Perturbative Quantization of Split Chern-Simons Theory on Handlebodies and Lens Spaces by the BV-BFV Formalism

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Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Konstantin Wernli

von

Thalheim AG

Promotionskommission

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Zürich, 2019

Summary

Outline

This thesis is devoted to the study of low-order perturbative quantization in Chern-Simons theory with polarized coefficient Lie algebra ("split Chern-Simons theory") through methods of the BV-BFV formalism, and connections to theta invariants. The theta invariant is an invariant of framed rational homology spheres that arises in the study of perturbative quantization of Chern-Simons theory. The BV-BFV formalism is a tool which allows to compute the perturbative quantization of a gauge theory on manifolds with boundary, in a way compatible with cutting and gluing. The thesis is divided into six main chapters. After an introduction, in the second chapter we recall some preliminaries. The third chapter is devoted to a particular choice of gauge on product manifolds, the *axial* or *lightcone* gauge. The fourth chapter contains some facts about polarized Lie algebra. The fifth chapter applies the methods of the previous two chapters to the study of the perturbative quantization of split Chern-Simons theory on 3-manifolds, using the axial gauge. In the sixth chapter we discuss an alternative method to evaluate Feynman diagrams which can in principle be applied arbitrary Heegaard splittings. Several appendices are added that cover technical material, especially computations, some background material on de Rham currents and theta functions, and conventions.

Main results

Chapter 2 is just review and contains no original material. In Chapter 3 there are several results about the axial gauge. This gauge is not regular, in the sense that the associated propagator defined on $C_2^0(M) = M \times M - \Delta$ does not extend to a smooth form on the Fulton-MacPherson-Axelrod-Singer (FMAS) compactification of the configuration space $C_2(M) = \overline{C_2^0(M)}$. We show that the axial gauge can be seen as a certain limit of regular gauges defined by Riemannian metrics, and give a physical interpretation of this result. We use this approximation to define a regularization of the axial gauge, i.e. a way to define the products of the distributional propagator. We show that this regularization reproduces some regularizations in the literature. In Chapter 4 we consider polarized Lie Algebras and prove some new results concerning contractions of structure constants. In particular, we show that certain contractions are not independent under twists. In Chapter 5 we consider split Chern-Simons theory. We describe the Feynman graphs and rules of theory, and give an explicit description of the Feynman graphs of the theory on handlebodies. Using the results of chapter 3, we then present explicit computations in the case of Heegard splittings of genus one, i.e. the decomposition of a lens space into two solid tori. We show that this reproduces the result of Kuperberg-Thurston-Lescop that the theta invariant coincides with the Casson-Walker invariant, up to a framing dependent term. Interestingly, this depends on an assumption on the choice of polarization, which seems to indicate that the polarization has to be compatible with the algebraic structure on the space of fields. This follows from the results of Chapter 4. Finally, we describe another approach to compute the Feynman diagrams of the theory using the cohomology of configuration spaces and Jacobi theta functions on the torus, that might generalize to Heegard splittings of higher genera.

Acknowledgements

Completing this thesis has required support from many persons, both on a professional and a personal level. Some of them I want to thank in particular, but I am afraid the list is woefully incomplete.

Mathematics, like many things in life, shows its best side when shared and enjoyed with others. I (and this thesis) have profited immensely from discussions with many excellent senior, junior, and becoming scientists.

I am particularly grateful to my supervisor Alberto Cattaneo. He did not only provide me with interesting and challenging problems to work on, but was always helpful and patient when needed.

Special thanks I also owe to Pavel Mnëv, for his continued support throughout the various stages of this work and his patient and detailed explanations of a countless number of mathematical intricacies.

I am also very grateful for the support of and the many discussions with Kolya Reshetikhin. The project that Alberto, Pavel and Kolya started together carried me through my PhD and will also be a main focus of work in the future.

This thesis has also greatly improved thanks to a thorough and careful review by the referees. In particular, I want to thank Christine Lescop for the invitation to Grenoble and the hours that she spent with me going over all the details.

Somewhere between the professional and personal level was the support of my many colleagues at the Institute at University of Zurich. In particular I want to thank the changing cast of the office K54, in order of appearance[1]: Yaël Frégier, Vincent Schlegel, Michele Schiavina, Ricardo Campos, Santosh Kandel, Alessandro Valentino and Nima Moshayedi, and the regular guest stars Jonathan Lorand, Giovanni Canepa, Artem Kalmykov, Pavel Safronov, and Nicola Capacci, for endless discussions about all the topics imaginable and unimaginable (sometimes

[1]At least approximately

also mathematics) and the occasional drink or two.
A final thanks on the professional side goes to the staff of the institute and the university for providing an excellent work environment. Special thanks to Bettina Kurth, for taking care of many administrative questions and always having the answer to all my questions.
On a more personal level, I want to thank my entire family for their continuous support in all my endeavours, but especially my parents. Throughout my life I have been lucky to be surrounded by many good friends who have helped me always, I want to thank all of them, but especially Gian, Stefan and Rafaela.
The deepest gratitude of all I feel for the love I receive everyday from the love of my life, wife, partner, and best friend Ramona. Her unwavering support and trust are what makes everything possible.

Contents

Chapter 1

Introduction

This thesis is devoted to the study of the low-order perturbative quantization of Chern-Simons theory through methods of the BV-BFV formalism, and connections to invariants of framed 3-manifolds.

Let us briefly attempt to explain the concepts in this sentence and their origins. We will start with the BV-BFV formalism.

1.1 Path Integrals and BV-BFV formalism

A great challenge for mathematical physics is to explain the vast success that Feynman path integrals have had in physics, especially quantum field theory, by attributing them a rigorous meaning. Feynman path integrals are formal expressions of the form

$$Z = \int_{F_M} e^{\frac{i}{\hbar}S[\phi]}\mathcal{D}\phi, \tag{1.1}$$

for some manifold M that we think of as the d-dimensional space-time, F_M a space of fields associated to it and $S_M\colon F_M \to \mathbb{R}$ an action functional.

A subset of theories where this seems especially feasible, and which is related to many interesting mathematical problems, is given by *topological theories* (in the sense of Schwarz [Sch00b]), i.e. theories whose action functional does not depend on the metric of spacetime. Most known examples of topological theories, and the ones relevant for this work, are *gauge theories*, i.e. theories where there is a form of symmetry[1] acting on F_M that leaves S_M invariant. There are at least two well-known different ways to formalize expression (1.1):

[1]This symmetry should be a symmetry of F_M which is not induced by the symmetries of M, in physics language, a *local* rather than a *global* symmetry.

- *Perturbative Quantization*: Using that $\hbar$ is very small, replace (1.1) by the formal asymptotics of an oscillatory integral.

- *Functorial Quantization*: Axiomatize the properties that (1.1) is expected to have from locality (especially the Fubini theorem for integrals). In the easiest formulation this yields a (symmetric monoidal) functor

$$Z\colon \mathrm{Cob}^d \to \mathrm{Vect} \tag{1.2}$$

  from the d-dimensional cobordism category to the category of vector spaces.

Both these approaches are struggling with their own problems.
The problem of the first approach is that first of all it treats $\hbar$ as a formal parameter - rather than a physical constant -, and that it is also notoriously plagued with infinities that have to be made sense of - the problems of regularization and renormalization. However, it is also the approach which is closest to praxis in physics.
The second approach is mathematically more appealing and its proposal by Atiyah [Ati88] and Segal [Seg88] sparked a new field of mathematical research. However, if one is given a classical theory from physics in the form a space of fields F_M and an action functional S_M, it is quite hard (and indirect) to extract the corresponding functor, see [FQ93] for an example.
The BV-BFV formalism, proposed by Cattaneo, Mnev and Reshetikhin in [CMR14; CMR17], aims at closing the gap between the these two approaches by providing a perturbative quantization scheme for gauge theories on manifolds with boundary that is compatible with cutting and gluing, i.e. which can be made functorial in a certain sense.

1.2 Chern-Simons theory and topological invariants

This thesis is concerned with a particular example of the BV-BFV formalism, namely, its application to Chern-Simons theory. Chern-Simons theory (in its simplest form) is the 3-dimensional gauge theory with space of fields $F_M = \Omega^1(M, \mathfrak{g})$ and action functional

$$S_M[A] = \int \frac{1}{2}\langle A, \mathrm{d}A\rangle + \frac{1}{6}\langle A, [A, A]\rangle$$

where $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a quadratic Lie algebra, i.e a Lie algebra with a non-degenerate invariant symmetric bilinear pairing. In this thesis we will study the case where $\mathfrak{g} = V \oplus V^*$ decomposes as a sum of Lagrangian (i.e. maximally isotropic) subspaces ("split Chern-Simons theory") which are not necessarily Lie subalgebras. Such Lie algebras are studied more closely in Chapter 4.

After the seminal paper on Chern-Simons theory by Witten [Wit89] and related work by e.g. by Jeffrey [Jef92], Freed and Gompf [FG91] there were many related constructions of topological invariants (of 3-manifolds, or links and knots) in mathematics. A famous one is the Reshetikhin–Turaev construction [RT91] of a functorial quantum field theory (FQFT), which can be seen as an answer to the Chern-Simons path integral from the second approach above. In the realm of perturbative quantization (the first approach), the first mathematical constructions were given by Axelrod–Singer [AS91; AS94] and by Kontsevich [Kon94]. It was later shown by Kuperberg, Thurston and Lescop [KT99; Les04a; Les04b] that the lowest order coefficient in the perturbative expansion - on rational homology 3-spheres - evaluates, up to correctional terms, to the Casson-Walker invariant [Wal92]. These results, however, work only on closed manifolds, and only for the case of rational homology 3-spheres. In this thesis, we recover these results in particular cases, and suggest that extension to all 3-manifolds might be possible. A long-term goal of the BV-BFV program is to compare the perturbative quantization of Chern-Simons theory on manifolds with boundary with the asymptotics of the Reshetikhin-Turaev FQFT. In this thesis, a first step towards this goal is taken by analyzing the behavior of split Chern-Simons theory on manifolds with boundary.

1.3 Contents of this thesis

In this thesis, we describe the perturbative quantization of Chern-Simons theory on manifolds with boundary. We take on a rather naive approach, whose only inputs are the Chern-Simons action functional, which by assumption on the Lie algebra can be seen as a perturbation of the abelian BF functional, so we can apply the results of [CMR17]. In the example of lens spaces, we obtain results similar to the ones in the literature. Let us explain in detail how the thesis is laid out.

- In Chapter 2, we recall some concepts that this thesis relies upon, with the goal of a minimal amount of self-containedness. Technical details are avoided for the sake of examples. In Section 2.1 we review the perturbative quantization of gauge theories through the BV formalism. In Section 2.2 we sketch the methods of the BV-BFV formalism. In Section 2.3 we recall some constructions and results regarding perturbative Chern-Simons invariants.
- In Chapter 3, we analyze a particular gauge fixing for abelian BF theory known as the axial gauge. We recall first the "Riemann–Hodge" gauge-fixing, i.e. gauge fixing via a

Riemannian metric, on closed manifolds (Section 3.1) and manifolds with boundary (Section 3.2). We then define the axial gauge and show that it can be seen as a particular limit of Riemann-Hodge gauge fixings (Section 3.3). Using this, we propose a regularization of the axial gauge in Section 3.4.

- In Chapter 4, we investigate polarized Lie algebras, i.e quadratic Lie algebras endowed with a splitting $\mathfrak{g} = V \oplus V^*$ into Lagrangian subspaces. We show that the quadratic Casimir of these Lie algebras can be expressed in terms of invariants of the splitting. We give examples that show that these invariants are not invariants of the quadratic Lie algebra.

- Chapters 5 and 6 contain the bulk of the thesis. In Chapter 5 we analyze the split Chern-Simons theory mainly from the axial gauge. In Sections 5.1, 5.2 we present the perturbative quantization of split Chern-Simons theory on handlebodies, and list the Feynman diagrams up to 2-point order. In Section 5.3, we evaluate those Feynman diagrams in the solid torus in the axial gauge. In Sections 5.4 and 5.5, we explain how to compute the effective action on lens spaces . The implications for the weights of theta graphs for lens spaces are explained in Section 5.6.

- In Chapter 6, we pursue an alternative "cohomological" approach to the evaluation of the Feynman diagrams, which has the advantage that it might be generalized to 3-manifolds of higher Heegaard genus. The general idea is laid out in 6.1. Again, we consider the example of lens spaces in Section 6.2. Finally, we give some ideas as to how one might proceed for arbitrary 3-manifolds in Section 6.3.

- In Chapter 7 we review the results that were obtained, and some of the many questions that are left open.

There are five appendices that mainly contain computations and some background material. Namely,

- Appendix A contains computations for the effective action on the solid torus,

- Appendix B contains the computations necessary for the gluing of lens spaces,

- Appendix C contains some background material on Jacobi Theta functions, and computations that are used in the "cohomological" approach to the gluing of lens spaces,

- Appendix D contains some background material on de Rham currents,
- Appendix E contains a summary of conventions and notations.

Chapter 2

Preliminaries

In this chapter we review some concepts that are of central importance to this thesis. Namely, in Section 2.1 we discuss the perturbative quantization of gauge theories, i.e. how to compute a formal power series which serves as interpretation of the Feynman path integral, for a field theory with gauge symmetries. In Section 2.2, we review the BV-BFV formalism, a method to extend the perturbative quantization of gauge theories to manifolds with boundary in a way that is compatible with cutting and gluing. In Section 2.3 we review previous work on how the perturbative quantization of the Chern–Simons field theory gives rise to topological invariants. The aim of this chapter is not to provide complete and detailed reviews of these topics, but rather to make the thesis somewhat self-contained, while providing ample references to more detailed discussions of these subjects.

2.1 Perturbative quantization of gauge theories

In this section we specify what we mean by classical and quantum field theories, and gauge theories, in this work. Throughout this chapter we assume for simplicity that M is a compact oriented manifold without boundary.

2.1.1 Classical Field Theories

For the purpose of this work, we will say that a *classical field theory* in dimension d associates to each d-dimensional manifold M, possibly equipped with extra structure such as a Riemannian metric, a topological space F_M and a function $S_M \colon F_M \to \mathbb{R}$. We will call F_M the *space of fields* and S_M the *action functional*. The theories that we consider will always be *local*. This means that both the space of fields and the action functional should satisfy the physical concept of locality, whose precise mathematical implementation depends on the context. For this work it is sufficient to require that F_M is modeled on a space of sections of a bundle over M, and S_M is of the form

$$S_M[\phi] = \int_{x\in M} \mathcal{L}[\phi](x) \tag{2.1}$$

where $\mathcal{L}$ is a density-valued functional on F_M such that $\mathcal{L}[\phi](x)$ depends only on a finite number of jets of ϕ at x. For a more detailed discussion of locality, see e.g. [Cos11], [CMR11] or [Chr99].

Example 2.1.1. A standard example is free scalar field theory, which is defined on Riemannian manifolds (M, g) of any dimension d. Here the space of fields is $F_M = C^\infty(M)$ and the action functional is

$$S_M[\phi] = \int_M ||\mathrm{d}\phi||^2 dvol_g, \tag{2.2}$$

where $||\cdot||$ denotes the norm on $\Omega^\bullet(M)$ induced by g, and $dvol_g$ the Riemannian volume form.

2.1.2 Gauge theories

We will usually think of F_M as equipped with the structure of a Banach or Fréchet manifold. It this sense, one can say that gauge theories are theories where there is a distribution $\mathfrak{X}$ on F_M that annihilates the action S_M. In some cases this distribution might not be globally integrable, but we require that its restriction to the critical locus of S be integrable (i.e. that it be integrable "on-shell"). The most well-known cases are when F_M is the space of connections of a principal G-bundle $P \to M$, and the distribution on F_M is given the fundamental vector fields of the automorphism group of P.

Example 2.1.2. A famous example in both mathematics and physics is Yang-Mills theory with gauge group $G \subset GL(n)$, defined in 4-dimensional Riemannian manifolds. Here the space of fields is the space of connections on a principal G-bundle $P \to M$, and the action functional is

$$S_M[A] = \int_M \mathrm{tr}(F_A \wedge *F_A)$$

where F_A is the curvature of the connection A. This action is invariant under automorphisms of P by cyclicity of the trace.

The importance of gauge theories for mathematics was realized in the famous work of Donaldson [Don83] (see also [Jos11] for a more detailed exposition). In this work we will only be concerned with gauge theories where the distribution $\mathfrak{X}$ has this particular form, but there are other case. A well-known example of case where the distribution does not close is the Poisson Sigma Model ([SS94],[Ike94]).

2.1.3 Topological Field Theories

In this work we focus on topological field theories[1], i.e. field theories defined on manifolds without any extra structure[2] Two examples of topological field theories that will be important in this thesis are the BF and Chern–Simons theory.

Example 2.1.3 (BF theory)**.** BF theory[3] is defined for any dimension $d \geq 1$. The space of fields is given by $\Omega^1(M, \mathfrak{g}) \oplus \Omega^{d-2}(M, \mathfrak{g}^*) \ni (A, B)$, where $\mathfrak{g}$ is a Lie algebra and we think of $\Omega^1(M, \mathfrak{g})$ as the space of connections on a trivial principal G bundle, for some Lie group G integrating $\mathfrak{g}$ (and we set $\Omega^{-1}(M) = \{0\}$). The action functional is zero for $d = 1$, and for $d \geq 2$ is given by

$$S_M[A, B] = \int_M \langle B, F_A \rangle = \int_M \langle B, \mathrm{d}A \rangle + \frac{1}{2}\langle B, [A, A] \rangle \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$ extended to $\mathfrak{g}^*$ and $\mathfrak{g}$-valued forms, and $[\cdot, \cdot]$ the extension of the Lie bracket to $\mathfrak{g}$-valued forms. For further discussions of BF theory, see e.g. [Cat+95], [Mne08] or [Ros02].

Another theory which became famous after the breakthrough paper by Witten ([Wit89]) is Chern–Simons theory.

Example 2.1.4. Chern–Simons theory[4] is defined in dimension $d = 3$ only. Here the space of fields is $F_M = \Omega^1(M, \mathfrak{g})$ again interpreted as the space of connections on trivial principal bundle, and $\mathfrak{g}$ is a Lie algebra with an invariant bilinear symmetric pairing $\langle \cdot, \cdot \rangle$. The action functional is

$$S_M[A] = \int_M \frac{1}{2}\langle A, \mathrm{d}A \rangle + \frac{1}{6}\langle A, [A, A] \rangle. \tag{2.4}$$

Both examples discussed here are notably also gauge theories. In Chern–Simons theory, there are some subtleties to that statement, especially in the case of non-trivial principal bundles. A detailed discussion of these issues and related constructions in classical Chern–Simons theory can be found in [Fre95; Fre02].

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[1] In mathematics, there is a precise meaning associated with Topological Field Theory or TFT, which can be interpreted as generalization of our "definition".

[2] Although it should be noted that our manifolds are always oriented.

[3] The name stems from the fact that the action functional is the integral of BF, see below.

[4] Its name comes from the fact that the action functional is the integral of the Chern–Simons 3-form, named after [CS74].

2.1.4 Critical Points

Of particular importance in physics are the critical points of an action functional, usually called the Euler–Lagrange space

$$EL_M = \{\phi \in F_M : \delta S_\phi = 0\} \subset F_M \tag{2.5}$$

Here δ is the variational derivative, which we interpret as the de Rham differential on F_M. The equation

$$\delta S_\phi = 0 \tag{2.6}$$

is called the Euler–Lagrange equation. Elements of EL_M, i.e. solutions to the Euler–Lagrange equation, give the classical configurations of the theory. This is called the *principle of least action*. For example, the Euler–Lagrange space in scalar field theory (Example 2.1.1 above) is the space of harmonic functions on M (with respect to the Laplace–Beltrami operator).

2.1.5 Quantum Field Theory and Perturbative Quantization

There are many aspects to quantum field theory, and many mathematical formulations of various problems related to it (see e.g. the two volumes [Del99a; Del99b]). A particular question that arises in quantum field theory is the computation of expectation values of observables. For us, an observable is a functional $O\colon F_M \to R$. According to Feynman's interpretation of the principle of least action ([Fey42; Fey49; Fey50]), its expectation value $\langle O \rangle$ can be computed as a weighted average over all fields by

$$\langle O \rangle = \frac{1}{Z} \int_{F_M} e^{\frac{i}{\hbar} S_M} O(\phi) \mathcal{D}\phi \tag{2.7}$$

where

$$Z = \int_{F_M} e^{\frac{i}{\hbar} S_M} \mathcal{D}\phi \tag{2.8}$$

is called the partition function. These integrals, as they stand, are not defined: There is no measure $\mathcal{D}\phi$ on F_M that gives sensible answers. If the field theory is one-dimensional, then one can make sense of similar integrals using the Wiener measure [Wie27; WP34], but in higher dimensions these approaches have been unsuccessful, see [GJ87] for a detailed discussion. Hence, other approaches to define the integrals (2.7), (2.8) were required. The mathematical version of an approach that has been very successful in physics is called *perturbative quantization*, of which we give a very brief recollection here. The procedure is discussed in detail in [Mne17, Section

3], [Res10] and [Eti02]. A more leisurely exposition can be found in [Pol05]. For integral (2.8) it takes the following form. The idea is to start with a finite-dimensional oscillatory integral

$$I_\hbar = \int_F e^{\frac{\mathrm{i}}{\hbar}S}\mu \tag{2.9}$$

where F is a manifold of finite dimension $\dim F = n$ and μ some density on F. We are then interested in the $\hbar \to 0$ limit. We quote the following theorem (3.48 in Reference [Mne17]):

Theorem 2.1.1. *Assume that S has finitely many non-degenerate critical points. As $\hbar \to 0$, integral (2.9) has an asymptotic expansion of the form*

$$\begin{aligned} I_\hbar \sim \sum_{x_0\in\mathrm{Crit}(S)} & e^{\frac{i}{\hbar}S(x_0)}(2\pi\hbar)^{\frac{n}{2}}|\det(S''(x_0)|^{-\frac{1}{2}}e^{\frac{\pi\mathrm{i}}{4}\mathrm{sign}S''(x_0)}\mu_{x_0} \\ & \cdot\exp\left(\frac{\mathrm{i}}{\hbar}\sum_{\Gamma}\frac{(-\mathrm{i}\hbar)^{\#\mathrm{loops}(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\psi_\Gamma\right) \end{aligned} \tag{2.10}$$

where $S''(x_0)$ is the Hessian of S at x_0, $\mathrm{sign}S''(x_0)$ *denotes its signature, μ_{x_0} is the value of the density around x_0 (we assume we have chosen coordinates $y_1,\dots,y_n$ such that $\mu = \mu_{x_0}|dy_1\cdots dy_n|$ is constant on a neighborhood of x_0). The sum in the second line runs over connected graphs where every vertex has valence at least 3, and ψ_Γ is the* weight *of the graph (briefly explained below).*

The weight of a graph ψ_Γ is constructed by a state sum, roughly as follows:

1. Label all half-edges of the graph with indices $i_k \in \{1,\dots,n\}$.
2. To every vertex v with incident half-edges $\mathrm{i}_1,\dots i_{val(v)}$ (here $val(v)$ denotes the valence of v, i.e. the number of incident half-edges) associate the number
$$N_v := \left(\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_{val(v)}}}S\right)(x_0)$$
3. To every edge e connecting half-edges i_1, i_2, associate $N_e := (S''(x_0))^{-1}_{i_1 i_2}$ (here $S''(x_0))^{-1}$ is the matrix of the inverse of the Hessian of S in the given coordinate system)
4. For every labeling, multiply the corresponding numbers N_v and N_e over all vertices and edges,
5. sum over all labelings.

A labeling is also called a state of the graph, hence the weight ψ_Γ is called a state sum. Note that if S is a polynomial, every term is S of homogeneous degree $k \geq 3$ will give rise to vertices of valence k. Again, for more details we refer to the references above.

After having understood the behavior of finite-dimensional integrals of the form (2.9), we now define the value of the integrals (2.7) and(2.8) by an infinite-dimensional analogue of the asymptotic series (2.10). All actions above can be brought to the form

$$S[\phi] = \int_M (\phi, D\phi) + \textit{higher order terms in } \phi, \tag{2.11}$$

where D is a differential operator on sections of a bundle over M and $(\cdot, \cdot)$ some pairing on sections of this bundle. Hence, the asymptotic series will be constructed from the determinant of D, and the integral kernel of the inverse of D - the "propagator", and takes the form

$$Z = (\det D)^{-\frac{1}{2}} \exp\left(\sum_{\Gamma} \frac{(-\mathrm{i}\hbar)^{\#\mathrm{loops}(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\psi_\Gamma\right), \tag{2.12}$$

where, again, the sum goes over all connected graphs where all vertices have valence greater than 3. The graphs Γ appearing in the sum are known as *Feynman graphs*, and the rules for the evaluation of ψ_Γ are known as *Feynman rules.* In addition, we label the vertices by points $x_i \in M$, and integrate over all these points. Since integral kernels of inverses to differential operators are usually distributional, some extra care is needed in the multiplication (of distributions) and integration processes in the construction of the state sum. This is the problem of regularization and renormalization, see e.g. Section 3.4 or the discussion in [Mne17, Section 3.11.2] and references given there.

2.1.6 BV Formalism

Theorem 2.1.1 has the crucial assumptions that the function in question has finitely many critical points, and that they are non-degenerate. However, in gauge theories with non-trivial symmetries this is never the case: for any critical point $x \in F_M$, the orbit of x under the distribution $\mathfrak{X}$ consists of critical points, in particular, non of these is isolated. The perhaps most advanced solution to this problem goes under the name of *BV formalism*, after Batalin and Vilkovisky who introduced it in the early 1980's ([BV81; BV83], see also [HT94] for a general review of gauge fixings from a physical perspective). It consists of two steps, the classical and the quantum BV formalism. The classical BV formalism can be summarized as follows[5].

- Find a -1-shifted graded symplectic manifold [6] $(\mathcal{F}_M, \omega)$, such that $F_M \subset (\mathcal{F}_M)^0$.

[5] Detailed discussions and more pedagogical introductions can be found in [Mne08],[Mne17],[Cos11],[Sch93].

[6] All gauge fixing formalisms require the language of supergeometry, and sometimes graded manifolds. See [CS11] for an introduction to these concepts, and the Appendix of [CMR14] for an extension to the infinite-dimensional context.

- Find an action functional $\mathcal{S}_M \in \mathcal{O}(\mathcal{F}_M)$ such that $\mathcal{S}_M\big|_{F_M} = S_M$.

$\mathcal{S}_M$ is subject to the condition

$$\{\mathcal{S}_M, \mathcal{S}_M\} = 0 \tag{2.13}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket induced by ω. Equation (2.13) is called the *Classical Master Equation.* The Quantum BV formalism can be summarized as follows:

- Find a Lagrangian $\mathcal{L} \subset \mathcal{F}_M$ (called the *gauge-fixing Lagrangian*) such that $\mathcal{S}_M\big|_{\mathcal{L}}$ has isolated critical critical points.

- Define the partition function $Z_{\mathcal{L}}$ as of

$$Z_{\mathcal{L}} = \int_{\mathcal{L}\subset\mathcal{F}_M} e^{\frac{i}{\hbar}\mathcal{S}_M} \tag{2.14}$$

Under the assumption that $\mathcal{S}_M$ satisfies the Quantum Master Equation

$$\Delta e^{\frac{i}{\hbar}S_M} = 0 \tag{2.15}$$

the partition function $Z_{\mathcal{L}}$ is independent of the deformation class of $\mathcal{L}$. Here Δ denotes the "BV operator" or "BV Laplacian", which in Darboux coordinates (x,p) can be expressed by

$$\Delta = \sum_i \pm \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i},$$

see e.g. [Šev06] or [Khu04]. Again, all of this makes sense in the finite-dimensional case, and has to extended to the infinite-dimensional case with care. In particular the Quantum Master Equation (2.15) has to be regularized, and integral (2.14) has to defined by formal asymptotic expansion similar to Theorem 2.1.1, but extended to supermanifolds. Independence of the gauge-fixing choices has to be proven *a posteriori.*

The Batalin-Vilkovisky formalism and related conceptes feature in many mathematical approaches to Quantum Field Theory, we will only name a few of them here, for example the long line of work of Cattaneo, Felder and others on the Poisson Sigma Model[7] ([CF00; CF01c; CF01b; CF01a; CF01d; CFT02; CF04; CF11; BCM12]), the Costello-Gwilliam approach to Factorization algebras [CG16], its use in algebraic perturbative Quantum Field Theory by Fredenhagen and Rejzner [Rej11; FR12]. Worth mentioning is also the recent work by Felder and Khazdan [FK14] on the existence of BV actions given a classical action functional, and the uniqueness of

[7] The Poisson Sigma Model is a gauge theory where other gauge fixings fail, and one has to resort to the BV formalism.

the "BRST cohomology", i.e. the cohomology of the operator $\mathcal{Q} = \{S, \cdot\}$. More recently it was generalized by Cattaneo, Mnev and Reshetikhin [CMR14; CMR17] to a gauge-fixing formalism on manifolds with boundary compatible with cutting and gluing, see the next section.

Residual fields and effective actions

In topological theories, it is often the case that it is not possible to find a gauge-fixing Lagrangian $\mathcal{L} \subset \mathcal{F}_M$, i.e. such that the action restricted to it has a unique (or at least finitely many) isolated critical point(s). This happens e.g. when there is a continuous family of critical points of the original action functional unrelated by gauge transformations, or when the symmetry does not act freely. As an easy example, consider abelian BF theory, i.e. BF theory from example 2.1.3 with Lie algebra $\mathfrak{g} = \mathbb{R}$, say, in dimension 2. The space of fields is $F_M = \Omega^1(M) \oplus \Omega^0(M)$, the action functional is

$$S[A, B] = \int_M B \wedge \mathrm{d}A$$

and its critical points are given by $\mathrm{d}A = \mathrm{d}B = 0$. A gauge transformation is given by $A \mapsto A + \mathrm{d}c$, where $c \in \Omega^0(M)$. If M is a compact surface, then the critical points up to gauge transformations are $H^1(M) \oplus H^0(M)$. The solution in the BV formalism is to use the formalism of effective theories: Rather that integrating over all fields $\mathcal{F}_M$, one splits out a finite-dimensional BV subspace $\mathcal{V}_M$, called the space of *residual fields*, such that there exists a gauge-fixing Lagrangian $\mathcal{L}$ in a complement $\mathcal{Y}$ to $\mathcal{V}_M$, i.e $\mathcal{F}_M \cong \mathcal{V}_M \times \mathcal{Y}$, and one defines the partition function to be

$$Z[\mathsf{c}] = \int_{\gamma \in \mathcal{L} \subset \mathcal{Y}} e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_M[\mathsf{c}+\gamma]}. \tag{2.16}$$

The effective action is then defined by

$$\exp\left(\frac{\mathrm{i}}{\hbar} S_{eff}[\mathsf{c}]\right) = Z[\mathsf{c}]. \tag{2.17}$$

The partition function now is a BV cocycle on $\mathcal{V}_M$, i.e. $\Delta_{\mathcal{V}_M} Z[\mathsf{c}] = 0$, and only its cohomology class is invariant under deformations of the gauge fixing Lagrangian $\mathcal{L}$. See also [CM08; BCM12].

2.1.7 AKSZ construction

Usually, when given an action functional S_M, finding the correct BV extension is a long (but straightforward) procedure. The AKSZ construction [Ale+97], named after Alexandrov, Kontsevich, Schwarz and Zaboronsky, can be seen as a procedure that directly outputs a BV theory, i.e. the association of data $(\mathcal{F}_M, \Omega_M, \mathcal{S}_M)$ to any compact oriented manifold of dimension d. It can be summarized as follows.

- The input is dg Hamiltonian manifold of degree $d-1$, called the *target*, i.e.
  - a graded manifold $\mathcal{N}$,
  - a symplectic form $\omega = \delta\alpha$ of degree $d-1$,
  - a *Hamiltonian function* of degree $d-1$, i.e. a function Θ of degree $d-1$ satisfying $\{\Theta, \Theta\} = 0$.
- The output is the BV theory which associates to a manifold M the data
  - space of fields $\mathcal{F}_M = \mathrm{Map}(T[1]M, \mathcal{N})$
  - symplectic form $\Omega_M = \int_M \omega_{ab}\delta X^a \wedge \delta X^b$
  - action functional $S_M[X] = \int_M \alpha_a(X^a)\mathrm{d}X^a + \Theta(X)$.

Here X^a is the component of the map $X \in \mathrm{Map}(T[1]M, \mathcal{N})$ in coordinates $x^1, \ldots x^k$ on the target $\mathcal{N}$. $\mathrm{Map}(\mathcal{M}, \mathcal{N})$ denotes the internal Hom in the category of graded manifolds (see e.g. [CMR14, Appendix D]) and $T[1]N$ the shifted tangent bundle. See [Mne17] for a coordinate independent formulation and further references.
In the cases of relevance to us, $\mathcal{N}$ will be a graded vector space, so we can identify $\mathrm{Map}(T[1]M, \mathcal{N}) \cong \Omega^\bullet(M) \otimes \mathcal{N}$. Two examples are of particular interest:

Example 2.1.5 (BV extension of BF theory)**.** Let $\mathcal{N} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[d-2]$. Let e^i a basis for $\mathfrak{g}$ with dual basis e_i. Then $\sum_i \mathrm{d}e_i \wedge \mathrm{d}e^i$ defines a symplectic form of degree $d-1$ on $\mathcal{N}$, and the function $(a,b) \mapsto \langle b, [a,a]\rangle$ is Hamiltonian of degree d. The corresponding BV theory is given by

$$\begin{aligned}
\mathcal{F}_M &= \Omega^\bullet(M, \mathfrak{g})[1] \oplus \Omega^\bullet(M, \mathfrak{g}^*)[d-2] \ni (\mathsf{A}, \mathsf{B}) \\
\mathcal{S}_M[\mathsf{A}, \mathsf{B}] &= \frac{1}{2}\int_M \langle \mathsf{B}, \mathrm{d}\mathsf{A}\rangle + \frac{1}{2}\langle \mathsf{B}, [\mathsf{A}, \mathsf{A}]\rangle \\
\omega_M &= \int_M \langle \delta\mathsf{A}, \delta\mathsf{B}\rangle
\end{aligned}$$

Here $\Omega^\bullet(M)[l]$ means that we give k forms internal degree (or "ghost" number) $l-k$ (see also Appendix E.2). In particular, the fields of ghost number 0 are $F_M = \Omega^1(M, \mathfrak{g}) \oplus \Omega^{d-2}(M, \mathfrak{g}^*)$, and the action functional evaluated on those fields is the action functional of BF theory from example 2.1.3.

Example 2.1.6 (BV extension of Chern–Simons Theory)**.** Also Chern–Simons has an AKSZ formulation. Namely, $\mathcal{N} = \mathfrak{g}[1]$, where $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a quadratic Lie algebra, and let $\Theta(a) =$

$\langle a, [a,a]\rangle$. Let e^i be a basis of $\mathfrak{g}$, and $\omega = \frac{1}{2}\sum_{i,j}\langle e^i, e^j\rangle \mathrm{d}e_i \wedge \mathrm{d}e_j$ defines a 2-shifted symplectic form on $\mathcal{N}$. Then the output of the AKSZ construction with target $\mathcal{N}$ is

$$\begin{aligned}\mathcal{F}_M &= \Omega^\bullet(M,\mathfrak{g})[1] \ni \mathsf{A}\\ \mathcal{S}_M[\mathsf{A},\mathsf{B}] &= \frac{1}{2}\int_M \langle \mathsf{A}, \mathrm{d}\mathsf{A}\rangle + \frac{1}{6}\langle \mathsf{A}, [\mathsf{A},\mathsf{A}]\rangle \\ \omega_M &= \int_M \langle \delta\mathsf{A}, \delta\mathsf{A}\rangle\end{aligned} \tag{2.18}$$

2.2 BV-BFV formalism

The discussion above considers only the cases when the spacetime manifold M does not have a boundary. The case when the spacetime manifold does have a boundary requires extra considerations. In particular, gauge theories on manifolds with boundary are usually invariant under gauge transformations only up to boundary terms. A common approach is to fix boundary conditions on the fields, and in many cases this is enough to proceed with quantization. However, from the perspective of functorial quantum field theories (FQFTs) after Atiyah-Segal [Ati88; Seg88], it is not desirable to fix a particular boundary condition [8]. A formalism for the treatment of gauge theories which is compatible both with the cutting and gluing of manifolds, and perturbative quantization, is the *BV-BFV formalism*, developed by Cattaneo, Mnev and Reshetikhin in a series of papers [CMR11; CMR15; CMR14; CMR17] (a condensed introduction can be found in [CMR16]). Here BFV is short for Batalin, Fradkin and Vilkovisky, who developed an approach to deal with gauge theories in the Hamiltonian setting [BV77; BV83; BF86]. Like the BV formalism, it comes in a classical and quantum version.

Remark 2.2.1. From now on we will restrict the discussion to the case when the spaces of fields are vector spaces. This is the case for the examples that are of importance of this paper, and simplifies the discussion somewhat, especially for perturbative quantization. Extension to the non-linear case for the classical case can be found in [CMR14]. A version of the quantum BV-BFV formalism for AKSZ models with nonlinear target was developed in [CMW18a].

2.2.1 Classical BV-BFV formalism

The main reference for the classical case is [CMR14], an introduction can also be found in [Sch15]. First, we define a BFV vector space (for background on the BFV complex see [Sta97; Sch10]).

[8] The reason being that - in the formal Fubini theorem for path integrals which is encoded in the functoriality of FQFTs - one wants to “integrate” over the space of boundary fields.

Definition 2.2.1. A BFV vector space is a triple $(\mathcal{F}^\partial, \omega^\partial, \mathcal{Q}^\partial)$, where $\mathcal{F}^\partial$ is a $\mathbb{Z}$-graded vector space, ω^∂ is a symplectic form of degree 0 on $\mathcal{F}^\partial$, and $\mathcal{Q}^\partial$ is a degree +1 vector field which is symplectic and satisfies $(\mathcal{Q}^\partial)^2 = 0$.

By degree reasons $\mathcal{Q}^\partial$ is automatically Hamiltonian with Hamiltonian function $\mathcal{S}^\partial$.
We now also formalize the idea of a BV vector space that was implicit in the discussion above.

Definition 2.2.2. A BV vector space is a quadruple $(\mathcal{F}, \omega, \mathcal{Q}, \mathcal{S})$, where $\mathcal{F}^\partial$ is a $\mathbb{Z}$-graded vector space, ω^∂ is a symplectic form of degree -1 on $\mathcal{F}^\partial$, $\mathcal{Q}^\partial$ is a degree +1 vector field which is Hamiltonian with Hamiltonian function $\mathcal{S}$ and satisfies $(\mathcal{Q}^\partial)^2 = 0$.

Notice that $\mathcal{Q}^2 = 0$ is equivalent to $\{\mathcal{S}, \mathcal{S}\} = 0$, i.e. the CME (2.13). In accordance with the above, we say that a d-dimensional BV theory is an association of a BV vector space to every closed d-dimensional manifold. We now want to extend this to manifolds with boundary. The idea is to associate to the boundary of a manifold a BFV vector space, and to the bulk a suitable generalization of a BV vector space such that these data are compatible. The solution is the notion of BV-BFV vector space as introduced in [CMR14].

Definition 2.2.3. Let $\mathcal{F}^\partial = (\mathcal{F}^\partial, \omega^\partial = \delta\alpha^\partial, \mathcal{Q}^\partial)$ be a BFV vector space with exact symplectic form ω^∂. A BV-BFV vector space over $\mathcal{F}^\partial$ is a quintuple $(\mathcal{F}, \omega, \mathcal{Q}, \mathcal{S}, \pi)$, where $\mathcal{F}$ is a $\mathbb{Z}$-graded vector space, ω is a degree -1 symplectic form, $\mathcal{Q}$ is a degree +1 vector field, $\mathcal{S}$ is a degree 0 function on $\mathcal{F}$ and $\pi\colon \mathcal{F} \to \mathcal{F}^\partial$ is a surjective submersion such that $\mathcal{Q}^2 = 0$, $\delta\pi\mathcal{Q} = \mathcal{Q}^\partial$ and

$$\iota_Q \omega = \delta S + \pi^* \alpha^\partial. \tag{2.19}$$

Remark 2.2.2. Equation (2.19) should be thought of as a generalization of the CME (2.13). In fact, in the case $\mathcal{F}^\partial = \{0\}$ the definition of BV-BFV vector space reduces to that of an ordinary BV vector space.

We are now ready to define the notion of classical BV-BFV theory. Namely, a d-dimensional BV-BFV theory associates to closed $d-1$-dimensional manifold Σ an exact BFV vector space $\mathcal{F}^\partial_\Sigma$ and to a d-dimensional manifold M with boundary ∂M a BV-BFV vector space $\mathcal{F}_M$ over the BFV vector space $\mathcal{F}^\partial_{\partial M}$. Let us give some remarks on this construction.

Remark 2.2.3. i) Like the linearity condition, i.e. the fact that we consider only vector spaces, the exactness condition on ω^∂ can be dropped. See [CMR14, Remark 3.3] for a short discussion.

ii) We continue to require locality from BV-BFV theories. That means the vector spaces $\mathcal{F}, \mathcal{F}^\partial$ are given by sections of a sheaf and as such are typically infinite-dimensional (over $\mathbb{R}$ or $\mathbb{C}$). They can be equipped with natural Banach or Fréchet topologies depending on the situation. $\mathcal{S}$ is subject to similar locality conditions as above. Again, a good reference for further discussion is [Cos11].

iii) With enough care, a BV-BFV theory yields a functor from a cobordism category (perhaps equipped with extra structure) to a category of vector spaces where composition is given by (homotopy) fibered product. We refer to [CMR14, Section 4] for a more detailed discussion.

2.2.2 Quantum BV-BFV formalism

In [CMR17], the Quantum BV-BFV formalism was introduced, which allows to perform perturbative quantization of BV-BFV theories in a way that is still compatible with cutting and gluing. Roughly, the idea is to combine geometric quantization of $(\mathcal{F}^\partial, \omega)$ and perturbative quantization as discussed in Section 2.1. The main steps can be summarized as follows.

i) *Polarizing:* Choose a polarization $\mathcal{P}$ on $(\mathcal{F}^\partial, \omega)$ with smooth leaf space $\mathcal{B}^\mathcal{P}$. In our case it will be enough to find a splitting

$$\mathcal{F}^\partial = \mathcal{B}_1^\mathcal{P} \oplus \mathcal{B}_2^\mathcal{P}, \tag{2.20}$$

where both $\mathcal{B}_1^\mathcal{P}$ and $\mathcal{B}_2^\mathcal{P}$ are Lagrangian subspaces of $\mathcal{F}^\partial$. One can choose either $\mathcal{B}_i^\mathcal{P}$ as base space or fibers of the polarization respectively. If necessary, change α^∂ by an exact term such that it vanishes on the fibers of $\mathcal{P}$. To preserve Equation (2.19) one has to change $\mathcal{S}$ by a boundary term.

ii) *Extraction of boundary fields:* Choose a section σ of $\mathcal{F} \to \mathcal{F}^\partial \to \mathcal{B}^\mathcal{P}$, and a splitting $\mathcal{F} = \sigma(\mathcal{B}^\mathcal{P}) \times \mathcal{Y}$ (subject to certain requirements discussed in [CMR17]). We immediately suppress σ from the notation.

iii) *Choice of residual fields:* Proceed to split $\mathcal{Y} = \mathcal{V} \times \mathcal{Y}'$ into odd symplectic vector spaces, such that $\mathcal{V}$ is finite-dimensional and there is a Lagrangian $\mathcal{L} \subset \mathcal{Y}'$ on which the action $\mathcal{S}$ has isolated critical points. We now have a splitting $\mathcal{F} = \mathcal{B}^\mathcal{P} \times \mathcal{V} \times \mathcal{Y}'$, and accordingly we write $\mathsf{X} = \mathbb{X} + \mathsf{x} + \xi$ for $\mathsf{X} \in \mathcal{F}$.

iv) *Perturbative Quantization:* Finally, define the state or partition function formally by

$$\psi(\mathbb{X}, \mathsf{x}) = \int_{\xi \in \mathcal{L}_M \subset \mathcal{Y}} e^{\frac{i}{\hbar}\mathcal{S}[\mathbb{X}+\mathsf{x}+\xi]} \tag{2.21}$$

where again, (2.21) is to be computed by formally extending the superspace version of Theorem 2.1.1 to infinite dimensions.

In the finite-dimensional case, Equation (2.21) is an example of a "BV pushforward in families" introduced in [CMR17]. In the infinite-dimensional case, definition (2.21) has to be interpreted via the Feynman graphs and rules discussed in [CMR17] and [CMW17]. See Section 5.2.5 for a discussion of Feynman graphs and rules relevant in our case.

The state (2.21) is a functional on both $\mathcal{V}$ and $\mathcal{B}^{\mathcal{P}}$. We think of it as an element of $\widehat{\mathcal{H}}^{\mathcal{P}} = \widehat{S}\mathcal{V}^* \otimes \mathcal{H}^{\mathcal{P}}$, where $\mathcal{H}^{\mathcal{P}}$ is a certain space of functionals on $\mathcal{B}^{\mathcal{P}}$ that should be thought of as a geometric quantization of $(\mathcal{F}^{\partial}, \omega)$, and $\widehat{S}$ denotes the formal completion of the symmetric algebra. The geometric quantization of the boundary action $\mathcal{S}^{\partial}$ yields a coboundary operator $\Omega^{\mathcal{P}}$ on $\mathcal{H}^{\mathcal{P}}$. The precise construction of this space and the coboundary operator Ω are not relevant for this work, the interested reader is again referred to [CMR17, Section 4.1]. Also $\mathcal{V}$ carries a coboundary operator, the BV Laplacian $\Delta_{\mathcal{V}}$, and the state is a cocycle in the bicomplex $\widehat{\mathcal{H}}^{\mathcal{P}}$:

$$(\hbar^2 \Delta + \Omega^{\mathcal{P}})\psi = 0. \tag{2.22}$$

Equation (2.22) is called the modified Quantum Master Equation (mQME).

An important feature of the quantum BV-BFV formalism is that the perturbative expansions associated to manifolds with boundary can be glued together using a form of the BKS[9] pairing discussed in [CMR17]. We will briefly review how this works in Section 5.4 below.

2.2.3 BV-BFV quantization of abelian BF theory

As an example of BV-BFV quantization we briefly review the example of abelian BF theory that was discussed in [CMR17]. Namely, consider a manifold M with boundary ∂M. The space of fields is

$$\mathcal{F} = \Omega^\bullet(M)[1] \oplus \Omega^\bullet(M)[d-2]. \tag{2.23}$$

The space of boundary fields is

$$\mathcal{F}^{\partial} = \Omega^\bullet(\partial M)[1] \oplus \Omega^\bullet(\partial M)[d-2] \ni (\mathsf{A}^{\partial}, \mathsf{B}^{\partial}). \tag{2.24}$$

The symplectic form on the boundary fields is given by

$$\omega^{\partial} = \int_{\partial M} \delta \mathsf{A}^{\partial} \wedge \delta \mathsf{B}^{\partial}. \tag{2.25}$$

[9]For Blattner, Kostant, Sternberg. See [BW97]

There are two obvious polarizations of the space of boundary fields by declaring either the A^∂ or the B^∂ fields to be coordinates along the base. We will denote coordinates on the quotient by $\mathbb{A}$ and $\mathbb{B}$ respectively, and we call the first polarization the $\mathbb{A}$-representation and the second polarization the $\mathbb{B}$-representation. Compatibility with cutting and gluing motivates the following choice of polarization: Choose a decomposition $\partial M = \partial_1 M \sqcup \partial_2 M$. Then we choose the $\mathbb{A}$-representation on $\partial_1 M$ and the $\mathbb{B}$-representation on $\partial_2 M$. This means that we have

$$\mathcal{B}^{\mathcal{P}} = \Omega^\bullet(\partial_1 M)[1] \oplus \Omega^\bullet(\partial_2 M)[d-2]. \tag{2.26}$$

The next is the extraction of the boundary fields. Here we opt for a singular extension by fields which drop to zero immediately outside the boundary[10] This yields

$$\mathcal{Y} = \Omega^\bullet(M, \partial_1 M)[1] \oplus \Omega^\bullet(M, \partial_2 M). \tag{2.27}$$

Here for $\iota\colon V \hookrightarrow M$ a submanifold we denote by $\Omega^\bullet(M,V) = \{\omega \in \Omega^\bullet(M), \iota^*\omega = 0\}$ the subcomplex of forms whose pullback to the boundary vanishes. In the next step, we choose the space of residual fields

$$\mathcal{V}_M = H^\bullet(M, \partial_1 M)[1] \oplus H^\bullet(M, \partial_2 M)[d-2] \ni (\mathsf{a}, \mathsf{b}). \tag{2.28}$$

A gauge-fixing Lagrangian and the corresponding propagator $\eta \in \Omega^{d-1}(M \times M - \Delta_M)$[11] are constructed in [CMR17], see also section 3.2. Then one can compute the state ψ explicitly and it given by

$$\psi = T_M \exp\left(\frac{\mathrm{i}}{\hbar} S_{eff}\right) \tag{2.29}$$

where T_M is a constant related to the Reidemeister-Ray-Singer torsion of M and the effective action is given by

$$S_{eff} = (-1)^{d-1}\left(\int_{\partial_1 M} \mathsf{b}\mathbb{A} - \int_{\partial_2 M} \mathsf{a}\mathbb{B}\right) - \int_{\partial_1 M \times \partial_2 M} \pi_1^*\mathbb{B}\ \eta\ \pi_2^*\mathbb{A}. \tag{2.30}$$

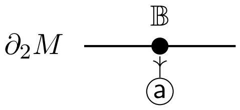
These three terms come from the very simple Feynman diagrams depicted in figure 2.1 below.

2.3 Perturbative Chern–Simons Invariants

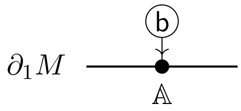
It was an idea of Schwarz that quantities such as the partition function, or expectation values of observables, computed for topological theories, should yield topological invariants of the

[10] This should be understood as a suitable limit, see [CMR17].

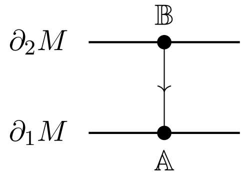
[11] Here $\Delta_M \subset M \times M$ denotes the diagonal.



(a) Diagram corresponding to $\int_{\partial_2 M} \mathsf{a}\mathbb{B}$.



(b) Diagram corresponding to $\int_{\partial_1 M} \mathsf{b}\mathbb{A}$.



(c) Diagram corresponding to $\int_{\partial_2 M \times \partial_1 M} \pi_1^* \mathbb{B}\ \eta\ \pi_2^* \mathbb{A}$.

Figure 2.1: Feynman graphs in abelian BF theory. The boundaries are depicted schematically. A black vertex denotes integration.

underlying space-time. He gave a first example of this fact in [Sch78], where he argued that the partition function of abelian BF theory is the Ray-Singer ([RS71]) torsion of the spacetime manifold. See also the review [Sch00b]. Applying this line of thought to perturbative quantization, the weights of Feynman graphs (or rather, the weighted sum over Feynman graphs of the same loop order) should yield topological invariants as well.

Let us turn to the theory which is of key interest for this work, the Chern–Simons theory. The seminal paper by Witten [Wit89], and, shortly after, the construction of the Reshetikhin–Turaev TQFT [RT91], sparked a huge interest in both the mathematical and physical aspects of this theory, see e.g. the collection [And+11], and its links to a wide variety of fields of mathematics and physics[12]. In this work we will restrict ourselves entirely to the perturbative formulation of Chern–Simons theory. The first investigations in that direction were probably done by Guadagnini, Martellini and Minchev ([GMM89]) and Bar-Natan [Bar91]. After that, there were two slightly different approaches which explained how the weights of Feynman graphs in Chern–Simons theory give rise to topological invariants of M, one by Axelrod and Singer [AS91; AS94] and the one by Kontsevich [Kon94]. In both approaches, one sees the framing anomaly observed by Witten in [Wit89], but it arises in different ways. We will briefly review these results[13].

[12]On a personal note, the author recalls N. Berkovits calling Chern–Simons theory "almost trivial" during a lecture series he gave at the Villa da Leyva summer school 2015, which - at least to the author's understanding - is exactly the intersection of "non-trivial" with "accessible" which allows for fruitful mathematical investigations.

[13]An important concept that we do not mention here is the idea of graph cohomology introduced by Kontsevich in the same paper, namely, that the perturbative invariants are indexed by cocycles in a certain graph complex, which has led to the study of other graph complexes with rather remarkable results e.g. by T. Willwacher [Wil14].

2.3.1 Perturbative Chern–Simons Invariants after Axelrod-Singer

In a sequence of two papers ([AS91; AS94]) Axelrod and Singer constructed a perturbative quantization of Chern–Simons theory along the following lines.

i) Fix a flat "reference" connection A_0 in a principal G-bundle whose associated local system is acyclic, i.e. $H^\bullet(M, \mathrm{d}_{A_0}) = 0$.

ii) Because A_0 is flat, one can then decompose the Chern–Simons action functional of any connection $A = A_0 + \alpha$ as $S_{CS}[A] = S_{CS}[A_0] + S_{CS}^{A_0}[\alpha]$, where $S_{CS}^{A_0}[\alpha]$ is the Chern–Simons functional twisted by A_0, i.e.

$$S_{CS}^{A_0}[\alpha] = \int_M \frac{1}{2}\langle \alpha, \mathrm{d}_{A_0}\alpha\rangle + \frac{1}{6}\langle \alpha, [\alpha,\alpha]\rangle. \tag{2.31}$$

iii) Gauge fix $S_{CS}^{A_0}[\alpha]$. In our language, pass to the BV-extended[14] twisted Chern–Simons functional, given by

$$S_{CS}^{A_0}[\mathsf{A}] = \int_M \frac{1}{2}\langle \mathsf{A}, \mathrm{d}_{A_0}\mathsf{A}\rangle + \frac{1}{6}\langle \mathsf{A}, [\mathsf{A},\mathsf{A}]\rangle. \tag{2.32}$$

iv) Pick a Riemannian metric g on M and consider the adjoint $\mathrm{d}^*_{A_0}$ of d_{A_0}. Then, a gauge-fixing Lagrangian is given by $\mathcal{L} = \Omega_{coex} = \{\mathsf{A} \in \Omega^\bullet(M,\mathfrak{g}), \mathrm{d}^*_{A_0}\mathsf{A} = 0\}$. Here the last equality follows from vanishing of d_{A_0}-cohomology. On $\mathcal{L}$, 0 is an isolated critical point of (2.32).

v) Let η be the integral kernel of $K = \mathrm{d}^*_{A_0} \circ \Delta^{-1}_{A_0}$, here $\Delta_{A_0} = \mathrm{d}_{A_0}\mathrm{d}^*_{A_0} + \mathrm{d}^*_{A_0}\mathrm{d}_{A_0}$ is the associated Laplacian. This is a left inverse of d_{A_0} on $_{\mathcal{L}}$. Also, let f_{abc} be the structure constants of $\mathfrak{g}$ in an orthonormal basis.

vi) Construct the perturbative series by Feynman graphs and rules: Roughly, Feynman graphs are connected 3-valent graphs. We label the half-edges by indices $1 \leq a, b, c \leq \dim \mathfrak{g}$, and vertices by a point $x_i \in M$. Then we associate $\frac{\mathrm{i}}{\hbar} f_{abc}$ to a vertex with half-edges labeled by a, b, c and to an edge between vertices x and y we associate $\frac{\hbar}{\mathrm{i}}\delta_{ab}\eta(x,y)$. Then multiply all this contributions and sum over all labelings of half-edges, and integrate over all points $x_i \in M$.

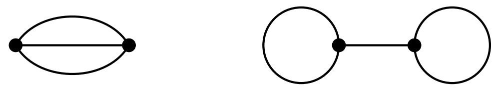
[14] Axelrod and Singer use the Faddeev-Popov description, but it is equivalent to this BV formulation, see e.g. [Mne17, Section 4].

They show that the integrals in question always converge, but depend on the choice of metric g. Explicitly, the lowest order contributions have the form

$$\Theta_g(M) = \frac{1}{12}\sum_{a,b,c} f_{abc}f_{abc}\int_{M\times M}\eta_{12}^3 \tag{2.33}$$

$$\mathrm{D}\ _g(M) = \frac{-1}{8}\sum_{a,b,c} f_{aab}f_{bcc}\int_{M\times M}\eta_{11}\eta_{12}\eta_{22} \tag{2.34}$$

coming from the Theta graph[15], and the "dumbbell graph", depicted below in figure 2.2a and 2.2b below. They show that the weights of these graphs depend on the metric g, how-



(a) The theta, or sunset, graph. (b) The dumbbell graph.

Figure 2.2: Lowest order graphs in Chern-Simons theory

ever, they argue that for any framing f the combination $I_2(M, A_0, f) = \Theta(M, f) + D_g(M) - \frac{\dim Gh}{48\pi}CS_{grav}(g, f)$ is independent of the metric g, where $CS_{grav}(g, f)$ is the Chern–Simons action evaluated on the Levi-Civita connection using the framing.

Later work

In two papers some years after, Bott and Cattaneo [BC98; BC99] described how the approach of Axelrod and Singer could be translated to the setting of the trivial connection on a trivial bundle, at least for homology 3-spheres. Here the "zero modes" need to be taken care of: In this case, the problem is that the trivial connection has a nontrivial stabilizer given by the constant function, and they discover a framing dependence similar to the one observed by Axelrod and Singer. Later, it was explained by Cattaneo and Mnev how to generalise this construction to arbitrary 3-manifolds ([CM08]) by using the language of BV effective actions, briefly mentioned in section 2.1.6 above.

2.3.2 Perturbative Chern–Simons Invariants after Kontsevich

Kontsevich proposed a different way of constructing the Chern–Simons perturbation series: Namely, using a framing of the manifold M in question, define the propagator on the boundary of $C_2(M)$ (which is isomorphic to the sphere bundle of the tangent bundle over M) by the standard

[15] Which Axelrod and Singer call the "sunset" graph - a rather more poetic terminology which unfortunately hasn't persisted.

volume form in S^2, and then extend it to all of M. The main difference is that the gauge is fixed from the beginning, but in a way that does not introduce Riemannian metrics. His construction was later rephrased by Kuperberg and Thurston [KT99]. They show that that one can extract from it an invariant of framed homology spheres $Z_n(M)$ (given, essentially, by configuration space integrals), for any $n \geq 1$. They prove that there exists a correction $\delta_n(M)$, which is also an invariant of framed homology spheres, such that the difference $J_n(M) = Z_n(M) - \delta_n(M)$ is independent of the framing, and identify $J_1(M)$ as a multiple of the Casson invariant (see [AM89]) for integral homology spheres. Building upon their work, Lescop shows in [Les04a; Les04b] that the same is true also for rational homology 3-spheres: Namely, the first of these invariants, given by $\int_{C_2(M)} \omega^3$, where ω is constructed using a choice of framing is the Casson-Walker invariant, up to a framing-dependent term. However, there is a particular framing in which the correction vanishes. In the present work, we shall find a similar statement from the BV-BFV formalism, in the case of lens spaces. See Section 5.6.

Chapter 3

Axial Gauge

In this chapter we review gauge-fixings in the BV formalism for abelian BF and abelian Chern–Simons theories on closed manifolds and manifolds with boundary. We discuss two different types of gauge fixings: The Riemann-Hodge gauge constructed by a choice of Riemannian metric, and the axial gauge on product manifolds. The main new result is that the propagator in the axial gauge can be seen as a limit of Riemann–Hodge propagators (Theorem 3.3.2) associated to degenerating metrics. Based on this, we propose a regularization of the axial gauge, and show that it has some desirable properties (Section 3.4).

3.1 Gauge fixing in BF and Chern–Simons Theories on closed manifolds

In this section we review in detail the ideas of gauge fixings for abelian BF and abelian Chern–Simons theories in the context of effective actions in the BV formalism and its relation to the homological algebra of the complexes of forms.

3.1.1 Some facts about contracting triples

We use the following conventions for contracting triples. For more information, see e.g. [CM08] and references therein.

Definition 3.1.1 (Contracting triple)**.** Let $V = (V^\bullet, \mathrm{d})$ be a cochain complex with cohomology $H = H^\bullet(V)$. Then a *contracting triple* for V is a triple of linear maps (ι, p, K), where $\iota\colon H \hookrightarrow V$ $p\colon V \twoheadrightarrow H$ are of degree 0 and $K\colon V \to V$ is of degree -1 satisfying

1. For all closed elements $\alpha \in V$, there exists some $\beta \in V$ such that

$$\iota([\alpha]) = \alpha + \mathrm{d}\beta \tag{3.1}$$

$$p(\alpha) = [\alpha] \tag{3.2}$$

2. (ι, p, K) satisfies the following relations[1]

$$\mathrm{d} \circ K + K \circ \mathrm{d} = id_V - \iota \circ p \tag{3.3}$$

$$K \circ \iota = p \circ K = K \circ K 0 \tag{3.4}$$

We think of K as a partial inverse of d, after making a choice of embedding of the cohomology and projection to the cohomology.

3.1.2 Contracting triples and Hodge decompositions

Any contracting triple (ι, p, K) on a cochain complex (V, d) defines a weak Hodge decomposition of this complex

$$V = \operatorname{im} \iota \oplus \operatorname{im} \mathrm{d} \oplus \operatorname{im} K. \tag{3.5}$$

Maybe this merits a short proof. Indeed, (3.3) implies that any $v \in V$ can be decomposed as $v = \mathrm{d}Kv + K\mathrm{d}V + \iota p v$. We thus have to prove that the intersections are trivial. First, if $v \in \operatorname{im} \iota \cap \operatorname{im} \mathrm{d}$, then $v = \iota[\alpha]$ (for some closed form α) and (using (3.1),(3.3)) $0 = [v] = pv = p\iota[\alpha] = [\alpha]$, whence $v = 0$. If $v \in \operatorname{im} K \cap \operatorname{im} \iota$, then $v = \iota[\alpha]$, but $[\alpha] = p\iota[\alpha] = pv = 0$ (since $pK = 0$), hence $v = 0$. Finally if $v \in \operatorname{im} K \cap \operatorname{im} \mathrm{d}$ we have $\mathrm{d}v = Kv = pv = 0$ (since $\mathrm{d}^2, K^2, pK = 0$) and then $v = \mathrm{d}Kv + K\mathrm{d}v + \iota p v = 0$. This completes the proof[2].
the Restricted to $\operatorname{im} K$, the operator $\mathrm{d} \colon \operatorname{im} K \to \operatorname{im} \mathrm{d}$ is invertible, with inverse K. Assume that V is equipped with nondegenerate symmetric pairing $(\cdot, \cdot) \colon V \otimes V \to \mathbb{R}$. Then the function

$$S \colon V \to \mathbb{R}$$
$$v \mapsto \frac{1}{2}(v, \mathrm{d}v)$$

has 0 as a unique non-degenerate critical point when restricted to $\operatorname{im} K$.
We can also consider the complex (V^*, d^*) dual to V. The contracting triple (ι, p, K) defines a

[1] In the literature, the requirements in Equation (3.4) are sometimes relaxed. However, any contracting triple is homotopic to one satisfying (3.4), see [CM08].

[2] Conversely, choosing a complement L of $\ker \mathrm{d}$ in V and a complement $H \cong H^\bullet(V)$ of $\operatorname{im} \mathrm{d}$ in $\ker d$ defines a weak Hodge depcomposition $V = H \oplus \operatorname{im} \mathrm{d} \oplus L$, one obtains a contracting triple with $\iota \colon H^\bullet(V) \cong H, p \colon V \twoheadrightarrow H$ and $K(\mathrm{d}\alpha) = p_L(\alpha)$ and $K = 0$ on the complement of $\operatorname{im} \mathrm{d}$.

Hodge decomposition

$$V^* = \operatorname{im} p^* \oplus \operatorname{im} K^* \oplus \operatorname{im} \mathrm{d}^*. \tag{3.6}$$

Consider the function

$$S\colon T^*[-1]V = V \oplus V^*[-1] \to \mathbb{R}$$
$$(a, b) \mapsto \langle b, \mathrm{d}a \rangle$$

where $\langle \cdot, \cdot \rangle\colon V \otimes V^* \to \mathbb{R}$ denotes the canonical pairing (notice that this pairing is nonzero only on the degree 0 term). Then this function has 0 as a unique critical point when restricted to $\operatorname{im} K \oplus \operatorname{im} K^*$ where the operator $\mathrm{d} \oplus \mathrm{d}^*$ is invertible. This is as a special case of the example above in the case where $V = (W \oplus W^*[-1], d \oplus d^*)$ and pairing the natural extension of the canonical pairing to V.

3.1.3 On closed manifolds: Gauge fixing in the BV formalism

Throughout this section we let M be a closed manifold of dimension d. The BV space of fields of abelian BF theory is $\mathcal{F}_M = \Omega^\bullet(M)[1] \oplus \Omega^\bullet(M)[d-2] \ni (\mathsf{A}, \mathsf{B})$ which we think of as $T^*[-1](\Omega^\bullet(M)[1])$ via the Poincaré pairing

$$(\mathsf{A}, \mathsf{B}) \mapsto \int_M \mathsf{A} \wedge \mathsf{B}. \tag{3.7}$$

Here the shift acts on the total degree, meaning that the $p-$form component of A has ghost number (internal degree) $1-p$ and the $p-$form component of B has ghost number $d-2-p$. The superfields A and B have total degree (ghost number plus form degree) 1 and $d-2$, respectively. It is equipped with the canonical symplectic structure of ghost number -1 given by

$$\omega = \int_M \delta B \wedge \delta A \tag{3.8}$$

where we denote δ the de Rham differential on the space of fields. Explictly, this means that we consider the bilinear pairing of ghost number -1 on $\mathcal{F}_M$ given by

$$\omega((\mathsf{A}_1, \mathsf{B}_1), (\mathsf{A}_2, \mathsf{B}_2)) = \int_M \mathsf{B}_1 \wedge \mathsf{A}_2 - \mathsf{B}_2 \wedge \mathsf{A}_1 \tag{3.9}$$

The BF action is

$$\mathcal{S}[\mathsf{A}, \mathsf{B}] = \int_M \mathsf{B} \wedge \mathrm{d}\mathsf{A} = (\mathsf{B}, \mathrm{d}\mathsf{A}). \tag{3.10}$$

For an operator L on differential forms, we will denote its (formal) adjoint with respect to the Poincaré pairing by L'. The notation L^* will be reserved for formal adjoints with respect to pairings induced by a metric.

To define the BV gauge-fixing we must find a splitting $\mathcal{F}_M = \mathcal{V}_M \times \mathcal{Y}$ and a Lagrangian $\mathcal{L} \subset \mathcal{Y}$ such that the action restricted to $\mathcal{L}$ has a unique non-degenerate critical point. According to the above, given any contracting triple (ι, p, K) of $\Omega^\bullet(M)$[3], we can define $\mathcal{V}_M = \operatorname{im}\iota \oplus \operatorname{im} p'$ and the gauge fixing Lagrangian $\mathcal{L} = \operatorname{im} K \oplus \operatorname{im} K'$ (the fact that $\mathcal{L}$ is Lagrangian follows from $K^2 = 0$).

In the case of abelian Chern–Simons theory we have $\dim M = 3$, $\mathcal{F}_M = \Omega^\bullet(M)[1] \ni \mathsf{A}$ and $S = \frac{1}{2}\int_M \mathsf{A} \wedge \mathsf{dA}$. In this case, a contracting triple for $\Omega^\bullet(M)$ defines a BV gauge-fixing via $\mathcal{V}_M = \iota(H), \mathcal{L}_K = \operatorname{im} K$.

Propagators for abelian BF theory

We endow $\Omega^\bullet(M)$ with its natural Fréchet topology. Contracting triples for $\Omega^\bullet(M)$ introduced above are closely related to propagators of abelian BF theory. Namely, if the maps in the contracting triple (ι, p, K) are continuous, then they can be represented by integral kernels which are currents on $M \times M$ (see Appendix D for a brief summary of the theory of currents). Denote by η the integral kernel of K, it is a current on $M \times M$ of degree $d-1$. We will call any such η an *abelian BF propagator*. We will say that such a propagator is *regular* if it can be represented by a smooth form, also denoted η, on $M \times M - \operatorname{diag}$, that extends smoothly to the FMAS[4] compactification of the configuration space $C_2(M)$.

Now let us fix some notation for the following. Namely, any $\iota\colon H^\bullet(M) \to \Omega^\bullet(M)$ satisfying (3.1) is specified by a choice of a system of representatives $R = \{\chi_1, \ldots, \chi_k\} \subset \Omega^\bullet(M)$ (via $\iota_R[\chi_i] = \chi_i$). The Poincaré pairing on $H^\bullet(M)$ defines an isomorphism $H^\bullet(M)^* \to H^\bullet(M)$ and via composition with ι an injective map $H^\bullet(M)^* \to \Omega^\bullet(M)$. The images under this map of the basis of $H^\bullet(M)^*$ dual to $[\chi_1], \ldots, [\chi_k]$ are denoted $\chi^1, \ldots, \chi^k$, these are forms with property that $\int_M \chi^i \chi_j = \delta^i_j$. This defines a projection $p_R\colon \Omega^\bullet(M) \to H^\bullet(M)$ by

$$p_R(\alpha) = \sum_{j=1}^{k} (-1)^{d \deg \chi_i} \chi_j \int_M \chi^j \wedge \alpha$$

which satisfies $p_R \circ \iota_R = \mathrm{id}_H$ (see Appendix E for the choice of signs). We will now specialize to contracting triples which are of the form (ι_R, p_R, K). The integral kernels of such chain contractions have special properties given in the next two Lemmata.

[3]Notice that a contracting triple for V is also a contracting triple for $V[1]$.

[4]For Fulton-MacPherson-Axelrod-Singer [FM94; AS94], see also Appendix E

Lemma 3.1.1. *Fix a system of representatives $R = \{\chi_1, \dots, \chi_k\}$. Then (ι_R, p_R, K) is a continuous contracting triple for $\Omega^\bullet(M)$ if and only if the integral kernel η of K satisfies*

$$\mathrm{d}\eta = \delta_M^{(d)} + (-1)^{d-1} \sum_i (-1)^{d \cdot \deg \chi_i} \chi_i \chi^i \tag{3.11}$$

in the sense of currents.

Proof. As usual, we will write pushforward of currents by integrals, see appendix D. Let $\eta(x, y)$ be the kernel of a map K. Then we have

$$\begin{aligned}\int_y (\mathrm{d}\eta) \wedge \omega(y) &= \int_y \mathrm{d}(\eta(x,y) \wedge \omega(y)) - (-1)^{d-1} \eta \wedge \mathrm{d}\omega(y) \\ &= (-1)^d \left(\mathrm{d} \int_y \eta(x,y) \wedge \omega(y) + \int_y \eta(x,y) \wedge \mathrm{d}\omega(y) \right) \\ &= (-1)^d (\mathrm{d}K\omega + K\mathrm{d}\omega).\end{aligned}$$

Here we have used that the pushforward commutes with the differential of currents up to a sign (equation (D.3)). This shows that $\mathrm{d}\eta$ is the integral kernel of $(-1)^d(dK + Kd)$. K therefore defines a chain contraction if and only if

$$\mathrm{d}\eta = \delta_M^{(d)} + (-1)^{d-1} \sum_i (-1)^{d \cdot \deg \chi_i} \pi_1^* \chi_i \pi_2^* \chi^i$$

which is the integral kernel of $(-1)^d(\mathrm{id} - \iota \circ p)$[5]. □

Lemma 3.1.2. *Let (ι_R, p_R, K) a continuous contracting triple for $\Omega^\bullet(M)$ such that its integral kernel η extends smoothly to $C_2(M)$. Then the integral kernel satisfies*

$$\mathrm{d}\eta = (-1)^{d-1} \sum_i (-1)^{d \cdot \deg \chi_i} \pi_1^* \chi_i \pi_2^* \chi^i \tag{3.12}$$

$$\pi_*^\partial \eta \equiv (-1)^{d-1} \tag{3.13}$$

where $\pi^\partial \colon \partial C_2(M) \to M$ is the natural projection. Conversely, any differential form satisfying (3.12), (3.13) for a system of reprentatives $R = \{\chi_1, \dots, \chi_k\}$ defines a continuous contracting triple (ι_R, p_R, K).

Proof. Let η be a form on $C_2(M)$ which is the integral kernel of a map K. Then, by the fiberwise Stokes' theorem (Equation (E.6)) we have

$$\begin{aligned}\int_y (\mathrm{d}\eta(x,y)) \wedge \omega(y) &= (-1)^d \left(\mathrm{d}_x \int_y \eta(x,y)\omega(y) + \int_y \eta(x,y)\mathrm{d}\omega(y) \right) + \left(\int_\partial \eta(x,y) \right) \omega(x) \\ &= (-1)^d (\mathrm{d}K\omega + K\mathrm{d}\omega) + \left(\int_\partial \eta(x,y) \right) \omega(x)\end{aligned}$$

[5] In our conventions, $\delta_M^{(d)}$ is the integral kernel of $(-1)^d$ the identity, which can be verified from the local expression, i.e. the case $M = \mathbb{R}^n$

If η satisfies (3.12), (3.13), then K is a chain contraction. On the other hand, if K is a chain contraction, then

$$\int_y (\mathrm{d}\eta(x,y)) \wedge \omega(y) - \left(\int_\partial \eta(x,y)\right)\omega(x) = (-1)^d(\omega(x) - \iota \circ p\omega(x)).$$

This is equivalent to

$$\int_y (\mathrm{d}\eta(x,y) + (-1)^d P(x,y))\omega(y)) = \left((-1)^d + \int_\partial \eta(x,y)\right)\omega(x).$$

Since this property holds for all ω we conclude that both sides vanish independently. □

3.1.4 Riemann–Hodge propagators

In this section we discuss a particular class of propagators used in Chern–Simons theory and abelian BF theory, the *Riemann–Hodge propagators*, introduced in [AS91]. As we will explain, they generalize the classical Lorenz gauge used in electromagnetism.

Definition

Let (M, g) be a compact[6] Riemannian manifold of dimension n and denote $*_g$ the Hodge star defined by g. Define the L^2-inner product on $\Omega^\bullet(M)$ by

$$(\omega, \eta) = \int_M \omega \wedge *_g \eta \tag{3.14}$$

for $\omega, \eta \in \Omega^\bullet(M)$. Forms of different degree are orthogonal to each other with respect to this inner product. Denote by d_g^* the codifferential on M associated to g, i.e. the formal adjoint of the de Rham differential d with respect to $(\cdot, \cdot)$. It satisfies

$$(\mathrm{d}\omega, \eta) = (\omega, \mathrm{d}^*\eta) \tag{3.15}$$

for all $\omega, \eta \in \Omega^\bullet(M)$.

Definition 3.1.2. For any $0 \leq p \leq n$, the *Hodge-de Rham Laplacian on p-forms* $\Delta_g^{(p)} : \Omega^p(M) \to \Omega^p(M)$ is defined by

$$\Delta_g^{(p)} = \mathrm{d}\mathrm{d}_g^* + \mathrm{d}_g^*\mathrm{d}.$$

[6]Compactness is only needed so that we do not have to take care of convergence in the integrals, all concepts in this section can be generalized to non-compact manifolds by making the right adjustments.

The Hodge–de Rham Laplacians on p-forms assemble into an operator, also known as Hodge–de Rham Laplacian, $\Delta_g\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$. The operator $\Delta_g^{(0)}$ is also known as Laplace-Beltrami operator, and reduces to (minus) the ordinary Laplace operator if (M, g) is $\mathbb{R}^n$ with the Euclidean metric. Also notice that both d and d^* commute with the Laplacian.

It follows directly from (3.15) that the Hodge-de Rham Laplacian on closed manifolds is symmetric with respect to the L^2-inner product on forms, i.e. for $\omega, \eta \in \Omega^\bullet(M)$ we have

$$(\Delta\omega, \eta) = (\eta, \Delta\omega). \tag{3.16}$$

Definition 3.1.3. The kernel of $\Delta_g^{(p)}$ is called *space of harmonic p-forms* and denoted

$$\ker \Delta_g^{(p)} =: \mathrm{Harm}_g^p(M).$$

We quote two essential theorems in Hodge theory (see e.g. [Rha84]):

Theorem 3.1.3. *The space of harmonic p-forms is isomorphic to degree p de Rham cohomology of M:*

$$\mathrm{Harm}_g^p(M) \cong H^p(M).$$

Theorem 3.1.4 (Hodge decomposition). $\Omega^\bullet(M)$ *has a decomposition given by*

$$\Omega^\bullet(M) \cong \ker \Delta_g \oplus (\ker \Delta_g)^\perp \cong \mathrm{Harm}^p(M) \oplus \mathrm{im}\, \mathrm{d} \oplus \mathrm{im}\, \mathrm{d}_g^* \tag{3.17}$$

which is orthogonal with respect to the L^2-inner product (3.14) *on forms.*

Denote ι_{harm} the inclusion $\mathrm{Harm}(M) \hookrightarrow \Omega^\bullet(M)$, where $\mathrm{Harm}(M) = \bigoplus_{p=0}^n \mathrm{Harm}^p(M)$, and p_{harm} the projection $\Omega^\bullet(M) \to \mathrm{Harm}(M)$ and Denote by $P_{harm} = \iota_{harm} \circ p_{harm}\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ the orthogonal projection onto harmonic forms. Consider the operator $(\Delta_g + P_{harm})$ on $\Omega^\bullet(M)$. Since the operator Δ_g is invertible when restricted to $(\ker \Delta_g)^\perp$, and P_{harm} is just the identity when restricted to harmonic forms, this operator is the direct sum of two invertible operators, and hence also invertible. We are now ready to define the Hodge chain contraction.

Definition 3.1.4. The *Riemann–Hodge contraction* is the operator $K_g\colon \Omega^\bullet \to \Omega^{\bullet-1}$ defined by

$$K_g := \mathrm{d}_g^* \circ (\Delta_g + P_{harm})^{-1}. \tag{3.18}$$

The following are standard properties of the Riemann–Hodge contraction (see e.g. [AS91; AS94; CM08]).

Proposition 3.1.5. *The Riemann–Hodge contraction satisfies* *i)* $dK_g + K_g d = \mathrm{id} - P_{harm}$,

ii) $p_{harm} \circ K_g = 0$,

iii) $K_g \circ \iota_{harm} = 0$,

iv) $K_g \circ K_g = 0$,

i.e. $(\iota_{harm}, p_{harm}, K_g)$ *is a continuous contracting triple. In addition, the Hodge contraction is symmetric with respect to the Poincaré pairing on forms, i.e. we have*

$$\int_M \omega \wedge K[\tau] = \int_M K[\omega] \wedge \tau \tag{3.19}$$

The Riemann–Hodge contraction specifies the gauge-fixing Lagrangian $\mathcal{L}_g = \mathrm{im}(K) = \mathrm{im}(d^*)$ for BV Chern–Simons theory, or $\mathcal{L}_g = \mathrm{im}(K) \oplus \mathrm{im}\, K' = \mathrm{im}\, \mathrm{d}^* \oplus \mathrm{im}\, \mathrm{d}^*$ in BF theory.

Relation to Lorenz gauge

Let us briefly explain the relation to the Lorenz gauge fixing in electromagnetism. There we have a 1-form electromagnetic potential $A = A_\mu dx^\mu$, say, on $\mathbb{R}^4$, with the standard metric. The action is $S = \int_{\mathbb{R}^4} F_{\mu\nu} F^{\mu\nu}$, where $F = dA = F_{\mu\nu} dx^\mu dx^\nu$ is the field strength. Hence the action is invariant under the gauge symmetry $A \to A + df$, where f is any smooth function. The traditional *Lorenz gauge fixing* is given by $\partial^\mu A_\mu = \mathrm{d}^* A = 0$. This is only a partial gauge fixing as one can still change A by a harmonic form, but e.g. imposing that we consider only gauge transformations vanishing at infinity fixes the gauge completely. In that case $\mathrm{d}^* A = 0 \Leftrightarrow A \in \mathrm{im}\, \mathrm{d}^*$, i.e. A lies in the gauge-fixing Lagrangian associated to the Riemann–Hodge contraction.

3.1.5 Integral kernel of the Riemann–Hodge contraction

The following result was proven by Axelrod and Singer ([AS91; AS94]):

Lemma 3.1.6. K_g *has an integral kernel* $\eta_g \in \Omega^{d-1}(C_2^0(M))$ *which extends smoothly to the compactified configuration space* $C_2(M)$*, and thus defines a smooth propagator for abelian BF theory.*

The integral kernel of the Riemann–Hodge contraction can be constructed from the inhomogeneous Green's form, i.e. the fundamental solution of

$$(\Delta_g + P_{harm})\omega = \tau, \tag{3.20}$$

where $\omega, \tau \in \Omega^\bullet(M)$ are inhomogeneous forms. It can be understood as the integral kernel of the operator G (see Appendix D.4) satisfying $\Delta_g G = G\Delta_g = 1 - P_{harm}$. If $\alpha \in \Omega^\bullet(M \times M - \Delta)$ is the Green's form, then

$$\eta_{12} = \mathrm{d}_1^* \alpha \tag{3.21}$$

is the integral kernel of K_g. Green's functions are usually quite hard to obtain, and for Green's forms this is even harder. But in some special cases there are explicit formulas for the Green's form, most notably in the case of $\mathbb{R}^n$ (or some particular subsets) with the standard metric, or $\mathbb{T}^n$. Let us briefly look at the example of the 2-torus $\mathbb{T}^2$.

3.1.6 Example: The 2-torus $\mathbb{T}^2$

On the 2-torus $\mathbb{T}^2$ we have the usual euclidean coordinates $(t, \theta) \in (\mathbb{R}/\mathbb{Z})^2$. Thinking of it as quotient of the plane, rather than a product of circles, we will often denote the coordinates also by $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$. Representing the 2-torus as a quotient of the complex plane by the lattice generated by $1, \tau$, where $\tau \in \mathbb{H} = \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ defines a complex structure on the torus, and hence - since $\mathbb{T}^2$ has complex dimension 1 - a conformal structure. The complex coordinate on the torus is $z = x + \tau y$, and a particular representative of the conformal structure is

$$g = dz \cdot d\bar{z} = (dx + \tau dy) \cdot (dx + \bar{\tau} dy) = dx^2 + 2 \operatorname{Re} \tau dx dy + |\tau|^2 dy^2. \tag{3.22}$$

In particular the metric associated to a purely imaginary $\tau = \mathrm{i}\beta$, for $\beta > 0$, is a product metric. The Green's function for this metric is known explicitly (see e.g. [Oog15] or [BL17]) and given by $g(z, w) = g(z - w)$ where

$$g(z) = \frac{-1}{4\pi} \log \left| \frac{\vartheta_1(z, \tau)}{\eta(\tau)} \right|^2 + \frac{1}{2} \frac{(\operatorname{Im} z)^2}{\operatorname{Im} \tau}. \tag{3.23}$$

Here ϑ_1 is the Jacobi theta function vanishing at integer lattice points, and $\eta(\tau)$ is the Dedekind eta function. See Appendix C for the conventions on theta functions. From this Green's function, in this example one can find the Green's form α by simply multiplying $\alpha = g(z, w)(d\bar{z} - d\bar{w})(dz - dw)$ (this is particular to the case of the torus). One can then compute (see Proposition C.2.2) the propagator[7] corresponding to the metric induced by τ as

$$\eta^\tau(z, w) = \frac{1}{2\pi} \mathrm{d} \arg(\vartheta_1(z - w, \tau)) - \frac{\operatorname{Im}(z - w)}{\operatorname{Im} \tau} \mathrm{d} \operatorname{Re}(z - w) \tag{3.24}$$

[7]The notation is slightly unfortunate but unambiguous: $\eta(\tau)$ denotes the Dedekind eta function while η^τ denotes the propagator associated to τ.

It comes as no surprise that this propagator behaves well with respect to changing to an equivalent complex structure, namely, if

$$\varphi = \begin{pmatrix} m & p \\ n & q \end{pmatrix} \in SL(2,\mathbb{Z})$$

we have (Proposition C.2.3)

$$\varphi^*\eta^\tau = \eta^{({}^T\varphi).\tau} \tag{3.25}$$

where on the left hand side we interpret φ as acting on $\mathbb{R}^2/\mathbb{Z}^2$ (by matrix multiplication), and on the right hand side we denote

$${}^T\varphi = \begin{pmatrix} q & p \\ n & m \end{pmatrix}$$

the anti-transpose of φ, and by $({}^T\varphi).\tau$ the standard action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$:

$$({}^T\varphi).\tau = \frac{q\tau + p}{m\tau + n}. \tag{3.26}$$

3.1.7 Expression in terms of the heat kernel

Below it will be useful to express the Riemann–Hodge propagator in terms of the *heat kernel*. For us it will be convenient to use the inhomogeneous heat kernel on differential forms, and unless explicitly stated otherwise, if we speak of the heat kernel associated to a Riemannian metric we mean the associated heat kernel on inhomogeneous forms. Let us recall some basic facts about heat kernels. A good reference for general heat kernel techniques is [BGV03].

Heat kernel on differential forms

The heat kernel on differential forms was considered e.g. by Patodi ([Pat71]), but it appeared first in the work of Conner ([Con56]). It can be seen as the fundamental solution of the heat equation on $\Omega^\bullet(M)$. Let $\omega \in \Omega^\bullet(M)$. Then $u(x,t) \in C^\infty(\mathbb{R}_{>0}, \Omega^\bullet(M))$ solves the heat equation with initial condition ω if

$$\begin{cases} (\partial_t + \Delta)u(x,t) = 0, & x \in M, t \in \mathbb{R}, \\ \lim_{t\to 0} u(x,t) = \omega(x) & x \in M. \end{cases} \tag{3.27}$$

The fundamental solution $p(t,x,y) \in C^\infty(\mathbb{R}_{>0}, \Omega^\bullet(M) \otimes \Omega^\bullet(M))$ of the heat equation satisfies

$$\begin{cases} (\partial_t + \Delta_x)p(t,x,y) = 0, & x,y \in M, t \in \mathbb{R}, \\ \lim_{t\to 0} p(t,x,y) = \delta(x,y) & x,y \in M \end{cases} \tag{3.28}$$

where the second limit is understood in the distributional sense. As a fundamental solution, it has the property that if $\omega \in \Omega^\bullet(M)$, then

$$u(t,x) = \int_{y \in M} p(t,x,y) \wedge \omega(y) \tag{3.29}$$

solves the heat equation with initial condition ω.

The Hodge-de Rham Laplacian is a generalized Laplacian in the sense of [BGV03]. As established in loc. cit., its symmetry (3.16) implies that its heat kernel is self-adjoint and hence it is essentially self-adjoint. Denoting its unique self-adjoint extension also by Δ, it follows that the heat kernel p_t is the kernel of the operator $P_t = \exp(-t\Delta)$. On compact manifolds, the Laplacian has discrete spectrum[8], and P_t has an expansion in terms of eigenforms $\{\phi_j\}_j$ of the Laplacian[9]

$$p_t(x,y) = \sum_j e^{-t\lambda_j} \phi_j(x) \wedge *\phi_j(y). \tag{3.30}$$

It follows that

$$\lim_{t\to\infty} P_t\omega = P_{harm}\omega. \tag{3.31}$$

The heat kernel can be used to construct the kernel of the inverse of the Laplace operator needed in the definition of the Hodge contraction. Namely, the operator G defined on the orthogonal complement of harmonic forms by

$$(G\omega)(x) = \int_0^\infty P_t\omega(y)dt \tag{3.32}$$

satisfies

$$\begin{aligned}
(\Delta G\omega)(x) &= \Delta_x \int_0^\infty \int_{y\in M} p(t,x,y)\omega(y)dt \\
&= \int_{y\in M} \int_0^\infty -\partial_t p(t,x,y)\omega(y)dt \\
&= \int_{y\in M} \lim_{t\to 0} p(t,x,y)\omega(y) - \lim_{t\to\infty} p(t,x,y)\omega(y) = \omega(y),
\end{aligned}$$

i.e. G inverts the Laplace on the orthogonal complement of harmonic forms. If we define G to vanish on harmonic forms, then we claim that $(\Delta + P_{harm})^{-1} = G + P_{harm}$. Indeed,

$$(\Delta + P_{harm})(G + P_{harm}) = \Delta G + P_{harm}G + \Delta P_{harm} + P_{harm}^2 = \mathrm{id} - P_{harm} + P_{harm} = \mathrm{id}, \tag{3.33}$$

since $P_{harm}G = \Delta P_{harm} = 0$.

[8] More precisely, the spectrum of its unique self-adjoint extension is discrete, and its eigenforms are smooth by elliptic regularity.

[9] The appearance of the Hodge star is due to our conventions on kernels, which, in the language of de Rham ([Rha84]), are topological rather than metric (even though that book uses different conventions for topological kernels).

The propagator in terms of the heat kernel

Equipped with these facts we can now express the propagator in terms of the heat kernel. Namely, since $(\Delta + P_{harm})^{-1} = G + P_{harm}$, we can write

$$K = \mathrm{d}^* \circ (\Delta + P_{harm})^{-1} = \mathrm{d}^* \circ G, \tag{3.34}$$

since $\mathrm{d}^* \circ P_{harm} = 0$. Thus we can write the integral kernel of the propagator as

$$\eta(x, y) = \int_0^\infty \mathrm{d}_x^* p(t, x, y). \tag{3.35}$$

Expanding in terms of Eigenforms, this yields

$$\eta(x, y) = \int_0^\infty \sum_j e^{-t\lambda_j} \mathrm{d}^* \phi_j(x) \wedge *\phi(y) \tag{3.36}$$

The realization that Heat kernels can be used to give expansions of propagators goes at least back to Feynman [Fey42]. An extensive list of references on the use of heat kernel techniques in physics can be found in [Vas03].

Example: The propagator on the circle

The heat kernel on the circle (with respect to the standard metric) is

$$p_{S^1}(t, \theta_1, \theta_2) = \sum_{k \in \mathbb{Z}} e^{-(2\pi k)^2 t} e^{2\pi i k(\theta_1 - \theta_2)} (d\theta_1 - d\theta_2). \tag{3.37}$$

The codifferential sends 0-forms to 0 and acts on 1-forms as $\mathrm{d}^* f(\theta) d\theta = -f'(\theta)$. Hence, the codifferential of the heat kernel is

$$\mathrm{d}^*_{\theta_1} p_{S^1}(t, \theta_1, \theta_2) = \sum_{k \in \mathbb{Z}} (2\pi i k) e^{-(2\pi k)^2 t} e^{2\pi i k(\theta_1 - \theta_2)}. \tag{3.38}$$

Integrating over t, we get

$$\int_0^\infty \sum_{k \in \mathbb{Z} \setminus \{0\}} (2\pi i k) e^{-(2\pi k)^2 t} e^{2\pi i(\theta_1 - \theta_2)} dt = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{-1}{2\pi i k} e^{2\pi i k(\theta_1 - \theta_2)}. \tag{3.39}$$

Notice that this series converges in the L^2 sense. We claim that this is precisely the Fourier series of

$$\eta_{S^1}(\theta_1, \theta_2) = \Theta(\theta_1 - \theta_2) - (\theta_1 - \theta_2) - \frac{1}{2} \tag{3.40}$$

where $0 \leq \theta_1, \theta_2 \leq 1$ and $\Theta(x)$ is the Heaviside function defined by

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases} \tag{3.41}$$

Indeed, we can rewrite the propagator as

$$\eta_{S^1}(\theta_1,\theta_2) = \theta_2 - \theta_1 - \lfloor \theta_2 - \theta_1 \rfloor - \frac{1}{2} = ((\theta_1 - \theta_2)), \tag{3.42}$$

where $((x)) = x - \lfloor x \rfloor - 1/2, x \notin \mathbb{Z}$, is the sawtooth function (usually defined to vanish at integer points). The Fourier coefficients of the sawtooth function are given by

$$a_k = \int_0^1 ((\theta))e^{-2\pi i k\theta}d\theta = \int_0^1 (\theta - 1/2)e^{-2\pi i k\theta}d\theta = \begin{cases} 0 & k = 0 \\ \frac{-1}{2\pi i k} & k \neq 0 \end{cases}$$

as claimed, i.e. the series (3.39) converges to the propagator (3.42) in the sense of distributions. This example was considered also in [GG14].

Example: The torus, again

The heat kernel expansion allows us to give a different representation of the propagator on the torus, at least for the standard metric, for which the heat kernel on forms is $p_{\mathbb{T}^2}(t,(x_1,y_1),(x_2,y_2) = p_{\mathbb{T}^2}(t,(x_1-x_2,y_1-y_2),0)$ and

$$p_{\mathbb{T}^2}(t,(x,y),0) \equiv p_{\mathbb{T}^2}(t,(x,y)) = \sum_{k,l\in\mathbb{Z}^2} e^{-(2\pi)^2(k^2+l^2)t}e^{2\pi\mathrm{i}(kx+ly)}dxdy. \tag{3.43}$$

The codifferential of the heat kernel is given by

$$\mathrm{d}^*p(t,(x,y)) = \sum_{k,l\in\mathbb{Z}^2}(2\pi\mathrm{i}k)e^{-(2\pi)^2(k^2+l^2)t}e^{2\pi\mathrm{i}(kx+ly)}dy - (2\pi\mathrm{i}l)e^{-(2\pi)^2(k^2+l^2)t}e^{2\pi\mathrm{i}(kx+ly)}dx. \tag{3.44}$$

Integrating over t we obtain

$$\eta^{std}_{\mathbb{T}^2}((x,y),0) = -\frac{1}{2\pi\mathrm{i}}\sum_{k,l\in\mathbb{Z}^2-\{(0,0\}}\frac{k}{k^2+l^2}e^{2\pi\mathrm{i}(kx+ly)}dy - \frac{l}{k^2+l^2}e^{2\pi\mathrm{i}(kx+ly)}dx. \tag{3.45}$$

From the uniqueness of distributional kernels it follows that $\eta^{std}_{\mathbb{T}^2} = \eta^i$ in the sense of distributions on $\mathbb{T}^2$, where η^i is defined by (3.24) for $\tau = i$.

3.1.8 Regularization and renormalization

In this short section we briefly explain the view on regularization and renormalization taken in this work.

To evaluate Feynman diagrams one wants to take products of the distributional forms introduced above. If done naively this quickly leads to divergent integrals. The treatment of the divergencies goes under the name *regularization*. Usually, in quantum field theory there is also the problem of *renormalization*: How the theory and the interactions depend on the length or energy scales that are considered. Often, these two notions interact, since a common source of divergencies is trying to take into account all energy or length scales. One way of regularization corresponds to cutting off the heat kernel expansion below low times ε (corresponding to high energies and so-called ultra-violet (UV) divergences). ε is sometimes called a *regulator*. One then subtracts the divergent parts as $\varepsilon \to 0$ according to a choice of *renormalization scheme*. Another cut-off is introduced at long length scales to take care of "infrared" (IR) divergencies. Renormalization in this language is analyzing how the theory depends on the cut-offs. Again, we refer to the review articles [Vas03] and [Avr02] for further references. A write-up of renormalization using the heat kernel from the mathematical point of view can be found in [Cos11]. Working on compact manifolds there is no need for a cut-off at high length scales. The only possible problem on compact manifolds is posed by the zero-energy eigenmodes of the Laplacian, but in the BV effective action approach, these are taken care of using the residual fields, which can be interpreted as our choice of infrared cutoff. The UV divergences in principle persist in topological theories. However, it is a crucial feature of Chern–Simons and BF theories that the integrals appearing in the perturbative expansion are finite. This can be proven using the method of compactified configuration spaces as developed by Bott and Taubes ([BT94]) and Axelrod-Singer ([AS91; AS94]) based on compactification as defined by Fulton and MacPherson ([FM94]). The key result is that the metric propagator is regular, i.e. extends to the compactification $C_2(M)$ of the two-point configuration space $C_2^0(M) = M \times M - \Delta$. This provides an immediate proof of the fact that all integrals are finite, since they are now integrals of smooth forms over compact manifolds.

Remark 3.1.7 (Renormalization in BV-BFV formalism). In principle, in the BV-BFV formaluation of abelian BF theory, we are free to choose any finite-dimensional space of residual fields containing the kernel of the Laplacian. One can pass between different spaces of residual fields using BV fiber integration, see [CM08; CMR11]. This is a shadow of renormalization group methods in Quantum Field Theory, see also the discussion in [CMR17, Appendix F]. This is important when cutting and gluing manifolds. The "naive" space of residual fields is usually larger than the minimal one, and depends on choices. Only the minimal one - in case of abelian BF theory this is the cohomology of the manifold - is canonically associated to the manifold. Therefore if one wishes to obtain topological invariants from cutting and gluing, "renormalizing" to the minimal space of residual fields is strictly necessary. An example of this are the results of Chapter 5 in this thesis. The weights of the Feynman graphs directly after gluing are not invariants of framed lens spaces. However, after reduction of the residual fields, we obtain invariants of framed lens spaces.

3.2 Gauge fixing in BF theory on manifolds with boundary

The Riemann-Hodge propagators introduced above for closed manifolds have a generalization to manifolds with boundary. However, the introduction of boundaries complicates the analysis of the Hodge-de Rham Laplacian considerably. We therefore start by a brief discussion of boundary conditions. When working with Riemannian metrics on manifolds with boundary, we always make the assumption that the metric has product structure near the boundary, i.e. there exists a collar $U \cong \partial M \times [0, \varepsilon)$ on which the metric takes the form $g = g_{\partial M} + dt^2$.

3.2.1 A short digression on boundary conditions

We will review parts of the material in [CMR17, Appendix A], to which we refer for proofs of the statements. Consider a manifold M with boundary ∂M (possibly empty, in which case the entire discussion below reduces to the closed case). Let $\iota_\partial \colon \partial M \hookrightarrow M$ and denote by $\Omega^\bullet(M, \partial M) \subset \Omega^\bullet(M)$ the complex of forms with which vanish on the boundary, i.e.

$$\Omega^\bullet(M, \partial M) = \{\omega \in \Omega^\bullet(M), \iota_\partial^* \omega = 0\}.$$

We will also call this the space of forms satisfying vanishing Dirichlet boundary conditions, and denote it by $\Omega^\bullet_D(M)$. By Ω_N we denote the subspace of forms satisfying Neumann boundary conditions:

$$\Omega^\bullet_N = \{\alpha \in \Omega^\bullet(M), \iota_\partial^*(*\alpha) = 0.\} \tag{3.46}$$

Note that $*\Omega^\bullet_D(M) = \Omega^\bullet_N(M)$. Notice that $\Omega^\bullet_D(M)$ is not closed with respect to d^*. Looking for a subspace of $\Omega^\bullet(M)$ of forms vanishing on the boundary which is closed with respect to both d and d^*, we are led to introduce the subspaces of ultra-harmonic, ultra-Dirichlet, and ultra-Neumann boundary conditions:

Definition 3.2.1 ([CMR17])**.** Let $\alpha \in \Omega^\bullet(M)$.

i) α is called *ultra-harmonic* if it is closed with respect to both d and d^*. The subspace of ultra-harmonic forms is denoted $\widehat{Harm}^\bullet(M)$.

ii) α is called *ultra-Dirichlet* if the pullbacks to the boundary of all even normal derivatives of α and all odd normal derivatives of $*\alpha$ vanish. The subspace of ulta-Dirichlet forms is denoted $\Omega^\bullet_{\widehat{D}}(M)$.

iii) α is called *ultra-Neumann* if the pullbacks to the boundary of all even normal derivatives of $*\alpha$ and all odd normal derivatives of α vanish. The subspace of ultra-Neumann forms is denoted $\Omega_{\widehat{N}}(M)$.

Notice that harmonic forms are ultra-harmonic on closed manifolds, but the converse is not true. Again we have $*\Omega^\bullet_{\widehat{D}}(M) = \Omega^\bullet_{\widehat{N}}(M)$. Also, $\Omega^\bullet_{\widehat{D}}(M)$ is closed with respect to both d and d^*, and they are formally adjoint with respect to the L^2-inner product. It follows that the Hodge–de Rham Laplacian is symmetric on this space. There are several ways to see that this operator is essentially self-adjoint: One of them is to construct a self-adjoint heat kernel (see below) as done e.g. in [Con56; RS71].

3.2.2 Riemann–Hodge Contracting triples on manifolds with boundary

The space of fields is

$$\mathcal{F}_M = \Omega^\bullet(M, \partial M)[1] \oplus \Omega^\bullet(M)[d-2].$$

Again the propagator has a formulation in terms of a contracting triple. Now, we will be interested in a contracting triple (ι, p, K) of $\Omega^\bullet(M, \partial M)$ such that its dual (with respect to the Poincaré pairing) (p', ι', K') forms a contracting triple of $\Omega^\bullet_N(M)$. The construction of this goes via a small detour: First, one constructs a contracting triple $(\iota_R, p_R, K^{\widehat{D}})$ for $\Omega^\bullet_{\widehat{D}}(M)$ by letting K be the Riemann–Hodge chain contraction given by

$$K^{\widehat{D}} = \mathrm{d}^*_g \circ (\Delta_g + P_{harm})^{-1}. \tag{3.47}$$

and R the system of ultra-harmonic representatives[10] of the relative cohomology. Here $(\Delta_g + P_{harm})^{-1}$ is the Green's operator for the ultra-Dirichlet problem (with vanishing boundary condition) on M:

$$\begin{cases} (\Delta_g + P_{harm})\omega & = \tau \\ \iota^*_\partial \omega & = 0 \end{cases} \tag{3.48}$$

[10]This are representatives of the cohomology of $\Omega^\bullet_{\widehat{D}}(M)$, a complex which is quasi-isomorphic to $\Omega^\bullet_D(M)$ as remarked in [CMR17].

for $\omega, \tau \in \Omega^\bullet_{\widehat{D}}(M)$. The integral kernel η_g associated with this chain contraction via $K^{\widehat{D}} = \pi_{1,*}\eta_{g,12}\pi_2^*(\bullet)$ extends to smooth $d-1$ form on the compactified configuration space[11] $C_2(M)$. The chain contraction $(K^{\widehat{D}})'$ of $\Omega^\bullet_{\widehat{N}}(M)$ is given by the same formula, but with the Green's form of the Neumann problem for vanishing boundary condition on M. Its integral kernel η'_{12} satisfies $T^*\eta_{g,12} = (-1)^d\eta'_{g,12}$. Here T is the extension to $C_2(M)$ of the restriction of $(x, y) \to (y, x)$ to $M \times M \setminus \Delta$. Therefore, $\eta_{g,12}$ satisfies ultra-Dirichlet boundary condition in the first and ultra-Neumann boundary condition in the second argument. Now, we define the chain contraction $K\colon \Omega^\bullet_D(M) \to \Omega^{\bullet-1}_D(M)$ by the same integral kernel η_g: For $\alpha \in \Omega^\bullet_D(M)$, we let

$$K\alpha = \pi_{1,*}\eta_{g,12}\pi_2^*\alpha \tag{3.49}$$

and the other two maps in the contracting triple are given by $\iota = \iota_R, p = p_R$, where R is still the system of ultra-harmonic representatives. This has the desired property that (p'_R, ι'_R, K') is a contracting triple of $\Omega^\bullet(M)$.

3.2.3 Heat kernel expansion

Like the propagator on closed manifolds, the propagator on manifolds with boundary admits a heat kernel expansion. The heat kernel on differential forms on manifolds with boundary is constructed e.g in [Con56] and [RS71]. It is the fundamental solution of the initial value problem

$$\begin{cases} (\partial_t + \Delta)\omega(x,t) &= 0 \\ \lim_{t\to 0}\omega(x,t) &= \omega(x) \\ \iota_\partial^*\omega(\cdot, t) &= 0 \end{cases} \tag{3.50}$$

If $\{\phi_j\}_{j=1}^\infty \subset \Omega^\bullet(M, \partial M)$ is a system of eigenforms for the Laplacian with vanishing Dirichlet boundary conditions, where ϕ_j has eigenvalue λ_j, then the heat kernel $p_t(x, y)$ has the expansion

$$p_t(x,y) = \sum_{j=1}^\infty e^{-t\lambda_j}\phi_j(x) \wedge *\phi_j(y). \tag{3.51}$$

Note that $*\phi_j$ is also an eigenform of the Laplacian of eigenvalue λ_j satisfyingv vanishing Neumann boundary conditions. Similar to the case without boundary we have that

$$(\Delta + P_{harm})^{-1} = G + P_{harm}, \tag{3.52}$$

where G is the operator with integral kernel $\int_0^\infty p_t dt$.

[11]This space can be identfied with a certain subspace of the usual FMAS compactified configuration space of the double of M, see [CMR17; Cam+18].

3.2.4 Example: The disk

We consider the example of the unit disk D and give two representations of the propagator corresponding to the standard metric. First, we use the Green's form approach. In the case of the circle and the torus, determining the Green's form from the Green's function for the Laplacian on functions was easy. The disk is a good example to see the additional complications that come with boundaries. First, on the disk we have to specify the boundary conditions: We can impose either Dirichlet or Neumann boundary conditions. The Green's function on the unit disk for Dirichlet boundary conditions is well known to be

$$g^D(z,w) = \frac{1}{2\pi}(\log|z-w| - \log|1-z\bar{w}|). \tag{3.53}$$

The Green's function for Neumann boundary conditions is

$$g^N(z,w) = \frac{1}{2\pi}\left(\log|z-w| + \log|1-z\bar{w}| - \frac{1}{2}|z|^2\right), \tag{3.54}$$

this is the Green's function whose radial derivative vanishes on the boundary and whose Laplacian satisfies

$$\Delta_z g^N(z,w) = \delta(z-w) - \frac{1}{\pi}. \tag{3.55}$$

To see the relation to the discussion above, let $\Omega^0_N(D)$ denote 0-forms with vanishing normal derivative on the boundary. Here the Laplacian has a kernel given by constant functions. If L denotes the operator on $\Omega^0_N(D)$ with integral kernel $g_N\mu$ (where μ is a normalized volume form on the disk), then one can check that $(\Delta + P)Lf = f$ for all $f \in \Omega^0_N(D)$ (where $Pf = \int f\mu$ is the projection to constant functions).
We denote by $G^{(j)}$ the part of the Green's form G that has form degree j in the first argument, this is the integral kernel of the Laplacian on j-forms. Note that we have

$$G^{(0)}(z,w) = -g^D(z,w)\frac{d\bar{w}dw}{2i} \tag{3.56}$$

$$G^{(2)}(z,w) = -g^N(z,w)\frac{d\bar{z}dz}{2i} \tag{3.57}$$

since the Dirichlet problem on 2-forms is Hodge dual to the Neumann problem on 0-forms (the minus sign comes from our conventions on the Laplacian: The Hodge-de Rham Laplacian of the standard metric is minus the usual Laplacian). In two dimensions, the Green's form on 1-forms can be obtained from Hodge decomposition. Since we are interested in the propagator

$$\eta(z,w) = \mathrm{d}^*_z G = \mathrm{d}^*_z G^{(1)} + \mathrm{d}^*_z G^{(2)} \tag{3.58}$$

it is enough to invert the Laplacian on exact one-forms, since coexact one-forms are killed by the codifferential. Since the de Rham differential commutes with the Laplace operator, and is invertible on exact 1-forms, we can write

$$(\mathrm{d}^{(0)})^{-1} = K^{(1)} = (\mathrm{d}^*)^{(1)} \circ (\Delta^{(1)})^{-1} = (\Delta^{(0)})^{-1} \circ (\mathrm{d}^*)^{(1)} \tag{3.59}$$

where $\mathrm{d}^{(j)} \colon \Omega^j(D) \to \Omega^{j+1}(D)$, $(\mathrm{d}^*)^{(j)} \colon \Omega^j(D) \to \Omega^{j-1}(D)$, $(\Delta)^{(j)} \colon \Omega^j(D) \to \Omega^j(D)$ are the components of the de Rham and Laplace operators. Also notice that in this degrees there are no harmonic forms, so the Laplacian is invertible on the nose. Equation (3.59) implies that

$$\begin{aligned}
\mathrm{d}^*_z G^{(1)}(z,w) &= \mathrm{d}^*_w G^{(0)}(z,w) = *_w \mathrm{d}_w *_w G^{(0)}(z,w) = *_w \mathrm{d}_w g^D(z,w) \\
&= \frac{1}{4\pi} *_w \left(\mathrm{d}_w \frac{\partial}{\partial w} + d\bar{w}\frac{\partial}{\partial \bar{w}}\right)(\log(z-w) + \log(\bar{z}-\bar{w}) - \log(1-z\bar{w}) - \log(1-\bar{z}w)) \\
&= *_w \frac{1}{4\pi}\left(-\frac{dw}{z-w} - \frac{d\bar{w}}{\bar{z}-\bar{w}} + \frac{\bar{z}dw}{1-\bar{z}w} + \frac{zd\bar{w}}{1-z\bar{w}}\right) \\
&= -\frac{1}{4\pi\mathrm{i}}\left(\frac{dw}{z-w} + \frac{\bar{z}dw}{1-\bar{z}w} - \frac{d\bar{w}}{\bar{z}-\bar{w}} - \frac{zd\bar{w}}{1-z\bar{w}}\right)
\end{aligned}$$

where we have used that on the disk, we have $\mathrm{d}^* = - * \mathrm{d} *$[12], $*1 = \frac{1}{2i} d\bar{z}dz$, $*dz = -\mathrm{i}dz = dz/\mathrm{i}$ and $*d\bar{z} = \mathrm{i}d\bar{z} = -d\bar{z}/\mathrm{i}$. Also, we have

$$\begin{aligned}
\mathrm{d}^*_z G^{(2)}(z,w) &= *_z \mathrm{d}_z *_z G^{(2)}(z,w) = *_z \mathrm{d}_z g^N(z,w) \\
&= \frac{1}{4\pi} *_z \left(\mathrm{d}_z \frac{\partial}{\partial z} + d\bar{z}\frac{\partial}{\partial \bar{z}}\right)(\log(z-w) + \log(\bar{z}-\bar{w}) + \log(1-z\bar{w}) - \log(1-\bar{z}w) - \bar{z}z) \\
&= \frac{1}{4\pi} *_z \left(\frac{dz}{z-w} - \frac{\bar{w}dz}{1-z\bar{w}} - \bar{z}dz + \frac{d\bar{z}}{\bar{z}-\bar{w}} - \frac{wd\bar{z}}{1-\bar{z}w} - zd\bar{z}\right) \\
&= \frac{1}{4\pi\mathrm{i}}\left(\frac{dz}{z-w} - \frac{\bar{w}dz}{1-z\bar{w}}\bar{z}dz - \frac{d\bar{z}}{\bar{z}-\bar{w}} + \frac{wd\bar{z}}{1-\bar{z}w} + zd\bar{z}\right).
\end{aligned}$$

Taking the sum, we obtain

$$\eta(z,w) = \frac{1}{2\pi}(\mathrm{d}\arg(z-w) + \mathrm{d}\arg(1-z\bar{w})) - \frac{\bar{z}dz - zd\bar{z}}{4\pi\mathrm{i}}. \tag{3.60}$$

This is the propagator presented in [CF11], and also the one we will use in this work when computing Feynman diagrams on the disk. On the disk one can also find an explicit set of eigenforms given in terms of Bessel functions: Namely, the degree 0 Dirichlet eigenfunctions are given by

$$\phi^{(0),D}_{m,n}(z) = C_{mn} J_m(k_{mn} r) e^{2\pi i m\theta}, m \geq 0, n \geq 1 \tag{3.61}$$

[12] In general the sign of the codifferential is $\mathrm{d}^* = (-1)^{nk+n+1} * \mathrm{d} *$, so it -1 for all degrees on manifolds of even dimension.

where J_m is the m-th Bessel function, k_{mn} is its n-th zero, and C_{mn} is a normalizing constant. Likewise, the degree 0 eigenfunctions Neumann eigenfunctions are given by

$$\phi_{m,n}^{(0),N}(z)C'_{mn}J_m(k'_{mn}r)e^{2\pi i m\theta}, m \geq 0, n \geq 1 \tag{3.62}$$

where k'_{mn} is the n-th zero of the derivative J'_m of the m-th Bessel function, and C'_{mn} is another normalizing constant. Notice that $k'_{01} = 0$, and hence $\phi_{0,1}^{(0),N} \equiv 1$ is the constant (the only eigenfunction of eigenvalue 0). From these eigenfunctions it is possible to obtain a complete system of eigenforms by applying the operators $*, \mathrm{d}, *\mathrm{d}$. It is a unique feature of two dimensions that one can (in principle) solve the Laplace equation in all degrees from the solution in degree 0.

3.3 Axial gauge on product manifolds

We now come to the first main section of this chapter, where we introduce the axial gauge on product manifolds and show how it (that is, the corresponding chain contraction) can be approximated arbitrarily well by Riemann–Hodge chain contraction. This involves taking a limit where the volume of one of the two manifolds becomes arbitrarily large. The axial gauge, a well-known tool in the physics literature (often under the name *light-cone gauge*) featured under this name in [BCM12], but it was already used for the construction of knot invariants through quantization of Chern–Simons theory by Fröhlich and King in [FK89], and later implicitly by Kontsevich in [Kon93] and then Bar-Natan in [Bar95]. The link with contracting triples was explained in [Mne08].

3.3.1 Definition

The following is a standard construction on contracting triples.

Proposition 3.3.1. *Let $V_1 = (V_1^\bullet, d_1)$ and $V_2 = (V_2^\bullet, d_2)$ be two finite dimensional cochain complexes with contracting triples (ι_1, p_1, K_1) and (ι_2, p_2, K_2) respectively. Then*

$$\iota = \iota_1 \otimes \iota_2$$

$$p = p_1 \otimes p_2$$

$$K = K_1 \otimes id_2 + \iota_1 \circ p_1 \otimes K_2$$

is a contracting triple for $V = (V_1^\bullet \otimes V_2^\bullet, d_1 \otimes id_2 + id_1 \otimes d_2)$[13]

This is the chain contraction that arises from the composition of the quasi-isomorphisms

$$V_1 \otimes V_2 \to H_1 \otimes V_2 \to H_1 \otimes H_2. \tag{3.63}$$

The goal of this section is to extend and study this construction for the case when the complexes $V^\bullet$ are the de Rham complexes of manifolds. Let $M = M_1 \times M_2$ be the direct product of two manifolds. On manifolds of this type we have a special choice of gauge fixing, called the axial gauge. In this gauge the propagator no longer is a smooth form on the compactified configuration space, but rather a form with distributional coefficients on $M \times M$. The goal of this section is to show how axial gauge propagators can be approximated by smooth ones. After recalling some generalities we give the construction on closed manifolds and then extend to the case with boundary.

3.3.2 Axial gauge construction

Now let $M = M_1 \times M_2$. Suppose that we have continuous contracting triples (ι_1, p_1, K_1) and (ι_2, p_2, K_2) for $\Omega^\bullet(M_1)$ and $\Omega^\bullet(M_2)$ respectively. Mimicking the construction of proposition 3.3.1, we can define two contracting triples for $\Omega := \Omega^\bullet(M_1) \otimes \Omega^\bullet(M_2)$, the *axial* one with $K^{ax} = id_1 \otimes K_2 + K_1 \otimes \iota_2 \circ p_2$ and the *horizontal* one with $K^{hor} = K_1 \otimes id_2 + \iota_1 \circ p_1 \otimes K_2$[14]. These are continuous on Ω, since $\Omega \subset \Omega^\bullet(M_1 \times M_2)$ is dense, they extend to continuous contracting triples of $\Omega^\bullet(M_1 \times M_2)$ and as such have integral kernels η^{ax}, η^{hor} which are currents on $M \times M$.

Example 3.3.1. As an example let us consider the 2-torus $\mathbb{T}^2 = S^1 \times S^1$ which we give coordinates $(t, \theta) \in (\mathbb{R}/\mathbb{Z})^2$. The axial gauge propagator is

$$\eta^{ax}_{\mathbb{T}^2}((t_1, \theta_1), (t_2, \theta_2)) = \eta_{S^1}(\theta_1, \theta_2)\delta^{(1)}(t_1, t_2) + \eta_{S^1}(t_1, t_2)(d\theta_2 - d\theta_1). \tag{3.64}$$

The horizontal propagator is

$$\eta^{hor}_{\mathbb{T}^2}((t_1, \theta_1), (t_2, \theta_2)) = \eta_{S^1}(t_1, t_2)\delta_{S^1}(\theta_1, \theta_2) + \eta_{S^1}(\theta_1, \theta_2)(dt_2 - dt_1). \tag{3.65}$$

See [BCM12] for a computation using this propagator in the Poisson Sigma Model.

[13] We are using Koszul sign rules for tensor products of graded complexes: Explicitly, in this case $id_1 \otimes d_2(v_1 \otimes v_2) = (-1)^{|v_1|} v_1 \otimes d_2 v_2$, which implies that this differential squares to zero (the sign arises from exchanging the degree 1 linear map d_2 with v_1).

[14] Terminology should correspond to [CMR17].

3.3.3 Approximation of axial gauge propagators by Riemann–Hodge propagators

Now suppose that the contracting triples used before are Hodge contracting triples coming from Riemannian metrics g_1, g_2 on M_1 and M_2 respectively. We will denote by $d_1^*, d_2^*, \Delta_1, \Delta_2, P_1, P_2$ the corresponding codifferentials, Laplacians and projections to harmonic forms. Then also M carries a natural Riemannian metric $g = g_1 + g_2$. This will give rise to a Hodge contracting triple on M. For $\lambda > 0$ we can also consider the family of Riemannian metrics

$$g^\lambda = g_1 + \lambda^{-1} g_2. \tag{3.66}$$

In the limit $\lambda \to 0$ the second component gets "infinitely large". Intuitively this should correspond to the fields not propagating along that component. The next proposition makes this intuition more precise.

Theorem 3.3.2. *As $\lambda \to 0$, $K_{g^\lambda}\alpha \to K^{hor}\alpha$ and hence also $\eta_{g^\lambda} \to \eta^{hor}$ in the sense of distributions.*

Before the proof, let us state some remarks on the intuitions behind this result:

Remark 3.3.3. The physical interpretation that - even when the second factor of spacetime is "infinitely large" - the fields that are spread out infinitely far over the first factor (namely, the fields of zero energy, a.k.a. the zero modes) still "propagate" along the second factor.

Remark 3.3.4. In the physical context, the axial gauge is often used to set the components of the fields along the "axis" (the second component) to 0. In scale invariant theories, the theorem shows that this can be achieved by sending the volume of the second factor to infinity, but only if there are no zero modes *in the first component*, which at first is somewhat counter-intuitive, but explained by the remark above.

Using the expression for the propagator in terms of heat kernels developed in the previous section one can give an elegant proof[15] of Theorem 3.3.2. This uses two simple properties of the heat kernel (see e.g. [BGV03], or [RS71]):

Proposition 3.3.5. *Let (M, g) be a Riemannian manifold (with or without boundary) with heat kernel $p(t, x, y)$.*

i) If $(M, g) = (M_1 \times M_2, g_1 + g_2)$, where (M_1, g_1) and (M_2, g_2) are Riemannian manifolds with heat kernels p_1, p_2, then

$$p(t, (x_1, x_2), (y_1, y_2)) = p_1(t, x_1, y_1) \wedge p_2(t, x_2, y_2). \tag{3.67}$$

[15]However, the theorem can be proven without referring to heat kernel techniques, see Appendix D.5.

ii) For $\lambda > 0$, the heat kernel p_λ associated to $(M, \lambda g)$ is

$$p_\lambda(t,x,y) = p(\lambda^{-1}t,x,y). \tag{3.68}$$

Proof of theorem 3.3.2. Let p_1, p_2 be the heat kernels of (M_1, g_1) and (M_2, g_2). Consider the metric $g^\lambda = g_1 + \lambda^{-1} g_2$ on $M_1 \times M_2$. By Proposition 3.3.5, the heat kernel p_λ of g^λ is

$$p_\lambda(t,(x_1,x_2),(y_1,y_2)) = p_1(t,x_1,y_1)p_2(\lambda t,x_2,y_2), \tag{3.69}$$

hence the propagator is given by

$$\begin{aligned} K^\lambda &= \int_0^\infty dt \mathrm{d}^*_x p_1(t,x_1,y_1)p_2(\lambda t,x_2,y_2) = \int_0^\infty dt(\mathrm{d}^*_{x_1} + \lambda \mathrm{d}^*_{x_2})p_1(t,x_1,y_1)p_2(\lambda t,x_2,y_2) \\ &= \int_0^\infty dt \mathrm{d}^*_{x_1} p_1(t,x_1,y_1)p_2(\lambda t,x_2,y_2) + \lambda p_1(t,x_1,y_1)\mathrm{d}^*_{x_2} p_2(\lambda t,x_2,y_2) \end{aligned} \tag{3.70}$$

We are interested in the $\lambda \to 0$ limit. Consider the first term of the expression above. Here we have

$$\lim_{\lambda\to 0}\int_0^\infty dt\mathrm{d}^*_{x_1}p_1(t,x_1,y_1)p_2(\lambda t,x_2,y_2) = \int_0^\infty dt\mathrm{d}^*_{x_1}p_1(t,x_1,y_1)\delta_{M_2}(x_2,y_2) = \eta_1(x_1,y_1)\delta_{M_2}(x_2,y_2). \tag{3.71}$$

Here, δ_{M_2} denotes the integral kernel of the identity map on differential forms. In the second term we do a change of variables $s = t\lambda$:

$$\int_0^\infty dt\lambda\mathrm{d}^*_{x_2}p_1(t,x_1,y_1)p_2(\lambda t,x_2,y_2) = \int_0^\infty ds p_1(\lambda^{-1}s,x_1,y_1)\mathrm{d}^*_{x_2}p_2(s,x_2,y_2) \overset{\lambda\to 0}{\to} p_{harm}(x_1,y_1)\eta_2(x_2,y_2).$$

where $p_{harm,1}$ denotes the integral kernel of $P_{harm,1}$ and is the limit of $p(t,x_1,y_1)$ as $t\to\infty$. $\square$

Notice the obvious symmetry in the proof above: Instead of sending $\lambda \to 0$, we could have equally well sent $\lambda \to \infty$, implying the following corollary:

Corollary 3.3.6. *Let K^λ be the Riemann–Hodge contraction of $g^\lambda = g_1 + \lambda^{-1} g_2$, then if $\lambda \to \infty$ we have that $K^\lambda \to K^{ax}$.*

3.3.4 Example: The torus, yet again

Let us consider again the example of the torus. The convergence becomes clear if we consider the expansion in terms of eigenforms. From the discussion above it follows that

$$\eta^\lambda_{\mathbb{T}^2}((x,y),0) = \frac{1}{2\pi \mathrm{i}}\sum_{k,l\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{k}{k^2+\lambda l^2}e^{2\pi\mathrm{i}(kx+ly)}dy - \frac{l\lambda}{k^2+\lambda l^2}e^{2\pi \mathrm{i}(kx+ly)}dx. \tag{3.72}$$

As $\lambda \to 0$, the first term converges to

$$\frac{1}{2\pi \mathrm{i}} \sum_{k,l \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{k}{k^2 + \lambda l^2} e^{2\pi \mathrm{i}(kx+ly)} dy \overset{\lambda \to 0}{\to} \frac{1}{2\pi \mathrm{i}} \sum_{k \in \mathbb{Z} - \{0\}, l \in \mathbb{Z}} \frac{1}{k} e^{2\pi \mathrm{i}(kx+ly)} dy$$

$$= \sum_{k \neq 0} \frac{e^{2\pi \mathrm{i} kx}}{2\pi \mathrm{i} k} \sum_{l \in \mathbb{Z}} e^{2\pi \mathrm{i} ly} dy = \eta_{S^1}(x) \delta^{(1)}_{S^1}(y)$$

and the second term converges to 0 unless $k = 0$, in which case we can cancel one λl and we get

$$-\sum_{l \neq 0} \frac{1}{2\pi i l} e^{2\pi \mathrm{i}(ly)} dx = -dx \eta_{S^1}(y).$$

This recovers the axial gauge propagator of (3.64) if applied to $x_1 - x_2$ and $y_1 - y_2$.

Remark 3.3.7. Indirectly, this shows that the propagator $\eta^{\beta \mathrm{i}}$, the propagator associated to the complex structure on the torus by $\tau = \beta \mathrm{i}$ converges to η^{hor} as $\beta \to \infty$ and η^{ax} as $\beta \to 0$, in the sense of distributions. It would be nice to have a proof of this fact starting directly from the theta function formulation of propagators on the torus.

3.3.5 Extension to manifolds with boundary

The axial gauge propagator can be defined similarly for products of manifolds with boundary. In this work, we will consider only manifolds of the form $M = N_1 \times N_2$ where N_1 is closed and N_2 possibly has a boundary, so that M is a manifold with boundary. The axial and horizontal gauges are given by the same formulae. In particular, theorem 3.3.2 still holds: Since both the formula for the Hodge contraction and the heat kernel expansion generalize to the case with boundary, both proofs of theorem 3.3.2 generalize to the case with boundary.

Example 3.3.2. Let us consider the example of the solid torus $M = S^1 \times D$ with boundary $\partial M = S^1 \times S^1$ that will be of central importance in this paper. Here, the horizontal propagator for the $\mathbb{A}$-representation (i.e. Dirichlet boundary conditions in the first argument) is

$$\eta^{hor}_H = \eta_D \delta^{(1)}_{S^1} + \mu_1 \eta_{S^1}, \tag{3.73}$$

where μ_1 is the volume form on the disk which in this case is also the integral kernel of the projection, η_D is a propagator on the disk, $\delta^{(1)}_{S^1}$ is the integral kernel of minus the identity on S^1 and η_{S^1} a propagator on the circle.

3.4 Regularization

The key point of theorem 3.3.2 is that it allows us to define a regularization for the axial gauge. For the axial gauge, the proof of finiteness via compactified configuration space is not possible, since the support d-dimensional delta current $\delta^{(d)}_M$ defined by integration over the diagonal in $M \times M$ is the diagonal. Hence there exists no smooth form on $M \times M - \Delta$ representing the delta current (such a form would have to vanish everywhere, leading to a contradiction). In this section we will propose a different regularization of the axial gauge, and show that it behaves well in certain easy examples.

3.4.1 Definition of the regularization

Let $M = N_1 \times N_2$ be a product manifold with or without boundary with a family of metrics $g^\lambda = g_1 + \lambda^{-1} g_2$. In the case with boundary, we will assume that it is endowed with a choice of boundary conditions on each boundary component. Consider now the family of propagators η^λ, converging to the horizontal gauge for $\lambda \to 0$ and to the axial gauge for $\lambda \to \infty$. Recall that to a Feynman graph Γ we associate a differential form ω_Γ as a product of propagators and residual fields. In particular, denote ω^λ_Γ the product of the propagators η^λ and residual fields corresponding to Γ. We are now interested in taking the pushforward of this form over a subset $S \subset V(\Gamma)$ of the vertices of Γ:

$$\psi_\Gamma = \pi_{S,*}\omega_\Gamma. \tag{3.74}$$

We now have the following definition.

Definition 3.4.1. Let Γ be a Feynman graph and let $S \subset V(\Gamma)$. The we say that (Γ, S) is *horizontal gauge regularizable* if the limit

$$\psi^{hor}_{\Gamma,S} := \lim_{\lambda \to 0} \psi^\lambda_{\Gamma,S} = \lim_{\lambda \to 0} \pi_{S,*}\omega^\lambda_\Gamma \tag{3.75}$$

exists (in the sense of distributions). In that case, we say that the (regularized) weight of (Γ, S) in the horizontal gauge is ψ^{hor}_Γ. Similarly, we say that (Γ, S) is *axial gauge regularizable* if

$$\psi^{ax}_{\Gamma,S} := \lim_{\lambda \to \infty} \psi^\lambda_{\Gamma,S} = \lim_{\lambda \to \infty} \pi_{S,*}\omega^\lambda_\Gamma \tag{3.76}$$

exists, and $\psi^{ax}_{\Gamma,S}$ is called the (regularized) weight of (Γ, S) in the axial gauge.

In BV-BFV quantization, the Feynman graphs will have bulk and boundary vertices, and the subset S will usually be the set of all bulk vertices.

Remark 3.4.1. It might be that the limit as $\lambda \to 0$ does not exist. In that case, there should be a refined definition of the regularization above where one identifies the terms that are divergent as $\lambda \to 0$ and subtracts them according to a certain renormalization scheme. This problem should be addressed with more care in the future.

3.4.2 Some special cases

At the moment, there is no general result as to what (Γ, S) are axial (or horizontal) gauge regularizable but there are some special cases where regularizability can be proven, and we have explicit results for the regularized weight. The first case is when the graph is a tree.

Proposition 3.4.2. *For any tree T the product of the axial or horizontal gauge propagators is well-defined, i.e. $(T, \emptyset)$ is both axial and horizontal gauge regularizable.*

Proof. The key observation is that the wavefront sets of the distributions one needs to multiply all intersect transversally, hence, the Hörmander product [Hör03] of the distributions can be defined (we will not explain details of this here: A nice introduction can be found in [BDH14]). The best way to see this is inductively on the number of vertices in the tree. Indeed, if the tree has just one vertex there is nothing to prove. Suppose the result holds true for trees with n vertices. Now let T' be a tree with $n+1$ vertices, and write it as $T \cup \{v_1\}$. Then the statement holds for T. Since the singular support of the delta distributions and propagators are the diagonal, their wavefront sets are contained in $\{(x, x, k, k) : x \in M, k \in T^*_x M - \{0\}\} \subset M_0 \times M_1$. Since T' is a tree, the wavefront set of the distribution ω_T is nonzero only along the cotangent direction of M_2. See also the example 19 in [BDH14]. □

An approach that is often used in the literature on the axial gauge (e.g. [BCM12] or [IM18]) is degree counting. Notice that if $M = N_1 \times N_2$ and N_1, N_2 have dimensions d_1, d_2 respectively, then both the axial gauge and the horizontal gauge propagator have a term of bidegree $(d-1, d)$ and $(d, d-1)$. We have the following easy consequence of the heat kernel expansion (3.70) of η^λ. Consider a product coordinate chart $U = U_1 \times U_2 \subset M$. In this chart, any current T can be written as a sum of products of distributions $f_{T,I}$ (currents of degree 0) and coordinate forms dx^I: $T = \sum_I f_{T,I} dx^I$. We call the differential form $\widehat{T} = \sum_{I, f_{T,I} \neq 0} dx^I$ the *form part* of T in this coordinate system.

Proposition 3.4.3. *If the form part of a product of axial (or horizontal) gauge propagators vanishes in some (and hence all) coordinate system then so does the regularized product of the axial (or horizontal) gauge propagators.*

Proof. The form part of the heat kernel expansion (3.70) is the same as the form part of the axial gauge propagators, but the coefficients are smooth, hence the product vanishes. □

While the proof of this fact is very simple, this has some very direct and useful consequences.

Example 3.4.1. Take the horizontal gauge on a manifold $S^1 \times M$, $\eta = \eta_M \delta_{S^1}(1) + P_M \eta_{S(1)}$. Then the product of the propagator with itself contains the square of $\delta^{(1)}_{S^1}$. Since this is a 1-form, its square vanishes in the horizontal gauge regularization. Hence, any product that contains a square of $\delta^{(1)}_{S^1}$ vanishes in the regularization since its form part vanishes.

Remark 3.4.4. Essentially, this argument was used already in [BCM12], where it was argued that one could "smear out" the distributional coefficient to make this statement precise. The regularization through Riemann–Hodge gauges provides such a smearing, but with the important benefit that all the "smeared out" forms are propagators as well.

Remark 3.4.5 ("Universal" regularization). Instead of trying to regularize the axial gauge by approximating it with Riemann–Hodge gauges, for the case where one manifold is a circle factor there is a simpler but somewhat *ad hoc* regularization. Denote $\eta = \eta_M \delta^{(1)}_{S^1} + P_M \eta_{S^1} = \eta^I + \eta^{II}$. Namely, any regularization compatible with degree counting should regularize $\eta^I \eta^I$ to 0. After integrating, $\eta^I \eta^{II}$ yields a factor proportional to the ill-defined constant $\eta_{S^1}(0)$. One can argue heuristically that since the circle propagator is antisymmetric away from the diagonal, the only meaningful value one could associate to this constant is 0. These two prescriptions provide a complete regularization of the axial gauge on manifolds with a circle factor, thus they provide a somehow "universal" regularization of axial gauge on manifolds with a circle factor. This is compatible with the examples studied in Section 5.3.4. It would be interesting to understand the relationship between this rather *ad hoc* regularization and the one proposed in Remark 3.4.1 above. Concerning this we have the following slightly vague conjecture.

Conjecture 3.4.6. *The universal regularization of the axial gauge on manifolds with a circle factor can be obtained by adding a single counterterm to the action which cancels the divergencies coming from* $\eta_{S^1}(0)$.

Chapter 4

Polarized Lie algebras

In this section we explore the notion of *polarized Lie algebra* which plays a central role in split Chern-Simons theory. After the basic definitions we will treat in more detail the possible algebraic structure of polarized Lie algebras, especially, certain contractions of tensors which are naturally associated with this structure and that we call the eccentricity and complementary eccentricity. These will turn out to be important in the study of split Chern-Simons theory. Then we will look at a couple of examples which imply that the eccentricity and the complementary eccentricity are not invariant under twists, i.e. deformations of the splitting.

4.1 Definitions

The first important notion is that of a *quadratic Lie algebra*[1]. Let $\mathfrak{g}$ be a Lie algebra over $k = \mathbb{R}$ or $k = \mathbb{C}$.

Definition 4.1.1. A k-bilinear $\langle \cdot, \cdot \rangle$ form on $\mathfrak{g}$ is called *invariant* if for all $x, y, z \in \mathfrak{g}$ we have that

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle. \tag{4.1}$$

Definition 4.1.2. A Lie algebra $\mathfrak{g}$ over k together with k-bilinear symmetric invariant non-degenerate form is called *quadratic.*

We will usually suppress the bilinear form from the notation and just say that $\mathfrak{g}$ is a quadratic Lie algebra.

Notice that also in the complex case, we require the bilinear form to be symmetric, not sesquilinear. However, given the considerable differences between real and complex bilinear forms, and

[1] Sometimes also called *symmetric* ([Gom00]), *symmetric self dual* ([FS96]) or *metrised* ([Bor97]).

between real and complex Lie algebras, it comes as no surprise that complex and real quadratic Lie algebras differ considerably. For this reason we will distinguish between real and complex polarized Lie algebras. Recall that, in a vector space V with a bilinear form $\langle \cdot, \cdot \rangle$, a vector $v \in V$ is called *isotropic* if $\langle v, v \rangle = 0$. A subspace $W \subset V$ is called *isotropic* if it consists of isotropic vectors, or, equivalently, if $W \subset W^{\perp}$. An isotropic subspace of maximal dimension is said to be *Lagrangian.* If the bilinear form is non-degenerate, in the complex case the dimension of any Lagrangian subspace satisfies $\dim W = \lfloor \dim V/2 \rfloor$. Over $\mathbb{R}$, the maximal dimension of an isotropic subspace is an invariant of the bilinear form known as the Witt index (of the corresponding quadratic form).

Definition 4.1.3. A real (resp. complex) *polarized* Lie algebra $(\mathfrak{g}, V, W)$ is a real (resp. complex) quadratic Lie algebra together with a splitting $\mathfrak{g} = V \oplus W$ into Lagrangian subspaces.

We list some immediate consequences of this definition. A simple dimension counting shows that a real (resp. complex) polarized Lie algebra must have even real (resp. complex) dimension. In the complex case, even dimension together with non-degeneracy of $\langle \cdot, \cdot \rangle$ already implies the existence of a splitting. In the real case, such splittings exist if and only if the bilinear form has signature $(\dim_{\mathbb{R}} \mathfrak{g}/2, \dim_{\mathbb{R}} \mathfrak{g}/2)$, such bilinear forms are sometimes called *split* ([HM13, Chapter I]). In both cases, the bilinear form induces an isomorphism $W \cong V^*$.

Remark 4.1.1. Let us explain briefly why we call such Lie algebras polarized. The symmetric bilinear form endows the shifted Lie algebra $\mathfrak{g}[1]$ with an 2-shifted symplectic form given by $\omega(A, B) = \langle A, B \rangle$. A splitting into Lagrangian subspaces is a particular choice of polarization of this 2-symplectic vector space. Thus, we can use polarized Lie algebra as the target of an AKSZ construction (see Section 2.1.7) to obtain a polarized AKSZ theory (see also [CMW18b]). Also, notice that if the bilinear form is not split, then this is an example of a symplectic vector space not admitting polarizations (and, in fact, not even Darboux coordinates).

4.2 Algebraic structures related to polarized Lie algebras

In the definition of polarized Lie algebras, we did not make any assumption on the Lie algebraic nature of the Lagrangian subspaces V and W. In particular, one of them or even both of them could be Lie subalgebras. These are cases which have been studied in the literature, and as such have their own names.

Definition 4.2.1. Let $(\mathfrak{g}, V, W)$ be a polarized Lie algebra.

i) If V is a subalgebra of $\mathfrak{g}$, then (g, V) is called a *Manin pair*, and $(\mathfrak{g}, V, W)$ is called a *quasi-Manin triple.*

ii) If both V and W are subalgebras of $\mathfrak{g}$, then (g, V, W) is called a *Manin triple.*

Equivalently, one can look at the restriction of the Lie bracket of $\mathfrak{g}$ to the subspaces V and W, and compose the restrictions with the projection to V and W. This yields a total of four structure maps associated to the polarized Lie algebra. To analyze them, let us introduce a structure sometimes called the "big bracket" [Kos92; Kos95].

4.2.1 The big bracket

Let V be a vector space and consider the graded algebra $E_V = \bigwedge^\bullet(V \oplus V^*)$ where we define the bidegree of an element of $E^{(p,q)} = \wedge^{p+1}V \otimes \wedge^{q+1}V^*$ to be (p, q). Also let $E^{(k)} = \bigoplus_{p+q=k} E^{(p,q)}$, elements $\sigma \in E^{(k)}$ are said to have degree k and we write $|\sigma| = k$. With this bidegree this algebra is graded isomorphic to $C^\infty(T^*\Pi V)$, where Π denotes the parity shift. As such, we can equip the algebra E_V with the natural (even) graded Poisson bracket of $T^*\Pi V$, which for this section we will denote $\{\cdot, \cdot\}$. This bracket can be described in terms of elements of V and V^* as follows.

If $e_1, \ldots, e_n$ is a basis of V and $\varepsilon^1, \ldots, \varepsilon^n$ the dual basis of V^*, we can expand any element of $\sigma \in E^{(p,q)}$ as[2]

$$\sigma = \frac{1}{(p+1)!(q+1)!} \sigma^{i_1 \cdots i_{p+1}}_{j_1 \cdots j_{q+1}} e_{i_1} \wedge \cdots \wedge e_{i_{p+1}} \otimes \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_{q+1}}$$

where the upper and lower indices are totally antisymmetric. Fixing a basis of V, we can represent tensors by their components $\sigma^{i_1 \cdots}_{j_1 \cdots}$. With this representation we can give an explicit formula for the Poisson bracket, see Appendix E.

Example 4.2.1. Let us look in detail at an example. Consider an element $\mu \in \wedge^2 V^* \otimes V$. Picking a basis e_i of V and the dual basis ε^i, expand $\mu = \mu^k_{ij} \varepsilon^i \varepsilon^j e_k$. The μ has bidegree $(1, 0)$, so the equation $\{\mu, \mu\} = 0$ is nontrivial. In fact we claim that it is equivalent to the Jacobi

[2] We always use Einstein notation, meaning that repeated indices are summed over.

identity if we think of μ as a Lie bracket on V. Indeed, notice that

$$\begin{aligned}\{\mu,\mu\} &= \mu^k_{ij}\mu^n_{lm}\{\varepsilon^i\varepsilon^j e_k, \varepsilon^l\varepsilon^m e_n\}\\ &= \mu^k_{ij}\mu^n_{lm}\left(\varepsilon^i\varepsilon^j(\delta^l_k\varepsilon^m - \delta^m_k\varepsilon^l)e_n - (\delta^i_n\varepsilon^j - \delta^j_n\varepsilon^i)\varepsilon^l\varepsilon^m e_k\right)\\ &= -4\mu^m_{ij}\mu^n_{km}\varepsilon^i\varepsilon^j\varepsilon^k e_n.\end{aligned}$$

Here we have used the graded Leibniz rule in the second equation and relabeled indices in the last line. Now notice that

$$\mu^m_{ij}\mu^n_{km}\varepsilon^i\varepsilon^j\varepsilon^k = \mu^m_{[ij}\mu^n_{k]m}\varepsilon^i\varepsilon^j\varepsilon^k, \tag{4.2}$$

where the square brackets denote antisymmetrization of the indices: In particular,

$$\begin{aligned}\mu^m_{[ij}\mu^n_{k]m} &= \frac{1}{3!}\left(\mu^m_{ij}\mu^n_{km} - \mu^m_{ji}\mu^n_{km} + \mu^m_{ki}\mu^n_{jm} - \mu^m_{ik}\mu^n_{jm} + \mu^m_{jk}\mu^n_{im} - \mu^m_{kj}\mu^n_{im}\right)\\ &= \frac{1}{3}\left(\mu^m_{ij}\mu^n_{km} + \mu^m_{ki}\mu^n_{jm} + \mu^m_{jk}\mu^n_{im}\right)\end{aligned}$$

whose vanishing is indeed the Jacobi identity.

4.2.2 Big bracket and polarized Lie algebras

Let $(\mathfrak{g}, V, W)$ be a polarized Lie algebra. The Lie bracket $[\cdot,\cdot]$ on $\mathfrak{g}$ induces four tensors in E_V by restriction to V, W and composition with the projections p_V, p_W. Namely we have the maps

$$\begin{aligned}\mu\colon V\times V &\to V\\ (v_1,v_2) &\mapsto p_V[v_1,v_2]\\ \psi\colon V\times V &\to W\\ (v_1,v_2) &\mapsto p_W[v_1,v_2]\\ \lambda\colon W\times W &\to W\\ (w_1,w_2) &\mapsto p_W[w_1,w_2]\\ \varphi\colon W\times W &\to V\\ (w_1,w_2) &\mapsto p_V[w_1,w_2]\end{aligned} \tag{4.3}$$

Identifying $W \cong V^*$, this yields four ensors $\mu \in V\otimes\wedge^2V^*, \psi\in\wedge^3V^*, \lambda\in\wedge^2V\otimes V^*, \varphi\in\wedge^3V$, which have bidegrees $(0,1), (-1,2), (1,0)$ and $(2,-1)$ respectively (antisymmetry follows from the fact that the bilinear form is invariant). In particular, the element $M = \mu+\psi+\lambda+\varphi$ has total degree 1.

Definition 4.2.2. M is called a *structure* on V if M satisfies $\{M, M\} = 0$, i.e. M is a Maurer-Cartan element in the graded Lie algebra E_V.

In [WL88; LR90] it is proven that $\{M,M\}=0$ if and only if $[\cdot,\cdot]$ is a Lie bracket on $\mathfrak{g}$. Splitting the condition $\{M,M\}=0$ up into its various bidegrees gives a total of five structural equations:

$$\frac{1}{2}\{\mu,\mu\}+\{\lambda,\psi\}=0 \tag{4.4a}$$

$$\{\mu,\lambda\}+\{\varphi,\psi\}=0 \tag{4.4b}$$

$$\frac{1}{2}\{\lambda,\lambda\}+\{\mu,\varphi\}=0 \tag{4.4c}$$

$$\{\mu,\psi\}=0 \tag{4.4d}$$

$$\{\lambda,\varphi\}=0 \tag{4.4e}$$

The tuple $(V,\mu,\lambda,\psi,\varphi)$ is called a *proto-Lie-bialgebra* ([Kos92]). From the discussion above it follows that proto-Lie-bialgebras are in 1-to-1 correspondence with polarized Lie algebras $(\mathfrak{g},V,V^*)$ ($\mathfrak{g}$ is called the double of the proto-Lie-bialgebra). Again there are subcases: If $\psi=0$, then equation (4.4a) implies by the results of Example 4.2.1 that V is a Lie algebra. In this case one speaks of a *Lie quasi-bialgebra* (or Jacobian quasi-bialgebra), and $(\mathfrak{g},V)$ is a Manin pair. In $\varphi=0$, then V^* is a Lie algebra, and V is called a quasi-Lie bialgebra (or co-Jacobian quasi bialgebra), and $(\mathfrak{g},V^*)$ is a Manin pair. If $\varphi=\psi=0$, then equations (4.4a - 4.4c) imply that V is a Lie bialgebra, and $(\mathfrak{g},V,V^*)$ is a Manin triple.

4.2.3 Twists

In this paragraph, we introduce a notion of twist following [Kos11; ABM13] that generalizes the twists of Lie bialgebras introduced by Drinfeld ([Dri90]). We follow closely the exposition of [Kos11].

Let $\sigma\in E_V$ be a homogeneous element of bidegree $(1,-1)$ or $(-1,1)$ (i.e. $\sigma\in\wedge^2V$ or $\sigma\in\wedge^2V^*$). Consider the right adjoint action of σ, $ad_\sigma=\{\cdot,\sigma\}$. This is a degree 0 derivation of the algebra E_V, and its exponential e^{-ad_σ} (which is a finite sum for degree reasons) is a homomorphism of the bracket, i.e. $e^{-ad_\sigma}\{\tau,\tau'\}=\{e^{-ad_\sigma}\tau,e^{-ad_\sigma}\tau'\}$. It follows that for an $M\in E_V^1$ with $\{M,M\}=0$, also $M':=e^{-ad_\sigma}M$ satisfies $\{M',M'\}=0$. In the language of deformation theory, $e^{-\sigma}$ is a gauge transformation in E_V which takes Maurer-Cartan elements to Maurer-Cartan elements, see e.g. [DMZ07, Section 5].

Definition 4.2.3. The structure $e^{-ad_\sigma}M$ on V is said to be obtained from the structure M by twisting via σ.

In terms of the double $\mathfrak{g}$, twists can be interpreted as a deformation of the splitting $\mathfrak{g} = V \oplus W$ of g. Let M be the corresponding structure on V. Now, let e_i be a basis of V with dual basis ε^i. Then, the twist of M by an element $s = s^{ij} e_i e_j$ of bidegree $(1, -1)$ corresponds to the structure induced by the splitting $(e_i', (\varepsilon^i)')$ via

$$\begin{aligned} e_i' &= e_i, \\ (\varepsilon^i)' &= \varepsilon^i + s^{ij} e_j. \end{aligned}$$

Compare e.g. the discussion of twists in [AK00]. Similarly, a twist of M by an element $t = t_{ij}\varepsilon^i\varepsilon^j$ of bidegree $(-1, 1)$ corresponds to transforming the splitting as

$$\begin{aligned} e_i' &= e_i + t_{ij}\varepsilon^j \\ (\varepsilon^i)' &= \varepsilon^i. \end{aligned}$$

The composition of two twists of same bidegree is given simply by their sum. In general, the composition of twists is defined by the Baker-Campbell-Hausdorff formula (again, see e.g. [DMZ07]).

We can expand the twisted structure $M' = e^{-ad_s}$ in terms of bidegrees. First, let $s = s^{ij} e_i e_j$ be an element of bidegree $(1, -1)$. Then the components of M' are ([Roy02; Kos11])

$$\begin{aligned} \varphi' &= \varphi - \{\lambda, s\} + \{\{\mu, s\}, s\} - \{\{\{\psi, s\}, s\}, s\} \\ \lambda' &= \lambda - \{\mu, s\} + \{\{\psi, s\}, s\} \\ \mu' &= \mu - \{\psi, s\} \\ \psi' &= \psi \end{aligned} \tag{4.5}$$

and similar formulae hold in the case of twisting by an element of bidegree $(-1, 1)$.

4.3 Invariants of polarized Lie algebras

In this section we will introduce certain numbers naturally associated with polarized Lie algebras, namely, the eigenvalue of the quadratic Casimir, which is an invariant of quadratic Lie algebras, and two contractions that we call the *eccentricity* and *complementary eccentricity* of a polarized Lie algebra. We will discuss to what extent they form invariants of polarized Lie algebras, and invariants under twists.

4.3.1 Some identities on the structure constants

Let $(\mathfrak{g}, V, W)$ be a polarized Lie algebra. Then the bilinear form identifies $W \cong V^*$. Let $e_1, \ldots, e_n \in V$ be a basis of V and let $\varepsilon^1, \ldots, \varepsilon^n \in W$ be the dual basis of W. Then $(e_1, \ldots,_n, \varepsilon^1, \ldots, \varepsilon^n)$ is a basis of $\mathfrak{g}$ that we denote collectively by $(\xi_1, \ldots, \xi_{2n})$, i.e. $\xi_i = e_i$ for $1 \le i \le n$ and $\xi_i = \varepsilon^{i-n}$, for $n+1 \le i \le 2n$. Let $\mu, \lambda, \psi, \varphi$ be the maps induced by the Lie bracket on $\mathfrak{g}$ introduced in (4.3) above. We adopt the convention that letters from the beginning of the alphabet $a, b, c, \ldots$... run from 1 to $2n$ and letters $i, j, k, \ldots$ run from 1 to n. We have the following sets of structure constants:

$$\begin{aligned} [\xi_a, \xi_b] &= f^c_{ab}\xi_c \\ \mu(e_i, e_j) &= \mu^k_{ij} e_k \\ \psi(e_i, e_j) &= \psi_{ijk}\varepsilon^k \\ \varphi(\varepsilon^i, \varepsilon^j) &= \varphi^{ijk} e_k \\ \lambda(\varepsilon^i, \varepsilon^j) &= \lambda^{ij}_k \varepsilon^k \end{aligned} \tag{4.6}$$

Notice that the constants agree when the arguments agree, e.g. for $i, j \le n$ we have

$$[\xi_i, \xi_j] = [e_i, e_j] = \mu(e_i, e_j) + \psi(e_i, e_j) = \mu^k_{ij} e_k + \psi_{ijk}\varepsilon^k.$$

The mixed brackets can be expressed using μ and λ.

Proposition 4.3.1. *We have*

$$[e_i, \varepsilon^j] = \lambda^{jk}_i e_k - \mu^j_{ik}\varepsilon^k.$$

Proof. We can write

$$\langle [e_i, \varepsilon^j], e_k \rangle = \langle [e_k, e_i], \varepsilon^j \rangle = -\mu^j_{ik}$$

and

$$\langle [e_i, \varepsilon^j], \varepsilon^k \rangle = \langle [\varepsilon^j, \varepsilon^k], e_i \rangle = \lambda^{jk}_i$$

from where we conclude

$$[e_i, \varepsilon^j] = \lambda^{jk}_i e_k - \mu^j_{ik}\varepsilon^k.$$

□

Using the metric $B_{ab} = \langle \xi_a, \xi_b \rangle$ we can form the totally antisymmetric structure constants $f_{abc} = f^d_{ab} B_{cd}$.

Definition 4.3.1. The *quadratic Casimir invariant* of $(\mathfrak{g}, B)$ is

$$c_2(\mathfrak{g}, B) = f_{abc} f^{abc}. \tag{4.7}$$

It is well known that the quadratic Casimir invariant does not depend on the basis and is invariant under isometries of quadratic Lie algebras, hence it is an invariant of quadratic Lie algebras. However, the quadratic Casimir does not have any particular information about polarized Lie algebras. This leads to the definition of *eccentricity* and *complementary eccentricity* that we give now.

Definition 4.3.2. Let $(\mathfrak{g}, V, W)$ be a polarized Lie algebra with structure maps $(\mu, \lambda, \psi, \varphi)$.

1. We define the *eccentricity* $E(\mathfrak{g}, V, W)$ of $(\mathfrak{g}, V, W)$ to be

$$E(\mathfrak{g}, V, W) \equiv E(\mathfrak{g}) := \mu^i_{ki} \lambda^{kj}_j. \tag{4.8}$$

2. $(\mathfrak{g}, V, W)$ is called *regular* if $E(\mathfrak{g}, V, W) = 0$ and *eccentric* otherwise.

3. The *complementary eccentricity* $\widetilde{E}(\mathfrak{g}, V, W)$ is defined to be

$$\widetilde{E}(\mathfrak{g}, V, W) \equiv \widetilde{E}(\mathfrak{g}) := \langle \varphi, \psi \rangle = \varphi^{ijk} \psi_{ijk} \tag{4.9}$$

   where $\langle \cdot, \cdot \rangle$ is the extension of the pairing $V \times V^* \to k$ to E_V.

Clearly, these numbers do not depend on the basis of V. They are therefore an invariant of the polarized Lie algebra $(\mathfrak{g}, V, W)$. There are a few immediate consequences of this definition.

Proposition 4.3.2. *i) Let $(\mathfrak{g}, V, W)$ be a quasi-Manin triple (i.e. either V or W is a subalgebra). Then the complementary eccentricity of $(\mathfrak{g}, V, W)$ vanishes.*

ii) Suppose either μ or λ is unimodular (i.e. satisfies $\mu^i_{ik} = 0$ resp $\lambda^i_{ik} = 0$ for all k). Then the eccentricity of $(\mathfrak{g}, V, W)$ vanishes, i.e. $(\mathfrak{g}, V, W)$ is a regular polarized Lie algebra.

Proof. i) If V (resp. W) is a subalgebra, then ψ (resp. φ) vanishes, and in particular $\widetilde{E}(\mathfrak{g}) = 0$.

ii) Immediate from the definition, since $E = \mu^i_{ki} \lambda^{kj}_j = \mu^i_{ik} \lambda^{jk}_j$ which clearly vanishes if V or W is unimodular.

□

The following proposition gives an interesting formula for the quadratic Casimir in polarized Lie algebras.

Proposition 4.3.3. *Let $(\mathfrak{g}, V, W)$ be a polarized Lie algebra, then we have*

$$c_2(\mathfrak{g}) = 2\varphi^{ijk}\psi_{ijk} + 6\mu_{ij}^k\lambda_k^{ij} = 2\langle\varphi, \psi\rangle + 6\langle\mu, \lambda\rangle$$

where $\langle\cdot,\cdot\rangle$ is the extension of the pairing $V \times V^* \to k$ to E_V.

Proof. We can compute

$$\begin{aligned} f_{abc}f^{abc} &= \sum_{i,j,k=1}^{n} f_{ijk}f^{ijk} + 3\sum_{i,j=1}^{n}\sum_{k=n+1}^{2n} f_{ijk}f^{ijk} + 3\sum_{i=1}^{n}\sum_{j,k=n+1}^{2n} f_{ijk}f^{ijk} + \sum_{i,j,k=n+1}^{2n} f_{ijk}f^{ijk} \\ &= 2\sum_{i,j,k=1}^{n} f_{ijk}f^{ijk} + 6\sum_{i,j=1}^{n}\sum_{k=n+1}^{2n} f_{ijk}f^{ijk} \\ &= 2\varphi_{ijk}\psi^{ijk} + 6\mu_{ij}^k\lambda_k^{ij} \end{aligned}$$

In the first equality we have used total antisymmetry and in the second equality the fact that lifting an index is just shifting it by n (modulo $2n$). □

The following theorem allows us to go even further and express the quadratic Casimir of g in terms of the eccentricity and complementary eccentricity.

Theorem 4.3.4. *Let (g, V, W) be a polarized Lie algebra with invariant bilinear form B. Then*

$$c_2(\mathfrak{g}, B) = 12E(\mathfrak{g}, V, W) + 8\widetilde{E}(\mathfrak{g}, V, W). \tag{4.10}$$

The proof of the theorem follows from two Lemmas:

Lemma 4.3.5. *If $\mathfrak{g}$ is a quadratic Lie algebra, then*

$$\langle[x, y], [z, w]\rangle = \langle[x, z], [y, w]\rangle - \langle[x, w], [y, z]\rangle$$

Proof. The proof is a straightforward application of invariance and the Jacobi identity:

$$\begin{aligned} \langle[x, y], [z, w]\rangle &= \langle x, [y, [z, w]]\rangle = \langle x, [[y, z], w] + [z, [y, w]]\rangle \\ &= \langle[x, z], [y, w]\rangle - \langle[x, w], [y, z]\rangle. \end{aligned}$$

□

We can express $\varphi_{ijk}\psi^{ijk}$ in terms of other structure constants:

Lemma 4.3.6. *We have*

$$\langle\mu, \lambda\rangle = 2E(\mathfrak{g}, V, W) + \widetilde{E}(\mathfrak{g}, V, W).$$

Proof. We compute $\langle [e_i, e_j], [\varepsilon^i, \varepsilon^j]\rangle$ in two different ways, once directly and once using Lemma 4.3.5. First, we have

$$\langle [e_i, e_j], [\varepsilon^i, \varepsilon^j]\rangle = \psi_{ijk}\varphi^{ijk} + \mu_{ij}^k \lambda_k^{ij}.$$

On the other hand, using Lemma 4.3.5 and proposition 4.3.1, we get

$$\begin{aligned}\langle [e_i, e_j], [\varepsilon^i, \varepsilon^j]\rangle &= \langle [e_i, \varepsilon^i], [e_j, \varepsilon^j]\rangle - \langle [e_i, \varepsilon^j], [e_j, \varepsilon^i]\rangle \\ &= \langle \lambda_i^{ik} e_k - \mu_{ik}^i \varepsilon^k, \lambda_j^{jk} e_k - \mu_{jk}^j \varepsilon^k\rangle - \langle \lambda_i^{jk} e_k - \mu_{ik}^j \varepsilon^k, \lambda_j^{ik} e_k - \mu_{jk}^i \varepsilon^k\rangle \\ &= 2\mu_{ij}^k \lambda_k^{ij} - 2\mu_{ik}^i \lambda_j^{jk}.\end{aligned}$$

□

The proof of the theorem is now simple.

Proof. We simply combine Lemma 4.3.6 with Proposition 4.3.3 to get

$$\begin{aligned}f_{abc}f^{abc} &= 2\psi_{ijk}\varphi^{ijk} + 6\mu_{ij}^k \lambda_k^{ij} \\ &= 8\psi_{ijk}\varphi^{ijk} + 12\mu_{ik}^i \lambda_k^{jk}\end{aligned}$$

□

While the proof of this theorem is simple, it has several immediate corollaries which are quite interesting.

Corollary 4.3.7. *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra.*

1. *The eccentricities of all quasi-Manin triples $(\mathfrak{g}, V, W)$ are identical and satisfy*

$$E(\mathfrak{g}, V, W) = \frac{1}{12} c_2(\mathfrak{g}, B). \tag{4.11}$$

2. *The complementary eccentricities of all regular splittings $(\mathfrak{g}, V, W)$ agree, and are given by*

$$\widetilde{E}(\mathfrak{g}, V, W) = \frac{1}{8} c_2(g, B). \tag{4.12}$$

3. *Suppose $(\mathfrak{g}, B)$ admits a regular Manin triple $(\mathfrak{g}, V, W)$. Then*

$$c_2(\mathfrak{g}) = E(\mathfrak{g}, V, W) = \widetilde{E}(\mathfrak{g}, V, W) = \langle \mu, \lambda\rangle = 0. \tag{4.13}$$

The last case justifies the name *eccentric*, since regular Manin triples are quite common (see also the discussion below).

One might ask whether $E, \widetilde{E}$ are not in fact invariants of the quadratic Lie algebra itself. The example in Section 4.4.2 below shows that this is not the case. Notice that for a quasi-Manin triple, we have $2E = \langle \mu, \lambda \rangle$. Hence a good candidate for eccentric Manin triples are ones that correspond to self-dual Lie bialgebras $\mathfrak{b}$ (in the sense that the Lie algebra structures on $\mathfrak{b}, \mathfrak{b}^*$ are isomorphic). We consider such examples in 4.4.2, and 4.4.3.

Let us relate this to some other discussions in the literature. In [Kos11], a *Poisson (resp. pre-symplectic) map* with respect to a structure M is defined to be an element $\sigma \in E_V^{(1,-1)}$ (resp. $\tau \in E_V^{(-1,1)}$ such that after twisting one has $\varphi' = 0$ (resp. $\psi' = 0$). In this context we have the following result.

Proposition 4.3.8. *For* regular *structures (in the sense of Definition 4.3.2), the complementary eccentricity provides an obstruction to the existence of Poisson (or pre-symplectic) maps.*

On another hand, in [AK00], quasi-Manin triples $(\mathfrak{g}, V, W)$ and their twists are considered. Here one requires that V is a subalgebra and considers only twists of W. In this context we have the following result.

Proposition 4.3.9. *Let g be a quadratic Lie algebra which allows polarizations.*

i) The eccentricity is an invariant of the Manin pair $(\mathfrak{g}, V)$ which does not depend on the choice of isotropic complement W. In particular, it is invariant under twists (of W).

ii) The quadratic Casimir of a quadratic Lie algebra $\mathfrak{g}$ is an obstruction for the existence of a unimodular Lagrangian subalgebra.

Proof.

1. The complementary eccentricity of every quasi-Manin triple $(\mathfrak{g}, V, W)$, where V is a Lie subalgebra, vanishes. Hence the eccentricity of this Manin pair is equal to a multiple of the quadratic Casimir of $\mathfrak{g}$, and hence does not depend on W.

2. If there exists a unimodular Lagrangian subalgebra, then there exists a splitting for with both the eccentricity and the complementary eccentricity vanish. Hence the quadratic Casimir of the Lie algebra vanishes.

□

4.4 Examples

In this section we will consider various examples of polarized Lie algebras and their associated invariants. The lowest dimension a polarized Lie algebra can have is 2, however in this case all structure maps vanish, which follows from equations (4.4a - 4.4e) or simple computations.

4.4.1 A 4-dimensional example

As an example, we can consider the Manin triple $\mathfrak{g}$ associated to the Lie bialgebra $[x, y] = x$ with dual Lie bracket $[x^*, y^*] = x^*$. In this case we have $\mu_{12}^1 = 1 = \lambda_1^{12}$, so $\mu_{ij}^k \lambda_k^{ij} = 2$ (since $\mu_{21}^1 = \lambda_1^{21} = -1$). Also, $\mu_{ik}^i = \delta_{k2}$ (in particular this Lie algebra is not unimodular). We see that the eccentricity $E(\mathfrak{g}) = 1$. The totally antisymmetric structure constants of $\mathfrak{g}$ are $f_{123} = 1 = f_{134}$, implying that $f_{abc}f^{abc} = 12 = 6g_{ij}^k h_k^{ij} = 12E(\mathfrak{g})$. Notice that for any 4-dimensional split Lie algebra, the structure maps φ, ψ vanish (since they are totally antisymmetric). However, these maps can be nontrivial for a 6-dimensional Lie algebra, as we will see in the next example. This example also shows that the eccentricity and the complementary eccentricity are not invariants of quadratic Lie algebras, but depend on the splitting.

4.4.2 Example of a bad Lie algebra splitting

This is an example of a "bad" splitting of a Lie algebra where, starting from a splitting into Lie subalgebras, a sequence of twists produces a splitting in which the complementary eccentricity does not vanish. This shows that the complementary eccentricity is not an invariant of quadratic Lie algebras.

The Lie bialgebra

Consider the 3-dimensional Lie algebra[3] $\mathfrak{g}(\rho)$ given by

$$[e_0, e_1] = e_1, [e_0, e_2] = \rho e_2 \text{ and } [e_1, e_2] = 0.$$

Consider $\delta \colon \mathfrak{g} \to \wedge^2 \mathfrak{g}$ given by

$$\delta(e_0) = 0, \delta e_1 = e_0 \wedge e_1, \delta e_2 = -\rho e_0 \wedge e_2.$$

The first observation is that $(\mathfrak{g}(\rho), [,], \delta)$ is a Lie bialgebra. Indeed, the cocycle condition

$$\delta([X, Y]) = ad_X \delta(Y) - ad_Y \delta(X)$$

is satisfied:

$$\delta([e_0, e_1]) = \delta(e_1) = e_0 \wedge e_1,$$

[3] This is one of the two self-dual 3-dimensional Lie bialgebras discovered in [Gom00], from where we adapt the notation.

while

$$ad_{e_0}\delta(e_1) - ad_{e_1}\delta(e_0) = ad_{e_0}(e_0 \wedge e_1) = e_0 \wedge e_1,$$

similarly for $[e_0, e_2]$, while

$$\begin{aligned} ad_{e_1}\delta(e_2) - ad_{e_2}\delta(e_1) &= -\rho ad_{e_1} e_0 \wedge e_2 - ad_{e_2} e_0 \wedge e_1 \\ &= -\rho[e_1, e_0] \wedge e_2 - [e_2, e_0] \wedge e_1 \\ &= \rho(e_1 \wedge e_2 + e_2 \wedge e_1) = 0 = \delta([e_1, e_2]). \end{aligned}$$

In fact, the Lie bracket on the dual $\mathfrak{g}^*$ induced by δ is the one identical to $\mathfrak{g}(-\rho)$. Let us define structure constants μ and λ by

$$\begin{aligned} [e_a, e_b] &= \mu^c_{ab} e_c \\ [\varepsilon^a, \varepsilon^b] &= \lambda^{ab}_c \varepsilon^c \end{aligned}$$

(here ε^a is the basis dual to e_a). Then we have

$$\mu^1_{01} = 1, \mu^2_{02} = \rho, \lambda^{01}_1 = 1, \lambda^{02}_2 = -\rho$$

and all other structure constants not related by symmetry vanish. The contraction of interest $\mu^c_{ab}\lambda^{ab}_c = 2(1-\rho^2)$ vanishes precisely if $\rho = \pm 1$. Alternatively, we could compute $\mu^i_{ki} = \delta_{k0}(1+\rho)$, $\lambda^{ki}_i = \delta_{k0}(1-\rho)$ which implies that the eccentricity of this splitting is $(1-\rho^2)$.

The Drinfeld double

Consider the double $\mathfrak{d} = \mathfrak{d}(\rho) = \mathfrak{g}(\rho) \oplus \mathfrak{g}(-\rho)$ of $\mathfrak{g}$. We denote the bilinear form on $\mathfrak{d}$ induced by the canonical pairing by B. The new brackets are given by $[e_a, \varepsilon^b] = -\mu^b_{ac}\varepsilon^c + \lambda^{bc}_a e_c$ so that

$$\begin{aligned} &[e_0, \varepsilon^0] = 0, \quad [e_0, \varepsilon^1] = -\varepsilon^1, \quad [e_0, \varepsilon^2] = -\rho\varepsilon^2, \\ &[e_1, \varepsilon^0] = e_1, \quad [e_1, \varepsilon^1] = \varepsilon^0 - e_0, \quad [e_2, \varepsilon^1] = 0, \\ &[e_2, \varepsilon^0] = -\rho e_2, \quad [e_2, \varepsilon^1] = 0, \quad [e_2, \varepsilon^2] = \rho(\varepsilon^0 + e_0). \end{aligned} \tag{4.14}$$

We claim that $(\mathfrak{d}, B)$ is isometric to $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$, where each of the two factors carries a certain multiple of the Killing form. Let us make this isometry explicit. Define a basis of $sl_2(\mathbb{R})$ by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.15}$$

The commutators are $[h, x] = 2x, [h, y] = -2y, [x, y] = h$. The Killing form is given by $K(A, B) = 4tr(AB)$, hence, for this basis we have $K(h, h) = 8, K(x, y) = 4$, and $K(y, h) =$

$K(x,h) = K(x,x) = K(y,y) = 0$. Define a new basis $u = h/2, v = (x+y)/2, w = (x-y)/2$. In this new basis, we have $[u,v] = w, [u,w] = v, [v,w] = -u$, u, v, w are orthogonal with respect to the Killing form and $K(u,u) = K(v,v) = 2 = -K(w,w)$.

Now, consider the elements e_a^+, e_a^- of $\mathfrak{d}$ given by $e_a^{\pm} = e_a \pm \varepsilon^a$. Then this is a B-orthogonal family and $B(e_a^{\pm}, e_a^{\pm}) = \pm 2$. Now, the commutation relations (4.14) imply that

$$[e_0^-, e_1^+] = 2e_1^-, [e_0^-, e_1^-] = 2e_1^+, [e_1^-, e_1^+] = -2e_0^-. \tag{4.16}$$

So $\mathfrak{g}_1 = \langle e_0^-, e_1^-, e_1^+ \rangle \subset \mathfrak{d}$ is a Lie subalgebra and $e_0^- \mapsto 2u, e_1^- \mapsto 2v, e_1^+ \mapsto 2w$ defines an isometry $(g_1, B) \to (sl_2(\mathbb{R}), -\frac{1}{4}K)$. Similarly, $\mathfrak{g}_2 = \langle e_0^+, e_2^+, e_2^- \rangle$ is a subalgebra of $(\mathfrak{g})$ which is isometric to $(sl_2(\mathbb{R}), \frac{1}{4\rho^2}K)$. It is a straightforward check that $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$. Hence, $(\mathfrak{d}, B) \cong (sl_2(\mathbb{R}), \frac{1}{4\rho^2}K) \oplus (sl_2(\mathbb{R}), -\frac{1}{4}K)$. We can also compute the structure constants $f_{ijk} = f_{ij}^l B_{lk}$ and $f^{ijk} = B^{il}B^{jm}f_{lm}^k$ in the basis $(u_1, v_1, w_1, u_2, v_2, w_2)$. Namely, f_{ijk} is nonzero only if (i,j,k) is a permutation of $(1,2,3)$ or $(4,5,6)$, and we have $f_{123} = -\frac{1}{2\rho^2}, f_{456} = \frac{1}{2}, f^{123} = 4\rho^4, f^{456} = +4$. In particular, $f_{ijk}f^{ijk} = 12(1-\rho^2) = 12E(\mathfrak{g})$.

Twisting to a bad splitting

We will now consider a sequence of twists: $(\mathfrak{d}, g(\rho), g(-\rho)) \rightsquigarrow (\mathfrak{d}, V, g(-\rho)) \rightsquigarrow (\mathfrak{d}, V, W)$ such that the last splitting is "bad", i.e. has non-vanishing contraction $\psi_{ijk}\varphi^{ijk}$, where $\psi\colon V \times V \to W, \varphi\colon W \times W \to V$ are the restrictions of the bracket to V resp. W composed with projection to W resp. V as defined in (4.3), i.e. the measure the failure of V, W to be subalgebras.

The first twist is $e_a \mapsto e'_a = e_a + t_{ab}\varepsilon^b$, where $t_{12} = \lambda = -t_{21}$ and all other entries 0, where $\lambda > 0$. The new structure maps can be computed using formula (4.5), but it is simpler to do it directly. Since $[e'_1, e'_2] = \lambda([\varepsilon^2, e_2] - [e_1, \varepsilon^1]) = \lambda((1-\rho)e_0 - (1+\rho)\varepsilon^0)$ we have

$$\psi'_{012} = \langle e'_0, [e'_1, e'_2] \rangle = -\lambda(1+\rho)$$

and

$$(\mu')^0_{12} = \langle \varepsilon^0, [e'_1, e'_2] \rangle = \lambda(1-\rho).$$

These are the only structure constants which change. In the second twist, we map $\varepsilon^a \mapsto (\varepsilon^a)' = \varepsilon^a + s^{ab}e'_a$ where $s^{12} = 1 = -s^{21}$ and all other entries 0. This results in the new bases (e'_a) of

V with dual bases $(\varepsilon^a)'$ of W given by

$$e_0' = e_0, \qquad (\varepsilon^0)' = \varepsilon^0,$$
$$e_1' = e_1 + \lambda\varepsilon^2, \qquad (\varepsilon^1)' = (1-\lambda)\varepsilon^1 + e_2,$$
$$e_2' = e_2 - \lambda\varepsilon^1, \qquad (\varepsilon^2)' = (1-\lambda)\varepsilon^2 - e_1$$

Then, we have

$$\begin{aligned}
[(\varepsilon^1)', (\varepsilon^2)'] &= [(1-\lambda)\varepsilon^1 + e_2, (1-\lambda)\varepsilon^2 - e_1] = (1-\lambda)([e_1, \varepsilon^1] + [e_2, \varepsilon^2]) \\
&= (1-\lambda)((1-\rho)e_0 + (1+\rho)\varepsilon^0), \\
[(\varepsilon^0)', (\varepsilon^1)'] &= (1-\lambda)\varepsilon^1 + \rho e_2 \\
[(\varepsilon^0)', (\varepsilon^2)'] &= -\rho(1-\lambda)\varepsilon^2 + e_1
\end{aligned}$$

The following structure constants change:

$$\begin{aligned}
(\varphi'')^{012} &= \langle (\varepsilon^0)', [(\varepsilon^1)', (\varepsilon^2)'] \rangle = (1-\lambda)(1-\rho) \\
(\lambda'')_0^{12} &= \langle e_0, [(\varepsilon^1)', (\varepsilon^2)'] \rangle = (1-\lambda)(1+\rho) \\
(\lambda'')_1^{01} &= \langle e_1', [(\varepsilon^0)', (\varepsilon^1)'] \rangle = (1-\lambda) + \lambda\rho \\
(\lambda'')_2^{02} &= \langle e_2', [(\varepsilon^0)', (\varepsilon^2)'] \rangle = -\rho(1-\lambda) - \lambda \\
(\mu'')_{01}^1 &= \langle (\varepsilon^1)', [e_0, e_1'] \rangle = (1-\lambda) - \lambda\rho \\
(\mu'')_{02}^2 &= \langle (\varepsilon^2)', [e_0, e_2'] \rangle = \rho(1-\lambda) - \lambda.
\end{aligned}$$

From this we compute $(\mu'')_{ki}^i = \delta_{k0}(1 - 2\lambda - 2\rho\lambda + \rho) = \delta_{k0}(1+\rho)(1-2\lambda)$, $(\lambda'')_i^{ki} = \delta_{k0}(1 - 2\lambda + 2\lambda\rho - \rho) = \delta_{k0}(1-2\lambda)(1-\rho)$. Hence the eccentricity $E(\mathfrak{g}, V, W)$ of this splitting is $(1-2\lambda)^2(1-\rho^2)$. The complementary eccentricity of this splitting is

$$\widetilde{E}(\mathfrak{g}, V, W) = 6\psi_{012}\varphi^{012} = 6\lambda(1-\lambda)(1-\rho^2). \tag{4.17}$$

Notice that both eccentricities are non-zero only if $\lambda \neq 1, \rho \neq 1$. In passing, note that we have

$$8\widetilde{E}(\mathfrak{g}, V, W) + 12E(\mathfrak{g}, V, W) = 48\lambda(1-\lambda)(1-\rho^2) + 12(1 - 4\lambda + 4\lambda^2)(1-\rho^2) = 12(1-\rho^2) = c_2(\mathfrak{g}, B)$$

in agreement with Theorem 4.3.4.

4.4.3 Different invariant forms on $sl_2(\mathbb{C})$.

In this subsection we provide an example of how the invariants defined above depend on the invariant bilinear form. Namely, we will construct splittings of $sl_2(\mathbb{C})$ (considered as a real Lie algebra) with respect to different invariant bilinear forms, such that the eccentricity (and all other invariants) associated with the first bilinear form vanish, while they do not for the second one, which comes from presenting $sl_2(\mathbb{C})$ as a sum of Bianchi VII_a factors (see e.g. [Gla+14]).

Imaginary part of the Killing form

Consider $sl_2(\mathbb{C})$ with invariant bilinear form $B_0(X,Y) = \operatorname{Im} 2\operatorname{tr}(XY)$. This is a multiple of the imaginary part of the Killing form on $sl_2(\mathbb{C})$. The matrices h, x, y introduced above span $sl_2(\mathbb{C})$ over $\mathbb{C}$. From the fact that $2\operatorname{tr}(h^2) = 4, 2\operatorname{tr}(xy) = 2$ and all other traces vanish, we see that the splitting $(\mathfrak{g}, V, W)$ given by $V = \langle h, x, y\rangle_{\mathbb{R}}, W = \langle ih, iy, ix\rangle$ is a splitting of $sl_2(\mathbb{C})$ as regular quasi-Manin triple, since $V \cong sl_2(\mathbb{R})$ as Lie algebras and $sl_2(\mathbb{R})$ is unimodular. Hence, by Corollary 4.3.7 all eccentricities vanish. Another possibility is to split $sl_2(\mathbb{C}) = su_2 \oplus \mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra, this is a splitting as a regular Manin triple (since su_2 is unimodular).

Splitting into Bianchi VII_a Lie subalgebras

It is a non-trivial fact that $sl_2(\mathbb{C})$, endowed with a particular invariant bilinear form, admits a splitting into isomorphic subalgebras (i.e. can be seen as the double of a self-dual Lie bialgebra). The algebra that these subalgebras are isomorphic to is called type VII_a in the Bianchi classification of 3-dimensional Lie algebras. This was proven in [Gom00]. Let $a \in \mathbb{R}$ and consider the 3-dimensional Lie algebra $\mathfrak{s}(a)$ with basis e_0, e_1, e_2 and commutation relations

$$\begin{aligned} [e_0, e_1] &= ae_1 - e_2 \\ [e_0, e_2] &= e_1 + ae_2 \\ [e_1, e_2] &= 0. \end{aligned} \tag{4.18}$$

Define a Lie coalgebra structure λ on $\mathfrak{s}(a)$ by

$$\begin{aligned} \lambda e_0 &= 0, \\ \lambda e_1 &= e_0 \wedge (e_1 + ae_2) \\ \lambda e_2 &= e_0 \wedge (-ae_1 + e_2). \end{aligned}$$

It is straightforward to check the cocycle condition, e.g.

$$\begin{aligned} \lambda([e_0, e_1]) &= a\lambda e_1 - \lambda e_2 = e_0 \wedge (2ae_1 + (a^2 - 1)e_2) \\ ad_{e_0}\lambda e_1 - ad_{e_1}\lambda e_0 &= e_0 \wedge ([e_0, e_1] + a[e_0, e_2]) = e_0(\wedge 2ae_1 + (a^2 - 1)e_2), \end{aligned}$$

and so on (the fact that this is a Lie bialgebra is proven in more general terms in [Gom00]). Let us compute the eccentricity of the corresponding Manin triple: We have $\mu^i_{ki} = 2a\delta_{k0}$ and $\lambda^{kj}_j = 2\delta_{k0}$. It follows that the eccentricity of the Drinfeld double $\mathfrak{d}(a)$ of $\mathfrak{s}(a)$ is $E(\mathfrak{d}, \mathfrak{s}(a), \mathfrak{s}(a)^*) = 4a$. It follows that the quadratic Casimir of the double is $c_2(\mathfrak{d}(a)) = 48a$. We now have the following claim.

Claim 4.4.1. *$\mathfrak{d}(a)$ is isomorphic to $sl_2(\mathbb{C})$ as a Lie algebra for all a. For $a = 0$, this is an isometry with B_0 above.*

It follows that there is a family of non-isometric invariant bilinear forms B_a on $sl_2(\mathbb{C})$ which allows eccentric splittings.

Proof. This claim is proven indirectly in [Gom00], but here we construct an explicit isomorphism. For this, we explicitly spell out the brackets. The brackets on $\mathfrak{s}(a)^*$ spanned by $\varepsilon^0, \varepsilon^1, \varepsilon^2$ are

$$[\varepsilon^0, \varepsilon^1] = \varepsilon^1 - a\varepsilon^2, \quad , [\varepsilon^0, \varepsilon^2] = a\varepsilon^1 + \varepsilon^2, [\varepsilon^1, \varepsilon^2] = 0. \tag{4.19}$$

The mixed brackets can be deduced from Proposition 4.3.1 and are

$$\begin{aligned}
&[\varepsilon^0, e_0] = 0, \quad [\varepsilon^0, e_1] = -e_1 - ae_2, \quad [\varepsilon^0, e_2] = ae_1 - e_2 \\
&[\varepsilon^1, e_0] = a\varepsilon^1 + \varepsilon^2, \quad [\varepsilon^1, e_1] = -a\varepsilon^0 + e_0, \quad [\varepsilon^1, e_2] = -\varepsilon^0 - ae_0 \\
&[\varepsilon^2, e_0] = -\varepsilon^1 + a\varepsilon^2, \quad [\varepsilon^2, e_1] = \varepsilon^0 + ae_0, \quad [\varepsilon^2, e_2] = -a\varepsilon^0 + e_0.
\end{aligned}$$

Remember that $h, x, y, \mathrm{i}h, \mathrm{i}x, \mathrm{i}y$ is a real basis for $sl_2(\mathbb{C})$. Consider first the case $a = 0$. In that case, define a map $\mathfrak{d}(0) \to sl_2(\mathbb{C})$ by

$$\begin{aligned}
e_0 \mapsto \frac{\mathrm{i}}{2}h, e_1 \mapsto \frac{1}{\sqrt{2}}y, e_2 \mapsto \frac{\mathrm{i}}{\sqrt{2}}y \\
\varepsilon^0 \mapsto \frac{1}{2}h, \varepsilon^1 \mapsto \frac{1}{\sqrt{2}}x, \varepsilon^2 \mapsto \frac{\mathrm{i}}{\sqrt{2}}x.
\end{aligned}$$

Using the commutation relations of $[h, x] = 2x, [h, y] = 2y$ it is a straightforward check that this is a Lie algebra homomorphism on $s(0)$ and $s(0)^*$. Also, since this map is an isometry between the canonical invariant bilinear form on $\mathfrak{d}$ and B_0 and, it defines a Lie (bi)algebra isomorphism $\mathfrak{d}(0) \to (sl_2(C), B_0)$.

Next, consider the case $a \neq 0$. The first guess is to deform the images of e_0 and ε^0, such that one obtains the commutation relations (4.18) in $\mathfrak{s}(a)$ (resp. (4.19) $s(a)^*$). One quickly finds that sending $e_0 \mapsto \mathrm{i}/2h - a/2h$, $\varepsilon^0 \mapsto h/2 + a\mathrm{i}/2h$ does the job. However, this destroys the commutation relations $[\varepsilon^i, e_j]$ for $i = 1, 2$. Hence we also have to deform the image of these vectors. The trick is that multiplying e_i, ε^j by the same *complex* factor does not destroy the commutation relations on the subalgebras, since the bracket is a actually complex linear. The commutators of $[\varepsilon^i, e_j]$ of interest are either $[\varepsilon^2, e_1] = -[\varepsilon^1, e_2] = ae_0 + \varepsilon^0 = a\mathrm{i}h + (1-a^2)/2h = (1+a\mathrm{i})^2/2h$ or $[\varepsilon_1, e_1] = [\varepsilon^2, e_2] = e_0 - a\varepsilon^0 = \mathrm{i}(1+a\mathrm{i})^2/2h$. Hence, simply scaling $e_1, e_2, \varepsilon^1, \varepsilon^2$ by $(1+a\mathrm{i})$ does the job. □

From the proof we can extract the following statement.

Proposition 4.4.2. *The map* $\mathfrak{d}(a) \to sl_2(\mathbb{C})$ *defined by*

$$e_0 \mapsto \frac{\mathrm{i}-a}{2}h, e_1 \mapsto \frac{1+\mathrm{i}a}{\sqrt{2}}y, e_2 \mapsto \frac{i(1+\mathrm{i}a)}{\sqrt{2}}y$$
$$\varepsilon^0 \mapsto \frac{1+\mathrm{i}a}{2}h, \varepsilon^1 \mapsto \frac{\mathrm{i}(1+ia)}{\sqrt{2}}x, \varepsilon^2\frac{1+\mathrm{i}a}{2}x$$

is a Lie algebra isomorphism for all a. For $a=0$, it is an isometry with $1/2$ the imaginary part of the Killing form of $sl_2(\mathbb{C})$.

Interestingly, the results of [Gom00] imply that the bilinear form on $sl_2(\mathbb{C})$ induced by the isomorphism with $\mathfrak{d}(1)$ is (proportional to) the Killing form of $so(1,3)$, the Lie algebra of the Poincaré group. Therefore B_a interpolates continuously between the Killing form and the Poincaré group and the imaginary part of the Killing form of $sl_2(\mathbb{C})$: $(sl_2(\mathbb{C}), B_1) \cong (so(1,3), \lambda\kappa_{so(1,3)})$ and $(sl_2(\mathbb{C}), B_0) \cong (sl_2(\mathbb{C}), 1/2\,\mathrm{Im}\,\kappa_{sl_2(\mathbb{C})})$, where $\kappa_\mathfrak{g}$ denotes the Killing form of $\mathfrak{g}$. Also, this shows that these two quadratic Lie algebras are not isometric since their Casimir invariants are different: $c_2(sl_2(\mathbb{C}), \mathrm{Im}\,\kappa_{sl_2(\mathbb{C})}) = 0$ while $c_2(so(1,3), \lambda\kappa_{so(1,3)} = 48$.

4.4.4 Splittings associated to compact simple groups

The example of $sl_2(\mathbb{C})$ above can be generalized to complexifications of compact simple groups. The usual choice of gauge group in Chern-Simons theory is a compact simple group, the first of which is $su(2)$. However, it is well-known that on simple groups the space of invariant bilinear forms is spanned by the Killing form, and that the Killing form of a compact group is negative definite. This implies that compact simple Lie algebras can never be polarized, since a polarization of a real Lie algebra requires the invariant bilinear form to have split signature, i.e. signature $(\dim\mathfrak{g}/2, \dim\mathfrak{g}/2)$. However, one can ask the question whether one can associate a polarized Lie algebra with a compact simple Lie algebra such that certain invariants are the same (e.g. the quadratic Casimir) or at least related.

There are some standard constructions of Manin triples associated to compact simple Lie algebras with Killing form $(\mathfrak{g}, \kappa_\mathfrak{g})$. Namely, one can complexify the Lie algebra and consider the imaginary part of the complexified bilinear form (which is the Killing form of complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$). It is well known that this form has split signature, and the corresponding Manin triples have been classified by Delorme ([Del01]). However, one can easily see that these quadratic algebras admit a Manin triple of the form $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}, \mathfrak{b})$ (see e.g. [Kos97]). Since $\mathfrak{g}$ is a compact simple Lie algebra, it is unimodular, hence the quadratic Casimir and the eccentricities vanish.

One can also consider the Lie algebra $\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$ with the difference of the Killing forms. This admit Manin triples of the form $(\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_C, \mathfrak{h})$, where $g_C \subset \mathfrak{g}_C \oplus \mathfrak{g}_{\mathbb{C}}$ is the diagonal subalgebra. But since $\mathfrak{g}$ is unimodular, so is $\mathfrak{g}_{\mathbb{C}}$, and these are also regular Manin triples, so the associated invariants vanish.

Hence, none of these standard constructions yields a polarized Lie algebra which is of interest to us. An idea for future investigation is the following. Considering Chern-Simons theory with polarized target is an attempt to consider Chern-Simons theory as an AKSZ theory with polarized target. Rather than polarizing the target, we should consider the entire space of fields with its algebraic structure (a dg-Frobenius Lie algebra, as in [CM08]). One should then consider also the algebra of boundary fields, and consider polarizations therein (not just in the Lie algebra of coefficients), and their compatibility with the algebraic structure in the space of bulk fields.

Chapter 5

Split Chern-Simons Theory in the Axial Gauge

In this chapter we turn towards split Chern-Simons Theory. There are two main results. The first is that in the case where $(\mathfrak{g}, V, W)$ is a Manin triple, the weight of the theta graph on lens spaces is compatible with the results of Kuperberg-Thurston-Lescop reviewed in 2.1. Namely, the weight coincides with the Casson-Walker invariant of $\lambda_{CW}(L_{p,q}) = \frac{1}{2}s(q,p)$, plus a term which changes by $\frac{1}{12}$ if the framing of the lens space is changed by one unit.
The other main result is that if $(\mathfrak{g}, V, W)$ fails to be a Manin triple, the weights of the two possible orientations of the theta graph do not agree. This shows that the theory is anomalous. Possible reasons for this are discussed in Section 5.6.

The main tool that we use is the BV-BFV formalism developed by Cattaneo, Mnev and Reshetikhin in [CMR14; CMR17], reviewed in Section 2.2. Let us give a brief overview of the chapter. In section 5.1 we recall the definition of split Chern-Simons theory and of its BV-BFV extension. In section 5.2 we consider the perturbative quantization of split Chern-Simons theory, and explicitly give the Feynman diagrams at low orders on handlebodies. We discuss the evaluation of these diagrams in the axial gauge on the solid torus in Section 5.3), explicit computations can be found in Appendix A. In Section 5.4 we discuss how to glue lens spaces from solid tori. Explicit results for the low-order effective action on lens spaces are discussed in Section 5.5, while the longer computations are carried out in Appendix B. The implications for the weights of theta graphs are discussed in Section 5.6. In Section 6.1, we discuss an alternative procedure to evaluate the weights of Feynman diagrams, and evaluate the effective action on lens spaces from this point of view in Section 6.2. Finally, in Section 6.3 we give some remarks about how one could extend the methods of this chapter to arbitrary 3-manifolds, and comment on the various obstacles that present themselves.

5.1 Definition of Split Chern-Simons Theory

5.1.1 Split Chern-Simons Theory as an AKSZ theory

Let $(\mathfrak{g}, V, W)$ be a polarized Lie algebra, i.e. a quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ with a choice of two Lagrangian subspaces V, W such that $\mathfrak{g} = V \oplus W$. Split Chern-Simons theory is an example of an AKSZ theory (see Section 2.3). The target is the 2-shifted Hamiltonian manifold

$$\mathcal{M} = (\mathfrak{g}[1], \omega_{\mathfrak{g}}, \Theta) \tag{5.1}$$

where

$$\omega_{\mathfrak{g}}(A, B) = \langle A, B\rangle \tag{5.2}$$

is a 2-shifted symplectic form on $\mathfrak{g}[1]$, and Θ is the function

$$\Theta(a) = \frac{1}{3}\langle a, [a, a]\rangle \tag{5.3}$$

which satisfies

$$\{\Theta, \Theta\} = 0 \tag{5.4}$$

with respect to the Poisson structure induced by ω, i.e. Θ is a solution of the CME ((2.13)). Equation (5.4) follows from invariance of the inner product and the Jacobi identity. Alternatively, applying the splitting $\mathfrak{g} = V \oplus W \cong V \oplus V^*$, the Poisson bracket becomes that of the cotangent bundle of $V[1]$, i.e. the big bracket of Section 4.2.1, and equation (5.4) becomes equivalent to the system of equations (4.4).

Definition 5.1.1. BV-extended split Chern-Simons theory is the BV-BFV theory arising from the AKSZ construction with target $\mathcal{M} = (\mathfrak{g}[1], \omega_{\mathfrak{g}}, \Theta)$ for a polarized Lie algebra $\mathfrak{g}$.

Let us recall what this means explicitly. We use the superfield notation: $\mathsf{C} \in \Omega^\bullet(M, \mathfrak{g})[1]$ means that C is an inhomogeneous differential form with values in $\mathfrak{g}$. The shift [1] denotes the total degree, hence the p-form component of C has ghost number $1 - p$, in particular, the one-form component has ghost number 0, i.e. is the classical field. By $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ we denote (by abuse of notation) extensions of the bilinear form and the Lie bracket to Lie algebra-valued differential forms.

We briefly recall the output of the AKSZ construction of classical Chern-Simons theory on manifolds with boundary for an arbitrary Lie algebra $\mathfrak{g}$ from [CMR14]. The BFV space of boundary fields associated to a 2-manifold Σ is $(\mathcal{F}^\partial_\Sigma, \omega^\partial = \delta\alpha^\partial_\Sigma, Q^\partial_\Sigma)$ where

$$\begin{aligned}
\mathcal{F}^\partial_\Sigma &= \Omega^\bullet(\Sigma, \mathfrak{g})[1] \ni \mathsf{C} \\
\omega^\partial_\Sigma &= \int_\Sigma \delta\mathsf{C} \wedge \delta\mathsf{C} \\
\alpha^\partial_\Sigma &= \int_\Sigma \mathsf{C} \wedge \delta\mathsf{C} \\
Q^\partial_\Sigma &= \int_\Sigma \left\langle \mathsf{C}, \frac{\delta}{\delta\mathsf{C}} \right\rangle + \frac{1}{2}\left\langle [\mathsf{C},\mathsf{C}], \frac{\delta}{\delta\mathsf{C}} \right\rangle .
\end{aligned}$$

Here ω^∂_Σ has degree 0, and Q^∂_Σ is a Hamiltonian cohomological vector field. The BV-BFV space $\mathcal{F}_M$ associated to a 3-manifold is $(\mathcal{F}_M, \omega_M, \mathcal{S}_M, Q_M, \pi)$ where

$$\begin{aligned}
\mathcal{F}_M &= \Omega^\bullet(M, \mathfrak{g})[1] \ni \mathsf{C} \\
\omega_M &= \int_M \delta\mathsf{C} \wedge \delta\mathsf{C} \\
\mathcal{S}_M &= \int_M \frac{1}{2}\langle \mathsf{C}, \mathrm{d}\mathsf{C}\rangle + \frac{1}{6}\langle \mathsf{C}, [\mathsf{C},\mathsf{C}]\rangle \\
Q_M &= \int_M \mathrm{d}\mathsf{C} \wedge \frac{\delta}{\delta\mathsf{C}} + \frac{1}{2}\left\langle [\mathsf{C},\mathsf{C}], \frac{\delta}{\delta\mathsf{C}} \right\rangle
\end{aligned}$$

and $\pi\colon \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ is the projection to the BFV space of boundary fields $\mathcal{F}^\partial_{\partial M}$ which is given by restriction of forms. These data satisfy the BV-BFV compatibility axioms

$$\begin{aligned}
\iota_{Q_M}\omega_M &= \delta\mathcal{S}_M + \pi^*\alpha^\partial_{\partial M} \\
\delta\pi(Q_M) &= Q^\partial_{\partial M}
\end{aligned}$$

Note that so far this is just the classical AKSZ formulation of Chern-Simons theory and we have not yet made use of the splitting. This will be the next step.

5.1.2 Formulation as a BF-like theory

Let us recall from [CMR17] how to formulate split Chern-Simons theory as a BF-like theory. Consider the BV-extended Chern-Simons action functional

$$\mathcal{S}[\mathsf{C}] = \int_M \frac{1}{2}\langle \mathsf{C}, \mathrm{d}\mathsf{C}\rangle + \frac{1}{6}\langle \mathsf{C}, [\mathsf{C},\mathsf{C}]\rangle = \int_M \frac{1}{2}B_{ab}\mathsf{C}^a\mathsf{C}^b + \frac{1}{6}f_{abc}\mathsf{C}^a\mathsf{C}^b\mathsf{C}^c$$

which arises from the AKSZ construction as explained in the paragraph above. Here, in the second expression we pick a basis e_a of $\mathfrak{g}$ and define $B_{ab} = \langle e_a, e_b\rangle$, $f_{abc} = \langle e_a, [e_b, e_c]\rangle$ and expand the field as $\mathsf{C} = \mathsf{C}^a e_a$. Since $\mathfrak{g}$ admits a splitting $\mathfrak{g} = V \oplus W$ into maximal isotropic subspaces, we can choose a basis ξ_i of V and a dual basis ξ^i of W. Then the space of fields splits as $\Omega^\bullet(M,\mathfrak{g}) = \Omega^\bullet(M,V) \oplus \Omega^\bullet(M,W)$ and the superfield C splits as $\mathsf{A}+\mathsf{B} = \mathsf{A}^i\xi_i + \mathsf{B}_i\xi^j$. By $(\mu,\lambda,\varphi,\psi)$ we denote the structure maps of the polarized Lie algebra as defined in 4.3. Integrating by parts one can rewrite the action as

$$\mathcal{S}[\mathsf{A},\mathsf{B}] = \int_M \langle \mathsf{B}, \mathrm{d}\mathsf{A}\rangle + \frac{1}{6}\langle \mathsf{A},[\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\langle \mathsf{B},[\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\langle \mathsf{A},[\mathsf{B},\mathsf{B}]\rangle + \frac{1}{6}\langle \mathsf{B},[\mathsf{B},\mathsf{B}]\rangle. \tag{5.5}$$

The quadratic term is

$$\mathcal{S}_{M,0} = \int_M \mathsf{B}_i \mathrm{d}\mathsf{A}^i, \tag{5.6}$$

hence the theory is "BF-like" in the sense of [CMR17] with interaction term

$$\begin{aligned}\mathcal{V}(\mathsf{A},\mathsf{B}) &= \frac{1}{6}\langle \mathsf{A},[\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\langle \mathsf{B},[\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\langle \mathsf{A},[\mathsf{B},\mathsf{B}]\rangle + \frac{1}{6}\langle \mathsf{B},[\mathsf{B},\mathsf{B}]\rangle \\ &= \frac{1}{6}\psi_{ijk}\mathsf{A}^i\mathsf{A}^j\mathsf{A}^k + \frac{1}{2}\mu^i_{jk}\mathsf{B}_i\mathsf{A}^j\mathsf{A}^k + \frac{1}{2}\lambda^{jk}_i\mathsf{A}^i\mathsf{B}_j\mathsf{B}_k + \frac{1}{6}\varphi^{ijk}\mathsf{B}_i\mathsf{B}_j\mathsf{B}_k\end{aligned}$$

where we introduced the structure constants $\psi_{ijk}, \mu^i_{jk}, \lambda^{jk}_i, \varphi^{ijk}$ defined in (4.6). If $(\mathfrak{g}, V, W)$ form a a quasi-Manin triple (i.e. V is a subalgebra), by isotropy the interaction term simplifies to

$$\mathcal{V}(\mathsf{A},\mathsf{B}) = \frac{1}{2}\langle \mathsf{B},[\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\langle \mathsf{A},[\mathsf{B},\mathsf{B}]\rangle + \frac{1}{6}\langle \mathsf{B},[\mathsf{B},\mathsf{B}]\rangle. \tag{5.7}$$

In the special case where $(\mathfrak{g}, V, W)$ is a Manin triple (i.e. V, W are subalgebras), by isotropy we get that the interaction term simplifies to

$$\mathcal{V}(\mathsf{A},\mathsf{B}) = \frac{1}{2}\langle \mathsf{B},[\mathsf{A},\mathsf{A}]\rangle + \frac{1}{2}\langle \mathsf{A},[\mathsf{B},\mathsf{B}]\rangle \tag{5.8}$$

Remark 5.1.1. We can go even further, and assume that W has the trivial bracket. In that case, V can be an arbitrary Lie algebra $\mathfrak{h}$, since any Lie algebra admits the trivial bialgebra struture $\gamma = 0$, which leads to the Manin triple $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*, \mathfrak{h}, \mathfrak{h}^*)$ ($\mathfrak{g}$ is called T^*-extension of $\mathfrak{h}$, see e.g. [Bor97]). In that case, the interaction term is simply

$$\mathcal{V}(\mathsf{A},\mathsf{B}) = \frac{1}{2}\langle B,[A,A]\rangle \tag{5.9}$$

which is the interaction term of non-abelian BF theory, i.e. non-abelian BF theory in dimension 3 is a special case of split Chern-Simons theory.

5.2 Perturbative Quantization of split Chern-Simons Theory

In this section we consider the perturbative quantization of split Chern-Simons theory on handlebodies H_g. Since handlebodies are manifolds with boundary, we will use the perturbative quantization method of the BV-BFV formalism explained in section 2.2.

5.2.1 Polarization

In split Chern-Simons theory, the space of boundary fields splits as

$$\mathcal{F}^{\partial} = \Omega^{\bullet}(\partial M, V)[1] \oplus \Omega^{\bullet}(\partial M, W)[1]. \tag{5.10}$$

By the isotropy condition this is a splitting into Lagrangian subspaces, so we can use either of them as base or fibers of a polarization[1]. The coordinate on the base is denoted by a blackboard bold letter $\mathbb{A}$ or $\mathbb{B}$, and we speak of $\mathbb{A}$- or $\mathbb{B}$-representation respectively. This terminology comes from the p- and q-representations in Quantum Mechanics.
Recall from section 2.2.3 that the a choice of decomposition $\partial M = \partial_1 M \sqcup \partial_2 M$ defines a polarization by choosing the $\mathbb{A}$-representation on $\partial_1 M$ and the $\mathbb{B}$-representation on $\partial_2 M$. If ∂M is connected, the only choice is between the $\mathbb{A}$- or the $\mathbb{B}$-representation on ∂M. For computations we will use the $\mathbb{A}$-representation, i.e. we will split the A-field as

$$\mathsf{A} = \widehat{\mathsf{A}} + \widetilde{\mathbb{A}}$$

where $\widetilde{\mathbb{A}}$ is an extension of $\mathbb{A} = \iota^*\mathsf{A}$ to the bulk. However, as we will explain below, for split Chern-Simons theory the computations in $\mathbb{A}$- and $\mathbb{B}$-representations are analogous.

5.2.2 Residual fields

After choosing the polarization and a decomposition $\mathcal{F}_M \cong \mathcal{B}_M^{\mathcal{P}} \times \mathcal{Y}$, we must choose a decomposition $\mathcal{Y} \cong \mathcal{V}_M \times \mathcal{Y}'$ such that there exists a Lagrangian $\mathcal{L} \subset \mathcal{Y}'$ on which the action $\mathcal{S}_M$ has a unique non-degenerate critical point. For the case of abelian BF theory this has been carried out in [CMR17], see the brief review in section 2.2.3. This is the construction that we will also use in split Chern-Simons theory, as we consider it as a perturbation of abelian BF theory.

Residual fields on Handlebodies

We briefly recall the definition of a handlebody.

Definition 5.2.1. i) A *handlebody* H is a 3-dimensional manifold with boundary such that there is a proper[2] embedding of a disjoint union of disks $f\colon \bigsqcup_i D_i \hookrightarrow H$ such that $H \backslash \mathrm{im}\, f \cong B^3$, where B^3 denotes a 3-ball.

[1] If one thinks of Chern-Simons theory as an AKSZ theory as explained above, this amounts to lifting a target polarization instead of using a source polarization.

[2] Here by proper we mean that $f(\bigsqcup_i \partial D_i) \subset \partial H$.

ii) If H is a handlebody, then an embedding f of a disjoint union of disks is called a *system of disks* for H. It is minimal of the number of disks is minimal among all system of disks.

iii) The genus of a handlebody is the number of disks in a minimal system of disks.

It is a standard result (see e.g. [Joh]) that there is a unique (up to diffeomorphism) handlebody of genus g, obtained from the 3-ball by attaching g handles, for all $g \geq 0$, and that its boundary is a closed surface of genus g. The handlebody of genus 0 is a 3-ball.

Since the boundary of a handlebody is connected, we can define either $\partial H_g = \partial_1 H_g$ or $\partial H_g = \partial_2 H_g$, i.e. choose either $\mathbb{A}$ or $\mathbb{B}$ representation on the boundary. We will discuss only the first case, the other one works analogously. In that case we have

$$\mathcal{V}_{H_g} = (H^\bullet(H_g, \partial H_g) \oplus H^\bullet(H_g))[1] \ni (\mathsf{a}, \mathsf{b}).$$

In the case $g = 0$, we have $H^0(H_0) \cong H^3(H_0, \partial H_0)$, while other cohomology groups vanish. Let v be a normalised volume form on B^3, then we can define a basis for residual fields by $\chi_0 = v, \chi^0 = 1$. The residual fields a, b are then defined by

$$\begin{aligned} \mathsf{a} &= z^0 v \\ \mathsf{b} &= z_0^+ \end{aligned} \tag{5.11}$$

where the coordinate z^0, (resp. z_0^+,) is a linear V (resp. W) valued functions on $H^\bullet(M, \partial M)$ (resp. $H^\bullet(M)$).

Now let us consider the case $g > 0$. We have that $H^0(H_g) \cong H^3(H_g, \partial H_g) \cong \mathbb{R}$ and $H^1(H_g) \cong H^2(H_g, \partial H_g) \cong \mathbb{R}^g$. A useful choice of basis is the following. Let $D_1, \ldots, D_g$ be a (minimal) system of disks for H_g. Let $\mu_1, \ldots, \mu_2$ be the dual basis of $H^2(H_g, \partial H_g)$ whose existence is asserted by the universal coefficient theorem (i.e. we have $\int_{D_i} \mu_j = \delta_{ij}$). Choose Poincaré-Lefschetz dual 1-forms $dt_1, \ldots, dt_g$ to $\mu_1, \ldots, \mu_g$ (i.e. $\int_{H_g} \mu_i dt_j = \delta_{ij}$). We can then pick as a volume form $v = \sum_{i=1}^g \mu_i dt_i$. Its total volume is g. We can hence define the basis of residual fields as, for $i = 1, \ldots, g$

$$\begin{aligned} \chi_0 &= v \qquad \chi^0 = \frac{1}{g} \\ \chi_i &= \mu_i \qquad \chi^i = dt_i \end{aligned} \tag{5.12}$$

We then have

$$\mathsf{a} = z^0 v + z^i \mu_i \tag{5.13}$$

$$\mathsf{b} = z_0^+ \frac{1}{g} + z_i^+ dt_i \tag{5.14}$$

where the coordinates z^0, z^i (resp. z_0^+, z_i^+) are again linear V (resp. W) valued functions on $H^\bullet(M, \partial M)$ (resp. $H^\bullet(M)$).

Remark 5.2.1 (Genus 1 case)*.* The notation is suggestive for the genus 1 case $H_1 \cong S^1 \times D$, in that we can choose $dt_1 = dt$, the differential of the longitudinal coordinate. Dual to it we have a volume form $\mu \in H^2(D, \partial D)$ with total volume 1. These are the ultra-harmonic forms with respect to the product metric on the solid torus (where S^1, D carry the euclidean metric).

Remark 5.2.2 (Cohomology of ∂H_g)*.* Let $g \neq 0$. Restricting the representatives of the absolute cohomology to the the boundary of the handlebody $\partial H_g \cong \Sigma_g$ we obtain representatives for cohomology classes in Σ_g. We can choose 1-forms $d\theta_1, \ldots, d\theta_g$ such that $\int_{\Sigma_g} dt_i d\theta_j = \delta_{ij}$. The $d\theta_i$'s are Poincaré dual of the g 1-cycles in Σ_g that generate the first homology group of the handlebody after composing with the inclusion. There is a volume form v_∂ on Σ_g given by $v_\partial = \sum_{i=1}^g dt_i d\theta_i$ of volume g. In that way a choice of generators of $H^\bullet(H_g, \partial H_g) \oplus H^\bullet(H_g)$ yields a basis of $H^1(\Sigma_g)$ which is a basis in which the Poincaré pairing is the standard symplectic form.

Remark 5.2.3*.* The residual fields on handlebodies in the $\mathbb{A}$-representation have the particular property that $\mathsf{a} \wedge \mathsf{a} = 0$, since a only has 2- and 3-form components. Similarly, any power of b fields only has 0- and 1-form components, since b has only components of form degree 0 or 1. This will rule out some the Feynman diagrams that could in principle appear.

5.2.3 Gauge fixing

After choosing the polarization and the space of residual fields, one also has to choose a gauge-fixing Lagrangian $\mathcal{L} \subset \mathcal{Y}'$. In the case of abelian BF theory (which we use to gauge fix here) such a Lagrangian can be obtained from a contracting triple (ι, p, K) for the complex $\Omega^\bullet_{\hat{D}}(M)$, as discussed in chapter 3. Here $\Omega_{\hat{D}}(M)$ are forms which satisfy an "ultrified" set of boundary condition (see [CMR17]), which is quasi-isomorphic to $\Omega^\bullet(M, \partial M)$. The Lagrangian is then $\mathcal{L} = \operatorname{im} K' \oplus \operatorname{im} K$ (see section 3.2). The propagator η is the integral kernel of K, and can be used to extend the chain contraction K to the complex $\Omega(M, \partial M)$ by the formula

$$K[\alpha] = \pi_{1,*}(\eta_{12}\pi_2^*\alpha). \tag{5.15}$$

If (ι, p, K) is a normal contracting triple, denote by χ_i the image of a basis of $H^\bullet(M, \partial M)$ under ι, and by χ^i the image of the dual basis of $H^\bullet(M)$ under p^*. The properties $K \circ \iota = 0, K^2 = 0$

of the normal contracting triple translate into the following properties of η (cf. [CM08]):

$$\int_2 \eta_{12}\chi_i = \int_1 \eta_{12}\chi^1 = 0, \tag{5.16}$$

$$\int_2 \eta_{12}\eta_{23} = 0 \tag{5.17}$$

$$\tag{5.18}$$

We then define the state $\psi \in \mathcal{H}_M^{\mathcal{P}}$ by

$$\psi(\mathbb{A}, \mathsf{a}, \mathsf{b}) = \int_{\mathcal{L}} e^{\frac{i}{\hbar}\mathcal{S}_M} \tag{5.19}$$

where the integral on the right hand side is understood as the power series expansion of the formal $\hbar \to 0$ limit of the integral, as explained in Section 2.1.

5.2.4 Feynman graphs and rules

After integration over $\mathcal{L}$, we can label the terms in the perturbative expansion by graphs as follows. Fix $k, l, m \in \mathbb{N}_0$. We consider graphs Γ with three types of vertices (see also [CMW17]:

- *Boundary background vertices:* There are k of these distributed on ∂M. They are labeled by $\mathbb{B}\mathsf{a}$ if they lie on $\partial_2 M$ and $\mathsf{b}\mathbb{A}$ if they lie on $\partial_1 M$.
- *Boundary source vertices:* There are m boundary source vertices distributed on $\partial M = \partial_1 M$ labeled by $\mathbb{A}\beta$ with an arrowhead pointing towards them.
- *Internal interaction vertices:* There are l internal vertices. They come with three half-edges which are labeled by γ_i's in $\{\mathsf{a}, \alpha, \mathsf{b}, \beta\}$. These half-edges are either marked as leaves if they are labeled by a background, as an arrow tail if they are labeled by α, or an arrowhead if they are labeled by β.
- Divide by the number of automorphisms $|\mathrm{Aut}(\Gamma)|$ of Γ.

If it is possible to connect every arrow tail α to an arrowhead β (possibly at the same vertex), then the graph resulting from this procedure is called an *admissible graph.* To such a graph we can associate a functional on the space of boundary fields as follows (we set $\varepsilon = \frac{\hbar}{\mathrm{i}} = -\mathrm{i}\hbar$) to simplify the power counting):

- For every background boundary vertex, multiply by $\frac{1}{\varepsilon} = (i/\hbar)$ times the label and integrate over the corresponding boundary point.

- For every internal vertex multiply by $\frac{1}{\varepsilon} = (i/\hbar)$ times the correct structure constants (specified by the half-edge labels) and integrate over M. See figure 5.1.

- For every leaf, multiply by the corresponding background field evaluated at the point.

- For every arrow between vertices in different positions $i \neq j$, with tail labeled by α^k and head β_l, multiply by a propagator $\varepsilon\delta_l^k\eta(x_i, y_j)$.

- For every short loop (also called tadpole), i.e. an arrow starting and ending at the same vertex i ,with tail labeled by α^k and head β_l, multiply by $\varepsilon\delta_l^k\alpha(x_i)$, where $\alpha \in \Omega^2(M)$ is a so-called "tadpole form".[3]

- For every source boundary vertex, we multiply by $\frac{1}{\varepsilon}$ times the corresponding boundary field and integrate over the corresponding boundary point.

We denote the result by $\widehat{\psi}_\Gamma$. Denoting the set of all admissible graphs for k, l, m by $\Lambda_{k,l,m}$, we get

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = T_M \sum_{k,l,m} \sum_{\Gamma \in \Lambda_{k,l,m}} \widehat{\psi}_\Gamma.$$

Remark 5.2.4. We can factor out the non-interacting diagram parts (background boundary vertices and source boundary vertices connecting to other source boundary vertices). This will yield a prefactor of $e^{\frac{i}{\hbar}\mathcal{S}_0^{\text{eff}}}$ where $\mathcal{S}_0^{\text{eff}}$ is the free effective action

$$\mathcal{S}_0^{\text{eff}} = -\left(\int_{\partial_1 M} \langle \mathsf{b}, \mathbb{A} \rangle\right) \tag{5.20}$$

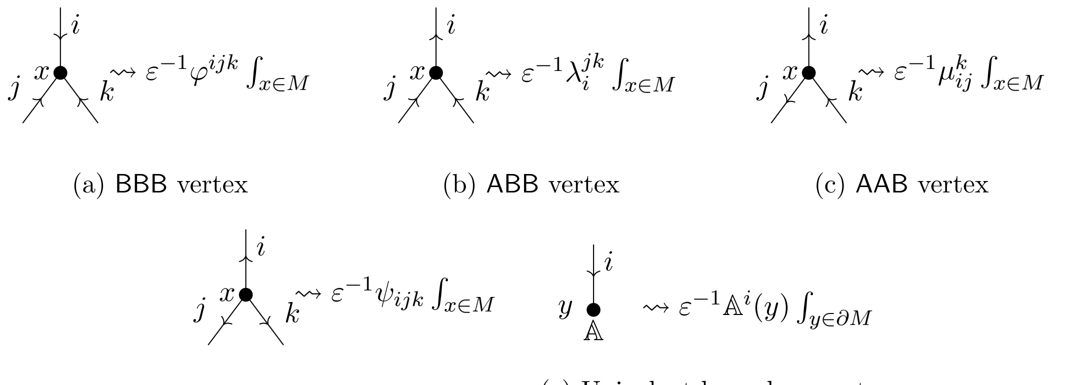
i.e. the effective action of the unperturbed theory.

The remaining interaction diagrams have $l \geq 1$ internal vertices and $m \leq 3l$ boundary vertices. Denoting the set of admissible interaction diagrams by $\Lambda_{l,m}^{int}$, the above expression becomes

$$\widehat{\psi}_M(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = T_M e^{\frac{i}{\hbar}\mathcal{S}_0^{\text{eff}}} \left(1 + \sum_{l=1}^{\infty} \sum_{m=0}^{3l} \sum_{\Gamma \in \Lambda_{l,m}^{int}} \widehat{\psi}_\Gamma\right).$$

Definition 5.2.2. We say that a graph is of n-point order if there are exactly n interaction vertices.

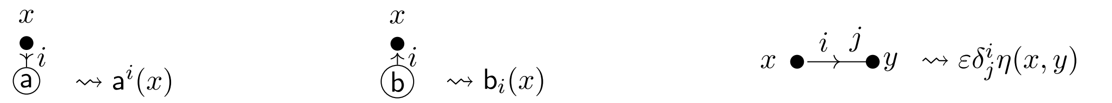
[3]These contributions can be ignored if the vertex coefficients are unimodular or the Euler characteristic of M is 0, since in this case the tadpole form can be chosen to vanish.



(a) BBB vertex (b) ABB vertex (c) AAB vertex

(d) AAA vertex (e) Univalent boundary vertex

Figure 5.1: Feynman graphs and rules for split Chern-Simons theory: Vertices



(a) A half-edge ending in an a residual field. (b) A half-edge coming from a b residual field. (c) An edge between vertices x and y

Figure 5.2: Feynman graphs and rules for split Chern-Simons theory: Decorations and edges

5.2.5 Graphs on handlebodies up to 2-point order

In this subsection we explicitly give the connected Feynman graphs appearing in split Chern-Simons order appearing on handlebodies, up to 2-point order (in the $\mathbb{A}$ polarization). The single boundary of the handlebody is depicted as a circle, and the interaction vertices are drawn inside the circle.

0-point order

At this order there is just a single graph depicted in corresponding to a single residual field at the boundary. Since we consider only one boundary component, this coincides with the free effective action.

1-point order

At the next order, we will see graphs with one interaction vertex. In principle, it could be any of the four possible vertices, i.e. either $\mathsf{A}^3, \mathsf{A}^2\mathsf{B}, \mathsf{A}\mathsf{B}^2$ or B^3. However, the last vertex is ruled out since it has only arriving arrows, but there are no source terms, so there can be no arrows arriving there. Hence the only possibility is to attach three b residual fields at this vertex.

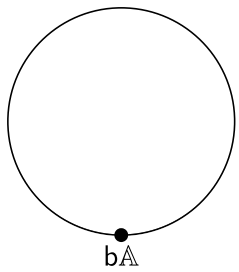
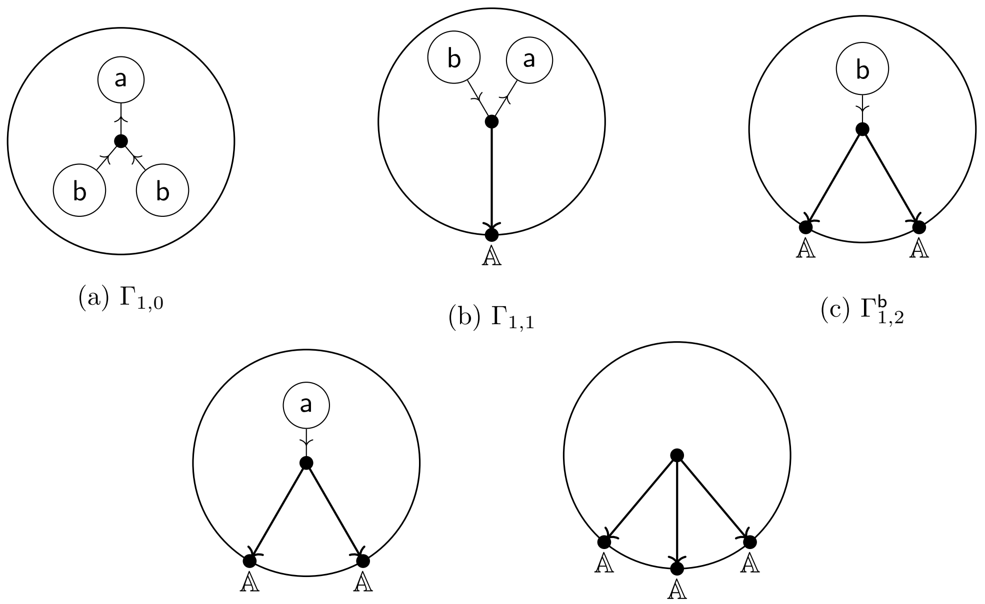


Figure 5.3: Single diagram Γ_0 contributing to zero-point effective action

As explained in Remark 5.2.3, the resulting form has form degree one, so integration over the 3-dimensional manifold vanishes. Also, again invoking Remark 5.2.3, we can place at most one a residual field at this vertex. The resulting tree graphs are depicted in 5.4. In principle,

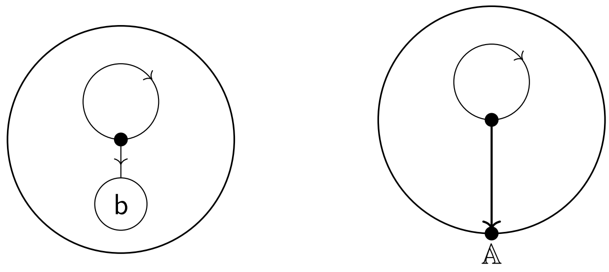


(a) $\Gamma_{1,0}$ (b) $\Gamma_{1,1}$ (c) $\Gamma^{\mathsf{b}}_{1,2}$ (d) $\Gamma^{\mathsf{a}}_{1,2}$ (e) $\Gamma_{1,3}$

Figure 5.4: Trees graphs in the handlebody H_g with one interaction vertex. A bullet denotes a point we integrate over, a long arrow denotes a propagator.

there can be a short loop starting and ending at the interaction vertex. This results in the graphs shown in Figure 5.5. The usual way to exclude tadpoles is to require that the structure constants at each vertex satisfy a unimodularity condition. However, as we saw in the discussion in the last chapter, requiring unimodularity of the maps μ and λ requires the eccentricity of the polarized Lie algebra to vanish (Proposition 4.3.2). So, we will formally consider tadpoles, and talk about other ways of regularizing them later. There are two graphs with a tadpole at this order, depicted in Figure 5.5. Notice that for degree reasons we cannot put a tadpole and an a

residual field at the same vertex (both have form degree ≥ 2).



(a) $\Gamma^{\circ}_{1,0}$ (b) $\Gamma^{\circ}_{1,1}$

Figure 5.5: Graphs with one interaction vertex and a tadpole

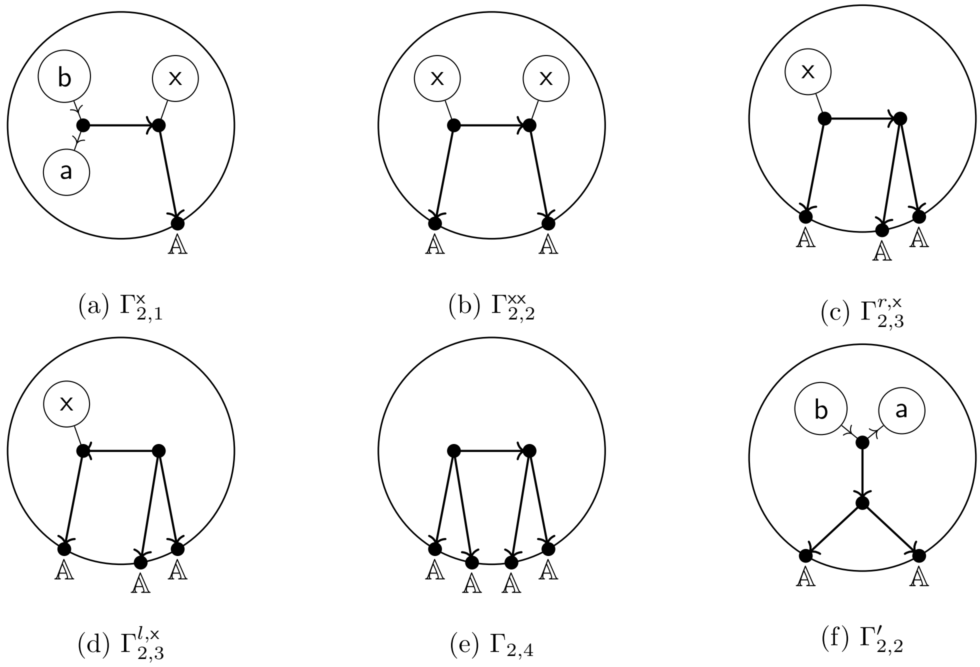
2-point order

At this order there is already a considerable number of Feynman diagrams. Let us first consider tree diagrams only. We have the following Lemma.

Lemma 5.2.5. *With the choice of residual fields as in 5.12 and a propagator satisfying condition* (5.16)*,the contribution of a 2-point tree diagrams with a* B^3 *vanishes after integration.*

Proof. In a 2-pt tree diagram, there can be at most one edge between the two interaction vertices. It has to end at the B^3 vertex. The remaining two half-edges have to be connected to b residual fields. But $\mathsf{b} \wedge \mathsf{b}$ is again a linear combination of the χ^i. Hence, after integrating over that vertex, the contribution vanishes by assumption 5.16 on the propagator. □

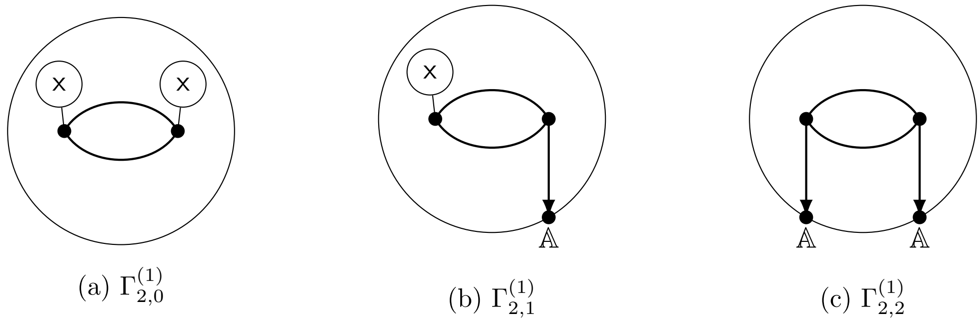
The remaining diagrams are listed in Figure 5.6 below. Here the label x can mean one of a, b. At this order one also sees loop graphs appearing. There can be both one- and two loop graphs. One-loop graphs are listed in figure 5.7. Notice that here we can have any orientation in the arrows between the two vertices, so they are drawn unoriented. There are two possible two-loop graphs (counting orientation), depicted in figure 5.8. These correspond to the theta invariant of the handlebody itself. Also, there can be graphs with a tadpole, they are listed in figure 5.9.



(a) $\Gamma^{\mathsf{x}}_{2,1}$ (b) $\Gamma^{\mathsf{xx}}_{2,2}$ (c) $\Gamma^{r,\mathsf{x}}_{2,3}$

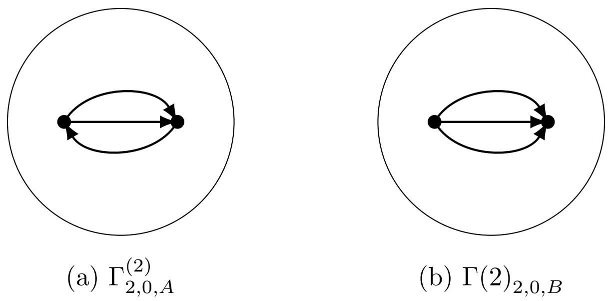
(d) $\Gamma^{l,\mathsf{x}}_{2,3}$ (e) $\Gamma_{2,4}$ (f) $\Gamma'_{2,2}$

Figure 5.6: Tree Graphs with 2 interaction vertices. A bullet denotes a point we integrate over, long arrow denotes a propagator. $\mathsf{x} \in \{\mathsf{a}, \mathsf{b}\}$.



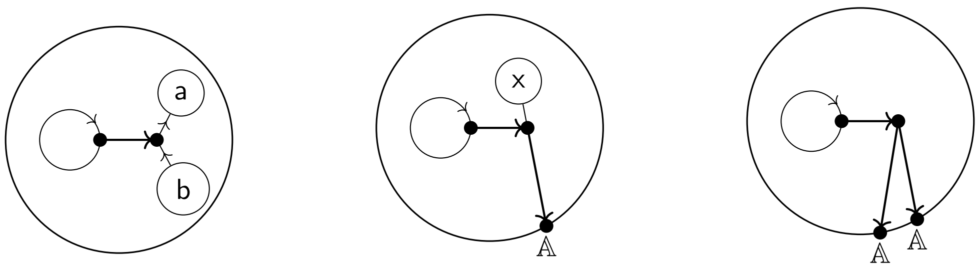
(a) $\Gamma^{(1)}_{2,0}$ (b) $\Gamma^{(1)}_{2,1}$ (c) $\Gamma^{(1)}_{2,2}$

Figure 5.7: One-Loop diagrams in the 2-point effective action



(a) $\Gamma^{(2)}_{2,0,A}$ (b) $\Gamma(2)_{2,0,B}$

Figure 5.8: 2-Loop diagrams in the 2-point effective action.



(a) $\Gamma^{\circ}_{2,0}$ (b) $\Gamma^{\circ}_{2,1}$ (c) $\Gamma^{\circ}_{2,2}$

Figure 5.9: Tadpoles diagrams at 2-point order

5.3 Evaluation of Feynman diagrams on the solid torus in the axial gauge

In this section we explain how use the axial gauge introduced in Chapter 3 to explicitly compute the weights of the Feynman graphs collected in the last section on the solid torus, i.e. the handlebody of genus 1. Since the computations (and the results) are still quite lengthy, the bulk of them is moved to Appendix A.

5.3.1 Axial gauge propagator

On the solid torus $M = S^1 \times D$, which is a product manifold, we have an axial gauge propagator as explained in Chapter 3.3. The identity these distributional propagators satisfy is

$$\mathrm{d}\eta_M = \delta^{(d)}_M(x_1, x_2) + (-1)^{d-1} \sum_i (-1)^{d \cdot \deg \chi_i} \pi_1^* \chi_i \pi_2^* \chi^i \tag{5.21}$$

On the solid torus, we have two choices for a distributional propagator. The horizontal propagator is

$$\eta^{hor}((z_1, t_1), (z_2, t_2)) = \eta_D(z_1, z_2) \delta^{(1)}_{S^1}(t_1, t_2) + \mu_1 \eta_{S^1}(t_1, t_2) \tag{5.22}$$

while the axial propagator is

$$\eta^{ax}((z_1, t_1), (z_2, t_2)) = \delta^{(2)}_D(z_1, z_2) \eta_{S^1}(t_1, t_2) + \eta_D(z_1, z_2)(dt_1 - dt_2) \tag{5.23}$$

where η_D and η_{S^1} are propagators on the the disk and the circle, respectively.

Lemma 5.3.1. *These propagators satisfy*

$$\mathrm{d}\eta = \delta^{(3)}_M(x_1, x_2) + \sum_i (-1)^{\deg \chi_i} \pi_1^* \chi_i \pi_2^* \chi^i = \delta^{(3)}_M(x_1, x_2) - \mu_1 dt_1 + \mu_1 dt_2. \tag{5.24}$$

Proof. First note that in the distributional sense we have

$$\begin{aligned} \mathrm{d}_D \eta_D &= \delta^{(2)}_D(z_1, z_2) - \mu_1 \\ \mathrm{d}_{S^1} \eta_{S^1} &= \delta^{(1)}_{S^1}(t_1, t_2) - (dt_1 - dt_2) \\ \mathrm{d}_{S^1} \delta^{(1)}_{S^1} &= 0 \end{aligned}$$

where the first two identities are (5.21) for D and S^1 respectively, and the third is for dimensional reasons. Using this, we evaluate (omitting the arguments)

$$\begin{aligned}
\mathrm{d}\eta^{hor} &= \mathrm{d}\eta_D \delta_{S^1} + \mu_1 \mathrm{d}\eta_{S^1} \\
&= (\delta_D - \mu_1)\delta_{S^1} + \mu_1(\delta_{S^1} - (dt_1 - dt_2)) \\
&= \delta_D \delta_{S^1} - \mu_1(dt_1 - dt_2) \\
&= \delta_M - \mu_1 dt_1 + \mu_1 dt_2
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}\eta^{ax} &= \delta_D(\delta_{S^1} - dt_1 + dt_2) + (\delta_D - \mu_1)(dt_1 - dt_2) \\
&= \delta_M - \mu_1 dt_1 + \mu_1 dt_2.
\end{aligned}$$

□

We will now state some other desirable properties the propagators have which allow us to simplify the computations considerably.

Proposition 5.3.2. *Suppose the disk propagator η_D satisfies*

1. $\int_{D_2} \eta_{D,12}\mu_2 = 0$,
2. $\int_{D_2} \eta_{D,12}\eta_{D,23} = 0$ *and*
3. $\int_{\partial D_2} \eta_{D,12} = 1$.

Suppose the circle propagator satisfies $\int_{S^1_2} \eta_{S^1,12} dt_2 = 0$ and $\eta_{S^1}(t_1, t_2) = -\eta_{S^1}(t_2, t_1)$. Then the following identities hold for $\eta \in \{\eta^{ax}, \eta^{hor}\}$:

i) $\int_1 \eta_{12} = 0$,

ii) $\int_1 dt_1 \eta_{12} = 0$,

iii) $\int_2 \eta_{12} dt_2 = 0$,

iv) $\int_2 \eta_{12}\mu_2 = 0$,

v) $\int_2 \eta_{12}\mu_2 dt_2 = 0$,

vi) $\int_2 \eta_{12}\eta_{23} = 0$,

vii) $\int_{2,\partial} \eta_{12} = 1$.

Proof. Notice that a form needs to have degree 2 on the disk and degree 1 on the circle to contribute to the integral. After expanding the axial and horizontal propagator, the assertions become straightforward verifications by degree counting and using the assumptions. As an example, let us prove (vi) in detail. For the horizontal propagator, we get

$$
\begin{aligned}
&(\eta_{D,12}\delta^{(1)}_{S^1,12} + \mu_1\eta_{S^1,12})(\eta_{D,23}\delta^{(1)}_{S^1,23} + \mu_2\eta_{S^1,23}) \\
&= -\eta_{D,12}\eta_{D,23}\delta^{(1)}_{S^1,12}\delta^{(1)}_{S^1,23} + \eta_{D,12}\delta^{(1)}_{S^1,12}\mu_2\eta_{S^1,23} + \mu_1\eta_{S^1,12}\eta_{D,23}\delta^{(1)}_{S^1,23} + \mu_1\eta_{S^1,12}\mu_2\eta_{S^1,23}.
\end{aligned}
$$

Now integrate over point 2. The first term vanishes by assumption 2 on the disk propagator. The second term vanishes by assumption 1 on the disk propagator. The third term vanishes for degree reasons, since it only has form degree 1 at point 2 in the disk. The last term vanishes also for degree reasons, since it has form degree 0 in the circle. For the axial propagator, we get

$$
\begin{aligned}
&(\eta_{S^1,12}\delta^{(2)}_{D,12} + (dt_1 - dt_2)\eta_{D,12})(\eta_{S^1,23}\delta^{(2)}_{D,23} + (dt_2 - dt_3)\eta_{D,23}) \\
&= \eta_{S^1,12}\delta^{(2)}_{D,12}(dt_2 - dt_3)\eta_{D,23} + \eta_{S^1,12}\delta^{(2)}_{D,12}\eta_{S^1,23}\delta^{(2)}_{D,23} \\
&+ (dt_1 - dt_2)\eta_{D,12}\eta_{S^1,23}\delta^{(2)}_{D,23} + (dt_1 - dt_2)\eta_{D,12}(dt_2 - dt_3)\eta_{D,23}.
\end{aligned}
$$

The first term vanishes by the assumption on the circle propagator, the second term vanishes since it has form degree 0 in the circle, the third term also vanishes by assumption on the circle propagator, and the last by the second assumption on the disk propagator. □

5.3.2 Gauge fixing on the disk and on the circle

In this section we fix the residual gauge freedom on the circle and the disk. On the circle there is a unique[4] propagator satisfying the assumptions of Proposition 5.3.2, namely

$$
\eta_{S^1}(t_1, t_2) = \frac{1}{2}\operatorname{sgn}(t_1 - t_2) - t_1 + t_2. \tag{5.25}
$$

This propagator can be seen as the periodic extension of the first Bernoulli polynomial $B_1(x) = 1/2 - x$ defined on the open interval $(0, 1)$ to the real line, which coincides with minus the sawtooth function

$$
((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & x \in \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} \tag{5.26}
$$

at non-integer points.

On the disk, we will use the propagator given in Example in Section 3.2.4 - the Riemann-Hodge

[4] See e.g. the discussions in [BCM12; CMR17; AM11].

propagator for the Euclidean metric, given by the formula

$$\eta_D^{std}(z,w) = \eta(z,w) = \frac{1}{2\pi}\left(\mathrm{d}\arg(z-w) + \mathrm{d}\arg(1-z\bar{w})\right) - \frac{\bar{z}dz - zd\bar{z}}{4\pi \mathrm{i}}. \tag{5.27}$$

As a metric propagator, it satisfies the assumptions of Proposition 5.3.2, and hence the axial gauge propagator has all the properties listed there.

5.3.3 Computation of the effective action in $\mathbb{A}$-representation on solid torus

In this thesis we used a particular method to compute the weights of Feynman graphs on the solid torus. The weights of the Feynman graphs are functionals on residual and boundary fields[5] given, in the case of the axial gauge, by a pushforward of distributions. Let Γ be a graph with k bulk vertices and l boundary vertices, then

$$\psi_\Gamma = \pi_*\omega_\Gamma = \int_{M^k\times \partial M^l} \omega_\Gamma \tag{5.28}$$

where the pushforward along the canonical map

$$\pi\colon M^k \times \partial M^l \to \partial M^l \tag{5.29}$$

is often denoted by an integral as in (5.28). For tree diagrams, the computation of the pushforward is carried out along the following four steps.

1) Use the decomposition

$$\eta = \eta_{D,12}\delta_{S^1,12} + \mu_1\eta_{S^1,12} =: \eta^I_{12} + \eta^{II}_{12} \tag{5.30}$$

and the splitting of the residual fields to decompose ω_Γ into disk and circle contributions. This is carried out in Appendix A.3.

2) Compute the pushforwards of the currents on the circle. This is done in Appendix A.8.

3) Decompose the disk propagator $\eta_{D,12}$ into "elementary 1 forms" defined in (A.61) as $\eta_{D,12} = \phi_{12} + \tau_{12} - \psi_1$. This is done in Appendix A.4.

4) Use complex analysis (residue calculus) to compute integrals of products of elementary 1-forms. This is the content of Appendix A.5.

[5]To be more precise, formal $\frac{1}{2}$-densities on residual fields with values in a certain space of functionals on boundary fields, see [CMR17].

Loop diagrams have to be regularized using the definition made in section 3.4. In the next subsection we explain in detail how this is done. As a result we obtain explicitly the functional ψ_Γ. The computation is carried out for all 1-point graphs. For 2-point graphs we only perform the computation for graphs that come from the Manin triple condition, i.e. graphs with only $\mathsf{A}^2 B$ and AB^2 vertices. For the explicit formulae we refer to the appendix. Here we only give some remarks about the structure.

5.3.4 Regularization of loop diagrams

We want to show that the 2-point loop graphs, with S the subset of bulk vertices, are horizontal gauge regularizable, and compute their regularized weights. First, we have the following Lemma:

Lemma 5.3.3. *The regularized weight of both orientations of the two-loop graph (figure 5.8a and figure 5.8b) vanishes.*

Proof. This is an easy consequence of degree counting and hence Proposition 3.4.3. Namely, notice that η^I has bidegree $(1,1)$ and η^{II} has bidegree $(2,0)$. The only possibility to get a form of bidegree $(4,2)$ is $\eta^I\eta^I\eta^{II}$, where orientations are arbitrary. However, $\eta^I\eta^I$ regardless of orientation of the two factors contains the square of $\delta^{(1)}_{S^1}$, whose regularization is 0 by Example 3.4.1. □

Next, let us turn to the 1-loop graphs shown in figure 5.7.

Lemma 5.3.4. *The regularized weight of the 1-loop graph with two residual fields (figure 5.7a) is nonzero if and only if the propagators are oriented in the opposite way and the residual field is* b*. In that case, its regularized weight is*

$$\psi_{\Gamma^{(1)}_{2,0}} = \frac{-1}{12}\gamma^{ij}_k\gamma^{kl}_j z^+_{2,i}z^+_{2,l}. \tag{5.31}$$

Proof. Again, the proof goes by degree counting. Since the propagators give a total degree of 4, the two residual fields must give a total degree of 2. This is possible if either both are b fields and contribute a dt or one of them is an a field and contributes μ. However, that case is ruled out by looking at the possible bidegrees (the only possibility is to combine it with two η^I, but this vanishes as above). If both residual fields give a dt, then again bidegree counting shows that the only term that survives is $dt_1\eta^{II}_{12}\eta^{II}_{21}dt_2$. This gives the contribution

$$\begin{aligned}\int_{S^1\times D\times S^1\times D} dt_1\mu_1\eta_{S^1}(t_1,t_2)\eta_{S^1}(t_2,t_1)\mu_2 dt_2 &= \int_{S^1\times S^1}\eta_{S^1}(t_1,t_2)\eta_{S^1}(t_2,t_1)dt_1dt_2\\ &= \frac{1}{2}\int_{S^1}B_2(0)dt = -\frac{1}{12}.\end{aligned}$$

Here $B_2(t)$ is the second Bernoulli polynomial, defined by equations (5.39),(5.40),(5.41), and where the equality between the first and the second line follows from Lemma 5.3.9 below. □

Now let us turn to the next one-loop diagrams, $\Gamma^{(2)}_{2,1}$ given in figure 5.7b respectively. We claim that there similarly to the graph $\Gamma^{(2)}_{2,0}$ above, the only term that survives in the regularization is the one with two η_{II} oriented in the opposite way.

Lemma 5.3.5. *The diagram $\Gamma^{(2)}_{2,1}$ given in figures 5.7b is horizontal gauge regularizable. The regularized weight of $\Gamma^{(2)}_{2,1}$ is nonzero only if the corresponding residual field is* b *and given by*

$$\psi_{\Gamma^{(1)}_{2,1}} = \frac{-1}{12}\gamma^{ij}_k \mu^{jl}_j z^+_{2,i} \int_{\partial M} d\theta \mathbb{A}^l. \tag{5.32}$$

Proof. Denote the bulk points by x_1 and x_2 and the boundary points by x_3 and x_4. We have already established that the regularized weight is zero if there two η^I's between the same points. Also, $\eta^{II}_{12}\eta^{II}_{12} = 0$ for degree reasons and $\eta^{II}_{12}\eta^{II}_{21}$ after regularization gives $-\mu_1\mu_2\eta_{S^1}(t_1,t_2)^2$. It remains to look at terms of the form $\eta^I_{12}\eta^{II}_{12}$. First let us look at $\Gamma^{(1)}_{2,1}$. If the residual field in the graph is a, then the contribution vanishes for degree reasons: The total degree in the disk between the two bulk points is 5 (recall that η^I has bidegree $(1,1)$ and η^{II} has bidegree $(2,0)$. If the residual field is b, this does not add any degree to the disk. Now suppose we have an η^{II}_{23} between the bulk and the boundary point. In the $\lambda \to 0$ limit this corresponds to placing μ at point 2, which gives the situation of the Lemma above, therefore this contribution vanishes. Consider therefore the product $\eta^I_{12}\eta^{II}_{12}\eta^I_{13}$ (if η_{II} is oriented in the other way then the disk degree at 1 is 1, hence integration over 1 gives 0). Expanding the corresponding terms of η^λ in terms of eigenfunctions with Dirichlet boundary conditions $(\phi^D_i)^\infty_{i=0}$ with eigenvalues λ_i on the disk, where ϕ_0 is the harmonic form μ, we get

$$\sum_{i,j,k=0}^{\infty} \sum_{l_1,l_2,l_3\in Z} \frac{a_{l_2}\lambda}{(\lambda_i+\lambda a^2_{l_1})(\lambda_j+\lambda a^2_{l_2})(\lambda_k+\lambda a^2_{l_3})}$$
$$\mathrm{d}^*\phi^D_i(z_1)(*\phi^D_i(z_2))\phi^D_j(z_1)(*\phi^D_j(z_2))\mathrm{d}^*\phi^D_k(z_2)(*\phi^D_k(z_3))\alpha_{S^1}.$$

Here $a_l = 2\pi i l$, and α is a product of eigenfunctions on the circle, and z_i are the disk coordinates of x_i The only terms that possibly survive[6] in the $\lambda \to 0$ limit are the ones where one of $\lambda_i, \lambda_j, \lambda_k = 0$ and we can cancel the λ in the numerator. However, λ_i and λ_k cannot be 0 since these eigenforms are killed by d^*. Therefore, we need to look at the terms where $j = 0$. In that case, $*\phi^D_j = 1$. In z_2, we then have the terms $*\phi^D_i(z_2)\mathrm{d}^*\phi^D_k(z_2)$. By Hodge decomposition, integration over z_2 can only be nonzero if ϕ^D_i is d^*-exact: But in that case, the form d^ϕ_i which

[6]This is not completely trivial - one has to take into account integration over both 1 and 2 to prove it.

is appears at z_1 in the product above is zero.

□

Remark 5.3.6. This proof relies on the fact that, disregarding the divergent factors on the circle, the product of the disk contributions in the axial gauge vanishes after integration, because it contains the integral $\int_D \eta_{D,12}\eta_{D,23}$. For the last diagram in question, $\Gamma^{(1)}_{2,2}$ given in figure 5.7c, this is in fact not true. At the moment, it is unknown whether this diagram is horizontal gauge regularizable or not according to Definition 3.4.1, or whether one can extract a term that is divergent as $\lambda \to 0$ as suggested in Remark 3.4.1. This does not affect our computation of the 2-loop function on lens spaces since this diagram does not survive after reducing the residual fields if one glues against 0-point fields. However, at the next loop order there is a contribution where one glues this diagram to its counterpart in the $\mathbb{B}$-representation. These issues need to be investigated with more care in the future.

5.3.5 Some remarks on the 2-point effective action

The evaluation of the weights of tree diagrams is quite lengthy, and deferred to Appendix A. However, there are already some interesting phenomena that one can observe, on which we briefly comment in this subsection.

Appearance of boundary propagator

From graph 5.4c after splitting the residual field we get the contribution (Lie algebra indices suppressed)

$$\psi_{\Gamma_{12}\mathsf{b}} = \int_{M_1\times\partial M_2\times\partial M_3} \eta_{12}\eta_{13}\mathbb{A}_2\mathbb{A}_3 = \int_{\partial M_2\times\partial M_3} \eta^T_{23}\mathbb{A}_2\mathbb{A}_3 \tag{5.33}$$

where η^T_{23} is the axial gauge propagator on the torus, given by

$$\eta^T_{23} = \eta_{S^1}(\theta_2,\theta_3)\delta^{(1)}_{S^1}(t_2,t_3) + (d\theta_2 - d\theta_3)\eta_{S^1}(t_2,t_3). \tag{5.34}$$

The same phenomenon can be observed on the disk (follows from equations (A.64), where we have

$$\int_{D,1} \eta_{D,12}\eta_{D,13} = \eta_{S^1}(\theta_2,\theta_3). \tag{5.35}$$

Theorem 6.1.1 shows that under certain assumptions this can be adapted to more general manifolds with boundary.

Feynman diagrams on the disk

The weights of all diagrams in the 1-point effective action on the disk can be expressesd through complex logarithms and polynomials. However, at the 2-point level we start to dilogarithms appearing, see equation (A.39). In general, we expect the following conjecture to hold:

Conjecture 5.3.7. *The weights of n-point tree graphs on the disk in cubic perturbations of abelian BF theory with gauge fixing given by the Riemann-Hodge propagator can be expressed using multiple polylogarithms and polynomials.*

By definition, a multiple polylogarithm ([Gon97]) is of the form

$$Li_{k_1,\ldots,k_n}(z_1,\ldots,z_n) = \sum_{\substack{m_1,\ldots,m_n\in\mathbb{Z}\\ 0<m_1<m_2<\ldots<m_n}} \frac{z_1^{m_1}\cdots z_n^{m_n}}{m_1^{k_1}\cdots m_n^{k_n}} \tag{5.36}$$

and its weight is $w = k_1 + \ldots + k_n$. A stronger conjecture is the following.

Conjecture 5.3.8. *The weight of a k-point graph with l boundary points denoted $z_1,\ldots,z_l$ can be expressed in terms of the $l-1$ quotients $z_i\bar{z}_{i+1}$ through polylogarithms with at most $l-1$ arguments and at most weight k, and polynomials.*

Weights of Feynman diagrams on the disk (or on on the upper half plane) have been of mathematical interest since the Formality Theorem by Kontsevich ([Kon03]) and the subsequent realization by Cattaneo and Felder that the weights of the Formality morphism are weights of Feynman diagrams of the Poisson Sigma model on the disk ([CF00]). In this approach the weights are numbers - and not functions on the circle - but the weights computed in the appendix might play in the role in S^1-equivariant L_∞-morphism ([CF11]). Recently[7] the weights of such diagrams have been of interest also as coefficients of Drinfeld associators. In particular, similar computations were performed in [Ale+16] and [RW14], where it was argued that a the "logarithmic propagator[8]"

$$\eta^{\log}(z_1,z_2) = \frac{1}{2\pi\mathrm{i}}\mathrm{d}\log\left(\frac{z_1-z_2}{\bar{z}_1-z_2}\right) \tag{5.37}$$

might have better number-theoretic properties. In principle, the machinery developed in Appendix A.7 should allow to perform computations also with this propagator. It might allow for a "logarithmic equivariant " L_∞-morphism.

Even more recently (Summer 2018), Pym, Banks and Panzer announced a proof of the fact

[7]The author was made aware of this fact by M. Felder.

[8]This is a propagator for the upper halfplane, but of course there is a corresponding one on the disk.

that all weights of Feynman diagrams can be expressed as linear combinations of even-weight multiple zeta values (Video of a talk available, see [Pan18]). Since the multiple zeta values are just values of polylogarithms

$$\zeta(k_1, \ldots, k_n) = Li_{k_1,\ldots k_n}(1, 1, \ldots, 1), \tag{5.38}$$

this can be seen as supporting the conjectures above.

Another remark on Feynman diagrams in the disk is that the complexity seems to increase due to two things: First, the inclusion of zero modes makes the propagator not closed. The primitive of the volume form which is added to the propagator for that purpose increases the complexity of the computations. The second reason seems to be vertices with three outgoing arrows. Notice that these do not appear in the Poisson Sigma model. Understanding the deeper reason for these phenomena might help to understand the polarization anomaly that results as a direct consequence of these vertices (see 5.6).

Bernoulli Polynomials and numbers

On the circle, the propagator coincides with the periodic extension of the first Bernoulli polynomial. This allows to compute all pushforwards explicitly by the following relation. Remember that the Bernoulli polynomials are defined by

$$b_0(x) = 1, \tag{5.39}$$

$$\frac{d}{dx} b_n(x) = n b_{n-1}(x), \tag{5.40}$$

$$\int_0^1 b_n(x) = 0 \qquad (n \geq 1). \tag{5.41}$$

Lemma 5.3.9. *Denote B_n the periodic extension to the real line of the restriction of the $n-$th Bernoulli polynomial to $(0,1)$. Then for $n \geq 2$ we have for $x_1, x_3 \in \mathbb{R}$ that*

$$\int_0^1 B_{n-1}(x_1 - x_2) B_1(x_2 - x_3) dx_2 = -\frac{1}{n} B_n(x_1 - x_3). \tag{5.42}$$

Proof. By periodicity we can shift the integral to

$$\int_0^1 B_{n-1}(x_1 - x_2) B_1(x_2 - x_3) dx_2 = \int_0^1 B_{n-1}(x_1 - x_3 - x_2) B_1(x_2) dx_2.$$

Now since $x_2 \in (0,1)$ we have $B_1(x_2) = b_1(x_2) = x_2 - 1/2$. Note that (5.41) implies $\int_a^{a+1} B_n(x)dx = 0$ for any $a \in \mathbb{R}$, hence, the integral above is (setting $y = x_1 - x_3$)

$$\begin{aligned}\int_0^1 B_{n-1}(y - x_2) B_1(x_2) dx_2 &= \int_0^1 x_2 B_{n-1}(y - x_2) dx_2 \\ &= \left[-x_2 \frac{1}{n} B_n(y - x_2) \right]_0^1 + \int_0^1 \frac{1}{n} B_n(y - x_2) = \frac{1}{n} B_n(y)\end{aligned}$$

where we have used (5.40) (which holds for B_n on all of $\mathbb{R}$ for $n \geq 3$, and on $\mathbb{R}\backslash\mathbb{Z}$ for $n = 2$.) $\square$

Hence the pushforwards of currents on the circle can be expressed in terms of Bernoulli polynomials, numbers, and delta functions. See also Appendix A.8.

5.3.6 Effective action on solid torus in $\mathbb{B}$-representation

So far we have only considered the $\mathbb{A}$ representation. However, since the theory is symmetric under the exchange $\mathsf{A} \leftrightarrow \mathsf{B}$, the $\mathbb{B}$-representation can be inferred directly from the $\mathbb{A}$-representation. More precisely, for every diagram in the $\mathbb{A}$-representation there is a diagram in the $\mathbb{B}$-representation obtained by exchanging the direction of all arrows, exchanging all residual fields and all boundary fields. See figure 5.10. We will call this diagram $\Gamma^{\mathbb{B}}$ the *dual diagram* of $\Gamma^{\mathbb{A}}$. However, since we have

(a) $\Gamma^{\mathbb{A}}$ (b) Corresponding $\Gamma^{\mathbb{B}}$

Figure 5.10: Corresponding diagrams in $\mathbb{A}$- and in $\mathbb{B}$-representations.

$$\eta^{\mathbb{A}}(x_1, x_2) = -\eta^{\mathbb{B}}(x_2, x_1),$$

and the form parts of $\mathsf{a}^{\mathbb{A}}$ and $\mathsf{b}^{\mathbb{B}}$ (resp. $\mathsf{a}^{\mathbb{B}}$ and $\mathsf{b}^{\mathbb{A}}$) are the same, the form parts of $\Gamma^{\mathbb{A}}$ and $\Gamma^{\mathbb{B}}$ are the same up to a sign. The structure constants and coordinates on the space of residual fields have to replaced with their "dual" counterparts. This amounts to the following prescription. To compute the state $\psi_{\Gamma^{\mathbb{B}}}$ from the state $\psi_{\Gamma^{\mathbb{A}}}$,

- replace $z^{i,k,\mathbb{A}}$ by $z^{+,\mathbb{B}}_{i,k}$, $z^{+,\mathbb{A}}_{i,k}$ by $z_{i,k,\mathbb{B}}$,
- replace μ^i_{jk} by γ^{jk}_i, ψ_{ijk} by φ^{ijk} and vice versa,
- replace every $\mathbb{A}^i$ by $\mathbb{B}_i$,
- multiply by $(-1)^{\#E(\Gamma_{\mathbb{A}})}$, where $E(\Gamma_{\mathbb{A}})$ denotes the set of edges of $\Gamma_{\mathbb{A}}$.

5.4 Gluing of lens spaces from axial gauge fixing on solid tori

We now know the 1- and 2-point functions on the solid torus in both the $\mathbb{A}$ and the $\mathbb{B}$-representation. In this section we will describe how to compute the 2-point effective action on lens spaces by applying the gluing procedure described in [CMR17].

5.4.1 Lens spaces

Let us describe the conventions we use for lens spaces. Consider two solid tori $M_1 = S^1 \times D = M_2$. The boundary is $S^1 \times S^1$ with coordinates $(t, \theta) \in (\mathbb{R}/\mathbb{Z})^2$. Pick two coprime integers p and q. Since they are coprime, there exist m, n such that $mq - np = 1$. Let $\varphi \in \mathrm{Diff}(S^1 \times S^1)$ be defined by

$$\varphi\colon \begin{pmatrix} t \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} m & p \\ n & q \end{pmatrix} \begin{pmatrix} t \\ \theta \end{pmatrix} = \begin{pmatrix} mt + p\theta \\ nt + q\theta \end{pmatrix} \tag{5.43}$$

Then we define the lens space $L_{p,q}$ by

$$L_{p,q} = M_1 \cup_\varphi M_2. \tag{5.44}$$

Note that with this convention $L_{1,0} \cong S^3$ and $L_{0,1} \cong S^1 \times S^2$.

Remark 5.4.1 (Dependence on choices). It is well known that the diffeomorphism type of $L_{p,q}$ is independent of the choice of m and n and also independent of the choice of q (mod p). Hence the diffeomorphism type of $L_{p,q}$ is well-defined by (5.44). However, the *framing* of the resulting lens space depends on the gluing diffeomorphism. Namely, we can either change $m \to m + kp, n \to n + kq$ or $q \to q + kp, n \to n + km$. Let

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then the former change corresponds to multiplying φ with T^k from the right, while the latter corresponds to multiplying with T^k from the left. Since in the gluing we identify $\partial M_1 \ni x \sim \varphi(x) \in \partial M_2$, geometrically the second operation corresponds to gluing after performing k Dehn twists around the meridian in ∂M_2, while the first operation corresponds to performing k *inverse* Dehn twists around the meridian in ∂M_1. Dehn twists around the longitude extend to the solid torus: Using polar coordinates on the disk a possible representation[9] is[10]

$$T\colon (t, r, \theta) \mapsto (t, r, \theta + t).$$

[9]Remember that only the isotopy class of a Dehn twist is well-defined.

[10]The following formula can be extended to $r = 0$ by the identity.

Such a Dehn twist changes the homotopy class of the framing on the solid torus by one generator. The first operation will change the framing of $L_{p,q}$ by $+k$ units, the second operation by $-k$ units. This is discussed in detail in [FG91, Appendix B].

Remark 5.4.2 (The case $p = 0$). If $p = 0$, then $qm = 1$, so we have $q = m = \pm 1$. That case needs to be considered separately: The resulting space $S^1 \times S^2$ is not a rational homology 3-sphere, unlike all other lens spaces. In the following we will always assume $p \neq 0$ unless otherwise stated.

5.4.2 Gluing perturbative expansions in BV-BFV

The gluing procedure discussed in [CMR17] amounts to the following prescription:

- Take a diagram Γ_1 in the $\mathbb{A}$-representation and a diagram Γ_2 in the $\mathbb{B}$-representation with the same number n of legs and multiply $\psi_{\Gamma_1} = \int_{(\partial M_1)^n} \omega_{\Gamma_1} \mathbb{A}_1 \cdots \mathbb{A}_n$ and $\psi_{\Gamma_2} = \int_{(\partial M_2)^n} \omega_{\Gamma_2} \mathbb{B}_1 \cdots \mathbb{B}_n$. Here the diagrams can be non connected, since the state is the exponential of the effective action.
- Sum over all ways of contracting $\mathbb{A}$ and $\mathbb{B}$ fields to a delta form $\delta^{(d-1)}_{\partial M}(x, \varphi(x))$.
- Perform the integration over $(\partial M_1)^n \times (\partial M_2)^n$.
- Reduce the residual fields.

Equivalently, the state glued from Ψ_{Γ_1} and Ψ_{Γ_2} can de defined as follows. Let $\sigma \in S_n$ be a permutation and denote $\Phi_\sigma \colon (\partial M)^n \to (\partial M)^n$ the map defined by

$$(x_1, \ldots, x_n) \mapsto (\varphi(x_{\sigma(1)}), \ldots, \varphi(x_{\sigma(n)})).$$

Then the above prescription results in

$$\psi_{\Gamma_1} * \psi_{\Gamma_2} := \sum_{\sigma \in S_n} \int_{(\partial M)^n} \omega_{\Gamma_1} \Phi_\sigma^* \omega_{\Gamma_2}.$$

In this integral only the top degree part survives. Sometimes it will be convenient to use the reformulation

$$\psi_{\Gamma_1} * \psi_{\Gamma_2} = \sum_{\sigma \in S_n} \int_{(\partial M)^n} (\Phi_\sigma^{-1})^* (\omega_{\Gamma_1} \Phi_\sigma^* \omega_{\Gamma_2}) = \sum_{\sigma \in S_n} \int_{(\partial M)^n} (\Phi_\sigma^{-1})^* (\omega_{\Gamma_1}) \omega_{\Gamma_2}. \tag{5.45}$$

In all the examples that we consider, the graphs are invariant with respect to permuting the boundary points. Hence, the sum is a constant times the pairing computed using Φ_{id}, which we

will also denote φ, abusing notation.
Notice also that for graphs not depending on boundary fields the gluing procedure is trivial, i.e. the corresponding contributions are simply multiplied with the rest, or, equivalently, added to the effective action.

5.4.3 The effective action on M_2

On M_2 we will choose the opposite polarization, namely, $\partial M = \partial_2 M$ (the $\mathbb{B}$-representation). To avoid confusion, from now on we decorate objects with a superscript depending on which representation they are computed in, e.g. $\psi^{\mathbb{A}}$ (resp. $\psi^{\mathbb{B}}$) denotes the state in the $\mathbb{A}$-representation ($\mathbb{B}$-representation). The residual fields change roles: $\mathsf{a}^{\mathbb{B}} = z_1^{\mathbb{B}} 1 + z_2^{\mathbb{B}} dt$, $\mathsf{b}^{\mathbb{B}} = z_1^{+,\mathbb{B}} \mu dt + z_2^{+,\mathbb{B}} \mu$. The effective action can be computed from the procedure described in section 5.3.6.

5.4.4 Reducing the residual fields

Naively, after pairing the states ψ_{M_1}, ψ_{M_2}, the new state is a function on the direct sum of the spaces of residual fields $\widetilde{\mathcal{V}_M} = \mathcal{V}_{M_1} \oplus \mathcal{V}_{M_2}$. This produces a valid state in the sense that it satisfies the mQME. However, in most cases it is not the minimal possible space of residual fields and it is possible to reduce it using the methods of [CMR17]. We will discuss the reduction first since it will allow us to simplify the computations later.
In the case of lens spaces there is a significant difference between the cases $p \neq 0$ and the case $p = 0$, and we will discuss these separately.

Case $p \neq 0$

We will first discuss the example of $M = S^3 = L_{1,0}$. Recall that $M_1 = M_2 = D^2 \times S^1$, $\partial_1 M_1 = \partial_2 M_2 = S^1 \times S^1 =: \mathbb{T}^2$ and $\partial_2 M_1 = \partial_1 M_2 = \emptyset$. Now, let $M = S^3$ and $\varphi \colon \mathbb{T}^2 \to \mathbb{T}^2$ be the diffeomorphism given by $t \mapsto -\theta, \theta \mapsto t$ so that $M = M_1 \cup_\varphi M_2$. Denote

$$\Omega_{Dj}(M_i) = \Omega(M_i, \partial_j M_i).$$

Then we have $H^\bullet_{Di}(M_i) = H^\bullet(M_i, \partial M_i) \cong H^\bullet(D^2, S^1) \otimes H^\bullet(S^1)$ and $H^\bullet_{Dj}(M_i) \cong H^\bullet(S^1)$ for $i \neq j$. We have the following spaces of residual fields

$$\begin{aligned}
\mathcal{V}_{M_1} &\cong \left(H^\bullet(D^2, \partial D^2) \otimes H^\bullet(S^1)\right) \oplus \left(H^\bullet(S^1)\right) \\
\mathcal{V}_{M_2} &\cong \left(H^\bullet(S^1)\right) \oplus \left(H^\bullet(D^2, \partial D^2) \otimes H^\bullet(S^1)\right) \\
\mathcal{V}_M &= H^\bullet(S^3) \oplus H^\bullet(S^3)
\end{aligned}$$

Now, we have generators $\mu \in H^2(D^2, S^1), dt \in H^1(S^1), v \in H^3(S^3)$ so that

$$\begin{aligned}
\mathcal{V}_{M_1} &\cong \langle \mu, \mu \wedge dt \rangle \oplus \langle 1, dt \rangle \\
\mathcal{V}_{M_2} &\cong \langle 1, dt \rangle \oplus \langle \mu, \mu \wedge dt \rangle \\
\mathcal{V}_M &= \langle 1, v \rangle \oplus \langle 1, v \rangle
\end{aligned}$$

Obviously, the "naive" space of backgrounds after gluing $\tilde{\mathcal{V}}_M = \mathcal{V}_{M_1} \oplus \mathcal{V}_{M_2}$ is not isomorphic to $\mathcal{V}_M$. We need to reduce the μ's and dt's. We will follow the method in [CMR17]. Let

$$\begin{aligned}
\tau_1 &\colon H^\bullet_{D2}(M_1) \to H^\bullet(\Sigma) \\
\tau_2 &\colon H^\bullet_{D1}(M_2) \to H^\bullet(\Sigma)
\end{aligned}$$

be the restrictions induced by the inclusion maps $\Sigma \hookrightarrow M_i$. Since $\partial_2 M_1 = \partial_1 M_2 = \emptyset$ they are maps $\tau_i \colon H^\bullet(S^1) \hookrightarrow H^\bullet(\mathbb{T}^2)$ given by (denoting the generators of the torus cohomology by dt and $d\theta$)

$$\begin{aligned}
\tau_1 \colon 1 &\mapsto 1 \\
dt &\mapsto dt \\
\tau_2 \colon 1 &\mapsto 1 \\
dt &\mapsto \varphi^*(dt) = -d\theta
\end{aligned}$$

(since we glue $\Sigma \subseteq M_1$ to $\varphi(\Sigma) \subseteq M_2$). Denoting $L_i = \operatorname{im} \tau_i$, we get $L_1 = \langle 1, dt \rangle$ and $L_2 = \langle 1, d\theta \rangle$. Using the Poincaré pairing, we get $L_1^\perp = L_1 = \langle 1, dt \rangle$ and $L_2^\perp = L_2 = \langle 1, d\theta \rangle$ so that we can choose as the complement inside L_1 to $L_1 \cap L_2^\perp = \langle 1 \rangle$ the space $L_1^\times = \langle dt \rangle$. The same argument goes through for L_2, yielding $L_2^\times = \langle d\theta \rangle$.
Using the other formula proposed in [CMR17], $L_i^\perp = \tau_i\left(H^\bullet(M_i, \partial M_i \setminus \Sigma)\right) = \tau_i(H^\bullet(M_i)) = L_i$, yields the same answer and we can still choose $L_1^\times = \langle dt \rangle$, $L_2^\times = \langle d\theta \rangle$.
For these choices of the $L_i^\perp$, we indeed get $\int_\Sigma \mathsf{b}_1 \mathsf{a}_2 = \int_\Sigma \mathsf{b}_1^\times \mathsf{a}_2^\times$. The reduced state is defined by

$$\check{\psi} = \int_{\mathcal{L}^\times} \widetilde{\psi} \tag{5.46}$$

where $\mathcal{L}^\times$ is the zero section of $T^*[-1](L_1^\times \oplus L_2^\times)$. This amounts to contracting a pair of $z_2^{+,\mathbb{A}}$ and $z^{2,\mathbb{B}}$ coordinates to the number

$$V = \Lambda^{-1} = \left(\int_{\mathbb{T}^2} dt\varphi^*(dt)\right)^{-1},$$

while setting $z^{2,\mathbb{A}} = z_2^{+,\mathbb{B}} = 0$. For the 3-sphere S^3 we have $V = -1$. Now let us check what the gluing of cohomology of the backgrounds produces. For this, we choose the sections $\sigma_i \colon L_i \to H^\bullet(M_i)$ by $\sigma_i(1) = 1, \sigma_1(dt) = dt, \sigma_2(d\theta) = dt$. We recall the following notation from [CMR17]. Since $L_1 \cap L_2^\perp = L_1^\perp \cap L_2 = L_1 \cap L_2 = \langle 1 \rangle$ and $\ker \tau_1 = \ker \tau_2 = 0$, we get

$$\begin{aligned}
H^\bullet_{D2}(M_1)' &:= \sigma_1(L_1 \cap L_2^\perp) = \langle 1 \rangle \\
H^\bullet_{D1}(M_2)' &:= \sigma_2(L_1^\perp \cap L_2) = \langle 1 \rangle \\
H^\bullet_{D1}(M_1)^\circ &:= \sigma_1(L_1 \cap L_2^\perp)^* = \langle \mu \wedge dt \rangle \\
H^\bullet_{D2}(M_2)^\circ &:= (\sigma_2(L_1^\perp \cap L_2)^* = \langle \mu \wedge dt \rangle.
\end{aligned}$$

Therefore

$$\begin{gathered}
\widetilde{H^\bullet_{D1}}(M_1, M_2) := H^\bullet_{D1}(M_1)^\circ \oplus H^\bullet_{D1}(M_2)' = \langle \mu_1 \wedge dt_1, 1_{M_2} \rangle \\
\check{\mathsf{a}}_1 = z^{1,\mathbb{A}} \mu_1 \wedge dt_1, \check{\mathsf{b}}_1 = \mathsf{b}_1^\cap = z_1^{+,\mathbb{A}} 1_{M_1} \\
\widetilde{H^\bullet_{D2}}(M_1, M_2) := H^\bullet_{D2}(M_1)' \oplus H^\bullet_{D2}(M_2)^\circ = \langle \mu_2 \wedge dt_2, 1_{M_1} \rangle \\
\check{\mathsf{b}}_2 = z_1^{+,\mathbb{B}} \mu_2 \wedge dt_2, \check{\mathsf{a}}_2 = \mathsf{a}_2^\cap = z^{1,\mathbb{B}} 1_{M_2}
\end{gathered}$$

There are maps

$$\begin{aligned}
h_1 \colon \widetilde{H^\bullet_{D1}}(M_1, M_2) \to H^\bullet(S^3) \\
h_2 \colon \widetilde{H^\bullet_{D2}}(M_1, M_2) \to H^\bullet(S^3)
\end{aligned}$$

given by extending the classes in M_1, M_2 respectively to the other manifold. We can see in our case that the maps are isomorphisms, but it is proven in [CMR17] that this is a general fact. The extensions can be computed as $\mathsf{a}_2^{\text{ext}} = z^{1,\mathbb{B}} 1_{M_1}, \mathsf{b}_1^{\text{ext}} = z_1^{\mathbb{A},+} 1_{M_2}$ and therefore

$$\begin{aligned}
\check{\mathsf{a}}\big|_{M_2} &= z^{1,\mathbb{B}} 1_{M_2} \\
\check{\mathsf{a}}\big|_{M_1} &= z^{1,\mathbb{B}} 1_{M_1} + z^{1,\mathbb{A}} \mu_1 \wedge dt_1 \\
\check{\mathsf{b}}\big|_{M_1} &= z_1^{+,\mathbb{A}} 1_{M_1} \\
\check{\mathsf{b}}\big|_{M_2} &= z_1^{+,\mathbb{A}} 1_{M_2} + z_1^{+,\mathbb{B}} \mu_2 \wedge dt_2
\end{aligned}$$

so we have induced representatives of the cohomology on the sphere

$$\chi_1 = 1_{S^3}$$
$$\chi_2 = \begin{cases} \mu_1 \wedge dt_1 & \text{on } M_1 \\ 0 & \text{on } M_2 \end{cases}$$

with coordinates $z^1 = z^{1,\mathbb{B}}, z^2 = z^{1,\mathbb{A}}$ and dual basis (with respect to the Poincare pairing)

$$\chi^1 = \begin{cases} 0 & \text{on } M_1 \\ \mu_2 \wedge dt_2 & \text{on } M_2 \end{cases}$$
$$\chi^2 = 1_{S^3}$$

with coordinates $z_1^+ = z_1^{+,\mathbb{B}}, z_2^+ = z_1^{+,\mathbb{A}}$. Now consider the diffeomorphism $\psi = \begin{pmatrix} m & p \\ n & q \end{pmatrix}$ of the torus. Then we get $\tau_2(dt) = \psi^*(dt) = mdt + pd\theta$ for some $a, b \in \mathbb{R}$, $L_2 = \langle 1, adt + bd\theta \rangle$ and hence

$$L_1 \cap L_2^\perp = L_1^\perp \cap L_2 = L_1 \cap L_2 = \langle 1, dt \rangle \cap \langle 1, mdt + pd\theta \rangle = \langle 1 \rangle$$

if $p \neq 0$. We still have $L_1^\times = \langle dt \rangle$, but now $L_2^\times = \langle mdt + pd\theta \rangle$. The glued representatives of the cohomology are exactly the same as above, replacing 1_{S^3} by $1_{L_{p,q}}$. We now have $\Lambda = \int_{\mathbb{T}^2} dt(mdt + pd\theta) = p$ and hence $V = 1/p$, but otherwise the reduction procedure is the same: Contract a pair of $z_2^{+,\mathbb{A}}$ and $z^{2,\mathbb{B}}$ coordinates to V and set $z^{2\mathbb{A}} = z_2^{+,\mathbb{B}} = 0$. We can use this to simplify the calculation of the state on $L_{p,q}$ by ignoring pairings of diagrams that would vanish after reducing residual fields.

The case $p = 0$

In this case the spaces $L_1^\times$ and $L_2^\times$ are empty and we do not have to perform the reduction. The resulting manifold is $M = S^2 \times S^1$. Sticking to the same conventions as above, we will get representatives of cohomology

$$\chi_1 = 1$$

$$\chi_2 = dt$$

$$\chi_3 = \begin{cases} \mu & \text{on } M_1 \\ 0 & \text{on } M_2 \end{cases}$$

$$\chi_4 = \begin{cases} \mu \wedge dt & \text{on } M_1 \\ 0 & \text{on } M_2 \end{cases}$$

with coordinates $z^1 = z^{1,\mathbb{B}}, z^2 = z^{2,\mathbb{B}}, z^3 = z^{2,\mathbb{A}}, z^4 = z^{1,\mathbb{A}}$. The dual basis is

$$\chi^1 = \begin{cases} 0 & \text{on } M_1 \\ \mu \wedge dt & \text{on } M_2 \end{cases}$$

$$\chi^2 = \begin{cases} 0 & \text{on } M_1 \\ \mu & \text{on } M_2 \end{cases}$$

$$\chi^3 = dt$$

$$\chi^4 = 1$$

with coordinates $z_1^+ = z_1^{+,\mathbb{B}}, z_2^+ = z_2^{+,\mathbb{B}}, z_3^+ = z_1^{+,\mathbb{A}}, z_4^+ = z_2+, \mathbb{A}$.

5.5 The effective action on $L_{p,q}$

Since we are interested in the two-point effective action after gluing, we have to consider all pairs of diagrams with a total of at most two interaction vertices. Also, since we are interested in the state only after reduction of the residual fields, only pairings with no residual fields or the same number of $z_2^{+,\mathbb{A}}$ and $z^{2,\mathbb{B}}$ survive, all others can be ignored. We now compute all the relevant pairings for the case where $(\mathfrak{g}, V, W)$ form a Manin triple.

5.5.1 2-point tree contribution

In principle, tree diagrams with two points could contribute to the 2-point effective action after gluing. However, in this subsection we will argue that it is not so. The point is that in the gluing process 2-point graphs can only be paired with 0-point graphs on the other side, i.e. graphs from the free effective action. As explained below in section 5.5.2, pairing against a

0-point diagram on the other side amounts to placing a linear combination of 1, dt and $d\theta$ at this point and integrating over it. The claim then follows from the following Lemmata:

Lemma 5.5.1. *For $x_2 \in \partial M$, we have*

$$\int_2 \eta_{12} dt_2 = dt_1$$
$$\int_2 \eta_{12} d\theta_2 = -\psi = \frac{z d\bar{z} - \bar{z} dz}{4\pi i}.$$

Proof. We have

$$\int_2 \eta_2 dt_2 = \int_2 (\eta_{D,12}\delta_{S^1,12} + \mu_1 \eta_{S^1,12}) dt_2 = \int_{\partial D,2} \eta_{D,12} \int_{S^1,2} \delta_{S^1,12} dt_2 = dt_1$$

since $\int_{\partial D,2} \eta_{D,12} = 1$ and

$$\int_2 \eta_2 d\theta_2 = (\eta_{D,12}\delta_{S^1,12} + \mu_1 \eta_{S^1,12}) d\theta_2 = \int_{\partial D,2} \eta_{D,12} d\theta_2$$
$$= \int_{\partial D,2} 2(\phi_{12} - \psi_1) d\theta_2 = -\psi_1$$

since $\int_{\partial D,2} \phi_{12} d\theta_2 = 0$, as can easily be checked by the residue theorem. □

Lemma 5.5.2. *Integrating η against dt, ψ or ψdt placed either at head or boundary vanishes.*

Proof. This follows from the fact that η_{S^1} (resp. η_D) vanish when integrated against dt (resp. ψ). For the disk propagator this follows directly from (A.62a),(A.62b) and $\psi^2 = 0$. □

Now consider a two-point tree diagram as in figure 5.11a. It consists of a single arrow between the two points, some legs on the boundary, and maybe some residual fields. Now integrate all the legs against dt or $d\theta$. The result is a graph consisting of single arrow with a product γ_i of residual fields, dt's and $d\theta$'s on both ends (figure 5.11b). From the two Lemmata above together with Proposition 5.3.2 it now follows that the contribution of such a graph is zero after gluing.

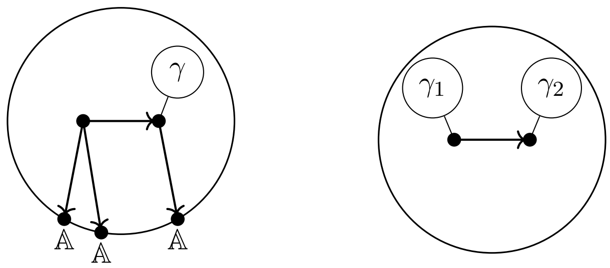
5.5.2 Case of a Manin triple

In this case, the A^3 and B^3 vertices vanish and the number of diagrams to be considered is considerably reduced.

Pairing against order 0 diagram

First we consider all pairings against the single order 0 diagram on M_2. Its contribution to the state, since M_2 is in the $\mathbb{B}$-representation, is

$$\psi^{\mathbb{B}}_{\Gamma_0} = -z_1^{k,\mathbb{B}} \int_{\partial M} \mathbb{B}_k - z_2^{k,\mathbb{B}} \int_{\partial M} dt \mathbb{B}_k.$$



(a) Example of a two-point tree diagram

(b) After integrating all legs against dt, $d\theta$

Figure 5.11: Two-point tree diagrams

We have $\varphi^* dt = mdt + pd\theta$. Hence (as used already in section 5.5.1) gluing a boundary point on the $\mathbb{A}$ side against the 0-point action up to constants corresponds to multiplying the corresponding form with $1, dt$ or $d\theta$ and integrating over that boundary point. Often, it is best to perform this integration first - the results are known from Lemma 5.5.1 - and then compute the integral. Notice also that this diagram does not pair to diagrams with no boundary vertices.

We can pair the two 0-point terms on either side, see figure 5.12. The result is

$$\psi^{\mathbb{A}}_{\Gamma_0} * \psi^{\mathbb{B}}_{\Gamma_0} = \int_{\partial M} \mathsf{b}^{\mathbb{A}} \varphi^* \mathsf{a}^{\mathbb{B}} = z^{+,\mathbb{A}}_{2,k} z^{2,\mathbb{B}} \int_{\partial M} dt \varphi^* dt = p z^{+,\mathbb{A}}_{2,k} z^{2,\mathbb{B}}. \tag{5.47}$$

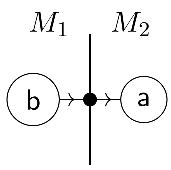
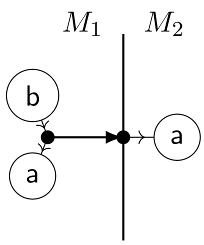


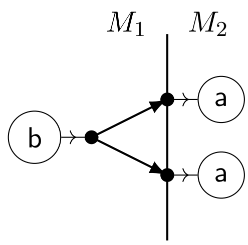
Figure 5.12: Pairing order 0 diagrams on either side.

These are precisely the fields that we ought to reduce in the reduction of the redshirt residual fields as discussed in section 5.4.4. We will perform this reduction in 5.5.2.
Let us turn to the 1-point diagrams. The first one ($\Gamma^{\mathbb{A}}_{1,0}$) does not contain $\mathbb{A}$ fields and hence does not pair to $\Gamma^{\mathbb{B}}_0$. The two possible pairings are shown in figure 5.13. Using Lemma 5.5.1 one quickly sees that the pairing in figure 5.13b vanishes. Alternatively one can use that $\int_{S^1,i} \eta_{S^1}(t_1, t_2) dt_i = 0$ for $i = 1, 2$, Together with degree counting this implies the same. The remaining pairing shown in figure 5.13a evaluates to

$$\psi^{\mathbb{A}}_{1,1} * \psi^{\mathbb{B}}_{\Gamma_0} = \mu^i_{jk} (z^{+,\mathbb{A}}_{1i} z^{1j,\mathbb{A}} - z^{+,\mathbb{A}}_{2i} z^{2j\mathbb{A}}) z^{1k,\mathbb{B}} + m \mu^i_{jk} z^{+,\mathbb{A}}_{1i} z^{2j,\mathbb{A}} z_{1k,\mathbb{B}}. \tag{5.48}$$



(a) $\psi_{\Gamma^{\mathbb{A}}_{1,1}} * \psi_{\Gamma^{\mathbb{B}}_0}$



(b) $\psi_{\Gamma^{\mathbb{A}}_{1,1}} * (\psi_{\Gamma^{\mathbb{B}}_0})^2$

Figure 5.13: Pairing 1-point diagrams on M_1 to 0-point diagram on M_2.

We can also pair against the loop diagrams, see figure 5.14 to obtain

$$\psi^{\mathbb{A}}_{\Gamma_{2,1}} * \psi^{\mathbb{B}}_{\Gamma_0} = \frac{m}{12} z^{+,\mathbb{A}}_{2i} z^{2,l,\mathbb{B}} \gamma^{ij}_k \mu^k_{jl} \tag{5.49}$$

$$\psi^{\mathbb{A}}_{\Gamma_{2,2}} * \psi^{\mathbb{B}}_{\Gamma_0} = C z^{2,k,\mathbb{B}} z^{2,l,\mathbb{B}} \gamma^{ij}_k \mu^k_{jl} \tag{5.50}$$

Here C is an unknown constant which is possibly ill-defined. See Remark 5.3.6.

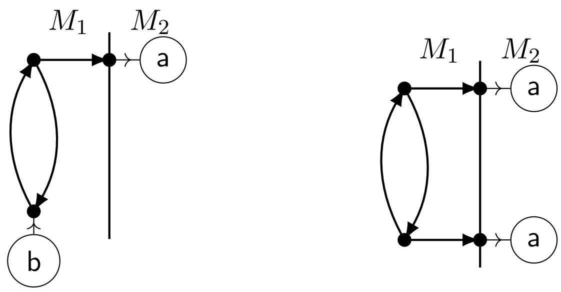
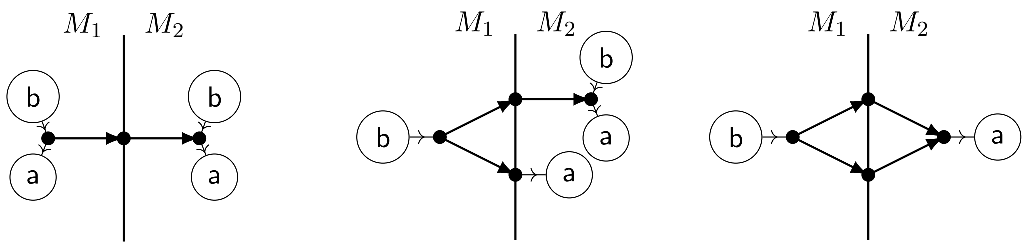


Figure 5.14: Pairing 2-point diagrams on M_1 to 0-point diagram on M_2.

Pairing the 1-point functions

Pairing the 1-point functions is computationally more intense, since we have to take the pull-back of non-constant forms. The graph $\Gamma_{1,0}$ with no legs does not depend on boundary, hence its contribution and the one of its dual diagram simply add to the effective action. Part of it survives after reducing residual fields, and in fact we will show this is the only one-point contribution to the effective action.

Under the Manin triple assumption, the other pairings are the ones described in figure 5.15. The next diagram $\Gamma_{1,1}$ has only constant form coefficients and the pairings are (see Appendix



(a) $\psi_{\Gamma^{\mathbb{A}}_{1,1}} * \psi_{\Gamma^{\mathbb{B}}_{1,1}}$ (b) $\psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,2}} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,1}}\psi_{\Gamma^{\mathbb{B}}_{0}}\right)$ (c) $\psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathsf{b},\mathbb{B}}_{1,2}}$

Figure 5.15: Pairing 1-point diagrams on M_1 to 1-point diagrams on M_2.

B for the computations)

$$\psi_{\Gamma^{\mathbb{A}}_{1,1}} * \psi_{\Gamma^{\mathbb{B}}_{1,1}} = -n\mu^i_{jk}\gamma^{kl}_m z^{+,\mathbb{A}}_{1i} z^{2j,\mathbb{A}} z^{1m,\mathbb{B}} z^{+,\mathbb{B}}_{2l} \tag{5.51}$$

$$\left(\psi_{\Gamma^{\mathbb{A}}_{1,1}}\psi_{\Gamma^{\mathbb{A}}_{0}}\right) * \psi_{\Gamma^{\mathsf{b},\mathbb{B}}_{1,2}} = n\mu^i_{jk}\varphi^{klm} z^{+,\mathbb{A}}_{1i} z^{2j,\mathbb{A}} z^{+,\mathbb{A}}_{1,l} z^{+,\mathbb{B}}_{2,m} \tag{5.52}$$

$$\psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,2}} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,1}}\psi_{\Gamma^{\mathbb{B}}_{0}}\right) = n\psi_{ijk}\gamma^{kl}_m z^{2i,\mathbb{A}} z^{1j,\mathbb{B}} z^{1m,\mathbb{B}} z^{+,\mathbb{B}}_{2,l} \tag{5.53}$$

$$\psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathsf{b},\mathbb{B}}_{1,2}} = \mu^i_{jk}\gamma^{jk}_l z^{+,\mathbb{A}}_{2i} z^{2l,\mathbb{B}} \begin{cases} \frac{-p}{2}s(q,p) & p \neq 0 \\ \frac{q}{12} & p = 0 \end{cases} \tag{5.54}$$

Reducing the residual fields

If $p \neq 0$, we have to reduce the residual fields as discussed in 5.4.4. We recall that this amounts to pairing $z^{+,\mathbb{A}}_{2i}$ with $z^{2,j,\mathbb{B}}$ to $\delta^j_i \cdot 1/p$ and setting their conjugates variables $z^{2,i,\mathbb{A}} = z^{+,\mathbb{B}}_{2,j} = 0$. This eliminates many of the pairings above, namely, (5.50),(5.51),(5.52) and (5.53). We denote the resulting effective action with S^{MT}_{eff}, where MT stands for Manin triple. We have that

$$S^{MT}_{eff} = S^{MT,(1)}_{eff} + S^{MT,(2)}_{eff} \tag{5.55}$$

where

$$S^{MT,(1)}_{eff} = \varepsilon\mu^i_{jk}\left(z^{+,\mathbb{A}}_{1i} z^{1j,\mathbb{A}} z^{1k,\mathbb{B}} + \frac{1}{2} z^{+,\mathbb{B}}_{1i} z^{1j,\mathbb{B}} z^{1k,\mathbb{B}}\right) + \gamma^{jk}_i\left(z^{1i,\mathbb{B}} z^{+,\mathbb{A}}_{1j} z^{+,\mathbb{B}}_{1k} + \frac{1}{2} z^{+,\mathbb{A}}_{1i} z^{1j,\mathbb{A}} z^{1k,\mathbb{A}}\right) \tag{5.56}$$

and

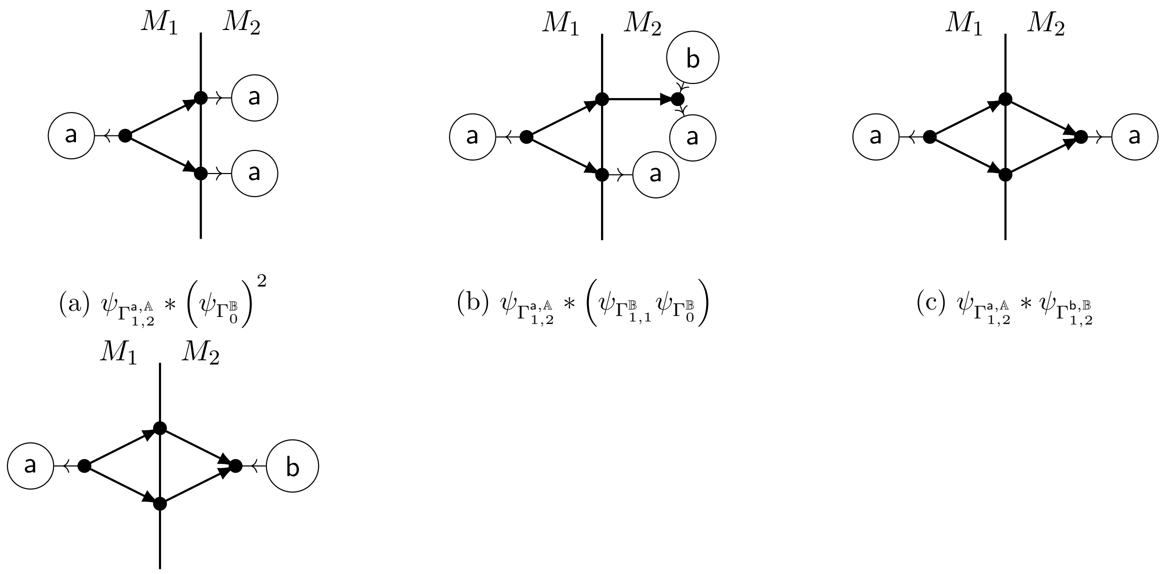
$$S^{MT,(2)}_{eff} = \varepsilon^2\mu^i_{jk}\gamma^{jk}_i\left(\frac{1}{2}s(q,p) + \frac{q+m}{12p}\right) \tag{5.57}$$

The powers of ε here are understood with the convention that the state on $L_{p,q}$ is

$$\psi_{L_{p,q}}(\mathsf{a},\mathsf{b}) = \exp\left(\frac{1}{\varepsilon}S_{eff}(\mathsf{a},\mathsf{b})\right) \tag{5.58}$$

5.5.3 The general case

At this order, the Manin triple condition amounts to ignoring diagrams $\Gamma^{\mathsf{a}}_{1,2}$ and $\Gamma_{1,3}$. Considering them amounts to computing the additional pairings described in figures 5.16 and 5.17, respectively. To simplify the computations we will now drop terms that vanish after reducing fields, i.e. we keep only terms that contain no $z^{2,i,\mathbb{A}}$- and $z^{+\mathbb{B}}_{2,j}$-variables and exactly the same number of $z^{+,\mathbb{A}}_{2,i}$ and $z^{2,j,\mathbb{B}}$ variables. In figure 5.16 this eliminates the diagrams 5.16b, 5.16c, 5.16d, together with degree counting: Ths dimension of the domain of integration is 3+2+2+3=10 in all cases, but the corresponding form degrees are 12, 11, and 14 respectively. On the other hand, diagram 5.16a yields a contribution corresponding to the $\check{\mathsf{a}}^3$ vertex on the glued lens space, while its dual will contribute the $\check{\mathsf{b}}^3$ vertex.



(a) $\psi_{\Gamma^{\mathsf{a},\mathbb{A}}_{1,2}} * \left(\psi_{\Gamma^{\mathbb{B}}_{0}}\right)^2$

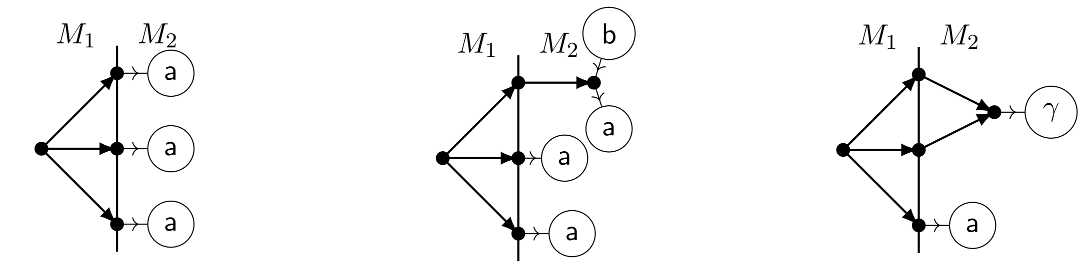
(b) $\psi_{\Gamma^{\mathsf{a},\mathbb{A}}_{1,2}} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,1}}\psi_{\Gamma^{\mathbb{B}}_{0}}\right)$

(c) $\psi_{\Gamma^{\mathsf{a},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathsf{b},\mathbb{B}}_{1,2}}$

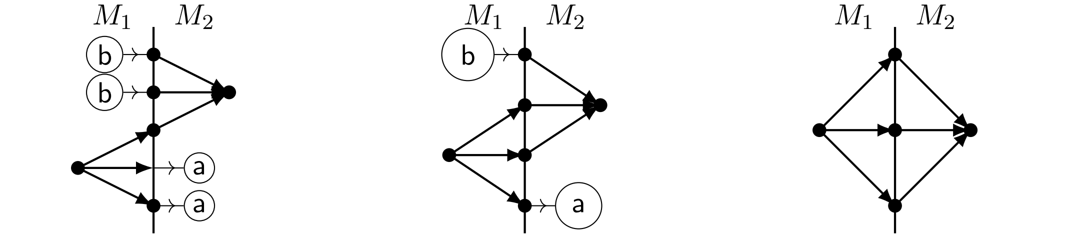
(d) $\psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathsf{b},\mathbb{B}}_{1,2}}$

Figure 5.16: Pairing diagram $\Gamma^{\mathsf{a},\mathbb{A}}_{1,2}$ on M_1 to diagrams on M_2. Here we excluded the diagrams including $\Gamma^{\mathbb{B}}_{1,3}$, we will get them from symmetry from the ones for $\Gamma^{\mathbb{A}}_{1,3}$.

Now let us look at the diagrams in figure 5.17. The first two diagrams in figures 5.17a,5.17b, and also the one in figure 5.17d do not contribute, even if we do not reduce residual fields, this follows from the discussion on vanishing of two-point tree contributions after gluing. After reducing residual fields, diagram 5.17c vanishes for degree reasons: only the zero-form part of a survives, so the total form degree is 10, while integration is over a 12-dimensional space. By degree counting, the only nonzero term contains the one-form parts of $\mathsf{b}^{\mathbb{A}}$ and $\mathsf{a}^{\mathbb{B}}$. After reducing residual fields, this corresponds to part of a theta diagram for the glued propagator, together with the last diagram 5.17f. Below we list the weights of the glued graphs, we only list the ones



(a) $\psi_{\Gamma^{\mathbb{A}}_{1,3}} * \left(\psi_{\Gamma^{\mathbb{B}}_0}\right)^3$ (b) $\psi_{\Gamma^{\mathbb{A}}_{1,3}} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,1}}(\psi_{\Gamma^{\mathbb{B}}_0})^2\right)$ (c) $\psi_{\Gamma^{\mathbb{A}}_{1,3}} * \left(\psi_{\Gamma^{\mathsf{a},\mathbb{B}}_{1,2}}\psi_{\Gamma^{\mathbb{B}}_0}\right)$



(d) $\psi_{\Gamma^{\mathbb{A}}_{1,3}}\psi^2_{\Gamma^{\mathbb{A}}_0} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,3}}\psi^2_{\Gamma^{\mathbb{B}}_0}\right)$ (e) $\psi_{\Gamma^{\mathbb{A}}_{1,3}}\psi_{\Gamma^{\mathbb{A}}_0} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,3}}\psi_{\Gamma^{\mathbb{B}}_0}\right)$ (f) $\psi_{\Gamma^{\mathbb{A}}_{1,3}} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,3}}\right)$

Figure 5.17: Pairing diagram $\Gamma^{\mathbb{A}}_{1,3}$ on M_1 to diagrams on M_2. Here $\gamma \in \{\mathsf{a}, \mathsf{b}\}$.

yielding a non-zero contribution after reducing the residual fields.

$$\psi_{\Gamma^{\mathsf{a},\mathbb{A}}_{1,2}} * \left(\psi_{\Gamma^{\mathbb{B}}_0}\right)^2 = \frac{1}{2}\psi_{ijk} z^{1i\mathbb{A}} z^{1j\mathbb{B}} z^{1k\mathbb{B}} \tag{5.59}$$

$$\psi_{\Gamma^{\mathbb{A}}_{1,3}}\psi_{\Gamma^{\mathbb{A}}_0} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,3}}\psi_{\Gamma^{\mathbb{B}}_0}\right) = \varphi^{ijk}\psi_{ljk} z^{+,\mathbb{A}}_{2i} z^{2l,\mathbb{B}} \frac{1}{4}\int_{(\partial M)^4} (\omega_{\Gamma_{1,3}})_{123} dt_4 (\varphi^{\times 4})^* (\omega_{\Gamma_{1,3}})_{234} dt_1) \tag{5.60}$$

$$\psi_{\Gamma^{\mathsf{b},\mathbb{B}}_{1,3}} * \psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,3}} = \frac{1}{6}\varphi^{ijk}\psi_{ijk}\int_{(\partial M)^3} (\omega_{\Gamma_{1,3}})(\varphi^{\times 3})^*(\omega_{\Gamma_{1,3}}) \tag{5.61}$$

The contribution of (5.60) is computed in Appendix B.1.3. The contribution of (5.61) is computed in Appendix B.1.4 The resulting effective action is

$$S_{eff} = S^{MT}_{eff} + S^{NMT}_{eff} = S^{MT}_{eff} + S^{NMT,(1)}_{eff} + S^{NMT,(2)}_{eff} \tag{5.62}$$

where

$$S^{NMT,(1)}_{eff} = \frac{1}{2}\psi_{ijk} z^{1i\mathbb{A}} z^{1j\mathbb{B}} z^{1k\mathbb{B}} + \frac{1}{2}\varphi^{ijk} z^{+,\mathbb{B}}_{1i} z^{+,\mathbb{A}}_{1j} z^{+,\mathbb{A}}_{1k} \tag{5.63}$$

and

$$S^{NMT,(2)}_{eff} = \frac{1}{2}\psi_{ijk}\varphi^{ijk}\left(s(q,p) + \sum_{k=0}^{p-1}\eta_{S^1}(k/p) f(qk/p) + \eta_{S^1}(k/p) f(mk/p) + \frac{H_{1/p}(q+m)}{2\pi^2}\right) \tag{5.64}$$

where the function $f\colon S^1 \to \mathbb{R}$ is given by

$$f(\theta) = \cos(2\pi\theta)\eta_{S^1}(\theta) - \frac{1}{\pi}\sin 2\pi\theta \log 2|\sin \pi\theta|$$

for $\theta \notin \mathbb{Z}$, and $f(k) = 0$ for $k \in \mathbb{Z}$.

5.6 Weights of theta graphs on lens spaces

The effective action on $L_{p,q}$ after gluing has the rough form that would be expected from computing the perturbative expansion of split Chern-Simons theory on lens spaces. Namely, the terms $S_{eff}^{MT,(1)}$ and $S_{eff}^{NMT,(1)}$ are given by corollas, i.e. 1-point graphs that contain no edges, just decorations with the various residual fields. These residual fields are the ones that stem from the presentation of the lens space as a solid torus. Then, there are the two-point terms. Since the reduced residual fields are concentrated in form degrees 0 and 3, a quick degree count shows that the only two-point terms with residual fields that survive are the products of corollas. The only connected two-point graphs appearing are the 2-loop graphs. Here, we are in for a surprise. With the non-symmetric propagator that arises from the gluing procedure, there are two different orientations for the 2-loop graphs depicted in Figure 5.18. Notice that the orientation given in Figure 5.18b can only appear if $(\mathfrak{g}, V, W)$ fails to be a Manin *pair* (i.e. both V, W fail to be subalgebras). As our computation shows, the weights of the two different orientations are not the same. This is unexpected, since on a closed manifold one could also choose a symmetric propagator, and then one would expect the two contributions to agree.

Remark 5.6.1. We could have computed the effective action on $L_{p,q}$ by starting directly with the formula for the glued propagator for reduced residual fields from [CMR17, Appendix D]. One can check that this produces exactly the same integrals over the boundary. However, for the weight of a particular diagram this is perhaps a more efficient way of computing it.

(a) Θ_A (b) Θ_B

Figure 5.18: Oriented two-loop diagrams. $\gamma \in \{\mathsf{a}, \mathsf{b}\}$.

Let us discuss the results in the two cases (Manin triple vs. General Case) in more detail.

5.6.1 The Manin triple case

In this case, there is only the orientation of theta graph shown in Figure 5.18a. The weight of this graph is given by

$$\psi_{\Theta_A} = \mu_{ij}^k \gamma_k^{ij} \left(\frac{1}{2} s(q,p) + \frac{q+m}{12p} \right). \tag{5.65}$$

This is coherent with the results of Kuperberg, Thurston and Lescop recalled in Section 2.1. Namely, in this case the weight of the theta graph is equal to the Casson-Walker invariant $\lambda_{CW}(L_{p,q}) = \frac{1}{2}s(q,p)$, plus a framing dependent term which changes by $\frac{1}{12}$ when the framing is changed by one unit (recall from Section 5.4 that changing q, m by ± 1 amounts to changing the framing by one unit). Up to this discrepancy, these results reproduce the known results for Theta invariants on lens spaces. However, at the moment it is unclear to the author how exactly the term $\frac{q+m}{12p}$ can be interpreted as a fraction of a Pontryagin number[11] associated with (M_1, M_2, φ). Also notice that in this case the Lie algebra coefficient is the quadratic Casimir $C_2(\mathfrak{g})$, as expected from the closed manifold computation.

The way that the Dedekind sum - i.e. the Casson-Walker invariant - in (5.65) arises in Appendix B.1.2 is encouraging. Namely, it comes from computing an integral involving a product of a pullback of delta currents. These have the effect of localizing the integral at the intersection of the meridian circle on M_1 and the image of the meridian circle under φ. Along these circles the integrand is the sawtooth function, summing over the intersection points yields precise the Dedekind Sum. This bears great resemblance with the combinatorial formulae for the Theta Invariant given in [Les15; Les16] in terms of Heegard diagrams. A possible generalization of this is the following conjecture, which at the moment has the status of wild speculation at best.

Conjecture 5.6.2. *Let M be a rational homology 3-sphere presented in terms of a genus g Heegard diagram $(\alpha_i(t), \beta_i(t))_{i=1}^g)$. Then, the Casson-Walker invariant is given by*

$$\sum_{x \in \alpha_i \cap \beta_j} ((x))_{\alpha_i} ((x))_{\beta_j} \tag{5.66}$$

where $((x))_\gamma = ((\gamma^{-1}(x)))$ is the sawtooth function along a curve γ.

A problem with this conjecture is that in higher genus, there is no good choice for the parametrization of the curves α_i, β_i which on the torus is simply given by the meridians where the longitudinal coordinate is an integer (this can be achieved rotational invariance). See also the discussion in section 6.3.

[11] In most treatments of perturbative Chern-Simons theory in the literature, the correctional term is controlled by some version of a Pontryiagin number.

5.6.2 The general case

In the general case, we can also have other orientation of the theta graph (Figure 5.18b). The weight of this graph is

$$\frac{1}{2}\psi_{ijk}\varphi^{ijk}\left(s(q,p)+\sum_{k=0}^{p-1}\eta_{S^1}(k/p)f(qk/p)+\eta_{S^1}(k/p)f(mk/p)+\frac{H_{1/p}(q+m)}{2\pi^2}\right). \qquad (5.67)$$

In particular, it does not agree with the weight of the other orientation of the theta graph (5.65). Let us discuss in more detail why this result is curious.

In Chern-Simons theory on a closed manifold, the propagator can be chosen to be symmetric, and of course in that case the weights of both orientations of the Theta graph agree. Of course in our computations we use an asymmetric propagator, but in principle this should not change the weight of the graph.

Notice that in principle (5.67) also is an invariant of framed lens spaces, but one that changes by an irrational number as one changes the framing. There are two related facts that might play a role here.

A general remark is that, if we require the eccentricity of the splitting $(\mathfrak{g}, V, W)$ to be non-zero, the vertex tensors are not unimodular[12] This might in principle destroy the Quantum Master Equation, which ensures that the overall state changes coherently under gauge transformations. The fact that the different orientations of the same diagram are different might be a consequence of this fact. If one forces all structure maps to be unimodular, then the the eccentricity of $(\mathfrak{g}, V, W)$ is zero, and the weight (5.65) is zero, and the discrepancy is no longer there.

On another hand, letting (V, W) be arbitrary means that the polarization we choose is incompatible with the algebraic structure on the space of fields, i.e. the map $\mathcal{F}^\partial \to \mathcal{B}^\mathcal{P}$ associated to the polarization is not a Lie algebroid morphism (notice that it is when both base and fibers are subalgebras). This might be related to the influence of the polarization on the quantization that was observed in the geometric quantization of symplectic groupoids integrating Poisson manifolds, see e.g. [Wei91] and [Haw08]. So, if one imposes the condition that the splitting should respect the algebraic structure, the weight (5.67) vanishes and therefore the discrepancy between the weights of the two graphs disappears. Notice that in the completely regular case (i.e. V, W are unimodular subalgebras), both of the weights are zero, and there is not much left to discuss.

[12] Meaning that $\mu^i_{ik}, \lambda^{ik}_i$ does not vanish for all k.

Chapter 6

A Cohomological Approach to Feynman Diagrams

6.1 Evaluation through cohomological approach

In this section we discuss an alternative approach to evaluate Feynman diagrams on manifolds with boundary. This method can be summarized as follows. Let M be a manifold with only one boundary component, which we give the $\mathbb{A}$ representation, and let us choose a propagator η and residual fields χ^i, χ_i induced by a contracting triple as explained in section 3.2. Let Γ be a Feynman graph of the theory with associated differential form ω_Γ (a product of k propagators and residual fields). By Lemma 3.1.2, the differential $\mathrm{d}\omega_\Gamma$ also can be expressed in terms of propagators and residual fields, but there will be one edge less, i.e. there will only be $k-1$ propagators in the product. So, assuming that we know the pushforwards over the bulk of all diagrams with one edge less, we can compute the differential of the pushforward by the Stokes' theorem for pushforwards along fibers with boundary,

$$\pi_*\mathrm{d}\omega = \pi_*^\partial\omega + (-1)^{\dim F}\mathrm{d}\pi_*\omega \tag{6.1}$$

where F denotes the fiber of the projection π (see e.g. [BT94], Appendix E). The restriction to the boundary of ω_Γ in this case (only one boundary component) contains less propagators since by the boundary conditions for the propagator the only boundary components which contribute are the ones where two bulk points with a propagator between them collide. Apart from knowing the differential of $\pi_*\omega_\Gamma$, we also have information about its behavior when integrated against cohomology classes of the boundary (see Lemma 6.1.4). Together with some information on the cohomology of the boundary configuration space, this can help to identify weights of low-order diagrams. In this section we will restrict ourselves to splittings of $\mathfrak{g}$ respecting the Manin triple condition.

6.1.1 Propagators on the boundary from propagator in the bulk

We will start by analyzing how chain contractions on differential forms can relate relative cohomology classes and cohomology classes on the boundary. This will allow us to extract conditions on when one can extract a boundary propagator from the bulk propagator.
We assume that M has a non-empty boundary ∂M so that the short exact sequence of complexes

$$0 \to \Omega^\bullet(M, \partial M) \xrightarrow{i} \Omega^\bullet(M) \xrightarrow{j} \Omega^\bullet(\partial M) \to 0 \tag{6.2}$$

is nontrivial. Let $\chi_i \in \Omega(M, \partial M)$ be representatives of a basis of $H^\bullet(M, \partial M)$, and χ^j representatives of the dual basis of $H^\bullet(M)$, yielding a system R of representatives. Let (ι_R, p_R, K) be a continuous contracting triple (see 3.1)) of $\Omega^\bullet(M, \partial M)$. Denote by $\eta \in \Omega^{d-1}(C_2(M))$ the integral kernel of K. The conventions on the propagator are that

$$K\omega = \pi_{1,*}\pi_2^*\omega\eta_{12} = \int_{M_2} \eta_{12}\omega_2. \tag{6.3}$$

The purpose of this section is to prove the following result.

Theorem 6.1.1. *Suppose that the map i_* in the short exact sequence 6.2 vanishes identically. Then*

i) The set $\{j(\chi^i), \psi_i := j(K[i\chi_i])\}$ gives representatives of a basis of $H^k(\partial M)$,

ii) $j(\chi^i)$ is Poincaré dual to ψ_i,

iii) The form[1]

$$\widehat{\eta}_{12} := \int_{M_0} \eta_{01}\eta_{02} \tag{6.4}$$

defines a propagator for abelian BF theory on ∂M with choice of residual fields $j(\chi^i), \psi_j$.

The condition of this theorem is rather strong, but it holds e.g. in the case of handlebodies - the main focus of interest in this work. We postpone the proof of this theorem and establish some facts that are needed for the proof first.

[1]Again, the notation means that we integrate the point labelled 0 over M.

Relating cohomology classes

Define a map $\tilde{K}\colon \Omega^k(\partial M) \to \Omega^k(M)$ by

$$\tilde{K}\alpha = \int_{\partial M_2} \eta_{12}\alpha = \pi_{1,*}\eta_{12}\pi_2^*\alpha \tag{6.5}$$

Proposition 6.1.2. *The map $\tilde{K}\colon \Omega^\bullet(\partial M) \to \Omega^\bullet(M)$ is a section of j, i.e. the short exact sequence* (6.2) *is split by $\tilde{K}$.*

Proof. Let $\tilde{\alpha} = \tilde{K}\alpha$. We want to show that $j(\tilde{\alpha}) = \iota^*_{\partial M}\tilde{\alpha} = \alpha$, so let $x_1 \in \partial M$. For $x_1 \in \partial M$, the fiber of π_1 has two strata: Stratum $S_I = \partial M - \{x_1\}$ on which the propagator vanishes due to boundary conditions, and stratum S_{II}, the "infinitesimal" half-sphere H_{x_1} around x_1 on which the propagator integrates to ± 1 (see [CMR17, Section 3.3]). Therefore

$$\begin{aligned}\tilde{K}(\alpha)_{x_1} &= \pi_{1,*}(\eta_{12}\alpha_2)\\ &= \int_{S_I} \underbrace{\eta_{12}}_{0}\alpha_2 + \int_{S_{II}} \underbrace{\eta(x_1,x_2)}_{\equiv\pm 1}\alpha(x_1)\\ &= 0 + \alpha(x_1).\end{aligned}$$

□

Denote by $\delta\colon H^k(M) \to H^{k+1}(M, \partial M)$ the induced map in the long exact sequence

$$\ldots \xrightarrow{i_*} H^k(M) \xrightarrow{j_*} H^k(\partial M) \xrightarrow{\delta} H^{k+1}(M,\partial M) \xrightarrow{i_*} H^{k+1}(M) \xrightarrow{j_*} \ldots \tag{6.6}$$

The section above allows to give explicit representatives for this map.

Proposition 6.1.3. *If $\alpha \in \Omega^k(\partial M)$ is closed then $\mathrm{d}\tilde{K}\alpha \in \Omega^{k+1}(M, \partial M)$ and this induces the map δ in cohomology.*

Proof. This is a a completely standard argument in homological algebra. □

The interesting statement for us is the following.

Lemma 6.1.4. *Suppose that j_* is injective on $H^k(M)$. Let $[\chi^{i_1}], \ldots, [\chi^{i_l}]$ be a basis of $H^k(M)$ and extend $j_*([\chi^{i_1}]), \ldots, j_*([\chi^{i_l}])$ to a basis of $H^\bullet(\partial M)$ with dual basis (with respect to Poincaré pairing) $[\psi_{i_1}], \ldots, [\psi_{i_l}], \ldots$. Then*

$$\mathrm{d}\tilde{K}\psi_{i_j} = \chi_{i_j}, \tag{6.7}$$

where χ_{i_j} is the Poincaré dual to χ^{i_j}.

Proof. Notice that the fibers of $\pi_1 \colon C_{1,1}(M, \partial M) \to M$ have empty boundaries over bulk points. Hence, we can apply Stokes theorem to compute

$$\begin{aligned}
\mathrm{d}\tilde{K}\psi_{i_j} &= \mathrm{d}\pi_{1,*}\eta_{12}\pi_2^*\psi_{i_j} \\
&= \pi_{1,*}\mathrm{d}\eta_{12}\pi_2^*\psi_{i_j} \\
&= \pi_{1,*}\sum_i \chi_{i,1}\chi_2^i\psi_{i_j,2} \\
&= \sum_i \delta^i_{i_j}\chi_{i,1} \\
&= \chi_{i_j,1}
\end{aligned}$$

where we have applied the notation $\pi_k^*\chi_i = \chi_{i,k}$ and used that by definition $\pi_{1,*}\chi^i\psi_j = \delta^i_j$. □

On the other hand, we have the following statement.

Lemma 6.1.5. *Suppose that the map j_* is injective on $H^{d-k}(M)$. Then*

$$\int_{\partial M} j(\chi^i) j(K[i\chi_j]) = \pm\delta^i_j \tag{6.8}$$

for those representatives χ^i of form degree k and χ_i of form degree $d-k$ respectively.

Proof. Again, we use Stokes' theorem to compute

$$\begin{aligned}
\int_{\partial M} j(\chi^i) j(K[i(\chi_j)]) &= \int_M \mathrm{d}(\chi^i K[\chi_j]) \\
&= \int_M \chi^i \mathrm{d}K[i(\chi_j)] = \int_M \chi^i (K\mathrm{d}i(\chi_j) \pm i(\chi_j) \pm p[i(\chi_j)]).
\end{aligned}$$

Since the χ_j are closed we have $\mathrm{d}\chi_j = 0$. On the other hand, the assumption that j_* is injective at level $d-k$ implies that i_* is zero, which implies that $p \circ i$ vanishes. Hence the integral equals $\pm\int_M \chi^i\chi_j = \pm 1$. □

We now have enough material to prove the theorem.

Proof of theorem 6.1.1. By assumption, there is a short exact sequence

$$0 \to H^k(M) \to H^k(\partial M) \to H^{k+1}(M, \partial M) \to 0.$$

First, notice that the assumption of the theorem makes jKi a morphism of complexes[2]:

$$\mathrm{d}jKi = j\mathrm{d}Ki = jK\mathrm{d}i \pm \underbrace{ji}_{0} \pm \underbrace{jpi}_{0} = jKi\mathrm{d}$$

[2]We suppress composition symbols for brevity.

where in the third equality, the second term vanishes trivially, and the third by assumption. Therefore the map descends to cohomology. We first prove that $(jKi)_*$ is injective. Indeed, suppose $\alpha \in \Omega^\bullet(M, \partial M)_{cl}$ satisfies that $jKi\alpha = \mathrm{d}\beta$ is exact. Then, we have

$$\int_{\partial M} j(\chi^i) jKi\alpha = \int_{\partial M} j(\chi^i)\mathrm{d}\beta = 0$$

for all χ^i. Since α is closed, it has a decomposition $\alpha = \lambda^i \chi_i + \mathrm{d}\gamma$. But now Lemma 6.1.5 implies that all λ_i vanish, hence α is exact. We conclude that $(jKi)_*$ is injective.

Next, we claim that $jKi(H^\bullet(M, \partial M)) \cap j(H^\bullet(M)) = \{0\}$. This follows immediately from the nondegeneracy of the Poincaré pairing and the fact that the Poincaré pairing vanishes on $j(H^\bullet(M))$ (by Stokes' theorem). It follows that $j(\chi^i), \psi_i = jKi(\chi_i)$ form a basis of $H^\bullet(\partial M)$. This is point i) of the theorem. Lemma 6.1.5 also implies point ii). Finally let us prove point iii). First, compute the differential of $\widehat{\eta}$ (here π_{12} denotes projection to the boundary point in $C_{1,2}(M, \partial M)$):

$$\begin{aligned} \mathrm{d}\pi_{12,*}\eta_{01}\eta_{02} &= \pi_{12,*}\mathrm{d}\eta_{01}\eta_{02} \pm \eta_{01}\mathrm{d}\eta_{02} \\ &= \sum_i \pi_{12,*}\chi_{i,0}\chi^i_1\eta_{02} \pm \eta_{01}\chi_{i,0}\chi^i_2 \\ &= \sum_i \pm\chi^i_1\psi_{i,2} \pm \psi_{i,1}\chi^i_2. \end{aligned}$$

Second, we have to check that $\widehat{\eta}$ behaves correctly on the boundary of the configuration space $\partial C_2(\partial M)$. Namely, we want to prove that the pushforward of $\widehat{\eta}_{12}$ along the fibration

$$\partial C_2(\partial M) \to \partial M_1 \tag{6.9}$$

is the constant function 1 (i.e. we want to fix the first point and integrate over the second). To this end, notice $\widehat{\eta}$ is itself a pushforward along the fibration

$$C_{1,2}(M, \partial M) \to C_2(\partial M) \tag{6.10}$$

whose pullback to the boundary of $C_2(\partial M)$ is

$$\partial C_2(\partial M) \times_{\partial M} C_{1,1}(M, \partial M) \to \partial C_2(\partial M). \tag{6.11}$$

Hence we are interested in the pushforward of $\eta_{01}\eta_{02}$ along the composition of fibrations (6.9) and (6.11), namely

$$\pi_1^\partial \colon \partial C_2(\partial M) \times_{\partial M} C_{1,1}(M, \partial M) \to \partial M. \tag{6.12}$$

Now, consider the fibration

$$\pi \colon C_2(\partial M) \times_{\partial M} C_{1,1}(M, \partial M) \to \partial M \tag{6.13}$$

where we consider the first factor as a fiber bundle over ∂M via the map $p_2\colon C_2(\partial M) \to \partial M$ which projects to the second point. The boundaries of the fibers of π are disjoint unions of the fibers of π_1^∂ and the fibers of

$$\pi_2^\partial\colon \partial C_2(\partial M) \times_{\partial M} \partial C_{1,1}(M, \partial M) \to \partial M. \tag{6.14}$$

Hence, we can apply Stokes' theorem for integration along the fiber as

$$\mathrm{d}\pi_*\eta_{01}\eta_{02} = \pi_*\mathrm{d}\eta_{01}\eta_{02} + \pi_1^\partial\eta_{01}\eta_{02} + \pi_2^\partial\eta_{01}\eta_{02}. \tag{6.15}$$

We claim that

$$\begin{aligned}\mathrm{d}\pi_*\eta_{01}\eta_{02} &= 0,\\ \pi_*\mathrm{d}\eta_{01}\eta_{02} &= \pm 1,\\ \pi_2^\partial\eta_{01}\eta_{02} &= 0.\end{aligned}$$

The last equality follows from the fact that η_{02} vanishes for $x_0 \in \partial M$. The first equality follows from degree reasons: Integrating $\eta_{01}\eta_{02}$ over point $x_2 \in \partial M$ yields η_{01} by normalization of the propagator. Integrating this over point $x_0 \in M$ yields 0, since η is a $d-1$ form. So we are left with proving the second equation. To see this, recall that

$$\mathrm{d}\eta_{01} = \sum_i \chi_{i,0}\chi^{i,1}.$$

Since M is compact and connected, the cohomology $H^0(M)$ is one-dimensional and spanned by the constant function 1. We can assume that the corresponding residual fields (denoted χ^0, χ_0) are normalized:

$$\chi^0 \equiv 1, \int_M \chi_0 = 1. \tag{6.16}$$

Now, notice that the only terms which survives in $\pi_*\mathrm{d}\eta_{01}\eta_{02}$ are the ones of form degree d in the bulk and form degree $d-1$ at point 2 in the boundary. Also, notice that the total degree of the form is $2d-1$, so there is a nonzero contribution only if there is no form degree concentrated at point 0. There are only two terms for which this is possible: The first arises when the differential hits the first propagator and produces a volume form at the bulk point and the constant function at the first boundary point: $\chi_1^0\chi_{0,0}\eta_{02}$. The pushforward of this is 1, by normalization of the propagator (notice that normalization of the volume form does not matter). The other possibility is that there is a class in $H^{d-1}(M)$ which restricts to a volume form on the boundary. However this is ruled out by the assumptions: We know that there is a short exact sequence

$$0 \to H^{d-1}(M) \to H^{d-1}(\partial M) \to H^d(M, \partial M) \to 0$$

in which the two right hand terms are 1-dimensional. Hence $H^{d-1}(M)$ vanishes. This finishes the proof that $\widehat{\eta}$ is normalized on the boundary, and therefore the proof of theorem 6.1.1. □

Remark 6.1.6. The proof, together with Lemma 6.1.4 shows a little more, namely:

- The exact sequence

$$0 \to H^{k-1}(M) \to H^{k-1}(\partial M) \to H^k(M, \partial M) \to 0$$

  is split by the map $(jKi)_*$.

- $j(H^{k-1})(M)$ is a Lagrangian subspace of $H^{k-1}(\partial M)$.

- $jKi(H^k(M, \partial M))$ is Lagrangian if the product $\chi_i\chi_j = 0$ for all i, j (this happens e.g. in the case of handlebodies, for degree reasons).

The second point maybe requires some justification. That the Poincaré pairing vanishes on $j(H^\bullet(M))$ is a consequence of Stokes' theorem:

$$\int_{\partial M} \chi^i \chi^j = \int_M \mathrm{d}\chi^i \chi^j = 0.$$

Similarly, for $\psi_i = jKi(\chi_i)$ we have

$$\int_{\partial M} \psi_i \psi_j = \int_{M_0} \mathrm{d} \int_{M_{12}} \chi_{i,1}\eta_{10}\chi_{j,2}\eta_{20}$$

All terms appearing in the differential are proportional to $\chi_i\chi_j$.

6.1.2 Invariant differential forms on the solid torus

The cohomological approach to the evaluation of Feynman diagrams gets greatly simplified by its symmetries. Namely, the solid torus M has an action of the usual 2-torus $T^2 = \partial M$ by rotations (this action comes from identifying the solid torus with $S^1 \times D$ and realizing D as the unit disk). We denote a point in the two-torus T^2 by $(t, \theta) \in (\mathbb{R}/\mathbb{Z})^2$ and a point in the solid torus by $(t, z) \in \mathbb{R}/\mathbb{Z} \times D$. Then, the action is given by

$$(t, \theta).(t', z) = (t + t', \exp(2\pi \mathrm{i}\theta)z). \tag{6.17}$$

On the boundary torus of the solid torus, this action reduces to the multiplication in the Lie group. Now, we can use the fact that on the solid torus there exist T^2-invariant propagators (e.g. the ones induced by T^2-invariant metrics, but also the axial gauge propagator of sections 5.3.1, 5.3) to simplify our computations, from the following elementary fact:

Proposition 6.1.7. *Let $\omega \in C_{n,m}(M, \partial M)$ be a T^2 invariant form. Then for any $n_2 \leq n_1, m_2 \leq m_1$ its pushforward along the fibers of $C_{n_1,m_1}(M, \partial M) \to C_{n_1,m_1}(\partial M)$ is also T^2-invariant.*

Proof. This follows immediately from the fact that the pushforward commutes with the pullback: Let L_g be the diffeomorphism induced by $g \in T^2$ (on both $C_{n,m}(M, \partial M)$ and $C_m(\partial M)$, then $L_g^* \pi_* \omega = \pi_* L_g^* \omega = \pi_* \omega$. □

The invariant forms on T^2 are of course dual to the Lie algebra of the torus: It is the linear span of the constant forms $1, dt, d\theta, dtd\theta$. On the solid torus, the invariant forms are the ones only depending on the radial coordinate. Of course, on $C_n(T^2)$ or $C_n(M)$ there are a lot more invariant forms for $n > 1$, e.g., all forms that depend only on the difference $t_i - t_j$, $\theta_i - \theta_j$.

6.1.3 Weights of Feynman diagrams from invariant propagators

In this section we will assume that we have a propagator on the solid torus for a choice of residual fields $\chi_1 = \mu dt, \chi_2 = \mu, \chi^2 = dt, \chi^1 = 1$, which also satisfies the assumptions

$$\int_2 \eta_{12}\mu_2 = \int_2 \eta_{12}\mu_2 dt_2 = 0, \tag{6.18}$$

$$\int_1 dt_1 \eta_{12} = 0, \tag{6.19}$$

$$\int_2 \eta_{12}\eta_{23} = 0, \tag{6.20}$$

These assumptions are satisfied e.g. by Riemann-Hodge propagators of T^2-invariant metrics, or the axial gauge propagator. It turns out that we can evaluate low-order Feynman diagrams just from the assumptions on the propagator.

Proposition 6.1.8. *Let η be an invariant propagator on $M = S^1 \times D$ satisfying the assumptions above. Then, there exists a function $f\colon [0,1] \to \mathbb{R}$ and a 1-form $\alpha \in \Omega^1(M)$ such that we have*

$$\int_1 \mu_1 \eta_{12} = \psi_2 + \mathrm{d}f(r_2), \tag{6.21}$$

$$\int_1 \mu_1 dt_1 \eta_{12} = \psi_2 dt_2 + \mathrm{d}\alpha, \tag{6.22}$$

$$\int_2 \eta_{12} dt_2 = 0, \tag{6.23}$$

where we put $\psi = \frac{1}{4\pi \mathrm{i}}(\bar{z}dz - zd\bar{z})$. *Moreover, if* $x_2 \in \partial M$ *we have*

$$\int_1 \mu_1 \eta_{12} = d\theta_2 \tag{6.24}$$

$$\int_1 \mu_1 dt_1 \eta_{12} = d\theta_2 dt_2 \tag{6.25}$$

$$\int_2 \eta_{01} dt_1 = dt_0 \tag{6.26}$$

Proof. Let us start with (6.21). Here we consider $\int_1 \mu_1\eta_{12} = K[\mu]$. The properties of the propagator imply that $dK\mu = \mu$. Since $\mathrm{d}\psi = \mu$, the 1-form $\beta = K\mu - \psi$ is closed and hence of the form $\beta = \lambda dt + df$, for some function f on M and a constant $\lambda \in \mathbb{R}$. Since

$$\int_M \beta\mu = \int_M (\beta + \psi)\mu = \int_{M,12} \mu_1\eta_{12}\mu_2 = 0,$$

where the last equality follows from assumption (6.18), we have $\lambda = 0$ and hence $\beta = \mathrm{d}f$. But since β is T^2-invariant, so is f, since for $X \in \{\partial_t, \partial_\theta\}$,

$$0 = L_X\beta = L_X\mathrm{d}f = \mathrm{d}L_x f$$

which implies that $L_X f$ is constant (since f is periodic in t and θ). This means that $f \equiv f(r)$. Consequently $\iota^*_{\partial M} df = 0$, which proves (6.24).

Next let us look at (6.22). This is proven exactly in the same way. Note that $\iota^*_{\partial M}\mathrm{d}\alpha$ is an exact invariant form. This implies that it is zero.

To see (6.23), we first observe that $\int_2 \eta_{12} dt_2$ is closed. This follows from the Stokes' theorem for integration along the fiber:

$$\mathrm{d}\int_2 \eta_{12}dt_2 = \mathrm{d}\pi_{1,*}\eta_{12}dt_2 = \pi_{1,*}\mathrm{d}\eta_{12}dt_2 + \pi^\partial_{1,*}\eta_{12}dt_2 = \pi_{1,*}(\mu_1 dt_1 dt_2) - dt_2 = 0.$$

Hence, it is a constant function. But by the boundary conditions on the propagator in vanishes on the boundary, hence it is zero.

To prove equation (6.26), take the differential of equation (6.23). This yields

$$0 = \mathrm{d}\int_2 \eta_{12}dt_2 = \int_{12} \mathrm{d}\eta_{12}dt_2 + dt_1 - \int_{\partial M_2} \eta_{01}dt_1.$$

Here the last two terms are the integral over the two boundary faces, one where points 1 and 2 collapse in the bulk and one where the second point is on the boundary. Now the claim follows from the fact that the first term on the left hand side vanishes for degree reasons. □

Equations (6.24) and (6.25) tells us that the choice of residual fields induced on the torus by our bulk gauge fixing is $\chi^1 = 1 = \chi_4, \chi^2 = dt = \chi_3, \chi_1 = d\theta dt = \chi^4, \chi_2 = d\theta = -\chi^3$. Next, we are interested in the forms $K^*\chi_i \in \Omega^\bullet(M)$ defined by (6.5). By theorem 6.1.1 we know that the diagram with two legs evaluates to a propagator $\widehat{\eta}$ on the torus for that choice of residual fields, moreover that propagator is invariant under the T^2 action on the boundary. We will need one further assumption on the bulk propagator. Denote by T_{S^1} the map $C_2(M) \to C_2(M)$ given by extending the map that exchanges the two copies of S^1 in $M \times M$. More precisely, let T_{S^1} be the extension of the restriction of $\bar{T}_{S^1} : M \times M \to M \times M$ to $M \times M - \Delta$, where $\bar{T}_{S^1}$ is defined by

$$\bar{T}_{S_1} : (S^1 \times D) \times (S^1 \times D) \to (S^1 \times D) \times (S^1 \times D)$$
$$((t_1, z_1), (t_2, z_2)) \mapsto ((t_2, z_1), (t_1, z_2))$$

Then we say that the bulk propagator is S^1-odd if

$$T^*_{S^1}\eta = -\eta.$$

Examples of S^1-odd propagators are e.g. the ones defined by product metrics[3], or the axial gauge propagator (with the definition adapted to distributional propagators). We now make the following claim:

Proposition 6.1.9. *If the propagator on the solid torus is S^1-odd, the induced propagator on the torus vanishes when integrated against the induced residual fields. More precisely, we have*

$$\int_2 \widehat{\eta}_{12} dt_2 = \int_2 \widehat{\eta}_{12} d\theta_2 = \int_2 \widehat{\eta}_{12} d\theta_2 dt_2 = 0. \tag{6.27}$$

Proof. Since the forms $\int_2 \widehat{\eta}_{12} dt_2, \int_2 \widehat{\eta}_{12} d\theta_2, \int_2 \widehat{\eta}_{12} d\theta_2 dt_2$ are all invariant, they are linear combinations of 1 resp dt and $d\theta$. Hence the claim follows by showing that

$$\int_{0,1,2} \eta_{01}\eta_{02} dt_1 dt_2 d\theta_2 = \int_{0,1,2} \eta_{01}\eta_{02} d\theta_1 dt_2 d\theta_2 = 0,$$

since these are the projections to the subspace spanned by 1 of $\int_2 \widehat{\eta}_{12} dt_2, \int_2 \widehat{\eta}_{12} d\theta_2$ or the projections to the subspaces spanned by $dt, d\theta$ of $\int_2 \widehat{\eta}_{12} dt_2 d\theta_2$. Vanishing of the first integral follows directly from (6.26): Performing first integral over point 1 we get

$$\int_{0,1,2} \eta_{01}\eta_{02} dt_1 dt_2 d\theta_2 = \int_{0,2} \eta_{02} dt_0 dt_2 d\theta_2 = 0.$$

For the second integral we have to work a bit more, and the proof relies on the assumption of S^1-oddity. By Lemma 6.1.4, we have know that $\alpha = K^* d\theta = \int_{\partial M_2} \eta_{02} d\theta_2$ and $\beta =$

[3] This follows from the heat kernel expansion 3.70.

$K^*d\theta dt = \int_{\partial M_2} \eta_{02} d\theta_2 dt_2$ are primitives of μ and μdt respectively. Moreover, since $\int_M \mu\beta = \int_{M_1\times\partial M_2} \mu\eta_{12} d\theta_2 = \int_{\partial M_1} d\theta_1 \wedge d\theta_1 = 0$, we know that up to exact invariant forms we have $\alpha = \psi$ and $\beta = \psi dt$, where $\psi = (4\pi \mathrm{i})^{-1}(\bar{z}dz - zd\bar{z}) = r^2 d\theta$ (note that in our normalization $\mu = 2rdrd\theta$). We have

$$\int_{0,1,2} \eta_{01}\eta_{02} d\theta_1 dt_2 d\theta_2 = \int_M \alpha \wedge \beta = \int_M (\psi + df)(\psi dt + d\tau)$$

where $\mathrm{d}f, \mathrm{d}\tau$ are invariant forms. We now claim that this integral vanishes. Since $\psi^2 = 0$ and $d\tau$ vanishes on the boundary, this is equivalent to saying that

$$\int_M \psi \mathrm{d}\tau = -\int_M \mathrm{d}f \psi dt.$$

For this, we first claim that actually a possible choice for f is $f(x_0) = \int_1 \eta_{01}\psi_1$. Indeed, by the fiberwise Stokes theorem we have

$$\int_1 \mathrm{d}(\eta_{01}\psi_1) = \mathrm{d}\int_1 \eta_{01}\psi_1 + \psi_1 - \int_{\partial M_1} \eta_{01}\psi_1.$$

The first term on the right hand side is df and the last is α. On the other hand

$$\int_1 \mathrm{d}(\eta_{01}\psi_1) = \int_1 (\mathrm{d}\eta_{01})\psi_1 + \int_1 \eta_{01}\mu_1.$$

The first term vanishes for degree reasons and the second term vanishes by assumption 6.18. Hence $\alpha = \psi + \mathrm{d}\int_{01} \eta_{01}\psi_1$. Since f vanishes on the boundary we have

$$\int_M df \psi dt = \int_M f\mu dt = \int_{M,0,1} \eta_{01}\psi_1\mu_0 dt_0 = -\int_{M,0,1} \mu_0 dt_0 \eta_{01}\psi_1.$$

Similarly, we can show that a possible choice for τ is $\tau = \int_{01} \eta_{01}\psi_1 dt_1$ and then

$$\int_M \psi d\tau = \int_M \mu\tau = \int_{M,0,1} \mu_0\eta_{01}\psi_1 dt_1.$$

By our assumption that the propagator is S^1-odd, the claim follows.

□

With the material already collected in this section, we can compute almost all 1-point diagrams in the Manin triple case. We are only missing one single piece, namely the diagram with two legs and an b residual field (figure 5.4c on page 80). Here we get a term $\int_0 dt_0\eta_{01}\eta_{02}$. For this term, we have the following claim.

Proposition 6.1.10. *Let $\widehat{\eta}$ be the boundary propagator induced by a S^1-odd bulk propagator η_0. Then $\int_0 dt_0\eta_{01}\eta_{02}$ is cohomologous to $\widehat{\eta}\frac{dt_1+dt_2}{2}$, i.e we have*

$$\int_0 dt_0\eta_{01}\eta_{02} = \widehat{\eta}\frac{dt_1 + dt_2}{2} + d\lambda \tag{6.28}$$

where λ is a 1-form on $C_2(M)$ that satisfies $\pi_*^\partial\lambda \equiv 0$.

Proof. First, it is a straightforward check that the differentials of the two forms agree. Hence their difference is a closed form. We want to show that its cohomology class is zero. But this follows from the well-known fact that for any compact manifold the cohomology of the 2-point configuration space satisfies

$$H^\bullet(C_2(M)) = \frac{H^\bullet(M) \otimes H^\bullet(M)}{(\Delta)}, \tag{6.29}$$

where Δ is the diagonal class. For the case of the torus this means that a cohomology class vanishes if its Poincaré pairing with any product of cohomology classes dt_i and $d\theta_i$ vanishes (where $i = 1, 2$). In our case, proposition 6.1.9 shows that both sides of (6.28) vanish when integrated against any such product. For the last statement notice that

$$\pi_*^\partial \lambda = \pi_{2,*} \mathrm{d}\lambda = \pi_{2,*} \int_0 dt_0 \eta_{01}\eta_{02} - \widehat{\eta}\frac{dt_1 + dt_2}{2}$$

but both these integrals vanish independently (the first by degree reasons, the second by Proposition 6.1.9.) □

6.1.4 Summarizing weights of Feynman graphs in the cohomological approach

Let us summarize the findings for the weights of one-point Feynman diagrams. Namely, we find

$$\psi_{\Gamma_{1,0}} = \frac{1}{2}\gamma_i^{jk}(z^{0i}z^+_{0j}z^+_{0k} + 2z^{1i}z^+_{0j}z^+_{1k}) \tag{6.30}$$

$$\psi_{\Gamma_{1,1}} = -\mu_{ij}^k(z^{0,i}z^+_{0,j} - z^{1,i}z^+_{1,j})\int_{\partial M_2} d\theta_2 dt_2 \mathbb{A}_2^k - \mu_{ij}^k z^{1,i}z^+_{0,j}\int_{\partial M_2} d\theta_2 \mathbb{A}_2^k \tag{6.31}$$

$$\begin{aligned}\psi_{\Gamma^\flat_{1,2}} &= \frac{1}{2}\int_{\partial M \times \partial M} \mu^i_{jk} z^+_{1i}\eta_{\mathbb{T}^2,12}\mathbb{A}_1^j\mathbb{A}_2^k \\ &\quad + \frac{1}{2}\int_{\partial M\times\partial M} \mu^i_{jk} z^+_{2i}\left(\eta_{\mathbb{T}^2,12}\frac{dt_1+dt_2}{2} + d\lambda\right)\mathbb{A}_1^j\mathbb{A}_2^k \end{aligned} \tag{6.32}$$

Remark 6.1.11. Even though we showed that the induced propagator on the torus is smooth, we can evaluate (6.32) with the axial gauge propagator (3.64) on the torus. This does not agree with the result (A.12). Indeed, the difference is

$$\frac{1}{4}\int_{\partial M\times\partial M} \mu^i_{jk} z^+_{2i}\eta_{S^1}(t_1,t_2)(d\theta_2(dt_2 - dt_1) + d\theta_1(dt_2 - dt_1))\mathbb{A}_1^j\mathbb{A}_2^k,$$

which is

$$\frac{1}{4}\int_{\partial M\times\partial M} \mu^i_{jk} z^+_{2i}\mathrm{d}\zeta(t_1 - t_2)\mathbb{A}_1^j\mathbb{A}_2^k$$

where ζ is defined by

$$\zeta(t) = 1/2(|t| - t^2)(d\theta_1 + d\theta_2)$$

for $-1 \leq t \leq 1$, and extended periodically[4]. Note that the distributional derivative of $|x|$ is $sgn(x)$ so that $\mathrm{d}\zeta(t_1 - t_2) = (\frac{1}{2}sgn(t_1 - t_2) - t_1 + t_2)(dt_1 - dt_2)(d\theta_1 + d\theta_2)$. This gives some evidence for the expectation[5] that the difference of two states should be $(\hbar^2\Delta + \Omega)$-exact, since the first term in Ω is the lift of the de Rham differential $\Omega_0 = \int_{\partial M} \mathrm{d}\mathbb{A}\frac{\delta}{\delta\mathbb{A}}$.

6.2 Gluing Lens Spaces, Again

We can use the results of the section above to give another computation of (parts of) the effective action on lens spaces. Namely, we can pick a propagator $\eta_{\mathbb{T}^2}$ on the torus and evaluate

$$P_*(\psi_{\Gamma^{\mathfrak{b},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathfrak{a},\mathbb{B}}_{1,2}}) = \frac{1}{2p}\mu^i_{jk}\gamma^{jk}_i \int_{C_2(\mathbb{T}^2)} \left(\eta_{\mathbb{T}^2,12}\frac{dt_1 + dt_2}{2} + d\lambda_{12}\right) \varphi^* \left(\eta_{\mathbb{T}^2,12}\frac{dt_1 + dt_2}{2} + d\lambda_{12}\right). \tag{6.33}$$

Here we already reduced the residual fields. This integral can be rewritten as

$$\begin{aligned} &\int_{C_2(\mathbb{T}^2)} \left(\eta_{\mathbb{T}^2,12}\frac{dt_1 + dt_2}{2}\right) \varphi^* \left(\eta_{\mathbb{T}^2,12}\frac{dt_1 + dt_2}{2}\right) \\ &+ \int_{C_2(\mathbb{T}^2)} \eta_{\mathbb{T}^2,12}\frac{dt_1 + dt_2}{2} \left(\varphi^*\mathrm{d}\lambda - \left(\varphi^{-1}\right)^* \mathrm{d}\lambda\right) \\ &+ \int_{C_2(\mathbb{T}^2)} \lambda \mathrm{d}\lambda. \end{aligned}$$

One possible choice for the propagator on the torus is the "standard" one, i.e. the one corresponding to the standard metric, derived in C, which can be expressed in terms of Jacobi theta functions by

$$\eta^{std}_{\mathbb{T}^2}(z_1, z_2) \equiv \eta^{\mathrm{i}}(z_1, z_2) = \frac{1}{2\pi}\mathrm{d}\arg\vartheta_1(z_1 - z_2|\mathrm{i}) \tag{6.34}$$

(see Appendix C for the conventions on Theta functions). Another possible choice is the axial gauge propagator 3.64, given by

$$\eta^{ax}_{\mathbb{T}^2}((t_1, \theta_1), (t_2, \theta_2) = \eta_{S^1}(\theta_1, \theta_2)\delta^{(1)}(t_1, t_2) + \eta_{S^1}(t_1, t_2)(d\theta_2 - d\theta_1). \tag{6.35}$$

For these propagators it is possible to compute the integrals above not containing $d\lambda$. For the standard propagator on the solid torus we cannot evaluate those, but we have the following conjectures ordered by increasing strength:

[4]Note that $\zeta(t)$ satisfies $\zeta(t + 1) = \zeta(t)$ for $-1 \leq t \leq 0$.

[5]For propagators that extend to configuration spaces it was proven in [CMR17], but for distributional propagators there is no general result yet.

Conjecture 6.2.1. *There is a propagator on the solid torus inducing the standard propagator for which the form* λ *satisfies:*

$$\iota_\partial^* \lambda_{12} = 0 \tag{6.36}$$

$$\int_{M_{1,2}} \lambda_{12}\gamma_1\delta_2 = 0 \tag{6.37}$$

where $\gamma, \delta \in \{dt, d\theta, d\theta dt\}$*.*

A stronger version of this is:

Conjecture 6.2.2. *There is a propagator on the solid torus which induces the standard propagator on the boundary and for which* λ *vanishes.*

A yet stronger version of this conjecture is:

Conjecture 6.2.3. *This propagator is the one corresponding to the standard metric on the solid torus.*

Notice however that Conjecture 6.2.1 is enough to ensure vanishing of the terms in (6.33) involving γ, which can be seen integrating by parts.

6.2.1 Weight of the theta graph

For a bulk propagator satisfying Conjecture 6.2.1 we can now compute the weight of the theta graph on the glued lens space, up to terms that come from loops on the solid torus. Possibly, there are also contributions from 2-point trees after gluing. For them we have the following result.

Lemma 6.2.4. *Contributions of two-point trees to the theta graph vanish after gluing and reducing the residual fields, if the bulk propagator is* S^1*-odd.*

Proof. The theta graph has degree 0 in the reduced residual fields. Since for two-point trees there is an $\mathsf{a}^{\mathbb{B}}$ residual field glued to every boundary vertex, so we need a two-point tree exactly the same number of $\mathsf{b}^{\mathbb{A}}$ residual fields and boundary vertices and no $\mathsf{a}^{\mathbb{A}}$ residual fields. A quick check of figure 5.6 on 82 shows that the only possible 2-point tree is 5.6b with two $\mathsf{b}^{\mathbb{A}}$ residual fields. The only possibly nonzero contribution appears when we take the dt component of both $\mathsf{b}^{\mathbb{A}}$ fields and the $d\theta$ component of both $\mathsf{a}^{\mathbb{B}}$ fields. This yields the integral

$$\int_{M_0,1} K^*[d\theta]_0 dt_0 \eta_{01} dt_1 K^*[d\theta]_1.$$

S^1-oddity now provides an orientation reversing involution *à la Kontsevich* which leaves the integrand invariant (there is an additional minus sign from exchanges the two dt's). □

With this we can finally present the following computation.

Proposition 6.2.5. *Let η be an S^1-odd propagator on the torus satisfying Conjecture 6.2.1. Then we have*

$$I := P_* \psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathsf{a},\mathbb{B}}_{1,2}} = \frac{1}{2}\left(s(q,p) - \frac{m+q}{12p} + \frac{1}{2} + \frac{1}{2\pi}\arg(n-p-\mathrm{i}(m+q))\right) \tag{6.38}$$

Proof. By our assumption that the propagator satisfies Conjecture 6.2.1 we have

$$I = \frac{1}{2p}\int_{C_2(\mathbb{T}^2)} \left(\eta^{std}_{\mathbb{T}^2,12}\frac{dt_1+dt_2}{2}\right)\varphi^*\left(\eta^{std}_{\mathbb{T}^2,12}\frac{dt_1+dt_2}{2}\right).$$

First, we claim that

$$I = \frac{1}{2}\int_{\mathbb{T}^2} \eta^{std}_{\mathbb{T}^2,12}\varphi^*\eta^{std}_{\mathbb{T}^2,12}.$$

This follows from the fact that the standard propagator depends only on the differences $t_{12} = t_1 - t_2, \theta_{12} = \theta_1 - \theta_2$ and is periodic in both t and θ. The factor of p comes from the pullback of $d\theta$. This integral can be computed using the method explained in Appendix C, where we prove first

$$\varphi^*\eta^\tau = \eta^{\tau'}, \tag{6.39}$$

where $\tau' = \left(\left({}^T\varphi\right).\tau\right)$, $\left(\left({}^T\varphi\right)\right.$ is the transpose about the anti-diagonal and $\varphi.\tau$ is the standard $SL(2,\mathbb{Z})$-action on the upper halfplane $\mathbb{H}$. Then we have by proposition C.3.3

$$\int_T \eta^\tau\eta^{\tau'} = s(q,p) - \frac{m+q}{12p} + \frac{1}{2} + \frac{1}{2\pi}\arg(n|\tau|^2 + m\bar{\tau} - q\tau - p). \tag{6.40}$$

In our case $\tau = \mathrm{i}$, so the proof follows. □

Remark 6.2.6. Equation (6.40) implies that if $\tau = \mathrm{i}\beta$ for $\beta > 0$, and we take the limit $\beta \to \infty$, the term $\frac{1}{2\pi}\arg(n|\tau|^2 + m\bar{\tau} - q\tau - p)$ vanishes. In this case the propagator $\eta^{\beta\mathrm{i}}$ approaches the axial gauge propagator, by Remark 3.3.7. Indeed, the integral I can also be computed with the axial gauge propagator, and this yields the same result.

Comparing with the weight of the theta graph computed from the axial gauge on the solid torus given in (5.65), this computation shows that the contribution of the loop diagrams on the solid torus to the weight of the theta graph for a propagator satisfying Conjecture 6.2.1 is

$$\frac{m+q}{12p} + \frac{1}{2} + \frac{1}{2\pi}\arg(n|\tau|^2 + m\bar{\tau} - q\tau - p) \tag{6.41}$$

6.3 Remarks on gluing of arbitrary 3-manifolds

Let us end this Chapter with some very sketchy remarks on how one could extend these methods to the case of arbitrary 3-manifolds. Any compact 3-manifold M can be expressed in terms of a Heegard splitting $M = H_g \cup_\varphi H_g$ (see e.g. [Joh; Sch00a]).

6.3.1 Reducing residual fields

Consider now a Heegard splitting $M = H_g \cup_\varphi H_g$ of arbitrary genus $g \geq 1$, with $\varphi \in \mathrm{MCG}(\Sigma_g)$. The action of φ on $H^1(\Sigma_g)$ by pullback is symplectic, and we denote by

$$S_\varphi = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

its representation in the basis induced by the choice of residual fields on the handlebody. In the same notation as above, we have

$$\begin{aligned} L_1 = L_1^\perp &= \langle 1, dt_1, \ldots, dt_g \rangle \\ L_2 = L_2^\perp &= \langle 1, \varphi^* dt_1, \ldots, \varphi^* dt_g \rangle \\ &= \left\langle 1, \sum_{j=1}^{g} A_{1j} dt_j + C_{1j} d\theta_j \right\rangle \end{aligned}$$

and we want to determine the spaces $L_i^\times$ as complements to $L_1 \cap L_2$ inside L_i. We claim that $L_1 \cap L_2 = A^T \ker C^T$. Indeed, let $\alpha \in L_1 \cap L_2$. Then

$$\alpha = \lambda_1 dt_1 + \ldots + \lambda_g dt_g = \sum_{i,j=1}^{g} \mu_i \left(A_{ij} dt_j + C_{ij} d\theta_j\right).$$

It follows that $\sum_i \nu_i C_{ij} = 0$ for all $j = 1, \ldots, g$, i.e. $(\nu, \ldots, \nu_g) \in \ker C^T$. Then, we have $\lambda_j = \sum_i \nu_i A_{ij}$, i,e. $\lambda = A^T \nu$.

Since φ is invertible, A^T restricted to $\ker C^T$ does not have a kernel, i.e. $\dim L_1 \cap L_2 = \dim \ker C$ and $\dim L_i^\times = \mathrm{rk} C$. In particular, the pairing for the reduction of residual fields is induced by the restriction of C to $L_1^\times$.

6.3.2 Weight of Theta graph

The weight of the theta graph on arbitrary 3-manifolds can be expressed by the Feynman diagrams on handlebodies, given in Section 5.2.5. In the genus 1 case, the main information about the weight was contained in the gluing of diagram $\Gamma^{\mathsf{b}}_{1,2}$ to its counterpart in the $\mathbb{B}$-representation. We no longer have the axial gauge available in higher genus [6] However, from theorem 6.1.1 we know that the weight of this graph can be expressed in terms of a boundary propagator, on a handlebody of higher genus, it should have the form

$$\psi_{\Gamma^{\mathsf{b}}_{1,2}} = \int_{\Sigma_g} \eta_{\Sigma_g}(z,w)\alpha(z,w)\mathbb{A}_1\mathbb{A}_2 \tag{6.42}$$

where Σ_g denotes the Riemann surface of genus g, and α can be expressed as product of the representatives of cohomology on Σ_g induced by restriction from $H^1(H_g)$. Propagators on Riemann surfaces are known explicitly, see[7] e.g. [Fer08], they can be constructed through the Riemann theta Functions on $\mathbb{C}^g$. Explicitly, they take a form similar to (C.23), namely,

$$\eta_{\Sigma_g}(z,w) = \frac{1}{2\pi}\mathrm{d}\arg\Theta(j(z),j(w),\Omega) + \beta(z,w) \tag{6.43}$$

where Θ denotes the Riemann theta function on $\mathbb{C}^g \times \mathcal{G}_g$, $\mathcal{G}_g$ is the Siegel upper half space of complex matrices with positive imaginary part, $\Omega \in \mathcal{G}_g$, $j\colon \Sigma_g \to \mathbb{C}^g$ is the Abel-Jacobi map and β is a smooth form which serves only make the combination of the two terms periodic, and for which explicit expressions are also available. Inspired by the discussion of 6.2, one might ask what is the relation between the integral

$$\int_{\Sigma_g} \eta_{\Sigma_g}(z,w)\alpha(z,w)\varphi^*\left(\eta_{\Sigma_g}(z,w)\alpha(z,w)\right) \tag{6.44}$$

to the Casson-Walker-Lescop invariant of $M = H_g \cup_\Sigma H_g$, and whether it can be computed similarly through residue calculus. A first obstacle is the absence of the symmetry that allowed us to reduce this integral to the integral of a single variable on the solid torus.

[6] However, one might try to investigate propagators that arise from degenerate metrics similar to the axial gauge.

[7] In [Fer08], different boundary conditions were used (on the boundary of a fundamental polygon of Σ_g), in our case periodic boundary conditions are required

Chapter 7

Conclusions and Outlook

It is the opinion of the author that this thesis raises many more questions than it answers. As a conclusion of this thesis, let us summarize some answers that have been given, and some questions that require further investigation.

7.1 Questions that have been answered

7.1.1 How can one approximate the axial gauge?

One previously open question that has found a satisfying answer is whether the axial gauge propagator can be approximated by propagators that extend smoothly to the compactified configuration space, and if yes, what the corresponding family is[1]. The answer given in Theorem 3.3.2 is that on any product manifold, the axial gauge propagator can be obtained by "blowing up" one of the factors.

7.1.2 What is the weight of the theta graph on a lens space?

Another question that has been answered is what the weight of theta graph on lens spaces is, at least in the gauge that corresponds to gluing it from two solid tori in the axial gauge. Explicit formulae have been presented in Equations (5.65) and (5.67), for the two different orientations of the graph. In the case of a Manin triple, this weight behaves very similarly to what would have been expected by the observations of Kuperberg-Thurston-Lescop: Namely, there is an invariant of framed lens spaces $(m+q)/(12p)$ such that the difference of the weight and this other invariant is the Casson-Walker invariant. This answer was first obtained for the case of a

[1]This question was posed to the author by P. Mnev while on a visit to Bonn.

Manin triple. Naturally, it lead to the next question.

7.1.3 Does the weight depend on the type of polarization used in the gluing?

The answer here is a clear yes. In the general case, the weight of one orientation of the Theta graph is irrational and transforms by irrational shifts under change of framing (as induced by the gluing diffeomorphism). The results of Chapter 4 rule out that this dependence is somehow cancelled by the quadratic Lie algebra: We have seen explicit examples of quadratic Lie algebras that admit splittings as Manin triples, but also splittings in which the complentary eccentricity (the Lie algebra coefficient of the other orientation of the Theta graph) are non-zero, and these splittings are related via a sequence of twists.

7.2 Questions that have not been answered

The author believes that most of the results in this work are not final, but give rise to more questions. Some of them are discussed below.

7.2.1 Some questions about the axial gauge

One question about the axial gauge was answered in this thesis. However, this immediately yields to a flurry of other questions, non of which was addressed in this work.

What is the homotopy between regular gauges and axial gauges?

Since the axial gauge can be seen as a limit of smooth gauges, one can ask about the homotopy between the propagators. Understanding better how the state defined using the regular gauge approaches the (regularized) state in the axial gauge is important in understanding whether the fact that the axial gauge is singular has an effect on the polarization anomaly (see below).

Can the axial gauge on product manifolds with more factors be approximated similarly to Theorem 3.3.2?

In this work, we only prove theorem 3.3.2 for at most two factors. However, a similar construction is relatively straightforward for products of two or more manifolds with or without boundary, to define a propagator on a manifold with corners: see, e.g. [Mne08, Section 6], where a propagator for the D-cube is considered. It can be speculated that a generalization of

3.3.2 can be proven for these constructions by "blowing up" or "degenerating" the gauge-fixing metrics on the factors successively in the right order.

Can the axial gauge be extended to non-trivial fiber bundles?

In this work, we only consider manifolds which are products, i.e. trivial fiber bundles. A natural question is whether one can extend the construction of the axial gauge to non-trivial fiber bundles, e.g. in the simple case of an S^1-bundle $S^1 \hookrightarrow M \twoheadrightarrow B$. One challenge is to send to volume of the circle fibers to infintiy in a compatible way. A variant of this approach has been explored by Blau and Thompson ([BT06]). It would be interesting to see whether there are connections between the axial gauge approach discussed here and their work.

Does our regularization of the axial gauge lead to a well-defined quantum theory?

Here, by well-defined theory we mean that the theory satisfies a version of the modified Quantum Master Equation. This has not been achieved. In particular, there is no good criterion as to when diagrams are axial gauge regularizable or not, and, in the case when diagram are divergent as one approaches the axial gauge, what a good way to extract a convergent term could be. However, this seems like a rather hard problem, at least on general manifolds. It might be best to start by investigating examples with a 1-dimensional factor, in particular, the "universal" regularization of the axial gauge proposed in 3.4.

7.2.2 What is the precise nature of the framing correction?

We showed that in the case of a Manin triple, the weight of a Theta graph is the Casson-Walker invariant up to a framing correction. However, at the moment we do not have a geometric interpretation of this term. According to the literature, it should be given by the Chern-Simons invariant of a residual connection, or - equivalently - by a Pontryagin number associated to the data (M_1, M_2, φ). This issue could certainly be clarified with a some more work.

7.2.3 What is the origin of the "Polarization anomaly"?

We have seen that the weight of the Theta graph on lens spaces in the "glued gauge" depends on the type of polarization that was chosen before gluing. We offered several possible reasons for this: the non-unimodularity of the vertex coefficients, or the fact that general polarizations are not compatible with the algebraic structure on the space of fields. It would be interesting to compare with the complex polarizations that are usually used in Chern-Simons theory, however, these seem to be harder to accomodate in the BV-BFV formalism, especially in the view of perturbative quantization (some results have been obtained in [ABM13].

7.2.4 Is there a general relation between Theta graphs on 3-manifolds and Casson-Walker-Lescop invariants?

Using the theory of effective BV actions, one can - in principle - define the weight of the theta graph on any 3-manifold, as a function on residual fields. An interesting question - of which this work has barely touched the beginnings - is what the relation between this graph and the Casson-Walker-Lescop invariant is. Notice that the Casson-Walker-Lescop invariant depends rather strongly on the dimension of $H^1(M)$. This could be interpreted as the dependence on residual fields.

Appendix A

Computation of Feynman weights on the solid torus

A.1 Some definitions and conventions

We usually consider the disk as the unit disk in the complex plane and use its complex coordinate $z = re^{2\pi i\theta}$. Its boundary is a circle S^1 with a coordinate $0 \le \theta \le 1$ but formulas will usually be valid for $-1 \le \theta \le 1$ which is useful when considering differences. We often make use of the elementary formulas

$$\operatorname{Im} z_1 \operatorname{Im} z_2 = \frac{1}{2}\operatorname{Re}(z_1\bar{z}_2 - z_1 z_2) \tag{A.1}$$

$$\operatorname{Im} z_1 \operatorname{Im} z_2 \operatorname{Im} z_3 = \frac{1}{4}\operatorname{Im}(z_1 z_2\bar{z}_3 - z_1 z_2 z_3 + z_1\bar{z}_2 z_3 - z_1\bar{z}_2\bar{z}_3) \tag{A.2}$$

$$= \frac{1}{4}\operatorname{Im}(z_1 z_2\bar{z}_3 - z_1 z_2 z_3 + z_1\bar{z}_2 z_3 + \bar{z}_1 z_2 z_3) \tag{A.3}$$

By log we always denote the principal branch given by $\log(1-z) = -\sum_{k=1}^{\infty} z^k/k$ for $|z| < 1$, and by $Li_2(z)$ the corresponding branch of the dilogarithm

$$Li_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt = \sum_{k=1}^{\infty}\frac{z^k}{k^2} \qquad (|z| < 1).$$

We also use the *double logarithm* $(|xy| < 1, |y| < 1)$

$$Li_{1,1}(x,y) = \sum_{0<m<n}\frac{x^m y^n}{mn}. \tag{A.4}$$

Also we will use the elementary fact that if

$$\iint_D f(z,\bar{z})d\bar{z}dz = 2\pi i K$$

then

$$\iint_D \overline{f(z,\bar z)} d\bar z dz = -\overline{\iint_D f(z,\bar z) d\bar z dz} = 2\pi \mathrm{i}\overline{K}.$$

We introduce the following condensed notation.

$$z_{ij} := z_i - z_j, L_{ij} = \log z_{ij}, u_{ij} = 1 - z_i \bar z_j, K_{ij} = \log u_{ij}, M_{ij} = \log\left(\frac{u_{ij}}{|z_{ij}|^2}\right) \tag{A.5}$$

In passing we note that $z_{ji} = -z_{ij}$ and $u_{ji} = \bar u_{ij}$, and hence $dL_{ji} = dL_{ij}$ while $dK_{ji} = \overline{dK_{ij}}$. We also define

$$\phi_{12} := \frac{1}{2\pi} d\arg z_{12} = \frac{1}{2\pi}\operatorname{Im} d\log(z_{12}) = \frac{1}{4\pi i}\left(\frac{dz_{12}}{z_{12}} - \frac{d\bar z_{12}}{\bar z_{12}}\right) \tag{A.6}$$

$$\tau_{12} := \frac{1}{2\pi} d\arg u_{12} = \frac{1}{2\pi}\operatorname{Im} dK_{12} = \frac{1}{4\pi i}\left(\frac{z_2 d\bar z_1 + \bar z_1 dz_2}{\bar u_{12}} - \frac{z_1 d\bar z_2 + \bar z_2 dz_1}{u_{12}}\right) \tag{A.7}$$

$$\psi_1 := \frac{1}{4\pi i}(\bar z dz - z d\bar z) = \frac{1}{2\pi}\operatorname{Im}(\bar z dz) \tag{A.8}$$

$$\mu_1 := \frac{1}{2\pi i}(d\bar z dz) = d\psi_1 \tag{A.9}$$

and call them "elementary 1-forms": all Feynman diagrams on the disk can be expressed in terms of those forms.

A.2 Results

In this section we present in a compact form the weights of the Feynman diagrams before and after the pushforward over the bulk vertices. We present the results in tables for each order. Each graph carries a label, the corresponding figure can be found in section 5.2.5. The computations are carried out over the next 2 sections. First we split the contribution into contributions on the circle and contributions on the disk in Section A.3. The contributions of the circle can be immediately evaluated. For the integrals over the disk, a systematic way to compute the resulting parameter integrals over the disk is developed in A.5. Also, some words of warning on the notation. There are four different types of indices. The indices $i, j, k, \ldots$ are Lie algebra indices. The *first* index on z and z^+ coordinates denotes the index of the background. Finally, a differential form ω can carry one or two indices ω_i or ω_{ij}: This means the form is placed at vertex i, i.e. $\omega_i = \pi_i^*\omega$, or $\omega_{ij} = \pi_{ij}^*\omega$. Finally, the indices on $M = D \times S^1$ and $\partial M = S^1 \times S^1$ denote the label of the vertex there, and projections to the corresponding factor carry that index. We will often abbreviate $\partial M_i \times \partial M_j = (\partial M)^2_{ij}$, and similarly for higher powers (the need for abbreviations is abundant).

A.2.1 0-point order

Graph label	Figure	Before Pushforward	After pushforward
Γ_0	5.3	$-\int_{\partial M}\langle \mathsf{b}, \mathbb{A}\rangle$	$-z^+_{0,i}\int_{\partial M}\mathbb{A} - z^+_{1,i}\int_{\partial M} dt\mathbb{A}^i$

Table A.1: Contributions of 0-point graphs

A.2.2 1-point order

Graph label	Figure	Before Pushforward	After pushforward
$\Gamma_{1,0}$	5.4a	$\int_M \langle \mathsf{b}, [\mathsf{a}, \mathsf{a}]\rangle$	$\frac{1}{2}\gamma_i^{jk}(z^{0i}z^+_{0j}z^+_{0k} + 2z^{1i}z^+_{0j}z^+_{1k})$
$\Gamma_{1,1}$	5.4b	$-\int_{M_1\times\partial M_2}\mu^i_{jk}\mathsf{b}_{1,i}\mathsf{a}^j_1\eta_{12}\mathbb{A}^k_2$	$-\mu^i_{jk}(z^+_{0i}z^{0j} - z^+_{1i}z^{1j})\int_{\partial M_2} dtd\theta\mathbb{A}^k_2$ $-\mu^i_{jk}z^+_{0i}z^{1j}\int_{\partial M_2} d\theta\mathbb{A}^k_2$
$\Gamma^{\mathsf{b}}_{1,2}$	5.4c	$\frac{1}{2}\mu^i_{jk}\int_{M_1\times(\partial M)^2_{23}} b_{1,i}\eta_{12}\eta_{13}\mathbb{A}^j_2\mathbb{A}^k_3$	(A.12)
$\Gamma^{\mathsf{a}}_{1,2}$	5.4d	$\frac{1}{2}\psi_{ijk}\int_{M_1\times(\partial M)^2_{23}} a^i_1\eta_{12}\eta_{13}\mathbb{A}^j_2\mathbb{A}^k_3$	(A.14)
$\Gamma_{1,3}$	5.4e	$\frac{1}{6}\psi_{ijk}\int_{M_1\times(\partial M)^2_{234}}\eta_{12}\eta_{13}\eta_{14}\mathbb{A}^i_2\mathbb{A}^j_3\mathbb{A}^k_4$	(A.16)

Table A.2: Contributions of 1-point graphs

A.2.3 2-point order

Trees, one boundary vertex

Graph label	Figure	Before Pushforward	After pushforward
$\Gamma^{\mathsf{b}}_{2,1}$	5.6a	$\mu^k_{ij}\gamma^{jl}_m \int_{M_1\times M_2\times\partial M_3} \mathsf{a}^i_1 \mathsf{b}_{1,k}\eta_{12}\mathsf{b}_{2,l}\eta_{13}\mathbb{A}^m_3$	0
$\Gamma^{\mathsf{a}}_{2,1}$	5.6a	$\mu^k_{ij}\mu^j_{lm} \int_{M_1\times M_2\times\partial M_3} \mathsf{a}^i_1 \mathsf{b}_{1,k}\eta_{12}\mathsf{a}^l_2\eta_{13}\mathbb{A}^m_3$	0

Table A.3: Contribution of 2-point trees with 1 boundary vertex

Trees, two boundary vertices

Graph label	Figure	Before Pushforward	After pushforward
$\Gamma^{\mathsf{ab}}_{2,2}$	5.6b	$\psi_{ijk}\gamma^{jl}_m \int_{M_1\times M_2\times\partial M_3\times\partial M_4} \mathsf{a}^i_1\eta_{13}\eta_{12}\mathsf{b}_{2,l}\eta_{24}$	0
$\Gamma^{\mathsf{bb}}_{2,2}$	5.6b	$\mu^k_{ij}\gamma^{jl}_m \int_{M_1\times M_2\times\partial M_3\partial M_4} \mathsf{b}_{1,k}\eta_{13}\eta_{12}\mathsf{b}_{2,l}\eta_{14}\mathbb{A}^i_3\mathbb{A}^m_4$	(A.22)
$\Gamma^{\mathsf{ab}}_{2,1}$	5.6b	$\mu^k_{ij}\mu^j_{lm} \int_{M_1\times M_2\partial M_3\times\partial M_4} \mathsf{a}^i_1 \mathsf{b}_{1,k}\eta_{12}\mathsf{a}^l_2\eta_{13}\mathbb{A}^m_3$	(A.20)
$\Gamma'_{2,2}$	5.6f	$\mu^k_{ij}\mu^j_{lm} \int_{M_1\times M_2\times\partial M_3} \mathsf{a}^i_1 \mathsf{b}_{1,k}\eta_{12}\mathsf{a}^l_2\eta_{13}\mathbb{A}^m_3$	(A.24)

Table A.4: Contribution of 2-point trees with 2 boundary vertices

Trees, three boundary vertices

Graph label	Figure	Before Pushforward	After pushforward
$\Gamma^{l,\mathsf{b}}_{2,3}$	5.6d	$\gamma^{ij}_k \psi_{jlm} \int_{M^2_{12}\times(\partial M)^3_{345}} \mathsf{b}_{1,i}\eta_{13}\eta_{21}\eta_{24}\eta_{25}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5$	(A.25)
$\Gamma^{l,\mathsf{a}}_{2,3}$	5.6d	$\mu^j_{ik}\psi_{jlm}\int_{M^2_{12}\times(\partial M)^3_{345}} \mathsf{a}^{1,i}\eta_{13}\eta_{21}\eta_{24}\eta_{25}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5$	(A.26)
$\Gamma^{r,\mathsf{b}}_{2,3}$	5.6c	$\mu^{ij}_k\mu^j_{lm}\int_{M^2_{12}\times(\partial M)^3_{345}} \mathsf{b}_{1,i}\eta_{13}\eta_{12}\eta_{24}\eta_{25}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5$	(A.27)
$\Gamma^{r,\mathsf{a}}_{2,3}$	5.6c	$\psi_{ijk}\mu^k_{lm}\int_{M^2_{12}\times(\partial M)^3_{345}} \mathsf{a}^i_1\eta_{13}\eta_{21}\eta_{24}\eta_{25}\mathbb{A}^j_3\mathbb{A}^l_4\mathbb{A}^m_5$	(A.29)

Table A.5: Contribution of 2-point trees with 3 boundary vertices

Trees, four boundary vertices

Graph label	Figure	Before Pushforward	After pushforward
$\Gamma_{2,4}$	5.6e	$\int_{M^2_{12}\times(\partial M)^4_{3456}} \eta_{13}\eta_{14}\eta_{12}\eta_{25}\eta_{26}$	(A.30)

Table A.6: Contribution of the 2-point tree with four boundary vertices

Loop diagrams

The loop diagrams in the axial gauge are regularized by the approximation by metric propagators. The loop diagrams in question were analyzed in Section 5.3.4. The two-loop diagrams vanish in this regularization. For the one-loop graphs, only the oriented loops survive.

A.3 Splitting of contributions

In this appendix we will compute the splitting of contributions in Section 5.3 into disk and circle contributions. To this end we expand the propagator $\eta_{12} = \eta_{D,12}\delta_{S^1,12} + \mu_1\eta_{S^1,12}$. We will identify the integrals that have to be computed on the disk and directly evaluate the integrals on the circle (where the integrals are really pushforwards of currents). We give the explicit result for all weights, using the computation of the pushforwards in the next section.
For future use let us briefly compute the product $\mathsf{a}^1 \wedge \mathsf{b}_j$:

$$\mathsf{a}^i \wedge \mathsf{b}_j = (z^{0,i}\mu dt + z^{1,i}\mu)(z^+_{0,j}1 + z^+_{1,j}dt) = (-z^{0,i}z^+_{0,j} + z^{1,i}z^+_{1,j})\mu dt + z^{1,i}z^+_{0,j}\mu. \tag{A.10}$$

Here the minus sign in the first term comes from commuting the odd coordinate $z^+_{0,j}$ with the odd μdt.

A.3.1 1-point order

Let us proceed by number of boundary vertices.

One boundary vertex

The contribution of $\Gamma_{1,0}$ (figure 5.4b) is

$$\begin{aligned}
&-\int_{M_1\times\partial M_2}\mu^i_{jk}\mathsf{b}_{1,i}\mathsf{a}^j_1\eta_{12}\mathbb{A}^k_2\\
&=\mu^k_{ij}(z^{0,i}z^+_{0,j}-z^{1,i}z^+_{1,j})\int_{M_1\times\partial M_2}\mu_1 dt_1(\eta_{D,12}\delta_{S^1,12}+\mu_1\eta_{S^1,12})\mathbb{A}^k_2\\
&\qquad-\mu^k_{ij}z^{1,i}z^+_{0,j}\int_{M_1\times\partial M_2}\mu_1(\eta_{D,12}\delta_{S^1,12}+\mu_1\eta_{S^1,12})\mathbb{A}^k_2\\
&=-\mu^k_{ij}(z^{0,i}z^+_{0,j}-z^{1,i}z^+_{1,j})\int_{\partial M_2}\int_{D,1}\mu_1\eta_{D,12}\int_{S^1,1}\delta_{S^1,12}dt_1\mathbb{A}^k_2\\
&-\mu^k_{ij}z^{1,i}z^+_{0,j}\int_{\partial M_2}\int_{D_1}\mu\eta_{D,12}\int_{S^1,1}\delta_{S^1,12}\mathbb{A}^k_2\\
&=-\mu^k_{ij}(z^{0,i}z^+_{0,j}-z^{1,i}z^+_{1,j})\int_{\partial M_2}\psi_2 dt_2\mathbb{A}^k_2-\mu^k_{ij}z^{1,i}z^+_{0,j}\int_{\partial M_2}\psi_2\mathbb{A}^k_2
\end{aligned}\tag{A.11}$$

where we denoted $\psi_2 := \int_{D,1}\mu_1\eta_{D,12}\in\Omega^1(D)$.

Two boundary vertices

The contributions from $\Gamma^{\mathsf{b}}_{1,2}$ (figure 5.4c) is

$$\psi_{\Gamma^{\mathsf{b}}_{1,2}}=\mu^i_{jk}\int_{M_1\times\partial M_2\times\partial M_3}(z^+_{0,i}1+z^+_{1,i}dt_1)\eta_{12}\eta_{13}\mathbb{A}^j_2\mathbb{A}^k_3.$$

The first term is

$$\begin{aligned}
&\mu^i_{jk}z^+_{0,i}\int_{M_1\times\partial M_2\times\partial M_3}(\eta_{D,12}\delta_{S^1,12}+\mu_1\eta_{S^1,12})(\eta_{D,13}\delta_{S^1,13}+\mu_1\eta_{S^1,13})\\
&=\mu^i_{jk}z^+_{0,i}\int_{(\partial M)^2_{23}}\left(\int_{D,1}-\eta_{D,12}\eta_{D,13}\int_{S^1_1}\delta_{S^1,12}\delta_{S^1,13}+\left(\int_{D,1}\mu_1\eta_{D,12}\int_{S^1,1}\eta_{S^1,13}\delta_{S^1,12}+2\leftrightarrow 3\right)\right)\mathbb{A}^j_2\mathbb{A}^k_3\\
&=\mu^i_{jk}z^+_{0,1}\int_{(\partial M)^2_{23}}\left(-\omega_{23}\delta_{S^1,23}+(\psi_2-\psi_3)\eta_{S^1,23}\right)\mathbb{A}^j_2\mathbb{A}^k_3
\end{aligned}$$

where $\omega_{13} := \int_{D,1}\eta_{D,12}\eta_{D,13}\in\Omega_0(C_2(D))$, and similarly, for the second term we compute

$$\begin{aligned}
\int_1 dt_1\eta_{12}\eta_{13}&=-\int_{D,1}\eta_{D,12}\eta_{D,13}\int_{S^1_1}dt_1\delta_{S^1,12}\delta_{S^1,13}+\left(\int_{D,1}\mu_1\eta_{D,12}\int_{S^1,1}dt_1\eta_{S^1,13}\delta_{S^1,12}+2\leftrightarrow 3\right)\\
&=-\omega_{23}\delta_{S^1,23}\frac{dt_2+dt_3}{2}+(\psi_2 dt_2-\psi_3 dt_3)\eta_{S^1,23}.
\end{aligned}$$

The entire contribution is therefore

$$\psi_{\Gamma^{\mathsf{b}}_{1,2}} = \mu^i_{jk}\bigg(z_0^+\int_{M_1\times\partial M_2\times\partial M_3}\left(-\eta^1_S(\theta_2,\theta_3)\delta_{S^1,23}+\eta_{S^1}(t_2,t_3)(d\theta_2-d\theta_3)\right)\mathbb{A}^j_2\mathbb{A}^k_3$$
$$+z_1^+\int_{M_1\times\partial M_2\times\partial M_3}\left(-\eta^1_S(\theta_2,\theta_3)\delta_{S^1,23}\frac{dt_2+dt_3}{2}+\eta_{S^1}(t_2,t_3)(d\theta_2dt_2-d\theta_3dt_3)\right)\mathbb{A}^j_2\mathbb{A}^k_3 \tag{A.12}$$

The contribution of $\Gamma^{\mathsf{a}}_{1,2}$ (figure 5.4d) is

$$\psi_{\Gamma^{\mathsf{a}}_{1,2}} = \psi_{ijk}\int_{M_1\times\partial M_2\times\partial M_3}(z^{0,i}\mu_1 dt_1+z^{1,i}\mu_1)\eta_{12}\eta_{13}\mathbb{A}^j_2\mathbb{A}^k_3. \tag{A.13}$$

In the second term, we get

$$\begin{aligned}\int_1\mu_1\eta_{12}\eta_{13} &= \int_1\mu_1(\eta_{D,12}\delta_{S^1,12}+\mu_1\eta_{S^1,12})(\eta_{D,13}\delta_{S^1,13}+\mu_1\eta_{S^1,13})\\ &= -\int_{D,1}\mu_1\eta_{D,12}\eta_{D,13}\int_{S^1_1}\delta_{S^1,12}\delta_{S^1,13} = \nu_{23}\delta_{S^1,23}\end{aligned}$$

where $-\nu_{23} = \int_{D,1}\mu_1\eta_{D,12}\eta_{D,13}\in\Omega^2(C_2(D))$ and similarly, in the second term we have the contribution

$$\int_1\mu_1 dt_1\eta_{12}\eta_{13} = -\int_{D,1}\mu_1\eta_{D,12}\eta_{D,13}\int_{S^1_1}dt_1\delta_{S^1,12}\delta_{S^1,13} = \nu_{23}\delta_{S^1,23}\frac{dt_2+dt_3}{2}.$$

Overall we obtain

$$\psi_{\Gamma^{\mathsf{a}}_{1,2}} = \psi_{ijk}z^{0,i}\int_{M_1\times\partial M_2\times\partial M_3}\nu_{23}\delta_{S^1,23}\mathbb{A}^j_2\mathbb{A}^k_3+\psi_{ijk}z^{1,i}\int_{M_1\times\partial M_2\times\partial M_3}\nu_{23}\delta_{S^1,23}\frac{dt_2+dt_3}{2}\mathbb{A}^j_2\mathbb{A}^k_3 \tag{A.14}$$

with ν_{23} given by (A.33).

Three boundary vertices

The contribution of $\Gamma_{1,3}$ (figure 5.4e) is

$$\frac{1}{6}\psi_{ijk}\int_{M_1\times(\partial M)^3_{234}}\eta_{12}\eta_{13}\eta_{14}\mathbb{A}^i_2\mathbb{A}^j_3\mathbb{A}^k_4. \tag{A.15}$$

We expand the integrand:

$$\begin{aligned}\int_1\eta_{12}\eta_{13}\eta_{14} &= \int_1(\eta_{D,12}\delta_{S^1,12}+\mu_1\eta_{S^1,12})(\eta_{D,13}\delta_{S^1,13}+\mu_1\eta_{S^1,13})(\eta_{D,14}\delta_{S^1,14}+\mu_1\eta_{S^1,14})\\ &= -\int_{D,1}\eta_{D,12}\eta_{D,13}\eta_{D,14}\int_{S^1_1}\delta_{S^1,12}\delta_{S^1,13}\delta_{S^1,14}\\ &\quad -\left(\int_{D,1}\mu_1\eta_{D,13}\eta_{D,14}\int_{S^1_1}\eta_{S^1,12}\delta_{S^1,13}\delta_{S^1,14}+\text{cycl.}\right)\\ &= -\alpha_{234}\delta_{S^1,23}\delta_{S^1,34}+\left(\nu_{34}\eta_{S^1,23}\delta_{S^1,34}+\text{cycl.}\right)\end{aligned}$$

where $\alpha_{234} := \int_{D,1} \eta_{D,12}\eta_{D,13}\eta_{D,14} \in \Omega^1(C_3(D))$. Summarizing, we have

$$\psi_{\Gamma_{1,3}} = \frac{1}{6}\psi_{ijk}\int_{M_1(\partial M)^3_{234}} -\alpha_{234}\delta_{S^1,23}\delta_{S^1,34} + \left(\nu_{34}\eta_{S^1,23}\delta_{S^1,34} + \text{cycl.}\right)\mathbb{A}^i_2\mathbb{A}^j_3\mathbb{A}^4_k \tag{A.16}$$

where α is given by (A.35) and ν is given by (A.33).

A.3.2 2-point contribution

Next let us look at the 2-point contributions. We will first consider only the tree diagrams. By the properties of the propagator, and the fact that residual fields are closed under multiplication, the weights of all tree diagrams with no boundary vertices vanish.

One boundary vertex

For one boundary vertex, we have the diagrams $\Gamma^{\mathsf{b}}_{2,1}$ and $\Gamma^{\mathsf{a}}_{2,1}$ (figure 5.6a. $\Gamma^{\mathsf{b}}_{2,1}$ has the contribution

$$\psi_{\Gamma^{\mathsf{b}}_{2,1}} = \mu^i_{jk}\gamma^{kl}_m \int_{M_1\times M_2\times\partial M_3} \mathsf{b}_{1,i}\mathsf{a}^j_1\eta_{12}\mathsf{b}_{2,l}\eta_{23}\mathbb{A}^m_3 \tag{A.17}$$

and contains the following integrals:

$$\begin{aligned}\int_{1,2}\mu_1\eta_{12}\eta_{23} &= 0,\\ \int_{1,2}\mu_1\eta_{12}dt_2\eta_{23} &= \int_{1,2}\mu_1\eta_{D,12}\delta_{S^1,12}dt_2(\eta_{D,23}\delta_{S^1,23}+\mu_2\eta_{S^1,23})\\ &= -\int_{D,1,2}\mu_1\eta_{D,12}\eta_{D,23}\int_{S^1,1,2}\delta_{S^1,12}dt_2\delta_{S^1,23}\\ &\quad+\int_{D,1,2}\mu_1\eta_{D,12}\mu_2\int_{S^1,1,2}\delta_{S^1,12}dt_2\eta_{S^1,23} = 0\end{aligned}$$

where we have made use of Proposition 5.3.2 and the properties of the disk propagator (which satisfies the assumptions of 5.3.2. Similarly, the $\Gamma^{\mathsf{a}}_{2,1}$ contributions also vanish for degree reasons (the disk form degree at point 2 is too high).

Two boundary vertices

Next let us turn to the diagrams with 2 boundary vertices. We look at the first case $\Gamma^{\mathsf{x}_1\mathsf{x}_2}_{2,2}$ (figure 5.6b). If $\mathsf{x}_1 = \mathsf{x}_2 = \mathsf{a}$, then the contribution is zero for degree reasons (the total form degree of the pushfoward in the disk is at least $1+1+1+2+2-4 = 3 > 2 = \dim C_2(S^1)$). Let us look at the other diagrams. First, $\Gamma^{\mathsf{ab}}_{2,2}$ gives the contribution

$$\psi_{\Gamma^{\mathsf{ab}}_{2,2}} = \psi_{ijk}\gamma^{jl}_m \int_{M_1\times M_2\times\partial M_3\times\partial M_4} \mathsf{a}^i_1\eta_{13}\eta_{12}\mathsf{b}_{2,l}\eta_{24}\mathbb{A}^k_3\mathbb{A}^m_4 \tag{A.18}$$

and contains the integrals

$$\begin{aligned}
\int_{1,2} \eta_{13}\mu_1\eta_{12}\eta_{24} &= 0,\\
\int_{1,2} \eta_{13}\mu_1 dt_1\eta_{12}\eta_{24} &= 0,\\
\int_{1,2} \eta_{13}\mu_1\eta_{12}dt_2\eta_{24} &= \int_{1,2} \eta_{D,13}\delta_{S^1,13}\mu_1\eta_{D,12}\delta_{S^1,12}dt_2(\eta_{D,24}\delta_{S^1,24}+\mu_2\eta_{S^1,24})\\
&= -\underbrace{\int_{D,1,2}\eta_{D,13}\mu_1\eta_{D,12}\eta_{D,23}}_{0}\int_{S^1,1,2}\delta_{S^1,13}\delta_{S^1,12}dt_2\delta_{S^1,24}\\
&\quad+\underbrace{\int_{D,1,2}\eta_{D,13}\mu_1\eta_{D,12}\mu_2}_{0}\int_{S^1,1,2}\delta_{S^1,13}\delta_{S^1,12}dt_2\eta_{S^1,23} = 0\\
\int_{1,2} \eta_{13}\mu_1 dt_1\eta_{12}dt_2\eta_{24} &= -\underbrace{\int_{D,1,2}\eta_{D,13}\mu_1\eta_{D,12}\eta_{D,23}}_{0}\int_{S^1,1,2}\delta_{S^1,13}dt_1\delta_{S^1,12}dt_2\delta_{S^1,24} = 0
\end{aligned}$$

so all contributions from this diagram vanish. Next, $\Gamma^{\mathsf{bb}}_{2,2}$ gives

$$\psi_{\Gamma^{\mathsf{bb}}_{2,2}} = \mu^i_{jk}\gamma^{jl}_m \int_{M_1\times M_2\times\partial M_3\times\partial M_4} \mathsf{b}_{1,i}\eta_{13}\eta_{12}\mathsf{b}_{2,l}\eta_{24}\mathbb{A}^k_3\mathbb{A}^m_4 \tag{A.19}$$

and contains the integrals

$$\begin{aligned}
\int_{1,2} \eta_{13}\eta_{12}\eta_{23} &= 0,\\
\int_{1,2} dt_1\eta_{13}\eta_{12}\eta_{23} &= 0,\\
\int_{1,2} \eta_{13}\eta_{12}dt_2\eta_{24} &= \int_{1,2} (\eta_{D,13}\delta_{S^1,13}+\mu_1\eta_{S^1,13})(\eta_{D,12}\delta_{S^1,12}+\mu_1\eta_{S^1,12})dt_2(\eta_{D,24}\delta_{S^1,24}+\mu_2\eta_{S^1,24})\\
&= \int_{D,1,2}\mu_1\eta_{D,13}\mu_2\int_{S^1,1,2}\delta_{S^1,13}dt_2\eta_{S^1,12}\eta_{S^1,24} = \psi_3 h_{34}\\
\int_{1,2} \eta_{13}dt_1\eta_{12}dt_2\eta_{24} &= \int_{D,1,2}\mu_1\eta_{D,13}\mu_2\int_{S^1,1,2}dt_1\delta_{S^1,13}dt_2\eta_{S^1,12}\eta_{S^1,24} = \psi_3 dt_3 h_{34}
\end{aligned}$$

where $h_{34} \coloneqq \int_2 \eta_{S^1,32}\eta_{S^1,24}dt_2$. By Lemma 5.3.9, we have $h_{34} = \frac{1}{2}B_2(t_{34})$, where B_2 the second Bernoulli polynomial. The first two integrals vanish because of proposition 5.3.2, in the next two integrals almost all integrals over the disk vanish for degree reasons or properties of the disk propagator. So, we can give the weight of this diagram explicitly as

$$\begin{aligned}
\psi_{\Gamma^{\mathsf{bb}}_{2,2}} &= \frac{1}{2}\mu^i_{jk}\gamma^{jl}_m z^+_{0,i}z^+_{1,l}\int_{M\times(\partial M)^3_{234}} d\theta_3 B_2(t_{34})\mathbb{A}^k_3\mathbb{A}^m_4\\
&\quad+\frac{1}{2}\mu^i_{jk}\gamma^{jl}_m z^+_{1,i}z^+_{1,l}\int_{M\times(\partial M)^3_{234}} d\theta_3 dt_3 B_2(t_{34})\mathbb{A}^k_3\mathbb{A}^m_4
\end{aligned} \tag{A.20}$$

Finally, $\Gamma^{\mathsf{ba}}_{2,2}$ evaluates to

$$\psi_{\Gamma^{\mathsf{ba}}_{2,2}} = \mu^i_{jk}\mu^j_{lm}\int_{M_1\times M_2\times\partial M_3\times\partial M_4} \mathsf{b}_{1,i}\eta_{13}\eta_{12}\mathsf{a}^l_2\eta_{24}\mathbb{A}^k_3\mathbb{A}^m_4 \tag{A.21}$$

which contains the integral

$$\begin{aligned}
\int_{1,2}\eta_{13}\eta_{12}\mu_2\eta_{23} &= \int_{1,2}(\eta_{D,13}\delta_{S^1,13}+\mu_1\eta_{S^1,13})(\eta_{D,12}\delta_{S^1,12}+\mu_1\eta_{S^1,12})\mu_2(\eta_{D,24}\delta_{S^1,24}+\mu_2\eta_{S^1,24})\\
&= \int_{D,1,2}\eta_{D,13}\eta_{D,12}\mu_2\eta_{D,24}\int_{S^1,1,2}\delta_{S^1,13}\delta_{S^1,12}\delta_{S^1,24}\\
&+ \int_{D,1,2}\mu_1\eta_{D,12}\mu_2\eta_{D,24}\int_{S^1,1,2}\eta_{S^1,13}\delta_{S^1,12}\delta_{S^1,24}\\
&+ \int_{D,1,2}\mu_1\eta_{D,13}\mu_2\eta_{D,24}\int_{S^1,1,2}\eta_{S^1,12}\delta_{S^1,13}\delta_{S^1,24}\\
&= \beta_{34}\delta_{S^1,34}+\psi_3\psi_4\eta_{S^1,34},
\end{aligned}$$

where $\beta_{34} \coloneqq \int_{D,1,2}\eta_{D,13}\eta_{D,12}\mu_2\eta_{D,24} \in \Omega^1(C_2(D))$, and the second term vanishes for the following reason: The integral over the first point on the disk produces a 1-form at the second point, but we already placed the 2-form μ_2 there. The other diagrams in $\Gamma^{\mathsf{ba}}_{2,2}$ evaluate to

$$\begin{aligned}
\int_{1,2}dt_1\eta_{13}\eta_{12}\mu_2\eta_{23} &= \int_{D,1,2}\eta_{D,13}\eta_{D,12}\mu_2\eta_{D,24}\int_{S^1,1,2}dt_1\delta_{S^1,13}\delta_{S^1,12}\delta_{S^1,24}\\
&+ \int_{D,1,2}\mu_1\eta_{D,12}\mu_2\eta_{D,24}\int_{S^1,1,2}dt_1\eta_{S^1,13}\delta_{S^1,12}\delta_{S^1,24}\\
&+ \int_{D,1,2}\mu_1\eta_{D,13}\mu_2\eta_{D,24}\int_{S^1,1,2}dt_1\eta_{S^1,12}\delta_{S^1,13}\delta_{S^1,24}\\
&= \beta_{34}\delta_{S^1,34}\frac{dt_3+dt_4}{2}+\psi_3\psi_4\eta_{S^1,34}dt_3,\\
\int_{1,2}\eta_{13}\eta_{12}\mu_2 dt_2\eta_{23} &= \int_{D,1,2}\eta_{D,13}\eta_{D,12}\mu_2\eta_{D,24}\int_{S^1,1,2}\delta_{S^1,13}\delta_{S^1,12}dt_2\delta_{S^1,24}\\
&+ \int_{D,1,2}\mu_1\eta_{D,12}\mu_2\eta_{D,24}\int_{S^1,1,2}\eta_{S^1,13}\delta_{S^1,12}dt_2\delta_{S^1,24}\\
&+ \int_{D,1,2}\mu_1\eta_{D,13}\mu_2\eta_{D,24}\int_{S^1,1,2}\eta_{S^1,12}\delta_{S^1,13}dt_2\delta_{S^1,24}\\
&= \beta_{34}\delta_{S^1,34}\frac{dt_3+dt_4}{2}+\psi_3\psi_4\eta_{S^1,34}dt_4,\\
\int_{1,2}dt_1\eta_{13}\eta_{12}\mu_2 dt_2\eta_{23} &= \int_{D,1,2}\mu_1\eta_{D,13}\mu_2\eta_{D,24}\int_{S^1,1,2}dt_1\eta_{S^1,12}\delta_{S^1,13}dt_2\delta_{S^1,24}\\
&= \psi_3\psi_4\eta_{S^1,34}dt_3dt_4,
\end{aligned}$$

Putting everything together we obtain

$$
\begin{aligned}
\psi_{\Gamma^{\mathsf{ba}}_{2,2}} = \mu^i_{jk}\mu^j_{lm}\Big[& z^+_{0,i}z^{1,l}\int_{(\partial M)^2_{34}}(\beta_{34}\delta_{S^1,34} + d\theta_3 d\theta_4\eta_{S^1,34})\mathbb{A}^k_3\mathbb{A}^m_4 \\
& + z^+_{1,i}z^{1,l}\int_{(\partial M)^2_{34}}(\beta_{34}\delta_{S^1,34}(dt_3+dt_4)/2 + d\theta_3 d\theta_4\eta_{S^1,34}dt_3)\mathbb{A}^k_3\mathbb{A}^m_4 \\
& + z^+_{0,i}z^{0,l}\int_{(\partial M)^2_{34}}(\beta_{34}\delta_{S^1,34}(dt_3+dt_4)/2 + d\theta_3 d\theta_4\eta_{S^1,34})\mathbb{A}^k_3\mathbb{A}^m_4 \\
& + z^+_{1,i}z^{0,l}\int_{(\partial M)^2_{34}}(d\theta_3 d\theta_4\eta_{S^1,34}dt_3 dt_4)\mathbb{A}^k_3\mathbb{A}^m_4
\end{aligned}
\tag{A.22}
$$

The last diagram with two boundary vertices is $\Gamma'_{2,2}$ (figure 5.6f). It gives the contribution

$$
\psi_{\Gamma'_{2,2}} = \mu^i_{jk}\mu^k_{lm}\int_{M_1\times M_2\times\partial M_3\times\partial M_4}\mathsf{b}_{1,i}\mathsf{a}^j_1\eta_{12}\eta_{23}\eta_{24}\mathbb{A}^k_3\mathbb{A}^m_4 \tag{A.23}
$$

which contains the integrals

$$
\begin{aligned}
\int_{1,2}\mu_1\eta_{12}\eta_{23}\eta_{24} &= \int_{1,2}\mu_1(\eta_{D,12}\delta_{S^1,12} + \mu_1\eta_{S^1,12})(\eta_{D,23}\delta_{S^1,23} + \mu_2\eta_{S^1,23})(\eta_{D,24}\delta_{S^1,24} + \mu_2\eta_{S^1,24}) \\
&= -\int_{D,1,2}\mu_1\eta_{D,12}\eta_{D,23}\eta_{D,24}\int_{S^1,1,2}\delta_{S^1,12}\delta_{S^1,23}\delta_{S^1,24} \\
&\quad + \underbrace{\int_{D,1,2}\mu_1\eta_{D,12}\mu_2\eta_{D,23}}_{0}\int_{S^1,1,2}\delta_{S^1,12}\delta_{S^1,23}\eta_{S^1,24} + 3\leftrightarrow 4 \\
&= \gamma_{34}\delta_{34}
\end{aligned}
$$

Here $\gamma_{34} := \int_{D,1,2}\mu_1\eta_{D,12}\eta_{D,23}\eta_{D,24}$ and the second (and therefore also the third) term vanish for the same reason as above. Similarly,

$$
\begin{aligned}
\int_{1,2}\mu_1 dt_1\eta_{12}\eta_{23}\eta_{24} &= -\int_{D,1,2}\mu_1\eta_{D,12}\eta_{D,23}\eta_{D,24}\int_{S^1,1,2}dt_1\delta_{S^1,12}\delta_{S^1,23}\delta_{S^1,24} \\
&= \gamma_{34}\delta_{34}\frac{dt_3+dt_4}{2}
\end{aligned}
$$

so overall we obtain

$$
\begin{aligned}
\psi_{\Gamma'_{2,2}} = \mu^i_{jk}\mu^k_{lm}\Big[& z^+_{0,1}z^{1,j}\int_{\partial M_3\times\partial M_4}\gamma_{34}\delta_{S^1,34}\mathbb{A}^k_3\mathbb{A}^m_4 \\
& + (z^+_{1,1}z^{1,j} - z^+_{0,1}z^{0,j})\int_{\partial M_3\times\partial M_4}\gamma_{34}\delta_{S^1,34}(dt_3+dt_4)/2\mathbb{A}^k_3\mathbb{A}^m_4\Big]
\end{aligned}
\tag{A.24}
$$

Three boundary vertices

Let us now turn to diagrams containing three boundary vertices. We will start with $\Gamma^{l,\mathsf{b}}_{2,3}$ (figure 5.6d). This gives the contributions

$$
\psi_{\Gamma^{l,\mathsf{b}}_{2,3}} = \gamma^{ij}_k\psi_{jlm}\int_{M^2_{12}\times(\partial M)^3_{345}}\mathsf{b}_{1,i}\eta_{13}\eta_{21}\eta_{24}\eta_{25}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5.
$$

It contains the integralls

$$\int_{1,2} \eta_{13}\eta_{21}\eta_{24}\eta_{25} = 0,$$

$$\begin{aligned}\int_{1,2} dt_1\eta_{13}\eta_{21}\eta_{24}\eta_{25} &= \int_{D,1,2} \mu_1\mu_2\eta_{D,24}\eta_{D,25} \int_{S^1,1,2} dt_1\eta_{S^1,13}\eta_{S^1,21}\delta_{S^1,24}\delta_{S^1,25} \\ &= \nu_{34}\delta_{S^1,45}h_{34}\end{aligned}$$

while all terms above that involve an $\eta_{D,12}$ vanish upon integration. The result is

$$\psi_{\Gamma^{l,\mathsf{b}}_{2,3}} = \frac{1}{2}\gamma^{ij}_k\psi_{jlm}z^+_{1,i}\int_{(\partial M)^3_{345}} \nu_{34}\delta_{S^1,45}B_2(t_{34})\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5 \tag{A.25}$$

with ν given by (A.33). Continuing with $\Gamma^{l,\mathsf{a}}_{2,3}$, we have

$$\psi_{\Gamma^{l,\mathsf{a}}_{2,3}} = \mu^i_{jk}\psi_{ilm}\int_{M^2_{12}\times(\partial M)^3_{345}} \mathsf{a}^j_1\eta_{13}\eta_{21}\eta_{24}\eta_{25}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5.$$

which contains the integrals

$$\begin{aligned}\int_{1,2} \eta_{13}\eta_{14}\eta_{12}\mu_2\eta_{25} &= \int_{D,1,2} \mu_2\eta_{D,13}\eta_{D,14}\eta_{D,12}\eta_{D,25}\int_{S^1,1,2} \delta_{S^1,13}\delta_{S^1,14}\delta_{S^1,12}\delta_{S^1,25} \\ &\quad - \left(\int_{D,1,2} \mu_1\mu_2\eta_{D,14}\eta_{D,12}\eta_{D,25}\int_{S^1,1,2} \eta_{S^1,13}\delta_{S^1,14}\delta_{S^1,12}\delta_{S^1,25} + \text{similar}\right) \\ &\quad \int_{D,1,2} \eta_{D,13}\eta_{D,14}\mu_1\mu_2\eta_{D,25}\int_{S^1,1,2} \delta_{S^1,13}\delta_{S^1,14}\eta_{S^1,12}\delta_{S^1,25} \\ &= \epsilon^l_{345}\delta_{S^1,34}\delta_{S^1,45} - \nu_{34}\psi_5\delta_{S^1,34}\eta_{S^1,45} \\ \int_{1,2} \eta_{13}\eta_{14}\eta_{12}\mu_2 dt_2\eta_{25} &= \epsilon^l_{345}\delta_{S^1,34}\delta_{S^1,45}\frac{dt_3+dt_4+dt_5}{3} - \nu_{34}\psi_5\delta_{S^1,34}\eta_{S^1,45}dt_5\end{aligned}$$

where we introduced the notation $\epsilon^l_{345} := \int_{D,1,2}\mu_2\eta_{D,13}\eta_{D,14}\eta_{D,12}\eta_{D,25}$ and noticed that

$$\int_{D,1,2} \mu_1\mu_2\eta_{D,14}\eta_{D,12}\eta_{D,25}$$

and one of the similar terms vanish for degree reasons. In total we end up with

$$\begin{aligned}\psi_{\Gamma^{l,\mathsf{a}}_{2,3}} = \mu^i_{jk}\psi_{ilm}\Bigg[z^{0,j}\int_{(\partial M)^3_{345}} &\left(\epsilon^l_{345}\delta_{S^1,45}\frac{dt_3+dt_4+dt_5}{3} - \nu_{34}d\theta_5\delta_{S^1,34}\eta_{S^1,45}dt_5\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5\right) \\ + z^{1,j}\int_{(\partial M)^3_{345}} &\left(\epsilon^l_{345}\delta_{S^1,45} - \nu_{34}d\theta_5\delta_{S^1,34}\eta_{S^1,45}\right)\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5\Bigg]\end{aligned} \tag{A.26}$$

Next let us look at $\Gamma^{r,\mathsf{b}}_{2,3}$, where we have

$$\psi_{\Gamma^{r,\mathsf{b}}_{2,3}} = \mu^i_{jk}\mu^j_{lm}\int_{M^2_{12}\times(\partial M)^3_{345}} \mathsf{b}_{1,i}\eta_{13}\eta_{12}\eta_{24}\eta_{25}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5.$$

It contains the integral

$$\int_{1,2}\eta_{13}\eta_{14}\eta_{21}\eta_{25} = \int_{1,2}\left(-\eta_{D,13}\eta_{D,14}\delta_{S^1,13}\delta_{S^1,14} + \mu_1\eta_{D,14}\eta_{S^1,13}\delta_{S^1,14} + \mu_1\eta_{D,13}\eta_{S^1,14}\delta_{S^1,13}\right)$$
$$\left(-\eta_{D,21}\eta_{D,25}\delta_{S^1,21}\delta_{S^1,25} + \mu_2\eta_{D,25}\eta_{S^1,21}\delta_{S^1,25} + \mu_2\eta_{D,21}\eta_{S^1,25}\delta_{S^1,21}\right)$$

Notice that the disk part of the second term in the second bracket does not talk to the first bracket at all. Furthermore, products of the third term in the second bracket with the last two terms in the first bracket will vanish after integrating over 1 for the reason discussed above. In the terms with three propagators we recover the forms β and γ from above. The only new form is $\sigma_{345} := \int_{D,1,2}\eta_{D,13}\eta_{D,14}\eta_{D,21}\eta_{D,25}$. In total we obtain

$$\int_{1,2}\eta_{13}\eta_{14}\eta_{21}\eta_{25} = -\sigma_{345}\delta_{S^1,34}\delta_{S^1,45} + \omega_{34}\psi_5\delta_{S^1,34}\eta_{S^1,45} + \gamma_{34}\delta_{S^1,34}\eta_{S^1,45}$$
$$+\left(\beta_{53}\delta_{S^1,35}\eta_{S^1,45} + \psi_3\psi_5\eta_{S^1,34}\eta_{S^1,35} + 3 \leftrightarrow 4\right)$$

and then also

$$\int_{1,2}\eta_{13}\eta_{14}\eta_{21}dt_2\eta_{25} = -\sigma_{345}\delta_{S^1,34}\delta_{S^1,45}\frac{dt_3+dt_4+dt_5}{3} + \omega_{34}\psi_5\delta_{S^1,34}\eta_{S^1,45}dt_5 + \gamma_{34}\delta_{S^1,34}\eta_{S^1,45}dt_5$$
$$+\left(\beta_{53}\delta_{S^1,35}\frac{dt_3+dt_5}{2}\eta_{S^1,45} + \psi_3\psi_5\eta_{S^1,34}\eta_{S^1,35}dt_5 + 3 \leftrightarrow 4\right)$$

Summarizing, the contribution is

$$\begin{aligned}\psi_{\Gamma^{r,\mathsf{b}}_{2,3}} = \mu^i_{jk}\mu^j_{lm}\Bigg[&z^+_{0,i}\int_{(\partial M)^3_{345}}\Bigg(-\sigma_{345}\delta_{S^1,34}\delta_{S^1,45} + \eta_{S^1}(\theta_3,\theta_4)d\theta_5\delta_{S^1,34}\eta_{S^1,45} + \gamma_{34}\delta_{S^1,34}\eta_{S^1,45}\\
&+\beta_{53}\delta_{S^1,35}\eta_{S^1,45} + d\theta_3 d\theta_5\eta_{S^1,34}\eta_{S^1,35} + 3\leftrightarrow 4\Bigg)\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5\\
&+z^+_{1,i}\int_{(\partial M)^3_{345}}\Bigg(-\sigma_{345}\delta_{S^1,34}\delta_{S^1,45}\frac{dt_3+dt_4+dt_5}{3} + \eta_{S^1}(\theta_3,\theta_4)d\theta_5\delta_{S^1,34}\eta_{S^1,45}dt_5 + \gamma_{34}\delta_{S^1,34}\eta_{S^1,45}dt_5\\
&+\left(\beta_{53}\delta_{S^1,35}\frac{dt_3+dt_5}{2}\eta_{S^1,45} + d\theta_3 d\theta_5\eta_{S^1,34}\eta_{S^1,35}dt_5 + 3\leftrightarrow 4\right)\Bigg)\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5\end{aligned} \tag{A.27}$$

where σ_{345} is given by (A.39), β_{34} is given by (A.37), γ_{34} is given by (A.33). In $\Gamma^{r,\mathsf{a}}_{2,3}$, whose contribution is

$$\psi_{\Gamma^{r,\mathsf{a}}_{2,3}} = \psi_{ijk}\mu_{jlm}\int_{M^2_{12}\times(\partial M)^3_{345}}\mathsf{a}^i_1\eta_{13}\eta_{12}\eta_{24}\eta_{25}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5. \tag{A.28}$$

only a single term survives:

$$\int_{1,2} \eta_{13}\eta_{14}\eta_{21}\mu_2\eta_{25} = \int_{1,2} \left(-\eta_{D,13}\eta_{D,14}\delta_{S^1,13}\delta_{S^1,14} + \mu_1\eta_{D,14}\eta_{S^1,13}\delta_{S^1,14} + \mu_1\eta_{D,13}\eta_{S^1,14}\delta_{S^1,13}\right)$$
$$- mu_2\eta_{D,21}\eta_{D,25}\delta_{S^1,21}\delta_{S^1,25}$$
$$= \int_{D,1,2} \eta_{D,13}\eta_{D,14}mu_2\eta_{D,21}\eta_{D,25}int_{S^1,1,2}\delta_{S^1,13}\delta_{S^1,14}\delta_{S^1,21}\delta_{S^1,25}$$
$$= \epsilon^r_{345}\delta_{S^1,34}\delta_{45}\int_{1,2}\eta_{13}\eta_{14}\eta_{21}\mu_2 dt_2\eta_{25}$$
$$= \epsilon^r_{345}\delta_{S^1,34}\delta_{45}\frac{dt_3+dt_4+dt_5}{3}$$

All other terms vanish for degree reasons as in the contribution of $\Gamma^{l,\mathsf{a}}_{2,3}$. In total the contribution is

$$\psi_{\Gamma^{r,\mathsf{a}}_{2,3}} = \psi_{ijk}\mu^j_{lm}z^{0,i}\int_{(\partial M)^3_{345}} \epsilon^r_{345}\delta_{S^1,34}\delta_{45}\frac{dt_3+dt_4+dt_5}{3}\mathbb{A}^k_3\mathbb{A}^l_4\mathbb{A}^m_5. \tag{A.29}$$

4 boundary vertices

There is also a single diagram with 4 boundary vertices $\Gamma_{2,4}$ (figure 5.6e), whose contribution is

$$\psi_{\Gamma_{2,4}} = \mu^i_{jk}\psi_{ilm}\int_{M^2_{12}\times(\partial M)^4_{3456}} \eta_{13}\eta_{14}\eta_{12}\eta_{25}\eta_{26}\mathbb{A}^j_3\mathbb{A}^k_4\mathbb{A}^l_5\mathbb{A}^m_6.$$

It contains the new integral

$$-\int_{D,1,2}\eta_{D,13}\eta_{D,14}\eta_{D,12}\eta_{D,25}\eta_{D,26}\int_{S^1,12}\delta_{S^1,13}\delta_{S^1,14}\delta_{S^1,12}\delta_{S^1,25}\delta_{S^1,26}$$
$$= -z_{3456}$$

Then we have the terms

$$\int_{D,1}\mu_1\eta_{D,13}\eta_{D,14}\int_{D,2}\eta_{D,25}\eta_{D,26}\int_{S^1,12}\delta_{S^1,13}\delta_{S^1,14}\eta_{S^1,12}\delta_{S^1,25}\delta_{S^1,26}$$
$$= \nu_{34}\omega_{26}\delta_{S^1,34}\eta_{S^1,45}\delta_{S^1,56}$$
$$\int_{D,1,2}\mu_1\eta_{D,13}\eta_{D,12}\eta_{D,25}\eta_{D,26}\int_{S^1,12}\delta_{S^1,13}\eta_{S^1,14}\delta_{S^1,12}\delta_{S^1,25}\delta_{S^1,26} - (3\leftrightarrow 4) \qquad = \epsilon^r_{356}\delta_{35}\eta_{34}\delta_{56} - (3\leftrightarrow 4)$$
$$\int_{D,1,2}\eta_{D,13}\eta_{D,14}\eta_{D,12}\mu_2\eta_{D,26}\int_{S^1,12}\delta_{S^1,13}\delta_{S^1,14}\delta_{S^1,12}\eta_{S^1,25}\delta_{S^1,26} - (5\leftrightarrow 6)$$
$$= \epsilon^l_{346}\delta_{S^1,34}\delta_{S^1,46}\eta_{S^1,56} - (5\leftrightarrow 6)$$
$$\int_{D,1}\mu_1\eta_{D,13}\eta_{D,14}\int_{D,2}\mu_2\eta_{D,26}\int_{S^1,12}\delta_{S^1,13}\delta_{S^1,14}\eta_{S^1,12}\eta_{S^1,25}\delta_{S^1,26} - (5\leftrightarrow 6)$$
$$= \nu_{34}\psi_6 g_{35} - (5\leftrightarrow 6).$$

In total, we get

$$\psi_{\Gamma_{2,4}} = \mu^i_{jk}\psi_{ilm}\int_{(\partial M)^4_{3456}}\Big(- z_{3456} + \nu_{34}\eta_{S^1}(\theta_5,\theta_6)\delta_{S^1,34}\eta_{S^1,45}\delta_{S^1,56}$$
$$+ (\epsilon^r_{356}\delta_{35}\eta_{34}\delta_{56} - (3\leftrightarrow 4)) \tag{A.30}$$
$$+ \epsilon^l_{346}\delta_{S^1,34}\delta_{S^1,46}\eta_{S^1,56} + \nu_{34}B_2(t_{35})/2d\theta_6 - (5\leftrightarrow 6)\Big)\mathbb{A}^j_3\mathbb{A}^k_4\mathbb{A}^l_5\mathbb{A}^m_6$$

A.4 Pushforwards on the disk

In this section we compute the pushforwards that we collected before. To this end we expand the disk propagator $\eta_{D,12} = \phi_{12} + \tau_{12} - \psi_1$ and use the results of the next section. Here we understand the free points to lie on the boundary of the disk.

A.4.1 Results

If $z_i \in \partial D$ then we write $z_i = \exp(i2\pi\theta_i)$, for $\theta \in [0,1)$.

One bulk point

With this notation we have

$$\int_{D_1}\mu_1\eta_{D,12} = \psi_2 = \frac{1}{4\pi i}(\bar z_2 dz_2 - z_2 d\bar z_2) = d\theta_2 \tag{A.31}$$
$$\omega_{23} = \int_{D_1}\eta_{D,12}\eta_{D,13} = \eta_{S_1}(\theta_2,\theta_3) \tag{A.32}$$
$$\nu_{23} = \int_{D_1}\mu_1\eta_{D,12}\eta_{D,13} = -g(\theta_2,\theta_3)d\theta_2 d\theta_3 \tag{A.33}$$
$$\alpha_{234} = \int_{D_1}\eta_{D,12}\eta_{D,13}\eta_{D,14} = \frac{1}{2\pi}\log\left|\frac{\sin\pi\theta_{43}}{\sin\pi\theta_{23}}\right|\cot(\pi\theta_{23})d\theta_{23} \tag{A.34}$$
$$+\frac{1}{2\pi}\left(1 - f(\theta_2,\theta_3)\right)(d\theta_2 + d\theta_3)$$
$$+ \text{ cycl.}$$

where

$$f(\theta_2,\theta_3) = \cos(2\pi\theta_{23}))\pi\eta_{S^1}(\theta_2,\theta_3) - \sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23} \tag{A.35}$$

$$g(\theta_2,\theta_3) = 1 + 2\cos(2\pi\theta_{23})\log 2|\sin\pi\theta_{23}| + 2\pi\sin(2\pi\theta_{23})\eta_{S^1}(\theta_2\theta_3) \tag{A.36}$$

Two bulk points

$$\beta_{34} = \int_{D,1,2} \mu_2 \eta_{D,13}\eta_{D,12}\eta_{D,24} = \frac{1}{\pi}\eta_{S^1}(\theta_2,\theta_4)(\cos(2\pi i\theta_{24})-1) - \log 2|\sin\pi\theta_{24}|\sin(2\pi\theta_{24})]d\theta_4 \tag{A.37}$$

$$\gamma_{34} = \int_{D_{1,2}} \mu_2\eta_{D,21}\eta_{D,13}\eta_{D,14} = \frac{1}{\pi}\left(\pi\cos(2\pi\theta_{34})\eta_{S^1}(\theta_3,\theta_4) - \sin 2\pi\theta_{34}\log 2|\sin\pi\theta_{34}|\right)(d\theta_3+d\theta_4) \tag{A.38}$$

$$\sigma_{345} = \int_{D_{1,2}} \eta_{D,13}\eta_{D,14}\eta_{D,21}\eta_{D,25} = \frac{\pi}{2}\left(\theta_{34}(2\theta_{35}+2\theta_{45}-1)\right) + \frac{1}{2\pi}(\log|\sin\pi\theta_{34}|-1/4)(\cos 4\pi\theta_4 - \cos 4\pi\theta_3). \tag{A.39}$$

$$\epsilon^l_{345} = \int_{D_{1,2}} \mu_2\eta_{D,12}\eta_{D,13}\eta_{D,14}\eta_{D,25} \tag{A.40}$$

$$\begin{aligned} &= \frac{1}{2\pi}d\theta_3 d\theta_5\Bigg\{ -2\log|\sin\pi\theta_{34}|\sin 2\pi\theta_{34} - 2\pi\eta_{S^1}(\theta_3,\theta_4) \\ &+ \mathrm{Im}\left[\frac{K_{45}\bar{z}_{45}+K_{43}\bar{z}_{34}}{\bar{z}_{35}}\bar{z}_3 z_5 F(z_4\bar{z}_5, z_3\bar{z}_5)\right. \\ &+ (K_{54}u_{45} - K_{53}u_{35})\mathrm{i}\cot\pi\theta_{34} + K_{53}u_{35} \\ &+ z_3\bar{z}_5\left(K_{53}(K_{34}+K_{43}+K_{45}+u_{35}) - Li_2\left(\frac{\bar{z}_{35}}{\bar{z}_{45}}\right) + Li_2\left(\frac{1}{u_{54}}\right) - \frac{Li_2(z_3\bar{z}_5)}{2}\right)\Bigg\} - (3\leftrightarrow 4) \end{aligned} \tag{A.41}$$

A.4.2 Proofs

Proof of (A.31)

We have

$$\int_{D_1}\mu_1\eta_{D,12} = \int_{D_1}\mu_1(\phi_{12}+\tau_{12}-\psi_1) = \psi_2$$

by equations (A.62d), (A.62c).

Proof of (A.32)

We have

$$\begin{aligned} \int_{D_1}\eta_{D,12}\eta_{D,13} &= \int_{D_1}(\phi_{12}+\tau_{12}-\psi_1)(\phi_{13}+\tau_{13}-\psi_1) \\ &= \int_{D_1}\phi_{12}\phi_{13}+\phi_{12}\tau_{13}+\tau_{12}\phi_{13}+\tau_{12}\tau_{13}-\psi_1(\phi_{13}+\tau_{13}-\phi_{12}-\tau_{12}) \\ &= \frac{1}{4\pi}(\arg(u_{23})+\arg(u_{23})-\arg(u_{32})+\arg(u_{23})) = \frac{1}{\pi}\arg(u_{23}) = \eta_{S_1}(\theta_2,\theta_3) \end{aligned}$$

where we have used equations (A.62) and the last equality is Proposition A.6.1. Alternatively, one could directly apply equations (A.64). Alternatively, we can make use of the fact that for $z_2 \in \partial D$ we have $\tau_{12} = \phi_{12} - \psi_2$ to obtain the same result.

Proof of (A.33)

We have

$$\begin{aligned}\int_{D_1} \mu_1 \eta_{D,12}\eta_{D,13} &= \int_{D_1} \mu_1(\phi_{12} + \tau_{12} - \psi_1)(\phi_{13} + \tau_{13} - \psi_1) \\ &= \int_{D_1} \mu_1\phi_{12}\phi_{13} + \mu_1\phi_{12}\tau_{13} - \mu_1\phi_{13}\tau_{12} + \mu_1\tau_{12}\tau_{13} \\ &= \int_{D_1} 4\mu_1\phi_{12}\phi_{13} - \mu\phi_{12}\psi_3 - \mu_1\psi_2\phi_{13} - \mu\phi_{12}\psi_3 - \mu_1\psi_2\phi_{13} + \mu_1\psi_2\psi_3 \\ &= 4\cdot\frac{1}{2}d\theta_2 d\theta_3\left[1 - \cos(2\pi\theta_{23})\log(2|\sin\pi\theta_{23}|) - \pi\sin(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3)\right] - 3d\theta_2 d\theta_3 \\ &= -d\theta_2 d\theta_3\left[1 + 2\cos(2\pi\theta_{23})\log 2|\sin\pi\theta_{23}| + 2\pi\sin(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3)\right]\end{aligned}$$

using equation (A.65h).

Proof of (A.35)

$$\begin{aligned}\int_{D_1} \eta_{D,12}\eta_{D,13}\eta_{D,14} &= \int_{D_1} (\phi_{12} + \tau_{12} - \psi_1)(\phi_{13} + \tau_{13} - \psi_1)(\phi_{14} + \tau_{14} - \psi_1) \\ &= \int_{D_1} (2\phi_{12} - \psi_1 - \psi_2)(2\phi_{13} - \psi_1 - \psi_3)(2\phi_{14} - \psi_1 - \psi_4) \\ &= \int_{D_1} 8\phi_{12}\phi_{13}\phi_{14} - 4\psi_1(\phi_{12}\phi_{13} + \phi_{14}\phi_{12} + \phi_{13}\phi_{14}) \\ &\quad - 4(\phi_{12}\phi_{13}\psi_4 + \phi_{14}\phi_{12}\psi_3 + \phi_{13}\phi_{14}\psi_2)\end{aligned}$$

where terms with less than two ϕ's vanish for degree reasons or because $\int_{D_1}\psi_1\phi_{1j} = 0$ (this is equation (A.62a)). From (A.64b) we know that $\int_{D_1}\phi_{12}\phi_{13} = 1/4\eta_{S^1}(\theta_2,\theta_3)$, so the last term equals

$$\int_{D_1} -4(\phi_{12}\phi_{13}\psi_4 + \phi_{14}\phi_{12}\psi_3 + \phi_{13}\phi_{14}\psi_2) = -(\eta_{S^1}(\theta_2,\theta_3)d\theta_4 + \text{ cycl.})).$$

Also, we have

$$\int_{D_1} \psi_1\phi_{12}\phi_{13} = \frac{1}{4\pi}\left(\pi\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3) - \sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|\right)(d\theta_2 + d\theta_3)$$

so for the second term we get

$$\begin{aligned}\int_{D_1} 4\psi_1(\phi_{12}\phi_{13} + \phi_{14}\phi_{12} + \phi_{13}\phi_{14}) = &\left(\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3) - \frac{1}{\pi}\sin(2\pi\theta_{23})\log 2|\sin\pi\theta_{23}|\right)(d\theta_2 + d\theta_3) \\ &+ \text{ cycl.}\end{aligned}$$

Combining this with the expression (A.65a) for $\int_{D_1} \phi_{12}\phi_{13}\phi_{14}$, we arrive at

$$\begin{aligned}\alpha_{234} &= \frac{1}{2\pi}\log\left|\frac{\sin\pi\theta_{43}}{\sin\pi\theta_{23}}\right|\cot(\pi\theta_{23})d\theta_{23} \\ &+ \frac{1}{2\pi}\left((1-\cos(2\pi\theta_{23}))\pi\eta_{S^1}(\theta_2,\theta_3) + \sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|\right)(d\theta_2 + d\theta_3) \\ &+ \text{ cycl.}\end{aligned}$$

(here cycl. means one should sum over cylic permutations of both terms).

Proof of (A.37)

We start to compute $\int_1 \eta_{D,12}\eta_{D,13}$ if $z_3 \notin \partial D$. In that case we have

$$\begin{aligned}\int_1 \eta_{D,12}\eta_{D,13} &= \int_1 (2\phi_{12} - \psi_1 - \psi_2)(\phi_{13} + \tau_{13} + \psi_1) \\ &= 2\int_1 \phi_{12}\phi_{13} + 2\int_1 \phi_{12}\tau_{13} = \frac{1}{\pi}\arg(u_{23}).\end{aligned}$$

Hence we are now interested in computing (remember that $z_4 \in \partial D$)

$$\beta_{24} = \int_3 \arg(u_{23})\mu_3\eta_{D,34} = \int_3 \operatorname{Im} K_{23}\mu_3(2\phi_{34} + \psi_1 - \psi_3).$$

By (A.60a), this reduces to

$$\begin{aligned}\frac{1}{\pi}\int_3 \operatorname{Im} K_{23}\operatorname{Im} dL_{34}\mu_3 &= \frac{1}{2\pi}\operatorname{Re}\int_3 (K_{23}\overline{dL}_{34} - K_{23}dL_{34})\mu_3 \\ &= \frac{1}{2\pi}\operatorname{Re}\frac{1}{2\pi i}\int_3 -\frac{K_{23}d\bar{z}_3dz_3d\bar{z}_4}{\bar{z}_{34}} + \frac{K_{23}d\bar{z}_3dz_3dz_4}{z_{34}} \quad = \frac{1}{2\pi}\operatorname{Re}(\bar{z}_4 + K_{24}(\bar{z}_{24})dz_4)\end{aligned}$$

The first term vanishes by (A.60d) since $z_2 \in \partial D$, the second term gives, using equation (A.60e)

$$\begin{aligned}\frac{1}{2\pi}\operatorname{Re}\left[\bar{z}_4 + K_{24}\bar{z}_{24}\right]dz_4 &= \frac{1}{2\pi}\operatorname{Re} -K_{24}u_{42}\bar{z}_4dz_4 = \operatorname{Im} K_{24}u_{42}d\theta_4 \\ &= [\eta_{S^1}(\theta_2,\theta_4)(\cos(2\pi i\theta_{24}) - 1) - \log 2|\sin\pi\theta_{24}|\sin(2\pi\theta_{24})]d\theta_4.\end{aligned}$$

Proof of (A.38)

Since $\int_{D_1} \mu_1\eta_{12} = \psi_2$ we immediately get

$$\gamma_{34} = \int_{D_{1,2}} \mu_2\eta_{D,21}\eta_{D,13}\eta_{D,14} = \int_{D_1} \psi_1\eta_{D,13}\eta_{D,34}.$$

Using that $\eta_{D_{1,3}} = 2\phi_{13} + \psi_1 - \psi_3$ we get that this equals

$$\gamma_{34} = \int_{D,1} 4\psi_1\phi_{13}\phi_{14} = \frac{1}{\pi}\left(\pi\cos(2\pi\theta_{34})\eta_{S^1}(\theta_3,\theta_4) - \sin 2\pi\theta_{34}\log 2|\sin\pi\theta_{34}|\right)(d\theta_3 + d\theta_4).$$

Proof of (A.39)

We have

$$\sigma_{345} = \int_{D,1,2} \eta_{D,13}\eta_{D,14}\eta_{D,21}\eta_{D,25}$$
$$= \int_{D,1,2} \eta_{D,13}\eta_{D,14}2(\phi_{21}+\tau_{21})\phi_{25}$$

(other terms vanish for degree reasons or because of (A.62b), (A.62a)). Expanding further we get

$$\int_{D,1,2} 8\phi_{13}\phi_{14}(\phi_{21}+\tau_{21})\phi_{25} + \left(\int_{D,1,2} 4\psi_1\phi_{14}(\phi_{12}+\tau_{12})\phi_{25} - (3\leftrightarrow 4)\right)$$
$$= \frac{1}{2\pi}\operatorname{Re}\left[Li_{1,1}(\bar z_4 z_3, \bar z_5 z_4) + Li_2(\bar z_5 z_3) + K_{45}M_{43} - (3\leftrightarrow 4)\right]$$
$$+ \frac{1}{4\pi}\operatorname{Re}\left[\bar z_4^2 K_{34}(\bar z_3^2 z_4^2 - 1) + \bar z_3\bar z_4 + \bar z_4^2/2 - (3\leftrightarrow 4)\right]$$
$$= \frac{1}{2\pi}\operatorname{Re}[Li_{1,1}(\bar z_4 z_3, \bar z_5 z_4) - Li_{1,1}(z_4\bar z_3, \bar z_5 z_3) + Li_2(\bar z_5 z_3) - Li_2(\bar z_5 z_4) - K_{45}K_{34} + K_{35}K_{43}]$$
$$+ \frac{1}{4\pi}\operatorname{Re}[(\bar z_3^2 - \bar z_4^2)(K_{34}+K_{43}-1/2)]$$

To simplify it further, we use the relation ([Zag07])

$$Li_{1,1}(x,y) = Li_1(x)Li_1(y) + Li_2\left(\frac{-x}{1-x}\right) - Li_2\left(\frac{xy-x}{1-x}\right) \tag{A.42}$$

We apply this equation to $Li_{1,1}(x,y_1) - Li_{1,1}(x,y_2)$ appearing above with $x = z_3\bar z_4, y_1 = z_4\bar z_5, y_2 = \bar z_3 z_5$) (we conjugate the arguments of second double logarithm inside the real part to this end). Notice that $Li_1(z_i\bar z_j) = -\log(1-z_i\bar z_j) = -K_{ij}$, so that the K's above cancel, and we are left with

$$\operatorname{Re}[Li_{1,1}(\bar z_4 z_3, \bar z_5 z_4) - Li_{1,1}(z_4\bar z_3, \bar z_5 z_3) + K_{34}(K_{53} - K_{45})]$$
$$= \operatorname{Re}\left[Li_2\left(\frac{\bar z_4 z_5 - \bar z_3 z_5}{1 - z_3\bar z_4}\right) - Li_2\left(\frac{\bar z_5 z_3 - \bar z_5 z_4}{1 - z_3\bar z_4}\right)\right] = \operatorname{Re}\left[Li_2(-\bar z_3 z_5) - Li_2(-z_4\bar z_5)\right]$$

Now we can use the duplication formula

$$Li_2(z) + Li_2(-z) = \frac{1}{2}Li_2(z^2)$$

to simplify the first term above to

$$\frac{1}{4\pi}\operatorname{Re}[Li_2((z_3\bar z_5)^2) - Li_2((z_4\bar z_5)^2)].$$

The real part of the dilogarithm can be evaluated exactly on the unit circle: For $0 \le \theta \le 1$

$$\operatorname{Re} Li_2\left(e^{2\pi i\theta}\right) = \pi^2\left(\frac{1}{6} - \theta + \theta^2\right)$$

which follows e.g. from Fourier series expansion of the RHS, or from the more general equation

$$Li_n(e^{2\pi \mathrm{i} x}) + (-1)^n Li_n(e^{-2\pi \mathrm{i} x}) = -\frac{(2\pi \mathrm{i})^n}{n!} B_n(x). \tag{A.43}$$

So

$$\begin{aligned}
\mathrm{Re}[Li_2((\bar{z}_5 z_3)) - Li_2((\bar{z}_5 z_4)^2)] &= \mathrm{Re}\left[Li_2\left(e^{4\pi \mathrm{i}\theta_{35}}\right) - Li_2\left(e^{4\pi \mathrm{i}\theta_{45}}\right)\right] \\
&= \pi^2[B_2(2\theta_{35}) - B_2(2\theta_{45})] \\
&= \pi^2[-2(\theta_{35}) + 4(\theta_{35})^2 + 2\theta_{45} - 4\theta_{45}^2] = \pi^2[2\theta_{43} + 4\theta_3^2 - 4\theta_4^2 + 8\theta_5\theta_{43}] \\
&= 2\pi^2\theta_{34}(2\theta_{35} + 2\theta_{45} - 1)
\end{aligned}$$

Also, notice that $K_{34} + K_{43} = 2\,\mathrm{Re}\, K_{34} = 2\log|\sin \pi\theta_{34}|$ is purely real. With this, we can make the following simplifications:

$$\sigma_{345} = \frac{\pi}{2}\left(\theta_{34}(2\theta_{35} + 2\theta_{45} - 1)\right) + \frac{1}{2\pi}(\log|\sin \pi\theta_{34}| - 1/4)(\cos 4\pi\theta_4 - \cos 4\pi\theta_3).$$

Proof of (A.41)

We will first prove the following very technical lemma.

Lemma A.4.1. *For $z_3, z_4 \in \partial D$, (but not necessarily z_2) we have*

$$\begin{aligned}
\int_{D,1} \eta_{D,12}\eta_{D,13}\eta_{D,14} = \frac{1}{4\pi^2}\,\mathrm{Re}\Bigg[&2(K_{24} - K_{23})dL_{34} + (K_{34} + K_{43})(dL_{23} - dL_{24}) \\
&+ K_{42}\left(\frac{\bar{z}_2^2 dz_3 + d\bar{z}_2}{\bar{z}_2 u_{32}} - dL_{23}\right) - K_{32}\left(\frac{\bar{z}_2^2 dz_4 + d\bar{z}_2}{\bar{z}_2 u_{42}} - dL_{24}\right) \\
&+ K_{34}\left(z_3 d\bar{z}_3 + \bar{z}_4 dz_3 - \bar{z}_4 dz_4 - z_3 d\bar{z}_4\right)\Bigg] \\
&+ \frac{1}{8\pi^2}\,\mathrm{Re}\Bigg[\left(\bar{z}_3 K_{32} + \bar{z}_2 - \frac{1}{z_2} - \frac{z_3 K_{23}}{z_2^2}\right)dz_2 - \left(\bar{z}_2 K_{23} + \frac{K_{23}}{z_2}\right)dz_3 \\
&+ (1 - |z_2|^2)dL_{23} - \big(3 \leftrightarrow 4\big)\Bigg] + \frac{1}{\pi}\left[d\theta_3 \arg(u_{42}) - d\theta_4 \arg(u_{23})\right]
\end{aligned} \tag{A.44}$$

and

$$\begin{aligned}
\int_{D,1} \eta_{D,21}\eta_{D,13}\eta_{D,14} = \frac{1}{4\pi^2}\,\mathrm{Re}\Bigg[&(K_{43} - K_{34})(dL_{23} - dL_{24}) - K_{34}\left(\frac{d\bar{z}_3}{\bar{z}_3} - \frac{dz_4}{z_4}\right) \\
&- K_{42}\left(\frac{\bar{z}_2^2 dz_3 + d\bar{z}_2}{\bar{z}_2 u_{32}} + dL_{23}\right) + K_{32}\left(\frac{\bar{z}_2^2 dz_4 + d\bar{z}_2}{\bar{z}_2 u_{42}} + dL_{24}\right) \\
&- \left(\bar{z}_3 K_{32} + \bar{z}_2 + \frac{1}{z_2} + \frac{z_3 K_{23}}{z_2^2}\right)dz_2 + K_{23}\left(\frac{1}{z_2} - \bar{z}_2\right)dz_3 \\
&+ (1 - |z_2|^2)dL_{23} - (3 \leftrightarrow 4)\Bigg] - \psi_2 \frac{1}{\pi}\arg(u_{34}).
\end{aligned} \tag{A.45}$$

Notice that the first term reduces to (A.35) for $z_2 \in \partial D$, while the second vanishes for $z_2 \in \partial D$.

Proof. In the first case we have

$$\int_{D_1} \eta_{D,12}\eta_{D,13}\eta_{D,14} = \int_{D_1} (\phi_{12} + \tau_{12} - \psi_1)(2\phi_{13} - \psi_1 - \psi_3)(2\phi_{14} - \psi_1 - \psi_4)$$
$$= \int_{D_1} 4(\phi_{12}\phi_{13}\phi_{14} + \tau_{12}\phi_{13}\phi_{14}) - \psi_1(4\phi_{13}\phi_{14} + 2\phi_{14}\phi_{12} + 2\phi_{14}\tau_{12} + 2\phi_{12}\phi_{13} + 2\tau_{12}\tau_{13})$$
$$- 2\psi_3(\phi_{14}\phi_{12} + \phi_{14}\tau_{12}) - 2\psi_4(\phi_{12}\phi_{13} + \tau_{12}\phi_{13}).$$

The last two terms integrate to $+\frac{1}{\pi}d\theta_3 \arg(u_{24})$ and $-\frac{1}{\pi}d\theta_4 \arg(u_{23})$ respectively. Now let us look at the terms in middle. If $z_3 \in \partial D$ we get (equation (A.63e))

$$\int_{D_1} \psi_1\phi_{12}\phi_{13} = \frac{1}{16\pi^2} \operatorname{Re}\left[-(\bar z_3 M_{23} - \bar z_2)\, dz_2 + (\bar z_2 M_{32} - \bar z_3)\, dz_3 + \frac{|z_2|^2 - |z_3|^2}{\bar z_{23}} d\bar z_{32}\right]$$
$$= \frac{1}{16\pi^2} \operatorname{Re}\left[(\bar z_3 K_{32} + \bar z_2) dz_2 - \bar z_2 K_{23} dz_3 + (1 - |z_2|^2) dL_{23}\right] \tag{A.46}$$

(notice $\operatorname{Re} \bar z_3 dz_3 = 0$ if $|z_3| = 1$) and (equation (A.63g))

$$\int_{D_1} \psi_1\tau_{12}\phi_{13} = -\frac{1}{16\pi^2} \operatorname{Re}\left[\left(\bar z_3 + \frac{K_{23}}{z_2}\right) dz_3 + \left(\frac{|z_3|^2}{\bar z_2} + \frac{\bar z_3 K_{32}}{\bar z_2^2}\right) d\bar z_2 + (1 - |z_3|^2) dK_{23}\right]$$
$$= -\frac{1}{16\pi^2} \operatorname{Re}\left[\frac{K_{23}}{z_2} dz_3 + \left(\frac{1}{\bar z_2} + \frac{\bar z_3 K_{32}}{\bar z_2^2}\right) d\bar z_2\right] \tag{A.47}$$

Taking the sum of (A.46) and (A.47) we obtain

$$\psi_1(2\phi_{14}\phi_{12} + 2\phi_{14}\tau_{12} + 2\phi_{12}\phi_{13} + 2\tau_{12}\tau_{13}) = \frac{1}{8\pi^2} \operatorname{Re}\Bigg[\left(\bar z_3 K_{32} + \bar z_2 - \frac{1}{z_2} - \frac{z_3 K_{23}}{z_2^2}\right) dz_2$$
$$- K_{23}\left(\bar z_2 + \frac{1}{z_2}\right) dz_3$$
$$+ (1 - |z_2|^2) dL_{23} - (3 \leftrightarrow 4)\Bigg]$$

Now let us look at the first two terms. When $z_3, z_4 \in \partial D$ we have

$$\int_{D_1} \phi_{12}\phi_{13}\phi_{14} = \frac{1}{16\pi^2} \operatorname{Re}\left[(M_{24} - M_{34}) dL_{23} + \text{ cycl.}\right]$$
$$= \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{43} - K_{42}) dL_{23} + (K_{24} - K_{23}) dL_{34} + (K_{32} - K_{34}) dL_{42}\right] \tag{A.48}$$

(notice that $M_{ij} = -K_{ji}$ if one of $z_i, z_j \in \partial D$). Also

$$\int_{D_1} \tau_{12}\phi_{13}\phi_{14} = \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{24} - K_{23}) dL_{43} + M_{43}(dK_{42} - dK_{23}) + K_{42}\frac{\bar z_2^2 dz_3 + d\bar z_2}{\bar z_2 u_{32}} - K_{32}\frac{\bar z_2^2 dz_4 + d\bar z_2}{\bar z_2 u_{42}}\right]$$
$$= \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{24} - K_{23}) dL_{34} + K_{34}(dK_{23} - dK_{42}) + K_{42}\frac{\bar z_2^2 dz_3 + d\bar z_2}{\bar z_2 u_{32}} - K_{32}\frac{\bar z_2^2 dz_4 + d\bar z_2}{\bar z_2 u_{42}}\right]$$
$$= \frac{1}{16\pi^2} \operatorname{Re}\Bigg[(K_{24} - K_{23}) dL_{34} + K_{34}\left(dL_{23} + \frac{d\bar z_3}{\bar z_3} - \overline{dL_{42}} - \frac{dz_4}{z_4}\right)$$
$$+ K_{42}\frac{\bar z_2^2 dz_3 + d\bar z_2}{\bar z_2 u_{32}} - K_{32}\frac{\bar z_2^2 dz_4 + d\bar z_2}{\bar z_2 u_{42}}\Bigg] \tag{A.49}$$

Taking the sum of (A.48) and (A.49) we obtain

$$\int_{D_1} 4(\phi_{12}\phi_{13}\phi_{14} + \tau_{12}\phi_{13}\phi_{14}) = \frac{1}{4\pi^2}\,\mathrm{Re}\Bigg[2(K_{24} - K_{23})dL_{34} + (K_{34} + K_{43})(dL_{23} - dL_{24})$$
$$+ K_{34}\left(\frac{d\bar{z}_3}{\bar{z}_3} - \frac{dz_4}{z_4}\right) + K_{42}\left(\frac{\bar{z}_2^2 dz_3 + d\bar{z}_2}{\bar{z}_2 u_{32}} - dL_{23}\right)$$
$$- K_{32}\left(\frac{\bar{z}_2^2 dz_4 + d\bar{z}_2}{\bar{z}_2 u_{42}} - dL_{24}\right)\Bigg] \tag{A.50}$$

So, taking the sum of (A.50) and (A.48), in total we arrive at

$$\int_{D_1} \eta_{D,12}\eta_{D,13}\eta_{D,14} = \frac{1}{4\pi^2}\,\mathrm{Re}\Bigg[2(K_{24} - K_{23})dL_{34} + (K_{34} + K_{43})(dL_{23} - dL_{24})$$
$$+ K_{42}\left(\frac{\bar{z}_2^2 dz_3 + d\bar{z}_2}{\bar{z}_2 u_{32}} - dL_{23}\right) - K_{32}\left(\frac{\bar{z}_2^2 dz_4 + d\bar{z}_2}{\bar{z}_2 u_{42}} - dL_{24}\right)$$
$$+ K_{34}\left(z_3 d\bar{z}_3 + \bar{z}_4 dz_3 - \bar{z}_4 dz_4 - z_3 d\bar{z}_4\right)\Bigg]$$
$$+ \frac{1}{8\pi^2}\,\mathrm{Re}\Bigg[\left(\bar{z}_3 K_{32} + \bar{z}_2 - \frac{1}{z_2} - \frac{z_3 K_{23}}{z_2^2}\right)dz_2 - \left(\bar{z}_2 K_{23} + \frac{K_{23}}{z_2}\right)dz_3$$
$$+ (1 - |z_2|^2)dL_{23} - \big(3 \leftrightarrow 4\big)\Bigg] + \frac{1}{\pi}\left[d\theta_3 \arg(u_{42}) - d\theta_4 \arg(u_{23})\right].$$

In the second case we have

$$\int_{D_1} \eta_{D,21}\eta_{D,13}\eta_{D,14} = \int_{D_1} (\phi_{12} - \tau_{12} - \psi_2)(2\phi_{13} - \psi_1 - \psi_3)(2\phi_{14} - \psi_1 - \psi_4)$$
$$= \int_{D_1} 4(\phi_{12}\phi_{13}\phi_{14} - \tau_{12}\phi_{13}\phi_{14}) - 4\psi_2\phi_{13}\phi_{14} - 2\psi_1(\phi_{14}\phi_{12} - \phi_{14}\tau_{12} + \phi_{12}\phi_{13} - \tau_{12}\phi_{13})$$
$$- 2\psi_3(\phi_{14}\phi_{12} - \phi_{14}\tau_{12}) - 2\psi_4(\phi_{12}\phi_{13} - \tau_{12}\phi_{13}).$$

The last two terms integrate to zero by (A.62). Next, taking the difference of (A.46) and (A.47), we obtain

$$2\psi_1(\phi_{14}\phi_{12} - \phi_{14}\tau_{12} + \phi_{12}\phi_{13} - \tau_{12}\phi_{13})$$
$$= \frac{1}{8\pi^2}\,\mathrm{Re}\Bigg[\left(\bar{z}_3 K_{32} + \bar{z}_2 + \frac{1}{z_2} + \frac{z_3 K_{23}}{z_2^2}\right)dz_2 + K_{23}\left(\frac{1}{z_2} - \bar{z}_2\right)dz_3$$
$$+ (1 - |z_2|^2)dL_{23} - (3 \leftrightarrow 4)\Bigg].$$

The next term is

$$\int_{D,1} 4\psi_2\phi_{13}\phi_{14} = \psi_2 \frac{1}{\pi}\arg(u_{34}).$$

Finally, the difference of (A.48) and (A.49) is

$$\int_{D,1} \phi_{12}\phi_{13}\phi_{14} - \tau_{12}\phi_{13}\phi_{14} = \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{43} - K_{34})(dL_{23} - dL_{24}) - K_{34}\left(\frac{d\bar{z}_3}{\bar{z}_3} - \frac{dz_4}{z_4}\right)\right.$$
$$\left. - K_{42}\left(\frac{\bar{z}_2^2 dz_3 + d\bar{z}_2}{\bar{z}_2 u_{32}} + dL_{23}\right) + K_{32}\left(\frac{\bar{z}_2^2 dz_4 + d\bar{z}_2}{\bar{z}_2 u_{42}} + dL_{24}\right)\right]$$

Taking the sum we obtain

$$\int_{D,1} \eta_{D,21}\eta_{D,13}\eta_{D,14} = \frac{1}{4\pi^2} \operatorname{Re}\left[(K_{43} - K_{34})(dL_{23} - dL_{24}) - K_{34}\left(\frac{d\bar{z}_3}{\bar{z}_3} - \frac{dz_4}{z_4}\right)\right.$$
$$- K_{42}\left(\frac{\bar{z}_2^2 dz_3 + d\bar{z}_2}{\bar{z}_2 u_{32}} + dL_{23}\right) + K_{32}\left(\frac{\bar{z}_2^2 dz_4 + d\bar{z}_2}{\bar{z}_2 u_{42}} + dL_{24}\right)$$
$$- \left(\bar{z}_3 K_{32} + \bar{z}_2 + \frac{1}{z_2} + \frac{z_3 K_{23}}{z_2^2}\right) dz_2 + K_{23}\left(\frac{1}{z_2} - \bar{z}_2\right) dz_3$$
$$\left. + (1 - |z_2|^2) dL_{23} - (3 \leftrightarrow 4)\right] - \psi_2 \frac{1}{\pi} \arg(u_{34}).$$

□

Now we are interested in

$$\epsilon^l_{345} = \int_{D_{1,2}} \eta_{D,12}\eta_{D,13}\eta_{D,14}\mu_2\eta_{D,25} = \int_{D_{1,2}} \eta_{D,12}\eta_{D,13}\eta_{D,14}\mu_2(2\phi_{25} - \psi_5)$$
$$= \int_{D_2} \operatorname{Re}[X_{234}]\mu_2\left(\frac{1}{\pi}\operatorname{Im} dL_{25} - d\theta_5\right)$$

Notice that $\mu = \frac{d\bar{z}dz}{2\pi i} = \frac{dxdy}{\pi}$ is real, so we can take μ inside the real part to get

$$\epsilon^l_{345} = d\theta_5 \operatorname{Re}\int_{D_2} X_{234}\mu_2 + \frac{1}{2\pi}\operatorname{Im}\int_{D_2} X_{234}\mu_2 dL_{25} - \frac{1}{2\pi}\operatorname{Im}\int_{D_2} X_{234}\mu_2\overline{dL_{25}}$$

(since $\operatorname{Re} z \operatorname{Im} w = 1/2Im(zw - z\bar{w})$.) The product of X_{234} with μ_2 is

$$X_{234}\mu_2 = \frac{1}{4\pi^2}\left[2(K_{24} - K_{23})\mu_2 dL_{34} + (K_{34} + K_{43})\mu_2\left(\frac{dz_4}{z_{24}} - \frac{dz_3}{z_{23}}\right)\right.$$
$$\left. + \left(K_{42}\mu_2 dz_3\left(\frac{\bar{z}_2}{u_{32}} + \frac{1}{z_{23}}\right) - (3 \leftrightarrow 4)\right) + K_{34}\left(z_3 d\bar{z}_3 + \bar{z}_4 dz_3 - \bar{z}_4 dz_4 - z_3 d\bar{z}_4\right)\mu_2\right]$$
$$- \frac{1}{8\pi^2}\left[\mu_2\left(dz_3\left(\bar{z}_2 K_{23} + \frac{K_{23}}{z_2} + \frac{1 - |z_2|^2}{z_{23}}\right) - (3 \leftrightarrow 4)\right)\right] - \frac{i\mu_2}{\pi}(K_{24}d\theta_3 - K_{23}d\theta_4)$$

Now we integrate this over z_2. The first and last terms will drop out because $\int \mu_1 K_{12} = 0$. For the second term we use $\int \frac{\mu_1}{z_{12}} = -\bar{z}_2$, so integrates to

$$\int_2 (K_{34} + K_{43})\mu_2\left(\frac{dz_4}{z_{24}} - \frac{dz_3}{z_{23}}\right) = (K_{34} + K_{43})(\bar{z}_3 dz_3 - \bar{z}_4 dz_4).$$

For the next term, we have that $\int_{D_2}\mu_2\frac{K_{42}}{\bar z_2}u_{32} = 0$ and $\int_{D_2}\frac{K_{42}}{z_{23}} = \bar z_3 + K_{43}(\frac{1}{z_4}+\bar z_3)$. In the next, we simply get $\int_{D_2}\mu_2 = 1$. In the last row, we get $\int_{D_2}\mu_2(\bar z_2 K_{23}) = -\bar z_3/2$ and $\int_{D_2}\mu_2 K_{23}/z_2 = -\bar z_3$, but also $\int_{D_2}\mu_2\frac{1-|z_2|^2}{z_{23}} = -\bar z_3(1-1/2|z_3|^2) = -\bar z_3/2$, i.e. in total we get $-2\bar z_3$. So over all we have

$$\int_{D_2} X_{234}\mu_2 = \frac{1}{4\pi^2}\Big[(K_{34}+K_{43})(\bar z_3 dz_3 - \bar z_4 dz_4) + (\bar z_3 + K_{43}(\frac{1}{z_4}+\bar z_3))dz_3 - (3\leftrightarrow 4) + K_{34}(z_3 d\bar z_3 + \bar z_4 dz_3 - \bar z_4 dz_4 - z_3 d\bar z_4) + \bar z_3 dz_3 - \bar z_4 dz_4\Big]$$

Notice again that $\operatorname{Re}\bar z dz = 0$ if $|z| = 1$, so upon taking the real part the first and the last line vanish. The remaining second line results in (using $K_{34} = \overline{K}_{43}$)

$$\begin{aligned}\operatorname{Re}\int_{D_2} X_{234}\mu_2 &= \frac{1}{4\pi^2}\operatorname{Re}\left[K_{34}(z_4 d\bar z_3 + z_3 d\bar z_3 - \bar z_3 dz_4 - \bar z_4 dz_4 + z_3 d\bar z_3 + \bar z_4 dz_3 - \bar z_4 dz_4 - z_3 d\bar z_4\right]\\ &= \frac{1}{4\pi^2}\operatorname{Re}\left[K_{34}(z_4 d\bar z_3 + \bar z_4 dz_3 + 2z_3 d\bar z_3 - (3\leftrightarrow 4))\right]\\ &= \frac{1}{4\pi^2}\operatorname{Re}\left[K_{34}(4\pi\sin 2\pi\theta_{34}d\theta_3 - 4\pi i d\theta_3 - (3\leftrightarrow 4))\right]\\ &= \frac{1}{\pi}(\log|\sin\pi\theta_{34}|\sin 2\pi\theta_{34} + \pi\eta_{S^1}(\theta_3,\theta_4))(d\theta_3 + d\theta_4)\end{aligned}$$

Now let us turn to the next two terms. We have that

$$\int_{D_2} X_{234}\mu_2 dL_{25} = -\int_{D_2} X_{234}\mu_2\frac{dz_5}{z_{25}} = dz_5\int_{D_2}\frac{X_{234}\mu_2}{z_{25}}$$

and similarly for the other term. To compute it we take the above expression for $X_{234}\mu_2$, multiply with z_{25}, and intergrate over z_2. Let us look at the result line by line. The first line integrates to (ignoring the prefactor)

$$\begin{aligned}2dL_{34}\int_2\frac{(K_{24}-K_{23})\mu_2}{z_{25}} &+ (K_{34}+K_{43})\int_2\mu_2\left(\frac{dz_4}{z_{24}z_{25}} - \frac{dz_3}{z_{23}z_{25}}\right)\\ = 2dL_{34}\frac{1-|z_5|^2}{z_5}(K_{54}-K_{53}) &+ (K_{34}+K_{43})\left(\frac{\bar z_{35}}{z_{35}}dz_3 - \frac{\bar z_{45}}{z_{45}}dz_4\right)\end{aligned}$$

Note that the first term vanishes for $|z_5| = 1$. The first term in the next line is more complicated and we will look at it later. The second term simply integrates to

$$K_{34}(z_3 d\bar z_3 + \bar z_4 dz_3 - \bar z_4 dz_4 - z_3 d\bar z_4)\int_2\frac{\mu_2}{z_{25}} = K_{34}(z_3 d\bar z_3 + \bar z_4 dz_3 - \bar z_4 dz_4 - z_3 d\bar z_4)(-\bar z_5).$$

In the last line, the first term integrates to

$$dz_3 \int_2 \mu_2 \left(\frac{\bar{z}_2 K_{23}}{z_{25}} + \frac{K_{23}}{z_2 z_{25}} + \frac{1-|z_2|^2}{z_{23} z_{25}} \right) = dz_3 \left(\frac{1}{2} \left(\bar{z}_3 \bar{z}_5 |z_5|^2 + \left(\frac{1}{z_5^2} - \bar{z}_5^2 \right) (K_{53} + z_5 \bar{z}_3) \right) \right.$$
$$\left. + \bar{z}_3 \bar{z}_5 + \frac{1}{z_5^2} (1 - |z_5|^2)(K_{53} + \bar{z}_3 z_5) + \frac{1}{z_{35}} \left(\bar{z}_5 \left(1 - \frac{|z_5^2|}{2} \right) - \bar{z}_3 \left(1 - \frac{|z_3^2|}{2} \right) \right) \right)$$

Since $|z_5| = |z_3| = 1$, this simplifies to

$$dz_3 \left(\frac{\bar{z}_3 \bar{z}_5}{2} + \bar{z}_3 \bar{z}_5 - \frac{\bar{z}_{35}}{2 z_{35}} \right) = 2 \bar{z}_3 \bar{z}_5 dz_3$$

where the second equality follows from the fact that $\bar{z}_{35}/z_{35} = -\bar{z}_3 \bar{z}_5$ for $|z_3| = |z_5| = 1$. The last term in the last line integrates to 0 (for $|z_5| = 1$). Now let us look at the terms that we skipped before. We have

$$\int_2 \frac{K_{42} \mu_2}{z_{23} z_{25}} = \frac{1}{z_{35}} \left(\bar{z}_{35} + K_{45} \left(\bar{z}_5 - \frac{1}{z_4} \right) - K_{43} \left(\bar{z}_5 - \frac{1}{z_4} \right) \right)$$
$$= -\bar{z}_3 \bar{z}_5 + \frac{K_{45} \bar{z}_{54}}{z_{35}} - \frac{K_{43} \bar{z}_{34}}{z_{35}}$$

while the other term can be expressed in terms of the function

$$F(x,y) := Li_2 \left(\frac{x-y}{1-y} \right) - Li_2 \left(\frac{-y}{1-y} \right) - Li_2(x) = Li_{1,1}(x/y, y)$$

by

$$\int_2 \frac{K_{42} \bar{z}_2 \mu_2}{u_{32} z_{25}} = \frac{1}{z_3^2} \left(F(z_4 \bar{z}_5, z_3 \bar{z}_5) - z_3 \bar{z}_5 \left(1 - K_{45} + \frac{K_{45}}{z_4 \bar{z}_5} \right) \right)$$
$$= \bar{z}_3^2 F(z_4 \bar{z}_5, z_3 \bar{z}_5) - \bar{z}_3 \bar{z}_5 + \bar{z}_3 \bar{z}_5 K_{45} - \bar{z}_3 \bar{z}_4 K_{45}$$

where in the second line we have used $|z_3| = |z_4| = |z_5| = 1$. Let us put everything together. We obtain

$$\operatorname{Im} \int_{D_2} X_{234} \mu_2 dL_{25} = \operatorname{Im} \frac{dz_5}{4\pi^2} \Big[(K_{34} + K_{43})(\bar{z}_4 \bar{z}_5 dz_4 - \bar{z}_3 \bar{z}_5 dz_3) - K_{34} \bar{z}_5 (z_3 d\bar{z}_3 + \bar{z}_4 dz_3 - \bar{z}_4 dz_4 - z_3 d\bar{z}_4)$$
$$+ \bar{z}_4 \bar{z}_5 dz_4 - \bar{z}_3 \bar{z}_5 dz_3$$
$$+ \left(dz_3 \left(-\bar{z}_3 \bar{z}_5 + \frac{K_{45} \bar{z}_{54} - K_{43} \bar{z}_{34}}{z_{35}} + \bar{z}_3^2 F(z_4 \bar{z}_5, z_3 \bar{z}_5) - \bar{z}_3 \bar{z}_5 + \bar{z}_3 \bar{z}_5 K_{45} - \bar{z}_3 \bar{z}_4 K_{45} \right) - (3 \leftrightarrow$$

Notice that $\operatorname{Im} \bar{z}_3 \bar{z}_5 dz_3 dz_5 = \operatorname{Im} -(4\pi^2) d\theta_3 d\theta_5 = 0$ for $z_i = \exp(2\pi i \theta_i)$. I.e. the first term in the first line, the second line and two terms in the last line vanish. The remainder in the first line is

$$
\begin{aligned}
&\frac{-1}{4\pi^2}\operatorname{Im}[\bar{z}_5 dz_5 K_{34}(z_3 d\bar{z}_3 + \bar{z}_4 dz_3 - \bar{z}_4 dz_4 - z_3 d\bar{z}_4))] \\
&= \frac{-d\theta_5}{2\pi}\operatorname{Re}[K_{34}(z_3 d\bar{z}_3 + \bar{z}_4 dz_3 - \bar{z}_4 dz_4 - z_3 d\bar{z}_4)] \\
&= d\theta_5 \operatorname{Im}[K_{34}(-d\theta_3 + e^{2\pi i\theta_{34}} d\theta_3 - d\theta_4 + e^{2\pi i \theta_{34}} d\theta_4)] \\
&= (d\theta_3 + d\theta_4) d\theta_5 \operatorname{Im} K_{34}(1 - e^{2\pi i\theta_{34}}) \\
&= (d\theta_3 + d\theta_4) d\theta_5 (-\log|\sin \pi\theta_{34}| \sin 2\pi\theta_{34} + \pi\eta_{S^1}(\theta_3, \theta_4)(1 - \cos 2\pi\theta_{34})).
\end{aligned}
$$

Now let us look at the last line, we get

$$
\begin{aligned}
&\frac{1}{4\pi^2}\operatorname{Im}\left(dz_5 dz_3\left(\frac{K_{45}\bar{z}_{54} - K_{43}\bar{z}_{34}}{z_{35}} + \bar{z}_3^2 F(z_4\bar{z}_5, z_3\bar{z}_5) + \bar{z}_3\bar{z}_5 K_{45} - \bar{z}_3\bar{z}_4 K_{45}\right) - (3 \leftrightarrow 4)\right) \\
&\quad = d\theta_3 d\theta_5 \operatorname{Im} z_3 z_5 \left(\left(\frac{\bar{z}_3\bar{z}_5}{\bar{z}_{53}}\right)(K_{45}\bar{z}_{54} - K_{43}\bar{z}_{34}) + \bar{z}_3^2 F(z_4\bar{z}_5, z_3\bar{z}_5) + \bar{z}_3\bar{z}_5 K_{45} - \bar{z}_3\bar{z}_4 K_{45}\right) - (3 \leftrightarrow 4) \\
&\quad = d\theta_3 d\theta_5 \operatorname{Im}\left(K_{45}\frac{\bar{z}_{54}}{\bar{z}_{53}} + K_{43}\frac{\bar{z}_{34}}{\bar{z}_{35}} + \bar{z}_3 z_5 F(z_4\bar{z}_5, z_3\bar{z}_5) + K_{45}(1 - \bar{z}_4 z_5)\right) - (3 \leftrightarrow 4)
\end{aligned}
$$

so we get

$$
\operatorname{Im}\int_{D_2} X_{234}\mu_2 dL_{25} = d\theta_3 d\theta_5 \operatorname{Im}\left(K_{34}u_{34} + K_{45}\frac{\bar{z}_{54}}{\bar{z}_{53}} + K_{43}\frac{\bar{z}_{34}}{\bar{z}_{35}} + \bar{z}_3 z_5 F(z_4\bar{z}_5, z_3\bar{z}_5) + K_{45}u_{54}\right) - (3 \leftrightarrow 4)
$$

Then we have

$$
\int_{D_2} X_{234}\mu_2 \overline{dL_{25}} = d\bar{z}_5 \int_{D_2} \frac{X_{234}\mu_2}{\bar{z}_{25}}
$$

and as before, we take the above expression for $X_{234}\mu_2$ and integrate. We then obtain

$$
\begin{aligned}
d\bar{z}_5 \int_{D_2} \frac{X_{234}\mu_2}{\bar{z}_{25}} = \frac{d\bar{z}_5}{4\pi^2}\Bigg[&2\left(K_{54}\left(\frac{1}{\bar{z}_4} - z_5\right) - K_{53}\left(\frac{1}{\bar{z}_3} - z_5\right)\right) dL_{34} + (K_{34} + K_{43})(dz_4 M_{45} - dz_3 M_{35}) \\
&+ \left(dz_3\left(\frac{K_{45}(1 - |z_5|^2)}{u_{35}} + K_{45}M_{35} + Li_2\left(\frac{\bar{z}_3 z_4 - \bar{z}_5 z_4}{1 - \bar{z}_5 z_4}\right) - Li_2\left(\frac{-\bar{z}_5 z_4}{1 - z_4\bar{z}_5}\right)\right) - (3 \leftrightarrow 4)\right) \\
&- K_{34}(z_3 d\bar{z}_3 + \bar{z}_4 dz_3 - \bar{z}_4 dz_4 - z_3 d\bar{z}_4) z_5\Bigg] \\
&- \frac{d\bar{z}_5}{8\pi^2}\left[\left(dz_3(|z_5|^2 + K_{53}\left(\frac{\bar{z}_5}{\bar{z}_3} - |z_5|^2\right) + Li_2(z_3\bar{z}_5) + M_{35}(1 - z_3\bar{z}_5) + |z_3|^2 + |z_5|^2 - 1\right) - (3 \leftrightarrow\right. \\
&- \frac{id\bar{z}_5}{\pi}\left(d\theta_3\left(z_5 + K_{54}\left(\frac{1}{\bar{z}_4} - z_5\right)\right) - (3 \leftrightarrow 4)\right)
\end{aligned}
$$

Using that z_3, z_4, z_5 lie on the unit circle, we obtain

$$d\bar{z}_5 \int_{D_2} \frac{X_{234}\mu_2}{\bar{z}_{25}} = \frac{d\bar{z}_5}{4\pi^2}\Bigg[2\left(K_{54}z_{45} - K_{53}z_{35}\right)dL_{34} + (K_{34}+K_{43})(-dz_4K_{54} + dz_3K_{53})$$
$$\left(dz_3\left(-K_{45}K_{53} + Li_2\left(\frac{\bar{z}_{35}}{\bar{z}_{45}}\right) - Li_2\left(\frac{1}{u_{54}}\right)\right) - (3\leftrightarrow 4)\right)$$
$$- K_{34}(z_3d\bar{z}_3 + \bar{z}_4dz_3 - \bar{z}_4dz_4 - z_3d\bar{z}_4)z_5\Bigg]$$
$$- \frac{d\bar{z}_5}{8\pi^2}\Big[\left(dz_3(1 - K_{53}u_{35} + Li_2(z_3\bar{z}_5) - K_{53}u_{35} + 1\right) - (3\leftrightarrow 4)\Big]$$
$$+ \frac{id\bar{z}_5}{\pi}\left(d\theta_4\left(z_5 + K_{53}z_{35}\right)\right))$$

Notice that (for $z_j = e^{2\pi i\theta_j}$) we have $z_{ij} = z_iu_{ji} = -z_ju_{ij}$, $z_jd\bar{z}_j = -2\pi i d\theta_j = -\bar{z}_jdz_j$ and $dL_{ij} = \pi\cot\pi\theta_{ij} + i\pi(d\theta_i + d\theta_j)$. Applying these identities, we find that

$$d\bar{z}_5 \int_{D_2} \frac{X_{234}\mu_2}{\bar{z}_{25}} = \left(K_{54}u_{45} - K_{53}u_{35}\right)\left(d\theta_3 + d\theta_4 - i\cot\pi\theta_{34}d\theta_{34}\right)d\theta_5$$
$$+ (K_{34}+K_{43})(K_{54}e^{2\pi i\theta_{45}}d\theta_4 - K_{53}e^{2\pi i\theta_{35}}d\theta_3)d\theta_5$$
$$+ \left(e^{2\pi i\theta_{35}}d\theta_3d\theta_5\left(-K_{45}K_{53} + Li_2\left(\frac{\bar{z}_{35}}{\bar{z}_{45}}\right) - Li_2\left(\frac{1}{u_{54}}\right)\right) - (3\leftrightarrow 4)\right)$$
$$+ (d\theta_3 + d\theta_4)d\theta_5K_{34}(1 - e^{2\pi i\theta_{34}})$$
$$+ \left(e^{2\pi i\theta_{35}}d\theta_3d\theta_5\left(1 - K_{53}u_{35} + \frac{Li_2(z_3\bar{z}_5)}{2}\right) - (3\leftrightarrow 4)\right)$$
$$+ \left(2d\theta_3d\theta_5(1 - K_{54}u_{45}) - (3\leftrightarrow 4)\right)$$

Taking imaginary part, we find

$$\operatorname{Im} d\bar{z}_5 \int_{D_2} \frac{X_{234}\mu_2}{\bar{z}_{25}} = d\theta_3d\theta_5\operatorname{Im}\Bigg[\left(K_{54}u_{45} - K_{53}u_{35}\right)\left(1 - i\cot\pi\theta_{34}\right) - (K_{34}+K_{43})(K_{53}e^{2\pi i\theta_{35}})$$
$$+ \left(e^{2\pi i\theta_{35}}\left(-K_{45}K_{53} + Li_2\left(\frac{\bar{z}_{35}}{\bar{z}_{45}}\right) - Li_2\left(\frac{1}{u_{54}}\right)\right)\right) + K_{34}(1 - e^{2\pi i\theta_{34}})$$
$$+ \left(e^{2\pi i\theta_{35}}\left(1 - K_{53}u_{35} + \frac{Li_2(z_3\bar{z}_5)}{2}\right)\right) + 2 - K_{45}u_{54})\Bigg] - (3\leftrightarrow 4)$$

In total we have

$$
\begin{aligned}
\epsilon^l &= d\theta_5 \operatorname{Re} \int_2 X_{234}\mu_2 + \frac{1}{2\pi} \operatorname{Im} dz_5 \int_{D,2} \frac{X_{234}\mu_2}{z_{25}} - \frac{1}{2\pi} \operatorname{Im} d\bar{z}_5 \int_{D,2} \frac{X_{234}\mu_2}{\bar{z}_{25}} \\
&= \frac{1}{\pi} d\theta_5 (\log|\sin\pi\theta_{34}|\sin 2\pi\theta_{34} + \pi\eta_{S^1}(\theta_3,\theta_4))(d\theta_3 + d\theta_4) \\
&+ \frac{1}{2\pi} d\theta_3 d\theta_5 \operatorname{Im}\left[K_{34}u_{34} + K_{45}\frac{\bar{z}_{54}}{\bar{z}_{53}} + K_{43}\frac{\bar{z}_{34}}{\bar{z}_{35}} + \bar{z}_3 z_5 F(z_4\bar{z}_5, z_3\bar{z}_5) + K_{45}u_{54}\right) - (3 \leftrightarrow 4)] \\
&- \frac{1}{2\pi} d\theta_3 d\theta_5 \operatorname{Im}\left[(K_{54}u_{45} - K_{53}u_{35})(1 - i\cot\pi\theta_{34}) - (K_{34} + K_{43})(K_{53}e^{2\pi i\theta_{35}}) \right. \\
&+ \left(e^{2\pi i\theta_{35}}\left(-K_{45}K_{53} + Li_2\left(\frac{\bar{z}_{35}}{\bar{z}_{45}}\right) - Li_2\left(\frac{1}{u_{54}}\right)\right)\right) + K_{34}(1 - e^{2\pi i\theta_{34}}) \\
&+ \left. \left(e^{2\pi i\theta_{35}}\left(1 - K_{53}u_{35} + \frac{Li_2(z_3\bar{z}_5)}{2}\right)\right) + 2K_{45}u_{54})\right] - (3 \leftrightarrow 4) \\
&= \frac{1}{2\pi} d\theta_3 d\theta_5 \Big\{ -2\log|\sin\pi\theta_{34}|\sin 2\pi\theta_{34} - 2\pi\eta_{S^1}(\theta_3,\theta_4) + \operatorname{Im}\Big[\frac{K_{45}\bar{z}_{45} + K_{43}\bar{z}_{34}}{\bar{z}_{35}}\bar{z}_3 z_5 F(z_4\bar{z}_5, z_3\bar{z}_5) \\
&+ (K_{54}u_{45} - K_{53}u_{35})\mathrm{i}\cot\pi\theta_{34} + K_{53}u_{35} \\
&+ z_3\bar{z}_5\left(K_{53}(K_{34} + K_{43} + K_{45} + u_{35}) - Li_2\left(\frac{\bar{z}_{35}}{\bar{z}_{45}}\right) - Li_2\left(\frac{1}{u_{54}}\right) - \frac{Li_2(z_3\bar{z}_5)}{2}\right)\Big\} - (3 \leftrightarrow 4)
\end{aligned}
$$

A.5 Parameter integrals on the disk

The functions we look at will often have integrable singularities. We sort them first by number of parameters and then by number of singularities.

A.5.1 One Parameter

No singularity

$$\iint_{D_1} \frac{z_1 d\bar{z}_1 dz_1}{u_{12}} = 0 \tag{A.51}$$

One singularity

$$\iint_{D_1} \frac{d\bar{z}_1 dz_1}{z_{12}} = -2\pi\mathrm{i}\bar{z}_2 \tag{A.52a}$$

$$\iint_{D_1} \frac{z_1 d\bar{z}_1 dz_1}{z_{12}} = 2\pi\mathrm{i}(1 - |z_2|^2) \tag{A.52b}$$

$$\iint_{D_1} \frac{|z_1|^2 d\bar{z}_1 dz_1}{z_{12}} = -\pi\mathrm{i}\bar{z}_2|z_2|^2 \tag{A.52c}$$

A.5.2 Two parameters

No singularity

$$\iint_{D_1} \frac{d\bar{z}_1 dz_1}{\bar{u}_{12} u_{13}} = -2\pi i \frac{K_{23}}{z_2 \bar{z}_3} \tag{A.53a}$$

$$\iint_{D_1} \frac{z_1^2 d\bar{z}_1 dz_1}{u_{12} u_{13}} = 0 \tag{A.53b}$$

$$\iint_{D_1} \frac{z_1^2 d\bar{z}_1 dz_1}{u_{12} \bar{u}_{13}} = -\frac{2\pi \mathrm{i}}{\bar{z}_2^2} \left(\frac{K_{32}}{\bar{z}_2 z_3} + 1 + \frac{\bar{z}_2 z_3}{2} \right) \tag{A.53c}$$

$$\iint_{D_1} \frac{|z_1|^2 d\bar{z}_1 dz_1}{\bar{u}_{12} u_{13}} = -\frac{2\pi i \left(z_2 \bar{z}_3 + K_{23} \right)}{\left(z_2 \bar{z}_3 \right)^2} \tag{A.53d}$$

One singularity

$$\iint_{D_1} \frac{z_3 d\bar{z}_1 dz_1}{z_{12} \bar{u}_{13}} = 2\pi i K_{32} \tag{A.54a}$$

$$\iint_{D_1} \frac{\bar{z}_1 d\bar{z}_1 dz_1}{\bar{z}_{12} \bar{u}_{13}} = 2\pi \mathrm{i} \left(\frac{1 - |z_2|^2}{u_{23}} \right) \tag{A.54b}$$

$$\iint_{D_1} \frac{\bar{z}_1 d\bar{z}_1 dz_1}{z_{12} \bar{u}_{13}} = 2\pi \mathrm{i} \left(\frac{\bar{z}_2}{z_3} + \frac{K_{32}}{z_3^2} \right) \tag{A.54c}$$

$$\iint_{D_1} \frac{\bar{z}_1^2 d\bar{z}_1 dz_1}{\bar{z}_{12} \bar{u}_{13}} = 2\pi \mathrm{i} \left(\bar{z}_2 \frac{1 - |z_2|^2}{u_{23}} \right) \tag{A.54d}$$

$$\iint_{D_1} \frac{|z_1|^2 d\bar{z}_1 dz_1}{z_{12} \bar{u}_{13}} = 2\pi \mathrm{i} \left(\frac{|z_2|^2}{z_3} + \frac{z_2}{z_3^2} K_{32} \right) \tag{A.54e}$$

Two singularities

$$\iint_{D_1} \frac{d\bar{z}_1 dz_1}{z_{12} z_{13}} = -2\pi i \left(\frac{\bar{z}_{23}}{z_{23}} \right) \tag{A.55a}$$

$$\iint_{D_1} \frac{d\bar{z}_1 dz_1}{z_{12} \bar{z}_{13}} = 2\pi i M_{23} \tag{A.55b}$$

$$\iint_{D_1} \frac{z_1 d\bar{z}_1 dz_1}{z_{12} z_{13}} = 2\pi i \left(\frac{|z_2|^2 - |z_3|^2}{z_{23}} \right) \tag{A.55c}$$

$$\iint_{D_1} \frac{z_1 d\bar{z}_1 dz_1}{z_{12} \bar{z}_{13}} = 2\pi i \left(z_2 M_{23} - z_3 \right) \tag{A.55d}$$

A.5.3 Three parameters

No singularity

$$\iint_{D_1} \frac{-z_1 z_4 d\bar{z}_1 dz_1}{u_{12} u_{13} \bar{u}_{14}} = \frac{2\pi i}{\bar{z}_2 \bar{z}_3 \bar{z}_{23}} \left(\bar{z}_3 K_{42} - \bar{z}_2 K_{43} \right) \tag{A.56}$$

One singularity

$$\iint_{D_1} \frac{z_1 z_3 d\bar{z}_1 dz_1}{u_{12}\bar{u}_{13} z_{14}} = \frac{2\pi i}{u_{42}} \left(z_4 K_{34} - \frac{1}{\bar{z}_2} K_{32} \right) \tag{A.57a}$$

$$\iint_{D_1} \frac{z_1 d\bar{z}_1 dz_1}{u_{12} u_{13} (\bar{z}_{14})} = \frac{2\pi i}{\bar{z}_2 \bar{z}_3 (\bar{z}_{23})} \left(\bar{z}_3 K_{42} - \bar{z}_2 K_{43} \right) \tag{A.57b}$$

$$\iint_{D_1} \frac{z_3 d\bar{z}_1 dz_1}{u_{12}\bar{u}_{13} z_{14}} = \frac{2\pi i}{u_{42}} \left(K_{34} - K_{32} \right) \tag{A.57c}$$

$$\tag{A.57d}$$

Two singularities

$$\iint_{D_1} \frac{z_1 d\bar{z}_1 dz_1}{u_{12} z_{13} (\bar{z}_{14})} = \frac{2\pi i}{\bar{z}_2 (\bar{u}_{23})} \left(K_{42} + \bar{z}_2 z_3 M_{34} \right) \tag{A.58a}$$

$$\iint_{D_1} \frac{d\bar{z}_1 dz_1}{u_{12} z_{13} (\bar{z}_{14})} = \frac{2\pi i}{\bar{u}_{23}} \left(K_{42} + M_{34} \right) \tag{A.58b}$$

$$\iint_{D_1} \frac{\bar{z}_2 d\bar{z}_1 dz_1}{u_{12} \bar{z}_{13} (\bar{z}_{14})} = \frac{2\pi i}{\bar{z}_{34}} \left(K_{32} - K_{42} \right) \tag{A.58c}$$

Functions with three singularities

$$\iint_{D_1} \frac{d\bar{z}_1 dz_1}{z_{12} z_{13} (\bar{z}_{14})} = \frac{2\pi i}{z_{23}} \left(M_{24} - M_{34} \right) \tag{A.59}$$

Integrals involving a logarithm

$$\iint_{D_1} K_{12} d\bar{z}_1 dz_1 = 0 \tag{A.60a}$$

$$\iint_{D_1} \bar{z}_1 K_{12} d\bar{z}_1 dz_1 = -\pi i \bar{z}_2 \tag{A.60b}$$

$$\iint_{D_1} \frac{K_{12}}{z_1} d\bar{z}_1 dz_1 = -2\pi i \bar{z}_2 \tag{A.60c}$$

$$\iint_{D_1} \frac{K_{12}}{z_{13}} d\bar{z}_1 dz_1 = 2\pi i (1 - |z_3|^2) \frac{K_{32}}{z_3} \tag{A.60d}$$

$$\iint_{D_1} \frac{K_{21}}{z_{13}} d\bar{z}_1 dz_1 = 2\pi i \left(\bar{z}_3 + K_{23} \left(\frac{1}{z_2} - \bar{z}_3 \right) \right) \tag{A.60e}$$

$$\iint_{D_1} \frac{\bar{z}_1 K_{21}}{z_{13}} d\bar{z}_1 dz_1 = \pi i \left(\frac{1}{z_2^2} \left(K_{23} + z_2 \bar{z}_3 \frac{(z_2 \bar{z}_3)^2}{2} \right) - \bar{z}_3^2 K_{23} \right) \tag{A.60f}$$

$$\iint_{D_1} \frac{K_{12} z_1}{u_{13}} d\bar{z}_1 dz_1 = 0 \tag{A.60g}$$

$$\iint_{D_1} \frac{K_{12} d\bar{z}_1 dz_1}{z_{13} \bar{z}_{14}} = 2\pi \mathrm{i} \left(Li_{1,1}(\bar{z}_3 z_4, \bar{z}_2 z_3) \right) + Li_2(\bar{z}_2 z_4) + K_{32} M_{34} \tag{A.60h}$$

A.6 Pushforwards of forms on the disk

We are intersted in pushforwards of the following "elementary" forms:

$$\phi_{12} := \frac{1}{2\pi} d \arg z_{12} = \frac{1}{2\pi} \operatorname{Im} d \log(z_{12}) = \frac{1}{4\pi i} \left(\frac{dz_{12}}{z_{12}} - \frac{d\bar{z}_{12}}{\bar{z}_{12}} \right) \tag{A.61a}$$

$$\tau_{12} := \frac{1}{2\pi} d \arg u_{12} = \frac{1}{2\pi} \operatorname{Im} dK_{12} = \frac{1}{4\pi i} \left(\frac{z_2 d\bar{z}_1 + \bar{z}_1 dz_2}{\bar{u}_{12}} - \frac{z_1 d\bar{z}_2 + \bar{z}_2 dz_1}{u_{12}} \right) \tag{A.61b}$$

$$\psi_1 := \frac{1}{4\pi i} (\bar{z} dz - z d\bar{z}) = \frac{1}{2\pi} \operatorname{Im}(\bar{z} dz) \tag{A.61c}$$

$$\mu_1 := \frac{1}{2\pi i} (d\bar{z} dz) = d\psi_1 \tag{A.61d}$$

Again, in this section we will collect only the results, while the proofs are postponed to the next section.

A.6.1 Pushforwards in the bulk

Pushforwards of a product of two forms

We have the following identities.

$$\int_{D,1} \psi_1 \phi_{12} = 0 \tag{A.62a}$$

$$\int_{D,1} \psi_1 \tau_{12} = 0 \tag{A.62b}$$

$$\int_{D,1} \mu_1 \tau_{12} = 0 \tag{A.62c}$$

$$\int_{D,1} \mu_1 \phi_{12} = \psi_2 \tag{A.62d}$$

$$\int_{D,1} \phi_{12} \phi_{13} = \frac{1}{4\pi} \arg(u_{23}) \tag{A.62e}$$

$$\int_{D,1} \tau_{12} \tau_{13} = \frac{1}{4\pi} \arg(u_{23}) \tag{A.62f}$$

$$\int_{D,1} \phi_{12} \tau_{13} = \frac{1}{4\pi} \arg(u_{23}) \tag{A.62g}$$

Pushforwards of a product of three forms

We have the following identities:

$$\int_{D_1} \phi_{12}\phi_{13}\phi_{14} = \frac{1}{16\pi^2}\operatorname{Re}\left[(M_{24} - M_{34})\,\frac{dz_{23}}{z_{23}} + \text{ cycl.}\right] \tag{A.63a}$$

$$\int_{D_1} \phi_{12}\phi_{13}\tau_{14} = \frac{1}{16\pi^2}\operatorname{Re}\left[(K_{43} - K_{42})dL_{23} + M_{23}\,(dK_{43} - dK_{24}) + K_{34}\frac{\bar z_4^2 dz_2 + d\bar z_4}{\bar z_4 u_{24}} - K_{24}\frac{\bar z_4^2 dz_3 + d\bar z_4}{\bar z_4 u_{34}}\right] \tag{A.63b}$$

$$\int_{D_1} \phi_{12}\tau_{13}\tau_{14} = \frac{1}{16\pi^2}\operatorname{Re}\left[K_{42}\frac{\bar z_3 dz_2 + z_2 d\bar z_3}{u_{23}} - K_{32}\frac{\bar z_4 dz_2 + z_2 d\bar z_4}{u_{24}} - K_{43}\frac{\bar z_3^2 dz_2 + d\bar z_3}{u_{23}} \right. \tag{A.63c}$$

$$\left. + K_{34}\frac{\bar z_4^2 dz_2 + d\bar z_4}{u_{24}} + \frac{z_3 K_{42} - z_4 K_{32}}{z_3 z_4 z_{34}}(z_4 dz_3 - z_3 dz_4)\right]$$

$$\int_{D_1} \tau_{12}\tau_{13}\tau_{14} = \frac{1}{16\pi^2}\operatorname{Re}\left[\frac{\bar z_2 K_{43} - \bar z_3 K_{42}}{\bar z_2 \bar z_3 \bar z_{23}}(\bar z_3 d\bar z_2 - \bar z_2 d\bar z_3) + \text{ cycl.}\right] \tag{A.63d}$$

$$\int_{D_1} \psi_1\phi_{12}\phi_{13} = \frac{1}{16\pi^2}\operatorname{Re}\left[-\left(\bar z_3 M_{23} - \bar z_2\right) dz_2 + \left(\bar z_2 M_{32} - \bar z_3\right) dz_3 + \frac{|z_2|^2 - |z_3|^2}{\bar z_{23}} d\bar z_{32}\right] \tag{A.63e}$$

$$\int_{D_1} \psi_1\tau_{12}\tau_{13} = \frac{1}{16\pi^2}\operatorname{Re}\left[\left(\frac{1}{\bar z_3} + \frac{K_{23}}{\bar z_3^2 z_2}\right) d\bar z_3 - \left(\frac{1}{\bar z_2} + \frac{K_{32}}{\bar z_2^2 z_3}\right) d\bar z_2\right] \tag{A.63f}$$

$$\int_{D_1} \psi_1\phi_{12}\tau_{13} = \frac{1}{16\pi^2}\operatorname{Re}\left[\left(\bar z_2 + \frac{K_{32}}{z_3}\right) dz_2 + \left(\frac{|z_2|^2}{\bar z_3} + \frac{\bar z_2 K_{23}}{\bar z_3^2}\right) d\bar z_3 + (1 - |z_2|^2) dK_{32}\right] \tag{A.63g}$$

$$\int_{D_1} \mu_1\phi_{12}\phi_{13} = \frac{1}{8\pi^2}\operatorname{Re}\left[M_{23} dz_2 d\bar z_3 + \frac{\bar z_{23}}{z_{23}} dz_2 dz_3\right] \tag{A.63h}$$

$$\int_{D_1} \mu_1\tau_{12}\tau_{13} = -\frac{1}{8\pi^2}\operatorname{Re}\left[\frac{(z_2\bar z_3 + K_{23})}{(z_2\bar z_3)^2} dz_2 d\bar z_3\right] \tag{A.63i}$$

$$\int_{D_1} \mu_1\phi_{12}\tau_{13} = \frac{1}{8\pi^2}\operatorname{Re}\left[\left(\frac{\bar z_2}{z_3} + \frac{K_{32}}{z_3^2}\right) dz_2 dz_3 - \left(\frac{1 - |z_2|^2}{u_{23}}\right) dz_2 d\bar z_3\right] \tag{A.63j}$$

A.6.2 Restrictions to the boundary

We are mostly (but not always) interested in the case when one or several of the free points are on the boundary. In this section we show how the results of the pushforwards above simplify when restricted to the boundary.

Proposition A.6.1. *We have*

i) If $z = \exp(2\pi \mathrm{i}\theta) \in \partial D$*, then*

$$\frac{dz}{z} = -\frac{d\bar{z}}{\bar{z}} = 2\pi \mathrm{i} d\theta, \quad \psi = \frac{1}{4\pi i}(\bar{z}dz - zd\bar{z}) = d\theta.$$

ii) If $z_i = \exp(2\pi \mathrm{i}\theta_i) \in \partial D$*, then*

$$\frac{1}{\pi}\arg(u_{12}) = \frac{1}{\pi}\arg(1 - z_1\bar{z}_2) = \eta_{S^1}(\theta_1, \theta_2) = \frac{1}{2}\operatorname{sgn}(\theta_2 - \theta_1) + \theta_1 - \theta_2.$$

iii) If $z = \exp(2\pi \mathrm{i}\theta) \in \partial D$*, then*

$$|1 - z| = 2\left|\sin \pi\theta\right|.$$

iv) If $z_i = \exp(2\pi \mathrm{i}\theta_i) \in \partial D$ *then*

$$d\log(z_1 - z_2) = \pi \cot \pi(\theta_1 - \theta_2)(d\theta_1 - d\theta_2) + i\pi(d\theta_1 + d\theta_2)$$
$$d\log(1 - z_1\bar{z}_2) = \pi(\cot \pi(\theta_1 - \theta_2) + i)(d\theta_1 - d\theta_2)$$

v) If $z_1 \in \partial D$ *or* $z_2 \in \partial D$ *then*

$$M_{12} = -\overline{K_{12}} = -K_{21}.$$

vi) If $z_4 \in \partial D$ *then*

$$\frac{\bar{z}_4^2 dz_2 + d\bar{z}_4}{\bar{z}_4 u_{24}} = \frac{\bar{z}_4^2 dz_2 - \bar{z}_4^2 dz_4}{\bar{z}_4 u_{24}} = \frac{dz_2 - dz_4}{z_4 - z_2} = -d\log z_{24}.$$

Using these results, one obtains the following simplifications:

Pushforwards of a product of two forms restricted to the boundary

We have

$$\int_{D,1} \mu_1\phi_{12} = d\theta_2 \tag{A.64a}$$
$$\int_{D,1} \phi_{12}\phi_{13} = \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3) \tag{A.64b}$$
$$\int_{D,1} \tau_{12}\tau_{13} = \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3) \tag{A.64c}$$
$$\int_{D,1} \phi_{12}\tau_{13} = \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3) \tag{A.64d}$$

Pushforwards of a product of three forms restricted to the boundary

We have

$$\int_{D_1} \phi_{12}\phi_{13}\phi_{14} = \frac{1}{16\pi}\left[\log\left|\frac{\sin\pi\theta_{43}}{\sin\pi\theta_{42}}\right|\cot(\pi\theta_{23})d\theta_{23} + \pi(\eta_{S^1}(\theta_4,\theta_2) - \eta_{S^1}(\theta_4,\theta_3))(d\theta_2 + d\theta_3) + \text{ cycl.}\right] \tag{A.65a}$$

$$\int_{D_1} \phi_{12}\phi_{13}\tau_{14} = \int_{D_1} \phi_{12}\phi_{13}\phi_{14} - \frac{1}{4}\eta_{S^1}(\theta_2,\theta_3)d\theta_4 \tag{A.65b}$$

$$\int_{D_1} \phi_{12}\tau_{13}\tau_{14} = \int_{D_1} \phi_{12}\phi_{13}\phi_{14} - \frac{1}{4}\eta_{S^1}(\theta_2,\theta_3)d\theta_4 - \frac{1}{4}\eta_{S^1}(\theta_4,\theta_2)d\theta_3 \tag{A.65c}$$

$$\int_{D_1} \tau_{12}\tau_{13}\tau_{14} = \int_{D_1} \phi_{12}\phi_{13}\phi_{14} - \frac{1}{4}(\eta_{S^1}(\theta_2,\theta_3)d\theta_4 + \text{ cycl.}) \tag{A.65d}$$

$$\int_{D_1} \psi_1\phi_{12}\phi_{13} = \frac{1}{8\pi}\left(\pi\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3) - \sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|\right)(d\theta_2 + d\theta_3) \tag{A.65e}$$

$$\int_{D_1} \psi_1\tau_{12}\tau_{13} = \frac{1}{8\pi}((\pi\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3) - \sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|)(d\theta_2 + d\theta_3) \tag{A.65f}$$

$$\int_{D_1} \psi_1\phi_{12}\tau_{13} = \frac{1}{8\pi}((\pi\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3) - \sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|)(d\theta_2 + d\theta_3) \tag{A.65g}$$

$$\int_{D_1} \mu_1\phi_{12}\phi_{13} = \frac{1}{2}d\theta_2 d\theta_3\left(1 - \cos(2\pi\theta_{23})\log(2|\sin\pi\theta_{23}|) - \pi\sin(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3)\right) \tag{A.65h}$$

$$\int_{D_1} \mu_1\tau_{12}\tau_{13} = -\frac{1}{2}d\theta_2 d\theta_3\left(1 + \cos(2\pi\theta_{23})\log(2|\sin\pi\theta_{23}|) + \pi\sin(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3)\right) \tag{A.65i}$$

$$\int_{D_1} \mu_1\phi_{12}\tau_{13} = -\frac{1}{2}d\theta_2 d\theta_3\left(1 + \cos(2\pi\theta_{23})\log(2|\sin\pi\theta_{23}|) + \pi\sin(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3)\right) \tag{A.65j}$$

A.6.3 Pushforwards over 2 points

Here we always assume that points $z_3, z_4, \ldots \in \partial D$, and we integrate over points z_1 and z_2. Then, we have

$$\int_{D,1,2} \phi_{13}\phi_{12}\mu_2\phi_{24} = -\frac{1}{16\pi^2}\operatorname{Re} K_{34}\bar{z}_{34}dz_4 \tag{A.66}$$

$$\int_{D,1,2} \phi_{13}\phi_{12}\psi_2\phi_{24} = \frac{1}{32\pi}\operatorname{Re}\left[\bar{z}_4^2 K_{34}(\bar{z}_3^2 z_4^2 - 1) + \bar{z}_3\bar{z}_4 + \bar{z}_4^2/2\right] \tag{A.67}$$

$$\int_{D,1,2} \phi_{13}\phi_{14}\phi_{12}\phi_{25} = \frac{1}{32\pi}\operatorname{Re}\left[Li_{1,1}(\bar{z}_4 z_3, \bar{z}_5 z_4) + Li_2(\bar{z}_5 z_3) + K_{45}M_{43} - (3 \leftrightarrow 4)\right] \tag{A.68}$$

$$\int_{D,1,2} \phi_{13}\phi_{14}\tau_{21}\phi_{25} = \frac{1}{32\pi}\operatorname{Re}\left[Li_{1,1}(\bar{z}_4 z_3, \bar{z}_5 z_4) + Li_2(\bar{z}_5 z_3) + K_{45}M_{43} - (3 \leftrightarrow 4)\right] \tag{A.69}$$

A.7 Proofs

A.7.1 Computation of the Parameter integrals

The proofs of these identities can be done as follows. First, using partial fraction decomposition one can decompose an integrand of the form

$$f(z,\bar{z}) = \frac{z^k \bar{z}^l}{\prod_{i=1}^N (z-z_i) \prod_{j=1}^M (\bar{z}-\bar{w}_j)}$$

into a sum of products of rational functions of the $z_i, \bar{w}_j$ with terms of the form

$$z^k \bar{z}^l, \frac{\bar{z}^k}{z-z_i}, \frac{z^k}{\bar{z}-\bar{w}_j}, \text{ or } \frac{1}{(z-z_i)(\bar{z}-\bar{w})}.$$

An integral containing a logarithm $K_i j = \log(1-z_i\bar{z}_j)$ can be converted into a series of rational functions by using the power series

$$\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \tag{A.70}$$

In the next lemma we summarise the results of integrating these terms over the disk.

Lemma A.7.1. *We have for $k, l \geq 0$*

$$\iint_D z^k \bar{z}^l d\bar{z}dz = \delta_{kl} \frac{2\pi i}{k+1} \tag{A.71}$$

$$\iint_D \frac{\bar{z}^k}{z-z_i} d\bar{z}dz = \begin{cases} -\frac{2\pi i}{k+1} \bar{z}_i^{k+1} & z_i \in D \\ -\frac{2\pi i}{k+1} z_i^{-(k+1)} & z_i \notin D \end{cases} \tag{A.72}$$

$$\iint_D \frac{z^k}{\bar{z}-\bar{w}_j} d\bar{z}dz = \begin{cases} -\frac{2\pi i}{k+1} w_j^{k+1} & w_j \in D \\ -\frac{2\pi i}{k+1} \bar{w}_j^{-(k+1)} & w_j \notin D \end{cases} \tag{A.73}$$

$$\iint_D \frac{1}{(z-z_i)(\bar{z}-\bar{w}_j)} d\bar{z}dz = \begin{cases} -2\pi \mathrm{i} \log(1 - z_i^{-1}\bar{w}_j) & z_i \notin D, w_j \notin D \\ -2\pi \mathrm{i} \log(1 - \bar{z}_i \bar{w}_j^{-1}) & z_i \in D, w_j \notin D \\ -2\pi \mathrm{i} \log(1 - z_i^{-1} w_j) & z_i \notin D, w_j \in D \\ 2\pi \mathrm{i} \log\left(\frac{1-z_i\bar{w}_j}{|z_i-w_j|^2}\right) & z_i \in D, w_j \in D \end{cases} \tag{A.74}$$

Proof. (A.71) is standard in complex analysis, it can be proved e.g. by applying the Stokes theorem

$$\iint_D z^k \bar{z}^l d\bar{z}dz = \int_{\partial D} \frac{z^k \bar{z}^{l+1}}{l+1} dz = \int_{\partial D} \frac{z^{k-(l+1)}}{l+1} dz = \delta_{kl} \frac{2\pi i}{l+1}.$$

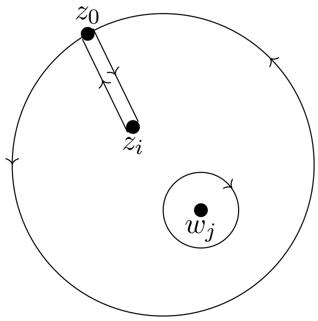


Figure A.1: Integration contours. The small disk around w_j has orientation opposite to the big disk.

(A.73) is of course the conjugate of (A.72), which again follows from standard complex analysis equalities. For example, the Cauchy-Pompeiu formula tells us that

$$\iint_D \frac{\bar{z}^k}{z-z_i} d\bar{z}dz = \frac{1}{k+1}\int_{\partial D}\frac{\bar{z}^{k+1}}{z-z_i} - 2\pi \mathrm{i}\chi_D(z_i)\frac{\bar{z}^{k+1}}{k+1} = \frac{1}{k+1}\left(\int_{\partial D}\frac{1}{z^{k+1}(z-z_i)} - 2\pi\mathrm{i}\bar{z}_i^{k+1}\right).$$

The claim now follows from the residue theorem by noting that the residues at 0 and z_i cancel out if $z_i \in D$.

(A.74) requires a bit more work. The hard case is the one where both $z_i, w_j \in D$, we prove only this one, the others follow along similar lines. In that case, choose $z_0 \partial D$ and cut the disk from z_i to z_0. Now, we can use Stokes' theorem to write

$$\begin{aligned}\iint_D \frac{1}{(z-z_i)(\bar{z}-\bar{w}_j)} d\bar{z}dz &= -\iint_D \frac{\mathrm{d}\log(z-z_i)d\bar{z}}{\bar{z}-\bar{w}_j} \\ &= -\int_{\partial D}\frac{\log(z-z_i)d\bar{z}}{\bar{z}-\bar{w}_j} + \int_{\partial D_{w_j,\varepsilon}}\frac{\log(z-z_i)d\bar{z}}{\bar{z}-\bar{w}_j} - 2\pi i\int_{z_0}^{z_i}\frac{d\bar{z}}{\bar{z}-\bar{w}_j}\end{aligned}$$

where $D_{w_j,\varepsilon}$ is a disk of radius ε around w_j. The sign of the integral along the cut follows from the fact that in the positive direction (from z_0 to z_1) the logarithm has acquired 2πi argument more. The last term evaluates to

$$\int_{z_0}^{z_i}\frac{d\bar{z}}{\bar{z}-\bar{w}_j} = \log(\bar{z}_i - \bar{w}_j) - \log(\bar{z}_0 - \bar{w}_j). \tag{A.75}$$

The second term is, using a parametrization $z = w_j + \varepsilon e^{i\theta}$,

$$\begin{aligned}\int_{\partial D_{w_j,\varepsilon}}\frac{\log(z-z_i)d\bar{z}}{\bar{z}-\bar{w}_j} &= \int_0^{2\pi}\frac{\log(w_j+\varepsilon e^{i\theta}-z_i) - \varepsilon \mathrm{i}e^{-\mathrm{i}\theta}d\theta}{\varepsilon e^{-\mathrm{i}\theta}} \\ &= -\mathrm{i}\int_0^{2\pi}\log(z_3+\varepsilon e^{i\theta})d\theta \overset{\varepsilon\to 0}{\to} -2\pi\mathrm{i}\log(w_j - z_i).\end{aligned} \tag{A.76}$$

We rewrite the first term using that $\bar{z} = z^{-1}$, when $z \in \partial D$, and get

$$\begin{aligned} -\int_{\partial D} \frac{\log(z - z_i) d\bar{z}}{\bar{z} - \bar{w}_j} &= \int_{\partial D} \frac{\log(z - z_i) dz}{z^2(\bar{z} - \bar{w}_j)} \\ &= \int_{\partial D} \frac{\log(z - z_i) dz}{z(1 - z\bar{w}_j)} \\ &= 2\pi \mathrm{i} \mathrm{Res}_0 \left(\frac{\log(z - z_i)}{z(1 - z\bar{w}_j)} \right) - 2\pi \mathrm{i} \int_{z_0}^{z_i} \frac{1}{z(1 - z\bar{w}_j)} \\ &= 2\pi \mathrm{i} \log(-z_i) + 2\pi i \left(\log(z_0) - \log(z_i) - \log(1 - z_0\bar{w}_j) + \log(1 - z_i\bar{w}_j) \right) \end{aligned} \tag{A.77}$$

The terms in which z_0 appears cancel out (this is obvious e.g. for $z_0 = 1$ and the fact that the integral does not depend on the cut). Next, note that $\log(-z_i) - \log(z_i) - \log(w_j - z_i) - log(\bar{z}_i - \bar{w}_j) = -\log(|z_i - w_j|)$ (this can be seen e.g. from taking z_i, w_j real and the fact that the derivatives agree). From this, the claim follows after taking the sum (with appropriate signs and prefactors) of (A.75),(A.76) and (A.77). □

We now give some examples of proofs of the parameter integrals over the disk.

Proof of (A.54c)

Consider e.g. (A.54c). We want to compute

$$\iint_D \frac{z_1 d\bar{z}_1 dz_1}{z_{12} z_{13}}.$$

First, perform partial fraction decomposition of the integrand:

$$\frac{z}{z_{12} z_{13}} = \frac{1}{z_{23}} \left(\frac{z_2}{z_{12}} - \frac{z_3}{z_{13}} \right).$$

Now apply Lemma A.7.1 to compute the integrals of the terms. We get

$$\begin{aligned} \iint_D \frac{z_1 d\bar{z}_1 dz_1}{z_{12} z_{13}} &= -2\pi \mathrm{i} \frac{1}{z_{23}} (z_2\bar{z}_2 - z_3\bar{z}_3) \\ &= -2\pi \mathrm{i} \left(\frac{|z_2|^2 - |z_3|^3}{z_{23}} \right). \end{aligned}$$

Proof of (A.60h)

As an example of a more complicated integral consider

$$\int_{D,1} \frac{K_{12} d\bar{z}_1 dz}{z_{13} z_{14}}.$$

First, expand K_{12} in a power series to obtain

$$\begin{aligned}
\int_{D,1} \frac{K_{12} d\bar{z}_1 dz}{z_{13}\bar{z}_{14}} &= -\sum_{k=1}^{\infty} \frac{\bar{z}_2^k}{k} \int_{D,1} \frac{z_1^k}{z_{13}\bar{z}_{14}} \\
&= -\sum_{k=1}^{\infty} \frac{\bar{z}_2^k}{k} \int_{D,1} \frac{\sum_{l=l}^{k} z_1^{k-l} z_3^{l-1}}{\bar{z}_{14}} + \frac{z_3^k}{z_{13}\bar{z}_{14}} \\
&= 2\pi \mathrm{i} \sum_{k=1}^{\infty} \left(\sum_{l=1}^{k} \frac{\bar{z}_2^k z_3^{l-1}}{k} \frac{z_4^{k-l+1}}{k-l+1} \right) - \frac{\bar{z}_2^k z_3^k}{k} M_{34}
\end{aligned}$$

where in the last line we applied equations (A.73) and (A.74). The last term sums up to $K_{32}M_{34}$. In the first term, relabel $n = k - l$ to obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(\sum_{l=1}^{k} \frac{\bar{z}_2^k z_3^{l-1}}{k} \frac{z_4^{k-l+1}}{k-l+1} \right) &= \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \frac{\bar{z}_2^{n+l} z_3^{l-1} z_4^{n+1}}{(n+l)(n+1)} \\
&= \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{(\bar{z}_2 z_3)^l (\bar{z}_2 z_4)^n}{(n+l)n}
\end{aligned}$$

This is almost a series for the double logarithm

$$Li_{1,1}(x/y, y) = \sum_{0<m<n} \frac{(x/y)^m y^n}{mn} = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^m y^l}{m(m+l)},$$

except for the fact that the first index starts at 0 rather than 1. However, the $l = 0$ sum over n is

$$\sum_{n=1}^{\infty} \frac{(\bar{z}_2 z_4)^n}{n^2} = Li_2(\bar{z}_2 z_4)$$

so that overall we get

$$\int_{D,1} \frac{K_{12} d\bar{z}_1 dz}{z_{13} z_{14}} = 2\pi \mathrm{i} \left(Li_{1,1}(\bar{z}_3 z_4, \bar{z}_2 z_3) + Li_2(\bar{z}_2 z_4) + K_{24} M_{34} \right)$$

A.7.2 Computation of the pushforwards of forms

Proof of (A.62a)

We will show (A.62a).

$$\begin{aligned}
\int_{D,1} \psi_1 \phi_{12} &= \frac{1}{4\pi i} \frac{1}{4\pi i} \int_{D,1} (\bar{z}_1 dz_1 - z_1 d\bar{z}_1) \left(\frac{dz_{12}}{z_{12}} - \frac{d\bar{z}_{12}}{\bar{z}_{12}} \right) \\
&= \left(\frac{1}{4\pi i} \right)^2 \int_{D,1} d\bar{z}_1 dz_1 \left(\frac{\bar{z}_1}{\bar{z}_{12}} - \frac{z_1}{z_{12}} \right) \\
&= \frac{1}{8\pi \mathrm{i}} \left(1 - |z_2|^2 - (1 - |z_2|^2 \right) = 0
\end{aligned}$$

using (A.52b).

Proof of (A.62b) **and** (A.62c)

We have

$$\int_{D,1} \psi_1 \tau_{12} = \left(\frac{1}{4\pi i}\right)^2 \int_{D,1} (\bar{z}_1 dz_1 - z_1 d\bar{z}_1) \left(\frac{z_2 d\bar{z}_1 + \bar{z}_1 dz_2}{\bar{u}_{12}} - \frac{z_1 d\bar{z}_2 + \bar{z}_2 dz_1}{u_{12}} \right)$$
$$= \left(\frac{1}{4\pi i}\right)^2 \int_{D,1} d\bar{z}_1 dz_1 \left(\frac{z_1 \bar{z}_2}{u_{12}} - \frac{\bar{z}_1 z_2}{\bar{u}_{12}} \right) = 0,$$

where we have used (A.51). (A.62c) leads to the same parameter integral.

Proof of (A.62d) **and** (A.64a)

We will compute (A.62d).

$$\int_{D_1} \mu\phi_{12} = \frac{1}{2\pi i}\frac{1}{4\pi i} \int_{D_1} d\bar{z}_1 dz_1 \left(\frac{dz_{12}}{z_{12}} - \frac{d\bar{z}_{12}}{\bar{z}_{12}} \right)$$
$$= \frac{1}{2\pi i}\frac{1}{4\pi i} \int_{D_1} d\bar{z}_1 dz_1 \left(\frac{d\bar{z}_2}{\bar{z}_{12}} - \frac{dz_2}{z_{12}} \right)$$
$$= \frac{1}{4\pi \mathrm{i}} (\bar{z}_2 dz_2 - z_2 d\bar{z}_2) = \psi_2$$

using (A.52a) and its conjugate.

If $z_2 = \exp(2\pi \mathrm{i}\theta_2)$ then we get

$$\int_{D_1} \mu\phi_{12} = \psi_2 = d\theta_2$$

by Proposition A.6.1.

Proof of (A.62e) **and** (A.64b)

We will prove (A.62e).

$$\int_{D_1} \phi_{12}\phi_{13} = \frac{1}{4\pi i}\frac{1}{4\pi i} \int_{D_1} \left(\frac{dz_{12}}{z_{12}} - \frac{d\bar{z}_{12}}{\bar{z}_{12}} \right) \left(\frac{dz_{13}}{z_{13}} - \frac{d\bar{z}_{13}}{\bar{z}_{13}} \right)$$
$$= \left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left(\frac{1}{z_{12}\bar{z}_{13}} - \frac{1}{\bar{z}_{12} z_{13}} \right)$$
$$= \frac{1}{8\pi i} (M_{23} - M_{32})$$
$$= \frac{1}{4\pi} \arg(u_{23})$$

using (A.55b) and noticing that the real number in the denominator does not change the argument.

If $z_i = \exp(2\pi \mathrm{i}\theta_i)$ then by proposition A.6.1 we have

$$\int_{D_1} \phi_{12}\phi_{13} = \frac{1}{4\pi} \arg(u_{23}) = \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3).$$

Proof of (A.62f)

Let us prove (A.62f).

$$\begin{aligned}
\int_{D_1} \tau_{12}\phi_{13} &= \frac{1}{4\pi i}\frac{1}{4\pi i}\int_{D_1}\left(\frac{dz_{12}}{z_{12}} - \frac{d\bar{z}_{12}}{\bar{z}_{12}}\right)\left(\frac{z_3 d\bar{z}_1 + \bar{z}_1 dz_3}{\bar{u}_{13}} - \frac{z_1 d\bar{z}_3 + \bar{z}_3 dz_1}{u_{13}}\right) \\
&= \left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 z_1 \left(\frac{\bar{z}_3}{\bar{z}_{12}u_{13}} - \frac{z_3}{z_{12}\bar{u}_{13}}\right) \\
&= \frac{1}{8\pi i}\left(K_{23} - K_{32}\right) \\
&= \frac{1}{4\pi}\arg(u_{23})
\end{aligned}$$

If $z_i = \exp(2\pi \mathrm{i}\theta_i)$ then by proposition A.6.1 we have

$$\int_{D_1} \tau_{12}\phi_{13} = \frac{1}{4\pi}\arg(u_{23}) = \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3).$$

Proof of (A.62g)

Let us prove (A.62g).

$$\begin{aligned}
\int_{D_1} \tau_{12}\phi_{13} &= \frac{1}{4\pi i}\frac{1}{4\pi i}\int_{D_1}\left(\frac{z_2 d\bar{z}_1 + \bar{z}_1 dz_2}{\bar{u}_{12}} - \frac{z_1 d\bar{z}_2 + \bar{z}_2 dz_1}{u_{12}}\right)\left(\frac{z_3 d\bar{z}_1 + \bar{z}_1 dz_3}{\bar{u}_{13}} - \frac{z_1 d\bar{z}_3 + \bar{z}_3 dz_1}{u_{13}}\right) \\
&= \left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left(\frac{z_3\bar{z}_2}{u_{12}\bar{u}_{13}} - \frac{z_2\bar{z}_3}{\bar{u}_{12}u_{13}}\right) \\
&= \frac{1}{8\pi i}\left(K_{23} - K_{32}\right) \\
&= \frac{1}{4\pi}\arg(u_{23})
\end{aligned}$$

If $z_i = \exp(2\pi \mathrm{i}\theta_i)$ then again by proposition A.6.1 we have

$$\int_{D_1} \tau_{12}\tau_{13} = \frac{1}{4\pi}\arg(u_{23}) = \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3).$$

Proof of (A.63a) **and** (A.65a)

We have

$$\begin{aligned}
\int_{D_1} \phi_{12}\phi_{13}\phi_{14} &= \left(\frac{1}{2\pi}\right)^3 \int_{D_1} \operatorname{Im} dL_{12} \operatorname{Im} dL_{13} \operatorname{Im} dL_{14} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4}\operatorname{Im}\int_{D_1} dL_{12}dL_{13}\overline{dL_{14}} - dL_{12}dL_{13}dL_{14} + dL_{12}\overline{dL_{13}}dL_{14} + \overline{dL_{12}}dL_{13}dL_{14} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4}\operatorname{Im}\int_{D_1} dL_{12}dL_{13}\overline{dL_{14}} + \text{ cycl.} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4}\operatorname{Im}\int_{D_1} d\bar{z}_1 dz_1 \left[\frac{dz_{23}}{z_{12}z_{13}\bar{z}_{14}} + \text{ cycl.}\right] \\
&= \frac{1}{16\pi^2}\operatorname{Re}\left[\left(M_{24} - M_{34}\right)\frac{dz_{23}}{z_{23}} + \text{ cycl.}\right]
\end{aligned}$$

In the second equation notice that the integral of the second term is zero because it contains only holomorphic differentials. Now assume that we have $z_i = \exp(2\pi i\theta_i)$. By proposition A.6.1 we get that the above equals

$$= \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{43} - K_{42}) d \log z_{23} + \text{ cycl.}\right]$$

(using $\operatorname{Re} z = \operatorname{Re} \bar{z}$ for the last term). Noticing that

$$K_{43} - K_{42} = \log \left| \frac{\sin \pi\theta_{43}}{\sin \pi\theta_{42}} \right| + i\pi(\eta_{S^1}(\theta_4, \theta_3) - \eta_{S^1}(\theta_4, \theta_2))$$

we obtain

$$\begin{aligned}
&\frac{1}{16\pi}\left[\log\left|\frac{\sin \pi\theta_{43}}{\sin \pi\theta_{42}}\right| \cot(\pi\theta_{23}) d\theta_{23} + \pi(\eta_{S^1}(\theta_4, \theta_2) - \eta_{S^1}(\theta_4, \theta_3))(d\theta_2 + d\theta_3) + \text{ cycl.}\right] \\
&\quad = \frac{1}{16\pi}\left[\log\left|\frac{\sin \pi\theta_{43}}{\sin \pi\theta_{42}}\right| \cot \pi\theta_{23} d\theta_{23} + \log\left|\frac{\sin \pi\theta_{32}}{\sin \pi\theta_{34}}\right| \cot \pi\theta_{24} d\theta_{24} + \log\left|\frac{\sin \pi\theta_{24}}{\sin \pi\theta_{23}}\right| \cot \pi\theta_{34} d\theta_{34}\right. \\
&\quad + \pi(\eta_{S^1}(\theta_2, \theta_3) d\theta_4 + \eta_{S^1}(\theta_3, \theta_4) d\theta_2 + \eta_{S^1}(\theta_4, \theta_2) d\theta_3) \\
&\quad \left. + \pi\left(\eta_{S^1}(\theta_2, \theta_3) + \eta_{S^1}(\theta_3, \theta_4) + \eta_{S^1}(\theta_4, \theta_2)\right)(d\theta_2 + d\theta_3 + d\theta_4)\right]
\end{aligned}$$

Proof of (A.63b) **and**(A.65b)

We have

$$\begin{aligned}
\int_{D_1} \phi_{12}\phi_{13}\tau_{14} &= \left(\frac{1}{2\pi}\right)^3 \int_{D_1} \operatorname{Im} dL_{12} \operatorname{Im} dL_{13} \operatorname{Im} dK_{14} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4} \operatorname{Im} \int_{D_1} dL_{12} dL_{13} \overline{dK_{14}} - dL_{12} dL_{13} dK_{14} + dL_{12} \overline{dL_{13}} dK_{14} - dL_{12} \overline{dL_{13} dK_{14}} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4} \operatorname{Im} \int_{D_1} d\bar{z}_1 dz_1 \left[\frac{z_4 dz_{32}}{z_{12} z_{13} \bar{u}_{14}} + \frac{\bar{z}_4 dz_2 + z_1 d\bar{z}_4}{z_{12} \bar{z}_{13} u_{14}} - \frac{z_4 d\bar{z}_3 + \bar{z}_1 dz_4}{z_{12} \bar{z}_{13} \bar{u}_{14}}\right] \\
&= \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{43} - K_{42}) \frac{dz_{23}}{z_{23}} + (K_{34} + M_{23}) \frac{\bar{z}_4 dz_2}{u_{24}}\right. \\
&+ (K_{34} + \bar{z}_4 z_2 M_{23}) \frac{d\bar{z}_4}{\bar{z}_4 u_{24}} - (K_{42} + M_{23}) \frac{z_4 d\bar{z}_3}{u_{43}} \\
&\left. - (K_{42} + z_4 \bar{z}_3 M_{23}) \frac{dz_4}{z_4 u_{43}}\right] \\
&= \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{43} - K_{42}) \frac{dz_{23}}{z_{23}} + M_{23}\left(\frac{\bar{z}_4 dz_2 + z_2 d\bar{z}_4}{u_{24}} - \frac{z_4 d\bar{z}_3 + \bar{z}_3 dz_4}{u_{43}}\right)\right. \\
&\left. + K_{34} \frac{\bar{z}_4^2 dz_2 + d\bar{z}_4}{\bar{z}_4 u_{24}} - K_{24} \frac{\bar{z}_4^2 dz_3 + d\bar{z}_4}{\bar{z}_4 u_{34}}\right] \\
&= \frac{1}{16\pi^2} \operatorname{Re}\left[(K_{43} - K_{42}) dL_{23} + M_{23}(dK_{43} - dK_{24}) + K_{34} \frac{\bar{z}_4^2 dz_2 + d\bar{z}_4}{\bar{z}_4 u_{24}} - K_{24} \frac{\bar{z}_4^2 dz_3 + d\bar{z}_4}{\bar{z}_4 u_{34}}\right]
\end{aligned}$$

where we used equations (A.58a), (A.58b), (A.58c) and also in the last equality also $\operatorname{Re} z = \operatorname{Re} \bar{z}$. Observe that for $z_4 \in \partial D$ we have

$$\frac{\bar{z}_4^2 dz_2 + d\bar{z}_4}{\bar{z}_4 u_{24}} = \frac{\bar{z}_4^2 dz_2 - \bar{z}_4^2 dz_4}{\bar{z}_4 u_{24}} = \frac{dz_2 - dz_4}{z_4 - z_2} = -d \log z_{24}$$

so on the boundary we get

$$\begin{aligned}
&\frac{1}{16\pi^2} \operatorname{Re} \left[(K_{43} - K_{42}) d \log z_{23} + K_{32}(dK_{24} - d \log u_{43}) + K_{24} d \log z_{34} - K_{34} d \log z_{24} \right] \\
&= \frac{1}{16\pi} \left[\log \left| \frac{\sin \pi\theta_{43}}{\sin \pi\theta_{42}} \right| \cot \pi\theta_{23} d\theta_{23} + \pi(\eta_{S^1}(\theta_4, \theta_2) - \eta_{S^1}(\theta_2, \theta_3))(d\theta_2 + d\theta_3) \right. \\
&+ \log(2|\sin \pi\theta_{32}|)(\cot \pi\theta_{24} d\theta_{24} - \cot \pi\theta_{43} d\theta_{43}) - \pi\eta_{S^1}(\theta_3, \theta_2)(d\theta_{24} - d\theta_{43}) \\
&+ \log(2|\sin \pi\theta_{24}|) \cot \pi\theta_{34} d\theta_{34} - \pi\eta_{S^1}(\theta_2, \theta_4)(d\theta_3 + d\theta_4) \\
&\left. - \log(2|\sin \pi\theta_{34}|) \cot \pi\theta_{24} d\theta_{24} + \pi\eta_{S^1}(\theta_3, \theta_4)(d\theta_2 + d\theta_4) \right] \\
&= \frac{1}{16\pi} \left[\log \left| \frac{\sin \pi\theta_{43}}{\sin \pi\theta_{42}} \right| \cot \pi\theta_{23} d\theta_{23} + \log \left| \frac{\sin \pi\theta_{32}}{\sin \pi\theta_{34}} \right| \cot \pi\theta_{24} d\theta_{24} + \log \left| \frac{\sin \pi\theta_{24}}{\sin \pi\theta_{23}} \right| \cot \pi\theta_{34} d\theta_{34} \right. \\
&+ \pi \left[(\eta_{S^1}(\theta_2, \theta_3) + \eta_{S^1}(\theta_3, \theta_4) + \eta_{S^1}(\theta_4, \theta_2))(d\theta_2 + d\theta_3 + d\theta_4) \right. \\
&\left. \left. + \eta_{S^1}(\theta_3, \theta_4) d\theta_2 + \eta_{S^1}(\theta_4, \theta_2) d\theta_3 - 3\eta_{S^1}(\theta_2, \theta_3) d\theta_4 \right] \right. \\
&= \int_{D_1} \phi_{12}\phi_{13}\phi_{14} - \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3) d\theta_4
\end{aligned}$$

Proof of (A.63c) **and**(A.65c)

We have

$$\begin{aligned}
\int_{D_1} \phi_{12}\tau_{13}\tau_{14} &= \left(\frac{1}{2\pi}\right)^3 \int_{D_1} \operatorname{Im} dL_{12} \operatorname{Im} dK_{13} \operatorname{Im} dK_{14} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4} \operatorname{Im} \int_{D_1} dL_{12} dK_{13} \overline{dK_{14}} - dL_{12} dK_{13} dK_{14} + dL_{12} \overline{dK_{13}} dK_{14} - dL_{12} \overline{dK_{13} dK_{14}} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4} \operatorname{Im} \int_{D_1} d\bar{z}_1 dz_1 \left[\frac{z_4(\bar{z}_3 dz_2 + z_1 d\bar{z}_3)}{z_{12} u_{13} \bar{u}_{14}} - \frac{z_3(\bar{z}_4 dz_2 + z_1 d\bar{z}_4)}{z_{12} \bar{u}_{13} u_{14}} - \frac{\bar{z}_1(z_4 dz_3 - z_3 dz_4)}{z_{12} \bar{u}_{13} \bar{u}_{14}} \right] \\
&= \frac{1}{16\pi^2} \operatorname{Re} \left[\frac{\bar{z}_3}{u_{23}} (K_{42} - K_{43}) \, dz_2 - \frac{\bar{z}_4}{u_{24}} (K_{32} - K_{34}) \, dz_2 \right. \\
&+ \frac{1}{u_{23}} \left(z_2 K_{42} - \frac{1}{\bar{z}_3} K_{43} \right) d\bar{z}_3 - \frac{1}{u_{24}} \left(z_2 K_{32} - \frac{1}{\bar{z}_4} K_{34} \right) d\bar{z}_4 \\
&\left. + \frac{z_3 K_{42} - z_4 K_{32}}{z_3 z_4 z_{34}} (z_4 dz_3 - z_3 dz_4) \right] \\
&= \frac{1}{16\pi^2} \operatorname{Re} \left[K_{42} \frac{\bar{z}_3 dz_2 + z_2 d\bar{z}_3}{u_{23}} - K_{32} \frac{\bar{z}_4 dz_2 + z_2 d\bar{z}_4}{u_{24}} - K_{43} \frac{\bar{z}_3^2 dz_2 + d\bar{z}_3}{u_{23}} + K_{34} \frac{\bar{z}_4^2 dz_2 + d\bar{z}_4}{u_{24}} \right. \\
&\left. + \frac{z_3 K_{42} - z_4 K_{32}}{z_3 z_4 z_{34}} (z_4 dz_3 - z_3 dz_4) \right]
\end{aligned}$$

Note that for $z_3, z_4 \in \partial D$ we have

$$\frac{z_4 dz_3}{z_3 z_{34}} - \frac{dz_4}{z_{34}} = -\frac{z_4 d\bar{z}_3}{\bar{z}_3 z_{34}} - \frac{\bar{z}_3 dz_4}{\bar{z}_3 z_{34}} = -\frac{z_4 d\bar{z}_3 + \bar{z}_3 dz_4}{u_{43}} = dK_{43}.$$

Using this for the last term, on the boundary we get

$$\begin{aligned}
&\frac{1}{16\pi^2} \operatorname{Re} [K_{32} dK_{24} - K_{42} dK_{23} + K_{43} d \log z_{23} - K_{34} d \log z_{24} - K_{32} dK_{43} + K_{42} dK_{34}] \\
&\quad = \frac{1}{16\pi^2} \operatorname{Re} \left[K_{32}(dK_{24} - dK_{43}) - K_{42}(dK_{23} - dK_{34}) + K_{34}(\overline{d \log z_{23}} - d \log z_{24})\right] \\
&\quad = \frac{1}{16\pi} [\log(2|\sin \pi\theta_{32}|)(\cot \pi\theta_{24} d\theta_{24} - \cot \pi\theta_{43} d\theta_{43}) - \pi\eta_{S^1}(\theta_3, \theta_2)(d\theta_{24} - d\theta_{43}) \\
&\quad + \log(2|\sin \pi\theta_{42}|)(\cot \pi\theta_{34} d\theta_{34} - \cot \pi\theta_{23} d\theta_{23}) + \pi\eta_{S^1}(\theta_4, \theta_2)(d\theta_{23} - d\theta_{34}) \\
&\quad + \log(2|\sin \pi\theta_{34}|)(\cot \pi\theta_{23} d\theta_{23} - \cot \pi\theta_{24} d\theta_{24}) + \pi\eta_{S^1}(\theta_3, \theta_4)(d\theta_2 + d\theta_3 + d\theta_2 + d\theta_4)] \\
&\quad = \frac{1}{16\pi} \left[\log \left|\frac{\sin \pi\theta_{43}}{\sin \pi\theta_{42}}\right| \cot \pi\theta_{23} d\theta_{23} + \log \left|\frac{\sin \pi\theta_{32}}{\sin \pi\theta_{34}}\right| \cot \pi\theta_{24} d\theta_{24} + \log \left|\frac{\sin \pi\theta_{24}}{\sin \pi\theta_{23}}\right| \cot \pi\theta_{34} d\theta_{34}\right. \\
&\quad + \pi \left[(\eta_{S^1}(\theta_2, \theta_3) + \eta_{S^1}(\theta_3, \theta_4) + \eta_{S^1}(\theta_4, \theta_2))(d\theta_2 + d\theta_3 + d\theta_4)\right. \\
&\quad \left.+ \eta_{S^1}(\theta_3, \theta_4) d\theta_2 - 3\eta_{S^1}(\theta_4, \theta_2) d\theta_3 - 3\eta_{S^1}(\theta_2, \theta_3) d\theta_4\right] \\
&\quad = \int_{D_1} \phi_{12}\phi_{13}\phi_{14} - \frac{1}{4}\eta_{S^1}(\theta_2, \theta_3) d\theta_4 - \frac{1}{4}\eta_{S^1}(\theta_4, \theta_2) d\theta_3
\end{aligned}$$

Proof of (A.63d) **and**(A.65d)

We have

$$\begin{aligned}
\int_{D_1} \tau_{12}\tau_{13}\tau_{14} &= \left(\frac{1}{2\pi}\right)^3 \int_{D_1} \operatorname{Im} dK_{12} \operatorname{Im} dK_{13} \operatorname{Im} dK_{14} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4} \operatorname{Im} \int_{D_1} dK_{12} dK_{13} \overline{dK_{14}} - dK_{12} dK_{13} dK_{14} + dK_{12} \overline{dK_{13}} dK_{14} + \overline{dK_{12}} dK_{13} dK_{14} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4} \operatorname{Im} \int_{D_1} dK_{12} dK_{13} \overline{dK_{14}} + \text{ cycl.} \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{1}{4} \operatorname{Im} \int_{D_1} d\bar{z}_1 dz_1 \left[\frac{z_1 z_4 (\bar{z}_3 d\bar{z}_2 - \bar{z}_2 d\bar{z}_3)}{u_{12} u_{13} \bar{u}_{14}} + \text{ cycl.}\right] \\
&= \frac{1}{16\pi^2} \operatorname{Re} \left[\frac{\bar{z}_2 K_{43} - \bar{z}_3 K_{42}}{\bar{z}_2 \bar{z}_3 \bar{z}_{23}} (\bar{z}_3 d\bar{z}_2 - \bar{z}_2 d\bar{z}_3) + \text{ cycl.}\right]
\end{aligned}$$

Here we have used equation (A.56). Restricting to the boundary we obtain

$$\begin{aligned}
&\frac{1}{16\pi^2}\operatorname{Re}\left[K_{43}dK_{32} - K_{42}dK_{23} + \text{ cycl. }\right] \\
&\quad = \frac{1}{16\pi^2}\operatorname{Re}\left[(K_{43} - K_{24})dK_{32} + \text{ cycl. }\right] \\
&\quad = \frac{1}{16\pi}\left[\log\left|\frac{\sin\pi\theta_{43}}{\sin\pi\theta_{42}}\right| - \pi\eta_{S^1}(\theta_4,\theta_3)d\theta_{32} + \pi\eta_{S^1}(\theta_2,\theta_4))d\theta_{32} + \text{ cycl.}\right] \\
&\quad = \int_{D_1}\phi_{12}\phi_{13}\phi_{14} - \frac{1}{4}(\eta_{S^1}(\theta_2,\theta_3)d\theta_4 + \text{ cycl.})
\end{aligned}$$

Proof of (A.63e) **and** (A.65e)

We have

$$\begin{aligned}
\int_{D_1}\psi_1\phi_{12}\phi_{13} &= \left(\frac{1}{2\pi}\right)^3\int_{D_1}\operatorname{Im}(\bar{z}dz)\operatorname{Im}dL_{12}\operatorname{Im}dL_{13} \\
&= \left(\frac{1}{2\pi}\right)^3\frac{1}{4}\operatorname{Im}\int_{D_1}(\bar{z}_1dz_1dL_{12}\overline{dL_{13}} - \bar{z}_1dz_1dL_{12}dL_{13} + \bar{z}_1dz_1\overline{dL_{12}}dL_{13} - \bar{z}_1dz_1\overline{dL_{12}dL_{13}}) \\
&= \left(\frac{1}{2\pi}\right)^3\frac{1}{4}\operatorname{Im}\int_{D_1}d\bar{z}_1dz\left[\frac{-\bar{z}_1dz_2}{z_{12}\bar{z}_{13}} + \frac{\bar{z}_1dz_3}{\bar{z}_{12}z_{13}} + \frac{\bar{z}_1(d\bar{z}_{32})}{\bar{z}_{12}\bar{z}_{13}}\right] \\
&= \frac{1}{16\pi^2}\operatorname{Re}\left[-\left(\bar{z}_3M_{23} - \bar{z}_2\right)dz_2 + \left(\bar{z}_2M_{32} - \bar{z}_3\right)dz_3 + \frac{|z_2|^2 - |z_3|^2}{\bar{z}_{23}}d\bar{z}_{32}\right]
\end{aligned}$$

using equations (A.55c) and (A.55d). If $z_i = \exp(2\pi\mathrm{i}\theta_i) \in \partial D$, the third term vanishes and the other terms evaluate to

$$\begin{aligned}
&\frac{1}{16\pi^2}\operatorname{Re}\left[-\left(\bar{z}_3M_{23} - \bar{z}_2\right)dz_2 + \left(\bar{z}_2M_{32} - \bar{z}_3\right)dz_3\right] \\
&= \frac{1}{16\pi^2}\operatorname{Re}2\pi\mathrm{i}\left[(\exp(2\pi i\theta_{23}K_{32} + 1)))\,d\theta_2 - (\exp(2\pi\mathrm{i}\theta_{32}K_{23} + 1)\,d\theta_3\right] \\
&= \frac{1}{8\pi}(\operatorname{Im}(\exp(2\pi\mathrm{i}\theta_{32})K_{23})d\theta_3) - \operatorname{Im}(\exp(2\pi\mathrm{i}\theta_{23})K_{32})\,d\theta_2) \\
&= \frac{1}{8\pi}(\operatorname{Im}(\exp(2\pi\mathrm{i}\theta_{32})K_{23}))\,(d\theta_2 + d\theta_3) \\
&= \frac{1}{8\pi}\left(\pi\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3) - \sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|\right)(d\theta_2 + d\theta_3)
\end{aligned}$$

Proof of (A.63f) **and** (A.65f)

We have

$$\begin{aligned}
\int_{D_1}\psi_1\tau_{12}\tau_{13} &= \left(\frac{1}{2\pi}\right)^3\int_{D_1}\operatorname{Im}(\bar{z}dz)\operatorname{Im}dK_{12}\operatorname{Im}dK_{13} \\
&= \left(\frac{1}{2\pi}\right)^3\frac{1}{4}\operatorname{Im}\int_{D_1}(\bar{z}_1dz_1dK_{12}\overline{dK_{13}} - \bar{z}_1dz_1dK_{12}dK_{13} + \bar{z}_1dz_1\overline{dK_{12}}dK_{13} - \bar{z}_1dz_1\overline{dK_{12}dK_{13}}) \\
&= \left(\frac{1}{2\pi}\right)^3\frac{1}{4}\operatorname{Im}\int_{D_1}d\bar{z}_1dz_1\left[\frac{|z_1|^2z_3}{u_{12}\bar{u}_{13}}d\bar{z}_2 - \frac{|z_1|^2z_2}{\bar{u}_{12}u_{13}}d\bar{z}_3 + \frac{\bar{z}_1^2}{\bar{u}_{12}\bar{u}_{13}}(z_2dz_3 - z_3dz_2)\right] \\
&= \frac{1}{16\pi^2}\operatorname{Re}\left[\left(\frac{1}{\bar{z}_3} + \frac{K_{23}}{\bar{z}_3^2z_2}\right)d\bar{z}_3 - \left(\frac{1}{\bar{z}_2} + \frac{K_{32}}{\bar{z}_2^2z_3}\right)d\bar{z}_2\right]
\end{aligned}$$

using equations (A.53b) and (A.53d). Let now $z_i = \exp(2\pi \mathrm{i}\theta_i) \in \partial D$, then we get

$$\begin{aligned}
&\frac{1}{16\pi^2}\operatorname{Re}\left[\left(\frac{1}{\bar z_3}+\frac{K_{23}}{\bar z_3^2 z_2}\right)d\bar z_3-\left(\frac{1}{\bar z_2}+\frac{K_{32}}{\bar z_2^2 z_3}\right)d\bar z_2\right]\\
&=\frac{1}{16\pi}\operatorname{Re}2\pi\mathrm{i}\left[\left(1+\frac{K_{32}}{\bar z_2 z_3}\right)d\theta_2-\left(1+\frac{K_{23}}{z_2\bar z_3}\right)d\theta_3\right]\\
&=\frac{1}{8\pi}\left[d\theta_3\operatorname{Im}\frac{K_{23}}{z_2\bar z_3}-d\theta_2\operatorname{Im}\frac{K_{32}}{\bar z_2 z_3}\right]\\
&=\frac{1}{8\pi}((\pi\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3)-\sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|)(d\theta_2+d\theta_3)
\end{aligned}$$

Proof of (A.63g) **and** (A.65g)

We have

$$\begin{aligned}
\int_{D_1}\psi_1\phi_{12}\tau_{13}&=\left(\frac{1}{2\pi}\right)^3\int_{D_1}\operatorname{Im}(\bar z dz)\operatorname{Im}dL_{12}\operatorname{Im}dK_{13}\\
&=\left(\frac{1}{2\pi}\right)^3\frac{1}{4}\operatorname{Im}\int_{D_1}(\bar z_1 dz_1 dL_{12}\overline{dK_{13}}-\bar z_1 dz_1 dL_{12}dK_{13}+\bar z_1 dz_1\overline{dL_{12}}dK_{13}-\bar z_1 dz_1\overline{dL_{12}dK_{13}})\\
&=\left(\frac{1}{2\pi}\right)^3\frac{1}{4}\operatorname{Im}\int_{D_1}d\bar z_1 dz\left[\frac{\bar z_1 z_3}{z_{12}\bar u_{13}}dz_2+\frac{|z_1|^2}{\bar z_{12}u_{13}}d\bar z_3-\frac{\bar z_1^2}{\bar z_{12}\bar u_{13}}dz_3-\frac{z_3\bar z_1}{\bar z_{12}\bar u_{13}}d\bar z_2\right]\\
&=\frac{1}{16\pi^2}\operatorname{Re}\left[\left(\bar z_2+\frac{K_{32}}{z_3}\right)dz_2+\left(\frac{|z_2|^2}{\bar z_3}+\frac{\bar z_2K_{23}}{\bar z_3^2}\right)d\bar z_3-\frac{\bar z_2(1-|z_2|^2)}{\bar u_{23}}dz_3-\frac{z_3(1-|z_2|^2)}{\bar u_{23}}d\bar z_2\right]\\
&=\frac{1}{16\pi^2}\operatorname{Re}\left[\left(\bar z_2+\frac{K_{32}}{z_3}\right)dz_2+\left(\frac{|z_2|^2}{\bar z_3}+\frac{\bar z_2K_{23}}{\bar z_3^2}\right)d\bar z_3-\frac{(1-|z_2|^2)}{\bar u_{23}}(\bar z_2dz_3+z_3d\bar z_2)\right]\\
&=\frac{1}{16\pi^2}\operatorname{Re}\left[\left(\bar z_2+\frac{K_{32}}{z_3}\right)dz_2+\left(\frac{|z_2|^2}{\bar z_3}+\frac{\bar z_2K_{23}}{\bar z_3^2}\right)d\bar z_3+(1-|z_2|^2)dK_{32}\right]
\end{aligned}$$

using equations (A.54b),(A.54c),(A.54d),(A.54e). If now $z_i = \exp(2\pi\mathrm{i}\theta_i)$, the last two terms vanish. The first two terms evaluate to

$$\begin{aligned}
&\frac{1}{16\pi^2}\operatorname{Re}\left[\left(\bar z_2+\frac{K_{32}}{z_3}\right)dz_2+\left(\frac{|z_2|^2}{\bar z_3}+\frac{\bar z_2K_{23}}{\bar z_3^2}\right)d\bar z_3\right]\\
&=\frac{1}{16\pi^2}\operatorname{Re}2\pi\mathrm{i}\left[(1+\exp(2\pi\mathrm{i}\theta_{23})K_{32})\,d\theta_2-(1+\exp(2\pi\mathrm{i}\theta_{32})K_{23})\,d\theta_3\right]\\
&=\frac{1}{8\pi}(\operatorname{Im}(\exp(2\pi\mathrm{i}\theta_{32})K_{23})d\theta_3)-\operatorname{Im}(\exp(2\pi\mathrm{i}\theta_{23})K_{32})\,d\theta_2)\\
&=\frac{1}{8\pi}((\pi\cos(2\pi\theta_{23})\eta_{S^1}(\theta_2,\theta_3)-\sin 2\pi\theta_{23}\log 2|\sin\pi\theta_{23}|)(d\theta_2+d\theta_3)
\end{aligned}$$

Proof of (A.63h) **and** (A.65h)

We have

$$\begin{aligned}\int_{D_1} \mu_1\phi_{12}\phi_{13} &= \frac{1}{2\pi i}\left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left(\frac{dz_{12}}{z_{12}} - \frac{d\bar{z}_{12}}{\bar{z}_{12}}\right)\left(\frac{dz_{13}}{z_{13}} - \frac{d\bar{z}_{13}}{\bar{z}_{13}}\right) \\ &= \frac{1}{2\pi i}\left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left[\frac{dz_2 dz_3}{z_{12}z_{13}} + \frac{d\bar{z}_2 d\bar{z}_3}{\bar{z}_{12}\bar{z}_{13}} \right. \\ &\qquad \left. - \frac{dz_2 d\bar{z}_3}{z_{12}\bar{z}_{13}} - \frac{d\bar{z}_2 dz_3}{\bar{z}_{12} z_{13}}\right] \\ &= \left(\frac{1}{4\pi i}\right)^2 \left[-\frac{\bar{z}_{23}}{z_{23}} dz_2 dz_3 - \frac{z_{23}}{\bar{z}_{23}} d\bar{z}_2 d\bar{z}_3 - M_{23} dz_2 d\bar{z}_3 - M_{32} d\bar{z}_2 dz_3\right] \\ &= \frac{1}{8\pi^2}\operatorname{Re}\left[M_{23} dz_2 d\bar{z}_3 + \frac{\bar{z}_{23}}{z_{23}} dz_2 dz_3\right]\end{aligned}$$

using (A.55a) and (A.55b). If now $z_i = \exp(2\pi \mathrm{i}\theta_i)$, we get (taking care of the signs)

$$\begin{aligned}\frac{1}{8\pi^2}\operatorname{Re}\left[M_{23} dz_2 d\bar{z}_3 + \frac{\bar{z}_{23}}{z_{23}} dz_2 dz_3\right] &= \frac{1}{8\pi^2}\operatorname{Re}\left[-K_{32} dz_2 d\bar{z}_3 + \frac{\frac{1}{z_2} - \frac{1}{z_3}}{z_2 - z_3} dz_2 dz_3\right] \\ &= \frac{1}{8\pi^2}\operatorname{Re}\left[(2\pi i)^2 \log(1 - \exp(2\pi i\theta_{32}))\exp(2\pi \mathrm{i}\theta_{23}) d\theta_2 d\theta_3 - \frac{dz_2 dz_3}{z_2 z_3}\right] \\ &= \frac{1}{2} d\theta_2 d\theta_3 \left(1 - \cos(2\pi\theta_{23})\log(2|\sin\pi\theta_{23}|) - \pi\sin(2\pi\theta_{23})\eta_{S^1}(\theta_2, \theta_3)\right)\end{aligned}$$

Proof of (A.63i) **and** (A.65i)

$$\begin{aligned}\int_{D_1} \mu_1\tau_{12}\tau_{13} &= \frac{1}{2\pi i}\left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left(\frac{z_2 d\bar{z}_1 + \bar{z}_1 dz_2}{\bar{u}_{12}} - \frac{z_1 d\bar{z}_2 + \bar{z}_2 dz_1}{u_{12}}\right)\left(\frac{z_3 d\bar{z}_1 + \bar{z}_1 dz_3}{\bar{u}_{13}} - \frac{z_1 d\bar{z}_3 + \bar{z}_3 dz_1}{u_{13}}\right) \\ &= \frac{1}{2\pi i}\left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left[\frac{\bar{z}_1^2 dz_2 dz_3}{\bar{u}_{12}\bar{u}_{13}} + \frac{z_1^2 d\bar{z}_2 d\bar{z}_3}{(u_{12})u_{13}} \right. \\ &\qquad \left. - \frac{|z_1|^2 d\bar{z}_2 dz_3}{u_{12}\bar{u}_{13}} - \frac{|z_1|^2 dz_2 d\bar{z}_3}{\bar{u}_{12} u_{13}}\right] \\ &= \left(\frac{1}{4\pi i}\right)^2 \left[\left(\frac{(z_3\bar{z}_2 + K_{32})}{(z_3\bar{z}_2)^2}\right) d\bar{z}_2 dz_3 + \left(\frac{(z_2\bar{z}_3 + K_{23})}{(z_2\bar{z}_3)^2}\right) dz_2 d\bar{z}_3\right] \\ &= -\frac{1}{8\pi^2}\operatorname{Re}\left[\frac{(z_2\bar{z}_3 + K_{23})}{(z_2\bar{z}_3)^2} dz_2 d\bar{z}_3\right]\end{aligned}$$

where we have used equations (A.53b) and (A.53d).

If now $z_i = \exp(2\pi \mathrm{i}\theta_i)$,

$$\begin{aligned}-\frac{1}{8\pi^2}\operatorname{Re}\left[\frac{(z_2\bar{z}_3 + K_{23})}{(z_2\bar{z}_3)^2} dz_2 d\bar{z}_3\right] &= -\frac{1}{8\pi^2}\operatorname{Re}\left[\frac{dz_2 d\bar{z}_3}{z_2\bar{z}_3}\left(1 + \frac{K_{23}}{z_2\bar{z}_3}\right)\right] \\ &= \frac{1}{2} d\theta_2 d\theta_3 \operatorname{Re}\left[1 + \exp(2\pi \mathrm{i}\theta_{32})\log(1 - \exp(2\pi \mathrm{i}\theta_{23}))\right] \\ &= -\frac{1}{2} d\theta_2 d\theta_3 \left(1 + \cos(2\pi\theta_{32})\log(2|\sin\pi\theta_{32}|) - \pi\sin(\theta_{32})\eta_{S^1}(\theta_2, \theta_3)\right) \\ &= -\frac{1}{2} d\theta_2 d\theta_3 \left(1 + \cos(2\pi\theta_{23})\log(2|\sin\pi\theta_{23}|) + \pi\sin(\theta_{23})\eta_{S^1}(\theta_2, \theta_3)\right)\end{aligned}$$

Proof of (A.63j)**and** (A.65j)

We have

$$\begin{aligned}
\int_{D_1} \mu_1\phi_{12}\tau_{13} &= \frac{1}{2\pi i}\left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left(\frac{dz_{12}}{z_{12}} - \frac{d\bar{z}_{12}}{\bar{z}_{12}}\right)\left(\frac{z_3 d\bar{z}_1 + \bar{z}_1 dz_3}{\bar{u}_{13}} - \frac{z_1 d\bar{z}_3 + \bar{z}_3 dz_1}{u_{13}}\right) \\
&= \frac{1}{2\pi i}\left(\frac{1}{4\pi i}\right)^2 \int_{D_1} d\bar{z}_1 dz_1 \left[\frac{z_1 dz_2 d\bar{z}_3}{z_{12}u_{13}} - \frac{\bar{z}_1 dz_2 dz_3}{z_{12}\bar{u}_{13}}\right. \\
&\qquad\qquad\left. + \frac{\bar{z}_1 d\bar{z}_2 dz_3}{\bar{z}_{12}\bar{u}_{13}} - \frac{z_1 d\bar{z}_2 d\bar{z}_3}{\bar{z}_{12}u_{13}}\right] \\
&= \left(\frac{1}{4\pi i}\right)^2 \left[\left(\frac{1-|z_2|^2}{u_{23}}\right) dz_2 d\bar{z}_3 + \left(\frac{1-|z_2|^2}{\bar{u}_{23}}\right) d\bar{z}_2 dz_3\right. \\
&\left. - \left(\frac{\bar{z}_2}{z_3} + \frac{K_{32}}{z_3^2}\right) dz_2 dz_3 - \left(\frac{z_2}{\bar{z}_3} + \frac{\log(\bar{u}_{32})}{\bar{z}_3^2}\right) d\bar{z}_2 d\bar{z}_3\right] \\
&= \frac{1}{8\pi^2}\operatorname{Re}\left[\left(\frac{\bar{z}_2}{z_3} + \frac{K_{32}}{z_3^2}\right) dz_2 dz_3 - \left(\frac{1-|z_2|^2}{u_{23}}\right) dz_2 d\bar{z}_3\right]
\end{aligned}$$

where we have used (A.54b) and (A.54c).

If now $z_i = \exp(2\pi \mathrm{i}\theta_i)$, the second term vanishes and the first equals

$$\begin{aligned}
\frac{1}{8\pi^2}\operatorname{Re}\left[\left(\frac{\bar{z}_2}{z_3} + \frac{K_{32}}{z_3^2}\right) dz_2 dz_3\right] &= \frac{1}{8\pi^2}\operatorname{Re}\left[(2\pi i)^2 (d\theta_2 d\theta_3 + \exp(2\pi \mathrm{i}(\theta_{23}))\log(1-\exp(2\pi\mathrm{i}\theta_{32}))d\theta_2 d\theta_3\right] \\
&= -\frac{1}{2} d\theta_2 d\theta_3 \left(1 + \cos(2\pi\theta_{23})\log(2|\sin\pi\theta_{23}|) + \pi\sin(\theta_{23})\eta_{S^1}(\theta_2,\theta_3)\right)
\end{aligned}$$

Proof of (A.66)

We have

$$\int_{D,1} \phi_{13}\phi_{12} = \frac{1}{4\pi}\operatorname{Im} K_{32}$$

and therefore

$$\begin{aligned}
\int_{D,1,2} \phi_{13}\phi_{12}\mu_2\phi_{24} &= \frac{1}{8\pi^2}\int_{D,2} \operatorname{Im}(K_{32}\mu_2)\operatorname{Im} dL_{24} \\
&= \frac{1}{16\pi^2}\operatorname{Re}\int_{D,2} (K_{32}\mu_2 dL_{24} - K_{32}\mu_2\overline{dL_{24}}) \\
&= \frac{1}{16\pi^2}\operatorname{Re}\int_{D,2} \frac{K_{32}\mu_2 d\bar{z}_4}{\bar{z}_{24}} - \frac{K_{32}\mu_2 dz_4}{z_{24}} \\
&= -\frac{1}{16\pi^2}\operatorname{Re}\left[(\bar{z}_4 + K_{34}(\bar{z}_3 - \bar{z}_4)dz_4\right] \\
&= -\frac{1}{16\pi^2}\operatorname{Re} K_{34}\bar{z}_{34} dz_4.
\end{aligned}$$

Here we used that $\int_1 K_{12}/z_{13} = 0$ for $z_3 \in \partial D$.

Proof of (A.67)

$$\int_{D,1,2} \phi_{13}\phi_{12}\psi_2\phi_{24} = \frac{1}{8\pi^2}\int_{D,2} \operatorname{Im}(K_{32})\operatorname{Im}(\bar{z}dz)\operatorname{Im} dL_{24}$$
$$= \frac{1}{32\pi^2}\operatorname{Im}\int_{D,2} K_{32}\bar{z}_2 dz_2 \overline{dL_{24}} + K_{32}z_2 d\bar{z}_2 dL_{24},$$

with two other terms vanishing because they contain no top form. This gives

$$\frac{1}{32\pi^2}\operatorname{Im}\int_{D,2} K_{32}\bar{z}_2 dz_2\overline{dL_{24}} + K_{32}z_2 d\bar{z}_2 dL_{24} = \frac{1}{32\pi^2}\operatorname{Im}\int_{D,2}\frac{\bar{z}_2 K_{32} dz_2 d\bar{z}_2}{\bar{z}_{24}} + \frac{\bar{z}_2 K_{32} d\bar{z}_2 dz_2}{z_{24}}$$
$$= \frac{1}{32\pi^2}\operatorname{Im}\int_{D,2}\frac{\bar{z}_2 K_{32} d\bar{z}_2 dz_2}{z_{24}} - \frac{z_2 K_{23} d\bar{z}_2 dz_2}{z_{24}}$$
$$= \frac{1}{16\pi}\operatorname{Re}\left[\frac{1}{2z_3^2}\left(K_{34} + z_3\bar{z}_4 + \frac{(z_3\bar{z}_4)^2}{2}\right) - \frac{\bar{z}_4^2}{2}K_{34} - (1-|z_4|^2)K_{42}\right]$$

by equations (A.60d) and (A.60f). Now we can use that $z_3, z_4 \in \partial D$ to obtain

$$\frac{1}{16\pi}\operatorname{Re}\left[\frac{1}{2z_3^2}\left(K_{34} + z_3\bar{z}_4 + \frac{(z_3\bar{z}_4)^2}{2}\right) - \frac{\bar{z}_4^2}{2}K_{34} - (1-|z_4|^2)K_{42}\right]$$
$$= \frac{1}{32\pi}\operatorname{Re}\left[\bar{z}_4^2 K_{34}(\bar{z}_3^2 z_4^2 - 1) + \bar{z}_3\bar{z}_4 + \bar{z}_4^2/2\right]$$

Proof of (A.68) **and** (A.69)

Here we first perform pushforward over z_2. Then we get

$$\int_{D,1,2} \phi_{13}\phi_{14}\phi_{12}\phi_{25} = \int_{D,1} \phi_{13}\phi_{14}\operatorname{Im}\frac{1}{4\pi}K_{15}$$
$$= \frac{1}{16\pi^2}\int_{D,1} \operatorname{Im} K_{15}\operatorname{Im} dL_{13}\operatorname{Im} dL_{14}$$
$$= \frac{1}{64\pi^2}\operatorname{Im}\int_{D,1} K_{15}(dL_{13}\overline{dL_{14}} + \overline{dL}_{13}dL_{14})$$
$$= \frac{1}{64\pi^2}\operatorname{Im}\int_{D,1}\frac{K_{15}d\bar{z}_1 dz_1}{\bar{z}_{13}z_{14}} - \frac{K_{15}d\bar{z}_1 dz_1}{\bar{z}_{14}z_{13}}$$
$$= \frac{1}{64\pi^2}\operatorname{Im}\int_{D,1}\frac{K_{15}d\bar{z}_1 dz_1}{\bar{z}_{13}z_{14}} - \frac{K_{15}d\bar{z}_1 dz_1}{\bar{z}_{14}z_{13}}$$
$$= \frac{1}{32\pi}\operatorname{Re}\left[Li_{1,1}(\bar{z}_4 z_3, \bar{z}_5 z_4) + Li_2(\bar{z}_5 z_3) + K_{45}M_{43} - (3 \leftrightarrow 4)\right]$$

using (A.60h). This implies the proof of (A.69), since the pushforwards $\int_1 \phi_{12}\phi_{13} = \int_1 \phi_{12}\tau_{13}$ are identical.

A.8 Pushforwards of currents on the circle

The pushforward of currents on the circle can be evaluated as follows. For products of delta functions

Lemma A.8.1. *We have*

$$\int_{S^1_1} \delta^{(1)}_{S^1}(t_1 - t_2)\delta^{(1)}_{S^1}(t_1 - t_3) = \delta^{(1)}_{S^1}(t_2 - t_3) \tag{A.78}$$

$$\int_{S^1_1} dt_1 \delta^{(1)}_{S^1}(t_1 - t_2)\delta^{(1)}_{S^1}(t_1 - t_3) = \delta^{(1)}_{S^1}(t_2 - t_3)\frac{dt_2 + dt_3}{2} \tag{A.79}$$

$$\int_{S^1_1} \delta^{(1)}_{S^1}(t_1 - t_2)\delta^{(1)}_{S^1}(t_1 - t_3)\delta^{(1)}_{S^1}(t_1 - t_4) = \delta^{(1)}_{S^1}(t_2 - t_3)\delta^{(1)}_{S^1}(t_3 - t_4) \tag{A.80}$$

$$\int_{S^1_1} dt_1 \delta^{(1)}_{S^1}(t_1 - t_2)\delta^{(1)}_{S^1}(t_1 - t_3)\delta^{(1)}_{S^1}(t_1 - t_4) = \delta^{(1)}_{S^1}(t_2 - t_3)\delta^{(1)}_{S^1}(t_3 - t_4)\frac{dt_2 + dt_3 + dt_4}{3} \tag{A.81}$$

Proof. These identities can be checked by the definition of the pushforward. E.g. for the first equation, we have to check that for all forms $\alpha \in \Omega^\bullet(S^1 \times S^1)$ we have

$$\int_{S^1,2,3} \delta^{(1)}_{S^1}(t_2 - t_3) \wedge \alpha = \int_{S^1,1,2,3} \delta^{(1)}_{S^1}(t_1 - t_2)\delta^{(1)}_{S^1}(t_1 - t_3) \wedge \alpha$$

In this case this is simple because both sides equal $\iota^*_\Delta \alpha$, where ι is the embedding of the diagonal in $S^1 \times S^1$. The other cases can be check by straighforward computation. □

Appendix B

Computations in the gluing of lens spaces

In this appendix we perform the explicit computations that appear when gluing two lens spaces to a solid torus. Often we will choose not to evaluate integrals that disappear when reducing the residual fields. Recall that reducing residual fields corresponds to setting $z^{2,i,\mathbb{A}} = z^{+,\mathbb{B}}_{2,i} = 0$ and pairing $z^{2,i,\mathbb{B}}$ with $z^{+,\mathbb{A}}_{2,i}$ to $\frac{1}{p}$, in particular, if in a product of residual fields does not contain the same amount of $z^{+,\mathbb{A}}_{2,i}$ and $z^{2,i,\mathbb{B}}$ fields, it vanishes after reducing residual fields.

B.1 Pairing the 1-point functions

B.1.1 Pairing $\Gamma_{1,1}$

As noted above $\Gamma_{1,1}$ pairs to zero against $\Gamma^{\mathsf{b}}_{1,2}$ and $\Gamma_{1,3}$ (notice that $\psi_{\Gamma^{\mathbb{A}}_X} * \psi_{\Gamma^{\mathbb{B}}_Y} = 0$ is equivalent to $\psi_{\Gamma^{\mathbb{A}}_Y} * \psi_{\Gamma^{\mathbb{B}}_X} = 0$). For the remaining pairings note that $\varphi^* dtd\theta = (\varphi^{-1})^*(dtd\theta) = dtd\theta$, and $\varphi^* d\theta = ndt + qd\theta, (\varphi^{-1})^* d\theta = -ndt + md\theta$. Then we get

$$\psi_{\Gamma^{\mathbb{A}}_{1,1}} * \psi_{\Gamma^{\mathbb{B}}_{1,1}} = \mu^i_{jk}\gamma^{kl}_m z^{+,\mathbb{A}}_{1i} z^{2j,\mathbb{A}} z^{1l,\mathbb{B}} z^{+,\mathbb{B}}_{2j} \int_{\partial M} d\theta \varphi^* d\theta = -n\mu^i_{jk}\gamma^{kl}_m z^{+,\mathbb{A}}_{1i} z^{2j,\mathbb{A}} z^{1m,\mathbb{B}} z^{+,\mathbb{B}}_{2l}$$

To pair against $\Gamma^{\mathsf{a}}_{1,2}$, take the disjoint union of $\Gamma_{1,1}$ with Γ_0. Since we integrate over 2 boundary points, only the 4-form part survives - this is the product of the 3-form part of $\psi_{\Gamma^{\mathsf{a}}_{1,2}}$ with the

1-form part of $\psi_{\Gamma_{1,1}}$ and the 0-form part of ψ_{Γ_0}. The result is

$$\begin{aligned}
\left(\psi_{\Gamma^{\mathbb{A}}_{1,1}}\psi_{\Gamma^{\mathbb{A}}_{0}}\right) * \psi_{\Gamma^{\mathbb{B}}_{1,2}} &= -\mu^i_{jk}\varphi^{klm} z^{+,\mathbb{A}}_{1i} z^{2j,\mathbb{A}} z^{+,\mathbb{A}}_{1,l} z^{+,\mathbb{B}}_{2,m} \int_{\partial M\times\partial M} (\varphi^{-1})^*(d\theta)_1 \nu_{12}\delta^{(1)}_{S^1}(t_1-t_2) \\
&= n\mu^i_{jk}\varphi^{klm} z^{+,\mathbb{A}}_{1i} z^{2j,\mathbb{A}} z^{+,\mathbb{A}}_{1,l} z^{+,\mathbb{B}}_{2,m} \\
\psi_{\Gamma^{\mathbb{A}}_{1,2}} * \left(\psi_{\Gamma^{\mathbb{B}}_{1,1}}\psi_{\Gamma^{\mathbb{B}}_{0}}\right) &= -\psi_{ijk} z^{2i,\mathbb{A}}\gamma^{kl}_m z^{1j,\mathbb{B}} z^{1m,\mathbb{B}} z^{+,\mathbb{B}}_{2,l} \int_{\partial M\times\partial M} \nu_{12}\delta^{(1)}_{S^1}(t_1-t_2)\varphi^*(d\theta)_1 \\
&= n\psi_{ijk}\gamma^{kl}_m z^{2i,\mathbb{A}} z^{1j,\mathbb{B}} z^{1m,\mathbb{B}} z^{+,\mathbb{B}}_{2,l}
\end{aligned}$$

All these terms vanish after reducing residual fields.

B.1.2 Pairing $\Gamma^{\mathsf{b}}_{1,2}$

The nonzero pairings of $\Gamma^{\mathsf{b}}_{1,2}$ are with itself or - in a product with Γ_0 - to $\Gamma_{1,3}$.

Pairing $\Gamma^{\mathsf{b}}_{1,2}$ with itself

In the pairing with itself, only the 2-form part contributes, and we get

$$\begin{aligned}
\psi_{\Gamma^{\mathsf{b},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathsf{b},\mathbb{B}}_{1,2}} = \frac{1}{2}\mu^i_{jk}\gamma^{jk}_l z^{+,\mathbb{A}}_{2i} z^{2l,\mathbb{B}} \Bigg(&\int_{\partial M\times\partial M} \eta_{S^1}(\theta_1,\theta_2)\delta_{S^1}(t_1-t_2)dt_1dt_2\varphi^*(\eta_{S^1}(\theta_1,\theta_2)\delta_{S^1}(t_1-t_2)dt_1dt_2) \\
+ &\int_{\partial M\times\partial M} \eta_{S^1}(\theta_1,\theta_2)\delta_{S^1}(t_1-t_2)dt_1dt_2)\varphi^*\left(\eta_{S^1}(t_1,t_2)(d\theta_1dt_1 - d\theta_2dt_2)\right) \\
+ &\int_{\partial M\times\partial M} \eta_{S^1}(t_1,t_2)(d\theta_1dt_1 - d\theta_2dt_2)\varphi^*(\eta_{S^1}(\theta_1,\theta_2)\delta_{S^1}(t_1-t_2)dt_1dt_2) \\
+ &\int_{\partial M\times\partial M} \eta_{S^1}(t_1,t_2)(d\theta_1dt_1 - d\theta_2dt_2)\varphi^*\left(\eta_{S^1}(t_1,t_2)(d\theta_1dt_1 - d\theta_2dt_2)\right)\Bigg)
\end{aligned}$$

The integral in the third line is equal to the integral in the second line if we replace φ by φ^{-1}, which in turn is equal to zero (the forms do not multiply to a top form).
For the other two integrals assume first $p=0$, then $m=q=\pm1$ and $\phi = q\mathrm{id}$. The integral in the first line is zero and the integral in the last line is $-2\int_{S^1\times S^1}(\eta^{S^1,12})^2dt_1dt_2 = \frac{1}{6}$.
Now assume $p\neq 0$. The integral in the last line is then (-2) times (we set $v = dt_1d\theta_1dt_2d\theta_2)$)

$$\begin{aligned}
\int_{\partial M\times\partial M} \eta_{S^1}(t_1,t_2)\varphi^*(\eta_{S^1}(t_1,t_2))v &= \int_{\partial M\times\partial M} \eta_{S^1}(t_1,t_2)\eta_{S^1}(mt_1+p\theta_1, mt_2+p\theta_2)v \\
&= \int_{\partial M\times\partial M} \eta_{S^1}(t_1,t_2)\eta_{S^1}(mt_1+p\theta_1-mt_2, p\theta_2)v \\
&= p\int_{\partial M\times\partial M} \eta_{S^1}(t_1,t_2)\eta_{S^1}(mt_1+p\theta_1-mt_2, \theta_2)v = 0.
\end{aligned}$$

Here we have used rotational invariance and periodicity of η_{S^1}, and in the last equality the fact that $\int_2 \eta_{S^1,12}d\theta_2 = 0$.

The integral in the first line is

$$-p^2 \int_{\partial M\times\partial M} \eta_{S^1}(\theta_1,\theta_2)\delta_{S^1}(t_1-t_2)\varphi^*(\eta_{S^1}(\theta_1,\theta_2)\delta_{S^1}(t_1-t_2))v.$$

To compute it, notice that δ_{S^1} is actually a Dirac Comb with period 1, whichwe denote III_1. This allows us to write

$$\begin{aligned}\varphi^*\delta_{S^1}(t_1,t_2) = III_1(m(t_1-t_2)+p(\theta_1-\theta_2)) &= III_1\left(p\left(\frac{m}{p}(t_1-t_2)+(\theta_1-\theta_2)\right)\right)\\ &= \frac{1}{p}III_{1/p}\left(\frac{m}{p}(t_1-t_2)+(\theta_1-\theta_2)\right)\\ &= \frac{1}{p}\sum_{k=0}^{p-1} III_1\left(\frac{m}{p}(t_1-t_2)+(\theta_1-\theta_2)+\frac{k}{p}\right)\end{aligned}$$

Plug this in the integral and integrate over θ_2, which leaves us with

$$\begin{aligned}\int_{\mathbb{T}^2\times S^1} \frac{1}{p}\sum_{k=0}^{p-1}&\eta_{S^1}\left(\theta_1,\frac{m}{p}(t_1-t_2)+\theta_1+\frac{k}{p}\right)\\ &\eta_{S^1}\left(nt_1+q\theta_1, nt_2+q\left(\frac{m}{p}(t_1-t_2)+\theta_1+\frac{k}{p}\right)\right)\delta(t_1-t_2)dt_1d\theta_1dt_2.\end{aligned}$$

Now, we integrate over t_2, which forces t_1 and t_2 to agree, leaving us with

$$\int_{S^1\times S^1}\frac{1}{p}\sum_{k=0}^{p-1}\eta_{S^1}\left(\theta_1,\theta_1+\frac{k}{p}\right)\eta_{S^1}\left(nt_1+q\theta_1,nt_1+q\theta_1+\frac{kq}{p}\right)dt_1d\theta_1.$$

Because of rotational invariance of η_{S^1} we find this equals

$$\frac{1}{p}\int_{S^1\times S^1}\sum_{k=0}^{p-1}\eta_{S^1}\left(0,\frac{k}{p}\right)\eta_{S^1}\left(0,\frac{kq}{p}\right)dt_1d\theta_1 = \frac{1}{p}\sum_{k=0}^{p-1}\eta_{S^1}\left(0,\frac{k}{p}\right)\eta_{S^1}\left(0,\frac{kq}{p}\right)$$

which is precisely $\frac{1}{p}$ times the Dedekind sum $s(q,p)$. Hence overall the pairing evaluates to

$$\psi_{\Gamma^{\mathfrak{b},\mathbb{A}}_{1,2}} * \psi_{\Gamma^{\mathfrak{b},\mathbb{B}}_{1,2}} = \mu^i_{jk}\gamma^{jk}_l z^{+,\mathbb{A}}_{2i} z^{2l,\mathbb{B}}\begin{cases}\frac{-p}{2}s(q,p) & p\neq 0\\ \frac{q}{12} & p=0\end{cases}\tag{B.1}$$

Pairing $\Gamma^{\mathfrak{b}}_{1,2}$ with $\Gamma_{1,3}$

To pair $\Gamma^{\mathfrak{b}}_{1,2}\sqcup\Gamma_0$ with $\psi_{\Gamma_{1,3}}$, notice that the form

$$\omega_{\Gamma_{1,3}} = \int_{M_0}\eta_{01}\eta_{02}\eta_{03},$$

which is the weight of the graph $\psi_{\Gamma_{1,3}}$ is a 3-form. Hence the only term contributing to the pairing is the one containing the product of the 2-form term of $\psi_{\Gamma^{\mathfrak{b}}_{1,2}}$ and the 1-form part of

ψ_{Γ_0}. We have

$$\left(\psi_{\Gamma_{1,2}^{\mathfrak{b},\mathbb{A}}}\psi_{\Gamma_0^{\mathbb{A}}}\right) * \psi_{\Gamma_{0,3}^{\mathbb{B}}} = z_{2i}^{+,\mathbb{A}} z_{2l}^{+,\mathbb{A}} \mu^i_{jk}\varphi^{jkl}\int_{(\partial M)^3}(\omega_{\Gamma_{1,2}})_{12}dt_3(\varphi^{\times 3})^*(\omega_{\Gamma_{1,3}})$$

$$\psi_{\Gamma_{1,3}^{\mathbb{A}}} * \left(\psi_{\Gamma_{1,2}^{\mathfrak{b},\mathbb{B}}}\psi_{\Gamma_0^{\mathbb{B}}}\right) = z^{2i,\mathbb{B}} z^{2l,\mathbb{B}} \mu_i^{jk}\varphi_{jkl}\int_{(\partial M)^3}\omega_{\Gamma_{1,3}}((\varphi^{\times 3})^*(\omega_{\Gamma_{1,2}})_{12}dt_3)$$

These terms vanish after reducing the background fields.

B.1.3 Pairing $\Gamma_{1,3}\Gamma_0$ with itself

In this section we compute the integral

$$J = \int_{(\partial M)^4}(\omega_{\Gamma_{1,3}})_{123}dt_4(\varphi^{\times 4})^*((\omega_{\Gamma_{1,3}})_{234}dt_1). \tag{B.2}$$

The first step is to rewrite $\omega_{\Gamma_{1,3}}$ as an integral over the bulk and then use Lemma 5.5.1 to compute integral over boundary points 1 and 4, like so:

$$\begin{aligned}
J &= \int_{(\partial M)^4}(\omega_{\Gamma_{1,3}})_{123}dt_4(\varphi^{\times 4})^*(\omega_{\Gamma_{1,3}})_{234}dt_1) \\
&= \int_{M_0\times M_5\times(\partial M)^4}\eta_{01}\eta_{02}\eta_{03}dt_4(\varphi^{\times 4})^*(\eta_{52}\eta_{53}\eta_{54})dt_1) \\
&= \int_{M_0\times M_5\times(\partial M)^4}\eta_{01}\eta_{02}\eta_{03}(\varphi^{\times 3})^*(\eta_{52}\eta_{53})\eta_{54}\varphi_*(dt_4))(mdt_1+pd\theta_1) \\
&= \int_{M_0\times M_5\times(\partial M)^2}\eta_{01}\eta_{02}(mdt_0+p\psi_0)\varphi^*(\eta_{15}\eta_{25}(qdt_5-p\psi_5)) \\
&= \int_{(\partial M)^2} mq\omega_{J_1}\varphi^*\omega_{J_1} - mp\omega_{J_1}\varphi^*\omega_{J_2} + qp\omega_{J_2}\varphi^*\omega_{J_1} - p^2\omega_{J_2}\varphi^*\omega_{J_2}
\end{aligned}$$

where we defined $\omega_{J_1} := \int_0 dt_0\eta_{01}\eta_{02}$ and $\omega_{J_2} := \int_0 \psi_0\eta_{01}\eta_{02}$. We already have the explicit expression

$$\omega_{J_1} = -\eta^\theta_{12}\delta_{S^1,12}dt_1dt_2 + (d\theta_2 dt_2 - d\theta_3 dt_3)\eta^t_{S^1,23}$$

and already computed that

$$\int_{(\partial M)^2}\omega_{J_1}\varphi^*\omega_{J_1} = -ps(q,p)$$

if $p \neq 0$. Let us compute ω_{J_2}. We have

$$\begin{aligned}
\omega_{I_2} &= \int_0 \psi_0\eta_{01}\eta_{02} = \int_0 \psi(\eta_{D,01}\delta^{(1)}_{S^1,01}\eta_{D,02}\delta^{(1)}_{S^1,02}) \\
&= \delta^{(1)}_{S^1,12}(4\int_{D_0}\psi_0\phi_{01}\phi_{02}) \\
&= \frac{1}{2}\delta^{(1)}_{S^1,12}f(\theta_1,\theta_2)(d\theta_1+d\theta_2)
\end{aligned}$$

where

$$f(\theta_1,\theta_2) = \cos(2\pi\theta_{12})\eta_{S^1}(\theta_1,\theta_2) - \frac{1}{\pi}\sin 2\pi\theta_{12}\log 2|\sin\pi\theta_{12}|.$$

Now let us compute

$$\begin{aligned}\int_{(\partial M)^2}\omega_{J_1}\varphi^*\omega_{J_2} &= \frac{1}{2}\int_{(\partial M)^2}\left(\eta^\theta_{12}\delta_{S^1,12}dt_1dt_2 + (d\theta_1dt_1 - d\theta_2dt_2)\eta^t_{S^1,12}\right)\varphi^*(\delta_{S^1}(t_{12})f(\theta_{12})dt_{12}(d\theta_1+d\theta_2)) \\ &= pq\int_{(\partial M)^2}\left(\eta^\theta_{12}\delta_{S^1,12}\varphi^*(\delta_{S^1}(t_{12})f(\theta_{12}))\right)v - \int_{(\partial M)^2}\eta^t_{S^1,12}\varphi^*(\delta_{S^1}(t_{12})f(\theta_{12}))v\end{aligned}$$

where $v = dt_1dt_2d\theta_1d\theta_2$. If $\varphi = \pm\mathrm{id}$ then this expression vanishes, so from now on we assume $\varphi \neq \pm\mathrm{id} \Leftrightarrow p \neq 0$. Now we use that

$$\varphi^*\delta_{S^1}(t_{12}) = \frac{1}{p}\sum_{k=0}^{p-1}\delta_{S^1}\left(\frac{m}{p}t_{12} + \theta_{12} + \frac{k}{p}\right)$$

to compute the first term above as

$$\sum_{k=0}^{p-1} q\int_{(\partial M)^2}\eta^\theta_{12}f(nt_{12}+q\theta_{12})\delta_{S^1}(t_{12})\delta_{S^1}\left(\frac{m}{p}t_{12}+\theta_{12}+\frac{k}{p}\right)v = q\sum_{k=0}^{p-1}\eta_{S^1}\left(\frac{k}{p}\right)f\left(\frac{qk}{p}\right)$$

similarly to before. The second term is a bit more complicated since we lack a second delta function. We get

$$\begin{aligned}\int_{(\partial M)^2}\eta^t_{12}\varphi^*(\delta_{S^1}(t_{12})f(\theta_{12}))v &= \sum_{k=0}^{p-1}\frac{1}{p}\int_{(\partial M)^2}\eta^t_{12}\delta\left(\frac{m}{p}t_{12}+\theta_{12}+\frac{k}{p}\right)f(nt_{12}+q\theta_{12})v \\ &= \sum_{k=0}^{p-1}\frac{1}{p}\int_{S^1_{t_1}\times S^1_{t_2}}\eta_{S^1}(t_{12})f((n-mq/p)t_{12}-qk/p)dt_1dt_2 \\ &= -\sum_{k=0}^{p-1}\frac{1}{p}\int_{S^1_{t_1}\times S^1_{t_2}}\eta_{S^1}(t_{12})f(1/pt_{12}+qk/p)dt_1dt_2\end{aligned}$$

To compute the last integral, note that

$$\begin{aligned}\eta_{S^1}(t) &= \frac{1}{\pi}\operatorname{Im}\log(1-e^{2\pi it}) = -\frac{1}{\pi}\operatorname{Im}\sum_{l\geq 1}\frac{e^{2\pi itl}}{l}, \\ f(t) &= \frac{1}{\pi}\operatorname{Im}(e^{-2\pi it}\log(1-e^{2\pi it}) = -\frac{1}{\pi}\operatorname{Im}\sum_{l\geq 1}\frac{e^{2\pi it(l-1)}}{l}\end{aligned}$$

where the sums converge conditionally for every $t \notin \mathbb{Z}$, so we can rewrite the integral using $\operatorname{Im} z \operatorname{Im} w = 1/2 \operatorname{Re}(\bar{z}w - zw)$. Further, we have

$$\begin{aligned}
\sum_{k=0}^{p-1} f(1/pt + qk/p) &= \sum_{k=0}^{p-1} f(1/pt + k/p) \\
&= -\frac{1}{\pi} \operatorname{Im} \sum_{l \geq 1} \sum_{k=0}^{p-1} \frac{e^{2\pi i(l-1)(1/pt+k/p)}}{l} \\
&= -\frac{1}{\pi} \operatorname{Im} \sum_{l \geq 0} \frac{e^{2\pi i l t/p}}{l+1} \sum_{k=0}^{p-1} e^{2\pi i k l/p} \\
&= -\frac{1}{\pi} \operatorname{Im} p \sum_{l \geq 0} \frac{e^{2\pi i l t}}{pl+1},
\end{aligned}$$

where we used that

$$\sum_{k=0}^{p-1} e^{2\pi i k l/p} = \begin{cases} p & l \mid p \\ 0 & l \nmid p \end{cases} \tag{B.3}$$

Plugging everything into the integral we find

$$\begin{aligned}
&-\sum_{k=0}^{p-1} \frac{1}{p} \int_{S^1_{t_1} \times S^1_{t_2}} \eta_{S^1}(t_{12}) f(1/pt_{12} + qk/p) dt_1 dt_2 \\
&= \frac{1}{\pi^2} \sum_{k \geq 1} \sum_{l \geq 0} \frac{1}{2} \operatorname{Re} \int_{S^1 \times S^1} \frac{e^{-2\pi i k t_{12}}}{k} \frac{e^{2\pi i l t_{12}}}{pl+1} - \frac{e^{2\pi i k t_{12}}}{k} \frac{e^{2\pi i l t_{12}}}{pl+1} dt_1 dt_2 \\
&= \frac{1}{2\pi^2} \sum_{k \geq 1} \frac{1}{k(kp+1)} = \frac{1}{2\pi^2} H_{\frac{1}{p}},
\end{aligned}$$

since $\int_{S^1 \times S^1} e^{2\pi i(k-l)t_{12}} dt_1 dt_2 = \delta_{kl}$. Here $H_{\frac{1}{p}}$ denotes analytic extension of the harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{n}$ evauluated at $1/p$. In total, we get

$$\int_{(\partial M)^2} \omega_{J_1} \varphi^* \omega_{J_2} = q \sum_{k=0}^{p-1} \eta_{S^1}\left(\frac{k}{p}\right) f\left(\frac{qk}{p}\right) + \frac{1}{2\pi^2} H_{\frac{1}{p}}.$$

Similarly, we can compute

$$\int_{(\partial M)^2} \omega_{J_2} \varphi^* \omega_{J_1} = \int_{(\partial M)^2} \omega_{J_1} (\varphi^{-1})^* \omega_{J_2} = m \sum_{k=0}^{p-1} \eta_{S^1}\left(\frac{k}{p}\right) f\left(\frac{mk}{p}\right) + \frac{1}{2\pi^2} H_{\frac{1}{p}}.$$

Finally, we can compute, exploiting again the delta functions

$$\begin{aligned}
\int_{(\partial M)^2} \omega_{J_2} \varphi^* \omega_{J_2} &= \frac{1}{4} \int_{(\partial M)^2} \delta^{(1)}_{S^1}(t_{12}) f(\theta_1, \theta_2)(d\theta_1 + d\theta_2) \varphi^* (\delta^{(1)}_{S^1}(t_{12}) f(\theta_1, \theta_2)(d\theta_1 + d\theta_2)) \\
&= n \sum_{k=0}^{p-1} f(k/p) f(qk/p)
\end{aligned}$$

Summing everything, we get

$$J = mqp\left(s(q,p) + \sum_{k=0}^{p-1} \eta_{S^1}\left(\frac{k}{p}\right) f\left(\frac{mk}{p}\right) + \eta_{S^1}\left(\frac{k}{p}\right) f\left(\frac{qk}{p}\right)\right) np^2 \sum_{k=0}^{p-1} f\left(\frac{k}{p}\right) f\left(\frac{qk}{p}\right) + (q-m)\frac{H_{1/p}}{2\pi^2}$$

$$= p\left(s(q,p) + \sum_{k=0}^{p-1} \eta_{S^1}\left(\frac{k}{p}\right) f\left(\frac{mk}{p}\right) + \eta_{S^1}\left(\frac{k}{p}\right) f\left(\frac{qk}{p}\right)\right)$$

$$+ np^2 \sum_{k=0}^{p-1} \left(\eta_{S^1}\left(\frac{k}{p}\right) + f\left(\frac{k}{p}\right)\right)\left(\eta_{S^1}\left(\frac{qk}{p}\right) + f\left(\frac{qk}{p}\right)\right) + (q+m)\frac{H_{1/p}}{2\pi^2} \tag{B.4}$$

where we used that $mq = 1 + np$ and

$$\sum_{k=0}^{p-1} \eta_{S^1}\left(\frac{k}{p}\right) f\left(\frac{mk}{p}\right) = \sum_{k=0}^{p-1} \eta_{S^1}\left(\frac{qk}{p}\right) f\left(\frac{k}{p}\right)$$

which follows from periodicity of η, f and the fact that $qm \equiv 1(\text{mod}p)$.

B.1.4 Pairing $\Gamma_{1,3}$

It only remains to compute the pairing of $\psi_{\Gamma^{\mathfrak{b}}_{1,3}}$ with itself. It is given by

$$\psi_{\Gamma^{\mathfrak{b},\mathbb{B}}_{1,3}} * \psi_{\Gamma^{\mathfrak{b},\mathbb{A}}_{1,3}} = \frac{1}{6} f_{ijk} f^{ijk} \int_{(\partial M)^3} \omega_{\Gamma_{1,3}} (\varphi^{\times 3})^* (\omega_{\Gamma_{1,3}}).$$

Recall that $\omega_{1,3}$ is given by

$$\omega_{1,3} = \alpha_{123} \delta^{(1)}_{S^1}(t_1, t_2) \delta^{(1)}_{S^1}(t_2, t_3) + \left(\nu_{12} \eta_{S^1}(t_2, t_3) \delta^{(1)}(t_1, t_2) + \text{cycl.}\right)$$

$$= \underbrace{(\alpha_{123}\delta_{12}\delta_{23} dt_{12} dt_{23})}_{\omega_I} + \underbrace{\left(h^\theta_{12} \eta^t_{23} \delta_{12} d\theta_1 d\theta_2 dt_{12} + \text{cycl.}\right)}_{\omega_{II}}$$

where we have condensed the notation a little: f^x_{ij} is short for $f(x_i, x_j)$, δ for δ^t and we have introduced h by $\nu_{ij} = h^\theta_{ij} d\theta_i d\theta_j$. Also note that $d\theta_1 d\theta_2 dt_{12} = v_1 d\theta_2 + v_2 d\theta_1$ where $v_i = dt_i d\theta_i$. Let us compute the pairings. We will do so by considering seperately

$$\int \omega_I \varphi^* \omega_I, \int \omega_I \left(\varphi^*(\omega_{II}) - (\varphi^{-1})^*(\omega_{II})\right), \int \omega_{II} \varphi^* \omega_{II}.$$

Let us start with the last one. We have to check what are the coefficients of a top form $v = v_1 v_2 v_3$. Since $\varphi^*(v_1 d\theta_2) = v_1(ndt + qd\theta)$, we get

$$\int \omega_{II} \varphi^* \omega_{II} = -n \int \left(h^\theta_{12} \eta^t_{23} \delta_{12} \varphi^*(h^\theta_{23} \eta^t_{31} \delta_{23} + h^\theta_{31} \eta^t_{12} \delta_{31}) + \text{cycl.}\right) v$$

(e.g. $v_1 d\theta_2$ conspires with $v_3(ndt_2)$ to give a top form, etc.). We claim that all terms integrate to 0. To see this, let us look closer at the first term. We have

$$\int h^\theta_{12} \eta^t_{23} \delta_{12} \varphi^*(h_{23} \eta^t_{31} \delta_{23}) v = \frac{1}{p} \sum_{k=0}^{p-1} \int h(\theta_{12} \eta(t_{23}) \delta(t_{12}) h(nt_{23} + q\theta_{23}) \eta(mt_{31} + p\theta_{31}) \delta\left(\frac{m}{p} t_{23} + \theta_{23} + \frac{k}{p}\right) v$$

where we treat the delta function as above. Now integrating over θ_3 forces $\theta_3 = \frac{m}{p}t_{23} + \theta_2 + \frac{k}{p}$. The integrand above then becomes

$$\int h(\theta_{12}\eta(t_{23})\delta(t_{12})h(nt_{23} - qm/pt_{23} - k/p)\eta(mt_{31} + mt_{23} + p\theta_{21} + k)v_1v_2dt_3$$
$$= \int h(\theta_{12}\eta(t_{13})h(1/pt_{13} + k/p)\eta(p\theta_{21})v_1d\theta_2dt_3$$

where in the second line we integrated over t_2. Now the claim follows from the following lemma:

Lemma B.1.1. *For any $p \in \mathbb{Z}, p \neq 0$ we have*

$$\int_0^1 h(\theta_{12})\eta_{S^1}(p\theta_{12})d\theta_1 = 0.$$

Proof. Since both h and η_{S^1} are periodic, we can shift the domain of integration and equivalently prove

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} h(\theta)\eta_{S^1}(p\theta) = 0,$$

which follows from the simple fact that h is even and η_{S^1} is odd. □

Now consider a pairing of the form $\int \omega_I\varphi^*\omega_{II}$. Rewriting $d\theta_1 d\theta_2 dt_{12} = \frac{1}{2}(d\theta_1 + d\theta_2)d\theta_{12}dt_{12}$ we see that $\varphi^*\omega_{II}$ always contains a term of the form dt_{ij} (since $\varphi^*(d\theta_{ij}dt_{ij}) = d\theta_{ij}dt_{ij}$). Any such term wedges to zero against $dt_{12}dt_{23}$, therefore the mixed terms all vanish.
Now consider

$$\omega_I\varphi^*\omega_I = (\alpha_{123}\delta_{12}\delta_{23}dt_{12}dt_{23})\,\varphi^*\,(\alpha_{123}\delta_{12}\delta_{23}dt_{12}dt_{23})$$
$$= p^2\,(\alpha_{123}\varphi^*(\alpha_{123})\delta_{12}\delta_{23}\varphi^*(\delta_{12}\delta_{23})dt_{12}dt_{23}d\theta_{12}d\theta_{23})\,.$$

Since α does not contain any dt's, this shows that the only surviving terms in α are the ones of the form $f(d\theta_i + d\theta_j)$. Inspection of α shows that that these terms are of the form $1/2(\eta^\theta_{ij} + f^\theta_{ij})(d\theta_i + d\theta_j)$. Hence the integral we have to evaluate is

$$np^2\int(\eta^\theta_{12} + f^\theta_{12} + \text{cycl.})\varphi^*(\eta^\theta_{12} + f^\theta_{12} + \text{cycl.})\delta_{12}\delta_{23}\varphi^*(\delta_{12}\delta_{23})v$$
$$= 3np^2\int(\eta^\theta_{12} + f^\theta_{12})\varphi^*(\eta^\theta_{12} + f^\theta_{12} + \text{cycl.})\delta_{12}\delta_{23}\varphi^*(\delta_{12}\delta_{23})v$$

where the equality follows from the cyclic invariance of α. Now the dependence of the integrand on the t variables in the pullback is immediately cancelled by the delta functions (here it is very helpful that the integrand is only a function of the differences t_{ij}!). To wit,

$$\int (\eta_{12}^{\theta} + f_{12}^{\theta})\varphi^*(\eta_{12}^{\theta} + f_{12}^{\theta} + \text{cycl.})\delta_{12}\delta_{23}\varphi^*(\delta_{12}\delta_{23})v$$

$$= \frac{1}{p^2}\sum_{k,l=0}^{p-1}\int (\eta(\theta_{12}) + f(\theta_{12}))(\eta(nt_{12} + q\theta_{12}) + f(nt_{12} + q\theta_{12}) + \text{cycl.})\delta(t_{12})\delta(t_{23})$$

$$\delta\left(\frac{m}{p}t_{12} + \theta_{12} + \frac{k}{p}\right)\delta\left(\frac{m}{p}t_{23} + \theta_{23} + \frac{l}{p}\right)v$$

$$= \frac{1}{p^2}\sum_{k,l=0}^{p-1}\int (\eta(\theta_{12}) + f(\theta_{12}))(\eta(q\theta_{12}) + f(q\theta_{12}) + \text{cycl.})\delta(\theta_{12} + k/p)\delta(\theta_{23} + l/p)d\theta_1 d\theta_2 d\theta_3$$

$$= \frac{1}{p^2}\sum_{k,l=0}^{p-1}\left(\eta\left(\frac{k}{p}\right) + f\left(\frac{k}{p}\right)\right)\left(\eta\left(\frac{qk}{p}\right) + f\left(\frac{qk}{p}\right) + \eta\left(\frac{ql}{p}\right) + f\left(\frac{ql}{p}\right) + \eta\left(\frac{q(k+l)}{p}\right) + f\left(\frac{q(k+l)}{p}\right)\right)$$

We can now perform the sum over l. Since both η and f are periodic and odd about half-periods, the terms depending on l sum to zero and we are left with

$$\frac{1}{p}\sum_{k=0}^{p-1}\left(\eta\left(\frac{k}{p}\right) + f\left(\frac{k}{p}\right)\right)\left(\eta\left(\frac{qk}{p}\right) + f\left(\frac{qk}{p}\right)\right)$$

so that in total we get

$$\psi_{\Gamma_{1,3}^{\mathfrak{b},\mathbb{B}}} * \psi_{\Gamma_{1,3}^{\mathfrak{b},\mathbb{A}}} = \frac{1}{2}\psi_{ijk}\varphi^{ijk} np\sum_{k=0}^{p-1}\left(\eta\left(\frac{k}{p}\right) + f\left(\frac{k}{p}\right)\right)\left(\eta\left(\frac{qk}{p}\right) + f\left(\frac{qk}{p}\right)\right). \tag{B.5}$$

Appendix C

Theta Functions and Propagators on the torus

C.1 Theta functions

In this Section we set up notation for theta functions and collect some results that are needed in the sequel. The main references are [**Mumford2007. Whittaker2009**], but we will deviate at times. Denote $\mathbb{H} \subset \mathbb{C}$ the upper half plane. We will use the following definition of the four Jacobi theta functions $\vartheta_i \colon \mathbb{C} \times \mathbb{H} \to \mathbb{C}$:

$$\vartheta_3(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z} = \vartheta(z,\tau) \tag{C.1}$$

$$\vartheta_0(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2 \tau + 2\pi i n z} = \vartheta\left(z + \frac{1}{2}, \tau\right) \tag{C.2}$$

$$\vartheta_1(z,\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i \left(n+\frac{1}{2}\right)^2 \tau + 2\pi i \left(n+\frac{1}{2}\right) z} = -e^{\pi i \tau/4 + \pi i \left(z+\frac{1}{2}\right)} \vartheta\left(z + \frac{\tau+1}{2}, \tau\right) \tag{C.3}$$

$$\vartheta_2(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i \left(n+\frac{1}{2}\right)^2 \tau + 2\pi i \left(n+\frac{1}{2}\right) z} = e^{\pi i \tau/4 + \pi i z} \vartheta\left(z + \frac{\tau}{2}, \tau\right) \tag{C.4}$$

$$\tag{C.5}$$

Here $\vartheta(z,\tau)$ is the theta function as defined by Mumford. More generally for $a, b \in \frac{1}{l}\mathbb{Z}$ one can define

$$\vartheta_{a,b}(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (a+n)^2 \tau + 2\pi i (n+a)(z+b)} \tag{C.6}$$

and recover

$$\vartheta(z,\tau) = \vartheta_{0,0}(z,\tau),$$
$$\vartheta_0(z,\tau) = \vartheta_{0,\frac{1}{2}}(z,\tau),$$
$$\vartheta_1(z,\tau) = -\vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau),$$
$$\vartheta_2(z,\tau) = \vartheta_{\frac{1}{2},0}(z,\tau)$$

Theta functions satisfy a wide range of interesting identities. We will collect some of them for future reference.

Proposition C.1.1 (Periodicity)**.** *The theta function is periodic in both variables:*

$$\vartheta(z+1,\tau) = \vartheta(z,\tau)$$
$$\vartheta(z,\tau+2) = \vartheta(z,\tau)$$

Remark C.1.2. The same holds for ϑ_0. ϑ_1 and ϑ_2 however are not periodic with respect to these periods, because of the exponential prefactor in their definition. In fact, one has

$$\vartheta_1(z+1,\tau) = -\vartheta_1(z,\tau)$$
$$\vartheta_1(z,\tau+2) = i\vartheta_1(z,\tau)$$

and the same holds for ϑ_2. If one writes the Theta functions in terms of the nome $q = e^{i\pi\tau}$, one has to interpret the multi-valued function q^λ as $e^{i\pi\tau\lambda}$. Otherwise ϑ_1 and ϑ_2 have branch cuts from 0 to -1 inside the unit q disk.

Proposition C.1.3 (Quasi-Periodicity)**.** *The theta function is quasiperiodic with respect to $z \mapsto z+\tau$, namely,*

$$\vartheta(z+\tau,\tau) = e^{-\pi i\tau-2\pi iz}\vartheta(z,\tau). \tag{C.7}$$

This implies the following quasiperiodicity for the other theta functions:

$$\vartheta_0(z+\tau,\tau) = -e^{-\pi i\tau-2\pi iz}\vartheta_0(z,\tau)$$
$$\vartheta_1(z+\tau,\tau) = -e^{-\pi i\tau-2\pi iz}\vartheta_1(z,\tau)$$
$$\vartheta_2(z+\tau,\tau) = e^{-\pi i\tau-2\pi iz}\vartheta(z,\tau)$$

Proof. We have

$$\vartheta(z+\tau,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau + 2\pi i n(z+\tau)} = \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau + 2\pi i n\tau + 2\pi i n z}$$
$$= \sum_{n=-\infty}^{\infty} e^{\pi i (n+1)^2\tau - \pi i\tau + 2\pi i (n+1)z - 2\pi i z} = e^{-\pi i\tau - 2\pi i z}\vartheta(z,\tau).$$

Then

$$\vartheta_0(z+\tau,\tau) = \vartheta(z+\tau+1/2,\tau) = e^{-\pi i\tau - 2\pi i(z+1/2)}\vartheta(z+1/2,\tau) = -e^{-\pi i\tau - 2\pi i z}\vartheta_0(z,\tau),$$

and

$$\vartheta_1(z+\tau,\tau) = e^{\pi i\tau/4 + \pi i\left(z+\tau+\frac{1}{2}\right)}\vartheta\left(z+\frac{\tau+1}{2}+\tau,\tau\right)$$
$$= e^{\pi i\tau}e^{-\pi i\tau - 2\pi i(z+\tau/2+1/2)}\vartheta_1(z,\tau) = -e^{-\pi i\tau - 2\pi i z}\vartheta_1(z,\tau),$$

and similarly for ϑ_2. ☐

Without proof we quote [WW09] a product formula for ϑ_1:

Proposition C.1.4. *Denoting $q = e^{i\pi\tau}$ we have*

$$\vartheta_1(z,\tau) = 2Gq^{1/4}\sin\pi z\prod_{n=1}^{\infty}(1-2q^{2n}\cos(2\pi z)+q^{4n}) \tag{C.8}$$

where

$$G = \prod_{n=1}^{\infty}(1-q^{2n}).$$

C.1.1 Zeroes

To find the zeroes of the theta function we can look at their logarithmic derivatives, but they will also be interesting on their own. They have the following property:

Proposition C.1.5. *Denote $L_i^\tau(z) = \vartheta_i'(z,\tau)/\vartheta(z,\tau)$. Then we have*

$$L_i^\tau(z+1) = L_i^\tau(z)$$
$$L_i^\tau(z+\tau) = L_i^\tau(z) - 2\pi i$$

Proof. This follows from propositions C.1.1 and C.1.3 (ϑ_1 and ϑ_2 are not periodic with respect to $z \mapsto z+1$ but change sign, this however does not influence the logarithmic derivative). ☐

By integrating the logarithmic derivative along the boundary of a fundamental domain of the lattice Λ, it follows that the ϑ_i have exactly one simple zero inside that domain. However, we do not know exactly where they are. For this we need another property of ϑ_1, namely $\vartheta_1(-z) = -\vartheta_1(z)$, i.e. ϑ_1 is odd. To see this, replace $z \mapsto -z$ and put $m = -n-1$ in the defining sum:

$$\begin{aligned}
\vartheta_1(-z,\tau) &= -i \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i \left(n+\frac{1}{2}\right)^2 \tau + 2\pi i \left(n+\frac{1}{2}\right)(-z)} \\
&= -i \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i \left(-n-\frac{1}{2}\right)^2 \tau + 2\pi i \left(-n-\frac{1}{2}\right) z} \\
&= -i \sum_{m=-\infty}^{\infty} (-1)^{-m-1} e^{\pi i \left(m+\frac{1}{2}\right)^2 \tau + 2\pi i \left(m+\frac{1}{2}\right) z} = -\vartheta_1(z,\tau).
\end{aligned}$$

(Using similar tricks one can show that $\vartheta_j, j \neq 1$, are even.) Oddness of ϑ_1 requires that it vanishes at 0. By quasi-periodicity, this implies that it vanishes at all elements of the lattice Λ, so we have proven the following proposition:

Proposition C.1.6 (Zeroes)**.** *The zeroes of $\vartheta_1(z,\tau)$ as a function of z are all simple and given by the integral lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$.*

From this one can also deduce the zeroes of the other ϑ_j.

There is another consequence of this which is important for us. Namely, one has the following formulas for the ratios of derivatives of theta functions to themselves, which again we quote without proof from [WW09]:

Proposition C.1.7. *Let $z = x + \tau y$ with $|y| < 1$. Then*

$$L_1^\tau(z) = \frac{\vartheta_1'(z,\tau)}{\vartheta_1(z,\tau)} = \pi\cot(\pi z) + 4\pi \sum_{n=1}^{\infty} \frac{e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}} \sin 2\pi n z \tag{C.9}$$

$$L_2^\tau(z) = \frac{\vartheta_2'(z,\tau)}{\vartheta_2(z,\tau)} = -\pi\tan(\pi z) + 4\pi \sum_{n=1}^{\infty} (-1)^n \frac{e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}} \sin 2\pi n z \tag{C.10}$$

$$L_3^\tau(z) = \frac{\vartheta_3'(z,\tau)}{\vartheta_3(z,\tau)} = 4\pi \sum_{n=1}^{\infty} (-1)^n \frac{e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}} \sin 2\pi n z \tag{C.11}$$

$$L_4^\tau(z) = \frac{\vartheta_4'(z,\tau)}{\vartheta_4(z,\tau)} = 4\pi \sum_{n=1}^{\infty} \frac{e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}} \sin 2\pi n z \tag{C.12}$$

C.1.2 Change of argument

Later it will be important to now exactly the change of argument as we move around the lattice. First, notice that the quasiperiodicity of the theta functions immediately implies the following lemma.

Lemma C.1.8. *Let $z = x + \tau y$ with $x, y \notin \mathbb{Z}$. Then*

$$\begin{aligned}\mathrm{d}\arg\vartheta_1(z+1,\tau) &= \mathrm{d}\arg\vartheta_1(z,\tau)\\ \mathrm{d}\arg\vartheta_1(z+\tau,\tau) &= \mathrm{d}\arg\vartheta_1(z,\tau) - 2\pi\mathrm{d}\,\mathrm{Re}\,z\end{aligned}$$

For the explicit change of argument along some path we have the following formulae:

Lemma C.1.9. *Let $\tau \in \mathbb{H}$, $x_0, y_0 \in \mathbb{R}$ and $z_0 = x_0 + \tau y_0$. Consider the paths $\gamma_1 \colon [0,1] \to \mathbb{C}, t \mapsto z_0 + t$ and $\gamma_2 \colon [0,1] \to \mathbb{C}, \mapsto z_0 + \tau t$. Then we have that if $y_0 \notin \mathbb{Z}$*

$$\int_{\gamma_1} \mathrm{d}\arg\vartheta_1(z,\tau) = -\pi - 2\pi\lfloor y_0 \rfloor \tag{C.13}$$

On the other hand, if $x_0 \notin \mathbb{Z}$, we have

$$\int_{\gamma_2} \mathrm{d}\arg\vartheta_1(z,\tau) = \pi(1 - \mathrm{Re}\,\tau - 2(x_0 - \lfloor x_0 \rfloor) - 2\,\mathrm{Re}\,\tau y_0). \tag{C.14}$$

Here $\lfloor y_0 \rfloor$ is the largest integer less or equal than y_0.

Proof. First, we use that

$$\mathrm{d}\arg\vartheta_1(x+\tau y,\tau) = \mathrm{Im}\,L_1^\tau(x+\tau y)dx + \mathrm{Im}(\tau L^\tau(x+\tau y))dy.$$

So

$$\int_{\gamma_1} \mathrm{d}\arg\vartheta_1(z,\tau) = \int_0^1 \mathrm{Im}\,L^\tau(x + x_0 + \tau y_0)dx.$$

Now we reduce to the case where $0 < x_0, y_0 < 1$. For this, notice that by Proposition C.1.5 we have

$$\mathrm{Im}\,L_1^\tau(x + x_0 + \tau y_0) = \mathrm{Im}\,L_1^\tau\left(x + (x_0 - \lfloor x_0 \rfloor) + \tau(y_0 - \lfloor y_0 \rfloor)\right) - 2\pi\lfloor y_0 \rfloor.$$

So, to prove (C.13) it is enough to prove that for x_0, y_0 in $(0,1)^2$ we have $\int_{\gamma_1} \mathrm{d}\arg\vartheta_1(z,\tau) = -\pi$. To see this, use equation (C.9) with $z(x) = x + x_0 + \tau y_0$:

$$\begin{aligned}\int_0^1 \mathrm{Im}\,L_1^\tau(z(x))dx &= \mathrm{Im}\left[\pi\int_0^1 \cot\pi(z(x)) + 4\sum\frac{q^{2n}}{1-q^{2n}}\sin(2\pi n z(x))dx\right]\\ &= \pi\,\mathrm{Im}\left[\int_0^1 \cot\pi(x + x_0 + \tau y_0)\right]\\ &= -\pi\end{aligned}$$

where the integrals of the sine vanish, and we have used that for any $y > 0$ we have

$$\int_0^1 \cot\pi(x+iy)dx = -i$$

(this follows from observing that the integral is independent of y and $\lim_{y\to\infty}\cot(x+iy) = -i$). Similarly, we have

$$\int_{\gamma_2} \mathrm{d}\arg\vartheta_1(z,\tau) = \int_0^1 \operatorname{Im}(\tau L_1^\tau(x_0+\tau(y+y_0))dy$$
$$= \int_0^1 \operatorname{Im}\tau(L_1^\tau(x_0-\lfloor x_0\rfloor+\tau(y+y_0))dy$$

So, we have to prove that for $0 < x_0 < 1$

$$\int_0^1 \operatorname{Im}\tau L_1^\tau(x_0+\tau(y_0+y))dy = \pi(1-\operatorname{Re}\tau-2x_0-2\operatorname{Re}\tau y_0).$$

By lemma C.1.8, we know that the difference between the two sides (divided by π) is an even integer which varies continuously in $(x_0,y_0) \in (0,1)\times\mathbb{R}$ (since both sides vary continuously in x_0, y_0 in that region). Hence, we can compute the constant at any point in that region, for example in $(1/2,-1/2)$. This reduces the problem to showing that

$$\int_{-1/2}^{1/2} \operatorname{Im}\tau L_1^\tau(1/2+\tau y)dy = 0.$$

Since $|y| < 1$, we can again apply formula (C.9). Now notice that $\cot(\pi/2+\tau y) = \tan\tau y$ is an odd function of y, as is $\sin 2\pi n(1/2+\tau y) = (-1)^n \sin 2\pi n\tau y$. Hence the integrals vanish and the claim is proven. □

We are interested in changes of argument between zeros of ϑ_1. We have the following result:

Lemma C.1.10. *We have*

$$\int_0^1 \mathrm{d}\arg\vartheta_1(z,\tau) = 0 \tag{C.15}$$
$$\int_0^1 \mathrm{d}\arg\vartheta_1(z\tau,\tau) = -\pi\operatorname{Re}\tau. \tag{C.16}$$
$$\tag{C.17}$$

Note that from this one can easily compute similar integrals between any two zeros.

Proof. The first claim follows from formula (C.9) and the fact that $\cot x$ is odd about $1/2$. To see the second, close the contour from $\varepsilon\tau$ to $(1-\varepsilon\tau)$ with quartercircles around 0 and τ of radius ε and a straight vertical line. The total integral is zero. By the previous lemma, the straight line contributes $\pi\operatorname{Re}\tau - \pi + 2\pi\varepsilon$. In the limit as $\varepsilon \to 0$, the quartercircles contribute with π, since the argument of the theta function behaves like

$$\mathrm{d}\arg\vartheta_1(z(\tau),\tau) \simeq \mathrm{d}\arg z(\tau)$$

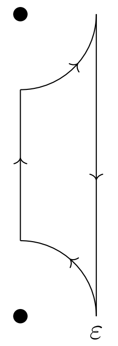


Figure C.1: Closing the contour

around 0 and by Lemma C.1.8 we can "transport" the quartercircle around 1 to 0 to form a half-circle there.

□

C.1.3 Transformation properties

As we have seen, the functions ϑ_i are quasi-periodic in the z variable with respect to both $z \mapsto z+1$ and $z \mapsto z+\tau$. They also satisfy interesting identities under modular transformations of τ. Consider the two generators of the modular group, $T\colon \tau \mapsto \tau+1$ and $S\colon \tau \mapsto -1/\tau$. The first of them e.g. exchanges ϑ_3 and ϑ_0:

$$\vartheta(z,\tau+1) = \vartheta(z+1/2,\tau) = \vartheta_0(z,\tau)$$

(this follows $n^2-n = n(n-1) \in 2\mathbb{Z}$). This already shows that the τ-transformations are slightly more intricate. However, the function ϑ_1 is the one we are concerned with most in this note, and it is carried to itself under both transformations:

Proposition C.1.11 (Modular transformations on ϑ_1)**.** *The function ϑ_1 satisfies*

$$\vartheta_1(z,\tau+1) = e^{\pi i/4}\vartheta_1(z,\tau) \tag{C.18}$$

$$\vartheta_1\left(\frac{z}{\tau},\frac{-1}{\tau}\right) = -i(-i\tau)^{1/2}e^{i\pi z^2/\tau}\vartheta_1(z,\tau). \tag{C.19}$$

where we choose the principal branch of the square root (notice that $\operatorname{Re} -i\tau > 0$*).*

Proof. For the first equality we can write

$$\begin{aligned}\vartheta_1(z,\tau+1) &= e^{\pi i(\tau+1)/4+\pi i(z+1/2)}\vartheta\left(z+\frac{(\tau+1)+1}{2},\tau+1\right)\\ &= e^{\pi i/4}e^{\pi i\tau/4+\pi i(z+1/2)}\vartheta\left(z+\frac{\tau}{2},\tau+1\right)\\ &= e^{\pi i/4}e^{\pi i\tau/4+\pi i(z+1/2)}\vartheta\left(z+\frac{\tau}{2}+\frac{1}{2},\tau\right) = e^{\pi i/4}\vartheta_1(z,\tau).\end{aligned}$$

For the proof of the second equality we refer to the literature, e.g. [WW09]. □

In particular, we can compute the transformation property of the logarithmic derivative:

Corollary C.1.12 (Modular transformation of L_1^τ)**.** *We have*

$$L_1^{\tau+1}(z) = L_1^\tau(z), \tag{C.20}$$

$$L_1^{-1/\tau}\left(\frac{z}{\tau}\right) = 2\pi i z + \tau L^\tau(z). \tag{C.21}$$

Proof. Equation (C.20) is a direct consequence of equation (C.18). For the second equation, deriving (C.19) we get

$$\begin{aligned}
\frac{1}{\tau}\vartheta_1'\left(\frac{z}{\tau},\frac{-1}{\tau}\right) &= \frac{d}{dz}\vartheta_1\left(\frac{z}{\tau},\frac{-1}{\tau}\right)\\
&= \frac{d}{dz}\left(-i(-i\tau)^{1/2}e^{i\pi z^2/\tau}\vartheta_1(z,\tau)\right)\\
&= \frac{2i\pi z}{\tau}\vartheta_1\left(\frac{z}{\tau},\frac{-1}{\tau}\right) - i(-i\tau)^{1/2}e^{i\pi z^2/\tau}\vartheta_1'(z,\tau)
\end{aligned}$$

so that

$$L_1^{-1/\tau}\left(\frac{z}{\tau}\right) = \frac{\vartheta_1'\left(\frac{z}{\tau},\frac{-1}{\tau}\right)}{\vartheta_1\left(\frac{z}{\tau},\frac{-1}{\tau}\right)} = 2\pi i z + \tau L^\tau(z). \tag{C.22}$$

□

C.2 Integral kernels of chain contractions on the torus

In this section we construct some particular integral kernels for chain contraction on the circle that depends on a the choice of modulus $\tau \in \mathbb{H}$.

C.2.1 Definition

Let $\tau \in \mathbb{H}$ and denote $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. We define a 1-form η^τ on $\mathbb{C} - \Lambda$ by

$$\eta^\tau(z) = \frac{1}{2\pi}\mathrm{d}\arg\vartheta_1(z,\tau) + \frac{1}{4i\operatorname{Im}\tau}(z-\bar z)\mathrm{d}(z+\bar z) \tag{C.23}$$

Proposition C.2.1. *η^τ is Λ-periodic, i.e. $\eta^\tau(z+1) = \eta^\tau(z+\tau) = \eta^\tau(z)$.*

Proof. We have

$$\mathrm{d}\arg\vartheta_1(z,\tau) = \operatorname{Im}\left(\mathrm{d}\log\vartheta_1(z,\tau)\right) = \frac{1}{2i}\left(L_1^\tau(z)dz - \overline{L_1^\tau(z)}d\bar z\right).$$

It follows that $\mathrm{d}\arg\vartheta_1$ is invariant under $z \mapsto z+1$, implying periodicity of $\eta^\tau(z)$ with respect to $z \mapsto z+1$. On the other hand, Proposition C.1.5 implies that

$$\mathrm{d}\arg\vartheta_1(z+\tau,\tau) = \mathrm{d}\arg\vartheta_1(z,\tau) - \pi(dz + d\bar z).$$

This is cancelled by the shift of $\frac{1}{4i\operatorname{Im}\tau}(z-\bar{z})\mathrm{d}(z+\bar{z})$. □

We denote $T = \mathbb{R}^2/\mathbb{Z}^2$ the real torus with coordinates x, y. we introduce the complex coordinate $z = x + \tau y$ on T, thus identifying $T \cong \mathbb{C}/\Lambda$.

Definition C.2.1. The 1-form $\eta^\tau(z) \in \Omega^1(T - \{(0,0\})$ is called the *τ-propagator* on T.

We can express the τ-propagator in real coordinates as

$$\eta^\tau(x,y) = \frac{1}{2\pi}\mathrm{d}\arg\vartheta_1(x+\tau y,\tau) + y(dx + \operatorname{Re}\tau dy) \tag{C.24}$$

$$= \frac{1}{2\pi}\left(\operatorname{Im} L_1^\tau(x+\tau y)(dx+\operatorname{Re}\tau dy) + \operatorname{Re} L_1^\tau(x+\tau y)\operatorname{Im}\tau dy\right) + y(dx+\operatorname{Re}\tau dy) \tag{C.25}$$

$$= \frac{1}{2\pi}(\operatorname{Im} L_1^\tau(x+\tau y)dx + \operatorname{Im}(\tau L_1^\tau(x+\tau y))dy) + y(dx+\operatorname{Re}\tau dy) \tag{C.26}$$

In the following, it is important to remember that the complex parameter z depends on the modular parameter τ, and sometimes we will write $z = z(\tau)$ to emphasize this.

C.2.2 Chain contraction

In this subsection we prove that the τ-propagators deserve their name, namely, they are propagators for abelian BF theory. This means that every τ-propagator is an integral kernel of a chain contraction, i.e. a linear map

$$K\colon \Omega^\bullet(T) \to \Omega^{\bullet-1}(T)$$

satisfying $dK + Kd = \mathrm{id} - \iota\circ P$, where

$$\iota\colon H^\bullet(T) \to \Omega^\bullet(T)$$

is a specific embedding of the de Rham cohomology of the torus into differential forms, and $P\colon \Omega^\bullet(M) \to H^\bullet(M)$ is a specific projection to cohomology. See also section 3.1 for a discussion of propagators in abelian BF theory. Both the embedding and the projection depend on the parameter τ. In particular, we claim that the τ-propagator is the Hodge propagator $K = d_g^* \circ (\Delta_g + P_{harm})^{-1}$ associated to the metric

$$dz\cdot d\bar{z} = dx^2 + 2\operatorname{Re}\tau dx\cdot dy + |\tau|^2 dy^2 \tag{C.27}$$

(here $\cdot$ denotes the symmetric product, and powers are powers in the symmetric product). In this metric, we have

$$\det g = \operatorname{Im}\tau, v_g = \frac{1}{2i} d\bar{z} \wedge dz$$
$$*(dx + \operatorname{Re}\tau dy) = \operatorname{Im}\tau dy$$
$$*(\operatorname{Im}\tau dy) = -(dx + \operatorname{Re}\tau dy)$$

or $*dz = -idz, *d\bar{z} = id\bar{z}$. From this facts, the codifferential can be expressed as $\mathrm{d}_z^* = -2(\partial_z \iota_{\partial_{\bar{z}}} + \partial_{\bar{z}} \iota_{\partial_z})$.

Proposition C.2.2. *Let $g = dz \cdot d\bar{z} = dx^2 + 2\operatorname{Re}\tau dx \cdot dy + |\tau|^2 dy^2$ and $K^\tau = d_g^* \circ (\Delta_g + P_{harm})^{-1}$. Then for all $\omega \in \Omega^\bullet(M)$ and $x \in T$, we have*

$$K^\tau \omega(z) = \int_{w \in T} \eta^\tau(z - w)\omega(w).$$

Proof. The Green's function for this metric is (see [Oog15], [BL17])

$$g^\tau(z, w) = -\frac{1}{4\pi} \log \left| \frac{\vartheta_1(z - w, \tau)}{\eta(\tau)} \right|^2 + \frac{(\operatorname{Im}(z - w))^2}{2\operatorname{Im}\tau}. \tag{C.28}$$

It satisfies $g^\tau(z, w) = g^\tau(z - w, 0) =: g^\tau(z - w)$ and

$$\Delta_g g^\tau(z, w) = \delta(z - w) - \frac{1}{\operatorname{Im}\tau}. \tag{C.29}$$

Its derivatives are, rewriting $\log|z|^2 = \log z + \log\bar{z}$ and $(\operatorname{Im} z)^2 = -1/4(z - \bar{z})^2$

$$\partial_z g^\tau(z) = -\frac{1}{4\pi}\frac{\vartheta'(z, \tau)}{\vartheta(z, \tau)} - \frac{z - \bar{z}}{4\operatorname{Im}\tau} \tag{C.30}$$

$$\partial_{\bar{z}} g^\tau(z) = -\frac{1}{4\pi}\overline{\left(\frac{\vartheta'(z, \tau)}{\vartheta(z, \tau)}\right)} + \frac{z - \bar{z}}{4\operatorname{Im}\tau} \tag{C.31}$$

In this case, the Green's form is simply

$$\alpha = \frac{1}{2i} g^\tau(z, w)(d\bar{z} - d\bar{w})(dz - dw) = \alpha(z - w, 0) =: \alpha(z).$$

The metric propagator is therefore, using (C.30) and (C.31)

$$\begin{aligned}
\eta(z, 0) = \mathrm{d}_z^*\alpha(z) &= -\frac{1}{\mathrm{i}}\partial_z g^\tau(z, 0)dz + \frac{1}{i}\partial_{\bar{z}} g^\tau(z)d\bar{z} \\
&= \left(\frac{1}{4\pi\mathrm{i}}\frac{\vartheta'(z, \tau)}{\vartheta(z, \tau)} + \frac{z - \bar{z}}{4\mathrm{i}\operatorname{Im}\tau}\right)dz + \left(-\frac{1}{4\pi\mathrm{i}}\overline{\left(\frac{\vartheta'(z, \tau)}{\vartheta(z, \tau)}\right)} + \frac{\bar{z}}{4\mathrm{i}\operatorname{Im}\tau}\right)d\bar{z} \\
&= \frac{1}{2\pi}\mathrm{d}\arg\vartheta_1(z, \tau) + \frac{1}{4\mathrm{i}\operatorname{Im}\tau}((z - \bar{z}))(dz + d\bar{z}) \\
&= \eta^\tau(z).
\end{aligned}$$

□

C.2.3 $SL(2,\mathbb{Z})$ action

For a matrix $\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)$ we define a diffeomorphism of the torus, also denoted φ, by

$$(x,y) \mapsto \varphi.(x,y) = (ax+by, cx+dy). \tag{C.32}$$

This provides an identification of $SL(2,\mathbb{Z})$ with the mapping class group of T. On the other hand, $SL(2,\mathbb{Z})$ acts on the upper halfplane via the standard modular action

$$\varphi.\tau = \frac{a\tau+b}{c\tau+d}. \tag{C.33}$$

$SL(2,\mathbb{Z})$ acts on τ-propagators in two ways: Once by pullback via the corresponding diffeomorphism of the torus, once by acting directly on τ. We will now describe how these actions are related.

For a matrix φ, we denote the transpose by φ^T, and the anti-transpose (reflection along the anti-diagonal) by ${}^T\varphi$, i.e.

$${}^T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}. \tag{C.34}$$

Equipped with this notation we can state the following proposition:

Proposition C.2.3. *For any $\tau \in \mathbb{H}$ and any $\varphi \in SL(2,\mathbb{Z})$ we have*

$$\varphi^* \eta^\tau = \eta^{{}^T\varphi.\tau}. \tag{C.35}$$

Proof. Like the transpose, the anti-transpose is an antihomorphism of the matrix algebra, i.e. it satisfies

$${}^T(\varphi_1\varphi_2) = {}^T\varphi_2 {}^T\varphi_1. \tag{C.36}$$

Therefore, both sides of (C.35) define right actions of $SL(2,\mathbb{Z})$ on 1-forms and it is enough to check equation(C.35) on generators of $SL(2,\mathbb{Z})$. First, notice that $-I$ acts trivially on both sides, so the action factors through $PSL(2,Z)$. Next, let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{C.37}$$

be the standard generators of $PSL(2,\mathbb{Z})$. Notice that ${}^TS = S$ and ${}^TT = T$. For this proof it is easier to work with real coordinates. We have $T.(x,y) = (x+y,y)$ and $S.(x,y) = (-y,x)$. On the other hand, we have $T.\tau = \tau+1, S.\tau = -1/\tau$. So, we have to prove that

$$\eta^\tau(x+y,y) = \eta^{\tau+1}(x,y) \tag{C.38}$$

and

$$\eta^\tau(-y,x) = \eta^{\tau'}(x,y) \tag{C.39}$$

The proof of equations (C.38) and (C.39) is by direct computation: Let $\tau' = \tau + 1$, then we have

$$\begin{aligned}
\eta^\tau(x+y,y) &= \frac{1}{2\pi}(\operatorname{Im} L_1^\tau(x+y+\tau y)(d(x+y) + \operatorname{Re}\tau dy) + \operatorname{Re} L_1^\tau(x+y+\tau y)\operatorname{Im}\tau dy \\
&+ y(d(x+y) + \operatorname{Re}\tau dy) \\
&= \frac{1}{2\pi}(\operatorname{Im} L_1^\tau(x+(\tau+1)y)(dx + \operatorname{Re}(\tau+1dy) + \operatorname{Re} L_1^\tau(x+(\tau+1)y)\operatorname{Im}(\tau+1)dy \\
&+ y(dx) + \operatorname{Re}(\tau+1)dy) \\
&= \frac{1}{2\pi}(\operatorname{Im} L_1^{\tau'}(x+(\tau')y)(dx + \operatorname{Re}(\tau' dy) + \operatorname{Re} L_1^{\tau'}(x+(\tau')y)\operatorname{Im}(\tau')dy \\
&+ y(dx) + \operatorname{Re}(\tau')dy) \\
&= \eta^{\tau'}(x,y)
\end{aligned}$$

where we have used that $L^\tau(z) = L^{\tau+1}(z)$ (equation (C.20)). For the the next equation denote $\tau' = -1/\tau$. Notice that by equation (C.21) we have

$$L_1^\tau(-y+\tau x) = L_1^\tau(\tau(x - y/\tau)) = \frac{1}{\tau}L_1^{\tau'}(x+\tau' y) - 2\pi i(x+\tau' y).$$

So

$$\begin{aligned}
\operatorname{Im} L_1^\tau(-y+\tau x) &= \operatorname{Im}\left(\frac{1}{\tau}L_1^{\tau'}(x+\tau' y) - 2\pi i(x+\tau' y)\right) \\
&= -\operatorname{Im}\left(\tau'\right)\operatorname{Re} L_1^{\tau'}(x+\tau' y) - \operatorname{Re}\left(\tau'\right)\operatorname{Im} L_1^{\tau'}(x+\tau' y) - 2\pi(x + \operatorname{Re}\tau' y) \\
&= -\operatorname{Im}(\tau' L_1^{\tau'}(x+\tau' y) - 2\pi(x+\operatorname{Re}\tau' y)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Im} L_1^\tau(-y+\tau x) &= \operatorname{Im}\left(L_1^{\tau'}(x+\tau' y) - \tau 2\pi i(x+\tau' y)\right) \\
&= \operatorname{Im}(L_1^{\tau'}(x+\tau' y)) - 2\pi\operatorname{Re}\tau x + 2\pi y,
\end{aligned}$$

since $\tau\tau' = -1$. Plugging this into equation (C.26) we obtain

$$\begin{aligned}
\eta^\tau(-y,x) &= \frac{1}{2\pi}\left(-\operatorname{Im} L_1^\tau(-y+\tau x)dy + \operatorname{Im}(\tau L_1^\tau(-y+\tau x))dx\right) + x(-dy + \operatorname{Re}\tau dx) \\
&= \frac{1}{2\pi}(\operatorname{Im}(\tau' L_1^{\tau'}(x+\tau' y))dy + \operatorname{Im} L_1^{\tau'}(x+\tau' y)dx) \\
&+ (x + \operatorname{Re}\tau' y)dy - \operatorname{Re}\tau x dx + y dx + x(-dy + \operatorname{Re}\tau dx) \\
&= \frac{1}{2\pi}(\operatorname{Im}(\tau' L_1^{\tau'}(x+\tau' y))dy + \operatorname{Im} L_1^{\tau'}(x+\tau' y)dx) + y(dx + \operatorname{Re}\tau' dy) \\
&= \eta^{\tau'}(x,y).
\end{aligned}$$

Remark C.2.4. Notice that for the corresponding equality in complex coordinates we also need to transform z. I.e. if we denote $\tau' = \left({}^T\varphi\right).\tau$, we have

$$\varphi^*(\eta^\tau(z(\tau))) = \eta^{\tau'}(z(\tau')).$$

□

C.3 Computation of an Integral

Fix some $\tau \in \mathbb{H}$, and let

$$\varphi = \begin{pmatrix} m & p \\ n & q \end{pmatrix} \in SL(2,\mathbb{Z})$$

as above. In this section we want to compute the integral

$$I = \int_T \eta^\tau \varphi^* \eta^\tau. \tag{C.40}$$

Note that this integral is zero if $\varphi = \pm\mathrm{id}$, which is equivalent to $p = 0$, so we exclude this case from now. We first rewrite the integral as, letting again $\tau' = \left({}^T\varphi\right).\tau$

$$I = \int_T \eta^\tau \varphi^* \eta^\tau = \int_T \eta^\tau \eta^{\tau'}.$$

This integral can be computed explicitly.

Proposition C.3.1. *Let $\tau, \tau' \in \mathbb{H}$. Then*

$$I(\tau,\tau') = \int_T \eta^\tau \eta^{\tau'} = \frac{1}{\pi}\left(\arg\eta(\tau') - \arg\eta(\tau)\right) + \frac{1}{2\pi}\arg(i(\bar\tau - \tau')). \tag{C.41}$$

Here $\arg\eta(\tau)$ is the imaginary part of the branch of $\log\eta(\tau)$ defined on the upper half plane (this branch exists because $\eta(\tau)$ never vanishes) normalized as $\arg\eta(ix) = 0$, for $x \in \mathbb{R}$.

Remark C.3.2. For this proposition to it is not necessary that $\tau' = \left({}^T\varphi\right).\tau$.

The proof of this proposition will be deferred to section C.3.1. For now, let us look at the case $\tau' = \left({}^T\varphi\right)\tau = \frac{q\tau+p}{n\tau+m}$.

Proposition C.3.3. *Let $\tau' = \frac{q\tau+p}{n\tau+m}$, with $n > 0$. Then*

$$\int_T \eta^\tau \eta^{\tau'} = \mathrm{sgn}(p)s(q,p) - \frac{m+q}{12p} + \frac{\mathrm{sgn}(q) + sgn(qp)}{4} + \frac{1}{2\pi}\arg(n|\tau|^2 + m\bar\tau - q\tau - p). \tag{C.42}$$

If $n = 0$, then

$$\int_T \eta^\tau \eta^{\tau'} = \frac{p}{12} - \frac{1}{2\pi}\arctan\frac{mp}{2\,\mathrm{Im}\,\tau}. \tag{C.43}$$

Here $s(b,c)$ is the *Dedekind sum*

$$s(b,c) = \sum_{k=1}^{b-1} \left(\left(\frac{b}{c}\right)\right)\left(\left(\frac{kb}{c}\right)\right) \tag{C.44}$$

and $((x))$ is the sawtooth function

$$((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$$

Proof. It is well known how the logarithm of the Dedekind eta function behaves under this transformation:

Proposition C.3.4 (Apostol). *Let $\tau' = \frac{q\tau+p}{n\tau+m}$, with $n > 0$. Then*

$$\log\eta(\tau') = \log\eta(\tau) + \pi i\left(\frac{q+m}{12n} + s(-m,n)\right) + \frac{1}{2}\log(-i(n\tau+m)). \tag{C.45}$$

On the other hand, if $n = 0$, we have $m = q = \pm 1$, $\tau' = \tau + mp$ and

$$\log\eta(\tau') = \log\eta(\tau) + \pi imp/12. \tag{C.46}$$

For $n < 0$ one can replace q, p, n, m by their negatives, which yields the same τ'. For $n = 0$, we obtain

$$I = \frac{mp}{12} + \frac{1}{2\pi}\arg(-imp + 2\operatorname{Im}\tau) = \frac{p}{12} - \frac{1}{2\pi}\arctan\frac{mp}{2\operatorname{Im}\tau} \tag{C.47}$$

If $n > 0$, we obtain

$$I = \frac{m+q}{12n} + s(-m,n) + \frac{1}{2\pi}\arg(-i(n\tau+m)) + \frac{1}{2\pi}\arg(i(\bar\tau - \tau')). \tag{C.48}$$

Since $\operatorname{Im} n\tau + m > 0$ and $\operatorname{Im}(\bar\tau - \tau') < 0$ we have

$$\begin{aligned} \arg(-i(n\tau+m)) + \arg(i(\bar\tau-\tau)) &= \arg((n\tau+m)(\bar\tau-\tau')) \\ &= \arg(n|\tau|^2 + m\bar\tau - q\tau - p). \end{aligned}$$

We can rewrite the Dedekind sum in terms of p and q by applying some properties of Dedekind sums (see e.g. [RG09]), namely that

$$s(b,c) = \pm s(b',c)$$

if $bb' \equiv \pm 1 (\operatorname{mod} c)$, and the *reciprocity law*

$$\operatorname{sgn}(b)s(b,c) + \operatorname{sgn}(c)s(c,b) = \frac{1}{12}\left(\frac{b}{c} + \frac{1}{bc} + \frac{c}{b}\right) - \frac{\operatorname{sgn}(bc)}{4} \tag{C.49}$$

if b and c are both nonzero. We can apply these two rules, and the fact that $mq \equiv 1 (\mathrm{mod}\, p)$ and $np \equiv -1 (\mathrm{mod}\, q)$, to rewrite (remember that $n > 0$)

$$\begin{aligned} s(-m,n) = -s(m,n) = -s(q,n) &= \operatorname{sgn}(q) s(n,q) - \frac{1}{12}\left(\frac{n}{q} + \frac{1}{nq} + \frac{q}{n}\right) + \frac{\operatorname{sgn}(q)}{4} \\ &= -\operatorname{sgn}(q) s(p,q) - \frac{1}{12}\left(\frac{n}{q} + \frac{1}{nq} + \frac{q}{n}\right) + \frac{\operatorname{sgn}(q)}{4} \\ &= \operatorname{sgn}(p) s(q,p) - \frac{1}{12}\left(\frac{q}{p} + \frac{1}{pq} + \frac{p}{q}\right) - \frac{1}{12}\left(\frac{n}{q} + \frac{1}{nq} + \frac{q}{n}\right) + \frac{\operatorname{sgn}(q) + \operatorname{sgn}(pq)}{4} \end{aligned}$$

We can rewrite $mq - np = 1$ as $n/q = m/p - 1/qp$ and $1/nq = m/n - p/q$, arriving at

$$s(-m,n) = \operatorname{sgn}(p) s(q,p) - \frac{1}{12}\left(\frac{q+m}{p} + \frac{m+q}{n}\right) + \frac{\operatorname{sgn}(q) + \operatorname{sgn}(pq)}{4}.$$

Plugging this into (C.48), we obtain (for $n > 0$)

$$I = \operatorname{sgn}(p) s(q,p) - \frac{m+q}{12p} + \frac{\operatorname{sgn}(q) + \operatorname{sgn}(pq)}{4} + \frac{1}{2\pi} \arg(n|\tau|^2 + m\bar{\tau} - q\tau - p) \tag{C.50}$$

□

C.3.1 Proof of Proposition C.3.1.

First, we expand

$$\begin{aligned} \int_T \eta^\tau \eta^{\tau'} &= \frac{1}{4\pi^2} \int_T \mathrm{d} \arg \vartheta_1(x + \tau y, \tau) \mathrm{d} \arg \vartheta_1(x + \tau' y, \tau') \\ &+ \frac{1}{2\pi} \int_T \mathrm{d} \arg \vartheta_1(x + \tau y, \tau) y(dx + \operatorname{Re} \tau' dy) \\ &- \frac{1}{2\pi} \int_T \mathrm{d} \arg \vartheta_1(x + \tau' y, \tau') y(dx + \operatorname{Re} \tau dy) \\ &+ \int_T y(dx + \operatorname{Re} \tau dy) y(dx + \operatorname{Re} \tau' dy) \\ &= I_1 + I_2 - I_3 + I_4. \end{aligned}$$

Now, we represent the torus as $0 \leq x, y \leq 1$. Then the last integral is immediately computed as

$$I_4 = (\operatorname{Re} \tau' - \operatorname{Re} \tau) \int_{[0,1]^2} y^2 dx dy = \frac{1}{3}(\operatorname{Re} \tau' - \operatorname{Re} \tau).$$

To compute I_2, expand

$$\begin{aligned} \mathrm{d} \arg \vartheta_1(x + \tau y, \tau) y(dx + \operatorname{Re} \tau' dy) &= (\operatorname{Im} L_1^\tau(x + \tau y) dx + \operatorname{Im}(\tau L_1^\tau(x + \tau y)) dy)\, y(dx + \operatorname{Re} \tau' dy) \\ &= y \left(\operatorname{Im} L_1^\tau(x + \tau y) \operatorname{Re} \tau' - \operatorname{Im}(\tau L_1^\tau(x + \tau y)) \right) dx dy. \end{aligned}$$

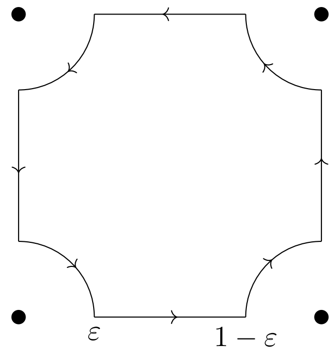


Figure C.2: The contour Γ_ε is made up of four straight segments and four quarter-circles of radius ε.

Now we can apply Lemma C.1.9 to evaluate the integral over x first, the result is independent of y so:

$$\int_{[0,1]^2} y\left(\operatorname{Im} L_1^\tau(x+\tau y)\operatorname{Re}\tau'\right)dxdy = -\frac{\pi\operatorname{Re}\tau'}{2}$$

$$\int_{[0,1]^2} y\left(\operatorname{Im}\tau L_1^\tau(x+\tau y)\right)dxdy = -\frac{\pi\operatorname{Re}\tau}{2}$$

$$I_2 = \frac{\pi}{2}(\operatorname{Re}\tau - \operatorname{Re}\tau')$$

Similarly $I_3 = \pi/2(\operatorname{Re}\tau' - \operatorname{Re}\tau)$. It remains to compute I_1. To do this we want to apply Stokes' theorem to rewrite the integral as a line integral. Since ϑ_1 has zeros on the corners of $[0,1]^2$ we have to choose a contour Γ_ε that makes quartercircles of radius ε around the corners and otherwise follows the boundary of the square. Then we have

$$\int_{[0,1]^2} \mathrm{d}\arg\vartheta_1(x+\tau y,\tau)\mathrm{d}\arg\vartheta_1(x+\tau' y,\tau') = \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\arg\vartheta_1(x+\tau y,\tau)\mathrm{d}\arg\vartheta_1(x+\tau' y,\tau').$$

First we consider the part Γ_ε^1 of Γ_ε that follows the boundary of the square. Parametrising, we obtain

$$\begin{aligned}
&\int_{\Gamma_\varepsilon^1}\arg\vartheta_1(x+\tau y,\tau)\mathrm{d}\arg\vartheta_1(x+\tau' y,\tau')\\
&=\int_\varepsilon^{1-\varepsilon}\arg\vartheta_1(t,\tau)\mathrm{d}\arg\vartheta_1(t,\tau') + \int_\varepsilon^{1-\varepsilon}\arg\vartheta_1(1+t\tau,\tau)\mathrm{d}\arg\vartheta_1(1+t\tau,\tau')\\
&+\int_{1-\varepsilon}^{\varepsilon}\arg\vartheta_1(t+\tau,\tau)\mathrm{d}\arg\vartheta_1(t+\tau',\tau') + \int_{1-\varepsilon}^{\varepsilon}\arg\vartheta_1(t\tau,\tau)\mathrm{d}\arg\vartheta_1(t\tau',\tau')
\end{aligned}$$

Now we can use quasi-periodicity of ϑ_1 to combine the integrals. Namely, we have Lemma C.1.9 which allows us to combine the second and the fourth integral into

$$\int_{\varepsilon}^{1-\varepsilon} -\pi \mathrm{d}\arg(t\tau', \tau') \overset{\varepsilon \to 0}{\longrightarrow} \pi^2 \operatorname{Re}\tau'$$

using also Lemma C.1.10. We can rewrite the third integral as

$$\begin{aligned}
&-\int_{\varepsilon}^{1-\varepsilon} [\arg\vartheta_1(t,\tau) - \pi(\operatorname{Re}\tau - 1) - 2\pi t]\left[\mathrm{d}\arg\vartheta_1(t,\tau') - 2\pi dt\right] \\
&= -\int_{\varepsilon}^{1-\varepsilon} \arg\vartheta_1(t,\tau)\mathrm{d}\arg\vartheta_1(t,\tau') + \pi(\operatorname{Re}\tau - 1)\int_{\varepsilon}^{1-\varepsilon} \mathrm{d}\arg\vartheta_1(t,\tau') + \int_{\varepsilon}^{1-\varepsilon} 2\pi t\mathrm{d}\arg\vartheta_1(t,\tau') \\
&+ 2\pi\int_{\varepsilon}^{1-\varepsilon} \arg\vartheta_1(t,\tau)dt - \pi(\operatorname{Re}\tau - 1)\int_{\varepsilon}^{1-\varepsilon} 2\pi dt - \int_{\varepsilon}^{1-\varepsilon} 2\pi t 2\pi dt
\end{aligned}$$

The first integral cancels with the first one above. The second vanishes by Lemma C.1.10. The last two terms together computed give $-2\pi^2 \operatorname{Re}\tau$. The fourth term can be integrated by parts and gives

$$\int_{\varepsilon}^{1-\varepsilon} \arg\vartheta_1(t,\tau)dt = \lim_{\varepsilon\to 0} \arg\vartheta_1(1-\varepsilon,\tau) - \int_{\varepsilon}^{1-\varepsilon} t\mathrm{d}\arg\vartheta_1(t,\tau).$$

The limit can of the argument can be evaluated using the product formula in Proposition. C.1.4: Namely, for $x \in \mathbb{R}$ we have

$$\vartheta(x,\tau) = 2Gq^{1/4}\sin x \prod_{n=1}^{\infty}(1 - 2q^{2n}\cos(2\pi x) + q^{4n})$$

with $G = \prod_{n=1}^{\infty}(1-q^{2n})$ and $q = e^{i\pi\tau}$, so that

$$\arg\vartheta(x,\tau) = \arg\left(Gq^{1/4}\prod_{n=1}^{\infty}(1 - 2q^{2n}\cos(2\pi x) + q^{4n})\right).$$

In the limit $x \to 1^-$ we get

$$\lim_{x\to 1}\arg(\vartheta_1(x,\tau)) = \arg\left(q^{1/4}G^3\right).$$

In terms of the q-Pochhammer symbol

$$(a,q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$$

we can express $G = (q^2, q^2)_\infty$, which in turn be expressed by the Dedekind eta function

$$\eta(\tau) = e^{i\pi\tau/12}\prod_{k=1}^{\infty}(1 - q^{2k}).$$

So, we find that

$$\lim_{x\to 1}\arg(\vartheta_1(x,\tau)) = \arg\eta(\tau)^3.$$

We are left with evaluating the integral

$$\int_0^1 t \mathrm{d} \arg \vartheta(t,\tau).$$

We can use again the expansion (C.9). Since $y = 0$, and we are only interested in the imaginary part, the first term with the cotangent drops out and we are left with

$$\begin{aligned}
\int_0^1 t \mathrm{d} \arg \vartheta(t,\tau) &= 4\pi \operatorname{Im} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \int_0^1 t \sin 2\pi n t dt \\
&= -2 \operatorname{Im} \sum_{n=1}^{\infty} \frac{q^{2n}}{n(1-q^{2n})} \\
&= -2 \operatorname{Im} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \frac{q^{2n(l+1)}}{n} \\
&= 2 \operatorname{Im} \sum_{l=0}^{\infty} \log(1-q^{2(l+1)}) = 2 \arg \prod_{l=0}^{\infty} (1-q^{2(l+1)}) = 2 \arg(q^2, q^2).
\end{aligned}$$

The equality in the last line is true up to a multiple of 2π, but since both sides vanish as $\operatorname{Im} \tau \to \infty$, they agree. In the equations above we have used a geometric series and the series expansion of the logarithm. Let us summarise the contribution of Γ_1^{ε}:

$$\begin{aligned}
&\int_{\Gamma_\varepsilon^1} \arg \vartheta_1(x+\tau y, \tau) \mathrm{d} \arg \vartheta_1(x+\tau' y, \tau') \\
&= \pi^2 \operatorname{Re} \tau' - 2\pi^2 \operatorname{Re} \tau + 2\pi \arg \eta(\tau)^3 + 4\pi \arg((q')^2, (q')^2) - 4\pi \arg(q^2, q^2).
\end{aligned}$$

Now let us turn to the quartercircles around the corners. The idea is to exploit the quasi-periodicity to piece them together to a small circle around 0. Notice that the quarter-circles in the boundary are traversed in clockwise direction. We parametrise a small circle around 0 from $-\pi$ to π and choose the branch of $\arg \vartheta_1$ defined on $[-1,1]^2 - \{(x,0) | x \leq 0\}$. The contribution of the quartercircles is then

$$\begin{aligned}
&\int_{-\pi}^{-\frac{\pi}{2}} \arg \vartheta_1(z(\tau)+\tau+1, \tau) \mathrm{d} \arg \vartheta_1(z(\tau')+\tau'+1, \tau') \\
&+ \int_{-\frac{\pi}{2}}^{0} \arg \vartheta_1(z(\tau)+\tau, \tau) \mathrm{d} \arg \vartheta_1(z(\tau')+\tau', \tau') \\
&+ \int_{0}^{\frac{\pi}{2}} \arg \vartheta_1(z(\tau), \tau) \mathrm{d} \arg \vartheta_1(z(\tau'), \tau') \\
&+ \int_{\frac{\pi}{2}}^{\pi} \arg \vartheta_1(z(\tau)+1, \tau) \mathrm{d} \arg \vartheta_1(z(\tau')+1, \tau')
\end{aligned}$$

where $z(\tau) = \varepsilon(\cos t + \tau \sin t)$. Using lemma C.1.9 we can express this as

$$\int_{-\pi}^{\pi} \arg \vartheta_1(z(\tau)) \mathrm{d} \arg \vartheta_1(z(\tau'))$$
$$+ \int_{-\pi}^{-\frac{\pi}{2}} (\pi(-\operatorname{Re}\tau - 2\operatorname{Re} z(\tau))(\mathrm{d}\arg\vartheta_1(z(\tau')) - 2\pi d \operatorname{Re} z(\tau')) - 2\pi \arg \vartheta_1(z(\tau)) d \operatorname{Re} z(\tau')$$
$$+ \int_{-\frac{\pi}{2}}^{0} (\pi(1 - \operatorname{Re}\tau - 2\operatorname{Re} z(\tau))(\mathrm{d}\arg\vartheta_1(z(\tau')) - 2\pi d \operatorname{Re} z(\tau')) - 2\pi \arg \vartheta_1(z(\tau)) d \operatorname{Re} z(\tau')$$
$$+ \int_{\frac{\pi}{2}}^{\pi} -\pi \mathrm{d} \arg \vartheta_1(z(\tau'))$$

In the limit $\varepsilon \to 0$, we can neglect all the terms containing $\operatorname{Re} z(\tau)$ or $\operatorname{Re} z(\tau')$, as they carry a factor of ε. Around 0, we have by proposition C.1.4 and according to the computations above,

$$\vartheta_1(z(\tau), \tau) = 2\eta(\tau)^3 z(\tau) + O(z^3) \tag{C.51}$$

So, in the limit we have $\mathrm{d} \arg \vartheta_1 z(\tau) = \mathrm{d} \arg z(\tau)$ and so

$$\int_{-\pi}^{-\frac{\pi}{2}} \mathrm{d} \arg \vartheta_1 z(\tau') = \arg \tau' - \pi - (-\pi) = \arg \tau'$$
$$\int_{-\frac{\pi}{2}}^{0} \mathrm{d} \arg \vartheta_1 z(\tau') = 0 - (\arg \tau' - \pi) = \pi - \arg \tau'$$
$$\int_{\frac{\pi}{2}}^{\pi} \mathrm{d} \arg \vartheta_1 z(\tau') = \pi - \arg \tau'.$$

(where we have used that $\operatorname{Im} \tau' > 0$ so that $\arg -\tau' = \arg \tau' - \pi$).so that the above evaluates to

$$\int_{-\pi}^{\pi} \arg \vartheta_1(z(\tau)) \mathrm{d} \arg \vartheta_1(z(\tau'))$$
$$+ \pi(-\operatorname{Re}\tau) \arg \tau' + \pi(1 - \operatorname{Re}\tau)(\pi - \arg\tau') - \pi(\pi - \arg \tau')$$
$$= \int_{-\pi}^{\pi} \arg \vartheta_1(z(\tau)) \mathrm{d} \arg \vartheta_1(z(\tau')) - \pi^2 \operatorname{Re}\tau$$

The remaining integral is

$$\int_{-\pi}^{\pi} \arg \vartheta_1(z(\tau)) \mathrm{d} \arg \vartheta_1(z(\tau')) \xrightarrow{\varepsilon \to 0} \int_{-\pi}^{\pi} \arg(\eta(\tau)^3 z(\tau)) \mathrm{d} \arg z(\tau').$$

Integrating by parts we get

$$\int_{-\pi}^{\pi} \arg(\eta(\tau)^3 z(\tau)) \mathrm{d} \arg z(\tau') = \left[\arg(\eta(\tau)^3 z(\tau)) \arg z(\tau')\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \arg(z(\tau')) \mathrm{d} \arg(z(\tau)).$$

Notice that for this, the two arguments need to have different branch cuts. The branch cut of $\arg \eta(\tau)^3 z(\tau)$ is with an angle $\arg \eta(\tau)^3 + \pi$, while the branch cut of $\arg z(\tau')$ is on the negative real axis. So, the part in brackets evaluates to

$$(\arg \eta(\tau)^3 + \pi)\pi - (\arg(\eta(\tau)^3) - \pi)(-\pi) = 2\pi \arg \eta(\tau)^3.$$

The computation of the last integral is the following Lemma:

Lemma C.3.5. *Denote $\tau = \theta + i\beta, \tau' = \theta' + i\beta'$. Then*

$$\int_{-\pi}^{\pi} \arg(z(\tau'))\mathrm{d}\arg(z(\tau)) = 2\pi \arg\left(1 + \frac{\beta}{\beta'} + i\frac{\theta - \theta'}{\beta'}\right). \tag{C.52}$$

Proof. We have

$$\mathrm{d}\arg z(\tau) = \frac{\beta dt}{(\cos t + \theta \sin t)^2 + \beta^2 \sin^2 t} = \frac{\beta dt}{\sin^2 t} \frac{1}{(\cot t + \theta)^2 + \beta^2}.$$

If we denote $\varphi = \arg(z(\tau'))$, we have

$$\cot \varphi = \frac{\cos t + \theta' \sin t}{\beta' \sin t} = \frac{1}{\beta'}(\cot t + \theta').$$

Hence, we have

$$\frac{\mathrm{d}\varphi}{\sin^2 \varphi} = -\mathrm{d}\cot\varphi = -\frac{1}{\beta'}\mathrm{d}\cot t = \frac{1}{\beta'}\frac{dt}{\sin^2 t}.$$

Substituting φ for t, we obtain the integral

$$\frac{\beta}{\beta'}\int_{-\pi}^{\pi} \frac{\varphi \mathrm{d}\varphi}{\sin^2\varphi\left((\cot\varphi + \frac{\theta-\theta'}{\beta'})^2 + \left(\frac{\beta}{\beta'}\right)^2\right)} = b\int_{-\pi}^{\pi} \frac{\varphi\mathrm{d}\varphi}{(\cos\varphi + a\sin\varphi)^2 + b^2\sin^2\varphi},$$

where we put $a = (\theta - \theta')/\beta', b = \beta/\beta'$. This integral can be computed using a keyhole contour. Let

$$f(z) = \frac{b \log z}{z\left(\frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{a}{2i}\left(z - \frac{1}{z}\right)\right)^2 + b^2\left(\frac{1}{2i}\left(z - \frac{1}{z}\right)\right)^2} = g(z)\log z.$$

Then if $z = e^{i\varphi}$ we get

$$f(z)dz = -\frac{\varphi\mathrm{d}\varphi}{(\cos\varphi + a\sin\varphi)^2 + b^2\sin^2\varphi}.$$

Let C be a keyhole contour with a cut on the negative real axis, then we have

$$\int_C f(z)dz = -\int_{-\pi}^{\pi} \frac{\varphi\mathrm{d}\varphi}{(\cos\varphi + a\sin\varphi)^2 + b^2\sin^2\varphi} + 2\pi i\int_{cut} g(z)dz + \int_{C_\varepsilon} f(z)dz.$$

Since $f(z) \simeq z\log z$ around 0, one immediately sees that the integral over the small circle around 0 vanishes in the limit $\varepsilon \to 0$. We rewrite

$$\begin{aligned} f(z) &= \frac{4bz\log z}{(z^2 + 1 - ia(z^2 - 1))^2 - b^2(z^2 - 1)^2} \\ &= \frac{4bz\log z}{(z^2 + 1 + (b - ia)(z^2 - 1))\,(z^2 + 1 - (b + ia)(z^2 - 1))} \\ &= \frac{4bz\log z}{(1 + \bar\gamma)(1 - \gamma)\left(z^2 - \frac{\bar\gamma - 1}{\bar\gamma + 1}\right)\left(z^2 - \frac{\gamma + 1}{\gamma - 1}\right)}, \end{aligned}$$

where we introduced $\gamma := b + ia$. Notice that $b = \beta/\beta' > 0$, so $|(\bar\gamma - 1)/(\bar\gamma + 1)| < 1$, while $|(\bar\gamma + 1)/(\bar\gamma - 1)| > 1$. So, f has two singularities inside the unit disk (except from 0), namely $z_{1,2} = \pm\sqrt{\frac{\bar\gamma-1}{\bar\gamma+1}}$. The residue at z_1 is

$$\frac{4bz_1 \log z_1}{(1+\bar\gamma)(1-\gamma)(z_1 - z_2)\left(z_1^2 - \frac{\gamma+1}{\gamma-1}\right)} = \frac{2b\log z_1}{(|\gamma+1|^2 - |\gamma-1|^2)} = \frac{\log z_1}{2},$$

and similarly, $\mathrm{Res}(f, z_2) = \log(z_2)/2$. Next we compute the integral along the cut, which is

$$\begin{aligned}
\int_{cut} g(z)dz &= \int_{-1}^{0} \frac{4btdt}{(1+\bar\gamma)(1-\gamma)\left(t^2 - \frac{\bar\gamma-1}{\bar\gamma+1}\right)\left(t^2 - \frac{\gamma+1}{\gamma-1}\right)} \\
&= \int_{-1}^{0} \frac{-t}{t^2 - \frac{\gamma+1}{\gamma-1}} + \frac{t}{t^2 - \frac{\bar\gamma-1}{\bar\gamma+1}} dt \\
&= \frac{1}{2}\left[\log\left(t^2 - \frac{\bar\gamma-1}{\bar\gamma+1}\right) - \log\left(t^2 - \frac{\gamma+1}{\gamma-1}\right)\right]_{t=-1}^{t=0} \\
&= \frac{1}{2}\left[\log(-z_1^2) - \log(-\bar z_1^{-2}) - \log\left(\frac{2}{1+\bar\gamma}\right) + \log\left(\frac{2}{1-\gamma}\right)\right]
\end{aligned}$$

So, applying the residue theorem, we arrive at

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{\varphi \mathrm{d}\varphi}{(\cos\varphi + a\sin\varphi)^2 + b^2\sin^2\varphi} \\
&= 2\pi i \int_{cut} g(z)dz - \int_C f(z)dz \\
&= \pi i\left[\log -z_1^2 - \log(-\bar z_1^{-2}) - \log\frac{1}{1+\bar\gamma} + \log\frac{1}{1-\gamma} - \log(z_1) - \log(-z_1)\right]
\end{aligned}$$

Notice that the real parts of the two integrals cancel out. In fact, we have

$$\log|z_1| = \frac{1}{2}(\log|\bar\gamma - 1| - \log|\bar\gamma + 1|)$$

so that the real parts of the two middle and two last terms add up to $-4\log|z_1$, while the real parts of the two first terms add up to $4\log|z_1|$. The imaginary parts of the two first terms cancel since $\arg \bar z^{-1} = \arg z$. The remaining arguments combine into $-2i\arg(1+\gamma)$, up to a integer multiple of $2\pi i$ that depends continuously on τ and τ'. But since the integral vanishes in the limit $\beta' \to \infty$, as does $2\pi\arg(1+\gamma)$, this constant is zero and the Lemma is proven. □

Hence, the total contribution of the quartercircles is

$$\int_{\Gamma_\varepsilon^2} \arg\vartheta_1(x + \tau y, \tau)\mathrm{d}\arg\vartheta_1(x + \tau' y, \tau') = \pi^2\,\mathrm{Re}\,\tau - 2\pi\arg\eta(\tau)^3 + 2\pi\arg(\beta + \beta' + i(\theta - \theta'))$$

where the total sign comes from the fact that we parametrised the quartercircles in the opposite direction. So, we obtain

$$\begin{aligned}I_1 &= \int_{[0,1]^2} \mathrm{d}\arg\vartheta_1(x+\tau y,\tau)\mathrm{d}\arg\vartheta_1(x+\tau' y,\tau') \\ &= \pi^2(\operatorname{Re}\tau' - \operatorname{Re}\tau) + 4\pi(\arg((q')^2,(q')^2) - \arg(q^2,q^2)) + 2\pi\arg(\beta+\beta'+i(\theta-\theta')). \quad \text{(C.53)}\end{aligned}$$

Here all arguments are given by the the principal branch of the argument. Now we can compute

$$\begin{aligned}&\frac{1}{4\pi^2}I_1 + \frac{1}{2\pi}I_2 - \frac{1}{2\pi}I_3 + I_4 \\ &= \frac{1}{4}(\operatorname{Re}\tau' - \operatorname{Re}\tau) + \frac{1}{\pi}(\arg((q')^2,(q')^2) - \arg(q^2,q^2))) + \frac{1}{2\pi}\arg(\beta+\beta'+i(\theta-\theta')) \\ &+ \frac{1}{4}(\operatorname{Re}\tau - \operatorname{Re}\tau') - \frac{1}{4}(\operatorname{Re}\tau' - \operatorname{Re}\tau) + \frac{1}{3}(\operatorname{Re}\tau' - \operatorname{Re}\tau) \\ &= \frac{1}{12}(\operatorname{Re}\tau + \operatorname{Re}\tau') + \frac{1}{\pi}(\arg((q')^2,(q')^2) - \arg(q^2,q^2))) + \frac{1}{2\pi}\arg(\beta+\beta'+i(\theta-\theta')) \\ &= \frac{1}{\pi}(\arg\eta(\tau') - \arg\eta(\tau)) + \frac{1}{2\pi}\arg(\beta+\beta'+i(\theta-\theta'))\end{aligned}$$

which concludes the proof of proposition C.3.1.

Appendix D

Currents

In this appendix we briefly review de Rham's theory of double forms and double currents as integral kernels of operators on forms explained in [Rha84], and establish the signs that appear in our conventions. We also explain when these currents extend to compactified configuration spaces.

We follow de Rham's book [Rha84], but we assume all manifolds to be compact and oriented. We endow $\Omega^\bullet(M)$ with a Fréchet topology as follows. Choose finitely many compact sets $K_1, \ldots, K_n$ that cover M and are contained in coordinate charts. The C^k-norm of a form ω in K_j is the maximum of the C^k-norms of all coefficients in that coordinate chart. The Fréchet topology on $\Omega^\bullet(M)$ is the one generated by all C^k-norms on K_j for $j = 1, \ldots n$ and $k = 0, \ldots, \infty$[1].

D.1 Double forms

Definition D.1.1. Let M, N be manifolds of dimensions m and n respectively. Then a double form on $M \times N$ is a form on M whose coefficients are forms on N.

Equivalently, it is a form on N with coefficients in forms on M. In local coordinates $x^1, \ldots x^m$ on M and local coordinates $y^1, \ldots y^n$ on N a double form γ can be written as

$$\gamma(x,y) = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_q}} c_{i_1\ldots i_p j_1 \ldots j_q} \left(dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right) \left(dy^{j_1} \wedge \cdots \wedge dy^{j_q}\right) = \sum_{I,J} c_{IJ} dx^I dy^J,$$

for smooth functions c_{IJ} on $M \times N$. The space of double forms on $M \times N$ will be denoted $\mathscr{D}(M \times N)$ and also carries a Fréchet topology defined analogously to the one on $\Omega^\bullet(M \times N)$.

[1] For non-compact manifolds one can just choose a countable cover of M with compact sets contained in coordinate charts.

The only difference to single forms $\Omega^\bullet(M \times N)$ is that the dx^i and dy^j commute instead of anticommuting. If γ is homogeneous of degrees (p, p') and γ' is homogeneous of degrees (q, q') then

$$\gamma \wedge \gamma' = (-1)^{pp'+qq'} \gamma' \wedge \gamma.$$

There are two commuting differentials on $\mathscr{D}(M \times N)$, the one on forms on M valued in forms on N and the one on forms on N valued in forms on M. They will be denoted d_x and d_y and satisfy

$$\mathrm{d}_x^2\gamma = \mathrm{d}_y^2\gamma = 0, \mathrm{d}_x\mathrm{d}_y\gamma = \mathrm{d}_y\mathrm{d}_x\gamma.$$

One can integrate double forms over a chain in a single factor, let e.g. C be a chain in M, then

$$\int_C \gamma(x, y) \in \Omega^\bullet(N).$$

If $\alpha(x)$ is a single form on M and $\beta(y)$ is a single form on N then $\alpha(x)\beta(y)$ is a double form on $M \times N$, this double form is called the *tensor product* of α and β. On the other hand, the wedge product $\alpha(x) \wedge \beta(y)$ is a single form on $M \times N$.

Theorem D.1.1 (de Rham [Rha84]). *Every double form on $M \times N$ is the limit of a sequence of finite sums of tensor products with respect to the Fréchet topology on $\mathscr{D}(M \times N)$.*

D.2 Currents

Definition D.2.1. A *current* on an n-dimensional manifold M is a linear functional on $\Omega^\bullet(M)$ that is continuous with respect to the Fréchet topology $\Omega^\bullet(M)$. The space of currents on M is denoted by $(\Omega^\bullet(M))'$.

If T is a current and ω is a form, we denote evaluation of T on ω by $T[\omega]$.
There are two main examples of currents.

Example D.2.1. If C is a chain in M, then

$$\omega \mapsto \int_C \omega \tag{D.1}$$

defines a current.

Example D.2.2. If α is a form on M, then

$$\omega \mapsto \int_M \alpha \wedge \omega \tag{D.2}$$

defines a current.

If C is homogeneous of dimension p, then the corresponding current vanishes on homogeneous forms not of degree p. On the other hand, if α is homogenoues of degree p, then the associated current vanishes on homogeneous forms of degree not equal to $n-p$. This motivates the following definition.

Definition D.2.2. Let T be a current that vanishes on homogeneous forms of degree not equal to p. Then we say that T has dimension p, or degree $n-p$.

Depending on whether one thinks of currents as a generalisation of chains, or forms, either the dimension or the degree is more natural. For us currents are a generalisation of forms, hence we will mostly work with the degree. If we think of a current of degree 0 as a generalised function[2], then locally, we can write any current as a form with coefficients generalised functions. Namely, let $x^1, \dots, x^n$ be local coordinates on $U \subset M$ and let T be a current of degree p on M. If f is a function on U, define

$$T_{i_1 \dots i_p}[f dx^1 \wedge \cdots \wedge dx^n] := \varepsilon^{i_1 \dots i_p j_1 \dots j_p} T[f dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}}].$$

Then we have that

$$T = \sum_{|I|=p} T_I dx^I.$$

D.2.1 Operations on currents

We briefly review the most important operations on currents. These definitions extend the operations that we already have on forms (or chains) to currents via the equations (D.1) and (D.2).

Wedge product

If T is a current, and α is a form, then we define the *exterior product* $T \wedge \alpha$ by

$$T \wedge \alpha[\omega] \colon = T[\alpha \wedge \omega].$$

Notice that if T is the current associated to a form β, then

$$T \wedge \alpha[\omega] = \int_M \beta \wedge (\alpha \wedge \omega) = \int_M (\beta \wedge \alpha) \wedge \omega,$$

so that $T \wedge \alpha$ is the current associated to $\beta \wedge \alpha$. In this spirit, we also define the current $\alpha \wedge T$ by asking that if T has degree p and α has degree q, then

$$\alpha \wedge T = (-1)^{pq} T \wedge \alpha.$$

[2]Currents of degree 0 are not exactly distributions, since they form the dual of top forms, not of functions. On compact oriented manifolds one can identify the two using a volume form.

Boundary and differential

One can both define the *boundary* or *differential* of a current, generalising the notions of boundaries of chains or differential on forms. Namely, let T be a current, then we define

$$\partial T[\omega] \colon = T[\mathrm{d}\omega],$$

generalising Stokes' theorem. If T is a homogeneous current of degree p, we can define the differential of T by

$$\mathrm{d}T[\omega] \colon = (-1)^p T[d\omega]$$

generalising integration by parts. Obviously, the differential of a degree p current is a current of degree $p+1$ and we have

$$\mathrm{d}T = (-1)^p \partial T$$

so that also the boundary operator increases the degree by 1, or decreases the dimension by 1. From the definition it is also clear that $\mathrm{d}^2 = \partial^2 = 0$. Both operators are compatible with the wedge product: Let T be a current of degree p, β be a current of degree q, then

$$\mathrm{d}(T \wedge \beta) = \mathrm{d}T \wedge \beta + (-1)^p T \wedge \mathrm{d}\beta$$
$$\partial(T \wedge \beta) = (-1)^q \partial T \wedge \beta + T \wedge \mathrm{d}\partial\beta$$

Pushforward and pullback

If $f \colon M \to N$ is a smooth map between manifolds and T is a current on M, then the *pushforward* of T is defined by

$$f_* T[\omega] := T[f^* \omega].$$

If f is a diffeomorphism then one can define also the pullback of a current as the pushforward of the inverse. The boundary always commutes with pushforward

$$\partial f_* T = f_* \partial T$$

but the differential only commutes up to a sign:

$$\mathrm{d} f_* T = (-1)^{(m+n)} f_* \mathrm{d}T \tag{D.3}$$

where m and n are the dimensions of M and N respectively.

Integral

One can also define the integral of a current by

$$\int_M T\colon = T[1].$$

This is the same as the pushforward of T along the map $M \to \{pt\}$. This generalises the integral of forms. In particular, it is zero unless T has a homogeneous component of top degree.

D.3 Double Currents

The main purpose of double currents is to serve as integral kernels for operators on forms. For us it will be interesting how to get a single current on $M \times M$ from the double current which is the integral kernel for the inverse of the de Rham differential.

D.3.1 Definition

Double currents are defined analogously to currents.

Definition D.3.1. Continuous linear functionals on $\mathcal{D}(M \times N)$ are called *double currents*. The space of double currents is denoted by $\mathcal{D}'(M \times N)$.

A double current L on $M \times N$ is called *homogeneous* of bidegree $(m-p, n-q)$ if it vanishes on homogeneus double forms which are not bidegree (p, q).

D.3.2 Operations on double currents

All operations on single currents have generalisations to double currents. The wedge product of a double current L with a double form α is defined by

$$L \wedge \alpha[\omega]\colon = L[\alpha \wedge \omega]$$

and we define boundaries and differentials $\partial_x, \mathrm{d}_x, \partial_y, \mathrm{d}_y$ by

$$\partial_x L[\omega]\colon L[\mathrm{d}_x \omega], \mathrm{d}_x L[\omega]\colon = (-1)^p L[\mathrm{d}_x \omega]$$

where the second definition is for homogeneous currents of degree p, and ∂_y, d_y are defined similarly.

If $T = T(x)$ is a single current on M, and $\gamma = \gamma(x, y)$ is a double form in $M \times N$, there is well-defined single form on N denoted by $T(x)[\gamma(x, y)]$ or $\int_{x \in M} T(x) \wedge \gamma(x, y)$, and the induced map $T\colon \mathcal{D}(M\times N)\to\Omega^\bullet(N)$ is continuous ([Rha84],Theorem 9). If S is a single current on N, we can thus define a double current ST, t as the composition of S with T. This double current is called the *tensor product* of T and S, and we have $ST=TS$ ([Rha84], Theorem 10).

D.3.3 Double currents and operators on forms

Let $L(x,y)$ be a double current on $M\times N$, homogeneous of degree p in M, and $\phi\in\Omega^\bullet(M),\psi\in\Omega^\bullet(N)$. Then we have

$$\int_{x\in M}\int_{y\in N}\phi(x)\wedge L(x,y)\wedge\psi(y)=(-1)^{mp+p}L(x,y)[\phi(x)\psi(y)]$$

and we can think of it as a map $\Lambda_L\colon(\Omega^\bullet(M)\to(\Omega^\bullet(N))'$, or vice versa, as a map $\Lambda_L^*\colon\Omega^\bullet(N)\to(\Omega^\bullet(M))'$. Combining de Rham's theory of currents with the Schwartz kernel theorem one can show that $L\mapsto\Lambda_L$ defines a bijective homeomorphism between double currents on $M\times N$ and continuous linear maps $(\Omega^\bullet(M)\to(\Omega^\bullet(N))'$.

D.3.4 Double and single currents

To a double current on $M\times N$ we can associate a single current on $M\times N$ as follows [Rha84]. One can easily associate to a single form ω on $M\times N$, represented in coordinates x^i on M and y^j on N by $\omega=\sum c_{I,J}dx^I\wedge dx^J$, the double form $A^*\omega\colon=\sum_{I,J}dx^Idx^J$. To a double current $L(x,y)$ one can then associate the single current AL by

$$AL[\omega]=L[A^*\omega].$$

Our propagators will be the single currents associated to double currents associated to certain maps on forms, as discussed below.

D.4 Currents and Hodge theory

Let g be a Riemannian metric on a compact manifold M. Denote $*$ the Hodge star of g and by d^* the codifferential, i.e. the formal adjoint of d with respect to the scalar product

$$(\alpha,\beta)=\int_M\alpha\wedge*\beta,$$

i.e. the unique operator satisfying

$$(\mathrm{d}\alpha,\beta)=(\alpha,\mathrm{d}^*\beta)$$

for all smooth forms α and β. On homogeneous forms of degree p one has

$$\mathrm{d}^*\alpha = (-1)^{np+p+1} * \mathrm{d} * \alpha.$$

The Hodge-de Rham Laplacian is

$$\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}.$$

If $\Lambda\colon \Omega\bullet(M) \to \Omega^\bullet(M)$ is a continuous linear operator, it has an integral kernel $L(x,y)$ which is a double current on $M \times M$ as explained above. It also has a *metric kernel* $l(x,y)$ defined by

$$\Lambda\phi(x) = \int_y l(x,y) \wedge *\phi(y),$$

it satisfies $l(x,y) = *_y^{-1} L(y,x)$. The following is a central theorem of de Rham about finding solutions to the equation $\Delta\phi = \psi$ ([Rha84], Theorem 23):

Theorem D.4.1. *There exist linear operators H and G on $\Omega(M)$, that both commute with d, d^* and $*$, with $GH = HG = 0$, such that*

$$\Delta G = G\Delta = 1 - H.$$

If $h_1, \ldots, h_n$ denotes a basis of harmonic forms on n, the metric kernel of H is given by the smooth double form

$$h(x,y) = \sum_i h_i(x)h_i(y).$$

The metric kernel $g(x,y)$of G is a double current which is smooth away from the diagonal and $O(r^{2-n})$ on the diagonal.

The last statement means the following. For a compact manifold M, and a chart U on M, there is $\varepsilon > 0$ such that the diagonal $D \subset U \times U$ has a neighbourhood V such that $V - D$ is isometric to $U \times (0,\varepsilon) \times S^{n-1}$. Here the second coordinate $r > 0$ measures the geodesic distance of the two points. We say that a form is $O(r^k)$ if all its coefficients f_I in all such charts satisfy that $r^{-k} f_I$ is bounded for $r \to 0$.

D.5 Alternative proof of Theorem 3.3.2

In this section we give an alternative proof of theorem 3.3.2 which does not require heat kernel techniques. For better readibility we briefly recall the definition of the Hodge propagator.

D.5.1 Hodge propagators

Let g be a Riemannian metric on M with codifferential d_g^* and Laplace–Beltrami operator Δ_g. Recall that it gives rise to a Hodge decomposition of $\Omega^\bullet(M) = \Omega_H \oplus \Omega_H^\perp$ with $\Omega_H := \ker \Delta_g$ the harmonic forms and that from these data we can form a contracting triple $(\iota_{harm}, p_{harm}, K_g)$ with ι_{harm} inclusion of harmonic forms, p_{harm} the projection onto harmonic forms, and

$$K_g \coloneqq d_g^* \circ (\Delta_g + \iota_{harm} \circ p_{harm})^{-1}. \tag{D.4}$$

This is the Riemann–Hodge contracting triple. We frequently denote $\iota_{harm} \circ p_{harm} =: P_{harm}$.

Remark D.5.1. Using the Hodge decomposition we can rewrite K_g as

$$K_g = d_g^* \circ \left(id_{\Omega_H} \oplus \left(\Delta_g\big|_{\Omega_H^\perp} \right)^{-1} \right) = d_g^* \circ \left(\Delta_g\big|_{\Omega_H^\perp} \right)^{-1} \circ P_{\Omega_H^\perp}.$$

D.5.2 Proof

We recall the statement of the theorem:

Theorem D.5.2. *Let $\alpha \in \Omega^\bullet(M)$. Then as $\lambda \to 0$, $K_{g^\lambda}\alpha \to K^{hor}\alpha$. It follows that $\eta_{g^\lambda} \to \eta^{hor}$ in the sense of distributions.*

Proof. We recall the following facts from Riemannian geometry: Scaling a Riemannian metric by a constant scales codifferential and Laplacian by the *inverse* of that constant[3]. Hence we have $d^*_{g^\lambda} = d_1^* + \lambda d_2^*$ and $\Delta_{g^\lambda} = \Delta_1 + \lambda\Delta_2$. The harmonic forms are constant and hence also the projection $P_{harm} = P_{g_1} \otimes P_{g_2}$. To show that $K_{g^\lambda} \to K^{hor}$ weakly, it is enough to check that $K_{g^\lambda}(\omega_1 \wedge \omega_2) \to K^{hor}(\omega_1 \wedge \omega_2)$ for every $\omega_1 \in \Omega^\bullet(M_1), \omega_2 \in \Omega^\bullet(M_2)$, since every form in $\Omega(M)$ can be approximated by finite sums of such forms. For such a form we have

$$\begin{aligned} K_g(\omega_1 \wedge \omega_2) &= (\Delta_1 + \lambda\Delta_2 + P_1 \otimes P_2)^{-1}(d^*\omega_1 \wedge \omega_2 + \lambda\omega_1 \wedge d^*\omega_2) \\ &= (\Delta_1 + \lambda\Delta_2 + P_1 \otimes P_2)^{-1}(d^*\omega_1 \wedge \omega_2) + (\Delta_1 + \lambda\Delta_2 + P_1 \otimes P_2)^{-1}(\lambda\omega_1 \wedge d^*\omega_2) \end{aligned}$$

since Δ and d^* commute. We claim that the first term converges to $K_1\omega_1 \wedge \omega_2$, and the second to $P_1\omega_1 \otimes K_2\omega_2$. Indeed, using remark D.5.1 the first term equals

$$(\Delta_1 + \lambda\Delta_2 + P_1 \otimes P_2)^{-1}(d^*\omega_1 \wedge \omega_2) = \left((\Delta_1 + \lambda\Delta_2)\big|_{\Omega_H^\perp} \right)^{-1} (d^*\omega_1 \wedge \omega_2)$$

[3] this can be seen e.g. from the expression in local coordinates for the Laplace-Beltrami operator (the degree 0 part of the Hodge-de Rham Laplacian)

$$\Delta_g = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j} \left(\sqrt{\det g} g^{ij} \right) \frac{\partial}{\partial x_i} \tag{D.5}$$

where g^{ij} denotes the *inverse* of g.

Notice that on the image of d_1^*, Δ_1 is invertible and in the limit $\lambda \to 0$ of the above expression we get $(\Delta_1)^{-1} d^* \omega_1 \wedge \omega_2$. Since on the image of d_1^* any operator of the form $P_1 \otimes A_2$ vanishes, we can add $P_1 \otimes id$ to this expression and get

$$(\Delta_1)^{-1} d^* \omega_1 \wedge \omega_2 = \left((\Delta_1)^{-1} + P_1 \otimes id_2\right) d^* \omega_1 \wedge \omega_2 = (K_1 \otimes id_2) \omega_1 \wedge \omega_2.$$

As for the other term, write $\omega_1 = P_1 \omega_1 + (id_1 - P_1)\omega_1$. Using again remark D.5.1, we can write

$$\begin{aligned}(\Delta_1 + \lambda\Delta_2 + P_1 \otimes P_2)^{-1}(\lambda\omega_1 \wedge d^*\omega_2) = & \left((\Delta_1 + \lambda\Delta_2)\big|_{\Omega_H^\perp}\right)^{-1} (\lambda P\omega_1 \wedge d^*\omega_2) \\ & + \left((\Delta_1 + \lambda\Delta_2)\big|_{\Omega_H^\perp}\right)^{-1} \lambda(id_1 - P_1)\omega_1 \wedge d^*\omega_2\end{aligned}$$

On $(\Omega_H(M_1) \otimes \Omega(M_2)) \cap \Omega_H^\perp(M)$, $\Delta_1 = 0$ and the operator $\left((\Delta_1 + \lambda\Delta_2)\big|_{\Omega_H^\perp}\right)^{-1}$ equals $\left(\lambda\Delta_2\big|_{\Omega_H^\perp}\right)^{-1}$. Hence, in the first term λ cancels, and the expression converges to $P_1 \otimes K_2 \omega_1 \wedge \omega_2$. On $\Omega_H(M_1)^\perp \otimes \Omega(M_2) \cap \Omega_H^\perp(M)$, $\left((\Delta_1 + \lambda\Delta_2)\big|_{\Omega_H^\perp}\right)^{-1}$ converges to $\left(\Delta_1\big|_{\Omega_H^\perp}\right)^{-1}$ as $\lambda \to 0$. Since the argument converges 0, the second term goes to 0. $\square$

Appendix E

Conventions

In this section we list the numerous conventions that we use, especially concerning signs and gradings.

E.1 Configuration Spaces and Pushforwards

E.1.1 Configuration spaces

If M is a compact manifold, then the open configuration space of n points is denoted

$$C_n^0(M) = \{(x_1, \ldots, x_n) \in M^n \colon x_i \neq x_j \text{ for all } i \neq j\}.$$

Its Fulton-MacPherson-Axelrod-Singer (FMAS) compactification is denoted $C_n(M)$ ([FM94; AS94], see also [Sin03]), and is a manifold with corners. In particular, the boundary of $C_2(M)$ is isomorphic to the tangent sphere bundle over M, $\partial C_2(M) \cong STM$. The restriction to $C_n^0(M)$ of the natural projection maps

$$\pi_i \colon M^n \to M$$

to the i-th factor extends to the compactification as a smooth map

$$\pi_i \colon C_n(M) \to M.$$

Similarly, projection to i-th and j-th factor yields a smooth map

$$\pi_{ij} \colon C_n(M) \to C_2(M).$$

We often denote the pullback of a form on M or $C_2(M)$ to $C_n(M)$ by the indices of the projection, that is, for $\alpha \in \Omega^\bullet(M)$,$\beta \in \Omega^\bullet(C_2(M))$, we denote

$$\alpha_i := \pi_i^* \alpha \in \Omega^\bullet(C_n(M)) \tag{E.1}$$

$$\beta_{ij} := \pi_i^* \beta \in \Omega^\bullet(C_n(M)) \tag{E.2}$$

Similar statements apply for projections $\pi_I \colon C_n(M) \to C_{|I|}(M)$, where $I \subset \{1, \ldots, n\}$. The orientation on $C_n(M)$ is induced by the product orientation on M^n. The orientation for the boundary is such that Stokes' theorem

$$\int_{C_n(M)} \mathrm{d}\omega = \int_{\partial C_n(M)} \omega \tag{E.3}$$

holds without extra signs.
Similarly, for M a manifold with boundary, we have the open configuration space

$$C_{n,m}(M, \partial M) = \{(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \in M^n \times \partial M^m \colon x_i \neq x_j \text{ for all } i \neq j\}.$$

It also has a FMAS compactification, which we denote by $C_{n,m}(M, \partial M)$ see e.g [Cam+18] and references given there. We use identical notations for the extensions of the projection maps π_i, π_{ij}.

E.1.2 Pushforwards

Since all the projection maps π_I have compact fibers, there is a well-defined notion of pushforward

$$\pi_{I,*} \colon \Omega^\bullet(C_n(M)) \to \Omega^{\bullet - |I| \dim M}(C_{|I|}(M)) \tag{E.4}$$

given by integration along the fibers. Consider now a general fiber bundle

$$F \hookrightarrow E \overset{\pi}{\twoheadrightarrow} B,$$

where F, E, B are manifolds, possibly with boundary. Our convention for orientations for fiber bundles in general is that which is locally $F \times B \overset{\pi}{\to} B$, i.e the pushforward of the volume form vol_F on the fiber times a basic form α equals the basic form, without sign:

$$\pi_*(vol_F \wedge \pi^* \alpha) = \alpha. \tag{E.5}$$

We sometimes denote integrals along the fiber by $\int_F$, in particular, in the case where π is the projection $\pi_{\{i\}^c} \colon C_n(M) \to C_{n-1}(M)$ on all but the i-th point, we will write $\int_{M_i}$, $\int_{x_i \in M}$ or sometimes simply $\int_i$ if there is no danger of confusion. The crucial identity is Stokes' theorem for pushforwards along the fiber (see e.g. [BT94]), which with this convention carries the following signs:

$$\pi_*(\mathrm{d}\omega) = \pi^\partial_*\omega + (-1)^{\dim F}\mathrm{d}\pi_*\omega \tag{E.6}$$

where π^∂ is the fiber bundle whose fibers are the boundaries of the fiber F. If B is a point, we recover Stokes' theorem. If the boundary of F is empty, we get

$$\int_F \mathrm{d} = (-1)^{\dim F}\mathrm{d}$$

which allows the interpretation of $\int_F$ as an object of degree $-\dim F$. All discussions in this subsection can be extended straightforward to the case where M has a boundary.

E.1.3 Integral kernels

The convention for operators $\Omega^\bullet(M) \to \Omega^\bullet(M)$ is that their integral kernels are single forms or currents (see Appendix D) that act on forms from the left. Concretely, if K is a form on $C_2(M)$, then the induced operator $\tilde{K}$ on forms is given by (in three different notations)

$$\tilde{K}[\omega] = \pi_{1,*}(K\pi_2^*\omega) = \int_{x_2\in M} K(x_1,x_2)\wedge\omega(x_2) = \int_2 K_{12}\omega_2 \tag{E.7}$$

where in the last we dropped the wedge product. We will do this very often, according to the fact that the wedge product is the natural product on differential forms which makes them into a commutative algebra.

E.2 The Graded Algebra of Differential forms

E.2.1 Different degrees on differential forms

If M is a smooth manifold, there is a natural identification $C^\infty(T[1]M) = \mathrm{Map}(T[1]M,\mathbb{R}) = \Omega^\bullet(M)$ (see e.g. [CS11]). Similarly, for a graded vector space $\mathcal{N}$, there is a natural identification $\mathrm{Map}(T[1]M,\mathcal{N}) = \Omega^\bullet(M)\otimes\mathcal{N}$. This is a bigraded algebra: An element $A\in\Omega^k(M)\otimes\mathcal{N}$ has a well-defined de Rham form degree k. However, it is also a map from the degree k component of $T[1]M$ to $\mathcal{N}$. In particular, the component A_l of A in $(\mathcal{N})_l$ has the degree $l-k$ as a map from $T[1]M\to\mathcal{N}$. In the AKSZ formalism (see 2.1.7), this degree is called the *ghost number*, and denoted by $gh(A)$. The total degree is the sum of the form degree and the ghost number. Note that in particular if $\mathcal{N}$ is concentrated in degree l, then forms of form degree k have ghost number $l-k$ and total degree l, i.e. all forms have the same total degree. This is the case in

Chern-Simons theory, where the target of the AKSZ construction is $\mathfrak{g}[1]$. By slight abuse of notation, we denote

$$\mathrm{Map}(T[1]M, \mathfrak{g}[1]) = \Omega(M, \mathfrak{g}[1]) = \Omega(M, \mathfrak{g})[1] \tag{E.8}$$

where the last shift acts on the *total degree.*

E.2.2 Lie algebra valued differential forms

E.3 Exterior algebra

We always work in characteristic 0. Denote $T(V)$ the tensor algebra on V and by $I \subset T(V)$ the two-sided ideal spanned by elements of the form $x \otimes x$, where $x \in V$. We identify the exterior algebra

$$\bigwedge(V) := T(V)/I \tag{E.9}$$

with the subspace of *totally antisymmetric tensors*, i.e. the image of the map

$$\begin{aligned} \mathrm{Alt}\colon T(V) &\to T(V) \\ x_1 \otimes \cdots \otimes x_n &\mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \end{aligned}$$

which is extended to all tensors by linearity. Here S_n denotes the symmetric group of a set with n elements. Exterior multiplication of two antisymmetric tensors is given by

$$\alpha \wedge \beta := \mathrm{Alt}(\alpha \otimes \beta) \tag{E.10}$$

This multiplication is graded commutative. We will often use index notation. Let $\alpha \in \bigwedge^k V$. Let $e_1, \ldots, e_n$ be a basis of V. Then we define the components $\alpha^{i_1 \ldots i_k}$ of α by

$$\alpha = \frac{1}{k!} \alpha^{i_1 \ldots i_k} e_{i_1} \otimes e_{i_k}. \tag{E.11}$$

Let α, β be homogeneous tensors of degree k and l respectively. Then we have that

$$\begin{aligned} (\alpha \wedge \beta)^{i_1 \ldots i_{k+l}} = \frac{1}{k!} \frac{1}{l!} \sum_{\sigma \in S_{k+l}} \alpha^{\sigma(i_1) \ldots \sigma(i_k)} \beta^{\sigma(i_{k+1}) \ldots \sigma(i_{k+l})} &= \frac{(k+l)!}{k! l!} \alpha^{[i_1 \ldots i_k} \beta^{i_{k+1} \ldots i_{k+l}]} \\ &= \binom{k+l}{k} \alpha^{[i_1 \ldots i_k} \beta^{i_{k+1} \ldots i_{k+l}]} \end{aligned} \tag{E.12}$$

where

$$\alpha^{[i_1 \ldots i_p]} = \frac{1}{p!} \sum_{\sigma \in S_p} \alpha^{i_{\sigma(1)} \cdots i_{\sigma(p)}} \tag{E.13}$$

is the antisymmetrisation, i.e. the action of the map Alt on tensor components. The factor comes from our convention of excluding the factor from the components.

E.3.1 Exterior algebra of $V \oplus V^*$

In the special case where we consider the exterior algebra $\bigwedge V \oplus V^*$, which we denote by A_V, we have the isomorphism of graded algebras

$$A_V = \bigwedge (V \oplus V^*) \cong \bigwedge V \otimes \bigwedge V^*. \tag{E.14}$$

Here the tensor product is the tensor product of graded algebras, meaning that

$$(\alpha_1 \otimes \alpha_2) \wedge (\beta_1 \otimes \beta_2) = (-1)^{|\alpha_2||\beta_1|} \alpha_1 \wedge \beta_1 \otimes \alpha_2 \wedge \beta_2. \tag{E.15}$$

Again we often use index notation: Picking a basis $e_1, \ldots, e_n$ of V with corresponding dual basis $\varepsilon^1, \ldots, \varepsilon^n$ of V^*, and letting $\alpha \in \bigwedge^p V \otimes \bigwedge^q V^*$, we write

$$\alpha = \frac{1}{p!q!} \alpha^{i_1 \ldots i_p}_{j_1 \ldots j_q}. \tag{E.16}$$

Let $\beta \in \bigwedge^{p'} V \otimes \bigwedge^{q'} V^*$. Then the wedge product in this notation is expressed by

$$(\alpha \wedge \beta)^{i_1 \ldots i_{p+p'}}_{j_1 \ldots j_{q+q'}} = \binom{p+p'}{p} \binom{q+q'}{q} \alpha^{[i_1 \ldots i_p}_{[j_1 \ldots j_q} \beta^{i_{p+1} \ldots i_{p+p'}]}_{j_{q+1} \ldots j'_{q+q}]}. \tag{E.17}$$

The important thing here is the combinatorial factor.

Interior product

The exterior algebra A_V of $V \oplus V^*$ is naturally a *bigraded* algebra via the isomorphism (E.14). We denote the part of bidegree (p, q) by $A^{(p,q)}$.

The algebra A_V carries another operation, called the interior product ι. It is the generalization of the usual interior product to of a form and a multi-vector to our setting, and can be defined as follows.

Definition E.3.1. The *interior product* is the bilinear operation

$$\iota \colon A_V^{(p,q)} \times A_V^{(p',q')} \to A_V^{(p+p'-1, q+q'-1)}$$
$$(\alpha, \beta) \mapsto \iota_\alpha \beta$$

defined by the following properties:

i) If $\alpha \in k$ or $\beta \in k$ then

ii) If $\alpha \in V$, then $\iota_\alpha \beta = 0$,

iii) If $\alpha \in V^*, \beta \in V$, then $\iota_\alpha \beta = \alpha(\beta)$,

iv) ι is a derivation in the first argument:

$$\iota_{\alpha_1\wedge\alpha_2}\beta = \alpha_1 \wedge \iota_{\alpha_2}\beta + (-1)^{|\alpha_1||\alpha_2|}\alpha_2 \wedge \iota_{\alpha_1}\beta, \tag{E.18}$$

v) ι is a derivation of degree $|\alpha|$ in the second argument:

$$\iota_\alpha(\beta_1 \wedge \beta_2) = \iota_\alpha(\beta_1 \wedge \beta_2) + (-1)^{|\alpha||\beta_1}\beta_1 \wedge \iota_\alpha\beta_2. \tag{E.19}$$

In index notation, the antisymmetry of the tensors allows us to write

$$(\iota_\alpha\beta)^{i_1\dots i_{p+p'-1}}_{j_1\dots j_{q+q'-1}} = \binom{p+p'-1}{p}\binom{q+q'-1}{q'}\alpha^{[i_1\dots i_p}_{[j_1\dots j_{q-1}k}\beta^{ki_{p+1}\dots i_{p+p'-1}]}_{j_q\dots j_{q+q'-1}]} \tag{E.20}$$

where by definition we sum over repeated indices and they are excluded from the antisymmetrization. The generalized interior product still satisfies the following property:

Lemma E.3.1. *Let $\alpha, \beta \in A_V$. Then the graded commutator of the the derivations $\iota_\alpha, \iota_\beta$ vanishes. In other words,*

$$\iota_\alpha\iota_\beta = (-1)^{|\alpha||\beta|}\iota_\beta\iota_\alpha. \tag{E.21}$$

Proof. This follows since the graded commutator is a derivation which vanishes on generators of the algebra A_V. □

Contractions

Another operation on A_V is given by the contraction. By definition, the (k,l) contraction of a (not necessarily antisymmetric) (p,q)-tensor t is the $(p-1,q-1)$ tensor $C^{(k,l)}t$ given by

$$(C^{(k,l)}t)^{i_1\dots i_{p-1}}_{j_1\dots j_{q-1}} \mapsto t^{i_1\dots i_{k-1}mi_k\dots i_{p-1}}_{j_1\dots j_{l-1}mi_l\dots i_{q-1}}.$$

On totally antisymmetric tensors all contractions agree up to a sign, and we can define a contration $C\colon A_V^{(p,q)} \to A_V^{(p-1,q-1)}$ by $C = \sum_{k,l}(-1)^{k+l}C^{(k,l)}$. By our convention on the normalization of the coefficients of a totally antisymmetric tensor, this means that

$$(C\alpha)^{i_1\dots i_{p-1}}_{j_1\dots j_{q-1}} = \alpha^{ki_1\dots i_{p-1}}_{kj_1\dots j_{q-1}}. \tag{E.22}$$

Definition E.3.2. Let $\alpha \in A_V^{(p,p)}$. Then the *total contraction* of α, denoted $\langle\alpha\rangle$, is the number $C^p\alpha \in k$.

By the equation (E.22) above, we have that

$$\langle\alpha\rangle = \alpha^{i_1\dots i_p}_{i_1\dots i_p}. \tag{E.23}$$

E.3.2 Shifted exterior algebra of $V \oplus V^*$

In Section 4 on polarized Lie algebras, we consider a shifted version E_V of this bigraded algebra where both degrees are shifted by -1, i.e. an element $\sigma \in \bigwedge^p V \otimes \bigwedge^q V^*$ has bidgree $(p-1, q-1)$. With shift notation we have $E_V = A_V[-2]$ as graded vector spaces or $E_V = A_V[(-1,-1)]$ as bigraded vector spaces. E_V is still a graded commutative algebra with respect to the wedge product.

Lemma E.3.2. *E_V carries a graded Poisson bracket $\{\cdot,\cdot\}$ (of degree 0) defined by*

$$\{\alpha, \beta\} = \iota_\alpha \beta - (-1)^{|\alpha||\beta|} \iota_\beta \alpha \tag{E.24}$$

(here $|\alpha|, |\beta|$ denote the degree in E_V!).

Proof. Itis clear that this bracket is bilinear and graded antisymmetric. By definition of ι, it satisifies the graded Leibniz rule. The Jacobi identity follows from Lemma □

Proposition E.3.3 (([Kos92]))**.** *The bracket $\{\cdot,\cdot\}$ on E_V satisfies the following properties:*

1. *If $\sigma, \sigma' \in k$, then $\{\sigma, \sigma'\} = 0$,*
2. *If $\sigma, \sigma' \in V$, then $\{\sigma, \sigma'\} = 0$,*
3. *If $\sigma, \sigma' \in V^*$, then $\{\sigma, \sigma'\} = 0$,*
4. *If $\sigma \in V, \sigma' \in V^*$, then $\{\sigma, \sigma'\} = \sigma'(\sigma)$,*
5. *$\{\cdot,\cdot\}$ satisfies a graded Leibniz rule with respect to the wedge product and the total degree on E_V, i.e.*

$$\{\sigma, \sigma' \wedge \sigma''\} = \{\sigma, \sigma'\} \wedge \sigma'' + (-1)^{|\sigma||\sigma'|} \sigma' \wedge \{\sigma, \sigma''\}. \tag{E.25}$$

Proof. This follows immediately from the properties of the interior product. □

Proposition E.3.3 implies that this Poisson bracket coincides with the Poisson bracket of the canonical symplectic structure by identifying $E_V = C^\infty(\Pi T^* V)$, where Π denotes the parity shift (both in the base and the fibers), since the two brackets agree on generators.

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