We would like to remark that the above result is correct only when all non-unit n_i are not equal to each other. If several n_i are equal to an integer that is not equal to 1, there will be a non-abelian symmetry between partons. In this case the edge states are described by non-abelian K-M algebras. (See Ref. 19 and Sec. 5.)

2.5. *Effective theory and the edge states*

In this section we will directly derive the macroscopic theory of the edge excitations from the effective theory of the bulk FQH states.[15,30] In this approach we do not rely on a specific construction of the FQH states. The relation between the bulk topological orders and edge states also becomes clear in this approach.

We know a hierarchical (or generalized hierarchical) FQH state contains many different condensates, the electron condensate forms the Laughlin state and the quasiparticle condensate on top of that forms a hierarchical state. Each condensate corresponds to one component of the incompressible fluid. The idea is to generalize the hydrodynamical approach in Sec. 2.1 to multi-component fluids and to obtain the low energy effective theory of the edge excitations. To accomplish this, we first would like to write down the low energy effective theory of the bulk FQH state. The effective theory should contain the information about the topological orders in the bulk states.

The different condensates in the FQH states are not independent. The particles in one condensate behave like a flux tube to the particles in other condensates. To describe such a coupling, it is convenient to use $U(1)$ gauge fields to describe the density and the current of each condensate. In this case the couplings between different condensates are described by Chern-Simons term of the gauge fields.[35,15,17,21] By some further consideration of the electron operators in the effective theory, it was shown[21] that the most general abelian FQH states of the electrons are classified by integer valued symmetry matrix K with odd diagonal elements and are described by the following effective theory[36]

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_I \partial a_J + a_I \cdot j_I + \frac{1}{2\pi} A \partial a_I + g_{IJ} f_I \cdot f_J \tag{2.55}$$

where $a_I \partial a_J$ is an abbreviated notation for $a_{I\mu} \partial_\nu a_{J\lambda} \varepsilon_{\mu\nu\lambda}$, $\mu, \nu, \lambda = 0, 1, 2$ and $f_{I\alpha\beta}$ is the field strength of the $U(1)$ gauge field $a_{I\mu}$. The FQH state described by (2.55) contains κ different condensates and there are κ kinds of different quasiparticle excitations, where κ is the rank of K. $j_{I\mu}$ is the density and the current of the Ith kind of excited quasiparticles (denoted by ψ_I) that behaves like vortices in the condensates. $j_{I\mu}$ are normalized such that $\int dx j_{I0}$ are integers. The density and the current of the Ith component of incompressible fluid (i.e., the Ith condensate) is given by

$$J_{I\mu} = \frac{1}{4\pi} \varepsilon_{\mu\alpha\beta} f_{I\alpha\beta} \ . \tag{2.56}$$

The filling fraction of the FQH state is

$$\nu = \sum_{IJ} (K^{-1})_{IJ} . \tag{2.57}$$

As we create a Jth quasiparticle ψ_J, it will induce a change in the density of the Ith condensates, δJ_{I0}. From the equation of motion we find that δJ_{I0} satisfy

$$\int d^2x \delta J_{I0} = (K^{-1})_{JI} . \tag{2.58}$$

(Note $\int d^2x j_{J0} = 1$ in the presence of the Jth quasiparticle.) The charge and the statistics of the Jth quasiparticle is given by

$$\theta_J = \pi (K^{-1})_{JJ}, \quad q_J = \sum_I (K^{-1})_{JI} . \tag{2.59}$$

A generic electron excitation can be written as a bound state of the quasiparticles:

$$j_e = \sum_{I,J} L_I K_{IJ} j_J \tag{2.60}$$

where L_I are integers satisfying $\sum_I L_I = 1$. From (2.55) we can show that electron excitations in (2.60) satisfy the following properties: A) They carry a unit charge (see (2.59)). B) They have the fermionic statistics. C) Moving an electron excitation defined in (2.60) around any quasiparticle excitations always induces a phase of multiple of 2π. D) The excitations defined in (2.60) are all the excitations satisfying the above three conditions. For a more detailed discussion, see Ref. 21.

We would like to point out that the effective theory (2.55) not only applies to the standard QH system in which all electrons are spin polarized and in the first Landau level, it also applies to the QH system in which electrons may occupy several Landau levels and/or occupy several layers and/or carry different spins. In this case the index I may label the condensates in different Landau levels, in different layers and/or with different spins. The edge excitations for the spin 1/2 electron system were discussed in Refs. 9, 37.

To understand the relation between the effective theory and the edge states, let us first consider the simplest FQH state of the filling fraction $\nu = 1/q$ and try to rederive the results in Sec. 2.1 from the bulk effective theory. Such a FQH state is described by $U(1)$ Chern-Simons theory with the action:[13,30]

$$S = \frac{q}{4\pi} \int a \partial a d^3x . \tag{2.61}$$

Suppose that our sample has a boundary. For simplicity we shall assume that the boundary is the x-axis and the sample is the lower half-plane. The Chern-Simons action is not invariant under gauge transformations $a_\mu \to a_\mu + \partial_\mu f$ due to the boundary effects. To solve this problem we will restrict the gauge transformations to be zero on the boundary $f(x, y = 0) = 0$. Due to this restriction some degrees of freedom of a_μ on the boundary become dynamical. We know the effective theory (2.61) is derived only for a bulk FQH state without boundary. We will take (2.61) with the restricted gauge transformation as the definition of the effective theory for a FQH state with boundary. Such a definition is definitely self-consistent. In the following we will show that such a definition reproduces the results obtained in Sec. 2.1.

One way to study the dynamics of a gauge theory is to choose the gauge condition $a_0 = 0$ and regard the equation of motion for a_0 as a constraint. For the Chern-Simons theory such a constraint becomes $f_{ij} = 0$. Thus we write a_i as $a_i = \partial_i \phi$. Plug this into (2.61), one obtains[38] an effective conformal theory on the edge with an action

$$S_{\text{edge}} = \frac{m}{4\pi} \int \partial_t \phi \partial_x \phi dx dt \ . \tag{2.62}$$

This approach, however has a setback. It is easy to see that a Hamiltonian associated with the action (2.62) is zero and the boundary excitations described by Eq. (2.62) have zero velocity. Therefore this action cannot be used to describe any physical edge excitations connected with the FQHE. The edge excitations in the FQH states always have finite velocities.

The appearance of finite velocities of edge excitations is a boundary effect. The bulk effective theory defined by Eq. (2.55) does not contain information about the velocities of the edge excitations. The edge velocities in the QH states are actually determined by the edge potentials. To determine the dynamics of the edge excitations from the effective theory we must find a way to input the information about the edge velocity. The edge velocities must be treated as the external parameters that are not contained in the bulk effective theory. The problem is how to put these parameters in the theory.

Let us now note that the condition $a_0 = 0$ is not a unique choice of the gauge fixing condition. More general gauge fixing condition has a form

$$a_\tau = a_0 + v a_x = 0 \ . \tag{2.63}$$

Here a_x are the component of the vector potential parallel to the boundary of the sample and v is a parameter that has a dimension of velocity.

It is convenient to choose new coordinates that satisfy

$$\begin{aligned} \tilde{x} &= x - vt \\ \tilde{t} &= t, \quad \tilde{y} = y \ . \end{aligned} \tag{2.64}$$

In these coordinates the components of the gauge field are given by

$$\begin{aligned} \tilde{a}_{\tilde{t}} &= a_t + v a_x \\ \tilde{a}_{\tilde{x}} &= a_x \\ \tilde{a}_{\tilde{y}} &= a_y \ . \end{aligned} \tag{2.65}$$

The gauge fixing condition becomes the one discussed before. It is easy to see that the form of the Chern-Simons action is preserved in the new coordinates:

$$S = \frac{q}{4\pi}\int d^3x\; a_\mu \partial_\nu a_\lambda \varepsilon^{\mu\nu\lambda} = \frac{q}{4\pi}\int d^3x\; \tilde{a}_{\tilde{\mu}} \partial_{\tilde{\nu}} \tilde{a}_{\tilde{\lambda}} \varepsilon^{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} \,. \tag{2.66}$$

Repeating the previous derivation, we find the edge action is given by

$$S = \frac{q}{4\pi}\int d\tilde{t} d\tilde{x} \partial_{\tilde{t}}\phi \partial_{\tilde{x}}\phi \,. \tag{2.67}$$

In terms of the original physical coordinates the above action acquires a form

$$S = \frac{q}{4\pi}\int dt dx (\partial_t + v\partial_x)\phi \partial_x \phi \tag{2.68}$$

which is a chiral boson theory. It is easy to see that the edge excitation described by (2.68) have a non-zero velocity. The quantization of chiral boson theory has been discussed in detail in Ref. 39. The canonical momentum $\pi(x)$ is equal to $\pi = \frac{\delta L}{\delta \phi_t} = \frac{q}{4\pi}\partial_x\phi$. The coordinate ϕ and momentum π obey the commutation relations:

$$\begin{aligned} [\pi(x), \phi(y)] &= \frac{1}{2}\delta(x-y) \\ [\phi(x), \phi(y)] &= \frac{\pi}{q}\,\mathrm{sgn}\,(x-y) \,. \end{aligned} \tag{2.69}$$

The Hamiltonian of the theory (2.4.11) is given by

$$H = -\frac{qv}{4\pi}\int dx \partial_x\phi \partial_x\phi \,. \tag{2.70}$$

The Hilbert space contains only left-moving degrees of freedom (or right-moving degrees of freedom if $v < 0$). The theory (2.69) and (2.70) describes free left (or right) moving phonons (i.e., the edge density waves). One can easily show that (2.69) and (2.70) are equivalent to the K-M algebra (2.7) by identifying $\rho = \frac{1}{2\pi}\partial_x\phi$.

In the following we would like to show that $\rho = \frac{1}{2\pi}\partial_x\phi$ can really be interpreted as the 1D electron density on the edge. First we notice that the coupling between the electrons and the external electromagnetic potential is given by $\int A_\mu J_\mu d^3x = \int \frac{1}{2\pi} A\partial a d^3x$ (see (2.56)). From $\tilde{a}_{\tilde{i}} = \partial_{\tilde{i}}\phi$ we see that

$$\int A_\mu J_\mu d^3x = \int d\tilde{x} d\tilde{t} \frac{1}{2\pi} A_{\tilde{t}} \partial_{\tilde{x}}\phi = \int dx dt \frac{1}{2\pi}(A_t + vA_x)\partial_x\phi \tag{2.71}$$

where we have used the equation of motion $(\partial_t + v\partial_x)\phi = 0$ and the transformation (2.64) and (2.65). (2.71) clearly indicates that the 1D edge electron density is given by $\frac{1}{2\pi}\partial_x\phi = \rho$.

The velocity of the edge excitations v enters our theory through the gauge fixing condition. Notice that under the restricted gauge transformations the gauge fixing conditions (2.63) with different v cannot be transformed into each other. They are physically inequivalent. This agrees with our result that v in the gauge fixing condition is physical and actually determines the velocity of the edge excitations.

The Hamiltonian (2.70) is bounded from below only when $vq < 0$. The consistency of our theory requires v and q to have opposite signs. Therefore the sign of the velocity (the chirality) of the edge excitations is determined by the sign of the coefficient in front of the Chern-Simons terms.

The above results can be easily generalized to the generic FQH states described by (2.55) because the matrix K can be diagonalized. The resulting effective edge theory has a form

$$S_{\text{edge}} = \frac{1}{4\pi}\int dt\, dx[K_{IJ}\partial_t\phi_I\partial_x\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J] \; . \tag{2.72}$$

The Hamiltonian is given by

$$H_{\text{edge}} = \frac{1}{4\pi}\int dt\, dx V_{IJ}\partial_x\phi_I\partial_x\phi_J \; . \tag{2.73}$$

Therefore V must a positive definite matrix. Using this result one can show that a positive eigenvalue of K corresponds to a left-moving branch and a negative eigenvalue corresponds to a right-moving one.

The effective theory of the $\nu = 2/5$ FQH state is given by[21]

$$K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \; . \tag{2.74}$$

Since K have two positive eigenvalues, the edge excitations of the $\nu = 2/5$ FQH state have two branches moving in the same direction. The $\nu = 1 - \frac{1}{n}$ FQH state is described by the effective theory with

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -n \end{pmatrix} \; . \tag{2.75}$$

The two eigenvalues of K now have opposite signs, hence the two branches of the edge excitations move in opposite directions. This prediction was suggested in Ref. 40 and has been confirmed by numerical calculations.[33,34]

3. Charged Excitations and Electron Propagator on the Edges of Generic FQH States

In the last chapter we studied dynamics of the edge excitations of the FQH effects at low energies. We found that the low lying edge excitations are described by a free phonon theory that is exactly soluble. In this section we will concentrate on the generic charge excitations. In particular we will calculate the propagators of the electrons and the quasiparticles for the most general (abelian) FQH state