# Sketches:
# Outline with References
# by Charles Wells
# Addendum 15 September 2009

This package contains the original article, written in December, 1993, and this addendum, which lists a few references to papers and books that have been written since 1993.

**Additional and Updated References**

Atish Bagchi and Charles Wells, *Graph Based Logic and Sketches.* May be downloaded from http://arxiv.org/PS_cache/arxiv/pdf/0809/0809.3023v1.pdf

Michael Barr and Charles Wells, **Toposes, Triples and Theories**. Revised and corrected edition now available online at http://www.case.edu/artsci/math/wells/pub/pdf/ttt.pdf

Michael Barr and Charles Wells, **Category Theory for Computing Science**, third edition (1999). Les Publications CRM, Montreal.

Zinovy Diskin and Uwe Wolter, *A diagrammatic logic for object-oriented visual modeling.* Electronic Notes in Theoretical Computer Science (2006). May be downloaded from http://www.cs.toronto.edu/ zdiskin/Pubs/ACCAT-07.pdf

René Guitart, *The theory of sketches.* Journées Faiscaux et Logique (1981). Online at http://pagesperso-orange.fr/rene.guitart/textespublications/guitart81theosketches.pdf

Peter Johnstone, **Sketches of an Elephant**, Vol. 2, Chapter D2. Oxford University Press (2003).

Yoshiki Kinoshita, John Power, and Makoto Takeyama. *Sketches.* In **Mathematical Foundations of Programming Semantics, Thirteenth Annual Conference**, Stephen Brookes and Michael W. Mislove, editors. Elsevier (1997).

Michael Makkai, *Generalized sketches as a framework for completeness theorems.* Journal of Pure and Applied Algebra 115 (1997): 49-79, 179-212, 214-274.

# Sketches:
## Outline with References*

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# 1 Introduction

### 1.1 Purpose

This document is an outline of the theory of sketches with pointers to the literature. An extensive bibliography is given. Some coverage is given to related areas such as algebraic theories, categorial model theory and categorial logic as well. An appendix beginning on page 14 provides definitions of some of the less standard terms used in the paper, but the reader is expected to be familiar with the basic ideas of category theory. A rough machine generated index begins on page 27.

I would have liked to explain the main ideas of all the papers referred to herein, but I am not familiar enough with some of them to do that. It seemed more useful to be inclusive, even if many papers were mentioned without comment. One consequence of this is that the discussions in this document often go into more detail about the papers published in North America than about those published elsewhere.

The DVI file for this article is available by anonymous FTP from `ftp.cwru.edu` in the directory `math/wells`. The BibTeX source for the bibliography is in a file in the same directory called `sketch.bib`.

#### 1.1.1 Addendum 8 December 1993

This version of this document contains a number of additions and corrections pointed out by M. Barr, C. Lair and P. Agéron and some I discovered myself. I do not intend to produce further revisions of this document. I understand that another, much more extensive document concerning sketches is in preparation, and I expect that the forthcoming text by Adámek and Rosičky[1994] will have a useful guide to the literature on categorical logic.

I will post any corrections anyone sends me in a file available by FTP in the directory mentioned above.

### 1.2 Terminology

This outline uses the terminology for sketches given in [Barr and Wells, 1990], Chapters 4, 7, 9 and 10. It is quite different from the usage of the French school beginning with Charles

Ehresmann [1968b], the inventor of sketches. The French usage is explained wherever there is a difference, but the terminology from [Barr and Wells, 1990] is the one used in discussions.

# 2 Sketches

In this section, sketches in the standard sense are defined. Generalizations are discussed in Section 9. An excellent short outline of sketch theory using North American terminology may be found in [Makkai and Paré, 1990], Chapter 3. More details are in [Barr and Wells, 1985] and [Barr and Wells, 1990]. The best source for the French version is by Coppey and Lair [1984], [1988].

## 2.1 Sketches

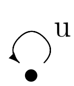
### 2.1.1 Definition

A **sketch** $\mathsf{S}$ consists of a graph $G_\mathsf{S}$, a set $D_\mathsf{S}$ of diagrams in $G_\mathsf{S}$, a set $L_\mathsf{S}$ of cones in $G_\mathsf{S}$ and a set $C_\mathsf{S}$ of cocones in $G_\mathsf{S}$.

Graphs, cones, cocones and other technical terms are defined in the Appendix, page 14. The phrase "distinguished diagram of $\mathsf{S}$" means a diagram in $D_\mathsf{S}$, and analogously for cones and cocones.

**2.1.2 Variations in terminology** The French replace the sets $G_\mathsf{S}$ and $D_\mathsf{S}$ with a compositive graph $M_\mathsf{S}$. The cones and cocones must be commutative cones and cocones in $M_\mathsf{S}$. These approaches are clearly equivalent.

Some North American variations: Barr and Wells [1985] add a function that specifies which arrows must become identities in a model. This was abandoned in [Barr and Wells, 1990] because one can force an arrow $u$ to be an identity by including the diagram




Makkai and Paré [1990] use "commutativity conditions" — pairs of paths in the graph with common source and target — instead of diagrams.

Many authors take the compositive graph to be simply a category. This is in fact not a restriction (Section 5).

## 2.2 Morphisms of sketches

If $\mathsf{S}$ and $\mathsf{S}'$ are sketches, a graph homomorphism $f : G_\mathsf{S} \to G_{\mathsf{S}'}$ is a **morphism of sketches** if it takes each diagram of $\mathsf{S}$ to a diagram of $\mathsf{S}'$, each cone of $\mathsf{S}$ to a cone of $\mathsf{S}'$, and each cocone of $\mathsf{S}$ to a cocone of $\mathsf{S}'$. This produces the category $\mathsf{Sk}$ of sketches.

Each small category $\mathcal{C}$ has an **underlying sketch** whose graph is the underlying graph of $\mathcal{C}$, whose diagrams are all the commutative diagrams in $\mathcal{C}$, whose cones are all the limit cones in $\mathcal{C}$ and whose cocones are all the limit cocones in $\mathcal{C}$.

## 2.3 Models of sketches

Let $\mathsf{S}$ be a sketch and $\mathcal{C}$ a category. $M : \mathsf{S} \to \mathcal{C}$ is a **model of $\mathsf{S}$ in $\mathcal{C}$** if $M$ is a sketch morphism from $\mathsf{S}$ to the underlying sketch of $\mathcal{C}$. If $M$ and $M'$ are two models of $\mathsf{S}$ in $\mathcal{C}$, $\mu : M \to M'$ is a **homomorphism of models** if $\mu$ is a natural transformation from $M$ to $M'$.

This produces a **category of models** of the sketch $\mathsf{S}$ in $\mathcal{C}$, using vertical composition of natural transformations as the composition.

This category is denoted $\mathrm{Mod}_{\mathcal{C}}(\mathsf{S})$. The category of models of $\mathsf{S}$ in the category of sets is denoted $\mathrm{Mod}(\mathsf{S})$, and normally the phrase "model of $\mathsf{S}$" without qualification means a model in the category of sets.

A model of a sketch in the category of sets associates a set with each node of the graph of the sketch, so that the nodes of the graph specify the *sorts* of the structure. The arrows correspond to mappings between sets that are values of the sorts, so they specify the *operations* of the structure. The cones and cocones formally specify *constructed sorts*. For example, a discrete cone whose diagram has $n$ nodes specifies a set of $n$-tuples, with the $i$th entry drawn from the value of the $i$th node in the diagram. General limit diagrams specify equationally defined subsorts, and colimit diagrams specify quotient structures (coequalizers), free products (coproducts), and amalgamated products (pushouts).

### 2.4 Remarks

Sketches were invented by Ehresmann to provide a mathematical way to specify a species of mathematical structure. For example, there is a sketch for groups; its models "are" groups and the homomorphisms between its models are exactly the group homomorphisms. There is a clear analogy with the way a path in a topological space is defined as a continuous map from a closed interval to the space. (And a loop is a map from the unit circle, and so on.) Of course, the traditional techniques of first order logic provide another way of specifying structures, and so does the method of signatures and equations used in universal algebra. More about this in Section 6.

Sketches show their superiority (in my opinion) particularly when you want to deal with multisorted structures, and when you want to deal with models in categories other than sets.

There is a second point of view concerning sketches, that a sketch is a presentation of a category in the usual sense of "presentation". This is discussed in Section 5.

## 3 Kinds of sketches

### 3.1 A listing of types

By restricting the kinds of cones and cocones that occur, we obtain a hierarchy of types of sketches. I list them from the most primitive to the most complex (the listing is not quite a total order).

**3.1.1** A **trivial** sketch consists of a graph only, with no diagrams, cones or cocones. For example, the category of sets and functions is the category of models of the trivial sketch with one node and no arrows, and the category of graphs is equivalent to the category of models of the trivial sketch whose graph is

$$g_1 \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} g_0$$

**3.1.2** A **linear** or **elementary** sketch (the latter is the French name) may have diagrams, but has no cones or cocones. The category of reflexive graphs is given by such a sketch [Barr and Wells, 1990], [Coppey and

Lair, 1988], Leçon 6. Models of linear sketches are algebraic structures whose operations are all unary.

**3.1.3** A **linear sketch with constants** has no cocones and cones only over the empty diagram. Thus they allow one to specify unary operations and also specific elements (nullary operations) in a model. See [Barr and Wells, 1990], Section 4.7.

**3.1.4** A **finite product** sketch, or FP sketch, has no cocones and all the cones are based on finite discrete diagrams. These correspond in expressive power to (multisorted, in general) universal algebras given by (finite) signatures and equations, so their models include well known algebraic structures such as semigroups, groups and rings. They do *not* include fields. See 6.2.1 below.

**3.1.5** A **finite discrete** sketch has only discrete cones and cocones. It is usually required that the models of a finite discrete sketch (and a finite sum sketch – see 3.1.7) be in a category with finite disjoint sums (see [Barr and Wells, 1990], page 219, or any book on topos theory). This is discussed in in Section 5.3. The category of fields is the category of models of a finite discrete sketch.

**3.1.6** A **finite limit** or FL sketch has no cocones and all the cones are based on finite diagrams. These correspond up to equivalence to **essentially algebraic** structures as defined by Freyd [1972]. See the discussion by Makkai and Paré [1990], page 2 (bottom). Finite limit sketches can sketch all structures expressible by Horn theories, and more [Barr, 1989]. Finite limit sketches are called "left exact" sketches in [Barr and Wells, 1985]. Cartmell [1986] provides a different (and illuminating) point of view concerning finite limit sketches, and Reichel [1987] develops an equivalent methodology in the style of universal algebra.

**3.1.7** A **finite sum** sketch or FS sketch has finite cones and finite discrete cocones.

**3.1.8** A **projective** sketch has any kind of cone but no cocones.

**3.1.9** A **regular** sketch has any kind of cone, and cocones that allow one to require an arrow to be an epimorphism in a model. In the literature, models are generally required to be in regular categories (why this is done is discussed in Section 5.3). Note: The French school uses the phrase "regular sketch" for a sketch in which no node is the vertex of more than one cone.

**3.1.10** A **coherent** sketch has finite cones and finite cocones that are either discrete or "regular epi specifications" (see [Makkai and Paré, 1990], page 42). The definition in terms of sieves in [Barr and Wells, 1985], page 294, is equivalent.

**3.1.11** A **geometric** sketch has finite cones and arbitrary cocones. The arbitrary cocones can be replaced by arbitrary discrete cocones and arbitrary sieves, since sieves can be expressed in terms of coequalizers of parallel sets of arrows and those and sums generate all colimits. See 5.3 and 7.

**3.1.12** A **mixed** sketch is the French name for any sketch.

### 3.2 Notation for the French School

The main focus of Ehresmann, Guitart, Lair, Coppey and others of the French school has been on properties of projective sketches and mixed sketches. In their modern papers, they use a systematic notation: $//\underline{\mathsf{S}}//$ denotes a mixed sketch. Its underlying projective sketch is $/\underline{\mathsf{S}}/$, its underlying compositive graph is $\underline{\mathsf{S}}$, and its underlying oriented graph is $\mathsf{S}$.

# 4 Examples and applications of sketches

## 4.1 Algebraic structures

Many examples of sketches of algebraic structures (groups, rings, fields, $M$-sets, ...) are given by Coppey and Lair [1988], Leçon 6. Some algebraic structures are sketched by Barr and Wells [1985] and [1990].

There are two ways of thinking of algebraic structures. I will illustrate them using semigroups as an example.

1. A semigroup is a set $S$ together with a function that takes each ordered pair of elements of $S$ to an element of $S$, such that ....
2. A semigroup consists of a function $m : S \times S \to S$ such that ....

Many mathematicians would say these are exactly the same, since $S \times S$ is the set of ordered pairs of elements of $S$. To a categorist, $S \times S$ is the product of $S$ with itself, so its elements can be *represented* as ordered pairs of elements of $S$, but in fact $S \times S$ is determined only by what you specify its first and second coordinate functions to be together with the universal property of products. For this reason, the category of models of a sketch for semigroups is *equivalent* but not *isomorphic* to the category of semigroups in sense (1) above. This is discussed in more detail in [Barr and Wells, 1990], pages 170–172.

## 4.2 Sketches for categories

The sketch for categories is given in detail in [Barr and Wells, 1990], Section 9.1.4 and in [Coppey and Lair, 1988], page 64. This was first done by Ehresmann [1966], [1968a] and [1968b]

Many types of categories with extra structure are essentially algebraic and can be sketched by an FL sketch (finite limit cones only). These include categories with various types of canonically-chosen limits and colimits, cartesian closed categories, toposes, and others. Constructions of this kind are given in [Lair, 1970], [Burroni, 1970a], [Burroni, 1970b], [Conduche, 1973], [Lair, 1975a], [Lair, 1977b], [Lair, 1979], [Burroni, 1981], [McDonald and Stone, 1984], [Barr and Wells, 1985], [Even and Agéron, 1987], [Coppey and Lair, 1985], [Coppey and Lair, 1988], and [Wells, 1990]. I have not seen many of these papers and so will not try to summarize their contents and relationships. A general framework for sketching structured categories is given by Lair [1987b], and a very different one with many examples by Makkai [1993a].

Barr and Wells [1992] indicate how to use regular sketches to sketch categories with limits and functors that preserve them but not necessarily on the nose.

A sketch for 2-categories is given in [Power and Wells, 1992]. I have been informed that sketches for 2-categories and double categories can be constructed using the monoidal closed structure on the category of sketches given in [Lair, 1975b].

## 4.3 Sketching sketches

Sketches and the morphisms between them can themselves be sketched using FL sketches. These are are discussed in [Burroni, 1970a], [Lair, 1974], [Coppey and Lair, 1984], Leçon 4 and in [Lair, 1987a].

## 4.4 Sketches in computer science

Gray [1989] describes a systematic and general approach to algebraic semantics using sketches. See also [Gray, 1987] and [Gray, 1990]. Duval and Reynaud [1994a], [1994b] introduce a systematic methodology for computing using sketches. In particular, they use finite sum sketches to compute in fields of arbitrary characteristic. See also [Duval and Sénéchaud, 1994]. Sketches and context-free grammars are discussed in [Wells and Barr, 1988] and (without actually mentioning sketches) [Walters, 1988] Examples of mathematical structures used in computer science also occur in the following papers, not all of which actually use sketches although they easily could have. I have not seen some of these papers. [Guitart, 1986], [Guitart, 1988], [Wells and Barr, 1988], [Lellahi, 1989], [MacDonald and Stone, 1990], [Cockett, 1990], [Gray, 1991a] and [Barr and Wells, 1990], Chapters 7, 9 and 10.

There is a large literature concerning algebraic specification of data structures and computer languages. Most of the constructions in that literature can be translated directly into the construction of finite product sketches. Ehrig and Mahr [1985] is a fundamental text in the field. Also, the book by Reichel [1987] develops an alternative formulation of finite limit theories, with many examples drawn from computer science. Other surveys and basic papers are [Goguen *et al.*, 1978], [Dybjer, 1986], [Manes and Arbib, 1986], [Wagner *et al.*, 1985] and [Goguen, 1990]. [Barr and Wells, 1990] has an extensive list of references in this area, updated in [Barr and Wells, 1993b].

Permvall [1991] provides a large bibliography of literature concerning sketches in computer science. They are not all included in this summary.

# 5 The Theory of a sketch

## 5.1 Doctrines

Let $\mathsf{E}$ be a type of sketch, determined by what sorts of cones and cocones are allowed in the sketch. Thus $\mathsf{E}$ could be any of the types listed in Section 3, and many others. Corresponding to each type $\mathsf{E}$ there is a type of category, required to have all limits, respectively colimits, of the type of cones, respectively cocones, allowed by $\mathsf{E}$. Likewise, there is a type of functor, required to preserve that type of limits or colimits.

For example, corresponding to finite product sketches are categories that have all finite products and functors that preserve finite products. Corresponding to projective sketches are categories with all small limits and functors that preserve small limits. Given a type $\mathsf{E}$, we will refer to $\mathsf{E}$-sketches, $\mathsf{E}$-categories and $\mathsf{E}$-functors. Following Lawvere, we will refer to $\mathsf{E}$ as a **doctrine**.[1]

## 5.2 The categorial theory of a sketch

Using the preceding notation, for any doctrine $\mathsf{E}$, every $\mathsf{E}$-sketch $\mathsf{S}$ has an **$\mathsf{E}$-theory**, which is an $\mathsf{E}$-category $\mathsf{Th}_\mathsf{E}(\mathsf{S})$ together with a sketch morphism $H_M : \mathsf{S} \rightarrow \mathsf{Th}_\mathsf{E}(\mathsf{S})$ with the following property: For every $\mathsf{E}$-category $\mathcal{C}$ and every model $M : \mathsf{S} \rightarrow \mathcal{C}$ there is an $\mathsf{E}$-functor $\mathsf{Th}_\mathsf{E}(M) : \mathsf{Th}_\mathsf{E}(\mathsf{S}) \rightarrow \mathcal{C}$, unique up to natural

[1] A doctrine can be a type of category requiring other structure besides limits and colimits – precisely, any type of category definable essentially algebraically over the category of categories.

isomorphism, for which

$$\begin{array}{ccc} \mathsf{S} & \xrightarrow{H_M} & \mathsf{Th}_{\mathsf{E}}(\mathsf{S}) \\ & {}_{M}\searrow & \downarrow \mathsf{Th}_{\mathsf{E}}(M) \\ & & \mathcal{C} \end{array} \qquad (1)$$

commutes. This determines $\mathsf{Th}_{\mathsf{E}}(\mathsf{S})$ up to equivalence of categories. The morphism $H_M$ is a model of $\mathsf{S}$ in $\mathsf{Th}_{\mathsf{E}}(\mathsf{S})$ called the **generic** $\mathsf{E}$-model of $\mathsf{S}$.

As a consequence of the definition, $H_M$ induces a natural equivalence between $\mathrm{Mod}_{\mathcal{C}}(\mathsf{S})$ and the category of $\mathsf{E}$-functors from $\mathsf{Th}_{\mathsf{E}}(\mathsf{S})$ to $\mathcal{C}$; the equivalence takes a model $M$ to the functor $\mathsf{Th}_{\mathsf{E}}(M)$ shown in Diagram 1.

An analogous construction holds for $\mathsf{E}$ categories with *designated* $\mathsf{E}$-limits and functors that preserve them on the nose; then the corresponding theory functor is a left adjoint to the underlying functor and the underlying functor is monadic [Agéron, 1991]. It is useful to call the theory in this sense the **strict** theory and the theory in the sense of the preceding paragraph the **loose** theory (following the French usage).

### 5.3 Construction of theories of sketches

The strict theory was first constructed in [Ehresmann, 1968b] and the loose theory (they called it the **type**) in citebastehres. Barr and Wells [1985] give a proof using the Yoneda Lemma that FL sketches generate FL theories. Another embedding construction produces the theory for many types of sketches with cocones[Barr and Wells, 1985], Chapter 8. In both cases, one gets an $\mathsf{E}$-embedding of the theory in a topos for free. This topos is called the **classifying topos**[2] of the sketch, and explains why geometric sketches have been studied specially among all sketches, and why, in much of the literature, sketches with cocones are required to have models in categories having some of the properties of toposes (for example, disjoint sums).

The unadorned phrase "generic model of $\mathsf{S}$" usually refers to the model of $\mathsf{S}$ in its classifying topos.

Other constructions for special kinds of sketches are given in [Peake and Peters, 1972], [Kelly, 1982b] and [Barr and Wells, 1993a]. The last paper is based on a direct (not inductive and not by embedding) construction of the free category with limits generated by a category.

The (strict) $\mathsf{E}$-theory of a sketch (for arbitrary $\mathsf{E}$) is the initial algebra for the finite limit sketch of $\mathsf{E}$-categories with constants added describing the sketch [Wells, 1990].

#### 5.3.1 Note concerning toposes

A topos is a category in which roughly speaking one can pretend the objects are sets and the arrows are functions, except that one must restrict the rules of inference in one's reasoning. Intuitionistic or constructive reasoning, defined precisely, is always appropriate in a topos. You may be able to use more powerful tools in particular toposes, such as the law of the excluded middle (in a Boolean topos) or the existence of only two truth values. Topos theory and its logic is expounded in [Johnstone, 1977], [Barr and Wells, 1985], [Mac Lane and Moerdijk, 1992], and [McLarty, 1992], among others. A brief but illuminating discussion of toposes is given in the review of

[2]If you know some topology, you will eventually notice that the arrow $H_M$ goes backward compared to the corresponding arrow for classifying spaces, so that the name "classifying topos" seems to be a false analogy. This is because in this case $H_M$ is the left adjoint of a geometric morphism that does indeed go *from* the the category $\mathcal{C}$ in which the models live *to* the classifying topos.

[Bell, 1988] by McLarty [1990]. Another useful discussion is given by Phoa [1993].

The word "topos" has two meanings. The first is the meaning originally given it by Grothendieck — a category of sheaves over a site. The second is a structure determined by axioms due to Lawvere and Tierney that they called an "elementary topos". Every Grothendieck topos is an elementary topos, but not conversely. I am using the word topos to mean elementary topos, but be warned: many authors use it to mean Grothendieck topos.

# 6 Categories as theories

## 6.1 Specification of mathematical structures

Mathematicians historically have created mathematical structures to be approximations of physical phenomena, extracting the salient properties of physical behavior and making them precise.

**6.1.1 Mathematical logic** Classical mathematical logic has done something similar: One creates formal mathematical objects (language, term, formula, rule of inference) that are special types of strings of symbols and sets thereof. These objects are precise mathematical constructions that extract and make formal certain properties of the statements and deductions that occur in mathematical theorems and proofs. Let's call this **string-based syntax**.

**6.1.2 Signatures and Equations** In universal algebra, one specifies structures using **tuple-based syntax** — signatures and equations. This formalism has been extended to essentially algebraic theories by Reichel [1987]. Of course, universal algebraists also use classical first order logic.

**6.1.3 Categorial theories** A third approach is that of regarding an $\mathsf{E}$-category (for a particular doctrine $\mathsf{E}$) as a **$\mathsf{E}$-theory**, often called a **categorial theory**[3] or **categorical theory**. Such phrases mean that the category is being thought of as analogous to a logical theory. In this setting, the semantics is given by an $\mathsf{E}$-functor to some $\mathsf{E}$-category $\mathcal{C}$. Of course, every $\mathsf{E}$-theory $\mathcal{C}$ is the theory of an $\mathsf{E}$-sketch, namely the underlying $\mathsf{E}$ sketch of $\mathcal{C}$.

A categorial theory has the advantage of being independent of any particular presentation, in much the same way that linear transformations have the advantage of being coordinate-free as compared to matrices.

The idea of categories as theories is discussed in the context of computer science by Fourman and Vickers [1986].

## 6.2 Background

The notion of category as theory and sketch as presentation of the theory originated in three streams of thought.

**6.2.1 Algebraic theories** In the 1960's, Lawvere [1963], [1968] introduced the idea that a category should be used to specify one-sorted algebraic structures such as groups

[3]In this paper, I use "categorial theory" because to logicians the phrase "categorical theory" means a theory with only one model up to isomorphism. On the other hand, "categorial" is confusing to linguists.

and rings. He called the category the **algebraic theory** of the type of structure; it is a category in which every object is a sum (coproduct) of a single object that corresponds to the underlying set of the algebra. The semantics is given by a contravariant functor that takes sums to products. Other early papers in this field are [Linton, 1966], [Linton, 1969b], and [Linton, 1969a].

An algebraic theory is equivalent to the opposite category of the theory of a finite product sketch for the same type of structure[4].

Manes [1975] treats algebraic theories and their relation with monads (see Section 7.2) in detail. They are also treated by Pareigis [1970].

Models in toposes are treated by Johnstone and Wraith [1978] and by Rosebrugh [1980]. Blackwell, Kelly and Power [1989] provide an important generalization.

**6.2.2 Classifying toposes** The concept of classifying topos is another example of a categorial theory, although in the early days it was thought of as the analog of a classifying space more than as a theory (see Section 5.3 above). Tierney [1976] clearly had in mind the construction of what I would call the classifying topos of a sketch, but I am not familiar with the early history of the relation between toposes, sketches and logic (see Section 7.5 below for more references in this area), and there may be earlier references to this idea.

**6.2.3 Sketches** The third strand is the introduction of sketches based on what are now called compositive graphs by Ehresmann [1968b], [1972]. He and his students developed the subject extensively, but I think it is fair to say that for the most part they studied sketches and their models directly and did not focus on the passage to the corresponding categorial theories.

# 7 Properties of model categories

## 7.1 Properties of model categories

Categories of models of particular doctrines of sketches (or of categories in the role of theories) have specific properties that have been studied in some detail. As one would expect the more one can specify in a sketch, the less nice is the category of models. I will mention two examples here. More detail is in [Barr and Wells, 1985], page 297 (that list omits to mention that all geometric theories have filtered colimits computed sortwise). The major paper [Guitart and Lair, 1980] also contains a lot of information about the properties of model categories for special kinds of sketches.

**7.1.1** The category of models of a linear sketch is a topos. (Two particular such toposes are discussed in [Lawvere, 1989].)

**7.1.2** The category of models of a finite product sketch has limits and filtered colimits computed "sortwise", is regular and has effective equivalence relations. These statements imply that the category of models is a variety.

[4] For some authors, an algebraic theory is the opposite category of what Lawvere called an algebraic theory, in which case it is equivalent to the finite product sketch. Note that finite product sketches in general sketch *multisorted* structures.

A limit or colimit constructed "sortwise" means that the underlying set functor(s) (there is one for each node of the graph of the sketch) preserve the limit or the colimit. The fact that the product of two groups is defined on the product of their underlying sets is an instance of sortwise construction. The free product of two groups is *not* defined on the direct sum of their underlying sets — and the direct sum is a colimit but not a filtered colimit. On the other hand, a coequalizer is a filtered colimit, and indeed one constructs quotient groups on the quotient of the underlying set.

Johnstone [1985], [1990], has given conditions on an algebraic theory for its category of models (the variety it generates) to be respectively a topos or a cartesian closed category.

Agéron [1992] gives sufficient conditions on a class $\mathcal{S}$ of sketches for the following category to be cartesian closed: Its objects are all categories Mod($\mathsf{S}$) with $\mathsf{S}$ in $\mathcal{S}$, and its arrows are all functors preserving sufficiently filtered colimits.

### 7.2 Connections with monads

A functor $U : \mathcal{C} \longrightarrow \mathcal{D}$ that has a left adjoint determines two categories, its category of **Eilenberg-Moore algebras** and its category of **Kleisli algebras**. $U$ is **monadic** or **tripleable** if $\mathcal{C}$ is equivalent to its category of Eilenberg-Moore algebras. Remarkably, monadic functors $U : \mathcal{C} \longrightarrow \mathcal{S}\rceil\sqcup$ are up to equivalence just the underlying functors from categories of algebraic structures defined by signatures (possibly with boundedly infinitary operations) and equations.

When $U$ is monadic, the operations and equations of the signature can be recovered from the natural endomorphisms of the underlying functors [Linton, 1969b], and the Kleisli algebras turn out to be the free algebras. In fact, Lawvere's algebraic theory is the opposite of the category of Kleisli algebras of the monad determined by the functor. See [Mac Lane, 1971], [Manes, 1975], and [Barr and Wells, 1985] for introductions to the subject of monads.

Algebraic structures in enriched categories (properly defined) are also monadic (propertly defined!) [Kelly and Power, 1991].

There are many theorems stating that some kind of categories with structure are monadic over $\mathsf{Cat}$ or over some category of sketches or graphs. See [Lair, 1975a], [Lair, 1979], [McDonald and Stone, 1984], [Coppey and Lair, 1985], [Agéron, 1991] (not all of which I have seen).

### 7.3 Characterization of model categories

Gabriel and Ulmer [1971] give a complete characterization of those categories that are Mod$\mathsf{S}$ for a finite limit sketch $\mathsf{S}$. [Ulmer, 1971] is a summary of that result in English.

Lair [1981] characterized sketchable categories (those that are equivalent to Mod$\mathsf{S}$ for some sketch $\mathsf{S}$) (see also [Mouen, 1984].) This result was rediscovered and elaborated by Makkai and Paré [1990], who called such categories **accessible categories**. I recommend the review by Gray [1991b].

Guitart and Lair [1980] showed that **axiomatizable** categories are the same as sketchable categories. The definition of "axiomatizable" involves "satisfying" certain cones in a specific sense (the cones correspond to cocones of the sketch). This is also a characterization of sketchable categories, though I think it is fair to say that it is not a *categorial* characterization.

If I understand it correctly, [Day and Street, 1990] can be interpreted as a kind of characterization of sketchable categories.

Barr [1986] has some results of a similar nature of this form: for certain $\mathsf{E}$ and $\mathsf{E}'$, if the category of models of an $\mathsf{E}$ sketch has certain properties of the category of models of an $\mathsf{E}'$ sketch (where $\mathsf{E}'$ is more restrictive), then it is in fact (equivalent to) the category of models of an $\mathsf{E}'$ sketch.

### 7.4 Initial algebras and locally free diagrams

**7.4.1 Initial algebras for FL sketches** Let $\mathsf{S}$ be a finite limit sketch. The category $\mathrm{Mod}(\mathsf{S})$ of its models in the category of sets has an **initial model** or **initial algebra**, a model $M$ with the property that there is exactly one homomorphism from $M$ to any model. This follows from the work of Gabriel and Ulmer [1971]. (That work actually implies the existence of initial algebras for any projective sketch whose cones are over diagrams with bounded cardinality.)

**7.4.2 Free algebras** When $\mathsf{S}$ is a finite limit sketch, $\mathrm{Mod}(\mathsf{S})$ has an underlying functor to $\mathcal{S}\rceil\sqcup^{G_0}$, where $G_0$ is the set of nodes of the graph of $\mathsf{S}$. This functor has a left adjoint $F$, which means that if $\mathcal{X}$ is any $G_0$-indexed family of sets, then $F(\mathcal{X})$ is a **free algebra** on $\mathcal{X}$. This actually follows from the existence of initial algebras for finite limit sketches, as follows: For each set $X_g$ in $\mathcal{X}$ (where $g$ is a node of $G$), adjoin one new constant of type $g$ for each element $x \in X_g$. The resulting sketch is still a finite limit sketch, and its initial model is the desired free algebra.

Such free algebras can be constructed inductively from the ground up, Herbrand style. (See [Ehresmann, 1969], [Kelly, 1980]. This construction, for finite product sketches (although they don't explicitly use sketches), has been used extensively to model data types in computing. See [Goguen *et al.*, 1978] and [Meseguer and Goguen, 1985].

When the sketch has cocones, initial algebras need not exist. If the cocones are all discrete, we have the case of a **localizable category** [Diers, 1977]. Then each component of the category of models has an initial algebra; for fields, these are the prime fields. These are constructed inductively in [Wells and Barr, 1988]. The construction there includes the construction of the initial algebra for a finite limit theory as a special case.

In the general case, instead of a free algebra, one has a **locally free diagram**, which in most important cases is small (but not necessarily unique). This is developed by Guitart and Lair [1980] (see also [Guitart and Lair, 1981] and [Guitart and Lair, 1982a]). A Galois group is an example of a locally free diagram [Lair, 1983].

### 7.5 Logic

There is a considerable literature that works out precise string-based languages and rules of inference that correspond in expressive power to certain specific types of categorial theories. Starting at the bottom, equational logic is the logic of finite product theories. Note that in the general categorial treatment, one allows some sorts of a multisorted theory to be empty in a model, which complicates the logic. The treatment of equational logic in [Goguen and Meseguer, 1985] illustrates this admirably.

Coste [1976] and McLarty [1986] provide a language and rules of inference for finite limit theories (see also [Keane, 1975]), and McLarty [1989] does the same for regular theories.

[Makkai and Reyes, 1977] is the classic source for the translation between geometric theories and first order string-based logic. This work is updated by Pitts [1989], which I recommend for its succinct technical introduction to the ideas. See also [Reyes, 1977] and [Paré, 1989].

The texts by Johnstone [1977], Lambek and Scott [1986] (which also develops the connection between cartesian closed categories and the typed $\lambda$-calculus), McLarty [1992], and Mac Lane and Moerdijk [1992] all describe the internal language of a topos. Other papers worth looking at in this area are [Boileau and Joyal, 1981], [Bunge, 1984], [Osius, 1975b], [Osius, 1975a], [Poigné, 1986a], [Fourman, 1977] and [Vickers, 1993]. Bell [1988] develops topos theory completely in terms of its language. Lawvere [1975] discusses some of the early history of the subject.

In a somewhat subtle sense, sketches are equivalent in expressive power to first order logic [Guitart and Lair, 1982b], [Makkai and Paré, 1990]. The subtlety may be seen by considering the category of connected graphs, which is sketchable by a finite sketch, but connected graphs cannot be specified in first order logic limited to finite formulas and terms (I don't have a reference to this). Finite sketches that contain formal coequalizers may require infinitary disjunctions to express the corresponding structure in first order logic. In the case of connected graphs, that a graph is connected is expressed by the requirement that the formal coequalizer of the source and target maps must be the formal terminator.

Another subtlety: The preceding discussion concerns the question of expressing the structure sketched by a sketch using a first order theory in such a way that a model of the first order theory is an example of the structure. Of course, you can incorporate all of Zermelo-Fränkel set theory in the logical theory and express the structure as an *element* of a model of the logical theory, but that is a different matter.

# 8 Categories of sketches

## 8.1 Properties of categories of sketches

The information in this subsection was provided in part by C. Lair and P. Agéron.

The category of sketches is studied in [Lair, 1975b], [Gray, 1989] and [Lair, 1988]. In the first paper, it is shown that the category of sketches is sketchable by a projective sketch, from which it follows that it is complete and cocomplete. He also shows that it is monoidally closed; applications of this are given in [Lair, 1977a] and [Agéron, 1992]. [Gray, 1989] gives several explicit constructions in the category of sketches, in particular showing that it is cartesian closed.

## 8.2 Institutions

Sketches form an **institution** [Goguen and Burstall, 1986]. This is a formalization of the idea that morphisms of sketches (equivalently, a morphism of theories) induce a morphism of the model categories going the other way. It is spelled out in [Barr and Wells, 1990], Section 10.3. In connection with this, one can ask whether, for a particular type of theory, a morphism of theories that induces an equivalence on the category of models must be an equivalence of theories (conceptual completeness). The answer is yes for pretoposes [Makkai and Reyes, 1977], Chapter 8, (see also [Pitts, 1989]) Some form of **Morita theory** can also explain when categories of models are equivalent (but in this case without necessarily being induced by a morphism of theories); for this, see [Freyd, 1966], [Lindner, 1974], [Elkins and Zilber, 1976], [Fisher-Palmquist and Palmquist, 1973], [Borceux and Vitale, 1991] and [McKenzie, 1992].

Lair [1979] gives precise conditions on a morphism of sketches for it to induce a *monadic* functor (see Section 7.2) between model categories.

# 9 Generalizations of the concept of sketch

The idea that sketches can be sketched using an FL sketch (see subsection 4.3) is the basis of a generalization of the concept of sketch in [Wells, 1990], developed for 2-sketches in [Power and Wells, 1992]. [Barr and Wells, 1992] contains other examples of their use. Because of the relationship between sketches and first order logic (see 7.5), these generalizations are appropriately referred to as **higher-order sketches**. One can conceive of a whole hierarchy of orders of sketches; this is the idea of both [Lair, 1987b] and [Makkai, 1993a] (see also [Makkai, 1993b]). Kelly [1982a] generalizes the concept of sketch to enriched categories, beginning on page 218. See also [Kelly, 1982c]. Obtułowicz [1992] describes a type of sketch that is intended to generate an infinite graph rather than a category.

# 10 Omissions

In preparation for this report, I have gradually become aware that the paper [Guitart and Lair, 1980] is of major importance and should be more widely known. It has not influenced this report as much as it should have because I have so far gained only a partial understanding of its contents. Similar comments could probably be made about [Isbell, 1972], but I understand it even less.

The following papers clearly should be included, but I either don't have them or have not become familiar enough with them to say anything. [Adámek and Rosičky, 1991], [Agéron, 1989], [Andréka and Németi, 1979], [Bénabou, 1972], [Bastiani, 1973], [Coppey, 1972], [Coppey, 1990], [Foltz and Lair, 1972], [Foltz *et al.*, 1980], [Henry, 1982], [Johnson and Walters, 1992], [Lair, 1971].

# 11 Acknowledgments

I am grateful to Michael Barr, Robin Cockett, Colin McLarty, Jim Otto and Robert Paré for conversations that clarified some of these ideas. Pierre Agéron, Michael Barr, G. M. Kelly and Christian Lair found errors and provided more references. The bibliography includes several items that I learned about from [Permvall, 1991], which has an excellent discussion of sketches and their applications to computer science. Not all of the items mentioned in [Permvall, 1991] are included here.

The diagrams in this paper were prepared using Michael Barr's `diagram.tex`. Thanks to Carol Larson for typing `sketch.bib`.

# Appendix: Some definitions

We define graphs, diagrams, cones and cocones in detail because the terminology is not standard.

## 11.1 Graphs

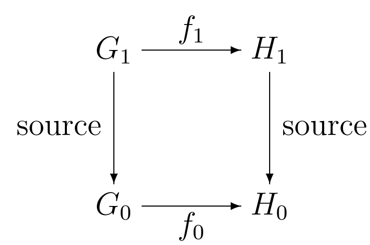
### 11.1.1 Definition

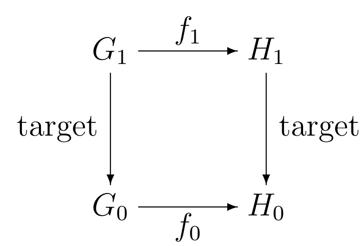
A **graph** $G$ consists of two sets $G_0$ and $G_1$ and two functions source $:G_1 \to G_0$ and target $:G_1 \to G_0$.

The elements of $G_0$ are called the **nodes** or **vertices** of $G$ and the elements of $G_1$ are the **arrows** of $G$. If an arrow $a$ has source $x$ and target $y$ we write $a : x \to y$. A graph is conventionally drawn using dots or labels for the nodes, and an arrow going from node $x$ to node $y$ for each element $a$ of $G_1$ with source $x$ and target $y$.

### 11.1.2 Definition

A **homomorphism of graphs** $f : G \to H$ is a pair of functions $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$ for which these diagrams commute:



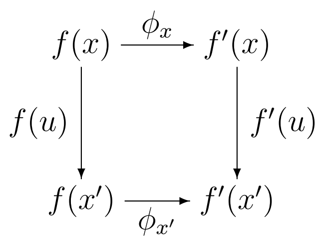




(Commutative diagrams are defined formally in Section 11.4 below. This one means that source $\circ f_1 = f_0 \circ$ source and target $\circ f_1 = f_0 \circ$ target.) The subscripts 0 and 1 are normally omitted when mentioning graph homomorphisms, in the same way they are for functors.

Every category has an underlying graph whose nodes are the objects of the category and whose arrows are the arrows of the category with the same source and target.

### 11.1.3 Definition

Let $f, f' : G \to \mathcal{C}$ be graph homomorphisms to a category $\mathcal{C}$. A **natural transformation** $\phi : f \to f'$ is a family of arrows $\phi_x : f(x) \to f'(x)$ indexed by the nodes of $G$ with the property that for every arrow $u : x \to x'$ of $\mathcal{C}$ the following diagram commutes:




This is just like the definition of natural transformation for functors between categories, which in any case makes no use of the composition in the domain of the functors.

A **path** of length $n$ from $x$ to $y$ in a graph $G$ is a sequence $\langle f_1, f_2, \ldots, f_n \rangle$ of arrows with the property that the target of $f_{j-1}$ is the source of $f_j$ for $j = 2, \ldots, n$, $\text{source}(f_1) = x$ and $\text{target}(f_n) = y$. If $p = \langle f_1, f_2, \ldots, f_n \rangle$ is a path from $x$ to $y$, we write $p : x \to y$. For each node $x$, there is one **empty path** $\langle\rangle : x \to x$.

We extend the definition of homomorphism $F : G \to H$ of graphs to paths in $G$ by mapping $F$ over the nodes in the path:

$$F\langle f_1, f_2, \ldots, f_n \rangle = \langle F(f_1), F(f_2), \ldots, F(f_n) \rangle$$

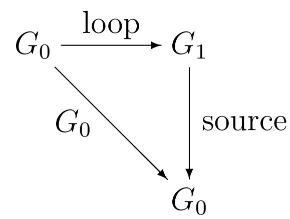
The nodes of $G$ and the paths form a category, the **free category** generated by the graph $G$, with the empty paths as identity arrows and composition by concatenation. See [Barr and Wells, 1990], Sections 2.1 and 2.6, for the details.

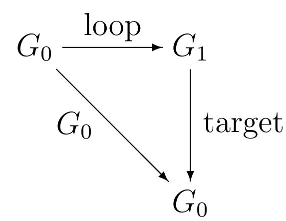
The set of paths of length $n$ in $G$ is denoted $G_n$. In particular, a path of length 2 is a pair $\langle f, g \rangle$ of arrows such that the source of $g$ is the target of $f$. Such a pair is called a **composable pair**.

## 11.2 Reflexive graphs

### 11.2.1 Definition

A **reflexive graph** is a graph with additional structure, namely a function $\text{loop} : G_0 \to G_1$ satisfying the equations




(In this text, I follow the categorial custom of using the name of an object also as the name of its identity arrow.) The arrows of the form $\text{loop}(x)$ for some node $x$ are called the **distinguished loops**.

A **homomorphism of reflexive graphs** is a homomorphism of graphs that takes distinguished loops to distinguished loops.

The underlying reflexive graph of a category is the underlying graph with the identity arrows as the distinguished loops.

## 11.3 Compositive graphs

The French school bases its definition of sketch on the concept of a **compositive graph** or **multiplicative graph**: A compositive graph $G$ is a reflexive graph with a partially defined binary operation $\kappa : C_G \to G$ satisfying the laws CG–1 through CG–3 below. If $\langle f, g \rangle$ is a composable pair and $\kappa(f,g)$ is defined, we write $g \circ f$ for $\kappa(f,g)$ (note the reversal).[5]

CG.1 $C_G$ is a subset of $G_2$.

CG.2 For all arrows $f$, $f \circ \mathrm{id}_{\mathrm{source}(f)}$ and $\mathrm{id}_{\mathrm{target}(f)} \circ f$ are both defined and equal to $f$.

CG.3 If $g \circ f$ is defined, $\mathrm{source}(g \circ f) = \mathrm{source}(f)$ and $\mathrm{target}(g \circ f) = \mathrm{target}(g)$.

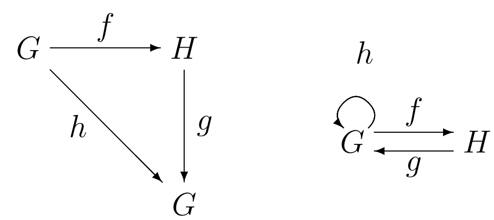
Associativity is not required. A category is a compositive graph $G$ for which $C_G = G_2$ and the composition is associative. See [Coppey and Lair, 1984], Leçon 2 for details. (Their "oriented graph" is our reflexive graph.) See Section 3.2 for notation concerning these ideas.

A **functor** between compositive graphs is defined just like functors between categories. In all cases, if $F$ is a functor and $g \circ f$ is defined, then $F(g) \circ F(f)$ is defined and equal to $F(g \circ f)$.

### 11.4 Diagrams

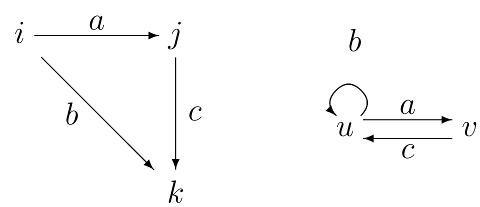
If $I$ and $G$ are graphs, a graph homomorphism $d : I \longrightarrow G$ is a **diagram** in $G$. In this case, $I$ is the **shape graph** of the diagram. If $I$ is a graph, $\mathcal{C}$ is a category, and $d$ is a diagram in the underlying graph of $\mathcal{C}$, we will write it as $d : I \longrightarrow \mathcal{C}$.

By convention, a diagram is drawn in such a way that its shape graph can be recovered (except for the actual names of the nodes and arrows) from the way it is drawn by replacing each node of the drawn graph by a different label (whether two nodes as drawn have the same label or not) and similarly for arrows. For example, the shape graphs corresponding to these two diagrams




(2)

are these two graphs respectively:




The diagrams in (2) are not the same!

A diagram $d : I \longrightarrow \mathcal{C}$ in a category $\mathcal{C}$ is **commutative** if for all pairs of paths $p, q : x \longrightarrow y$ in the shape graph $I$ between the same two nodes, $d(p)$ and $d(q)$ have the same composites in $\mathcal{C}$. For example, if the left diagram in (2) commutes, then $g \circ f = h$, whereas if the right diagram commutes, then $g \circ f = h = \mathrm{id}_G$ and $f \circ g = \mathrm{id}_H$. (The point is that there is an empty path from $G$ to $G$ and another one from $H$ to $H$.)

The concept of commutative diagram requires composition; it makes sense in a compositive graph but not in a graph in general.

In much of the categorial literature, a diagram $d : I \longrightarrow \mathcal{C}$ is defined to be a functor from a category (most commonly a small category) $I$ to $\mathcal{C}$. A graph-based diagram can be converted to an equivalent category-based one by basing it on the free category generated by the graph. Conversely, a functor defined on a category $\mathcal{I}$ is a diagram based on the underlying graph of $\mathcal{I}$. But be careful: there are in general diagrams based on the underlying graph of $\mathcal{I}$ that are

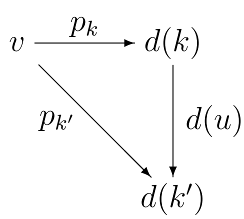
[5]Many computer scientists write $f;g$ in this case, a notation that has certain advantages. Many French authors use $g \circ f$ but draw the horizontal arrows in their diagrams from right to left so that the composite matches the path indicated by the arrows.

not functors from $\mathcal{I}$, because there is no way to enforce the preservation of composition.

### 11.5 Cones

#### 11.5.1 Definition

If $d : I \longrightarrow G$ is a diagram in a graph $G$, then a **cone** $p : v \longrightarrow d$ consists of an object $v$ of $\mathcal{C}$ and a set of arrows $p_k : v \longrightarrow d(k)$ indexed by the nodes of $I$. If $d : I \longrightarrow \mathcal{C}$ is a diagram in a category $\mathcal{C}$, the cone $p : v \longrightarrow d$ is **commutative** if for all arrows $u : k \longrightarrow k'$ in $I$,




commutes.

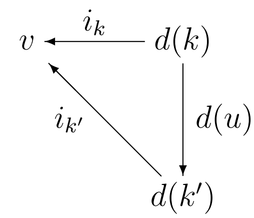
The diagram $d$ is called the **base diagram** of the cone and the arrows $p_k$ are the **projections** or sometimes **elements** of the cone.

Note that the concept of commutative cone does not make sense for a cone in a graph, although it does for a cone in a compositive graph. For most authors, a cone in a compositive graph (hence in a category) is commutative by definition.

The base diagram of a cone in a category is *not required to be commutative.*

If the shape graph of the base diagram has only identity arrows, the cone is **discrete**.

A **cocone** in $\mathcal{C}$ based on a diagram $d : I \longrightarrow \mathcal{C}$ consists of an object $v$ of $\mathcal{C}$ and arrows $i_k : d(k) \longrightarrow v$ indexed by the objects of $I$. It is **commutative** if for all $u : k \longrightarrow k'$ in $I$,




commute.

[Barr and Wells, 1990] has an extensive discussion of **limits** and **colimits** using the terminology introduced here. Mac Lane [1971] is an excellent source for limits and colimits defined in terms of functors based on small categories, with many mathematical examples. Poigné [1986b] introduces limits and colimits in the context of computer science.

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