are commutative diagrams

$$\begin{array}{ccc} FX & \xrightarrow{\Xi_X} & GX \\ Ff \downarrow & & \downarrow Gf \\ FY & \xrightarrow[\Xi_Y]{} & GY \end{array}$$

$$\begin{array}{ccc} F'X' & \xrightarrow{\Xi'_{X'}} & G'X' \\ {\scriptstyle F'f'}\downarrow & & \downarrow{\scriptstyle G'f'} \\ F'Y' & \xrightarrow[\Xi'_{Y'}]{} & G'Y'. \end{array}$$

In particular: The diagram

$$\begin{array}{ccc} FF'X & \xrightarrow{\Xi_{F'X}} & GF'X \\ {\scriptstyle F\Xi'_X}\downarrow & & \downarrow{\scriptstyle G\Xi'_X} \\ FG'X & \xrightarrow[\Xi_{G'X}]{} & GG'X \end{array}$$

commutes. This said, the claim is that

$$\Xi \otimes \Xi' \in \text{Nat}(F \circ F', G \circ G'),$$

i.e., that the diagram

$$\begin{array}{ccc} FF'X & \xrightarrow{(\Xi \otimes \Xi')_X} & GG'X \\ {\scriptstyle FF'f}\downarrow & & \downarrow{\scriptstyle GG'f} \\ FF'Y & \xrightarrow[(\Xi \otimes \Xi')_Y]{} & GG'Y \end{array}$$

commutes. In fact,

$$GG'f \circ (\Xi \otimes \Xi')_X$$

$$= GG'f \circ \Xi_{G'X} \circ F\Xi'_X$$

$$= GG'f \circ G\Xi'_X \circ \Xi_{F'X}$$

$$= G(G'f \circ \Xi'_X) \circ \Xi_{F'X}$$

$$= G(\Xi'_Y \circ F'f) \circ \Xi_{F'X}$$

$$= G\Xi'_Y \circ GF'f \circ \Xi_{F'X}$$

$$= G\Xi'_Y \circ \Xi_{F'Y} \circ FF'f$$

$$= \Xi_{G'Y} \circ F\Xi'_Y \circ FF'f$$

$$= (\Xi \otimes \Xi')_Y \circ FF'f.]$$

1.10 <u>LEMMA</u> Suppose that $\underline{C}$ is monoidal and let e,e' be units -- then e and e' are isomorphic.

[There is an isomorphism $\phi:e \rightarrow e'$ for which the diagrams

$$\begin{array}{ccc} X \otimes e & \xrightarrow{\text{id} \otimes \phi} & X \otimes e' \\ {\scriptstyle R_X}\downarrow & & \downarrow{\scriptstyle R'_X} \\ X & = & X \end{array} \quad , \quad \begin{array}{ccc} e \otimes X & \xrightarrow{\phi \otimes \text{id}} & e' \otimes X \\ {\scriptstyle L_X}\downarrow & & \downarrow{\scriptstyle L'_X} \\ X & = & X \end{array}$$

commute, viz.

$$\phi = L_{e'} \circ (R'_e)^{-1} \quad (e \rightarrow e \otimes e' \rightarrow e').]$$

§2. *MONOIDAL FUNCTORS*

Let $\underline{C}$, $\underline{C}'$ be monoidal categories -- then a monoidal functor is a triple (F,ξ,Ξ), where $F:\underline{C} \rightarrow \underline{C}'$ is a functor, $\xi:e' \rightarrow Fe$ is an isomorphism, and the

$$\Xi_{X,Y}:FX \otimes' FY \rightarrow F(X \otimes Y)$$

are isomorphisms, natural in X,Y, subject to the following assumptions.

(MF_1) The diagram

$$\begin{array}{ccccc} FX \otimes' (FY \otimes' FZ) & \xrightarrow{\text{id} \otimes \Xi} & FX \otimes' F(Y \otimes Z) & \xrightarrow{\Xi} & F(X \otimes (Y \otimes Z)) \\ {\scriptstyle A}\downarrow & & & & \downarrow{\scriptstyle FA} \\ (FX \otimes' FY) \otimes' FZ & \xrightarrow[\Xi \otimes \text{id}]{} & F(X \otimes Y) \otimes' FZ & \xrightarrow[\Xi]{} & F((X \otimes Y) \otimes Z) \end{array}$$

commutes.

(MF_2) The diagrams

$$\begin{array}{ccc} FX \otimes' e' & \xrightarrow{R'_{FX}} & FX \\ {\scriptstyle \text{id} \otimes \xi}\downarrow & & \uparrow{\scriptstyle FR_X} \\ FX \otimes' Fe & \xrightarrow[\Xi]{} & F(X \otimes e) \end{array} \qquad \begin{array}{ccc} e' \otimes' FX & \xrightarrow{L'_{FX}} & FX \\ {\scriptstyle \xi \otimes \text{id}}\downarrow & & \uparrow{\scriptstyle FL_X} \\ Fe \otimes' FX & \xrightarrow[\Xi]{} & F(e \otimes X) \end{array}$$

commute.

N.B. A monoidal functor is said to be strict if ξ and Ξ are identities.

2.1 EXAMPLE Write $\underline{FDHILB}$ for the full subcategory of $\underline{HILB}$ whose objects are finite dimensional -- then the forgetful functor

$$U:\underline{FDHILB} \rightarrow \underline{FDVEC}_{\underline{C}}$$

is strict monoidal.

[Take for

$$\Xi_{X,Y}:UX \otimes UY \to U(X \otimes Y)$$

the identity $id_{X \otimes Y}$ and let $\xi = id_{\underline{C}}$.]

[Note: A forgetful functor need not be monoidal, let alone strict monoidal. E.g.: Give $\underline{AB}$ its monoidal structure per the tensor product, give $\underline{SET}$ its monoidal structure per the cartesian product, and consider $U:\underline{AB} \to \underline{SET}$ — then the canonical maps

$$\begin{cases} UA \times UB \to U(A \otimes_{Z} B) \\ \{*\} \to Z \quad (* \to 0) \end{cases}$$

are not isomorphisms.]

Let

$$\begin{cases} (F,\xi,\Xi) \\ (G,\theta,\Theta) \end{cases}$$

be monoidal functors — then a <u>monoidal natural transformation</u>

$$(F,\xi,\Xi) \to (G,\theta,\Theta)$$

is a natural transformation $\alpha:F \to G$ such that the diagrams

$$\begin{array}{ccc} Fe & \xrightarrow{\alpha_e} & Ge \\ \xi \uparrow & & \uparrow \theta \\ e' & = & e', \end{array} \qquad \begin{array}{ccc} FX \otimes' FY & \xrightarrow{\Xi} & F(X \otimes Y) \\ \alpha_X \otimes' \alpha_Y \downarrow & & \downarrow \alpha_{X \otimes Y} \\ GX \otimes' GY & \xrightarrow[\Theta]{} & G(X \otimes Y) \end{array}$$

commute.

Write $[\underline{C},\underline{C}']^{\otimes}$ for the metacategory whose objects are the monoidal functors $\underline{C} \to \underline{C}'$ and whose morphisms are the monoidal natural transformations.

N.B. A monoidal natural transformation is a monoidal natural isomorphism if α is a natural isomorphism.

2.2 REMARK Some authorities assume outright that $Fe = e'$, the rationale being that this can always be achieved by replacing $F \in \text{Ob } [\underline{C},\underline{C}']^{\otimes}$ by an isomorphic $F' \in \text{Ob } [\underline{C},\underline{C}']^{\otimes}$ such that $F'e = e'$ (on objects $X \neq e$, $F'X = FX$).

2.3 LEMMA Let

$$\begin{cases} (F,\Xi,\xi) & (F:\underline{C} \to \underline{C}') \\ (F',\Xi',\xi') & (F':\underline{C}' \to \underline{C}'') \end{cases}$$

be monoidal functors -- then their composition $F' \circ F$ is a monoidal functor.

[Consider the arrows $e'' \xrightarrow{\xi'} F'e' \xrightarrow{F'\xi} F'Fe$ and

$$F'FX \otimes'' F'FY \xrightarrow{\Xi'_{FX,FY}} F'(FX \otimes' FY) \xrightarrow{F'\Xi_{X,Y}} F'F(X \otimes Y).]$$

Write MONCAT for the metacategory whose objects are the monoidal categories and whose morphisms are the monoidal functors.

2.4 RAPPEL Let $\underline{C}$, $\underline{D}$ be categories -- then a functor $F:\underline{C} \to \underline{D}$ is said to be an equivalence if there exists a functor $G:\underline{D} \to \underline{C}$ such that $G \circ F \approx \text{id}_{\underline{C}}$ and $F \circ G \approx \text{id}_{\underline{D}}$, the symbol $\approx$ standing for natural isomorphism.

2.5 <u>LEMMA</u> A functor $F:\underline{C} \to \underline{D}$ is an equivalence iff it is full, faithful, and has a representative image (i.e., for any $Y \in \text{Ob } \underline{D}$, there exists an $X \in \text{Ob } \underline{C}$ such that FX is isomorphic to Y).

<u>N.B.</u> Categories $\underline{C}$, $\underline{D}$ are said to be <u>equivalent</u> provided there is an equivalence $F:\underline{C} \to \underline{D}$. The object isomorphism types of equivalent categories are in a one-to-one correspondence.

2.6 <u>RAPPEL</u> Given categories $\left[\begin{array}{l} \underline{C} \\ \underline{D} \end{array}\right.$, functors $\left[\begin{array}{l} F:\underline{C} \to \underline{D} \\ G:\underline{D} \to \underline{C} \end{array}\right.$ are said to be an <u>adjoint pair</u> if the functors $\left[\begin{array}{l} \text{Mor} \circ (F^{OP} \times \text{id}_{\underline{D}}) \\ \\ \text{Mor} \circ (\text{id}_{\underline{C}^{OP}} \times \underline{G}) \end{array}\right.$ from $\underline{C}^{OP} \times \underline{D}$ to <u>SET</u> are naturally isomorphic, i.e., if it is possible to assign to each ordered pair $\left[\begin{array}{l} X \in \text{Ob } \underline{C} \\ Y \in \text{Ob } \underline{D} \end{array}\right.$ a bijective map $\Xi_{X,Y}:\text{Mor}(FX,Y) \to \text{Mor}(X,GY)$ which is functorial in X and Y. When this is so, F is a <u>left adjoint</u> for G and G is a <u>right adjoint</u> for F. Any two left (right) adjoints for G (F) are naturally isomorphic. In order that (F,G) be an adjoint pair, it is necessary and sufficient that there exist natural transformations $\left[\begin{array}{l} \mu \in \text{Nat}(\text{id}_{\underline{C}}, G \circ F) \\ \\ \nu \in \text{Nat}(F \circ G, \text{id}_{\underline{D}}) \end{array}\right.$ subject to $\left[\begin{array}{l} (\nu F) \circ (F\mu) = \text{id}_F \\ \\ (G\nu) \circ (\mu G) = \text{id}_G. \end{array}\right.$

The data (F,G,μ,ν) is referred to as an <u>adjoint situation</u>, the natural transformations $\begin{cases} \mu:\mathrm{id}_{\underline{C}} \to G \circ F \\ \nu:F \circ G \to \mathrm{id}_{\underline{D}} \end{cases}$ being the <u>arrows of adjunction</u>. An <u>adjoint equivalence</u> of categories is an adjoint situation (F,G,μ,ν) in which both μ and ν are natural isomorphisms.

2.7 <u>LEMMA</u> A functor $F:\underline{C} \to \underline{D}$ is an equivalence iff F is part of an adjoint equivalence.

Let $\underline{C}$, $\underline{C}'$ be monoidal categories — then $\underline{C}$, $\underline{C}'$ are monoidally equivalent if there are monoidal functors

$$\begin{cases} F:\underline{C} \to \underline{C}' \\ F':\underline{C}' \to \underline{C} \end{cases}$$

and monoidal natural isomorphisms

$$\begin{cases} F' \circ F \approx \mathrm{id}_{\underline{C}} \\ F \circ F' \approx \mathrm{id}_{\underline{C}'} \end{cases} .$$

2.8 <u>LEMMA</u> Suppose that $F:\underline{C} \to \underline{C}'$ is a monoidal functor. Assume: F is an equivalence -- then F is a monoidal equivalence.

2.9 <u>REMARK</u> Embed F in an adjoint situation (F,F',μ,μ'), where

$$\begin{cases} \mu:\mathrm{id}_{\underline{C}} \to F' \circ F \\ \mu':F \circ F' \to \mathrm{id}_{\underline{C}'} \end{cases}$$

are the arrows of adjunction (cf. 2.7) -- then one can equip F' with the structure of a monoidal functor in such a way that the natural isomorphisms μ, μ' are monoidal natural isomorphisms. Thus first specify $\xi':e \to F'e'$ by taking it to be the composition $e \xrightarrow{\mu_e} F'Fe \xrightarrow{F'\xi^{-1}} F'e'$. As for

$$\Xi'_{X',Y'}:F'X' \otimes F'Y' \to F'(X' \otimes' Y'),$$

build it in three stages:

1. $F'X' \otimes F'Y' \xrightarrow{\mu} F'F(F'X' \otimes F'Y')$;

2. $F'F(F'X' \otimes F'Y') \xrightarrow{F'\Xi^{-1}} F'(FF'X' \otimes' FF'Y')$;

3. $$\begin{cases} FF'X' \xrightarrow{\mu'_{X'}} X' \\ FF'Y' \xrightarrow{\mu'_{Y'}} Y' \end{cases}$$

$\Rightarrow$

$$\mu'_{X'} \otimes \mu'_{Y'}:FF'X' \otimes' FF'Y' \to X' \otimes' Y'$$

$\Rightarrow$

$$F'(\mu'_{X'} \otimes \mu'_{Y'}):F'(FF'X' \otimes' FF'Y') \to F'(X' \otimes' Y').$$

If $\underset{\sim}{C}$ is monoidal, then $\underset{\sim}{C}^{OP}$ is monoidal when equipped with the same $\otimes$ and e, taking

$$\begin{cases} R^{OP} = R^{-1} \\ L^{OP} = L^{-1} \\ A^{OP} = A^{-1}. \end{cases}$$

§3. *STRICTIFICATION*

A strictification of a monoidal category $\underline{C}$ is a strict monoidal category which is monoidally equivalent to $\underline{C}$.

3.1 EXAMPLE $\underline{MAT}_{\underline{k}}$ is a strictification of $\underline{FDVEC}_{\underline{k}}$.

[The equivalence $\underline{MAT}_{\underline{k}} \to \underline{FDVEC}_{\underline{k}}$ constructed in 1.7 is a monoidal functor, hence is a monoidal equivalence (cf. 2.8).]

3.2 THEOREM Every monoidal category $\underline{C}$ is monoidally equivalent to a strict monoidal category $\underline{C}_{\text{str}}$.

The proof is constructive and best broken up into steps.

Step 1: Let $\underline{S}$ be the class of all finite sequences $S = (X_1,\ldots,X_n)$ of objects of $\underline{C}$, including the empty sequence $\emptyset$. Given nonempty

$$\begin{cases} S = (X_1,\ldots,X_n) \\ T = (Y_1,\ldots,Y_m), \end{cases}$$

let

$$S * T = (X_1,\ldots,X_n,Y_1,\ldots,Y_m)$$

and write

$$S * \emptyset = S = \emptyset * S.$$

Step 2: The claim is that $\underline{S}$ is the object class of a strict monoidal category $\underline{C}_{\underline{str}}$, i.e., $\underline{S} = \text{Ob}\ \underline{C}_{\underline{str}}$. In any event, the multiplication

$$*:\underline{S} \times \underline{S} \to \underline{S}$$

is associative, so we can take A to be the identity. Also, $\emptyset$ serves as the unit and

$$\begin{bmatrix} R_S : S * \emptyset \to S \\ \\ L_S : \emptyset * S \to S \end{bmatrix}$$

are the identities.

Step 3: Given S, T, we need to specify Mor(S,T). For this purpose, define a map $\Gamma:\underline{S} \to \text{Ob}\ \underline{C}$ by $\Gamma\emptyset = e$, $\Gamma((X)) = X$, and $\Gamma(S * (X)) = \Gamma S \otimes X$, thus

$$\Gamma(X_1,\ldots,X_n)$$

$$= (\ldots(X_1 \otimes X_2) \otimes \ldots) \otimes X_n,$$

where all opening parentheses are to the left of X_1. Definition:

$$\text{Mor}(S,T) = \text{Mor}(\Gamma S, \Gamma T).$$

This prescription then gives rise to a category $\underline{C}_{\underline{str}}$ with $\text{Ob}\ \underline{C}_{\underline{str}} = \underline{S}$.

Step 4: We shall now define a functor $*:\underline{C}_{\underline{str}} \times \underline{C}_{\underline{str}} \to \underline{C}_{\underline{str}}$ that serves to render $\underline{C}_{\underline{str}}$ strict monoidal, the issue being the meaning of

$$u * u' = *(S \xrightarrow{u} T,\ S' \xrightarrow{u'} T')$$

$$\in \text{Mor}(S * S',\ T * T')$$

$$= \text{Mor}(\Gamma(S * S'),\ \Gamma(T * T')).$$

Bearing in mind that

$$\left[\begin{array}{l} \mathrm{Mor}(S,T) = \mathrm{Mor}(\Gamma S,\Gamma T) \\ \qquad u \longleftrightarrow f \\ \mathrm{Mor}(S',T') = \mathrm{Mor}(\Gamma S',\Gamma T') \\ \qquad u' \longleftrightarrow f', \end{array}\right.$$

let $u * u'$ be the composite

$$\Gamma(S * S') \to \Gamma S \otimes \Gamma S' \xrightarrow{f \otimes f'} \Gamma T \otimes \Gamma T' \to \Gamma(T * T'),$$

where the outer arrows are the obvious canonical morphisms in $\underline{C}$. Accordingly, with this agreement, $\underline{C}_{\underline{str}}$ is strict monoidal.

Step 5: It is clear from its very construction that $\Gamma:\underline{C}_{\underline{str}} \to \underline{C}$ is a functor which, moreover, is full, faithful, and is isomorphism dense. But $\Gamma\emptyset = e$ and there are isomorphisms

$$\Xi_{S,T}:\Gamma S \otimes \Gamma T \to \Gamma(S * T),$$

natural in S, T and satisfying $M\Gamma_1$, $M\Gamma_2$ of §2. Therefore Γ is monoidal. To finish, it remains only to quote 2.8.

[Note: It is not necessary to quote 2.8: Simply observe that there is an inclusion functor $\gamma:\underline{C} \to \underline{C}_{\underline{str}}$ and

$$\left[\begin{array}{l} \Gamma \circ \gamma = \mathrm{id}_{\underline{C}} \\ \gamma \circ \Gamma \approx \mathrm{id}_{\underline{C}_{\underline{str}}}. \end{array}\right.$$

Detail: From

$$\mathrm{Mor}(\gamma\Gamma S,S) = \mathrm{Mor}(\Gamma S,\Gamma S),$$

let

$$\alpha_S \longleftrightarrow \mathrm{id}_{\Gamma S},$$

thus $\alpha_S:\gamma\Gamma S \to S$ and $\alpha:\gamma \circ \Gamma \to \mathrm{id}_{\underline{C}_{\underline{str}}}$ is a monoidal natural isomorphism.]

3.3 REMARK Let $\underline{C}$, $\underline{C}'$ be monoidal categories -- then each monoidal functor $F:\underline{C} \to \underline{C}'$ induces a strict monoidal functor $F_{\underline{str}}:\underline{C}_{\underline{str}} \to \underline{C}'_{\underline{str}}$ and there is a commutative diagram

$$\begin{array}{ccc} \underline{C} & \xrightarrow{F} & \underline{C}' \\ \gamma\downarrow & & \downarrow\gamma' \\ \underline{C}_{\underline{str}} & \xrightarrow[F_{\underline{str}}]{} & \underline{C}'_{\underline{str}}. \end{array}$$

Here, on an object S,

$$F_{\underline{str}}S = (FX_1,\ldots,FX_n),$$

while on a morphism $u:S \to T$,

$$(FX_1,\ldots,FX_n) \xrightarrow{F_{\underline{str}}u} (FY_1,\ldots,FY_m)$$

is that element of $\mathrm{Mor}(\Gamma FS,\Gamma FT)$ defined by requiring commutativity of the square

$$\begin{array}{ccc} \Gamma FS & \longrightarrow & \Gamma FT \\ \approx\downarrow & & \downarrow\approx \\ F\Gamma S & \xrightarrow[Ff]{} & F\Gamma T, \end{array}$$

where $f \in \text{Mor}(\Gamma S, \Gamma T)$ corresponds to u.

[Note: Composition of monoidal functors is preserved by this construction.]

There are five ingredients figuring in the definition of a monoidal category: $\otimes$, e, R, L, A. Keeping track of R, L, A in calculations can be annoying and one way out is to pass from $\underline{C}$ to $\underline{C}_{\underline{str}}$. But this too has its downside since $\underline{C}_{\underline{str}}$ is a more complicated entity than $\underline{C}$. So, in what follows, we shall stick with $\underline{C}$ and determine to what extent R, L, A can be eliminated from consideration (i.e., are identities).

Suppose that

$$\begin{cases} (\otimes, E, R, L, A) \\ (\otimes', e', R', L', A') \end{cases}$$

are monoidal structures on $\underline{C}$ -- then these structures are deemed <u>isomorphic</u> if $\exists$ a monoidal equivalence of the form $(\text{id}_{\underline{C}}, \xi, \Xi)$ between them.

<u>N.B.</u> Therefore $\xi: e' \rightarrow e$ is an isomorphism and the

$$\Xi_{X,Y}: X \otimes' Y \rightarrow X \otimes Y$$

are isomorphisms, subject to the coherence conditions of §2.

3.4 <u>REMARK</u> The philosophy is that replacing a given monoidal structure on $\underline{C}$ by another isomorphic to it is of no consequence for the underlying mathematics.

3.5 <u>LEMMA</u> Let $(\otimes, e, R, L, A)$ be a monoidal structure on $\underline{C}$. Suppose given a map $\otimes': \text{Ob } \underline{C} \times \text{Ob } \underline{C} \rightarrow \text{Ob } \underline{C}$, an object $e' \in \text{Ob } \underline{C}$, an isomorphism $\xi': e \rightarrow e'$, and isomorphisms

$$\Xi'_{X,Y}: X \otimes Y \to X \otimes' Y.$$

Then there is a unique monoidal structure $(\otimes', e', R', L', A')$ on $\underline{C}$ such that

$$(\mathrm{id}_{\underline{C}}, \xi', \Xi'): (\underline{C}, \otimes', e', R', L', A') \to (\underline{C}, \otimes, e, R, L, A)$$

is an isomorphism.

PROOF Extend $\otimes'$ to a functor $\otimes': \underline{C} \times \underline{C} \to \underline{C}$ by the prescription

$$\begin{array}{ccc}
\otimes(X,Y) & \xrightarrow{\Xi'_{X,Y}} & \otimes'(X,Y) \\
{\scriptstyle \otimes(f,g)}\downarrow & & \downarrow{\scriptstyle \otimes'(f,g)} \\
\otimes(X',Y') & \xrightarrow[\Xi'_{X',Y'}]{} & \otimes'(X',Y'),
\end{array}$$

so $\otimes \approx \otimes'$ (via $\Xi' \in \mathrm{Nat}(\otimes, \otimes')$). This done, define R', L', A' by the diagrams

$$\begin{array}{ccc}
X \otimes' e' & \xrightarrow{R'} & X \\
{\scriptstyle \Xi'}\uparrow & & \uparrow{\scriptstyle R} \\
X \otimes e' & \xleftarrow[\mathrm{id} \otimes \xi']{} & X \otimes e
\end{array}
\qquad
\begin{array}{ccc}
e' \otimes' X & \xrightarrow{L'} & X \\
{\scriptstyle \Xi'}\uparrow & & \uparrow{\scriptstyle L} \\
e' \otimes X & \xleftarrow[\xi' \otimes \mathrm{id}]{} & e \otimes X
\end{array}$$

$$\begin{array}{ccc}
X \otimes' (Y \otimes' Z) & \xrightarrow{A'} & (X \otimes' Y) \otimes' Z \\
{\scriptstyle \Xi'}\uparrow & & \uparrow{\scriptstyle \Xi'} \\
X \otimes (Y \otimes' Z) & & (X \otimes' Y) \otimes Z \\
{\scriptstyle \mathrm{id} \otimes \Xi'}\uparrow & & \uparrow{\scriptstyle \Xi' \otimes \mathrm{id}} \\
X \otimes (Y \otimes Z) & \xrightarrow[A]{} & (X \otimes Y) \otimes Z.
\end{array}$$

3.6 THEOREM Let $(\otimes, e, R, L, A)$ be a monoidal structure on $\underline{C}$. Suppose that e' is an object isomorphic to e, say $\xi: e' \to e$ -- then there is an isomorphic monoidal structure $(\otimes', e', R', L', A')$ on $\underline{C}$ in which R', L' are identities.

PROOF Bearing in mind 3.5, put

$$X \otimes' Y = X \otimes Y \text{ if } X \neq e' \neq Y$$

and

$$X \otimes' Y = \begin{cases} Y \text{ if } X = e' \\ X \text{ if } Y = e'. \end{cases}$$

Define

$$\Xi'_{X,Y}: X \otimes Y \to X \otimes' Y$$

by stipulating that $\Xi'_{X,Y}$ is to be the identity if $X \neq e' \neq Y$, otherwise let

$$\begin{cases} \Xi'_{X,e'} = R_X \circ (\mathrm{id}_X \otimes \xi) \\ \Xi'_{e',Y} = L_Y \circ (\xi \otimes \mathrm{id}_Y). \end{cases}$$

To establish consistency, i.e., that

$$R_{e'} \circ (\mathrm{id}_{e'} \otimes \xi) = L_{e'} \circ (\xi \otimes \mathrm{id}_{e'}),$$

set $\xi' = \xi^{-1}$ -- then

$$e \otimes e \xrightarrow{\xi' \otimes \xi'} e' \otimes e'$$

is an isomorphism and due to the naturality of R, L, the diagrams

$$\begin{array}{ccc} e \otimes e & \xrightarrow{R_e} & e \\ {\scriptstyle \xi' \otimes id_e}\downarrow & & \downarrow{\scriptstyle \xi'} \\ e' \otimes e & \xrightarrow[R_{e'}]{} & e' \end{array} \qquad \begin{array}{ccc} e \otimes e & \xrightarrow{L_e} & e \\ {\scriptstyle id_e \otimes \xi'}\downarrow & & \downarrow{\scriptstyle \xi'} \\ e \otimes e' & \xrightarrow[L_{e'}]{} & e' \end{array}$$

commute. Therefore

$$R_{e'} \circ (id_{e'} \otimes \xi) \circ (\xi' \otimes \xi')$$

$$= R_{e'} \circ (\xi' \otimes id_e)$$

$$= \xi' \circ R_e = \xi' \circ L_e \quad (R_e = L_e)$$

$$= L_{e'} \circ (id_e \otimes \xi')$$

$$= L_{e'} \circ (\xi \otimes id_{e'}) \circ (\xi' \otimes \xi)$$

from which the contention. Finally, by construction (cf. 3.5), R', L' are identities. E.g.:

$$R'_X \circ \Xi'_{X,e'} \circ id_X \otimes \xi' = R_X$$

or still,

$$R'_X \circ R_X \circ (id_X \otimes \xi) \circ id_X \otimes \xi' = R_X$$

or still,

$$R'_X \circ R_X = R_X = \; R'_X = id_X.$$

[Note: If A is the identity and e' is not in the image of $\otimes$, then A' is also the identity. Proof:

$$\left[\begin{array}{l} e' \in \{X,Y,Z\} \Rightarrow A'_{XYZ} = \mathrm{id} \\ \\ e' \notin \{X,Y,Z\} \ \&\ e' \notin \mathrm{Im}\,\otimes \Rightarrow A'_{XYZ} = A_{XYZ}.\end{array}\right]$$

3.7 <u>REMARK</u> Take $e' = e$ — then the preceding result implies that by passing to an isomorphic monoidal structure, it is always possible to arrange that $\forall\, X \in \mathrm{Ob}\,\underline{C}$,

$$X \otimes e = X = e \otimes X.$$

The situation for the associativity constraint is more complicated and it will be necessary to impose some conditions on $\underline{C}$.

Definition: A <u>construct</u> is a pair $(\underline{C},U)$, where

$$U:\underline{C} \to \underline{SET}$$

is a faithful functor.

3.8 <u>EXAMPLE</u> Define a functor $Q:\underline{SET}^{OP} \to \underline{SET}$ as follows: On objects, $QX = 2^X$ and on morphisms, $Q(A \xrightarrow{f} B):QA \to QB$ sends $X \subset A$ to the inverse image $f^{-1}(X) \subset B$. In this connection, recall that

$$A \xrightarrow{f} B \in \mathrm{Mor}\ \underline{SET}^{OP}$$

means that

$$B \xrightarrow{f} A \in \mathrm{Mor}\ \underline{SET}.$$

Therefore $(\underline{SET}^{OP},Q)$ is a construct.

Let ($\underline{C}$,U) be a construct -- then ($\underline{C}$,U) is <u>amnestic</u> if a $\underline{C}$-isomorphism f is a $\underline{C}$-identity whenever Uf is a <u>SET</u>-identity, i.e., if $X,Y \in \text{Ob}\,\underline{C}$, if $f:X \to Y$ is an isomorphism, if Uf = id, then X = Y and f = id.

Let ($\underline{C}$,U) be a construct -- then ($\underline{C}$,U) is <u>transportable</u> if $\forall$ $\underline{C}$-object X and every bijection $UX \overset{\phi}{\to} S$, $\exists$ a $\underline{C}$-object Y with UY = S and an isomorphism $\Phi:X \to Y$ such that $U\Phi = \phi$.

3.9 <u>LEMMA</u> If ($\underline{C}$,U) is amnestic and transportable, then the pair (Y,Φ) is unique.

<u>PROOF</u> Say we have

$$Y_1 \xrightarrow{\Phi_1^{-1}} X \xrightarrow{\Phi_2} Y_2.$$

Then $\Phi_2 \circ \Phi_1^{-1}$ is an isomorphism and

$$U(\Phi_2 \circ \Phi_1^{-1}) = U\Phi_2 \circ U\Phi_1^{-1} = \phi \circ \phi^{-1} = \text{id}.$$

Therefore by amnesticity, $Y_1 = Y_2$ and $\Phi_2 \circ \Phi_1^{-1} = \text{id} \Rightarrow \Phi_2 = \Phi_1$.

3.10 <u>EXAMPLE</u> The construct $\underline{\text{FDVEC}}_{\underline{k}}$ is amnestic and transportable but the full subcategory of $\underline{\text{FDVEC}}_{\underline{k}}$ whose objects are the $\underline{k}^n$, while amnestic, is not transportable.

3.11 <u>LEMMA</u> If $\zeta:\underline{\text{SET}} \to \underline{\text{SET}}$ is an isomorphism and if ($\underline{C}$,U) is amnestic and transportable, then ($\underline{C}$,$\zeta \circ U$) is amnestic and transportable.

3.12 THEOREM Suppose that $(\underline{C},U)$ is amnestic and transportable. Let $(\otimes, e, R, L, A)$ be a monoidal structure on $\underline{C}$ -- then there is an isomorphic strict monoidal structure $(\otimes', e, R', L', A')$ on $\underline{C}$.

The proof is lengthy, the point of departure being 3.2:

$$\begin{cases} \Gamma:\underline{C}_{\underline{str}} \to \underline{C} \\ \gamma:\underline{C} \to \underline{C}_{\underline{str}}, \end{cases}$$

where

$$\begin{cases} \Gamma \circ \gamma = \text{id}_{\underline{C}} \\ \gamma \circ \Gamma \approx \text{id}_{\underline{C}_{\underline{str}}}. \end{cases}$$

Step 1: Given $S \in \text{Ob } \underline{C}_{\underline{str}}$, consider

$$\{S\} \times U\Gamma S \in \text{Ob } \underline{SET}.$$

Then the projection

$$\{S\} \times U\Gamma S \xrightarrow{\pi_S} U\Gamma S$$

is bijective, so there exists a unique $[S] \in \text{Ob } \underline{C}$ with $U[S] = \{S\} \times U\Gamma S$ and a unique isomorphism $\Pi_S:[S] \to \Gamma S$ such that $U\Pi_S = \pi_S$.

Step 2: There is a functor $\bar{\Gamma}:\underline{C}_{\underline{str}} \to \underline{C}$ which on objects is the prescription

$$\bar{\Gamma}S = [S]$$

and on morphisms is dictated by requiring that $\Pi \in \text{Nat}(\bar{\Gamma},\Gamma)$:

$$\begin{array}{ccc} \bar{\Gamma}S & \xrightarrow{\Pi_S} & \Gamma S \\ \bar{\Gamma}u \downarrow & & \downarrow \Gamma u \\ \bar{\Gamma}T & \xrightarrow[\Pi_T]{} & \Gamma T. \end{array}$$

Step 3: $\bar{\Gamma}:\underline{C}_{\underline{str}} \to \underline{C}$ is an equivalence of categories ($\Pi:\bar{\Gamma} \to \Gamma$ being a natural isomorphism). In addition, $\bar{\Gamma}$ is injective on objects.

Step 4: Define a functor $\bar{\gamma}:\underline{C} \to \underline{C}_{\underline{str}}$ on objects by taking $\bar{\gamma}X = \gamma X$ if X is not in the image of $\bar{\Gamma}$ and letting $\bar{\gamma}[S] = S$ otherwise. Next, define

$$\nu_X:\bar{\Gamma}\bar{\gamma}X \to X$$

by

$$[\gamma X] \xrightarrow{\Pi_{[\gamma X]}} \Gamma\gamma X = X$$

if X is not in the image of $\bar{\Gamma}$ and let $\nu_X = id_X$ if $X = [S]$ for some S. Since $\bar{\Gamma}$ is fully faithful, we can then define $\bar{\gamma}$ on morphisms by requiring that $\nu:\bar{\Gamma} \circ \bar{\gamma} \to id_{\underline{C}}$ be a natural isomorphism.

Step 5: The arrow

$$\mu = id:id_{\underline{C}_{\underline{str}}} \to \bar{\gamma} \circ \bar{\Gamma}$$

is a natural isomorphism.

Step 6: The data $(\bar{\Gamma},\bar{\gamma},\mu,\nu)$ is an adjoint situation:

$$\left[\begin{array}{l} (\nu\bar{\Gamma}) \circ (\bar{\Gamma}\mu) = id_{\bar{\Gamma}} \\ \\ (\bar{\gamma}\nu) \circ (\mu\bar{\gamma}) = id_{\bar{\gamma}} \end{array}\right. \qquad \text{(cf. 2.6).}$$

Explicated:

$$\left[\begin{array}{l} \nu_{\bar{\Gamma}S} \circ \bar{\Gamma}\mu_S = \mathrm{id}_{\bar{\Gamma}S} \\ \\ \bar{\gamma}\nu_X \circ \mu_{\bar{\gamma}X} = \mathrm{id}_{\bar{\gamma}X} . \end{array}\right.$$

Claim:

$$\left[\begin{array}{l} \nu_{\bar{\Gamma}S} = \mathrm{id}_{\bar{\Gamma}S} \\ \\ \bar{\gamma}\nu_X = \mathrm{id}_{\bar{\gamma}X} \end{array}\right. \quad \& \quad \left[\begin{array}{l} \bar{\Gamma}\mu_S = \mathrm{id}_{\bar{\Gamma}S} \\ \\ \mu_{\bar{\gamma}X} = \mathrm{id}_{\bar{\gamma}X} . \end{array}\right.$$

But

$$\bar{\Gamma}S = [S] \Rightarrow \nu_{\bar{\Gamma}S} = \mathrm{id}_{\bar{\Gamma}S} \ (\equiv \bar{\Gamma}\mu_S).$$

As for the relation

$$\bar{\gamma}\nu_X = \mathrm{id}_{\bar{\gamma}X} \ (\equiv \mu_{\bar{\gamma}X}),$$

since $\bar{\Gamma}$ is faithful, it suffices to show that

$$\bar{\Gamma}\bar{\gamma}\nu_X = \mathrm{id}_{\bar{\Gamma}\bar{\gamma}X}$$

for all $X \in \mathrm{Ob}\ \underline{C}$. But from the definitions, $\forall\ f \in \mathrm{Mor}(\bar{\Gamma}\bar{\gamma}X, X)$, there is a commutative diagram

$$\begin{array}{ccc} \bar{\Gamma}\bar{\gamma}\bar{\Gamma}\bar{\gamma}X & \xrightarrow{\bar{\Gamma}\bar{\gamma}f} & \bar{\Gamma}\bar{\gamma}X \\ {\scriptstyle \nu_{\bar{\Gamma}\bar{\gamma}X}} \downarrow & & \downarrow {\scriptstyle \nu_X} \\ \bar{\Gamma}\bar{\gamma}X & \xrightarrow[f]{} & X . \end{array}$$

Now take $f = \nu_X$ to get

$$\nu_X \circ \bar{\Gamma}\bar{\gamma}\nu_X = \nu_X \circ \nu_{\bar{\Gamma}\bar{\gamma}X}$$

or still,

$$\bar{\Gamma}\bar{\gamma}\nu_X = \nu_{\bar{\Gamma}\bar{\gamma}X}$$

or still,

$$\bar{\Gamma}\bar{\gamma}\nu_X = \mathrm{id}_{\bar{\Gamma}\bar{\gamma}X},$$

as desired.

Step 7: The adjoint situation $(\bar{\Gamma},\bar{\gamma},\mu,\nu)$ is an adjoint equivalence of categories (μ and ν are natural isomorphisms).

Step 8: Put

$$X \otimes' Y = \bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}Y)$$

and let $e' = \bar{\Gamma}\emptyset$ -- then

$$\begin{aligned} \bar{\gamma}(X \otimes' Y) &= \bar{\gamma}\bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}Y) \\ &= \bar{\gamma}X * \bar{\gamma}Y \end{aligned}$$

and

$$\bar{\gamma}e' = \bar{\gamma}\bar{\Gamma}\emptyset = \emptyset.$$

Step 9: We have

$$\begin{aligned} X \otimes' (Y \otimes' Z) &= \bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}(Y \otimes' Z)) \\ &= \bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}Y * \bar{\gamma}Z) \\ &= \bar{\Gamma}(\bar{\gamma}(X \otimes' Y) * \bar{\gamma}Z) \\ &= (X \otimes' Y) \otimes' Z, \end{aligned}$$

so $A' = \mathrm{id}$ will work.

Step 10: Let

$$\left[\begin{array}{l} R'_X = \nu_X \\ \\ L'_X = \nu_X \end{array}\right. \quad : \bar{\Gamma}\bar{\gamma}X \to X.$$

Then this makes sense:

$$\left[\begin{array}{l} X \otimes' e' = \bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}\bar{\Gamma}\emptyset) = \bar{\Gamma}(\bar{\gamma}X * \emptyset) = \bar{\Gamma}\bar{\gamma}X \\ e' \otimes' X' = \bar{\Gamma}(\bar{\gamma}\bar{\Gamma}\emptyset * \bar{\gamma}X) = \bar{\Gamma}(\emptyset * \bar{\gamma}X) = \bar{\Gamma}\bar{\gamma}X. \end{array}\right.$$

Furthermore, the diagram

$$\begin{array}{ccc} X \otimes' e' \otimes' Y & \xrightarrow{A' = \mathrm{id}} & X \otimes' e' \otimes' Y \\ {\scriptstyle \mathrm{id} \otimes' L'} \downarrow & & \downarrow {\scriptstyle R' \otimes' \mathrm{id}} \\ X \otimes' Y & = & X \otimes' Y \end{array}$$

commutes. To see this, note first that

$$\begin{aligned} X \otimes' e' \otimes' Y &= \bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}e' * \bar{\gamma}Y) \\ &= \bar{\Gamma}(\bar{\gamma}X * \emptyset * \bar{\gamma}Y) \\ &= \bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}Y) \\ &= X \otimes' Y. \end{aligned}$$

And the arrows

$$\left[\begin{array}{l} R'_X \otimes' \mathrm{id}_Y : X \otimes' e' \otimes' Y \to X \otimes' Y \\ \mathrm{id}_X \otimes' L'_Y : X \otimes' e' \otimes' Y \to X \otimes' Y \end{array}\right.$$

are identities. E.g.:

$$
\begin{aligned}
R'_X \otimes' id_Y &= \bar{\Gamma}(\bar{\gamma}\nu_X * \bar{\gamma}\, id_Y) \\
&= \bar{\Gamma}(id_{\bar{\gamma}X} * id_{\bar{\gamma}Y}) \\
&= \bar{\Gamma}(id_{\bar{\gamma}X * \bar{\gamma}Y}) \\
&= id_{\bar{\Gamma}(\bar{\gamma}X * \bar{\gamma}Y)} \\
&= id_{X \otimes' Y} .
\end{aligned}
$$

Step 11: It is clear that

$$\bar{\gamma}: (\underline{C}, \otimes', e', R', L', A') \to (\underline{C}_{str}, *, \emptyset, R, L, A)$$

is a monoidal equivalence (cf. 2.8), thus the same is true of

$$\Gamma\bar{\gamma}: (\underline{C}, \otimes', e', R', L', A') \to (\underline{C}, \otimes, e, R, L, A) \quad \text{(cf. 2.3)}.$$

But there is a monoidal natural isomorphism $\Gamma\bar{\gamma} \approx id_{\underline{C}}$: $\forall\, X \in \text{Ob}\, \underline{C}$,

$$\Gamma\bar{\gamma}X \xrightarrow{\Pi^{-1}_{\bar{\gamma}X}} \bar{\Gamma}\bar{\gamma}X \xrightarrow{\nu_X} X.$$

Therefore the monoidal structure $(\otimes', e', R', L', A')$ is isomorphic to $(\otimes, e, R, L, A)$.

Step 12: To complete the proof, it is necessary to fine tune $(\otimes', e', R', L', A')$ by an application of 3.6:

$$(\otimes', e', R', L', A') \to (\otimes'', e'', R'', L'', A''),$$

choosing $e'' = e$ (cf. 1.10). So, R'', L'' are identities. However, by construction, A' is the identity, thus if e is not in the image of $\otimes'$, then A'' is also the identity. To ensure that e is not in the image of $\otimes'$, it is enough that e is not in the image of $\bar{\Gamma}$. Suppose it were -- then

$$Ue = \{S\} \times U\Gamma S \quad (\exists\, S \in \text{Ob}\, \underline{C}_{\underline{str}}).$$

Now use 3.11 and replace U by ζU, where ζ has the property that ζUe is not a cartesian product of two sets.

3.13 EXAMPLE Consider the construct $\underline{\text{FDVEC}}_{\underline{k}}$ -- then the failure of the tensor product to be associative "on the nose" is an artifact of its definition by a universal property which determines it only up to isomorphism. While the usual procedures do not lead to an associative tensor product, the lesson to be drawn from 3.12 is that it is possible to find a tensor product on $\underline{\text{FDVEC}}_{\underline{k}}$ such that

$$\begin{cases} X \otimes_{\underline{k}} \underline{k} = X \\ \underline{k} \otimes_{\underline{k}} X = X \end{cases}$$

and

$$(X \otimes_{\underline{k}} Y) \otimes_{\underline{k}} Z = X \otimes_{\underline{k}} (Y \otimes_{\underline{k}} Z) = X \otimes_{\underline{k}} Y \otimes_{\underline{k}} Z.$$

§4. *SYMMETRY*

A symmetry for a monoidal category $\underline{C}$ is a natural isomorphism τ, where

$$\tau_{X,Y}:X \otimes Y \to Y \otimes X,$$

such that

$$\tau_{Y,X} \circ \tau_{X,Y}:X \otimes Y \to X \otimes Y$$

is the identity, $R_X = L_X \circ \tau_{X,e}$, and the diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{A} & (X \otimes Y) \otimes Z & \xrightarrow{\tau} & Z \otimes (X \otimes Y) \\ {\scriptstyle id \otimes \tau}\downarrow & & & & \downarrow{\scriptstyle A} \\ X \otimes (Z \otimes Y) & \xrightarrow[A]{} & (X \otimes Z) \otimes Y & \xrightarrow[\tau \otimes id]{} & (Z \otimes X) \otimes Y \end{array}$$

commutes. A symmetric monoidal category is a monoidal category $\underline{C}$ endowed with a symmetry τ. A monoidal category can have more than one symmetry (or none at all).

[Note: The "coherency" principle then asserts that "all" diagrams built up from instances of R, L, A, τ (or their inverses), and id by repeated application of $\otimes$ necessarily commute.]

N.B. Let

$$f:\underline{C} \times \underline{C} \to \underline{C} \times \underline{C}$$

be the interchange -- then f is an isomorphism and $\tau:\otimes \to \otimes \circ f$ is a natural isomorphism.

E.g.: $\underline{VEC}_{\underline{k}}$ and $\underline{HILB}$ are symmetric monoidal.

4.1 EXAMPLE Let $\underline{C^*ALG}$ be the category whose objects are the C*-algebras and whose morphisms are the *-homomorphisms — then under the minimal tensor product or the maximal tensor product, C*ALG is a symmetric monoidal category.

4.2 EXAMPLE Let CHX be the category of chain complexes of abelian groups and chain maps — then CHX is monoidal: Take $X \otimes Y$ to be the tensor product and let $e = \{e_n\}$ be the chain complex defined by $e_0 = \underline{Z}$ and $e_n = 0$ $(n \neq 0)$. Furthermore, if $\begin{cases} X = \{X_p\} \\ Y = \{Y_q\} \end{cases}$ and if $\begin{cases} x \in X_p \\ y \in Y_q \end{cases}$, then the assignment

$$\begin{cases} X \otimes Y \to Y \otimes X \\ x \otimes y \to (-1)^{pq} (y \otimes x) \end{cases}$$

is a symmetry for CHX.

4.3 REMARK In the strict situation, matters reduce to the relations $\tau_{e,X} = \tau_{X,e} = id_X$ and

$$\tau_{X \otimes Y,Z} = (\tau_{X,Z} \otimes id_Y) \circ (id_X \otimes \tau_{Y,Z}).$$

[Note: Therefore

$$\tau_{X \otimes Y,Z} \circ \tau_{Y \otimes Z,X} \circ \tau_{Z \otimes X,Y} = id.]$$

4.4 EXAMPLE Let $\mathcal{S}$ be the permutation category introduced in 1.6 -- then $\mathcal{S}$ is symmetric monoidal. To establish this, one must exhibit isomorphisms

$$\tau_{n,m} \in Mor(n \otimes m, m \otimes n)$$

$$= \mathcal{S}_{n+m}$$

fulfilling the various conditions. Definition:

$$\tau_{n,m} = \begin{bmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & n+m \\ m+1 & m+2 & \cdots & m+n & 1 & 2 & \cdots & m \end{bmatrix},$$

with the understanding that $\tau_{n,0} = id_n = \tau_{0,n}$, thus

$$\tau_{m,n} \circ \tau_{n,m} = id_{n \otimes m}.$$

As for the remaining details, it is simplest to work with permutation matrices, so take $n > 0$, $m > 0$, and note that

$$\tau_{n,m} = \begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}.$$

Then

$$(\tau_{n,p} \otimes id_m) \circ (id_n \otimes \tau_{m,p})$$

$$= \begin{bmatrix} 0 & I_p & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_p \\ 0 & I_m & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & I_p \\ I_n & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} = \tau_{n \otimes m,p}.$$

[Note:

$$\left[\begin{array}{l} \forall\, \sigma \in \mathcal{S}_n \\ \forall\, \tau \in \mathcal{S}_m, \end{array}\right.$$

$$\begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \tau \end{bmatrix} = \begin{bmatrix} 0 & \tau \\ \sigma & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \tau & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}.$$

Therefore naturality is manifest, i.e.,

$$T_{n,m} \circ (\sigma \otimes \tau) = (\tau \otimes \sigma) \circ T_{n,m}.]$$

Let $\underline{C}$, $\underline{C}'$ be symmetric monoidal categories — then a <u>symmetric monoidal functor</u> is a monoidal functor (F,ξ,Ξ) such that the diagram

$$\begin{array}{ccc} FX \otimes' FY & \xrightarrow{\Xi_{X,Y}} & F(X \otimes Y) \\ {\scriptstyle T'_{FX,FY}}\downarrow & & \downarrow{\scriptstyle FT_{X,Y}} \\ FY \otimes' FX & \xrightarrow[\Xi_{Y,X}]{} & F(Y \otimes X) \end{array}$$

commutes.

<u>N.B.</u> The monoidal natural transformations between symmetric monoidal functors are, by definition, "symmetric monoidal" (i.e., no further conditions are imposed

that reflect the presence of a symmetry).

[Note: Therefore the subcategory $[\underline{C},\underline{C}']^{\otimes,\tau}$ of $[\underline{C},\underline{C}']^{\otimes}$ whose objects are the symmetric monoidal natural transformations is, by definition, a full subcategory.]

4.5 EXAMPLE Recall that $\mathcal{S}_n$ has the following presentation: It is generated by $\sigma_1,\ldots,\sigma_{n-1}$ subject to the relations

$$\sigma_i^2 = 1,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| > 1).$$

Suppose now that $\underline{C}$ is symmetric strict monoidal and fix $X \in \text{Ob}\,\underline{C}$. Define automorphisms $\Pi^1,\ldots,\Pi^{n-1}$ of $X^{\otimes n}$ by

$$\Pi^i = \text{id}_{X^{\otimes(i-1)}} \otimes \tau_{X,X} \otimes \text{id}_{X^{\otimes(n-i-1)}}.$$

Then there exists a unique homomorphism

$$\Pi_n^X:\mathcal{S}_n \to \text{Aut}\,X^{\otimes n}$$

of groups such that

$$\Pi_n^X(\sigma_i) = \Pi^i \quad (i = 1,\ldots,n-1).$$

Combining the Π_n^X then leads to a symmetric monoidal functor $F:\mathcal{S} \to \underline{C}$ such that $Fn = X^{\otimes n}$.

4.6 LEMMA Let $F:\underline{C} \to \underline{C}'$ be a monoidal equivalence. Assume: $\underline{C}$ is symmetric -- then the symmetry τ on $\underline{C}$ can be transferred to a symmetry τ' on $\underline{C}'$ in such a way as to render F symmetric monoidal.

[Define $\tau_{FX,FY}$ by

$$FX \otimes' FY \xrightarrow{\Xi} F(X \otimes Y) \xrightarrow{F\tau} F(Y \otimes X) \xrightarrow{\Xi^{-1}} FY \otimes' FX$$

and recall that F has a representative image (cf. 2.5).]

4.7 <u>EXAMPLE</u> If $\underline{C}$ is symmetric monoidal, then $\underline{C}_{\underline{str}}$ is symmetric monoidal and $\gamma:\underline{C} \to \underline{C}_{\underline{str}}$ is a symmetric monoidal equivalence.

4.8 <u>LEMMA</u> Let $\underline{C}$, $\underline{C}'$ be symmetric monoidal and let (F,F',μ,μ') be an adjoint equivalence. Assume: F is symmetric monoidal — then F' is symmetric monoidal (cf. 2.9).

§5. *DUALITY*

Let $\underset{\sim}{C}$ be a monoidal category — then each $X \in \text{Ob } \underset{\sim}{C}$ defines functors

$$\begin{cases} - \otimes X : \underset{\sim}{C} \to \underset{\sim}{C} \\ X \otimes - : \underset{\sim}{C} \to \underset{\sim}{C}. \end{cases}$$

Definition: $\underset{\sim}{C}$ is

$$\begin{cases} \underline{\text{left closed}} \\ \underline{\text{right closed}} \end{cases}$$

if $\forall\, X \in \text{Ob } \underset{\sim}{C}$,

$$\begin{cases} - \otimes X \text{ admits a right adjoint, denoted lhom}(X,-) \\ X \otimes - \text{ admits a right adjoint, denoted rhom }(X,-). \end{cases}$$

[Note: $\underset{\sim}{C}$ is <u>closed</u> if it is both left closed and right closed.]

So:

$$\begin{cases} \underset{\sim}{C} \text{ left closed} \Rightarrow \text{Mor}(Y \otimes X, Z) \approx \text{Mor}(Y, \text{lhom}(X,Z)) \\ \underset{\sim}{C} \text{ right closed} \Rightarrow \text{Mor}(X \otimes Y, Z) \approx \text{Mor}(Y, \text{rhom}(X,Z)) \end{cases}$$

for all $Y,Z \in \text{Ob } \underset{\sim}{C}$.

<u>N.B.</u> The functor

$$\begin{cases} \text{lhom}(X,-) \\ \text{rhom}(X,-) \end{cases}$$

is called the $\begin{cases} \text{left} \\ \text{right} \end{cases}$ <u>internal hom functor</u> attached to X.

5.1 <u>REMARK</u> If $\underline{C}$ is symmetric monoidal, then left and right internal homs are naturally isomorphic and if $\underline{C}$ is left or right closed, then $\underline{C}$ is closed.

5.2 <u>EXAMPLE</u> Given a commutative ring $\underline{k}$, let $\underline{MOD}_{\underline{k}}$ be the category whose objects are the left $\underline{k}$-modules and whose morphisms are the $\underline{k}$-linear maps -- then $\underline{MOD}_{\underline{k}}$ is symmetric monoidal. Moreover, $\underline{MOD}_{\underline{k}}$ is closed and

$$\begin{cases} \text{lhom}(X,Z) \approx \text{Hom}_{\underline{k}}(X,Z) \\ \text{rhom}(X,Z) \approx \text{Hom}_{\underline{k}}(X,Z). \end{cases}$$

5.3 <u>LEMMA</u> Suppose that $\underline{C}$ is left closed -- then $\forall X \in \text{Ob } \underline{C}$, the functor $-- \otimes X$ preserves colimits (being a left adjoint) and the functor $\text{lhom}(X,--)$ preserves limits (being a right adjoint).

5.4 <u>LEMMA</u> Suppose that $\underline{C}$ is left closed -- then $\forall Z \in \text{Ob } \underline{C}$, the cofunctor $\text{lhom}(--,Z)$ converts colimits to limits.

<u>PROOF</u> Let $\underline{I}$ be a small category, $\Delta:\underline{I} \to \underline{C}$ a diagram for which $\text{colim}_{\underline{I}} \Delta_i$ exists -- then $\forall Y \in \text{Ob } \underline{C}$,

$$\begin{aligned} &\text{Mor}(Y,\text{lhom}(\text{colim}_{\underline{I}} \Delta_i,Z)) \\ &\approx \text{Mor}(Y \otimes \text{colim}_{\underline{I}} \Delta_i,Z) \\ &\approx \text{Mor}(\text{colim}_{\underline{I}} (Y \otimes \Delta_i),Z) \end{aligned}$$

$$\approx \lim_{\underline{I}} \operatorname{Mor}(Y \otimes \Delta_i, Z)$$

$$\approx \lim_{\underline{I}} \operatorname{Mor}(Y, \operatorname{lhom}(\Delta_i, Z))$$

$$\approx \operatorname{Mor}(Y, \lim_{\underline{I}} \operatorname{lhom}(\Delta_i, Z))$$

=>

$$\operatorname{lhom}(\operatorname{colim}_{\underline{I}} \Delta_i, Z) \approx \lim_{\underline{I}} \operatorname{lhom}(\Delta_i, Z).$$

Let $\underline{C}$ be a monoidal category. Given $X \in \operatorname{Ob} \underline{C}$, an object ${}^{\vee}X \in \operatorname{Ob} \underline{C}$ is said to be a <u>left dual</u> of X if $\exists$ morphisms

$$\begin{cases} \varepsilon_X : {}^{\vee}X \otimes X \to e \\ \eta_X : e \to X \otimes {}^{\vee}X \end{cases}$$

and commutative diagrams

$$\begin{cases}
\begin{array}{ccccc}
X & \xrightarrow{L^{-1}} & e \otimes X & \xrightarrow{\eta_X \otimes \mathrm{id}} & (X \otimes {}^{\vee}X) \otimes X \\
\Vert & & & & \downarrow A^{-1} \\
X & \xleftarrow[R]{} & X \otimes e & \xleftarrow[\mathrm{id} \otimes \varepsilon_X]{} & X \otimes ({}^{\vee}X \otimes X).
\end{array} \\[2em]
\begin{array}{ccccc}
{}^{\vee}X & \xrightarrow{R^{-1}} & {}^{\vee}X \otimes e & \xrightarrow{\mathrm{id} \otimes \eta_X} & {}^{\vee}X \otimes (X \otimes {}^{\vee}X) \\
\Vert & & & & \downarrow A \\
{}^{\vee}X & \xleftarrow[L]{} & e \otimes {}^{\vee}X & \xleftarrow[\varepsilon_X \otimes \mathrm{id}]{} & ({}^{\vee}X \otimes X) \otimes {}^{\vee}X.
\end{array}
\end{cases}$$

N.B. When $\underline{C}$ is strict, these diagrams reduce to the relations

$$\begin{cases} (\mathrm{id}_X \otimes \varepsilon_X) \circ (\eta_X \otimes \mathrm{id}_X) = \mathrm{id}_X \\ (\varepsilon_X \otimes \mathrm{id}_{{}^\vee X}) \circ (\mathrm{id}_{{}^\vee X} \otimes \eta_X) = \mathrm{id}_{{}^\vee X}. \end{cases}$$

5.5 LEMMA Suppose that ${}^\vee X$ is a left dual of X -- then the functor $— \otimes {}^\vee X$ is a right adjoint for the functor $— \otimes X$ and the functor ${}^\vee X \otimes —$ is a left adjoint for the functor $X \otimes —$.

In brief: $\forall\ Y,Z \in \mathrm{Ob}\ \underline{C}$,

$$\begin{cases} \mathrm{Mor}(Y \otimes X, Z) \approx \mathrm{Mor}(Y, Z \otimes {}^\vee X) \\ \mathrm{Mor}({}^\vee X \otimes Y, Z) \approx \mathrm{Mor}(Y, X \otimes Z). \end{cases}$$

PROOF It will be enough to show that $— \otimes {}^\vee X$ is a right adjoint for $— \otimes X$, the proof that ${}^\vee X \otimes —$ is a left adjoint for $X \otimes —$ being similar. So let

$$\begin{cases} F = — \otimes X \\ G = — \otimes {}^\vee X \end{cases} \qquad \text{(cf. 2.6)}$$

and to simplify the writing, take $\underline{C}$ strict. Define

$$\begin{cases} \mu \in \mathrm{Nat}(\mathrm{id}_{\underline{C}}, G \circ F) \\ \nu \in \mathrm{Nat}(F \circ G, \mathrm{id}_{\underline{C}}) \end{cases}$$

by

$$\left[\begin{array}{l} \mu_W \in \mathrm{Mor}(W, W \otimes X \otimes {}^{\vee}X) \\ \mu_W = \mathrm{id}_W \otimes \eta_X \end{array}\right.$$

$$\left[\begin{array}{l} \nu_W \in \mathrm{Mor}(W \otimes {}^{\vee}X \otimes X, W) \\ \nu_W = \mathrm{id}_W \otimes \epsilon_X. \end{array}\right.$$

Consider

$$(\nu F) \circ (F\mu).$$

Thus

$$((\nu F) \circ (F\mu))_W = (\nu F)_W \circ (F\mu)_W.$$

And

$$(\mu) \quad \left[\begin{array}{l} F\mu \in \mathrm{Nat}(F, FGF) \\ (F\mu)_W : FW \to FGFW \end{array}\right.$$

or still,

$$(F\mu)_W : W \otimes X \xrightarrow{\mathrm{id}_W \otimes \eta_X \otimes \mathrm{id}_X} W \otimes X \otimes {}^{\vee}X \otimes X.$$

$$(\nu) \quad \left[\begin{array}{l} \nu F \in \mathrm{Nat}(FGF, F) \\ (\nu F)_W : FGFW \to FW \end{array}\right.$$

or still,

$$(\nu F)_W : W \otimes X \otimes {}^{\vee}X \otimes X \xrightarrow{\mathrm{id}_W \otimes \mathrm{id}_X \otimes \epsilon_X} W \otimes X.$$

Therefore

$$(\nu F)_W \circ (F\mu)_W \in \text{Mor}(W \otimes X, W \otimes X)$$

is the composition

$$(\text{id}_W \otimes \text{id}_X \otimes \varepsilon_X) \circ (\text{id}_W \otimes \eta_X \otimes \text{id}_X)$$

$$= (\text{id}_W \circ \text{id}_W) \otimes ((\text{id}_X \otimes \varepsilon_X) \circ (\eta_X \otimes \text{id}_X))$$

$$= \text{id}_W \otimes \text{id}_X$$

$$= \text{id}_{W \otimes X}$$

$$= \text{id}_{FW}$$

$$= (\text{id}_F)_W.$$

I.e.:

$$(\nu F) \circ (F\mu) = \text{id}_F.$$

The verification that

$$(G\nu) \circ (\mu G) = \text{id}_G$$

is analogous.

5.6 <u>LEMMA</u> A left dual of X, if it exists, is unique up to isomorphism.

<u>PROOF</u> Suppose that

$$\begin{bmatrix} {}^{\vee}X_1 \\ {}^{\vee}X_2 \end{bmatrix}$$

are two left duals of X -- then the functors

$$\left[\begin{array}{l} - \otimes {}^{\vee}X_1 \\ \\ - \otimes {}^{\vee}X_2 \end{array}\right.$$

are naturally isomorphic (both being right adjoints for $— \otimes X$), so $\forall\ W \in \text{Ob}\ \underline{C}$,

$$W \otimes {}^{\vee}X_1 \approx W \otimes {}^{\vee}X_2.$$

Now specialize and take $W = e$ to get

$$e \otimes {}^{\vee}X_1 \approx e \otimes {}^{\vee}X_2$$

$\Rightarrow$

$${}^{\vee}X_1 \approx {}^{\vee}X_2.$$

[Note: Explicated,

$${}^{\vee}X_1 \xrightarrow{R^{-1}} {}^{\vee}X_1 \otimes e$$

$$\xrightarrow{\text{id} \otimes \eta_X^2} {}^{\vee}X_1 \otimes (X \otimes {}^{\vee}X_2)$$

$$\xrightarrow{A} ({}^{\vee}X_1 \otimes X) \otimes {}^{\vee}X_2$$

$$\xrightarrow{\epsilon_X^1 \otimes \text{id}} e \otimes {}^{\vee}X_2$$

$$\xrightarrow{L} {}^{\vee}X_2.]$$

5.7 <u>REMARK</u> Suppose that $({}^{\vee}X, \varepsilon_X, \eta_X)$ is a left dual of X. Let $\phi: {}^{\vee}X \to {}^{\vee}X'$ be an isomorphism and put

$$\begin{cases} \varepsilon_X' = \varepsilon_X \circ (\phi^{-1} \otimes id_X) \\ \eta_X' = (id_X \otimes \phi) \circ \eta_X. \end{cases}$$

Then the triple $({}^{\vee}X', \varepsilon_X', \eta_X')$ is a left dual of X.

[Consider first the case when $\underline{C}$ is strict, thus, e.g.,

$$(id_X \otimes \varepsilon_X') \circ (\eta_X' \otimes id_X)$$

$$= id_X \otimes (\varepsilon_X \circ (\phi^{-1} \otimes id_X)) \circ ((id_X \otimes \phi) \circ \eta_X) \otimes id_X$$

$$= id_X \otimes \varepsilon_X \circ id_X \otimes (\phi^{-1} \otimes id_X) \circ (id_X \otimes \phi) \otimes id_X \circ \eta_X \otimes id_X.$$

But

$$(id_X \otimes \phi) \otimes id_X = id_X \otimes (\phi \otimes id_X)$$

$\Rightarrow$

$$id_X \otimes (\phi^{-1} \otimes id_X) \circ (id_X \otimes \phi) \otimes id_X$$

$$= id_X \otimes (\phi^{-1} \otimes id_X) \circ id_X \otimes (\phi \otimes id_X)$$

$$= id_X \otimes (\phi^{-1} \otimes id_X) \circ (\phi \otimes id_X)$$

$$= id_X \otimes id_{{}^{\vee}X \otimes X}$$

$$= id_{X \otimes {}^{\vee}X \otimes X}$$

=>

$$\mathrm{id}_X \otimes \varepsilon'_X \circ \eta'_X \otimes \mathrm{id}_X$$

$$= (\mathrm{id}_X \otimes \varepsilon_X) \circ (\eta_X \otimes \mathrm{id}_X) = \mathrm{id}_X.$$

In general, the claim is that id_X equals

$$R \circ (\mathrm{id}_X \otimes \varepsilon'_X) \circ A^{-1} \circ (\eta'_X \otimes \mathrm{id}_X) \circ L^{-1}$$

or still,

$$R \circ \mathrm{id}_X \otimes (\varepsilon_X \circ (\phi^{-1} \otimes \mathrm{id}_X)) \circ A^{-1} \circ ((\mathrm{id}_X \otimes \phi) \circ \eta_X) \otimes \mathrm{id}_X \circ L^{-1}$$

or still,

$$R \circ \mathrm{id}_X \otimes \varepsilon_X \circ \mathrm{id}_X \otimes (\phi^{-1} \otimes \mathrm{id}_X) \circ A^{-1} \circ (\mathrm{id}_X \otimes \phi) \otimes \mathrm{id}_X \circ \eta_X \otimes \mathrm{id}_X \circ L^{-1}.$$

Here

$$A^{-1}:(X \otimes {}^{\vee}X') \otimes X \to X \otimes ({}^{\vee}X' \otimes X).$$

So, to complete the verification, one has only to show that the composition

$$(X \otimes {}^{\vee}X) \otimes X \xrightarrow{(\mathrm{id} \otimes \phi) \otimes \mathrm{id}} (X \otimes {}^{\vee}X') \otimes X$$

$$\xrightarrow{A^{-1}} X \otimes ({}^{\vee}X' \otimes X)$$

$$\xrightarrow{\mathrm{id} \otimes (\phi^{-1} \otimes \mathrm{id})} X \otimes ({}^{\vee}X \otimes X)$$

is

$$(X \otimes {}^{\vee}X) \otimes X \xrightarrow{A^{-1}} X \otimes ({}^{\vee}X \otimes X).$$

However, due to the naturality of the associativity constraint, there is a commutative diagram

$$\begin{array}{ccc} (X \otimes {}^{\vee}X) \otimes X & \xrightarrow{A^{-1}} & X \otimes ({}^{\vee}X \otimes X) \\ {\scriptstyle (id \otimes \phi) \otimes id}\downarrow & & \downarrow{\scriptstyle id \otimes (\phi \otimes id)} \\ (X \otimes {}^{\vee}X') \otimes X & \xrightarrow[A^{-1}]{} & X \otimes ({}^{\vee}X' \otimes X). \end{array}$$

And

$$(id \otimes (\phi \otimes id))^{-1} = id \otimes (\phi^{-1} \otimes id).]$$

A monoidal category $\underline{C}$ is said to be <u>left autonomous</u> if each object in $\underline{C}$ admits a left dual.

<u>N.B.</u> Suppose that $\underline{C}$ is left autonomous. Given $f \in Mor(X,Y)$, define ${}^{\vee}f \in Mor({}^{\vee}Y, {}^{\vee}X)$ by

$$\begin{aligned} {}^{\vee}Y & \xrightarrow{R^{-1}} {}^{\vee}Y \otimes e \\ & \xrightarrow{id \otimes \eta_X} {}^{\vee}Y \otimes (X \otimes {}^{\vee}X) \\ & \xrightarrow{A} ({}^{\vee}Y \otimes X) \otimes {}^{\vee}X \\ & \xrightarrow{(id \otimes f) \otimes id} ({}^{\vee}Y \otimes Y) \otimes {}^{\vee}X \\ & \xrightarrow{\varepsilon_Y \otimes id} e \otimes {}^{\vee}X \\ & \xrightarrow{L} {}^{\vee}X. \end{aligned}$$

Then the assignment

$$\begin{cases} X \to {}^{\vee}X \\ f \to {}^{\vee}f \end{cases}$$

defines a cofunctor $\underline{C} \to \underline{C}$.

[Note: The specific form of ${}^{\vee}f$ depends on the choices of ${}^{\vee}X$ and ${}^{\vee}Y$.]

5.8 REMARK If $\underline{C}$ is left autonomous and if $X,Y \in \text{Ob } \underline{C}$, then ${}^{\vee}(X \otimes Y)$ is isomorphic to ${}^{\vee}Y \otimes {}^{\vee}X$.

[We have

$$\begin{aligned} \text{Mor}({}^{\vee}(X \otimes Y) \otimes W, Z) &\approx \text{Mor}(W, (X \otimes Y) \otimes Z) \\ &\approx \text{Mor}(W, X \otimes (Y \otimes Z)) \\ &\approx \text{Mor}({}^{\vee}X \otimes W, Y \otimes Z) \\ &\approx \text{Mor}({}^{\vee}Y \otimes ({}^{\vee}X \otimes W), Z) \\ &\approx \text{Mor}(({}^{\vee}Y \otimes {}^{\vee}X) \otimes W, Z) \end{aligned}$$

$\Rightarrow$

$${}^{\vee}(X \otimes Y) \approx {}^{\vee}Y \otimes {}^{\vee}X.]$$

5.9 LEMMA Suppose that $\underline{C}$ is left autonomous — then $\underline{C}$ is left closed.

PROOF In fact, $\forall X \in \text{Ob } \underline{C}$,

$$\text{lhom}(X, —) = — \otimes {}^{\vee}X.$$

One can also introduce the notion of a right dual $X^{\vee}$ of X, where this time

$$\begin{cases} \varepsilon_X : X \otimes X^\vee \to e \\ \eta_X : e \to X^\vee \otimes X \end{cases}$$

subject to the obvious commutativity conditions. Here the functor $— \otimes X^\vee$ is a left adjoint for the functor $— \otimes X$ and the functor $X^\vee \otimes —$ is a right adjoint for the functor $X \otimes —$.

[Note: If X admits a left dual ${}^\vee X$ and a right dual $X^\vee$, then in general ${}^\vee X$ and $X^\vee$ are not isomorphic. On the other hand, it is true that

$$({}^\vee X)^\vee \approx X \approx {}^\vee(X^\vee).$$

E.g.:

$$\mathrm{Mor}(Y \otimes ({}^\vee X)^\vee, Z) \approx \mathrm{Mor}(Y, Z \otimes {}^\vee X) \approx \mathrm{Mor}(Y \otimes X, Z)$$

$$\Rightarrow$$

$$({}^\vee X)^\vee \approx X.]$$

The definition of "right autonomous" is clear and we shall term $\underline{C}$ <u>autonomous</u> if it is both left and right autonomous.

5.10 <u>LEMMA</u> Suppose that $\underline{C}$ is right autonomous — then $\underline{C}$ is right closed.

<u>PROOF</u> In fact, $\forall\ X \in \mathrm{Ob}\ \underline{C}$,

$$\mathrm{rhom}(X, —) = X^\vee \otimes — .$$

5.11 <u>REMARK</u> If $\underline{C}$ is autonomous, then $— \otimes —$ preserves colimits in both variables.

Suppose that $F:\underline{C} \to \underline{C}'$ is a monoidal functor. Assume: $X^\vee$ is a right dual of X — then $FX^\vee$ is a right dual of FX. Proof: Consider the arrows

$$\begin{cases} FX \otimes' FX^\vee \xrightarrow{\Xi} F(X \otimes X^\vee) \xrightarrow{F\varepsilon_X} Fe \xrightarrow{\xi^{-1}} e' \\ e' \xrightarrow{\xi} Fe \xrightarrow{F\eta_X} F(X^\vee \otimes X) \xrightarrow{\Xi^{-1}} FX^\vee \otimes' FX. \end{cases}$$

[Note: Assume that $\underline{C}$, $\underline{C}'$ are right autonomous — then there is an isomorphism

$$\Delta_X:FX^\vee \to (FX)^\vee,$$

namely the composition

$$\begin{aligned} FX^\vee &\xrightarrow{L} e' \otimes' FX^\vee \\ &\xrightarrow{\eta \otimes id} ((FX)^\vee \otimes' FX) \otimes' FX^\vee \\ &\xrightarrow{A^{-1}} (FX)^\vee \otimes' (FX \otimes' FX^\vee) \\ &\xrightarrow{id \otimes \Xi} (FX)^\vee \otimes' F(X \otimes X^\vee) \\ &\xrightarrow{id \otimes F\varepsilon} (FX)^\vee \otimes' Fe \\ &\xrightarrow{id \otimes \xi^{-1}} (FX)^\vee \otimes' e' \\ &\xrightarrow{R} (FX)^\vee, \end{aligned}$$

and the diagram

$$\begin{array}{ccc}
FX \otimes' FX^{\vee} & \xrightarrow{\ \mathrm{id} \otimes \Delta_X\ } & FX \otimes' (FX)^{\vee} \\
\Big\downarrow \Xi & & \Big\downarrow \varepsilon_{FX} \\
F(X \otimes X^{\vee}) & \xrightarrow[\ F\varepsilon_X\ ]{} & e'
\end{array}$$

commutes.]

N.B. One can, of course, work equally well with left duals.

5.12 LEMMA Let

$$\left[\begin{array}{l} (F,\xi,\Xi) \\ \\ (G,\theta,\Theta) \end{array}\right.$$

be monoidal functors and let $\alpha:F \rightarrow G$ be a monoidal natural transformation. Assume: The source $\underline{C}$ of F and G is autonomous — then α is a monoidal natural isomorphism.

PROOF The claim is that $\forall\ X \in \mathrm{Ob}\ \underline{C}$,

$$\alpha_X:FX \rightarrow GX$$

is an isomorphism. From the above, $FX^{\vee}$ $(GX^{\vee})$ is a right dual of FX (GX) or still, FX (GX) is a left dual of $FX^{\vee}$ $(GX^{\vee})$. This said, form

$$\alpha_{X^{\vee}}:FX^{\vee} \rightarrow GX^{\vee}$$

and consider

$${}^{\vee}(\alpha_{X^{\vee}}):GX \rightarrow FX.$$

[Note: Accordingly, if $\underline{C}$ is autonomous, then the metacategory $[\underline{C},\underline{C}']^{\otimes}$ is a groupoid.]

Suppose that $\underline{C}$ is symmetric monoidal and left autonomous — then $\underline{C}$ is right autonomous, hence $\underline{C}$ is autonomous. Proof: Given $X \in \text{Ob}\,\underline{C}$, take $X^{\vee} = {}^{\vee}X$ and define morphisms

$$\begin{cases} X \otimes X^{\vee} \to e \\ e \to X^{\vee} \otimes X \end{cases}$$

by

$$\begin{cases} \varepsilon_X \circ \tau_{X,{}^{\vee}X} \\ \tau_{X,{}^{\vee}X} \circ \eta_X. \end{cases}$$

5.13 <u>EXAMPLE</u> $\underline{\text{FDVECT}}_{\underline{k}}$ is autonomous. In fact, $\underline{\text{FDVECT}}_{\underline{k}}$ is symmetric monoidal, so it suffices to set up a left duality. Thus given X, let ${}^{\vee}X$ be its dual and define

$$\varepsilon_X : {}^{\vee}X \otimes X \to \underline{k}$$

by

$$\varepsilon_X(\lambda,x) = \lambda(x).$$

On the other hand, there is a canonical isomorphism

$$\phi : \text{Hom}(X,X) \to \text{Hom}(\underline{k},{}^{\vee}X \otimes X)$$

and we let

$$\eta_X = \phi(\text{id}_X).$$

[Note: An object X in $\underline{\text{VEC}}_{\underline{k}}$ admits a left dual iff it is finite dimensional.]

5.14 EXAMPLE The full subcategory of $\underline{\text{MOD}}_{\underline{k}}$ whose objects are finitely generated projective is autonomous (cf. 5.2).

Assume still that $\underline{C}$ is symmetric monoidal and left autonomous.

5.15 LEMMA There is a monoidal natural isomorphism

$$\text{id}_{\underline{C}} \to {}^{\vee\vee}(\text{—}).$$

[To see this, consider the composition

$$X \xrightarrow{R^{-1}} X \otimes e$$

$$\xrightarrow{\text{id} \otimes \eta} (X \otimes ({}^{\vee}X \otimes {}^{\vee\vee}X)$$

$$\xrightarrow{A} (X \otimes {}^{\vee}X) \otimes {}^{\vee\vee}X$$

$$\xrightarrow{T \otimes \text{id}} ({}^{\vee}X \otimes X) \otimes {}^{\vee\vee}X$$

$$\xrightarrow{\epsilon \otimes \text{id}} e \otimes {}^{\vee\vee}X$$

$$\xrightarrow{L} {}^{\vee\vee}X.]$$

N.B. Let

$$\delta_X : X \to {}^{\vee\vee}X$$

be the arrow constructed above -- then

$$(\delta_X)^{-1} = {}^{\vee}(\delta_{X^{\vee}}) \quad \text{(cf. 5.12)}.$$

But here $X^{\vee} = {}^{\vee}X$, so

$$(\delta_X)^{-1} = {}^{\vee}(\delta_{{}^{\vee}X}).$$

[Note: To make sense of this, recall that

$$\begin{cases} X \text{ is a left dual of } X^{\vee} \\ {}^{\vee\vee}X \text{ is a left dual of } {}^{\vee\vee}(X^{\vee}). \end{cases}$$

And

$$\delta_{X^{\vee}}: X^{\vee} \to {}^{\vee\vee}(X^{\vee})$$

=>

$$^{\vee}(\delta_{X^{\vee}}): {}^{\vee\vee}X \to X.]$$

§6. *TWISTS*

Let $\underline{C}$ be symmetric monoidal and left autonomous — then a twist Ω is a monoidal natural isomorphism of the identity functor $id_{\underline{C}}$ such that $\forall\ X \in \text{Ob}\ \underline{C}$,

$$(\Omega_X \otimes id_{{}^\vee X}) \circ \eta_X = (id_X \otimes \Omega_{{}^\vee X}) \circ \eta_X.$$

[Note: Tacitly, $id_{\underline{C}}$ is taken to be strict ($\xi = id$, $\Xi = id$), thus from the definitions

$$\Omega_{X \otimes Y} = \Omega_X \otimes \Omega_Y \text{ and } \Omega_e = id_e.]$$

To consolidate the terminology, a symmetric monoidal $\underline{C}$ which is left autonomous and has a twist Ω will be referred to as a ribbon category.

N.B. The choice $\Omega_X = id_X$ is permissible, in which case $\underline{C}$ is said to be even.

It was pointed out near the end of §5 that an even ribbon category is right autonomous. This fact is true in general. Proof: Given $X \in \text{Ob}\ \underline{C}$, take $X^\vee = {}^\vee X$ and define morphisms

$$\begin{cases} X \otimes X^\vee \to e \\ e \to X^\vee \otimes X \end{cases}$$

by

$$\begin{cases} \varepsilon_X \circ \tau_{X,{}^\vee X} \circ \Omega_X \otimes id_{{}^\vee X} \\ id_{{}^\vee X} \otimes \Omega_X \circ \tau_{X,{}^\vee X} \circ \eta_X. \end{cases}$$

6.1 LEMMA In the presence of a twist Ω,

$$X \approx {}^{\vee}({}^{\vee}X).$$

PROOF Consider the composition

$$X \xrightarrow{R^{-1}} X \otimes e$$

$$\xrightarrow{\Omega_X \otimes \eta_{{}^{\vee}X}} X \otimes ({}^{\vee}X \otimes {}^{\vee}({}^{\vee}X))$$

$$\xrightarrow{A} (X \otimes {}^{\vee}X) \otimes {}^{\vee}({}^{\vee}X)$$

$$\xrightarrow{\tau_{X,{}^{\vee}X} \otimes \mathrm{id}} ({}^{\vee}X \otimes X) \otimes {}^{\vee}({}^{\vee}X)$$

$$\xrightarrow{\varepsilon_X \otimes \mathrm{id}} e \otimes {}^{\vee}({}^{\vee}X)$$

$$\xrightarrow{L} {}^{\vee}({}^{\vee}X).$$

E.g.:

$${}^{\vee}e \approx {}^{\vee}e \otimes e \approx {}^{\vee}e \otimes {}^{\vee}({}^{\vee}e) \approx {}^{\vee}({}^{\vee}e \otimes e) \approx {}^{\vee}({}^{\vee}e) \approx e.$$

6.2 LEMMA In the presence of a twist Ω, the left and right dual of every morphism $f:X \to Y$ agree: ${}^{\vee}f = f^{\vee}$.

Let $\underline{C}$ be a ribbon category. Given $f \in \mathrm{Mor}(X,X)$, define the trace of f by

$$\mathrm{tr}_X(f) = \varepsilon_X \circ \tau_{X,{}^\vee X} \circ \Omega_X \otimes \mathrm{id}_{{}^\vee X} \circ (f \otimes \mathrm{id}_{{}^\vee X}) \circ \eta_X.$$

[Note:

$$\mathrm{tr}_X(f) \in \mathrm{Mor}(e,e) \ (= \underline{M}(\underline{C})).]$$

6.3 <u>LEMMA</u> We have

1. $\mathrm{tr}_X(f) = \mathrm{tr}_{{}^\vee X}({}^\vee f)$;

2. $\mathrm{tr}_X(g \circ f) = \mathrm{tr}_Y(f \circ g)$ $(f:X \rightarrow Y,\ g:Y \rightarrow X)$;

3. $\mathrm{tr}_{X_1 \otimes X_2}(f_1 \otimes f_2) = \mathrm{tr}_{X_1}(f_1)\ \mathrm{tr}_{X_2}(f_2)$.

Put

$$\dim X = \mathrm{tr}_X(\mathrm{id}_X),$$

the <u>dimension</u> of X.

So, on the basis of 6.3,

$$\dim X = \dim {}^\vee X$$

and

$$\dim (X \otimes Y) = (\dim X)(\dim Y).$$

<u>N.B.</u> Take $\Omega = \mathrm{id}$ -- then the <u>categorical dimension</u> of X is the arrow

$$e \xrightarrow{\eta_X} X \otimes {}^\vee X \xrightarrow{\tau_{X,{}^\vee X}} {}^\vee X \otimes X \xrightarrow{\varepsilon_X} e.$$

6.4 <u>EXAMPLE</u> Consider $\underline{\mathrm{FDVEC}}_{\underline{k}}$ (viewed as an even ribbon category (cf. 5.13)) — then the trace of $f:X \rightarrow X$ is the composition

$$\underline{k} \xrightarrow{\eta_X} X \otimes {}^\vee X \xrightarrow{f \otimes id_{{}^\vee X}} X \otimes {}^\vee X \xrightarrow{\tau_{X,{}^\vee X}} {}^\vee X \otimes X \xrightarrow{\epsilon_X} \underline{k}.$$

Therefore the abstract definition of $tr_X(f)$ is the usual one. In particular:

$$\dim X = (\dim_{\underline{k}} X) 1_{\underline{k}}.$$

E.g.:

$$\dim \underline{k}^n = n 1_{\underline{k}},$$

the distinction between $n \in N$ and $n1_{\underline{k}}$ being potentially essential if $\underline{k}$ has non-zero characteristic.

6.5 REMARK While evident, it is important to keep in mind that the definitions of trace and dimension depend on all the underlying assumptions, viz. that our monoidal $\underline{C}$ is symmetric, left autonomous, and has a twist Ω.

Suppose that $\underline{C}$, $\underline{C}'$ are ribbon categories with respective twists Ω, Ω' -- then a symmetric monoidal functor $F:\underline{C} \to \underline{C}'$ is twist preserving if $\forall\ X \in \text{Ob}\ \underline{C}$,

$$F\Omega_X = \Omega'_{FX}.$$

6.6 LEMMA If $F:\underline{C} \to \underline{C}'$ is twist preserving, then $\forall\ f \in \text{Mor}(X,X)$, the diagram

$$\begin{array}{ccc} e' & \xrightarrow{\xi} & Fe \\ {\scriptstyle tr_{FX}(Ff)} \downarrow & & \downarrow {\scriptstyle Ftr_X(f)} \\ e' & \xrightarrow[\xi]{} & Fe \end{array}$$

commutes.

Matters are invariably simpler if $\underline{C}$ is a strict ribbon category, which will be the underlying supposition in 6.7 - 6.9 below.

6.7 LEMMA The arrows

$$\begin{cases} \eta_e : e \to {}^{\vee}e \\ \varepsilon_e : {}^{\vee}e \to e \end{cases}$$

are mutually inverse isomorphisms.

PROOF Take X = e in the relation

$$(\mathrm{id}_X \otimes \varepsilon_X) \circ (\eta_X \otimes \mathrm{id}_X) = \mathrm{id}_X$$

to see that

$$\varepsilon_e \circ \eta_e = \mathrm{id}_e.$$

Now fix an isomorphism $\phi : e \to {}^{\vee}e$ — then

$$\begin{cases} \phi^{-1} \circ \eta_e \\ \varepsilon_e \circ \phi \end{cases} \in \underline{M}(\underline{C})$$

=>

$$(\phi^{-1} \circ \eta_e) \circ (\varepsilon_e \circ \phi) = (\varepsilon_e \circ \phi) \circ (\phi^{-1} \circ \eta_e) \quad \text{(cf. 1.4)}$$

$$= \varepsilon_e \circ \eta_e = \mathrm{id}_e$$

=>

$$\eta_e \circ \varepsilon_e = \mathrm{id}_e.$$

6.8 <u>LEMMA</u> $\forall\, s \in \underline{M}(\underline{C})$,

$$\mathrm{tr}_e(s) = s.$$

<u>PROOF</u> In fact,

$$\begin{aligned}
\mathrm{tr}_e(s) &= \varepsilon_e \circ \tau_{e,{}^{\vee}e} \circ \Omega_e \otimes \mathrm{id}_{{}^{\vee}e} \circ (s \otimes \mathrm{id}_{{}^{\vee}e}) \circ \eta_e \\
&= \varepsilon_e \circ \mathrm{id}_{{}^{\vee}e} \circ \mathrm{id}_{{}^{\vee}e} \circ (s \otimes \mathrm{id}_{{}^{\vee}e}) \circ \eta_e \\
&= \varepsilon_e \circ (s \otimes \mathrm{id}_{{}^{\vee}e}) \circ \eta_e \\
&= (\mathrm{id}_e \otimes \varepsilon_e)(s \otimes \mathrm{id}_e \otimes \mathrm{id}_{{}^{\vee}e})(\mathrm{id}_e \otimes \eta_e) \\
&= s \otimes (\varepsilon_e \circ \eta_e) \\
&= s \otimes \mathrm{id}_e \\
&= s.
\end{aligned}$$

[Note: Therefore

$$\dim e = \mathrm{tr}_e(\mathrm{id}_e) = \mathrm{id}_e.]$$

6.9 <u>LEMMA</u> $\forall\, X \in \mathrm{Ob}\,\underline{C}$,

$$\Omega_{{}^{\vee}X} = {}^{\vee}\Omega_X.$$

<u>PROOF</u> The compositions

$$\left[\begin{array}{l}
e \xrightarrow{\ \eta_X\ } X \otimes {}^{\vee}X \xrightarrow{\ \Omega_X \otimes \mathrm{id}\ } X \otimes {}^{\vee}X \\[2ex]
e \xrightarrow{\ \eta_X\ } X \otimes {}^{\vee}X \xrightarrow{\ \mathrm{id} \otimes \Omega_{{}^{\vee}X}\ } X \otimes {}^{\vee}X
\end{array}\right.$$

are equal, thus the compositions

$$\left[\begin{array}{l} {}^{\vee}X = {}^{\vee}X \otimes e \xrightarrow{\text{id} \otimes \eta_X} {}^{\vee}X \otimes X \otimes {}^{\vee}X \xrightarrow{\text{id} \otimes \Omega_X \otimes \text{id}} {}^{\vee}X \otimes X \otimes {}^{\vee}X \\ {}^{\vee}X = {}^{\vee}X \otimes e \xrightarrow{\text{id} \otimes \eta_X} {}^{\vee}X \otimes X \otimes {}^{\vee}X \xrightarrow{\text{id} \otimes \text{id} \otimes \Omega_{{}^{\vee}X}} {}^{\vee}X \otimes X \otimes {}^{\vee}X \end{array}\right.$$

are equal. Postcompose with $\varepsilon_X \otimes \text{id}_{{}^{\vee}X}$ — then the first line gives ${}^{\vee}\Omega_X$, while the second line is

$$\varepsilon_X \otimes \text{id}_{{}^{\vee}X} \circ \text{id}_{{}^{\vee}X} \otimes \text{id}_X \otimes \Omega_{{}^{\vee}X} \circ \text{id}_{{}^{\vee}X} \otimes \eta_X$$

or still,

$$\varepsilon_X \otimes \text{id}_{{}^{\vee}X} \circ \text{id}_{{}^{\vee}X \otimes X} \otimes \Omega_{{}^{\vee}X} \circ \text{id}_{{}^{\vee}X} \otimes \eta_X$$

or still,

$$(\text{id}_e \otimes \Omega_{{}^{\vee}X}) \circ (\varepsilon_X \otimes \text{id}_{{}^{\vee}X}) \circ \text{id}_{{}^{\vee}X} \otimes \eta_X$$

or still,

$$\Omega_{{}^{\vee}X} \circ \text{id}_{{}^{\vee}X} = \Omega_{{}^{\vee}X}.$$

6.10 <u>REMARK</u> Let $\underline{C}$ be a ribbon category -- then this structure can be transferred to $\underline{C}_{\text{str}}$. That the symmetry τ passes to a symmetry τ_{str} of $\underline{C}_{\text{str}}$ was noted already in 4.6. As for the left duality, a generic element of $\underline{C}_{\text{str}}$ is a finite sequence $(X_1,\ldots,X_n)$ and

$${}^{\vee}(X_1,\ldots,X_n) = ({}^{\vee}X_n,\ldots,{}^{\vee}X_1),$$

where ε and η are defined in the obvious way. It is also clear that the twist on $\underline{C}$ can be brought over to a twist on $\underline{C}_{str}$. Accordingly, $\gamma:\underline{C} \to \underline{C}_{str}$ is a symmetric monoidal equivalence which is twist preserving, i.e., $\gamma:\underline{C} \to \underline{C}_{str}$ is a ribbon equivalence.

§7. *-CATEGORIES

Let $\underline{k}$ be a commutative ring -- then a category $\underline{C}$ is $\underline{k}$-enriched if $\forall$ X,Y $\in$ Ob $\underline{C}$, Mor(X,Y) is a $\underline{k}$-module and if the composition of morphisms is $\underline{k}$-bilinear. A functor F between $\underline{k}$-enriched categories is $\underline{k}$-linear if the induced maps

$$\text{Mor}(X,Y) \rightarrow \text{Mor}(FX,FY)$$

are homomorphisms of $\underline{k}$-modules.

[Note: If $\underline{C}$ is $\underline{k}$-enriched and monoidal, then $\underline{C} \times \underline{C}$ is $\underline{k}$-enriched and the functor $\otimes:\underline{C} \times \underline{C} \rightarrow \underline{C}$ is assumed to be $\underline{k}$-bilinear.]

N.B. An object X in a $\underline{k}$-enriched category $\underline{C}$ is irreducible if $\text{Mor}(X,X) = \underline{k}\text{id}_X$.

7.1 EXAMPLE Suppose that $\underline{C}$ is Z-enriched and monoidal. Put

$$\underline{k} = \underline{M}(\underline{C}).$$

Then $\underline{k}$ is a unital commutative ring (cf. 1.4) and $\underline{C}$ is $\underline{k}$-enriched as a monoidal category (cf. 1.5).

[Note: Suppose in addition that $\underline{C}$ is a ribbon category -- then $\forall$ X $\in$ Ob $\underline{C}$,

$$\text{tr}_X:\text{Mor}(X,X) \rightarrow \underline{k}$$

is $\underline{k}$-linear and $\forall$ X,Y $\in$ Ob $\underline{C}$, the map

$$\begin{cases} \text{Mor}(X,Y) \otimes_{\underline{k}} \text{Mor}(Y,X) \rightarrow \underline{k} \\ f \otimes g \rightarrow \text{tr}_X(g \circ f) \end{cases}$$

is $\underline{k}$-bilinear.]

A *$*$-category* is a pair $(\underline{C},*)$, where $\underline{C}$ is a category enriched over the field of complex numbers and

$$*:\underline{C} \to \underline{C}$$

is an involutive, identity on objects, positive cofunctor. Spelled out: $\forall\ X,Y \in \text{Ob}\ \underline{C}$, $\text{Mor}(X,Y)$ is a complex vector space, composition

$$\text{Mor}(X,Y) \times \text{Mor}(Y,Z) \to \text{Mor}(X,Z)$$

is complex bilinear,

$$*:\text{Mor}(X,Y) \to \text{Mor}(Y,X)$$

subject to

$$(zf + wg)^* = \bar{z}f^* + \bar{w}g^*$$

and

$$\begin{cases} f^{**} = f \\ (g \circ f)^* = f^* \circ g^*. \end{cases}$$

Finally, the requirement that $*$ be positive means:

$$f^* \circ f = 0 \Rightarrow f = 0.$$

[Note: $\forall\ X \in \text{Ob}\ \underline{C}$, we have

$$\begin{aligned} \text{id}_X^* &= \text{id}_X \circ \text{id}_X^* \\ &= \text{id}_X^{**} \circ \text{id}_X^* \\ &= (\text{id}_X \circ \text{id}_X^*)^* \\ &= \text{id}_X^{**} \\ &= \text{id}_X. \end{aligned}$$]

N.B. A monoidal $*$-category is a $*$-category which is monoidal with

$$(f \otimes g)^* = f^* \otimes g^*$$

for all f,g.

[Note: A symmetric monoidal $*$-category is a monoidal $*$-category such that $\forall$ X,Y $\in$ Ob $\underline{C}$,

$$\tau_{X,Y}: X \otimes Y \to Y \otimes X$$

is unitary (see below).]

7.2 EXAMPLE FDHILB is a symmetric monoidal $*$-category.

[Note: For the record, FDHILB is a construct. As such, it is amnestic and transportable, thus there is no loss of generality in assuming that its monoidal structure is strict (cf. 3.12).]

7.3 REMARK Let A be a complex $*$-algebra -- then the involution is positive if $A^* \circ A = 0 \Rightarrow A = 0$ $(A \in A)$. To illustrate, take $A = M_2(\mathbb{C})$ and consider the involutions

$$A^{*_1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{*_1} = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{bmatrix}$$

$$A^{*_2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{*_2} = \begin{bmatrix} \overline{a_{22}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{11}} \end{bmatrix}.$$

Then $*_1$ is positive but $*_2$ is not positive since

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{*_2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

[Note: It is wellknown that if A is finite dimensional and if the involution is positive, then A is a semisimple algebra, hence "is" a multimatrix algebra.]

Let $f:X \to Y$ be a morphism in a $*$-category $\underline{C}$ -- then f is an <u>isometry</u> if $f^* \circ f = id_X$ and f is <u>unitary</u> if both f and f^* are isometries.

Let F be a $\mathbb{C}$-linear functor between $*$-categories -- then F is <u>$*$-preserving</u> if $\forall$ f, $F(f^*) = (Ff)^*$.

<u>N.B.</u> Suppose that F is a $*$-preserving monoidal functor between monoidal $*$-categories -- then F is <u>unitary</u> if the isomorphisms $\xi:e' \to Fe$ and

$$\Xi_{X,Y}:FX \otimes' FY \to F(X \otimes Y)$$

are unitary.

Let $p:X \to X$ be a morphism in a $*$-category $\underline{C}$ -- then p is a <u>projection</u> if $p = p^*$ and $p \circ p = p$.

[Note: If $g:Y \to X$ is an isometry, then $g \circ g^*:X \to X$ is a projection.]

Let $\underline{C}$ be a $*$-category and let $X,Y \in \text{Ob } \underline{C}$ -- then X is a <u>subobject</u> of Y if $\exists$ an isometry $f \in \text{Mor}(X,Y)$.

Definition: <u>$\underline{C}$ has subobjects</u> if for any $Y \in \text{Ob } \underline{C}$ and any projection $q \in \text{Mor}(Y,Y)$, $\exists$ $X \in \text{Ob } \underline{C}$ and an isometry $f \in \text{Mor}(X,Y)$ such that $f \circ f^* = q$.

Definition: <u>$\underline{C}$ has direct sums</u> if for all $X,Y \in \text{Ob } \underline{C}$, $\exists$ $Z \in \text{Ob } \underline{C}$ and isometries $f \in \text{Mor}(X,Z)$, $g \in \text{Mor}(Y,Z)$ such that $f \circ f^* + g \circ g^* = id_Z$.

E.g.: FDHILB has subobjects and direct sums.

7.4 RAPPEL A category $\underline{C}$ is essentially small if $\underline{C}$ is equivalent to a small category.

Suppose that $\underline{C}$ is a $*$-category which is essentially small -- then $\underline{C}$ is semisimple if the following conditions are met:

SS_1: $\forall\ X,Y \in \text{Ob}\ \underline{C}$,

$$\dim \text{Mor}(X,Y) < \infty.$$

SS_2: $\underline{C}$ has subobjects and direct sums.

SS_3: $\underline{C}$ has a zero object.

N.B. A monoidal $*$-category is semisimple if it is semisimple as a $*$-category and if in addition, e is irreducible.

7.5 EXAMPLE FDHILB is a semisimple strict monoidal $*$-category (cf. 7.2).

7.6 LEMMA Suppose that $\underline{C}$ is a semisimple $*$-category -- then every nonzero object in $\underline{C}$ is a finite direct sum of irreducible objects.

[$\forall\ X \in \text{Ob}\ \underline{C}$, $\text{Mor}(X,X)$ is a finite dimensional complex $*$-algebra and the involution $*:\text{Mor}(X,X) \to \text{Mor}(X,X)$ is positive (cf. 7.3).]

[Note: Conventionally, zero objects are not irreducible.]

Therefore a semisimple $*$-category is abelian.

Given a semisimple $*$-category $\underline{C}$, denote its set of isomorphism classes of irreducible objects by $I_{\underline{C}}$ and let $\{X_i : i \in I_{\underline{C}}\}$ be a set of representatives -- then

$$i \neq j \Rightarrow \text{Mor}(X_i, X_j) = \{0\}$$

and $\forall$ X $\in$ Ob C, $\exists$ a finite number of i such that

$$\text{Mor}(X_i, X) \neq \{0\},$$

thereby defining $I_X \subset I_{\underline{C}}$.

7.7 REMARK $\forall$ i $\in$ I_X, $\text{Mor}(X_i, X)$ is a finite dimensional Hilbert space with inner product

$$\langle \phi, \psi \rangle \, \text{id}_{X_i} = \phi^* \circ \psi.$$

7.8 LEMMA Let $\underline{C}$, $\underline{C}'$ be semisimple $*$-categories and suppose that $F:\underline{C} \to \underline{C}'$ is C-linear -- then F is faithful if FX is nonzero for every irreducible X.

PROOF Consider an $f \in \text{Mor}(X,Y): Ff = 0$, the claim being that f = 0. Fix orthonormal bases

$$\begin{cases} s_{ik} \in \text{Mor}(X_i, X) & (k = 1, \ldots, \dim \text{Mor}(X_i, X)) \\ t_{j\ell} \in \text{Mor}(Y_j, Y) & (\ell = 1, \ldots, \dim \text{Mor}(Y_j, Y)) \end{cases}$$

such that

$$\begin{cases} \sum_{ik} s_{ik} \circ s_{ik}^* = \text{id}_X \\ \sum_{j\ell} t_{j\ell} \circ t_{j\ell}^* = \text{id}_Y. \end{cases}$$

Write

$$f = \text{id}_Y \circ f \circ \text{id}_X$$

$$= \sum_{ik,j\ell} t_{j\ell} \circ t_{j\ell}^* \circ f \circ s_{ik} \circ s_{ik}^*$$

$$= \sum_{ik\ell} c_{ik\ell}\, t_{i\ell} \circ s_{ik}^* \quad (\exists\, c_{ik\ell} \in \mathbf{C}).$$

Then for indices m,μ,ν,

$$0 = F(t_{m\nu}^*) \circ Ff \circ F(s_{m\mu})$$

$$= \sum_{ik\ell} c_{ik\ell}\, F(t_{m\nu}^* \circ t_{i\ell} \circ s_{ik}^* \circ s_{m\mu})$$

$$= \sum_{k\ell} c_{mk\ell}\, F(t_{m\nu}^* \circ t_{m\ell} \circ s_{mk}^* \circ s_{m\mu})$$

$$= \sum_{k\ell} c_{mk\ell}\, F(\langle t_{m\nu}, t_{m\ell}\rangle\, \mathrm{id}_{X_m} \circ \langle s_{mk}, s_{m\mu}\rangle\, \mathrm{id}_{X_m})$$

$$= c_{m\mu\nu}\, F(\mathrm{id}_{X_m})$$

$$= c_{m\mu\nu}\, \mathrm{id}_{FX_m}.$$

But by assumption, $\mathrm{id}_{FX_m} \neq 0$, thus the $c_{m\mu\nu}$ vanish, so $f = 0$.

7.9 LEMMA Let $\underline{C}$, $\underline{C}'$ be semisimple $*$-categories and suppose that $F:\underline{C} \to \underline{C}'$ is $\mathbf{C}$-linear and faithful -- then F is full iff (a) $X \in \mathrm{Ob}\,\underline{C}$ irreducible => $FX \in \mathrm{Ob}\,\underline{C}'$ irreducible and (b) $X,Y \in \mathrm{Ob}\,\underline{C}$ irreducible and nonisomorphic => $FX,FY \in \mathrm{Ob}\,\underline{C}'$ irreducible and nonisomorphic.

§8. *NATURAL TRANSFORMATIONS*

Let $\underline{C}$, $\underline{C}'$ be $*$-categories and let $F:\underline{C} \to \underline{C}'$ be a $*$-preserving functor.

8.1 LEMMA Nat(F,F) is a unital $*$-algebra under the following operations:

$$\begin{cases} (a\alpha + b\beta)_X = a\alpha_X + b\beta_X \\ (\alpha \circ \beta)_X = \alpha_X \circ \beta_X \\ (\alpha^*)_X = (\alpha_X)^* \\ 1_X = \mathrm{id}_{FX}. \end{cases}$$

[To check the $*$-condition, observe that $\forall\ f \in \mathrm{Mor}(X,Y)$,

$$\begin{aligned} Ff \circ (\alpha^*)_X &= Ff \circ (\alpha_X)^* \\ &= (Ff^*)^* \circ (\alpha_X)^* \\ &= (\alpha_X \circ Ff^*)^* \\ &= (Ff^* \circ \alpha_Y)^* \\ &= (\alpha_Y)^* \circ (Ff^*)^* \\ &= (\alpha^*)_Y \circ Ff.] \end{aligned}$$

8.2 EXAMPLE Take $\underline{C}' = \underline{\mathrm{FDHILB}}$, put $\mathrm{Nat}_F = \mathrm{Nat}(F,F)$, and let $\underline{\mathrm{Rep}}_{fd}\ \mathrm{Nat}_F$ be the category whose objects are the finite dimensional $*$-representations of

Nat_F and whose morphisms are the intertwining operators. Define a $*$-preserving functor

$$\Phi:\underline{C} \to \underline{\mathrm{Rep}}_{fd}\,\mathrm{Nat}_F$$

as follows:

$$\begin{cases} \Phi X = (\pi_X, FX) & (X \in \mathrm{Ob}\,\underline{C}) \\ \Phi f = Ff & (f \quad \mathrm{Mor}(X,Y)). \end{cases}$$

Here

$$\pi_X(\alpha) = \alpha_X,$$

thus the diagram

$$\begin{array}{ccc} FX & \xrightarrow{\pi_X(\alpha)} & FX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Ff} \\ FY & \xrightarrow[\pi_Y(\alpha)]{} & FY \end{array}$$

commutes, so Ff is an intertwining operator.

[Note: If

$$U:\underline{\mathrm{Rep}}_{fd}\,\mathrm{Nat}\,F \to \underline{\mathrm{FDHILB}}$$

is the forgetful functor, i.e., $U(\pi,H) = H$, then $U \circ \Phi = \mathbf{F}$.]

8.3 THEOREM Let $\underline{C}$, $\underline{C}'$ be $*$-categories and let $F:\underline{C} \to \underline{C}'$ be a $*$-preserving functor. Assume: $\underline{C}$ is semisimple -- then there is an isomorphism

$$\Psi_F:\mathrm{Nat}(F,F) \to \prod_{i \in I_{\underline{C}}} \mathrm{Mor}(FX_i, FX_i)$$

of unital $*$-algebras.

<u>PROOF</u> The definition of Ψ_F is the obvious one:

$$\Psi_F(\alpha) = \prod_{i \in I_{\underline{C}}} \alpha_{X_i}.$$

<u>Ψ_F is injective:</u>

$$\alpha_{X_i} = 0 \ \forall \ i \in I_{\underline{C}} \Rightarrow \alpha_X = 0 \ \forall \ X \in \text{Ob } \underline{C}.$$

To see this, choose the $s_{ik} \in \text{Mor}(X_i, X)$ as in the proof of 7.8 -- then

$$\begin{aligned} \alpha_X &= \alpha_X \circ F\text{id}_X \\ &= \sum_{ik} \alpha_X \circ F(s_{ik} \circ s_{ik}^*) \\ &= \sum_{ik} \alpha_X \circ F(s_{ik}) \circ F(s_{ik}^*). \end{aligned}$$

But the diagram

$$\begin{array}{ccc} FX_i & \xrightarrow{\alpha_{X_i}} & FX_i \\ {\scriptstyle F(s_{ik})}\downarrow & & \downarrow{\scriptstyle F(s_{ik})} \\ FX & \xrightarrow[\alpha_X]{} & FX \end{array}$$

commutes, hence

$$\begin{aligned} \alpha_X &= \sum_{ik} F(s_{ik}) \circ \alpha_{X_i} \circ F(s_{ik}^*) \\ &= 0. \end{aligned}$$

<u>Ψ_F is surjective:</u>

$$\forall \ \{\alpha_i \in \text{Mor}(FX_i, FX_i) : i \in I_{\underline{C}}\}, \ \exists \ \alpha \in \text{Nat}(F,F) : \Psi_F(\alpha) = \prod_{i \in I_{\underline{C}}} \alpha_i.$$

Thus define $\alpha_X \in \mathrm{Mor}(FX,FX)$ by

$$\alpha_X = \sum_{ik} F(s_{ik}) \circ \alpha_i \circ F(s^*_{ik})$$

and define $\alpha_Y \in \mathrm{Mor}(FY,FY)$ by

$$\alpha_Y = \sum_{j\ell} F(t_{j\ell}) \circ \alpha_j \circ F(t^*_{j\ell}).$$

Then $\forall\ f \in \mathrm{Mor}(X,Y)$,

$$\begin{aligned}
Ff \circ \alpha_X &= \sum_{ik} F(f \circ s_{ik}) \circ \alpha_i \circ F(s^*_{ik}) \\
&= \sum_{ik,j\ell} F(t_{j\ell} \circ t^*_{j\ell} \circ f \circ s_{ik}) \circ \alpha_i \circ F(s^*_{ik}) \\
&= \sum_{ik\ell} F(t_{i\ell} \circ (t^*_{i\ell} \circ f \circ s_{ik})) \circ \alpha_i \circ F(s^*_{ik}) \\
&= \sum_{ik\ell} F(t_{i\ell}) \circ F(t^*_{i\ell} \circ f \circ s_{ik}) \circ \alpha_i \circ F(s^*_{ik}) \\
&= \sum_{ik\ell} F(t_{i\ell}) \circ \alpha_i \circ F(t^*_{i\ell} \circ f \circ s_{ik}) \circ F(s^*_{ik}) \\
&= \sum_{ik,j\ell} F(t_{j\ell}) \circ \alpha_j \circ F(t^*_{j\ell} \circ f \circ s_{ik} \circ s^*_{ik}) \\
&= \sum_{j\ell} F(t_{j\ell}) \circ \alpha_j \circ F(t^*_{j\ell} \circ f) \\
&= \alpha_Y \circ Ff.
\end{aligned}$$

Accordingly, the diagram

$$\begin{array}{ccc}
FX & \xrightarrow{\ \alpha_X\ } & FX \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Ff} \\
FY & \xrightarrow[\ \alpha_Y\ ]{} & FY
\end{array}$$

commutes, meaning that $\alpha \in \mathrm{Nat}(F,F)$. And, by construction, $\alpha_{X_i} = \alpha_i$, so

$$\Psi_F(\alpha) = \prod_{i \in I_{\underline{C}}} \alpha_i.$$

[Note: The isomorphism Ψ_F depends on the choice of the X_i.]

8.4 <u>EXAMPLE</u> Take $\underline{C}' = \underline{C}$ and let $F = \mathrm{id}_{\underline{C}}$ (the identity functor) -- then

$$\mathrm{Nat}(\mathrm{id}_{\underline{C}}, \mathrm{id}_{\underline{C}}) \approx \prod_{i \in I_{\underline{C}}} \mathbf{C}.$$

8.5 <u>EXAMPLE</u> Suppose that $\underline{C}$ is a semisimple monoidal $*$-category -- then $\underline{C} \times \underline{C}$ is a semisimple $*$-category with

$$I_{\underline{C} \times \underline{C}} = I_{\underline{C}} \times I_{\underline{C}}.$$

And

$$X_i \otimes X_j \approx \bigoplus_{k \in I_{\underline{C}}} N_{ij}^k X_k,$$

where

$$N_{ij}^k = \dim \mathrm{Mor}(X_k, X_i \otimes X_j),$$

so

$$\mathrm{Mor}(X_i \otimes X_j, X_i \otimes X_j) \approx \bigoplus_{k \in I_{\underline{C}}} M_{N_{ij}^k}(\mathbf{C}).$$

This said, let

$$F = \otimes : \underline{C} \times \underline{C} \to \underline{C}.$$

Then

$$\mathrm{Nat}(\otimes, \otimes) \approx \prod_{i,j \in I_{\underline{C}}} \mathrm{Mor}(\otimes(X_i, X_j), \otimes(X_i, X_j))$$

$$\approx \prod_{i,j\in I_{\underline{C}}} \text{Mor}(X_i \otimes X_j, X_i \otimes X_j)$$

$$\approx \prod_{\substack{i,j,k\in I_{\underline{C}} \\ N_{ij}^k \neq 0}} M_{N_{ij}^k}(\mathbb{C}).$$

Suppose that $\underline{C}$ is a semisimple $*$-category, let $F:\underline{C} \to \underline{\text{FDHILB}}$ be $*$-preserving and put

$$A_F = \bigoplus_{i\in I_{\underline{C}}} B(FX_i)$$

which, of course, can be embedded in

$$\prod_{i\in I_{\underline{C}}} B(FX_i) \quad (\approx \text{Nat}(F,F)).$$

Needless to say, A_F is a $*$-algebra, unital iff $I_{\underline{C}}$ is finite. The projections $p_i:A_F \to B(FX_i)$ are finite dimensional irreducible $*$-representations. Moreover, any finite dimensional nondegenerate $*$-representation of A_F is a direct sum of finite dimensional irreducible $*$-representations and every finite dimensional irreducible $*$-representation is unitarily equivalent to a p_i.

Define now a $*$-preserving functor

$$\Phi:\underline{C} \to \underline{\text{Rep}}_{fd}\, A_F$$

as in 8.2 -- then Φ is an equivalence of categories iff F is faithful. In fact, since Φ and F agree on morphisms, it is clear that

$$\Phi \text{ faithful} \iff F \text{ faithful}.$$

Assume therefore that F is faithful. From the definitions, $\pi_{X_i} = p_i$ (or still, $\forall\ \alpha \in A$, $\alpha_{X_i} = p_i(\alpha)$), which is a finite dimensional irreducible $*$-representation of A_F. Given an irreducible $X \in \text{Ob}\ \underline{C}$, $\exists\ i \in I_{\underline{C}}$ and an isomorphism $\phi_i : X_i \to X$. Since the diagram

$$\begin{array}{ccc} FX_i & \xrightarrow{\ p_i(\alpha)\ } & FX_i \\ {\scriptstyle F\phi_i}\downarrow & & \downarrow{\scriptstyle F\phi_i} \\ FX & \xrightarrow[\ \pi_X(\alpha)\ ]{} & FX \end{array}$$

commutes, π_X is also a finite dimensional irreducible $*$-representation of A_F. If $i \neq j$, then

$$\text{Mor}(p_i, p_j) = \{0\},$$

so if $X, Y \in \text{Ob}\ \underline{C}$ are irreducible and nonzero, then

$$\text{Mor}(\pi_X, \pi_Y) = \{0\}.$$

Because Φ is faithful (and $\underline{\text{Rep}}_{\text{fd}}\ A_F$ is a semisimple $*$-category), the foregoing considerations imply that Φ is full (cf. 7.9). Finally, Φ has a representative image. Indeed, as mentioned above, every finite dimensional irreducible $*$-representation of A_F is unitarily equivalent to a p_i.

To recapitulate:

8.6 <u>THEOREM</u> Let $\underline{C}$ be a semisimple $*$-category and let $F : \underline{C} \to \underline{\text{FDHILB}}$ be a $*$-preserving functor. Put

$$A_F = \bigoplus_{i \in I_{\underline{C}}} B(FX_i)$$

and define

$$\Phi:\underline{C} \to \underline{\mathrm{Rep}}_{fd}\ A_F$$

by

$$\begin{cases} \Phi X = (\pi_X, FX) & (X \in \mathrm{Ob}\ \underline{C}) \\ \Phi f = Ff & (f \in \mathrm{Mor}(X,Y)). \end{cases}$$

Then Φ is an equivalence of categories iff F is faithful.

Let $\underline{C}$ be a semisimple strict monoidal $*$-category.

Definition: An embedding functor (for $\underline{C}$) is a faithful unitary functor

$$F:\underline{C} \to \underline{\mathrm{FDHILB}}.$$

[Note: Recall from §7 that in this context, "unitary" means that F is a $*$-preserving monoidal functor for which the isomorphisms $\xi:\underline{e} \to Fe$ and

$$\Xi_{X,Y}:FX \underline{\otimes} FY \to F(X \otimes Y)$$

are unitary ($\underline{e}$ = standard unit in $\underline{\mathrm{FDHILB}}$, $\underline{\otimes}$ = strict monoidal structure in $\underline{\mathrm{FDHILB}}$ (cf. 7.5)).]

8.7 LEMMA There is an isomorphism

$$\Psi_F:\mathrm{Nat}(F,F) \to \prod_{i \in I_{\underline{C}}} B(FX_i)$$

of unital $*$-algebras (cf. 8.3).

8.8 LEMMA The map

$$\epsilon_F:\mathrm{Nat}(F,F) \to \mathrm{Mor}(Fe,Fe) \approx \mathbb{C}$$

that sends

$$\alpha = \{\alpha_X\} \text{ to } \alpha_e$$

is a unital *-homomorphism.

8.9 <u>SCHOLIUM</u> The map

$$\bar{\varepsilon}_F : \prod_{i \in I_{\underline{C}}} B(FX_i) \to C$$

that sends

$$T \text{ to } \varepsilon_F \circ \Psi_F^{-1}(T)$$

is a unital *-homomorphism.

Let

$$\varepsilon = \bar{\varepsilon}_F | A_F.$$

Then ε is a unital *-homomorphism, the <u>counit</u>.

8.10 <u>LEMMA</u> There is an isomorphism

$$\Psi_{F \circ \otimes} : \text{Nat}(F \circ \otimes, F \circ \otimes) \longrightarrow \prod_{i,j \in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j)$$

of unital *-algebras.

<u>PROOF</u> In fact,

$$\text{Nat}(F \circ \otimes, F \circ \otimes)$$

$$\approx \prod_{i,j \in I_{\underline{C}}} B(F(X_i \otimes X_j)) \quad \text{(cf. 8.3)}$$

$$\approx \prod_{i,j \in I_{\underline{C}}} B(FX_i \otimes' FX_j)$$

$$\prod_{i,j\in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j).$$

8.11 LEMMA The map

$$\Delta_F : \mathrm{Nat}(F,F) \to \mathrm{Nat}(F \circ \otimes, F \circ \otimes)$$

that sends

$$\alpha = \{\alpha_X\} \text{ to } \{\alpha_{X \otimes Y}\}$$

is a unital $*$-homomorphism.

8.12 SCHOLIUM The map

$$\bar{\Delta}_F : \prod_{i\in I_{\underline{C}}} B(FX_i) \to \prod_{i,j\in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j)$$

that sends

$$T \text{ to } \Psi_{F \circ \otimes} \circ \Delta_F \circ \Psi_F^{-1}(T)$$

is a unital $*$-homomorphism.

Let

$$\Delta = \bar{\Delta}_F | A_F.$$

Then Δ is a unital $*$-homomorphism, the coproduct.

Let

$$\left[\begin{array}{l} \pi_1 : A_F \to B(H_1) \\ \\ \pi_2 : A_F \to B(H_2) \end{array}\right.$$

be nondegenerate $*$-representations of A_F on finite dimensional Hilbert spaces

$\left[\begin{array}{l} H_1 \\ \\ H_2 \end{array}\right.$ (the zero representation is a possibility) — then we can form

$$\pi_1 \otimes \pi_2 : A_F \otimes_C A_F \to B(H_1) \otimes_C B(H_2) \approx B(H_1 \underline{\otimes} H_2).$$

Since

$$A_F \otimes_C A_F = \bigoplus_{i,j \in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j),$$

it follows that $\pi_1 \otimes \pi_2$ admits a unique extension to a unital *-homomorphism

$$\overline{\pi_1 \otimes \pi_2} : \prod_{i,j \in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j) \to B(H_1 \underline{\otimes} H_2).$$

This being so, put

$$\pi_1 \times \pi_2 = \overline{\pi_1 \otimes \pi_2} \circ \Delta.$$

Then $\pi_1 \times \pi_2$ is a nondegenerate *-representation of A_F on the finite dimensional Hilbert space $H_1 \underline{\otimes} H_2$.

8.13 LEMMA The data $(\times, \varepsilon, \ldots)$ is a monoidal structure on $\underline{\text{Rep}}_{\text{fd}}\, A_F$.

Therefore $\underline{\text{Rep}}_{\text{fd}}\, A_F$ is a semisimple monoidal *-category (the counit ε is the irreducible unit).

8.14 THEOREM Let $\underline{C}$ be a semisimple strict monoidal *-category and let

$$F : \underline{C} \to \underline{\textbf{FDHILB}}$$

be an embedding functor. Put

$$A_F = \bigoplus_{i \in I_{\underline{C}}} B(FX_i)$$

and define

$$\Phi : \underline{C} \to \underline{\mathrm{Rep}}_{fd}\, A_F$$

by

$$\begin{cases} \Phi X = (\pi_X, FX) & (X \in \mathrm{Ob}\ \underline{C}) \\ \Phi f = Ff & (f \in \mathrm{Mor}(X,Y)). \end{cases}$$

Then Φ is a monoidal equivalence.

PROOF By hypothesis, F is faithful, hence Φ is an equivalence of categories (cf. 8.6). So, in view of 2.8, it suffices to show that Φ is monoidal. There are two points. First

$$\Phi e = (\pi_e, Fe)$$

and $\forall\ \alpha \in A_F$, the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \varepsilon(\alpha)\ } & C \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ Fe & \xrightarrow[\ \pi_e(\alpha)\ ]{} & Fe \end{array}$$

commutes, i.e., ξ intertwines ε and π_e. Next, given $X, Y \in \mathrm{Ob}\ \underline{C}$, consider

$$\begin{cases} \Phi X \times \Phi Y = (\pi_X \times \pi_Y,\ FX \underline{\otimes} FY) \\ \Phi(X \otimes Y) = (\pi_{X \otimes Y},\ F(X \otimes Y)). \end{cases}$$

Then

$$\Xi_{X,Y} : FX \underline{\otimes} FY \to F(X \otimes Y)$$

is an intertwining operator: $\forall\ \alpha \in A_F$,

$$\Xi_{X,Y} \circ (\pi_X \times \pi_Y)(\alpha) = \pi_{X \otimes Y}(\alpha) \circ \Xi_{X,Y}.$$

The interchange $\sigma: A_F \otimes_C A_F \to A_F \otimes_C A_F$ $(\sigma(\alpha \otimes \beta) = \beta \otimes \alpha)$ is a nondegenerate $*$-homomorphism, thus has a unique extension to an involutive $*$-automorphism

$$\bar{\sigma}: \prod_{i,j \in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j) \to \prod_{i,j \in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j).$$

Let

$$\Delta^{OP} = \bar{\sigma} \circ \Delta.$$

Then A_F is said to be <u>cocommutative</u> if $\Delta = \Delta^{OP}$.

8.15 <u>LEMMA</u> Suppose that A_F is cocommutative -- then

$$\forall \left[\begin{array}{l} (\pi_1, H_1) \\ \\ (\pi_2, H_2) \end{array}\right. \in \text{Ob}\ \underline{\text{Rep}}_{fd}\ A_F,$$

the diagram

$$\begin{array}{ccc} H_1 \underline{\otimes} H_2 & \xrightarrow{\ \tau_{H_1,H_2}\ } & H_2 \underline{\otimes} H_1 \\ {\scriptstyle \pi_1 \times \pi_2} \big\downarrow & & \big\downarrow {\scriptstyle \pi_2 \times \pi_1} \\ H_1 \underline{\otimes} H_2 & \xrightarrow[\ \tau_{H_1,H_2}\ ]{} & H_2 \underline{\otimes} H_1 \end{array}$$

commutes.

PROOF Abbreviate τ_{H_1,H_2} to $\tau_{1,2}$ and note that

$$\forall\ T \in \prod_{i,j \in I_{\underline{C}}} B(FX_i) \otimes_C B(FX_j),$$

we have

$$\overline{(\pi_1 \otimes \pi_2)}\bar{\sigma}(T) = \tau_{2,1}\ \overline{(\pi_2 \otimes \pi_1)}(T)\tau_{1,2}.$$

So, $\forall\ \alpha \in A_F$,

$$\begin{aligned} &\tau_{1,2}(\pi_1 \times \pi_2)(\alpha) \\ &\quad = \tau_{1,2}\overline{(\pi_1 \otimes \pi_2)}(\Delta(\alpha)) \\ &\quad = \tau_{1,2}\overline{(\pi_1 \otimes \pi_2)}(\Delta^{OP}(\alpha)) \\ &\quad = \tau_{1,2}\overline{(\pi_1 \otimes \pi_2)}(\bar{\sigma}(\Delta(\alpha))) \\ &\quad = \tau_{1,2}\tau_{2,1}\overline{(\pi_2 \otimes \pi_1)}(\Delta(\alpha))\tau_{1,2} \\ &\quad = \overline{(\pi_2 \otimes \pi_1)}(\Delta(\alpha))\tau_{1,2} \\ &\quad = (\pi_2 \times \pi_1)(\alpha)\tau_{1,2}. \end{aligned}$$

Thus, if A_F is cocommutative, then $\underline{\text{Rep}}_{fd}\ A_F$ is a semisimple symmetric monoidal *-category.

8.16 REMARK Assume further that the category $\underline{C}$ is symmetric and that the embedding functor

$$F:\underline{C} \to \underline{\text{FDHILB}}$$

is symmetric monoidal -- then A_F is cocommutative and $\Phi:\underline{C} \to \underline{\mathrm{Rep}}_{fd}\, A_F$ is a symmetric monoidal equivalence.

§9. *CONJUGATES*

Suppose that $\underline{C}$ is a strict monoidal *-category which is left autonomous. Put $X^\vee = {}^\vee X$ -- then

$$\begin{cases} \varepsilon_X : {}^\vee X \otimes X \to e \\ \eta_X : e \to X \otimes {}^\vee X \end{cases}$$

=>

$$\begin{cases} \varepsilon_X^* : e \to {}^\vee X \otimes X \\ \eta_X^* : X \otimes {}^\vee X \to e \end{cases}$$

=>

$$\begin{cases} \eta_X^* : X \otimes X^\vee \to e \\ \varepsilon_X^* : e \to X^\vee \otimes X. \end{cases}$$

And

$$\begin{cases} (\mathrm{id}_X \otimes \varepsilon_X) \circ (\eta_X \otimes \mathrm{id}_X) = \mathrm{id}_X \\ (\varepsilon_X \otimes \mathrm{id}_{{}^\vee X}) \circ (\mathrm{id}_{{}^\vee X} \otimes \eta_X) = \mathrm{id}_{{}^\vee X} \end{cases}$$

=>

$$\begin{cases} (\eta_X^* \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \varepsilon_X^*) = \mathrm{id}_X \\ (\mathrm{id}_{X^\vee} \otimes \eta_X^*) \circ (\varepsilon_X^* \otimes \mathrm{id}_{X^\vee}) = \mathrm{id}_{X^\vee}. \end{cases}$$

I.e.: The left duality $({}^\vee X, \varepsilon_X, \eta_X)$ automatically leads to a right duality $(X^\vee, \eta_X^*, \varepsilon_X^*)$.

Now assume in addition that $\underline{C}$ is symmetric (hence that the $\tau_{X,Y}$ are unitary) -- then the left duality $({}^{\vee}X,\varepsilon_X,\eta_X)$ gives rise to another right duality, viz.

$$(X^{\vee},\varepsilon_X \circ \tau_{X,{}^{\vee}X}, \tau_{X,{}^{\vee}X} \circ \eta_X).$$

9.1 <u>COHERENCY HYPOTHESIS</u> $\forall\ X \in \text{Ob}\ \underline{C}$,

$$\varepsilon_X^* = \tau_{X,{}^{\vee}X} \circ \eta_X.$$

[Note: The asymmetry is only apparent. For

$$\eta_X = \tau_{X,{}^{\vee}X}^{-1} \circ \varepsilon_X^*$$

$$= \tau_{{}^{\vee}X,X} \circ \varepsilon_X^*$$

=>

$$\eta_X^* = \varepsilon_X \circ \tau_{{}^{\vee}X,X}^*$$

$$= \varepsilon_X \circ \tau_{{}^{\vee}X,X}^{-1}$$

$$= \varepsilon_X \circ \tau_{X,{}^{\vee}X}.]$$

In the presence of 9.1, let

$$\begin{bmatrix} \bar{X} = {}^{\vee}X\ (= X^{\vee}) \\ r_X = \varepsilon_X^* \\ \bar{r}_X = \tau_{\bar{X},X} \circ r_X, \end{bmatrix}$$

thus

$$\begin{cases} r_X : e \to \bar{X} \otimes X \\ \bar{r}_X : e \to X \otimes \bar{X} \end{cases}$$

and

$$\begin{cases} (\bar{X}, r_X^*, \bar{r}_X) \text{ is a left duality} \\ (\bar{X}, \bar{r}_X^*, r_X) \text{ is a right duality.} \end{cases}$$

Therefore

$$\begin{cases} \begin{cases} (id_X \otimes r_X^*) \circ (\bar{r}_X \otimes id_X) = id_X \\ (r_X^* \otimes id_{\bar{X}}) \circ (id_{\bar{X}} \otimes \bar{r}_X) = id_{\bar{X}} \end{cases} \\ \begin{cases} (\bar{r}_X^* \otimes id_X) \circ (id_X \otimes r_X) = id_X \\ (id_{\bar{X}} \otimes \bar{r}_X^*) \circ (r_X \otimes id_{\bar{X}}) = id_{\bar{X}}. \end{cases} \end{cases}$$

The relations

$$\begin{cases} (id_X \otimes r_X^*) \circ (\bar{r}_X \otimes id_X) = id_X \\ (id_{\bar{X}} \otimes \bar{r}_X^*) \circ (r_X \otimes id_{\bar{X}}) = id_{\bar{X}} \end{cases}$$

are called the <u>conjugate equations</u>, the triple $(\bar{X}, r_X, \bar{r}_X)$ being a <u>conjugate</u> for X.

<u>N.B.</u> The conjugate equations imply that

$$\begin{cases} (\bar{r}_X^* \otimes id_X) \circ (id_X \otimes r_X) = id_X \\ (r_X^* \otimes id_{\bar{X}}) \circ (id_{\bar{X}} \otimes \bar{r}_X) = id_{\bar{X}}. \end{cases}$$

Having made these points, matters can be turned around. So start with a symmetric strict monoidal *-category $\underline{C}$ -- then $\underline{C}$ has conjugates if one can assign to each $X \in \text{Ob } \underline{C}$ an object $\bar{X}$ and a morphism

$$r_X : e \to \bar{X} \otimes X$$

such that the triple $(\bar{X}, r_X, \bar{r}_X)$ satisfies the conjugate equations (here, of course, $\bar{r}_X = \tau_{\bar{X},X} \circ r_X$).

E.g.: $\underline{\text{FDHILB}}$ has conjugates.

9.2 REMARK If $\underline{C}$ has conjugates, then $\underline{C}$ is left autonomous (consider $(\bar{X}, r_X^*, \bar{r}_X)$) and right autonomous (consider $(\bar{X}, \bar{r}_X^*, r_X)$). Moreover, the coherency hypothesis is in force: $(r_X^*)^* = r_X$, while

$$\tau_{X,\bar{X}} \circ \bar{r}_X = \tau_{X,\bar{X}} \circ \tau_{\bar{X},X} \circ r_X = r_X.$$

9.3 LEMMA Suppose that $\underline{C}$ has conjugates.

- Under the identification

$$\text{Mor}(X \otimes Y, Z) \approx \text{Mor}(Y, \bar{X} \otimes Z),$$

the arrows

$$\begin{cases} f \to \text{id}_{\bar{X}} \otimes f \circ r_X \otimes \text{id}_Y \\ \\ g \to \bar{r}_X^* \otimes \text{id}_Z \circ \text{id}_X \otimes g \end{cases}$$

are mutually inverse.

- Under the identification

$$\text{Mor}(Y \otimes X, Z) \approx \text{Mor}(Y, Z \otimes \bar{X}),$$

the arrows

$$\begin{cases} f \to f \otimes \text{id}_{\bar{X}} \circ \text{id}_Y \otimes \bar{r}_X \\ g \to \text{id}_Z \otimes r_X^* \circ g \otimes \text{id}_X \end{cases}$$

are mutually inverse.

E.g.: $\forall\ X \in \text{Ob}\ \underline{C}$,

$$\text{Mor}(X,X) \approx \text{Mor}(e,\bar{X} \otimes X).$$

9.4 <u>LEMMA</u> If

$$\begin{cases} (\bar{X}, r_X, \bar{r}_X) \\ (\bar{X}', r_X', \bar{r}_X') \end{cases}$$

are conjugates for X, then

$$r_X^* \otimes \text{id}_{\bar{X}'} \circ \text{id}_{\bar{X}} \otimes \bar{r}_X' \in \text{Mor}(\bar{X},\bar{X}')$$

is unitary.

<u>PROOF</u> Put

$$U = r_X^* \otimes \text{id}_{\bar{X}'} \circ \text{id}_{\bar{X}} \otimes \bar{r}_X'$$

$$(= \text{id}_{\bar{X}'} \otimes \bar{r}_X^* \circ r_X' \otimes \text{id}_{\bar{X}} \ldots).$$

Then the claim is that

$$\begin{cases} U \circ U^* = \text{id}_{\bar{X}'} \\ U^* \circ U = \text{id}_{\bar{X}}. \end{cases}$$

And for this, it will be enough to consider $U \circ U^*$. So write

$$U \circ U^* = r_X^* \otimes id_{\bar{X}'} \circ id_{\bar{X}} \otimes \bar{r}_X' \circ U^*$$

$$= r_X^* \otimes id_{\bar{X}'} \circ (id_{\bar{X}} \otimes \bar{r}_X' \circ U^* \otimes id_e)$$

$$= r_X^* \otimes id_{\bar{X}'} \circ U^* \otimes \bar{r}_X'$$

$$= r_X^* \otimes id_{\bar{X}'} \circ (U^* \otimes id_{X \otimes \bar{X}'} \circ id_{\bar{X}'} \otimes \bar{r}_X')$$

$$= r_X^* \otimes id_{\bar{X}'}$$

$$\circ (r_X'^* \otimes id_{\bar{X}} \circ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'})$$

$$\circ id_{\bar{X}'} \otimes \bar{r}_X'$$

$$= r_X^* \otimes id_{\bar{X}'}$$

$$\circ (r_X'^* \otimes id_{\bar{X}} \circ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'} \circ id_{X \otimes \bar{X}'})$$

$$\circ id_{\bar{X}'} \otimes \bar{r}_X'$$

$$= r_X^* \otimes id_{\bar{X}'}$$

$$\circ (r_X'^* \otimes id_{\bar{X}} \otimes id_{X \otimes \bar{X}'} \circ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'})$$

$$\circ id_{\bar{X}'} \otimes \bar{r}_X'$$

$$= r_X^* \otimes id_{\bar{X}'} \circ r_X'^* \otimes id_{\bar{X}} \otimes id_{X \otimes \bar{X}'}$$

$$\circ\ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'} \circ id_{\bar{X}'} \otimes \bar{r}'_X$$

$$= id_e \otimes (r^*_X \otimes id_{\bar{X}'}) \circ r'^*_X \otimes id_{\bar{X} \otimes X \otimes \bar{X}'}$$

$$\circ\ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'} \circ id_{\bar{X}'} \otimes \bar{r}'_X$$

$$= r'^*_X \otimes r^*_X \otimes id_{\bar{X}'}$$

$$\circ\ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'} \circ id_{\bar{X}'} \otimes \bar{r}'_X$$

$$= r'^*_X \otimes id_{\bar{X}'} \circ id_{\bar{X}' \otimes X} \otimes r^*_X \otimes id_{\bar{X}'}$$

$$\circ\ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'} \circ id_{\bar{X}'} \otimes \bar{r}'_X$$

$$= r'^*_X \otimes id_{\bar{X}'}$$

$$\circ\ (id_{\bar{X}' \otimes X} \otimes r^*_X \otimes id_{\bar{X}'} \circ id_{\bar{X}'} \otimes \bar{r}_X \otimes id_{X \otimes \bar{X}'})$$

$$\circ\ id_{\bar{X}'} \otimes \bar{r}'_X$$

$$= r'^*_X \otimes id_{\bar{X}'}$$

$$\circ\ (id_{\bar{X}'} \otimes (id_X \otimes r^*_X) \otimes id_{\bar{X}'} \circ id_{\bar{X}'} \otimes (\bar{r}_X \otimes id_X) \otimes id_{\bar{X}'})$$

$$\circ\ id_{\bar{X}'} \otimes \bar{r}'_X$$

$$= r'^*_X \otimes id_{\bar{X}'}$$

$$\circ\ (\mathrm{id}_{\bar{X}'} \circ \mathrm{id}_{\bar{X}'} \otimes ((\mathrm{id}_X \otimes r_X^*) \otimes \mathrm{id}_{\bar{X}'} \circ (\bar{r}_X \otimes \mathrm{id}_X) \otimes \mathrm{id}_{\bar{X}'})$$

$$\circ\ \mathrm{id}_{\bar{X}'} \otimes \bar{r}_X'$$

$$= r_X'^* \otimes \mathrm{id}_{\bar{X}'}$$

$$\circ\ (\mathrm{id}_{\bar{X}'} \otimes ((\mathrm{id}_X \otimes r_X^*) \circ (\bar{r}_X \otimes \mathrm{id}_X)) \otimes \mathrm{id}_{\bar{X}'} \circ \mathrm{id}_{\bar{X}'}$$

$$\circ\ \mathrm{id}_{\bar{X}'} \otimes \bar{r}_X'$$

$$= r_X'^* \otimes \mathrm{id}_{\bar{X}'} \circ (\mathrm{id}_{\bar{X}'} \otimes \mathrm{id}_X \otimes \mathrm{id}_{\bar{X}'}) \circ \mathrm{id}_{\bar{X}'} \otimes \bar{r}_X'$$

$$= r_X'^* \otimes \mathrm{id}_{\bar{X}'} \circ \mathrm{id}_{\bar{X}' \otimes X \otimes \bar{X}'} \circ \mathrm{id}_{\bar{X}'} \otimes \bar{r}_X'$$

$$= r_X'^* \otimes \mathrm{id}_{\bar{X}'} \circ \mathrm{id}_{\bar{X}'} \otimes \bar{r}_X'$$

$$= \mathrm{id}_{\bar{X}'}.$$

[Note: Evidently,

$$\begin{bmatrix} r_X' = (U \otimes \mathrm{id}_X) \circ r_X \\ \bar{r}_X' = (\mathrm{id}_X \otimes U) \circ \bar{r}_X. \end{bmatrix}$$

Conjugates are therefore determined up to "unitary equivalence".

Put

$$\Omega_X = r_X^* \otimes \mathrm{id}_X \circ \mathrm{id}_{\bar{X}} \otimes \tau_{X,X} \circ r_X \otimes \mathrm{id}_X.$$

Then

$$\Omega_X \in \mathrm{Mor}(X,X)$$

is unitary and it can be verified by computation that the assignment $X \to \Omega_X$ defines a twist Ω. This fact, however, is a trivial consequence of the following result.

9.5 <u>LEMMA</u> $\forall\, X \in \mathrm{Ob}\,\underline{C}$,

$$\Omega_X = \mathrm{id}_X.$$

<u>PROOF</u> We have

$$(\bar{r}_X^* \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes r_X) = \mathrm{id}_X.$$

On the other hand, there is a commutative diagram

$$\begin{array}{ccc}
X = e \otimes X & \xrightarrow{\ \tau_{e,X}\ } & X \otimes e = X \\
{\scriptstyle r_X \otimes \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \otimes r_X} \\
\bar{X} \otimes X \otimes X & \xrightarrow[\ \tau_{\bar{X} \otimes X, X}\ ]{} & X \otimes \bar{X} \otimes X,
\end{array}$$

so

$$(\bar{r}_X^* \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes r_X)$$

$$= \bar{r}_X^* \otimes \mathrm{id}_X \circ \tau_{\bar{X} \otimes X, X} \circ r_X \otimes \mathrm{id}_X$$

$$= \bar{r}_X^* \otimes \mathrm{id}_X \circ \tau_{\bar{X},X} \otimes \mathrm{id}_X \circ \mathrm{id}_{\bar{X}} \otimes \tau_{X,X} \circ r_X \otimes \mathrm{id}_X.$$

And

$$\bar{r}_X^* \otimes \mathrm{id}_X \circ \tau_{\bar{X},X} \otimes \mathrm{id}_X$$

$$= (\tau_{\bar{X},X} \circ r_X)^* \otimes \mathrm{id}_X \circ \tau_{\bar{X},X} \otimes \mathrm{id}_X$$

$$= r_X^* \circ \tau_{X,\bar{X}} \otimes \mathrm{id}_X \circ \tau_{\bar{X},X} \otimes \mathrm{id}_X$$

$$= r_X^* \circ \tau_{X,\bar{X}} \circ \tau_{\bar{X},X} \otimes \mathrm{id}_X \circ \mathrm{id}_X$$

$$= r_X^* \otimes \mathrm{id}_X.$$

[Note: Therefore, in the terminology of §6, C is an even ribbon category.]

9.6 <u>REMARK</u> $\forall\ f \in \mathrm{Mor}(X,X)$, the diagram

$$\begin{array}{ccc}
\bar{X} \otimes X & \xrightarrow{\ \tau_{\bar{X},X}\ } & X \otimes \bar{X} \\
{\scriptstyle \mathrm{id}_{\bar{X}} \otimes f} \downarrow & & \downarrow {\scriptstyle f \otimes \mathrm{id}_{\bar{X}}} \\
\bar{X} \otimes X & \xrightarrow[\ \tau_{\bar{X},X}\ ]{} & X \otimes \bar{X}
\end{array}$$

commutes. Therefore

$$\bar{r}_X^* \circ f \otimes \mathrm{id}_{\bar{X}} \circ \bar{r}_X$$

$$= (\tau_{\bar{X},X} \circ r_X)^* \circ f \otimes \mathrm{id}_{\bar{X}} \circ (\tau_{\bar{X},X} \circ r_X)$$

$$= r_X^* \circ \tau_{X,\bar{X}} \circ f \otimes \mathrm{id}_{\bar{X}} \circ \tau_{\bar{X},X} \circ r_X$$

$$= r_X^* \circ \mathrm{id}_{\bar{X}} \otimes f \circ r_X.$$

Maintaining the supposition that $\underline{C}$ has conjugates, recall that $\underline{C}$ is left autonomous with left duality $(\bar{X}, r_X^*, \bar{r}_X)$ (cf. 9.2), thus by definition the categorical dimension of X is the arrow

$$e \xrightarrow{\bar{r}_X} X \otimes \bar{X} \xrightarrow{\tau_{X,\bar{X}}} \bar{X} \otimes X \xrightarrow{r_X^*} e \quad \text{(cf. §6).}$$

But

$$\bar{r}_X = \tau_{\bar{X},X} \circ r_X,$$

so the categorical dimension of X is the composition

$$r_X^* \circ \tau_{X,\bar{X}} \circ \tau_{\bar{X},X} \circ r_X$$

$$= r_X^* \circ r_X \in \mathrm{Mor}(e,e)$$

$$\equiv \dim X.$$

[Note: Since $\Omega = \mathrm{id}$, $\forall\, f \in \mathrm{Mor}(X,X)$,

$$\mathrm{tr}_X(f) = r_X^* \circ \tau_{X,\bar{X}} \circ \Omega_X \otimes \mathrm{id}_{\bar{X}} \circ (f \otimes \mathrm{id}_{\bar{X}}) \circ \bar{r}_X$$

$$= r_X^* \circ \tau_{X,\bar{X}} \circ \mathrm{id}_{X \otimes \bar{X}} \circ (f \otimes \mathrm{id}_{\bar{X}}) \circ \tau_{\bar{X},X} \circ r_X$$

$$= r_X^* \circ \tau_{X,\bar{X}} \circ (f \otimes \mathrm{id}_{\bar{X}}) \circ \tau_{\bar{X},X} \circ r_X$$

$$= r_X^* \circ \mathrm{id}_{\bar{X}} \otimes f \circ r_X \quad \text{(cf. 9.6).]}$$

<u>N.B.</u> dim X does not depend on the choice of a conjugate for X. Indeed, if $U:\bar{X} \to \bar{X}'$ is unitary, then

$$((U \otimes \mathrm{id}_X) \circ r_X)^* \circ ((U \otimes \mathrm{id}_X) \circ r_X)$$

$$= r_X^* \circ U^* \otimes \mathrm{id}_X \circ U \otimes \mathrm{id}_X \circ r_X$$

$$= r_X^* \circ \mathrm{id}_{\bar{X}} \otimes \mathrm{id}_X \circ r_X$$

$$= r_X^* \circ \mathrm{id}_{\bar{X} \otimes X} \circ r_X$$

$$= r_X^* \circ r_X.$$

9.7 <u>LEMMA</u> If

$$\begin{cases} (\bar{X}, r_X, \bar{r}_X) \text{ is a conjugate for } X \\ (\bar{Y}, r_Y, \bar{r}_Y) \text{ is a conjugate for } Y, \end{cases}$$

then

$$(\bar{Y} \otimes \bar{X}, r_{X \otimes Y}, \bar{r}_{X \otimes Y})$$

is a conjugate for $X \otimes Y$, where

$$\begin{cases} r_{X \otimes Y} = \mathrm{id}_{\bar{Y}} \otimes r_X \otimes \mathrm{id}_Y \circ r_Y \\ \bar{r}_{X \otimes Y} = \mathrm{id}_X \otimes \bar{r}_Y \otimes \mathrm{id}_{\bar{X}} \circ \bar{r}_X. \end{cases}$$

[The proof that

$$\begin{cases} (\mathrm{id}_{X \otimes Y} \otimes r_{X \otimes Y}^*) \circ (\bar{r}_{X \otimes Y} \otimes \mathrm{id}_{X \otimes Y}) = \mathrm{id}_{X \otimes Y} \\ (\mathrm{id}_{\bar{Y} \otimes \bar{X}} \otimes \bar{r}_{X \otimes Y}^*) \circ (r_{X \otimes Y} \otimes \mathrm{id}_{\bar{Y} \otimes \bar{X}}) = \mathrm{id}_{\bar{Y} \otimes \bar{X}} \end{cases}$$

will be left to the reader but we shall provide the verification that

$$\bar{r}_{X \otimes Y} = \tau_{\bar{Y} \otimes \bar{X}, X \otimes Y} \circ r_{X \otimes Y}.$$

Thus write

$$\tau_{\bar{Y} \otimes \bar{X}, X \otimes Y} \circ r_{X \otimes Y}$$

$$= \tau_{\bar{X}, X \otimes Y \otimes \bar{Y}} \circ \tau_{\bar{Y}, \bar{X} \otimes X \otimes Y} \circ r_{X \otimes Y} \quad \text{(cf. 4.3)}.$$

Then

$$\tau_{\bar{Y}, \bar{X} \otimes X \otimes Y} \circ r_{X \otimes Y}$$

$$= id_{\bar{X} \otimes X} \otimes \tau_{\bar{Y}, Y} \circ \tau_{\bar{Y}, \bar{X} \otimes X} \circ r_{X \otimes Y}$$

$$= id_{\bar{X} \otimes X} \otimes \tau_{\bar{Y}, Y} \circ \tau_{\bar{Y}, \bar{X} \otimes X} \circ id_{\bar{Y}} \otimes r_X \otimes id_Y \circ r_Y$$

$$= id_{\bar{X} \otimes X} \otimes \tau_{\bar{Y}, Y} \circ (r_X \otimes id_{\bar{Y}} \circ \tau_{\bar{Y}, e}) \otimes id_Y \circ r_Y$$

$$= id_{\bar{X} \otimes X} \otimes \tau_{\bar{Y}, Y} \circ r_X \otimes id_{\bar{Y} \otimes Y} \circ r_Y$$

$$= id_{\bar{X} \otimes X} \otimes \tau_{\bar{Y}, Y} \circ r_X \otimes id_{\bar{Y} \otimes Y} \circ id_e \otimes r_Y$$

$$= id_{\bar{X} \otimes X} \otimes \tau_{\bar{Y}, Y} \circ id_{\bar{X} \otimes X} \otimes r_Y \circ r_X$$

$$= id_{\bar{X} \otimes X} \otimes \tau_{\bar{Y}, Y} \circ r_Y \circ r_X = id_{\bar{X} \otimes X} \otimes \bar{r}_Y \circ r_X.$$

Therefore

$$\tau_{\bar{Y} \otimes \bar{X}, X \otimes Y} \circ r_{X \otimes Y}$$

$$= \tau_{\bar{X}, X \otimes Y \otimes \bar{Y}} \circ id_{\bar{X} \otimes X} \otimes \bar{r}_Y \circ r_X$$

$$= \tau_{\bar{X}, X \otimes Y \otimes \bar{Y}} \circ id_{\bar{X}} \otimes id_X \otimes \bar{r}_Y \circ r_X$$

$$= id_X \otimes \bar{r}_Y \otimes id_{\bar{X}} \circ \tau_{\bar{X}, X} \circ r_X$$

$$= \mathrm{id}_X \otimes \bar{r}_Y \otimes \mathrm{id}_{\bar{X}} \circ \bar{r}_X = \bar{r}_{X \otimes Y}.]$$

For all $X,Y \in \mathrm{Ob}\ \underline{C}$, the map

$$\mathrm{Mor}(X,Y) \to \mathrm{Mor}(\bar{Y},\bar{X})$$

that sends f to ${}^{\vee}f$ is a linear bijection.

<u>N.B.</u> Here, as will be recalled from §5,

$${}^{\vee}f = r_Y^* \otimes \mathrm{id}_{\bar{X}} \circ \mathrm{id}_{\bar{Y}} \otimes f \otimes \mathrm{id}_{\bar{X}} \circ \mathrm{id}_{\bar{Y}} \otimes \bar{r}_X.$$

Now put

$$f^+ = ({}^{\vee}f)^*,$$

thus

$$f^+ = \mathrm{id}_{\bar{Y}} \otimes \bar{r}_X^* \circ \mathrm{id}_{\bar{Y}} \otimes f^* \otimes \mathrm{id}_{\bar{X}} \circ r_Y \otimes \mathrm{id}_{\bar{X}}$$

and

$$f^+ \in \mathrm{Mor}(\bar{X},\bar{Y}).$$

Properties:

1. $\mathrm{id}_X^+ = \mathrm{id}_{\bar{X}}$;

2. $(f^+)^* = (f^*)^+$;

3. $(f \circ g)^+ = f^+ \circ g^+$.

9.8 <u>LEMMA</u> Given $f \in \mathrm{Mor}(X,Y)$, we have

$$f^+ \otimes \mathrm{id}_X \circ r_X = \mathrm{id}_{\bar{Y}} \otimes f^* \circ r_Y.$$

<u>PROOF</u> Start with the LHS and write

$$f^{+} \otimes id_X \circ r_X$$

$$= (id_{\bar{Y}} \otimes \bar{r}^{*}_{X} \circ id_{\bar{Y}} \otimes f^{*} \otimes id_{\bar{X}} \circ r_Y \otimes id_{\bar{X}}) \otimes id_X \circ r_X$$

$$= (id_{\bar{Y}} \otimes \bar{r}^{*}_{X} \circ id_{\bar{Y}} \otimes f^{*} \otimes id_{\bar{X}} \circ r_Y \otimes id_{\bar{X}}) \otimes (id_X \circ id_X \circ id_X) \circ r_X$$

$$= id_{\bar{Y}} \otimes \bar{r}^{*}_{X} \otimes id_X \circ id_{\bar{Y}} \otimes f^{*} \otimes id_{\bar{X}} \otimes id_X \circ r_Y \otimes id_{\bar{X}} \otimes id_X \circ r_X.$$

But

$$r_Y \otimes id_{\bar{X}} \otimes id_X \circ r_X$$

$$= r_Y \otimes id_{\bar{X} \otimes X} \circ id_e \otimes r_X$$

$$= id_{\bar{Y} \otimes Y} \otimes r_X \circ r_Y \otimes id_e$$

$$= id_{\bar{Y}} \otimes id_Y \otimes r_X \circ r_Y$$

=>

$$id_{\bar{Y}} \otimes f^{*} \otimes id_{\bar{X}} \otimes id_X \circ id_{\bar{Y}} \otimes id_Y \otimes r_X \circ r_Y$$

$$= id_{\bar{Y}} \otimes id_X \otimes r_X \circ id_{\bar{Y}} \otimes f^{*} \otimes id_e \circ r_Y$$

=>

$$f^{+} \otimes id_X \circ r_X$$

$$= id_{\bar{Y}} \otimes \bar{r}^{*}_{X} \otimes id_X \circ id_{\bar{Y}} \otimes id_X \otimes r_X \circ id_{\bar{Y}} \otimes f^{*} \circ r_Y$$

$$= id_{\bar{Y}} \otimes (\bar{r}^{*}_{X} \otimes id_X \circ id_X \otimes r_X \circ f^{*}) \circ r_Y$$

$$= id_{\bar{Y}} \otimes id_X \circ f^* \circ r_Y$$

$$= id_{\bar{Y}} \otimes f^* \circ r_Y.$$

9.9 <u>REMARK</u> Suppose that $T \in Mor(\bar{X},\bar{Y})$ satisfies the equation

$$T \otimes id_X \circ r_X = id_{\bar{Y}} \otimes f^* \circ r_Y.$$

Then

$$T = f^+.$$

Proof:

$$f^+ = id_{\bar{Y}} \otimes \bar{r}_X^* \circ id_{\bar{Y}} \otimes f^* \otimes id_{\bar{X}} \circ r_Y \otimes id_{\bar{X}}$$

$$= id_{\bar{Y}} \otimes \bar{r}_X^* \circ id_{\bar{Y}} \otimes f^* \circ r_Y \otimes id_{\bar{X}} \circ id_{\bar{X}}$$

$$= id_{\bar{Y}} \otimes \bar{r}_X^* \circ T \otimes id_X \circ r_X \otimes id_{\bar{X}}.$$

On the other hand,

$$T = T \circ id_{\bar{X}}$$

$$= T \circ id_{\bar{X}} \otimes \bar{r}_X^* \circ r_X \otimes id_{\bar{X}}$$

$$= T \otimes id_e \circ id_{\bar{X}} \otimes \bar{r}_X^* \circ r_X \otimes id_{\bar{X}}$$

$$= id_{\bar{Y}} \otimes \bar{r}_X^* \circ T \otimes id_{X \otimes \bar{X}} \circ r_X \otimes id_{\bar{X}}$$

$$= id_{\bar{Y}} \otimes \bar{r}_X^* \circ T \otimes id_X \otimes id_{\bar{X}} \circ r_X \otimes id_{\bar{X}}$$

$$= id_{\bar{Y}} \otimes \bar{r}_X^* \circ T \otimes id_X \circ r_X \otimes id_{\bar{X}} \circ id_{\bar{X}}$$

$$= \mathrm{id}_{\bar{Y}} \otimes \bar{r}_X^* \circ T \otimes \mathrm{id}_X \circ r_X \otimes \mathrm{id}_{\bar{X}}.$$

[Note: It is thus a corollary that if

$$\mathrm{id}_{\bar{Y}} \otimes f^* \circ r_Y = 0,$$

then $f^+ = 0$, so

$$(f^\vee)^* = 0 \Rightarrow (f^\vee)^{**} = 0 \Rightarrow f^\vee = 0 \Rightarrow f = 0.]$$

9.10 SCHOLIUM f^+ is the unique element of $\mathrm{Mor}(\bar{X},\bar{Y})$ such that

$$f^+ \otimes \mathrm{id}_X \circ r_X = \mathrm{id}_{\bar{Y}} \otimes f^* \circ r_Y.$$

[Note: ${}^\vee f$ is the unique element of $\mathrm{Mor}(\bar{Y},\bar{X})$ such that

$$\mathrm{id}_Y \otimes {}^\vee f \circ \bar{r}_Y = f \otimes \mathrm{id}_{\bar{X}} \circ \bar{r}_X,$$

so f^+ is the unique element of $\mathrm{Mor}(\bar{X},\bar{Y})$ such that

$$\bar{r}_Y^* \circ \mathrm{id}_Y \otimes f^+ = \bar{r}_X^* \circ f^* \otimes \mathrm{id}_{\bar{X}}.]$$

9.11 LEMMA Suppose that

$$F:\underline{C} \to \underline{\mathrm{FDHILB}}$$

is symmetric and unitary. Given $X \in \mathrm{Ob}\ \underline{C}$, put

$$\begin{cases} r_{FX} = (\Xi_{\bar{X},X})^{-1} \circ Fr_X \circ \xi \\ \bar{r}_{FX} = \tau_{F\bar{X},FX} \circ r_{FX}. \end{cases}$$

Then the triple $(F\bar{X}, r_{FX}, \bar{r}_{FX})$ is a conjugate for FX.

<u>PROOF</u> What we know is that

$$\begin{cases} (id_X \otimes r_X^*) \circ (\bar{r}_X \otimes id_X) = id_X \\ (id_{\bar{X}} \otimes \bar{r}_X^*) \circ (r_X \otimes id_{\bar{X}}) = id_{\bar{X}}, \end{cases}$$

hence

$$\begin{cases} F(id_X \otimes r_X^*) \circ F(\bar{r}_X \otimes id_X) = id_{FX} \\ F(id_{\bar{X}} \otimes \bar{r}_X^*) \circ F(r_X \otimes id_{\bar{X}}) = id_{F\bar{X}}, \end{cases}$$

and what we want to prove is that

$$\begin{cases} (id_{FX} \underline{\otimes} r_{FX}^*) \circ (\bar{r}_{FX} \underline{\otimes} id_{FX}) = id_{FX} \\ (id_{F\bar{X}} \underline{\otimes} \bar{r}_{FX}^*) \circ (r_{FX} \underline{\otimes} id_{F\bar{X}}) = id_{F\bar{X}}. \end{cases}$$

The LHS of the first of these is the composition

$$id_{FX} \underline{\otimes} \xi^{-1} \circ Fr_X^* \circ \Xi_{\bar{X},X}$$

$$\circ \, \tau_{F\bar{X},FX} \circ (\Xi_{\bar{X},X})^{-1} \circ Fr_X \circ \xi \underline{\otimes} id_{FX},$$

F being unitary. Write

$$\tau_{F\bar{X},FX} \circ (\Xi_{\bar{X},X})^{-1} \circ Fr_X \circ \xi \underline{\otimes} id_{FX}$$

$$= \ldots \underline{\otimes} id_{FX} \circ id_{FX} \circ id_{FX} \circ id_{FX}$$

$$= \tau_{F\bar{X},FX} \underline{\otimes} id_{FX} \circ (\Xi_{\bar{X},X})^{-1} \underline{\otimes} id_{FX} \circ Fr_X \underline{\otimes} id_{FX} \circ \xi \underline{\otimes} id_{FX}.$$

Taking into account the commutative diagrams

$$\begin{array}{ccc}
\underline{e} = \underline{e} \,\underline{\otimes}\, FX & =\!=\!=\!=\!= & FX \\
{\scriptstyle \xi \,\underline{\otimes}\, id_{FX}} \downarrow & & \| \\
Fe \,\underline{\otimes}\, FX & \xrightarrow[\Xi_{e,X}]{} & F(e \otimes X)
\end{array}$$

$$\begin{array}{ccc}
Fe \,\underline{\otimes}\, FX & \xrightarrow{\Xi_{e,X}} & F(e \otimes X) \\
{\scriptstyle Fr_X \,\underline{\otimes}\, id_{FX}} \downarrow & & \downarrow {\scriptstyle F(r_X \otimes id_X)} \\
F(\bar{X} \otimes X) \,\underline{\otimes}\, FX & \xrightarrow[\Xi_{\bar{X} \otimes X, X}]{} & F(\bar{X} \otimes X \otimes X),
\end{array}$$

we have

$$Fr_X \,\underline{\otimes}\, id_{FX} \circ \xi \,\underline{\otimes}\, id_{FX}$$

$$= (\Xi_{\bar{X} \otimes X, X})^{-1} \circ F(r_X \otimes id_X) \circ \Xi_{e,X} \circ \xi \,\underline{\otimes}\, id_{FX}$$

$$= (\Xi_{\bar{X} \otimes X, X})^{-1} \circ F(r_X \otimes id_X) \circ id_{FX}.$$

This leaves

$$\tau_{F\bar{X}, FX} \,\underline{\otimes}\, id_{FX} \circ (\Xi_{\bar{X}, X})^{-1} \,\underline{\otimes}\, id_{FX} \circ (\Xi_{\bar{X} \otimes X, X})^{-1} \circ F(r_X \otimes id_X) \circ id_{FX}.$$

Next

$$F(\bar{r}_X \otimes id_X) = F(\tau_{\bar{X}, X} \circ r_X \otimes id_X)$$

$$= F(\tau_{\bar{X}, X} \otimes id_X) \circ F(r_X \otimes id_X).$$

Since F is symmetric, there is a commutative diagram

$$
\begin{array}{ccc}
F\overline{X} \underline{\otimes} FX \underline{\otimes} FX & \xrightarrow{\quad \text{top} \quad} & F(\overline{X} \otimes X \otimes X) \\
\Big\downarrow {\scriptstyle \tau_{F\overline{X},FX} \underline{\otimes} \mathrm{id}_{FX}} & & \Big\downarrow {\scriptstyle F(\tau_{\overline{X},X} \otimes \mathrm{id}_X)} \\
FX \underline{\otimes} F\overline{X} \underline{\otimes} FX & \xrightarrow[\quad \text{bttm} \quad]{} & F(X \otimes \overline{X} \otimes X).
\end{array}
$$

Here "top" is the composition

$$F\overline{X} \underline{\otimes} FX \underline{\otimes} FX \xrightarrow{\Xi_{\overline{X},X} \underline{\otimes} \mathrm{id}_{FX}} F(\overline{X} \otimes X) \underline{\otimes} FX$$

$$\xrightarrow{\Xi_{\overline{X} \otimes X,X}} F(\overline{X} \otimes X \otimes X)$$

and "bttm" is the composition

$$FX \underline{\otimes} F\overline{X} \underline{\otimes} FX \xrightarrow{\Xi_{X,\overline{X}} \underline{\otimes} \mathrm{id}_{FX}} F(X \otimes \overline{X}) \underline{\otimes} FX$$

$$\xrightarrow{\Xi_{X \otimes \overline{X},X}} F(X \otimes \overline{X} \otimes X).$$

Therefore

$$\Xi_{X \otimes \overline{X},X} \circ \Xi_{X,\overline{X}} \underline{\otimes} \mathrm{id}_{FX} \circ \tau_{F\overline{X},FX} \underline{\otimes} \mathrm{id}_{FX}$$

$$= F(\tau_{\overline{X},X} \otimes \mathrm{id}_X) \circ \Xi_{\overline{X} \otimes X,X} \circ \Xi_{\overline{X},X} \underline{\otimes} \mathrm{id}_{FX}$$

$\Rightarrow$

$$\tau_{F\overline{X},FX} \underline{\otimes} \mathrm{id}_{FX} \circ (\Xi_{\overline{X},X})^{-1} \underline{\otimes} \mathrm{id}_{FX} \circ (\Xi_{\overline{X} \otimes X,X})^{-1}$$

$$= (\Xi_{X,\bar{X}})^{-1} \underline{\otimes}\ id_{FX} \circ (\Xi_{X \otimes \bar{X},X})^{-1} \circ F(\tau_{\bar{X},X} \otimes id_X)$$

=>

$$\tau_{F\bar{X},FX} \underline{\otimes}\ id_{FX} \circ (\Xi_{\bar{X},X})^{-1} \underline{\otimes}\ id_{FX} \circ (\Xi_{\bar{X} \otimes X,X})^{-1} \circ F(r_X \otimes id_X) \circ id_{FX}$$

$$= (\Xi_{X,\bar{X}})^{-1} \underline{\otimes}\ id_{FX} \circ (\Xi_{X \otimes \bar{X},X})^{-1} \circ F(\tau_{\bar{X},X} \otimes id_X) \circ F(r_X \otimes id_X) \circ id_{FX}$$

$$= (\Xi_{X,\bar{X}})^{-1} \underline{\otimes}\ id_{FX} \circ (\Xi_{X \otimes \bar{X},X})^{-1} \circ F(\bar{r}_X \otimes id_X) \circ id_{FX}.$$

Analogously,

$$id_{FX} \underline{\otimes}\ \xi^{-1} \circ Fr_X^* \circ \Xi_{\bar{X},X}$$

$$= id_{FX} \circ F(id_X \otimes r_X^*) \circ \Xi_{X,\bar{X} \otimes X} \circ id_{FX} \underline{\otimes}\ \Xi_{\bar{X},X}.$$

So, in summary,

$$(id_{FX} \underline{\otimes}\ r_{FX}^*) \circ (\bar{r}_{FX} \underline{\otimes}\ id_{FX})$$

$$= id_{FX} \circ F(id_X \otimes r_X^*)$$

$$\circ\ \Xi_{X,\bar{X} \otimes X} \circ id_{FX} \underline{\otimes}\ \Xi_{\bar{X},X} \circ (\Xi_{X,\bar{X}})^{-1} \underline{\otimes}\ id_{FX} \circ (\Xi_{X \otimes \bar{X},X})^{-1}$$

$$\circ\ F(\bar{r}_X \otimes id_X) \circ id_{FX},$$

thus to finish, it need only be shown that

$$\Xi_{X,\bar{X} \otimes X} \circ id_{FX} \underline{\otimes}\ \Xi_{\bar{X},X} \circ (\Xi_{X,\bar{X}})^{-1} \underline{\otimes}\ id_{FX} \circ (\Xi_{X \otimes \bar{X},X})^{-1}$$

$$= id_{F(X \otimes \bar{X} \otimes X)}.$$

This, however, follows from the commutative diagram

$$\begin{array}{ccc}
FX \,\underline{\otimes}\, F\bar{X} \,\underline{\otimes}\, FX & = & FX \,\underline{\otimes}\, F\bar{X} \,\underline{\otimes}\, FX \\
\Xi_{X,\bar{X}} \,\underline{\otimes}\, \mathrm{id}_{FX} \downarrow & & \downarrow \mathrm{id}_{FX} \,\underline{\otimes}\, \Xi_{\bar{X},X} \\
F(X \otimes \bar{X}) \,\underline{\otimes}\, FX & & FX \,\underline{\otimes}\, F(\bar{X} \otimes X) \\
\Xi_{X \otimes \bar{X},X} \downarrow & & \downarrow \Xi_{X,\bar{X} \otimes X} \\
F(X \otimes \bar{X} \otimes X) & = & F(X \otimes \bar{X} \otimes X).
\end{array}$$

9.12 REMARK We have

$$\bar{r}_{FX} = (\Xi_{X,\bar{X}})^{-1} \circ F\bar{r}_X \circ \xi.$$

In fact, the RHS equals

$$(\Xi_{X,\bar{X}})^{-1} \circ F\tau_{\bar{X},X} \circ Fr_X \circ \xi$$

and there is a commutative diagram

$$\begin{array}{ccc}
F\bar{X} \,\underline{\otimes}\, FX & \xrightarrow{\Xi_{\bar{X},X}} & F(\bar{X} \otimes X) \\
\tau_{F\bar{X},FX} \downarrow & & \downarrow F\tau_{\bar{X},X} \\
FX \,\underline{\otimes}\, F\bar{X} & \xrightarrow[\Xi_{X,\bar{X}}]{} & F(X \otimes \bar{X}).
\end{array}$$

§10. *TANNAKIAN CATEGORIES*

Let $\underline{C}$ be a symmetric strict monoidal *-category which is essentially small — then $\underline{C}$ is said to be <u>tannakian</u> if the following conditions are met:

<u>T_1:</u> $\forall\ X,Y \in \text{Ob}\ \underline{C}$,

$$\dim \text{Mor}(X,Y) < \infty.$$

<u>T_2:</u> $\underline{C}$ has subobjects, direct sums, and conjugates.

<u>T_3:</u> $\underline{C}$ has a zero object.

<u>T_4:</u> e is irreducible.

10.1 <u>REMARK</u> A tannakian category is necessarily semisimple, hence is abelian.

10.2 <u>EXAMPLE</u> Let <u>CPTGRP</u> be the category whose objects are the compact Hausdorff topological groups (in brief, the "compact groups") and whose morphisms are the continuous homomorphisms. Given an object G in this category, let <u>Rep</u> G be the category whose objects are the finite dimensional continuous unitary representations of G and whose morphisms are the intertwining operators — then <u>Rep</u> G is tannakian (define r and $\bar{r}$ by

$$\begin{cases} r\lambda = \lambda \sum\limits_i \bar{e}_i \otimes e_i \\ \bar{r}\lambda = \lambda \sum\limits_i e_i \otimes \bar{e}_i \end{cases} \quad (\lambda \in \mathbf{C}\ (= e)),$$

where $\{e_i\} \subset H$ is an orthonormal basis for the representation space and $\{\bar{e}_i\} \subset \bar{H}$ is its conjugate). In particular: FDHILB is tannakian (take $G = \{*\}$).

[Note: The construct Rep G is amnestic and transportable, so we can and will assume that its monoidal structure is strict (cf. 3.12).]

10.3 RAPPEL An additive functor $F: A \to B$ between abelian categories A and B is exact if it preserves finite limits and finite colimits.

Accordingly, since a tannakian category is not only abelian but also autonomous, $\forall\, X \in \text{Ob}\, \underline{C}$, the functors

$$\begin{cases} \text{---} \otimes X, \ \ell hom(X, \text{---}) \\ X \otimes \text{---}, \ rhom(X, \text{---}) \end{cases}$$

are exact.

If $\underline{C}$ is tannakian, then e is irreducible and

$$\dim : \text{Ob}\, \underline{C} \to \text{Mor}(e,e)$$

has the following properties.

1. $\dim X = \dim \bar{X}$.
2. $\dim(X \otimes Y) = (\dim X)(\dim Y)$.
3. $\dim(X \oplus Y) = \dim X + \dim Y$.
4. $\dim e = 1$, $\dim 0 = 0$.

10.4 LEMMA If X is not a zero object, then $\dim X \ (= r_X^* \circ r_X) \geq 1$.

PROOF First, from the positivity of the involution, $\dim X > 0$. But $X \otimes \bar{X}$ contains e as a direct summand, thus

$$(\dim X)^2 \geq 1 \Rightarrow \dim X \geq 1.$$

[Note: If dim $X = 1$, then $X \otimes \bar{X} \approx e \approx \bar{X} \otimes X$.]

Given $X \neq 0$ in Ob $\underline{C}$, define

$$\Pi_n^X : \mathcal{S}_n \to \operatorname{Aut} X^{\otimes n}$$

as in 4.5.

<u>N.B.</u> Π_n^X is a homomorphism from $\mathcal{S}_n$ to the unitary group of Mor(X,X).

Put

$$X^{\otimes 0} = e, \quad \begin{cases} \operatorname{Sym}_0^X = \operatorname{id}_e \\ \\ \operatorname{Alt}_0^X = \operatorname{id}_e, \end{cases}$$

and for $n \in \underline{N}$, put

$$\begin{cases} \operatorname{Sym}_n^X = \dfrac{1}{n!} \displaystyle\sum_{\sigma \in \mathcal{S}_n} \Pi_n^X(\sigma) \\ \\ \operatorname{Alt}_n^X = \dfrac{1}{n!} \displaystyle\sum_{\sigma \in \mathcal{S}_n} (\operatorname{sgn} \sigma) \Pi_n^X(\sigma). \end{cases}$$

Then

$$\begin{cases} \operatorname{Sym}_n^X \\ \\ \operatorname{Alt}_n^X \end{cases}$$

are projections.

10.5 LEMMA We have

$$\mathrm{tr}_{X^{\otimes n}}(\mathrm{Alt}_n^X)$$

$$= \frac{1}{n!} (\dim X)(\dim X - 1)\ldots(\dim X - n + 1).$$

PROOF The key preliminary is the observation that

$$\mathrm{tr}_{X^{\otimes n}}(\Pi_n^X(\sigma)) = (\dim X)^{\#\sigma},$$

where $\#\sigma$ is the number of cycles into which σ decomposes, thus

$$\mathrm{tr}_{X^{\otimes n}}(\mathrm{Alt}_n^X) = \frac{1}{n!} \sum_{\sigma \in S_n} (\mathrm{sgn}\ \sigma)\ (\dim X)^{\#\sigma}.$$

But for every complex number z,

$$\sum_{\sigma \in S_n} (\mathrm{sgn}\ \sigma) z^{\#\sigma} = z(z - 1)\ldots(z - n + 1).$$

10.6 THEOREM $\forall$ nonzero X in Ob $\underline{C}$,

$$\dim X \in \mathbb{N}.$$

PROOF Let $A_n(X)$ be the subobject of $X^{\otimes n}$ corresponding to Alt_n^X. Fix an isometry $f:A_n(X) \to X^{\otimes n}$ such that $f \circ f^* = \mathrm{Alt}_n^X$ -- then

$$\mathrm{tr}_{X^{\otimes n}}(\mathrm{Alt}_n^X)$$

$$= \mathrm{tr}_{X^{\otimes n}}(f \circ f^*)$$

$$= \mathrm{tr}_{A_n(X)}(f^* \circ f) \quad \text{(cf. 6.3)}$$

$$= \mathrm{tr}_{A_n(X)}(\mathrm{id}_{A_n(X)})$$

$$= \dim A_n(X) \geq 1 \quad (\text{cf. } 10.4).$$

On the other hand, thanks to 10.5,

$$\mathrm{tr}_{X^{\otimes n}}(\mathrm{Alt}_n^X)$$

is negative for some $n \in \mathbb{N}$ unless $\dim X \in \mathbb{N}$.

10.7 <u>LEMMA</u> Let $d = \dim X$ -- then

$$\dim A_d(X) = \mathrm{tr}_{X^{\otimes d}}\mathrm{Alt}_d^X = \frac{d!}{d!} = 1.$$

The isomorphism class of $A_d(X)$ is called the <u>determinant</u> of X (written $\det(X)$).

Properties:

1. $\det(\bar{X}) \approx \overline{\det(X)}$;

2. $\det(X \oplus Y) \approx \det(X) \otimes \det(Y)$;

3. $\det(X \oplus \bar{X}) \approx e$.

§11. FIBER FUNCTORS

Let $\underline{C}$ be a tannakian category — then a symmetric embedding functor

$$\mathcal{F}:\underline{C} \to \underline{\text{FDHILB}}$$

is called a fiber functor.

E.g.: Take $\underline{C} = \underline{\text{Rep}}\ G$ (cf. 10.2) — then the forgetful functor

$$U:\underline{\text{Rep}}\ G \to \underline{\text{FDHILB}}$$

is a fiber functor.

N.B. It is a nontrivial result that every tannakian category admits a fiber functor (proof omitted).

11.1 REMARK Let

$$\mathcal{F}:\underline{C} \to \underline{\text{FDHILB}}$$

be a fiber functor. Consider

$$A_{\mathcal{F}} = \underset{i \in I_{\underline{C}}}{\oplus} B(\mathcal{F}X_i),$$

viewed as a subset of $\underline{\text{Nat}}(\mathcal{F},\mathcal{F})$ -- then the coinverse is the map $S:A_{\mathcal{F}} \to A_{\mathcal{F}}$ defined by

$$S(\alpha)_X = \mathcal{F}(\text{id}_X \otimes r_X^*) \circ \text{id}_{\mathcal{F}X} \underline{\otimes} \alpha_{\bar{X}} \underline{\otimes} \text{id}_{\mathcal{F}X} \circ \mathcal{F}(\bar{r}_X \otimes \text{id}_X),$$

matters being slightly imprecise in that the identification

$$\mathcal{F}(X \otimes \bar{X} \otimes X) \approx \mathcal{F}X \underline{\otimes} \mathcal{F}\bar{X} \underline{\otimes} \mathcal{F}X$$

has been suppressed. It is not difficult to see that the equation defining $S(\alpha)_X$ is independent of the choice $(\bar{X},r_X,\bar{r}_X)$ of a conjugate for X and $\forall$ f $\in$ Mor(X,Y), the diagram

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{S(\alpha)_X} & \mathcal{F}X \\ {\scriptstyle \mathcal{F}f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}f} \\ \mathcal{F}Y & \xrightarrow[S(\alpha)_Y]{} & \mathcal{F}Y \end{array}$$

commutes. Algebraically, S is linear and antimultiplicative. Moreover,

$$S \circ * \circ S \circ * = \mathrm{id}_{A_{\mathcal{F}}},$$

hence S is invertible.

[Note: There are various relations among Δ, ε, S which, however, need not be detailed. Still, despite appearances, in general $(A_{\mathcal{F}}, \Delta, \varepsilon, S)$ is not a Hopf $*$-algebra but rather in the jargon of the trade is a "cocommutative discrete algebraic quantum group".]

Write $ff(\underline{C})$ for the full subcategory of

$$[\underline{C}, \underline{\mathrm{FDHILB}}]^{\otimes, \mathrm{T}}$$

whose objects are the fiber functors -- then $ff(\underline{C})$ is a groupoid (cf. 5.12).

11.2 <u>THEOREM</u> $ff(\underline{C})$ is a transitive groupoid, i.e., if $\mathcal{F}_1, \mathcal{F}_2$ are fiber functors, then $\mathcal{F}_1, \mathcal{F}_2$ are isomorphic.

Definition: Given fiber functors $\mathcal{F}_1, \mathcal{F}_2$, a <u>unitary</u> monoidal natural transformation $\alpha: \mathcal{F}_1 \to \mathcal{F}_2$ is a monoidal natural transformation such that $\forall\ X \in \mathrm{Ob}\ \underline{C}$,

$$\alpha_X:\mathcal{F}_1X \to \mathcal{F}_2X$$

is unitary.

Write $ff^*(\underline{C})$ for the category whose objects are the fiber functors and whose morphisms are the unitary monoidal natural transformations -- then $ff^*(\underline{C})$ is a subcategory of $ff(\underline{C})$.

11.3 THEOREM $ff^*(\underline{C})$ is a transitive groupoid, i.e., if $\mathcal{F}_1,\mathcal{F}_2$ are fiber functors, then $\mathcal{F}_1,\mathcal{F}_2$ are unitarily isomorphic.

Obviously,

$$11.3 \Rightarrow 11.2.$$

As for the proof of 11.3, there will be three steps.

Step 1: Construct a commutative unital *-algebra $A(\mathcal{F}_1,\mathcal{F}_2)$ whose dual space is in a one-to-one correspondence with the natural transformations $\mathcal{F}_1 \to \mathcal{F}_2$, to wit:

$$\text{Nat}(\mathcal{F}_1,\mathcal{F}_2) \longleftrightarrow A(\mathcal{F}_1,\mathcal{F}_2)^*.$$

Step 2: Under this bijection, prove that the monoidal natural transformations correspond to the nonzero multiplicative linear functionals on $A(\mathcal{F}_1,\mathcal{F}_2)$ and the unitary monoidal natural transformations correspond to the *-preserving multiplicative linear functionals on $A(\mathcal{F}_1,\mathcal{F}_2)$.

Step 3: Establish that $A(\mathcal{F}_1,\mathcal{F}_2)$ admits a C*-norm, thus is a pre-C*-algebra.

Therefore, since the structure space $\Delta(\bar{A}(\mathcal{F}_1,\mathcal{F}_2))$ of the C*-completion $\bar{A}(\mathcal{F}_1,\mathcal{F}_2)$ of $A(\mathcal{F}_1,\mathcal{F}_2)$ is not empty, it follows that $\text{Mor}(\mathcal{F}_1,\mathcal{F}_2)$ is also not empty, from which 11.3.

[Note: Here, of course, Mor is computed in $\mathbf{ff}^*(\underline{C})$.]

To fix notation, bear in mind that there are isomorphisms

$$\begin{bmatrix} \xi^1 : \underline{e} \to \mathcal{F}_1 e \\ \\ \xi^2 : \underline{e} \to \mathcal{F}_2 e \end{bmatrix}, \quad \begin{bmatrix} \Xi^1_{X,Y} : \mathcal{F}_1 X \underline{\otimes} \mathcal{F}_1 Y \to \mathcal{F}_1(X \otimes Y) \\ \\ \Xi^2_{X,Y} : \mathcal{F}_2 X \underline{\otimes} \mathcal{F}_2 Y \to \mathcal{F}_2(X \otimes Y) \end{bmatrix}$$

subject to the compatibility conditions enumerated in §2.

Let $A_0(\mathcal{F}_1,\mathcal{F}_2)$ be the complex vector space

$$\bigoplus_{X \in \mathrm{Ob}\, \underline{C}} \mathrm{Mor}(\mathcal{F}_2 X, \mathcal{F}_1 X).$$

Given $X \in \mathrm{Ob}\, \underline{C}$ and $\phi \in \mathrm{Mor}(\mathcal{F}_2 X, \mathcal{F}_1 X)$, write $[X,\phi]_0$ for the element of $A_0(\mathcal{F}_1,\mathcal{F}_2)$ that is ϕ at X and is zero elsewhere -- then $A_0(\mathcal{F}_1,\mathcal{F}_2)$ is simply the complex linear span of the $[X,\phi]_0$. Define a product in $A_0(\mathcal{F}_1,\mathcal{F}_2)$ by stipulating that

$$[X,\phi]_0 \cdot [Y,\psi]_0 = [X \otimes Y, u]_0,$$

where u is the composition

$$\mathcal{F}_2(X \otimes Y) \xrightarrow{(\Xi^2_{X,Y})^{-1}} \mathcal{F}_2 X \underline{\otimes} \mathcal{F}_2 Y$$

$$\xrightarrow{\phi \underline{\otimes} \psi} \mathcal{F}_1 X \underline{\otimes} \mathcal{F}_1 Y$$

$$\xrightarrow{\Xi^1_{X,Y}} \mathcal{F}_1(X \otimes Y).$$

11.4 <u>LEMMA</u> $A_0(\mathcal{F}_1,\mathcal{F}_2)$ is associative.

11.5 <u>LEMMA</u> $A_0(\mathcal{F}_1,\mathcal{F}_2)$ is unital.

<u>PROOF</u> Let

$$1_{A_0} = [e,\xi^1 \circ (\xi^2)^{-1}]_0.$$

Then 1_{A_0} is the unit. E.g.: Consider

$$[X,\phi]_0 \cdot [e,\xi^1 \circ (\xi^2)^{-1}]_0 = [X,u]_0,$$

the claim being that the composite

$$\mathcal{F}_2X = \mathcal{F}_2(X \otimes e) \xrightarrow{(\Xi^2_{X,e})^{-1}} \mathcal{F}_2X \underline{\otimes} \mathcal{F}_2e$$

$$\xrightarrow{\phi \underline{\otimes} (\xi^1 \circ (\xi^2)^{-1})} \mathcal{F}_1X \underline{\otimes} \mathcal{F}_1e$$

$$\xrightarrow{\Xi^1_{X,e}} \mathcal{F}_1(X \otimes e) = \mathcal{F}_1X$$

reduces to ϕ itself. To see this, recall that the composition

$$\mathcal{F}_1X = \mathcal{F}_1X \underline{\otimes} \underline{e} \xrightarrow{\mathrm{id}_{\mathcal{F}_1X} \underline{\otimes} \xi^1} \mathcal{F}_1X \underline{\otimes} \mathcal{F}_1e \xrightarrow{\Xi^1_{X,e}} \mathcal{F}_1(X \otimes e) = \mathcal{F}_1X$$

is the identity morphism of $\mathcal{F}_1X$ and the composition

$$\mathcal{F}_2X = \mathcal{F}_2X \underline{\otimes} \underline{e} \xrightarrow{\mathrm{id}_{\mathcal{F}_2X} \underline{\otimes} \xi^1} \mathcal{F}_2X \underline{\otimes} \mathcal{F}_2e \xrightarrow{\Xi^2_{X,e}} \mathcal{F}_2(X \otimes e) = \mathcal{F}_2X$$

is the identity morphism of $\mathcal{F}_2X$. Now write

$$\Xi^1_{X,e} \circ \phi \underline{\otimes} (\xi^1 \circ (\xi^2)^{-1}) \circ (\Xi^2_{X,e})^{-1}$$

$$= \mathrm{id}_{\mathcal{F}_1X} \circ \mathrm{id}_{\mathcal{F}_1X} \underline{\otimes} (\xi^1)^{-1} \circ \phi \underline{\otimes} (\xi^1 \circ (\xi^2)^{-1}) \circ \mathrm{id}_{\mathcal{F}_2X} \underline{\otimes} \xi^2 \circ \mathrm{id}_{\mathcal{F}_2X}$$

$$= \mathrm{id}_{\mathcal{F}_1 X} \circ (\mathrm{id}_{\mathcal{F}_1 X} \circ \phi \circ \mathrm{id}_{\mathcal{F}_2 X} \underline{\otimes} (\xi^{-1})^{-1} \circ (\xi^1 \circ (\xi^2)^{-1} \circ \xi^2) \circ \mathrm{id}_{\mathcal{F}_2 X}$$

$$= \mathrm{id}_{\mathcal{F}_1 X} \circ \phi \underline{\otimes} \mathrm{id}_{\underline{e}} \circ \mathrm{id}_{\mathcal{F}_2 X} = \mathrm{id}_{\mathcal{F}_1 X} \circ \phi \circ \mathrm{id}_{\mathcal{F}_2 X} = \phi.$$

Let $I_0(\mathcal{F}_1,\mathcal{F}_2)$ be the complex linear span of the

$$[X,a \circ \mathcal{F}_2 f]_0 - [Y,\mathcal{F}_1 f \circ a]_0,$$

where

$$f \in \mathrm{Mor}(X,Y),\ a \in \mathrm{Mor}(\mathcal{F}_2 Y,\mathcal{F}_1 X).$$

Then $I_0(\mathcal{F}_1,\mathcal{F}_2)$ is an ideal in $A_0(\mathcal{F}_1,\mathcal{F}_2)$.

Denote by $A(\mathcal{F}_1,\mathcal{F}_2)$ the quotient algebra

$$A_0(\mathcal{F}_1,\mathcal{F}_2)/I_0(\mathcal{F}_1,\mathcal{F}_2),$$

let

$$\mathrm{pr}:A_0(\mathcal{F}_1,\mathcal{F}_2) \to A(\mathcal{F}_1,\mathcal{F}_2)$$

be the projection, and put

$$[X,\phi] = \mathrm{pr}[X,\phi]_0.$$

11.6 <u>EXAMPLE</u> Let $f:X \to X$ be an isomorphism — then

$$[X,\phi] = [X,\mathcal{F}_1 f \circ \mathcal{F}_1 f^{-1} \circ \phi]$$

$$= [X,\mathcal{F}_1 f^{-1} \circ \phi \circ \mathcal{F}_2 f].$$

11.7 <u>EXAMPLE</u> Let

$$\Phi \in \mathrm{Mor}(\mathcal{F}_2(\bar{X} \otimes X),\mathcal{F}_1(\bar{X} \otimes X)).$$

Then

$$[\bar{X} \otimes X, \mathcal{F}_1 r_X \circ \mathcal{F}_1 r_X^* \circ \Phi]$$

$$= [e, \mathcal{F}_1 r_X^* \circ \Phi \circ \mathcal{F}_2 r_X].$$

[Note: We also have

$$[\bar{X} \otimes X, \mathcal{F}_1(r_X \circ r_X^*) \circ \Phi]$$

$$= [\bar{X} \otimes X, \Phi \circ \mathcal{F}_2(r_X \circ r_X^*)].]$$

11.8 REMARK Every $A \in A(\mathcal{F}_1, \mathcal{F}_2)$ can be written as $[X, \phi]$ for a suitable choice of X and ϕ. Thus suppose that $A = \sum_i [X_i, \phi_i]$, put $X = \bigoplus_i X_i$, and choose isometries $v_i : X_i \to X$ such that $\sum_i v_i \circ v_i^* = id_X$ — then

$$a_i = \phi_i \circ \mathcal{F}_2 v_i^* \in \text{Mor}(\mathcal{F}_2 X, \mathcal{F}_1 X_i)$$

=>

$$A = \sum_i [X_i, \phi_i]$$

$$= \sum_i [X_i, \phi_i \circ id_{\mathcal{F}_2 X_i}]$$

$$= \sum_i [X_i, \phi_i \circ \mathcal{F}_2(v_i^* \circ v_i)]$$

$$= \sum_i [X_i, \phi_i \circ \mathcal{F}_2 v_i^* \circ \mathcal{F}_2 v_i]$$

$$= \sum_i [X_i, a_i \circ \mathcal{F}_2 v_i]$$

$$= \sum_i [X, \mathcal{F}_1 v_i \circ a_i]$$

$$= \sum_i [X, \mathcal{F}_1 v_i \circ \phi_i \circ \mathcal{F}_2 v_i^*]$$

$$= [X, \sum_i \mathcal{F}_1 v_i \circ \phi_i \circ \mathcal{F}_2 v_i^*]$$

$$= [X, \phi],$$

where

$$\phi = \sum_i \mathcal{F}_1 v_i \circ \phi_i \circ \mathcal{F}_2 v_i^* \in \text{Mor}(\mathcal{F}_2 X, \mathcal{F}_1 X).$$

11.9 LEMMA $A(\mathcal{F}_1, \mathcal{F}_2)$ is commutative.

PROOF Let

$$\begin{bmatrix} [X,\phi]_0 & (\phi: \mathcal{F}_2 X \to \mathcal{F}_1 X) \\ [Y,\psi]_0 & (\psi: \mathcal{F}_2 Y \to \mathcal{F}_1 Y) \end{bmatrix}$$

be elements of $A_0(\mathcal{F}_1, \mathcal{F}_2)$ — then

$$[X,\phi]_0 \cdot [Y,\psi]_0 = [X \underline{\otimes} Y, \Xi^1_{X,Y} \circ \phi \underline{\otimes} \psi \circ (\Xi^2_{X,Y})^{-1}].$$

On the other hand,

$$[Y,\psi]_0 \cdot [X,\phi]_0 = [Y \underline{\otimes} X, \Xi^1_{Y,X} \circ \psi \underline{\otimes} \phi \circ (\Xi^2_{Y,X})^{-1}]$$

and there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_2 Y \underline{\otimes} \mathcal{F}_2 X & \xrightarrow{T_{\mathcal{F}_2 Y, \mathcal{F}_2 X}} & \mathcal{F}_2 X \underline{\otimes} \mathcal{F}_2 Y \\ {\scriptstyle \psi \underline{\otimes} \phi} \downarrow & & \downarrow {\scriptstyle \phi \underline{\otimes} \psi} \\ \mathcal{F}_1 Y \underline{\otimes} \mathcal{F}_1 X & \xrightarrow[T_{\mathcal{F}_1 Y, \mathcal{F}_1 X}]{} & \mathcal{F}_1 X \underline{\otimes} \mathcal{F}_1 Y. \end{array}$$

Thus

$$\Xi^1_{Y,X} \circ \psi \underline{\otimes} \phi \circ (\Xi^2_{Y,X})^{-1}$$

$$= \Xi^1_{Y,X} \circ T_{\mathcal{F}_1 X, \mathcal{F}_1 Y} \circ \phi \underline{\otimes} \psi \circ T_{\mathcal{F}_2 Y, \mathcal{F}_2 X} \circ (\Xi^2_{Y,X})^{-1}.$$

But there are also commutative diagrams

$$\begin{array}{ccc} \mathcal{F}_1 X \underline{\otimes} \mathcal{F}_1 Y & \xrightarrow{\Xi^1_{X,Y}} & \mathcal{F}_1 (X \otimes Y) \\ {\scriptstyle T_{\mathcal{F}_1 X, \mathcal{F}_1 Y}} \downarrow & & \downarrow {\scriptstyle \mathcal{F}_1 T_{X,Y}} \\ \mathcal{F}_1 Y \underline{\otimes} \mathcal{F}_1 X & \xrightarrow[\Xi^1_{Y,X}]{} & \mathcal{F}_1 (Y \otimes X) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{F}_2 Y \underline{\otimes} \mathcal{F}_2 X & \xrightarrow{\Xi^2_{Y,X}} & \mathcal{F}_2 (Y \otimes X) \\ {\scriptstyle T_{\mathcal{F}_2 Y, \mathcal{F}_2 X}} \downarrow & & \downarrow {\scriptstyle \mathcal{F}_2 T_{Y,X}} \\ \mathcal{F}_2 X \underline{\otimes} \mathcal{F}_2 Y & \xrightarrow[\Xi^2_{X,Y}]{} & \mathcal{F}_2 (X \otimes Y). \end{array}$$

Thus

$$\Xi^1_{Y,X} \circ T_{\mathcal{F}_1 X, \mathcal{F}_1 Y} \circ \phi \underline{\otimes} \psi \circ T_{\mathcal{F}_2 Y, \mathcal{F}_2 X} \circ (\Xi^2_{Y,X})^{-1}$$

$$= \mathcal{F}_1 T_{X,Y} \circ \Xi^1_{X,Y} \circ \phi \underline{\otimes} \psi \circ (\Xi^2_{X,Y})^{-1} \circ \mathcal{F}_2 T_{Y,X}.$$

Let $f = \tau_{X,Y}$ and put

$$a = \Xi^1_{X,Y} \circ \phi \underline{\otimes} \psi \circ (\Xi^2_{X,Y})^{-1} \circ \mathcal{F}_2\tau_{Y,X}.$$

Then

$$f \in \mathrm{Mor}(X \otimes Y, Y \otimes X)$$

and

$$a \in \mathrm{Mor}(\mathcal{F}_2(Y \otimes X), \mathcal{F}_1(X \otimes Y)).$$

Moreover

$$[Y,\psi]_0 \cdot [X,\phi]_0 = [Y \otimes X, \mathcal{F}_1 f \circ a]_0.$$

Meanwhile

$$\mathcal{F}_2\tau_{Y,X} \circ \mathcal{F}_2\tau_{X,Y}$$

$$= \mathcal{F}_2(\tau_{Y,X} \circ \tau_{X,Y})$$

$$= \mathcal{F}_2(\mathrm{id}_{X \otimes Y})$$

$$= \mathrm{id}_{\mathcal{F}_2(X \otimes Y)},$$

so

$$[X,\phi]_0 \cdot [Y,\psi]_0 = [X \otimes Y, a \circ \mathcal{F}_2 f]_0.$$

Therefore

$$[X,\phi]_0 \cdot [Y,\psi]_0 - [Y,\psi]_0 \cdot [X,\phi]_0 \in I_0(\mathcal{F}_1,\mathcal{F}_2)$$

$$\Rightarrow$$

$$[A_0(\mathcal{F}_1,\mathcal{F}_2), A_0(\mathcal{F}_1,\mathcal{F}_2)] \subset I_0(\mathcal{F}_1,\mathcal{F}_2).$$

And this implies that $A(\mathcal{F}_1,\mathcal{F}_2)$ is commutative.

Given $[X,\phi]_0$, choose a conjugate $(\bar{X}, r_X, \bar{r}_X)$ for X and let

$$[X,\phi]_0^* = [\bar{X},\bar{\phi}]_0,$$

where $\bar{\phi}$ is the composition

$$\begin{aligned}
\mathcal{F}_2\bar{X} = \underline{e} \,\underline{\otimes}\, \mathcal{F}_2\bar{X} &\xrightarrow{\xi^1 \,\underline{\otimes}\, \mathrm{id}} \mathcal{F}_1 e \,\underline{\otimes}\, \mathcal{F}_2\bar{X} \\
&\xrightarrow{\mathcal{F}_1 r_X \,\underline{\otimes}\, \mathrm{id}} \mathcal{F}_1(\bar{X} \otimes X) \,\underline{\otimes}\, \mathcal{F}_2\bar{X} \\
&\xrightarrow{(\Xi^1_{\bar{X},X})^{-1} \,\underline{\otimes}\, \mathrm{id}} \mathcal{F}_1\bar{X} \,\underline{\otimes}\, \mathcal{F}_1 X \,\underline{\otimes}\, \mathcal{F}_2\bar{X} \\
&\xrightarrow{\mathrm{id} \,\underline{\otimes}\, \phi^* \,\underline{\otimes}\, \mathrm{id}} \mathcal{F}_1\bar{X} \,\underline{\otimes}\, \mathcal{F}_2 X \,\underline{\otimes}\, \mathcal{F}_2\bar{X} \\
&\xrightarrow{\mathrm{id} \,\underline{\otimes}\, \Xi^2_{X,\bar{X}}} \mathcal{F}_1\bar{X} \,\underline{\otimes}\, \mathcal{F}_2(X \otimes \bar{X}) \\
&\xrightarrow{\mathrm{id} \,\underline{\otimes}\, \mathcal{F}_2\bar{r}_X^*} \mathcal{F}_1\bar{X} \,\underline{\otimes}\, \mathcal{F}_2 e \\
&\xrightarrow{\mathrm{id} \,\underline{\otimes}\, (\xi^2)^{-1}} \mathcal{F}_1\bar{X} \,\underline{\otimes}\, \underline{e} = \mathcal{F}_1\bar{X}.
\end{aligned}$$

<u>N.B.</u> We have

$$(\Xi^1_{\bar{X},X})^{-1} \,\underline{\otimes}\, \mathrm{id}_{\mathcal{F}_2\bar{X}} \circ \mathcal{F}_1 r_X \,\underline{\otimes}\, \mathrm{id}_{\mathcal{F}_2\bar{X}} \circ \xi^1 \,\underline{\otimes}\, \mathrm{id}_{\mathcal{F}_2\bar{X}}$$

$$= (\Xi^1_{\bar{X},X})^{-1} \circ \mathcal{F}_1 r_X \circ \xi^1 \,\underline{\otimes}\, \mathrm{id}_{\mathcal{F}_2\bar{X}}$$

$$= r_{\mathcal{F}_1 X} \,\underline{\otimes}\, \mathrm{id}_{\mathcal{F}_2\bar{X}} \qquad \text{(cf. 9.10)}$$

and

$$\mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} (\xi^2)^{-1} \circ \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \mathcal{F}_2\bar{r}_X^* \circ \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \Xi^2_{X,\bar{X}}$$

$$= \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} (\xi^2)^{-1} \circ \mathcal{F}_2\bar{r}_X^* \circ \Xi^2_{X,\bar{X}}$$

$$= \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \bar{r}^*_{\mathcal{F}_2X} \quad \text{(cf. 9.11).}$$

Therefore

$$\bar{\phi} = \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \bar{r}^*_{\mathcal{F}_2X} \circ \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \phi^* \underline{\otimes} \mathrm{id}_{\mathcal{F}_2\bar{X}} \circ r_{\mathcal{F}_1X} \underline{\otimes} \mathrm{id}_{\mathcal{F}_2\bar{X}}.$$

[Note: By definition, $\phi \in \mathrm{Mor}(\mathcal{F}_2X,\mathcal{F}_1X)$, so ${}^\vee\phi \in \mathrm{Mor}(\mathcal{F}_1\bar{X},\mathcal{F}_2\bar{X})$, where, as in §5,

$${}^\vee\phi = \varepsilon_{\mathcal{F}_1X} \underline{\otimes} \mathrm{id}_{\mathcal{F}_2\bar{X}} \circ \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \phi \underline{\otimes} \mathrm{id}_{\mathcal{F}_2\bar{X}} \circ \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \eta_{\mathcal{F}_2X}$$

or still,

$${}^\vee\phi = r^*_{\mathcal{F}_1X} \underline{\otimes} \mathrm{id}_{\mathcal{F}_2\bar{X}} \circ \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \phi \underline{\otimes} \mathrm{id}_{\mathcal{F}_2\bar{X}} \circ \mathrm{id}_{\mathcal{F}_1\bar{X}} \underline{\otimes} \bar{r}_{\mathcal{F}_2X}.$$

Therefore

$$\bar{\phi} = ({}^\vee\phi)^*.]$$

Replacing $(\bar{X},r_X,\bar{r}_X)$ by $(\bar{X}',r'_X,\bar{r}'_X)$ and using 9.4, one finds that

$$[\bar{X},\bar{\phi}]_0 - [\bar{X}',\bar{\phi}']_0 \in I_0(\mathcal{F}_1,\mathcal{F}_2).$$

Therefore the image of $[X,\phi]_0^*$ in $A(\mathcal{F}_1,\mathcal{F}_2)$ is independent of the choice of a conjugate for X.

11.10 <u>LEMMA</u> $I_0(\mathcal{F}_1,\mathcal{F}_2)$ is $*$-invariant.

Consequently, $*:A_0(\mathcal{F}_1,\mathcal{F}_2) \to A_0(\mathcal{F}_1,\mathcal{F}_2)$ induces a map $*:A(\mathcal{F}_1,\mathcal{F}_2) \to A(\mathcal{F}_1,\mathcal{F}_2)$.

11.11 <u>LEMMA</u> $A(\mathcal{F}_1,\mathcal{F}_2)$ is a $*$-algebra.

Summary: $A(\mathcal{F}_1,\mathcal{F}_2)$ is a commutative unital $*$-algebra.

Accordingly, to complete Step 1, it remains to construct an isomorphism between $A(\mathcal{F}_1,\mathcal{F}_2)^*$ and $\mathrm{Nat}(\mathcal{F}_1,\mathcal{F}_2)$.

On general grounds,

$$A_0(\mathcal{F}_1,\mathcal{F}_2)^* = \prod_{X\in \mathrm{Ob}\,\underline{C}} \mathrm{Mor}(\mathcal{F}_2X,\mathcal{F}_1X)^*.$$

But the pairing

$$\mathrm{Mor}(\mathcal{F}_2X,\mathcal{F}_1X) \times \mathrm{Mor}(\mathcal{F}_1X,\mathcal{F}_2X) \to \mathbb{C}$$

that sends $\phi \times \psi$ to $\mathrm{tr}(\phi \circ \psi)$ is nondegenerate, thus

$$A_0(\mathcal{F}_1,\mathcal{F}_2)^* \approx \prod_{X\in \mathrm{Ob}\,\underline{C}} \mathrm{Mor}(\mathcal{F}_1X,\mathcal{F}_2X).$$

On the other hand, $\mathrm{Nat}(\mathcal{F}_1,\mathcal{F}_2)$ consists of those elements

$$\alpha \in \prod_{X\in \mathrm{Ob}\,\underline{C}} \mathrm{Mor}(\mathcal{F}_1X,\mathcal{F}_2X)$$

such that $\forall\ f \in \mathrm{Mor}(X,Y)$,

$$\mathcal{F}_2f \circ \alpha_X = \alpha_Y \circ \mathcal{F}_1f,$$

and the dual of $A(\mathcal{F}_1,\mathcal{F}_2)$ is the subspace of $A_0(\mathcal{F}_1,\mathcal{F}_2)^*$ comprised of those elements that vanish identically on $I_0(\mathcal{F}_1,\mathcal{F}_2)$. To characterize the latter, take an

$$\alpha \in \prod_{X \in \mathrm{Ob}\,\underline{C}} \mathrm{Mor}(\mathcal{F}_1 X, \mathcal{F}_2 X)$$

and suppose that $\forall\, A \in I_0(\mathcal{F}_1,\mathcal{F}_2)$,

$$\langle A,\alpha\rangle = 0$$

or still,

$$\langle [X, a \circ \mathcal{F}_2 f]_0 - [Y, \mathcal{F}_1 f \circ a]_0, \alpha\rangle = 0$$

for all

$$f \in \mathrm{Mor}(X,Y),\ a \in \mathrm{Mor}(\mathcal{F}_2 Y, \mathcal{F}_1 X).$$

I.e.:

$$\mathrm{tr}_{\mathcal{F}_1 X}(a \circ \mathcal{F}_2 f \circ \alpha_X) = \mathrm{tr}_{\mathcal{F}_1 Y}(\mathcal{F}_1 f \circ a \circ \alpha_Y).$$

From the nondegeneracy of the trace, it then follows that

$$\mathcal{F}_2 f \circ \alpha_X = \alpha_Y \circ \mathcal{F}_1 f,$$

implying thereby that

$$\alpha \in \mathrm{Nat}(\mathcal{F}_1,\mathcal{F}_2).$$

11.12 <u>LEMMA</u> Under the bijection

$$\mathrm{Nat}(\mathcal{F}_1,\mathcal{F}_2) \longleftrightarrow A(\mathcal{F}_1,\mathcal{F}_2)^*,$$

the monoidal natural transformations correspond to the nonzero multiplicative linear functionals on $A(\mathcal{F}_1,\mathcal{F}_2)$.

<u>PROOF</u> To say that a linear functional on $A(\mathcal{F}_1,\mathcal{F}_2)$ corresponding to an $\alpha \in \mathrm{Nat}(\mathcal{F}_1,\mathcal{F}_2)$ is multiplicative amounts to saying that

$$\langle [X,\phi] \cdot [Y,\psi], \alpha\rangle$$

$$= <[X,\phi],\alpha> \cdot <[Y,\psi],\alpha>$$

for all

$$[X,\phi],[Y,\psi] \in A(\mathcal{F}_1,\mathcal{F}_2).$$

Since $< \text{---},\alpha>$ is null on $I_0(\mathcal{F}_1,\mathcal{F}_2)$, it suffices to work upstairs, hence explicated we have

$$\mathrm{tr}_{\mathcal{F}_1(X \otimes Y)}(\Xi^1_{X,Y} \circ \phi \underline{\otimes} \psi \circ (\Xi^2_{X,Y})^{-1} \circ \alpha_{X \otimes Y})$$

$$= \mathrm{tr}_{\mathcal{F}_1X}(\phi \circ \alpha_X)\mathrm{tr}_{\mathcal{F}_1Y}(\psi \circ \alpha_Y)$$

$$= \mathrm{tr}_{\mathcal{F}_1X \underline{\otimes} \mathcal{F}_1Y}((\phi \circ \alpha_X) \underline{\otimes} (\psi \circ \alpha_Y))$$

$$= \mathrm{tr}_{\mathcal{F}_1X \underline{\otimes} \mathcal{F}_1Y}(\phi \underline{\otimes} \psi \circ \alpha_X \underline{\otimes} \alpha_Y).$$

Therefore

$$\alpha_{X \otimes Y} = \Xi^2_{X,Y} \circ \alpha_X \underline{\otimes} \alpha_Y \circ (\Xi^1_{X,Y})^{-1},$$

the condition that α be monoidal.

[Note: Tacitly,

$$<1_{A_0},\alpha> = 1$$

or still,

$$<[e,\xi^1 \circ (\xi^2)^{-1}]_0,\alpha> = 1$$

or still,

$$\mathrm{tr}_{\mathcal{F}_1e}(\xi^1 \circ (\xi^2)^{-1} \circ \alpha_e) = 1,$$

from which the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{F}_1 e & \xrightarrow{\alpha_e} & \mathcal{F}_2 e \\ \uparrow \xi^1 & & \uparrow \xi^2 \\ \underline{e} & = & \underline{e} \end{array} \quad .]$$

11.13 LEMMA Under the bijection

$$\mathrm{Nat}(\mathcal{F}_1,\mathcal{F}_2) \longleftrightarrow A(\mathcal{F}_1,\mathcal{F}_2)^*,$$

the unitary monoidal natural transformations correspond to the $*$-preserving non-zero multiplicative linear functionals on $A(\mathcal{F}_1,\mathcal{F}_2)$.

PROOF Given $[X,\phi] \in A(\mathcal{F}_1,\mathcal{F}_2)$, the claim is that

$$<[X,\phi]^*,\alpha> = \overline{<[X,\phi],\alpha>} \quad (= <[X,\phi],\alpha>^* \ldots)$$

iff $\alpha_X^* = \alpha_X^{-1}$.

From the definitions,

$$<[X,\phi],\alpha> = \mathrm{tr}_{\mathcal{F}_1 X}(\phi \circ \alpha_X)$$

$\Rightarrow$

$$\overline{<[X,\phi],\alpha>} = \mathrm{tr}_{\mathcal{F}_1 X}(\phi^* \circ \alpha_X^*).$$

In the other direction,

$$\begin{aligned} <[X,\phi]^*,\alpha> &= <[\bar{X},\bar{\phi}],\alpha> \\ &= \mathrm{tr}_{\mathcal{F}_1 \bar{X}}(\bar{\phi} \circ \alpha_{\bar{X}}) \\ &= r^*_{\mathcal{F}_1 \bar{X}} \circ \mathrm{id}_{\mathcal{F}_1 X} \underline{\otimes} (\bar{\phi} \circ \alpha_{\bar{X}}) \circ r_{\mathcal{F}_1 \bar{X}} \end{aligned}$$

$$= \bar{r}^*_{\mathcal{F}_1 X} \circ id_{\mathcal{F}_1 X} \underline{\otimes} (\bar{\phi} \circ \alpha_{\bar{X}}) \circ \bar{r}_{\mathcal{F}_1 X}.$$

But

$$\bar{r}^*_{\mathcal{F}_2 X} \circ \phi^* \underline{\otimes} \alpha_{\bar{X}} \circ \bar{r}_{\mathcal{F}_1 X}$$

$$= \bar{r}^*_{\mathcal{F}_2 X} \circ \phi^* \underline{\otimes} id_{\mathcal{F}_2 \bar{X}} \circ id_{\mathcal{F}_1 X} \underline{\otimes} \alpha_{\bar{X}} \circ \bar{r}_{\mathcal{F}_1 X}$$

$$= r^*_{\mathcal{F}_1 X} \circ id_{\mathcal{F}_1 X} \underline{\otimes} \bar{\phi} \circ id_{\mathcal{F}_1 X} \underline{\otimes} \alpha_{\bar{X}} \circ \bar{r}_{\mathcal{F}_1 X} \quad \text{(cf. 9.10)}$$

$$= \bar{r}^*_{\mathcal{F}_1 X} \circ id_{\mathcal{F}_1 X} \underline{\otimes} (\bar{\phi} \circ \alpha_{\bar{X}}) \circ \bar{r}_{\mathcal{F}_1 X}.$$

I.e.:

$$tr_{\mathcal{F}_1 \bar{X}}(\bar{\phi} \circ \alpha_{\bar{X}}) = \bar{r}^*_{\mathcal{F}_2 X} \circ \phi^* \underline{\otimes} \alpha_{\bar{X}} \circ \bar{r}_{\mathcal{F}_1 X}.$$

Proceeding, write

$$\bar{r}^*_{\mathcal{F}_2 X} \circ \phi^* \underline{\otimes} \alpha_{\bar{X}} \circ \bar{r}_{\mathcal{F}_1 X}$$

$$= \bar{r}^*_{\mathcal{F}_2 X} \circ ((\alpha_X \circ \alpha_X^{-1} \circ \phi^*) \underline{\otimes} \alpha_{\bar{X}} \circ id_{\mathcal{F}_1 \bar{X}}) \circ \bar{r}_{\mathcal{F}_1 X}$$

$$= \bar{r}^*_{\mathcal{F}_2 X} \circ \alpha_X \underline{\otimes} \alpha_{\bar{X}} \circ \alpha_X^{-1} \circ \phi^* \underline{\otimes} id_{\mathcal{F}_1 \bar{X}} \circ \bar{r}_{\mathcal{F}_1 X}.$$

We then claim that

$$\bar{r}^*_{\mathcal{F}_2 X} \circ \alpha_X \underline{\otimes} \alpha_{\bar{X}} = \bar{r}^*_{\mathcal{F}_1 X}$$

implying thereby that

$$tr_{\mathcal{F}_1 \bar{X}}(\bar{\phi} \circ \alpha_{\bar{X}}) = tr_{\mathcal{F}_1 X}(\alpha_X^{-1} \circ \phi^*)$$

which, when combined with the initial observation, renders the contention of the lemma manifest. From the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_1 X \,\underline{\otimes}\, \mathcal{F}_1 \bar{X} & \xrightarrow{\Xi^1_{X,\bar{X}}} & \mathcal{F}_1 (X \otimes \bar{X}) \\ {\scriptstyle \alpha_X \,\underline{\otimes}\, \alpha_{\bar{X}}} \downarrow & & \downarrow {\scriptstyle \alpha_{X \otimes \bar{X}}} \\ \mathcal{F}_2 X \,\underline{\otimes}\, \mathcal{F}_2 \bar{X} & \xrightarrow[\Xi^2_{X,\bar{X}}]{} & \mathcal{F}_2 (X \otimes \bar{X}) \end{array}$$

we see that

$$\alpha_X \,\underline{\otimes}\, \alpha_{\bar{X}} = (\Xi^2_{X,\bar{X}})^{-1} \circ \alpha_{X \otimes \bar{X}} \circ \Xi^1_{X,\bar{X}}$$

and from the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_1 (X \otimes \bar{X}) & \xrightarrow{\alpha_{X \otimes \bar{X}}} & \mathcal{F}_2 (X \otimes \bar{X}) \\ {\scriptstyle \mathcal{F}_1 \bar{r}^*_X} \downarrow & & \downarrow {\scriptstyle \mathcal{F}_2 \bar{r}^*_X} \\ \mathcal{F}_1 e & \xrightarrow[\alpha_e]{} & \mathcal{F}_2 e \end{array}$$

we see that

$$\mathcal{F}_2 \bar{r}^*_X \circ \alpha_{X \otimes \bar{X}} = \alpha_e \circ \mathcal{F}_1 \bar{r}^*_X.$$

Recalling now that

$$\left[\begin{array}{l} \bar{r}_{\mathcal{F}_1 X} = (\Xi^1_{X,\bar{X}})^{-1} \circ \mathcal{F}_1 \bar{r}_X \circ \xi^1 \\ \\ \bar{r}_{\mathcal{F}_2 X} = (\Xi^2_{X,\bar{X}})^{-1} \circ \mathcal{F}_2 \bar{r}_X \circ \xi^2 \end{array}\right. \qquad \text{(cf. 9.12)}$$

we have

$$\bar{r}^*_{\mathcal{F}_2 X} \circ \alpha_X \underline{\otimes} \alpha_{\bar{X}}$$

$$= (\xi^2)^{-1} \circ \mathcal{F}_2 \bar{r}^*_X \circ \Xi^2_{X,\bar{X}} \circ (\Xi^2_{X,\bar{X}})^{-1} \circ \alpha_{X \otimes \bar{X}} \circ \Xi^1_{X,\bar{X}}$$

$$= (\xi^2)^{-1} \circ \mathcal{F}_2 \bar{r}^*_X \circ \alpha_{X \otimes \bar{X}} \circ \Xi^1_{X,\bar{X}}$$

$$= (\xi^2)^{-1} \circ \alpha_e \circ \mathcal{F}_1 \bar{r}^*_X \circ \Xi^1_{X,\bar{X}}$$

$$= (\xi^1)^{-1} \circ \mathcal{F}_1 \bar{r}^*_X \circ \Xi^1_{X,\bar{X}}$$

$$= \bar{r}^*_{\mathcal{F}_1 X},$$

as claimed.

The results embodied in 11.12 and 11.13 finish Step 2 of the program, which leaves Step 3 to be dealt with.

Put

$$A_{\mathcal{F}_1,\mathcal{F}_2} = \bigoplus_{i \in I_{\underline{C}}} \mathrm{Mor}(\mathcal{F}_2 X_i, \mathcal{F}_1 X_i).$$

11.14 <u>LEMMA</u> The linear map

$$\Psi : A_{\mathcal{F}_1,\mathcal{F}_2} \to A(\mathcal{F}_1,\mathcal{F}_2)$$

that sends

$$\phi_i \in \mathrm{Mor}(\mathcal{F}_2 X_i, \mathcal{F}_1 X_i)$$

to $[X_i, \phi_i]$ is an isomorphism of vector spaces.

PROOF Every $A \in A(\mathcal{F}_1,\mathcal{F}_2)$ is an $[X,\phi]$ (cf. 11.8) and every $[X,\phi]$ is a sum of elements $[X_i,\phi_i]$ with X_i irreducible. Therefore Ψ is surjective. That Ψ is injective is a consequence of the fact that

$$i \neq j \Rightarrow \mathrm{Mor}(X_i,X_j) = \{0\}.$$

Put

$$A_i = \Psi(\mathrm{Mor}(\mathcal{F}_2 X_i,\mathcal{F}_1 X_i)).$$

Then there is a direct sum decomposition

$$A(\mathcal{F}_1,\mathcal{F}_2) = \bigoplus_{i \in I_{\underline{C}}} A_i.$$

Define a linear functional

$$\omega : A(\mathcal{F}_1,\mathcal{F}_2) \rightarrow \mathbb{C}$$

by taking it to be zero on A_i if i does not correspond to e but on A_e, let

$$\omega([e,\phi]) = (\xi^1)^{-1} \circ \phi \circ \xi^2 \in \mathbb{C}.$$

11.15 LEMMA $\forall\, A \neq 0,\ \omega(A^*A) > 0.$

PROOF Write

$$A = \sum_i [X_i,\phi_i],$$

where the X_i are irreducible and distinct -- then

$$i \neq j \Rightarrow \omega([X_i,\phi_i]^* \cdot [X_j,\phi_j]) = 0.$$

In fact,

$$\mathrm{Mor}(e,\bar{X}_i \otimes X_j) \approx \mathrm{Mor}(X_i,X_j) = \{0\} \quad \text{(cf. 9.3)},$$

so e is not a subobject of $\bar{X}_i \otimes X_j$. One can therefore assume that $A = [X,\phi] \neq 0$ with X irreducible. Recall now that

$$r_X^* \circ r_X = \dim X = n_X \mathrm{id}_e \quad (n_X \in \mathbb{N}).$$

This said, let

$$p_X = \frac{r_X \circ r_X^*}{n_X}.$$

Then $p_X^* = p_X$ and

$$p_X \circ p_X = \frac{r_X \circ r_X^*}{n_X} \circ \frac{r_X \circ r_X^*}{n_X}$$

$$= \frac{1}{n_X^2} (r_X \circ r_X^* \circ r_X \circ r_X^*)$$

$$= \frac{1}{n_X^2} r_X \circ n_X \mathrm{id}_e \circ r_X^*$$

$$= \frac{r_X \circ r_X^*}{n_X} = p_X.$$

I.e.:

$$p_X \in \mathrm{Mor}(\bar{X} \otimes X, \bar{X} \otimes X)$$

is a projection. Write

$$A^*A = [X,\phi]^* \cdot [X,\phi]$$

$$= [\bar{X},\bar{\phi}] \cdot [X,\phi]$$

$$= [\bar{X} \otimes X, \Xi^1_{\bar{X},X} \circ \bar{\phi} \underline{\otimes} \phi \circ (\Xi^2_{\bar{X},X})^{-1}]$$

$$= [\bar{X} \otimes X, \mathcal{F}_1(p_X) \circ \Xi^1_{\bar{X},X} \circ \bar{\phi} \underline{\otimes} \phi \circ (\Xi^2_{\bar{X},X})^{-1}]$$

$$+ [\bar{X} \underline{\otimes} X, \mathcal{F}_1(\mathrm{id}_{\bar{X} \underline{\otimes} X} - p_X) \circ \Xi^1_{\bar{X},X} \circ \bar{\phi} \underline{\otimes} \phi \circ (\Xi^2_{\bar{X},X})^{-1}]$$

$$= [\bar{X} \underline{\otimes} X, \mathcal{F}_1(p_X) \circ \Xi^1_{\bar{X},X} \circ \bar{\phi} \underline{\otimes} \phi \circ (\Xi^2_{\bar{X},X})^{-1}]$$

$$= \frac{1}{n_X} [e, \mathcal{F}_1 r^*_X \circ (\Xi^1_{\bar{X},X} \circ \bar{\phi} \underline{\otimes} \phi \circ (\Xi^2_{\bar{X},X})^{-1} \circ \mathcal{F}_2 r_X] \quad \text{(cf. 11.7)}$$

$$= \frac{1}{n_X} ((\xi^1)^{-1} \circ \mathcal{F}_1 r^*_X \circ (\Xi^1_{\bar{X},X} \circ \bar{\phi} \underline{\otimes} \phi \circ (\Xi^2_{\bar{X},X})^{-1}) \circ \mathcal{F}_2 r_X \circ \xi^2)[e, \xi^1 \circ (\xi^2)^{-1}]$$

$$= \frac{1}{n_X} (r^*_{\mathcal{F}_1 X} \circ \bar{\phi} \underline{\otimes} \phi \circ r_{\mathcal{F}_2 X})[e, \xi^1 \circ (\xi^2)^{-1}]$$

$$= \frac{1}{n_X} (r^*_{\mathcal{F}_1 X} \circ \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} (\phi \circ \phi^*) \circ r_{\mathcal{F}_1 X})[e, \xi^1 \circ (\xi^2)^{-1}]$$

$$= \frac{1}{n_X} (\Phi^* \circ \Phi)[e, \xi^1 \circ (\xi^2)^{-1}],$$

where

$$\Phi = \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} \phi^* \circ r_{\mathcal{F}_1 X}.$$

Then

$$\Phi^* \circ \Phi : \underline{e} \to \underline{e},$$

when viewed as a constant, is nonnegative. But $\phi \neq 0 \Rightarrow \Phi \neq 0$. Proof: $\bar{\phi}$ is the unique element of $\mathrm{Mor}(\mathcal{F}_2\bar{X}, \mathcal{F}_1\bar{X})$ such that

$$\bar{\phi} \underline{\otimes} \mathrm{id}_{\mathcal{F}_2 X} \circ r_{\mathcal{F}_2 X} = \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} \phi^* \circ r_{\mathcal{F}_1 X} \quad \text{(cf. 9.10)},$$

so $\Phi = 0 \Rightarrow \bar{\phi} = 0$

$$\Rightarrow ({}^\vee\phi)^* = 0 \Rightarrow ({}^\vee\phi)^{**} = 0 \Rightarrow {}^\vee\phi = 0 \Rightarrow \phi = 0.$$

[Note: To justify the equation

$$\bar{\phi} \underline{\otimes} \phi \circ r_{\mathcal{F}_2 X} = \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} (\phi \circ \phi^*) \circ r_{\mathcal{F}_1 X},$$

write

$$\bar{\phi} \underline{\otimes} \phi = \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} \phi \circ \bar{\phi} \underline{\otimes} \mathrm{id}_{\mathcal{F}_2 X}.$$

Then

$$\begin{aligned}\bar{\phi} \underline{\otimes} \phi \circ r_{\mathcal{F}_2 X} &= \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} \phi \circ \bar{\phi} \underline{\otimes} \mathrm{id}_{\mathcal{F}_2 X} \circ r_{\mathcal{F}_2 X} \\ &= \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} \phi \circ \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} \phi^* \circ r_{\mathcal{F}_1 X} \\ &= \mathrm{id}_{\mathcal{F}_1 \bar{X}} \circ \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} (\phi \circ \phi^*) \circ r_{\mathcal{F}_1 X} \\ &= \mathrm{id}_{\mathcal{F}_1 \bar{X}} \underline{\otimes} (\phi \circ \phi^*) \circ r_{\mathcal{F}_1 X}.]\end{aligned}$$

Given $A, B \in A(\mathcal{F}_1, \mathcal{F}_2)$, let

$$\langle A, B \rangle = \omega(A^* B).$$

Then $\langle\ ,\ \rangle$ equips $A(\mathcal{F}_1, \mathcal{F}_2)$ with the structure of a pre-Hilbert space w.r.t. which the left multiplication operators

$$A(\mathcal{F}_1, \mathcal{F}_2) \to A(\mathcal{F}_1, \mathcal{F}_2)$$

are continuous. Denoting by $H(\mathcal{F}_1, \mathcal{F}_2)$ the Hilbert space completion of $A(\mathcal{F}_1, \mathcal{F}_2)$, it thus follows that $A(\mathcal{F}_1, \mathcal{F}_2)$ admits a faithful $*$-representation

$$L: A(\mathcal{F}_1, \mathcal{F}_2) \to B(H(\mathcal{F}_1, \mathcal{F}_2)),$$

hence $A(\mathcal{F}_1, \mathcal{F}_2)$ admits a C^*-norm as claimed in Step 3.

§12. *THE INTRINSIC GROUP*

Let $\underline{C}$ be a tannakian category and suppose that

$$\mathcal{F}:\underline{C} \to \underline{\text{FDHILB}}$$

is a fiber functor -- then its <u>intrinsic group</u> $G_{\mathcal{F}}$ is the group of unitary monoidal natural transformations $\alpha:\mathcal{F} \to \mathcal{F}$, i.e., in the notation of §11,

$$G_{\mathcal{F}} = \text{Mor}(\mathcal{F},\mathcal{F}),$$

where $\text{Mor}(\mathcal{F},\mathcal{F})$ is computed in $\text{ff}^*(\underline{C})$.

So

$$G_{\mathcal{F}} \subset \prod_{X \in \text{Ob } \underline{C}} \mathcal{U}(\mathcal{F}X),$$

$\mathcal{U}(\mathcal{F}X)$ the compact group of unitary operators $\mathcal{F}X \to \mathcal{F}X$. And $G_{\mathcal{F}}$ is closed if

$$\prod_{X \in \text{Ob } \underline{C}} \mathcal{U}(\mathcal{F}X)$$

is equipped with the product topology, thus $G_{\mathcal{F}}$ is a compact group.

<u>N.B.</u> Define

$$\pi_X:G_{\mathcal{F}} \to \mathcal{U}(\mathcal{F}X)$$

by $\pi_X(\alpha) = \alpha_X$ -- then

$$(\pi_X,\mathcal{F}X) \in \text{Ob } \underline{\text{Rep}}\ G_{\mathcal{F}}.$$

12.1 <u>LEMMA</u> $\exists$ a faithful symmetric monoidal $*$-preserving functor $\Phi:\underline{C} \to \underline{\text{Rep}}\ G_{\mathcal{F}}$ such that $U \circ \Phi = \mathcal{F}$, where

$$U:\underline{\text{Rep}}\ G_{\mathcal{F}} \to \underline{\text{FDHILB}}$$

is the forgetful functor.

PROOF Define Φ on objects by

$$\Phi X = (\pi_X, \mathcal{F}X)$$

and on morphisms $f:X \to Y$ by $\Phi f = \mathcal{F}f$ (cf. 8.2) and take for ξ, Ξ the corresponding entities per $\mathcal{F}$. To see that this makes sense for Ξ say, one must check that $\Xi_{X,Y}$ is a morphism in Rep $G_{\mathcal{F}}$, viz.:

$$\Xi_{X,Y} \circ (\pi_X(\alpha) \underline{\otimes} \pi_Y(\alpha)) = \pi_{X \otimes Y}(\alpha) \circ \Xi_{X,Y}.$$

But this is obvious since the diagram

$$\begin{array}{ccc} \mathcal{F}X \underline{\otimes} \mathcal{F}Y & \xrightarrow{\Xi_{X,Y}} & \mathcal{F}(X \otimes Y) \\ {\scriptstyle \alpha_X \underline{\otimes} \alpha_Y} \downarrow & & \downarrow {\scriptstyle \alpha_{X \otimes Y}} \\ \mathcal{F}X \underline{\otimes} \mathcal{F}Y & \xrightarrow[\Xi_{X,Y}]{} & \mathcal{F}(X \otimes Y) \end{array}$$

commutes. That Φ is symmetric is equally clear.

More is true: Φ is an equivalence of categories. Because Φ is faithful, it remains to establish that Φ is full and has a representative image (details below).

12.2 REMARK The category $\underline{\text{Rep}}_{\text{fd}}\, A_{\mathcal{F}}$ is a semisimple symmetric monoidal $\star$-category which can be shown to have conjugates, thus $\underline{\text{Rep}}_{\text{fd}}\, A_{\mathcal{F}}$ is "almost" tannakian. Specializing 8.14, it was pointed out in 8.16 that the "Φ" defined there is a symmetric monoidal equivalence $\underline{C} \to \underline{\text{Rep}}_{\text{fd}}\, A_{\mathcal{F}}$. Denote now by $\underline{\text{Rep}}_{\text{fd}}\, G_{\mathcal{F}}$ the category whose objects are the finite dimensional continuous representations of $G_{\mathcal{F}}$ and whose morphisms are the intertwining operators — then the inclusion functor

$$\underline{\text{Rep}}\ G_{\mathcal{F}} \to \underline{\text{Rep}}_{\text{fd}}\ G_{\mathcal{F}}$$

is an equivalence. On the other hand, there is a canonical functor

$$\underline{\text{Rep}}_{\text{fd}}\ A_{\mathcal{F}} \to \underline{\text{Rep}}_{\text{fd}}\ G_{\mathcal{F}}$$

and it too is an equivalence (a nontrivial fact).

12.3 <u>LEMMA</u> If $X \in \text{Ob}\ \underline{C}$ is irreducible, then the complex linear span of the $\pi_X(\alpha)$ $(\alpha \in G_{\mathcal{F}})$ is dense in $B(\mathcal{F}X)$.

12.4 <u>LEMMA</u> If $X,Y \in \text{Ob}\ \underline{C}$ are irreducible and nonisomorphic, then the complex linear span of the $\pi_X(\alpha) \oplus \pi_Y(\alpha)$ $(\alpha \in G_{\mathcal{F}})$ is dense in $B(\mathcal{F}X) \oplus B(\mathcal{F}Y)$.

12.5 <u>REMARK</u> If $X_1,\ldots,X_n$ are distinct elements of $I_{\underline{C}}$, then the complex linear span of the

$$\pi_{X_1}(\alpha) \oplus \cdots \oplus \pi_{X_n}(\alpha) \qquad (\alpha \in G_{\mathcal{F}})$$

is dense in

$$B(\mathcal{F}X_1) \oplus \cdots \oplus B(\mathcal{F}X_n).$$

To prove that Φ is full, we shall appeal to 7.9.

(a) X irreducible => ΦX irreducible. In fact, thanks to 12.3, the only $T \in B(\mathcal{F}X)$ that intertwine the $\pi_X(\alpha)$ $(\alpha \in G_{\mathcal{F}})$ are the scalar multiples of the identity.

(b) X,Y irreducible and nonisomorphic => $\Phi X, \Phi Y$ irreducible and nonisomorphic.

For suppose that $T:\mathcal{F}X \rightarrow \mathcal{F}Y$ intertwines π_X and π_Y, thus $T\pi_X(\alpha) = \pi_Y(\alpha)T$ $(\alpha \in G_{\mathcal{F}})$. But then $Tu = vT$ for all $u \in B(\mathcal{F}X)$, $v \in B(\mathcal{F}Y)$ (cf. 12.4). Now take $u = 0$, $v = 1$ to conclude that $T = 0$, hence $\Phi X, \Phi Y$ are nonisomorphic.

The final claim is that Φ has a representative image. To see this, consider the map

$$\gamma_{\mathcal{F}}:I_{\underline{C}} \rightarrow I_{\underline{\text{Rep}}\, G_{\mathcal{F}}}$$

defined by the rule

$$\gamma_{\mathcal{F}}(X_i) = (\pi_{X_i}, \mathcal{F}X_i).$$

Then $\gamma_{\mathcal{F}}$ is injective.

12.6 <u>LEMMA</u> $\gamma_{\mathcal{F}}$ is surjective.

<u>PROOF</u> The complex linear span of the matrix elements of the π_{X_i} as i ranges over $I_{\underline{C}}$ is a unital $\star$-subalgebra of $C(G_{\mathcal{F}})$ which separates the points of $G_{\mathcal{F}}$, thus is dense in $C(G_{\mathcal{F}})$. Accordingly, there can be no irreducible object in $\underline{\text{Rep}}\, G_{\mathcal{F}}$ which is not unitarily equivalent to a π_{X_i} for some i, so $\gamma_{\mathcal{F}}$ is surjective.

Therefore $\gamma_{\mathcal{F}}$ is bijective and Φ has a representative image.

12.7 <u>REMARK</u> Suppose that

$$\begin{cases} \mathcal{F}_1:\underline{C} \rightarrow \underline{\text{FDHILB}} \\ \mathcal{F}_2:\underline{C} \rightarrow \underline{\text{FDHILB}} \end{cases}$$

are fiber functors — then as objects of $\text{ff}^*(\underline{C})$, $\mathcal{F}_1, \mathcal{F}_2$ are isomorphic (cf. 11.3), so $G_{\mathcal{F}_1}, G_{\mathcal{F}_2}$ are isomorphic (in the category <u>CPTGRP</u>).

Let G be a compact group -- then the forgetful functor

$$U:\underline{\text{Rep}}\ G \to \underline{\text{FDHILB}}$$

is a fiber functor. Define a map $\Gamma:G \to G_U$ by sending $\sigma \in G$ to the string

$$\{\pi(\sigma):(\pi,H_\pi) \in \text{Ob}\ \underline{\text{Rep}}\ G\}.$$

That this is meaningful follows upon noting that if

$$\left[\begin{array}{l} (\pi_1,H_{\pi_1}) \\ \\ (\pi_2,H_{\pi_2}) \end{array}\right. \in \text{Ob}\ \underline{\text{Rep}}\ G,$$

then

$$\forall\ T \in \text{Mor}((\pi_1,H_{\pi_1}),(\pi_2,H_{\pi_2}))$$

there is a commutative diagram

$$\begin{array}{ccc} H_{\pi_1} & \xrightarrow{\pi_1(\sigma)} & H_{\pi_1} \\ T \downarrow & & \downarrow T \\ H_{\pi_2} & \xrightarrow[\pi_2(\sigma)]{} & H_{\pi_2}, \end{array}$$

thus the string

$$\{\pi(\sigma):(\pi,H_\pi) \in \text{Ob}\ \underline{\text{Rep}}\ G\}$$

defines an element

$$\alpha(\sigma) \in \text{Mor}(U,U),$$

where technically

$$\alpha(\sigma)_{(\pi,\mathcal{H}_\pi)} = \pi(\sigma).$$

12.8 LEMMA Γ is a continuous injective homomorphism.

[This is immediate from the definitions.]

In fact, Γ is surjective, hence G and G_U are isomorphic.

[If Γ were not surjective, replace G by ΓG and think of G as a proper closed subgroup of G_U -- then there would be an irreducible representation of G_U that contains a nonzero vector invariant under G but not under G_U. This, however, is impossible:

$$\gamma_U : I_{\underline{\text{Rep}}\, G} \to I_{\underline{\text{Rep}}\, G_U}$$

is bijective.]

12.9 THEOREM Up to isomorphism in CPTGRP, G is the "intrinsic group" of Rep G.

[If

$$\mathcal{F}:\underline{\text{Rep}}\, G \to \underline{\text{FDHILB}}$$

is a fiber functor, then $G_{\mathcal{F}} \approx G_U$ (cf. 12.7).]

12.10 REMARK Compact groups G,G' are said to be isocategorical if Rep G, Rep G' are equivalent as monoidal categories. In general, this does not mean that Rep G, Rep G' are equivalent as symmetric monoidal categories and G,G' may very well be isocategorical but not isomorphic.

§13. *CLASSICAL THEORY*

A character of a commutative unital C*-algebra A is a nonzero homomorphism $\omega: A \to \mathbf{C}$ of algebras. The set of all characters of A is called the structure space of A and is denoted by $\Delta(A)$.

N.B. We have

$$\begin{cases} \Delta(A) = \emptyset & (A = \{0\}) \\ \Delta(A) \neq \emptyset & (A \neq \{0\}). \end{cases}$$

13.1 LEMMA Let $\omega \in \Delta(A)$ -- then ω is necessarily bounded. In fact,

$$||\omega|| = 1 = \omega(1_A).$$

N.B. The elements of $\Delta(A)$ are the pure states of A, hence, in particular, are *-homomorphisms: $\forall \ A \in A$,

$$\omega(A^*) = \overline{\omega(A)}.$$

Given $A \in A$, define

$$\hat{A}: \Delta(A) \to \mathbf{C}$$

by

$$\hat{A}(\omega) = \omega(A).$$

Equip $\Delta(A)$ with the initial topology determined by the $\hat{A}$, i.e., equip $\Delta(A)$ with the relativised weak* topology.

13.2 LEMMA $\Delta(A)$ is a compact Hausdorff space.

If X is a compact Hausdorff space, then C(X) equipped with the supremum norm

$$||f|| = \sup_{x \in X} |f(x)|$$

and involution

$$f^*(x) = \overline{f(x)}$$

is a commutative unital C*-algebra. Moreover, $\forall\, x \in X$, the Dirac measure $\delta_x \in \Delta(C(X))$ and the arrow

$$\begin{cases} X \to \Delta(C(X)) \\ x \to \delta_x \end{cases}$$

is a homeomorphism.

13.3 <u>LEMMA</u> $\hat{A} \in C(\Delta(A))$ and the arrow

$$\begin{cases} A \to C(\Delta(A)) \\ A \to \hat{A} \end{cases}$$

is a unital *-isomorphism.

<u>N.B.</u> If $A = \{0\}$, then $\Delta(A) = \emptyset$ and there is exactly one map $\emptyset \to \mathbf{C}$, namely the empty function ($\emptyset = \emptyset \times \mathbf{C}$), which we shall take to be $\hat{0}$.

Notation: Let <u>CPTSP</u> be the category whose objects are the compact Hausdorff spaces and whose morphisms are the continuous functions.

Notation: Let <u>COMUNC*ALG</u> be the category whose objects are the commutative unital C*-algebras and whose morphisms are the unital *-homomorphisms.

Let X and Y be compact Hausdorff spaces. Suppose that $\phi: X \to Y$ is a continuous function — then ϕ induces a unital *-homomorphism

$$\phi^*: C(Y) \to C(X),$$

viz. $\phi^*(f) = f \circ \phi$. Therefore the association that sends X to C(X) defines a cofunctor

$$C:\underline{\text{CPTSP}} \to \underline{\text{COMUNC*ALG}}.$$

Let A and B be commutative unital C*-algebras. Suppose that $\Phi:A \to B$ is a unital *-homomorphism — then Φ induces a continuous function

$$\Phi^*:\Delta(B) \to \Delta(A),$$

viz. $\Phi^*(\omega) = \omega \circ \Phi$. Therefore the association that sends A to $\Delta(A)$ defines a cofunctor

$$\Delta:\underline{\text{COMUNC*ALG}} \to \underline{\text{CPTSP}}.$$

13.4 <u>THEOREM</u> The category <u>CPTSP</u> is coequivalent to the category <u>COMUNC*ALG</u>.

<u>PROOF</u> Define

$$\Xi_X:X \to \Delta(C(X))$$

by the rule $\Xi_X(x) = \delta_x$ — then Ξ_X is a homeomorphism and there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \Xi_X\ } & \Delta(C(X)) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi^{**}} \\ Y & \xrightarrow[\ \Xi_Y\ ]{} & \Delta(C(Y)). \end{array}$$

Define

$$\Xi_A:A \to C(\Delta(A))$$

by the rule $\Xi_A(A) = \hat{A}$ — then Ξ_A is a unital *-isomorphism and there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\Xi_A} & C(\Delta(A)) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi^{**}} \\ B & \xrightarrow[\Xi_B]{} & C(\Delta(B)). \end{array}$$

Therefore

$$\begin{cases} \text{id} \approx \Delta \circ C \\ \text{id} \approx C \circ \Delta. \end{cases}$$

The category <u>CPTSP</u> has finite products with final object $\{*\}$. Therefore the category <u>COMUNC*ALG</u> has finite coproducts with initial object $\mathbf{C}$. To explicate the latter, invoke the nuclearity of the objects of <u>COMUNC*ALG</u>, thus

$$A \otimes_{\max} B = A \otimes_{\min} B,$$

call it $A \underline{\otimes} B$ — then

$$A \coprod B = A \underline{\otimes} B$$

and there are arrows

$$\begin{cases} A \to A \underline{\otimes} B \\ A \to A \otimes 1_B, \end{cases} \qquad \begin{cases} B \to A \underline{\otimes} B \\ B \to 1_A \otimes B. \end{cases}$$

13.5 <u>EXAMPLE</u> We have

$$\begin{cases} C(\{*\}) \approx \mathbf{C} \quad \text{and} \quad C(X \times Y) \approx C(X) \underline{\otimes} C(Y) \\ \Delta(\mathbf{C}) \approx \{*\} \quad \text{and} \quad \Delta(A \underline{\otimes} B) \approx \Delta(A) \times \Delta(B). \end{cases}$$

13.6 <u>REMARK</u> Let $\mathcal{A}$ be a commutative unital C*-algebra — then the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$ can be viewed as an involutive subalgebra of $\mathcal{A} \underline{\otimes} \mathcal{A}$. Another point is this: Since $\mathcal{A} \underline{\otimes} \mathcal{A}$ is the coproduct, there is a canonical arrow $\mathcal{A} \underline{\otimes} \mathcal{A} \xrightarrow{m} \mathcal{A}$ with $m(A \otimes B) = AB$, i.e., the restriction of m to $\mathcal{A} \otimes \mathcal{A}$ is the multiplication in $\mathcal{A}$.

[Note: If $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}$ are commutative unital C*-algebras and if

$$\begin{cases} \Phi_1 : \mathcal{A}_1 \to \mathcal{B} \\ \Phi_2 : \mathcal{A}_2 \to \mathcal{B} \end{cases}$$

are unital *-homomorphisms, then the diagram

$$\begin{array}{ccccc} \mathcal{A}_1 & \longrightarrow & \mathcal{A}_1 \underline{\otimes} \mathcal{A}_2 & \longleftarrow & \mathcal{A}_2 \\ {\scriptstyle \Phi_1}\downarrow & & & & \downarrow{\scriptstyle \Phi_2} \\ \mathcal{B} & & = = = = = & & \mathcal{B} \end{array}$$

admits a unique filler

$$\Phi_1 \underline{\otimes} \Phi_2 : \mathcal{A}_1 \underline{\otimes} \mathcal{A}_2 \to \mathcal{B}$$

such that

$$(\Phi_1 \underline{\otimes} \Phi_2)(A_1 \otimes A_2) = \Phi_1(A_1)\Phi_2(A_2) \quad (A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2).]$$

13.7 <u>RAPPEL</u> Let $\underline{C}$ be a category with finite products and final object T -- then a <u>group object</u> in $\underline{C}$ consists of an object G and morphisms

$$\mu : G \times G \to G,\ \eta : T \to G,\ \iota : G \to G$$

such that the following diagrams commute:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times id_G} & G \times G \\ {\scriptstyle id_G \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G, \end{array}$$

$$\begin{array}{ccc} G \times T & \xrightarrow{id_G \times \eta} & G \times G \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \mu} \\ G & = & G, \end{array} \qquad \begin{array}{ccc} T \times G & \xrightarrow{\eta \times id_G} & G \times G \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ G & = & G, \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{!} & T \\ {\scriptstyle (id_G,\iota)}\downarrow & & \downarrow{\scriptstyle \eta} \\ G \times G & \xrightarrow[\mu]{} & G, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{!} & T \\ {\scriptstyle (\iota,id_G)}\downarrow & & \downarrow{\scriptstyle \eta} \\ G \times G & \xrightarrow[\mu]{} & G. \end{array}$$

There are obvious definitions of internal group homomorphism $G \to G'$, composition of internal group homomorphisms $G \to G'$, $G' \to G''$, and the identity internal group homomorphism $id_G:G \to G$. Accordingly, there is a category $\underline{GRP}(\underline{C})$ whose objects are the group objects in $\underline{C}$ and whose morphisms are the internal group homomorphisms.

[Note: If instead $\underline{C}$ is a category with finite coproducts and initial object I, then we put

$$\underline{COGRP}(\underline{C}) = \underline{GRP}(\underline{C}^{OP})^{OP}$$

and call the objects the cogroup objects in $\underline{C}$ and the morphisms the internal co-group homomorphisms.]

13.8 EXAMPLE Take $\underline{C} = \underline{SET}$ -- then

$$\underline{GRP}(\underline{SET}) = \underline{GRP}.$$

13.9 LEMMA We have

$$\underline{\mathrm{GRP}}(\underline{\mathrm{CPTSP}}) = \underline{\mathrm{CPTGRP}}.$$

13.10 REMARK The forgetful functor

$$\underline{\mathrm{CPTGRP}} \rightarrow \underline{\mathrm{SET}}$$

has a left adjoint. Proof: Given a set X, equip it with the discrete topology, form the associated free topological group $F_{gr}(X)$, and consider its Bohr compactification.

A commutative Hopf C*-algebra is commutative unital C*-algebra H together with unital *-homomorphisms

$$\Delta: H \rightarrow H \underline{\otimes} H,\ \epsilon: H \rightarrow \mathbb{C},\ S: H \rightarrow H$$

for which the following diagrams commute:

$$\begin{array}{ccc} H & \xrightarrow{\ \Delta\ } & H \underline{\otimes} H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle id_H \underline{\otimes} \Delta} \\ H \underline{\otimes} H & \xrightarrow[\Delta \underline{\otimes} id_H]{} & H \underline{\otimes} H \underline{\otimes} H, \end{array}$$

$$\begin{array}{ccc} H & = & H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle in_1} \\ H \underline{\otimes} H & \xrightarrow[id_H \underline{\otimes} \epsilon]{} & H \underline{\otimes} \mathbb{C}, \end{array} \qquad \begin{array}{ccc} H & = & H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle in_2} \\ H \underline{\otimes} H & \xrightarrow[\epsilon \underline{\otimes} id_H]{} & \mathbb{C} \underline{\otimes} H, \end{array}$$

$$\begin{array}{ccc} H & \xleftarrow{\;\;!\;\;} & C \\ {\scriptstyle (\mathrm{id}_H,S)}\uparrow & & \uparrow{\scriptstyle \varepsilon} \\ H \underline{\otimes} H & \xleftarrow[\Delta]{} & H, \end{array} \qquad \begin{array}{ccc} H & \xleftarrow{\;\;!\;\;} & C \\ {\scriptstyle (S,\mathrm{id}_H)}\uparrow & & \uparrow{\scriptstyle \varepsilon} \\ H \underline{\otimes} H & \xleftarrow[\Delta]{} & H. \end{array}$$

[Note: Such an H is not necessarily a Hopf algebra (in general, Δ takes values in $H \underline{\otimes} H$ rather than $H \otimes H$).]

<u>N.B.</u> Consider, e.g., (id_H,S) -- then in terms of the coproduct diagram

$$H \xrightarrow{\mathrm{in}_1} H \underline{\otimes} H \xleftarrow{\mathrm{in}_2} H,$$

the arrow

$$(\mathrm{id}_H,S):H \underline{\otimes} H \to H$$

is characterized by the condition that

$$\begin{cases} (\mathrm{id}_H,S) \circ \mathrm{in}_1 = \mathrm{id}_H \\ (\mathrm{id}_H,S) \circ \mathrm{in}_2 = S. \end{cases}$$

On the other hand, there is an arrow

$$\mathrm{id}_H \underline{\otimes} S:H \underline{\otimes} H \to H \underline{\otimes} H$$

characterized by the condition that

$$\begin{cases} \mathrm{id}_H \otimes S \circ \mathrm{in}_1 = \mathrm{in}_1 \circ \mathrm{id}_H \\ \mathrm{id}_H \otimes S \circ \mathrm{in}_2 = \mathrm{in}_2 \circ S \end{cases} \qquad \text{(cf. 13.6).}$$

And

$$m \circ \mathrm{id}_H \underline{\otimes} S = (\mathrm{id}_H,S).$$

Proof:

$$\begin{cases} m \circ \mathrm{id}_H \underline{\otimes} S \circ \mathrm{in}_1 = m \circ \mathrm{in}_1 \circ \mathrm{id}_H = \mathrm{id}_H \circ \mathrm{id}_H = \mathrm{id}_H \\ m \circ \mathrm{id}_H \underline{\otimes} S \circ \mathrm{in}_2 = m \circ \mathrm{in}_2 \circ S = \mathrm{id}_H \circ S = S. \end{cases}$$

Denote by COMHOPFC*ALG the category whose objects are the commutative Hopf C*-algebras and whose morphisms $f:H \to H'$ are the unital $\star$-homomorphisms such that $f \underline{\otimes} f \circ \Delta = \Delta' \circ f$, $\varepsilon = \varepsilon' \circ f$, $f \circ S = S' \circ f$.

13.11 LEMMA We have

COGRP(COMUNC*ALG) = COMHOPFC*ALG.

Let G be a compact group -- then the group operations in G induce operations Δ, ε, S in C(G) w.r.t. which C(G) acquires the structure of a commutative Hopf C*-algebra. And the association that sends G to C(G) defines a cofunctor

C:CPTGRP → COMHOPFC*ALG.

Let H be a commutative Hopf C*-algebra -- then the cogroup operations in H induce operations μ, η, ι in Δ(H) w.r.t. which Δ(H) acquires the structure of a compact group. And the association that sends H to Δ(H) defines a cofunctor

Δ:COMHOPFC*ALG → CPTGRP.

13.12 THEOREM The category CPTGRP is coequivalent to the category COMHOPFC*ALG (cf. 13.4).

13.13 RAPPEL Given a compact group G, let A(G) be its set of representative functions -- then A(G) is a unital $\star$-subalgebra of C(G) and when endowed with the restrictions of Δ, ε, S forms a commutative Hopf $*$-algebra.

[Note: Recall that $A(G)$ is dense in $C(G)$.]

- Let $\Delta(A(G))$ be the set of nonzero multiplicative linear functionals on $A(G)$.

- Let $\Delta^*(A(G))$ be the set of $*$-preserving nonzero multiplicative linear functionals on $A(G)$.

Then

$$\Delta^*(A(G)) \subset \Delta(A(G))$$

and the containment is proper in general.

Equip $\Delta(A(G))$ (and hence $\Delta^*(A(G))$) with the topology of pointwise convergence and introduce the following operations:

$$\text{(i)}\ (\omega_1 \cdot \omega_2) = (\omega_1 \underline{\otimes} \omega_2) \circ \Delta;\ \text{(ii)}\ 1_{A(G)} = \varepsilon;\ \text{(iii)}\ \omega^{-1} = \omega \circ S.$$

Then $\Delta(A(G))$ is a group containing $\Delta^*(A(G))$ as a subgroup (in this connection, note that $\Delta(f^*) = \Delta(f)^*$ and $S(f^*) = S(f)^*$).

13.14 <u>LEMMA</u> $\Delta^*(A(G))$ is a compact group.

13.15 <u>THEOREM</u> Define

$$ev: G \to \Delta^*(A(G))$$

by

$$ev(\sigma) = \delta_\sigma \quad (\delta_\sigma(f) = f(\sigma)).$$

Then ev is an isomorphism in <u>CPTGRP</u>.

Let

$$U:\underline{\text{Rep}}\ G \rightarrow \underline{\text{FDHILB}}$$

be the forgetful functor.

13.16 <u>LEMMA</u> The arrow

$$\rho:A(U,U) \rightarrow A(G)$$

that sends $[H_\pi,\phi]$ $(\phi:H_\pi \rightarrow H_\pi)$ to the representative function

$$\sigma \rightarrow \text{tr}(\pi(\sigma)\phi) \qquad (\sigma \in G)$$

is a linear bijection.

[Note: This can be sharpened in that $A(U,U)$ carries a canonical Hopf algebra structure which is preserved by ρ, i.e., ρ is an isomorphism of Hopf algebras.]