Views on the J-homomorphism

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K-theory and spherical fibrations

Recall: $KO(X) := \{\mathbb{R}\text{-vector bundles over } X, \oplus\}^{gp}$. Similarly,

Definition

$Sph(X) := \{\text{sectioned spherical fibrations over } X, \wedge_X\}^{gp}$. These are fibrations

$$\begin{array}{ccc} S^n & \longrightarrow & E \\ & & p\downarrow\ \uparrow\sigma \\ & & X \end{array} \qquad n \in \mathbb{N}.$$

Examples:

1. Trivial: $S^n \times X$.
2. Hopf: $\eta : S^3 \to S^2$ is *not* sectioned.
3. Unit sphere bundles: If $W \to X$ is a vector bundle, then $S(W) \to X$ is a spherical fibration. If W admits a nowhere-vanishing section (i.e., $W \cong V \oplus \mathbb{R}$), then $S(W)$ is sectioned.

Note: if V is a vector bundle, then $S(V \oplus \mathbb{R}) \to X$ is the fibrewise 1-point compactification of V.

J-homomorphism and representing spaces

Definition

The *J-homomorphism*

$$J : KO(X) \to Sph(X)$$

sends V to $S(V \oplus \mathbb{R})$.

Recall: KO is represented by $BO \times \mathbb{Z}$, since $O = \lim_n O(n)$ is the (stable) structure group for $\mathbb{R}$-vector bundles. That is, if X is compact,

$$KO(X) \cong [X, BO \times \mathbb{Z}].$$

Similarly: Sph is represented by $BF \times \mathbb{Z}$ for $F = \lim_n F(n)$, where

$$F(n) = \mathrm{hAut}_*(S^n) = \{f : S^n \to S^n, f \text{ is a based homotopy equivalence}\}$$

This is an associative monoid under composition of functions; $\pi_0 F(n) \cong \{\pm 1\}$.

Then J is represented by $J : BO \times \mathbb{Z} \to BF \times \mathbb{Z}$, induced by $O(n) \to F(n)$:

$$(M : \mathbb{R}^n \to \mathbb{R}^n) \mapsto (M \cup \{\infty\} : S^n \to S^n).$$

Unit spectra

Definition

If R is an E_∞ ring spectrum, define the unit space $\mathrm{GL}_1 R$ as the union of components of $\Omega^\infty R$ associated to $(\pi_0 R)^\times \subseteq \pi_0 R$.

$\mathrm{GL}_1 R$ is an E_∞-space with multiplication coming from the product on R. The unit spectrum $\mathrm{gl}_1 R$ is the connective spectrum with $\Omega^\infty \mathrm{gl}_1 R = \mathrm{GL}_1 R$.

Note: For $n \geq 1$, $\pi_n \mathrm{gl}_1 R = \pi_n \mathrm{GL}_1 R \cong \pi_n \Omega^\infty R = \pi_n R$.

Example

$R = S^0$. Then $\pi_0 S^0 = \mathbb{Z} \supseteq \{\pm 1\} = (\pi_0 S^0)^\times$. The zeroth space of the spectrum is $QS^0 = \lim_n \Omega^n S^n$.

$$\mathrm{GL}_1 S^0 = Q_{\pm 1} S^0 = \lim_{n \to \infty} \Omega^n_{\pm 1} S^n = \lim_{n \to \infty} F(n) = F.$$

The products on $\mathrm{GL}_1 S^0$ and F are *not* the same (smash product vs. composition), but do commute, so $B\,\mathrm{GL}_1 S^0 \simeq BF$:

$$Sph_{>0}(X) = [X, BF] = [X, B\,\mathrm{GL}_1 S^0] = [\Sigma^\infty X, \Sigma\,\mathrm{gl}_1 S^0].$$

Picard spectra

Let R be an E_∞-ring spectrum, and $(\mathrm{Mod}_R, \wedge_R)$ be the associated symmetric monoidal ∞-category of its (right) module spectra.

Definition (Ando-Blumberg-Gepner)

The **Picard space** $\mathrm{Pic}(R) \subseteq \mathrm{Mod}_R$ is the full subgroupoid spanned by the modules M which invertible with respect to $\wedge_R$. This is a grouplike E_∞ space; the **Picard spectrum** $\mathrm{pic}(R)$ is the associated connective spectrum.

Note: R is the unit of $\otimes_R$, so take $R \in \mathrm{Pic}(R)$ as a basepoint. Then

$$\Omega\,\mathrm{Pic}(R) = \mathrm{Aut}_R(R) = \mathrm{GL}_1(R)$$

In fact, this gives a connected cover $\Sigma\,\mathrm{gl}_1(R) \to \mathrm{pic}(R)$.

The J-homomorphism: in this language is

$$\begin{array}{ccc} bo & \xrightarrow{J} & \Sigma\,\mathrm{gl}_1(S^0) \\ \downarrow & & \downarrow \\ ko & \xrightarrow{J} & \mathrm{pic}(S^0) \end{array}$$

(**Here:** $bo = ko_{>0}$ is the connected cover)

induced by $\mathbb{R}$-Vect $\to \mathrm{Pic}(S^0)$, where $V \mapsto V \cup \{\infty\}$.

Remark: Dustin Clausen has formulated an analogous $(K\mathbb{Q}_p)_{>1} \to \mathrm{pic}(S^0)$.

Image in homotopy

Theorem (Bott periodicity)

k mod 8	1	2	3	4	5	6	7	8
$\pi_k ko$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

The induced map $\pi_* J : \pi_* ko \to \pi_* \operatorname{pic}(S^0) \cong \pi_{*-1}(S^0)$ for $* > 0$ is known:

Theorem (Adams, Quillen)

$\pi_* J$ is an injection if $* = 1, 2 \bmod 8$. Further, in dimension $* = 4n$, $\operatorname{im}(\pi_* J)$ is $\mathbb{Z}/m$ where m is the denominator of $B_{2n}/4n$.

Here, the Bernoulli numbers satisfy

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

Summary of p-torsion: ($p > 2$): If $* = 2(p-1)p^k m$, where m is coprime to p,

$${}_p\operatorname{im}(\pi_* J) = \mathbb{Z}/p^{k+1}$$

Otherwise, ${}_p\operatorname{im}(\pi_* J) = 0$. If $* = 2^k m$ with m odd, then ${}_2\operatorname{im}(\pi_* J) = \mathbb{Z}/2^{k+1}$.

Adams conjecture

For $k \in \mathbb{N}$, the k^{th} *Adams operation* is a natural transformation

$$\psi^k : KO(X) \to KO(X).$$

Properties:

1. For line bundles L, $\psi^k(L) = L^{\otimes k}$.
2. Each ψ^k is a ring homomorphism.
3. $\psi^k \circ \psi^\ell = \psi^{k\ell}$.

These are represented by maps $\psi^k : BO \to BO$.

Theorem (Quillen, Sullivan, Friedlander)

For a finite CW complex X and $V \in KO(X)$, there exists $e = e(k, V)$ so that $k^e J(V) = k^e J(\psi^k(V)) \in Sph(X)$.

Equivalently, on finite skeleta, the composite map

$$BO \xrightarrow{\psi^k - 1} BO \xrightarrow{J} BF \xrightarrow{loc} BF[\tfrac{1}{k}]$$

is null-homotopic. There exists a complex analogue (for BU), too.

Image of J space/spectra

Definition

For $p = 2$: Let $J_{(2)}$ be the homotopy fibre of the map

$$\psi^3 - 1 : BO_{(2)} \to BSpin_{(2)}.$$

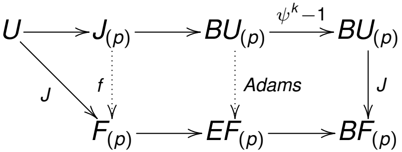
For $p > 2$: choose $k \in \mathbb{N}$ so that k mod p^2 is a generator of $(\mathbb{Z}/p^2)^\times$, and define $J_{(p)}$ to be the homotopy fibre of the map

$$\psi^k - 1 : BU_{(p)} \to BU_{(p)}$$

Write $j_{(2)}$ (respectively $j_{(p)}$) for the associated (ring) spectra. The unit of ko or ku lifts to $e : S^0 \to j_{(p)}$. This gives

$$e : SF \simeq Q_0 S^0 \to J_{(p)}.$$

The Adams conjecture gives us a commuting diagram of fibre sequences:



Note: $k \in \mathbb{Z}_{(p)}^\times$.

Computing the image of J in homotopy

Theorem (Mahowald; May-Tornehave)

The maps e and f split $J_{(p)}$ off of $Q_0 S^0_{(p)}$.

So: the p-torsion in $\mathrm{im}(\pi_* J : \pi_* ko \to \pi_{*-1} S^0)$ is isomorphic to $\pi_{*-1} J_{(p)}$:

$$\cdots \longrightarrow \pi_* J_{(p)} \longrightarrow \pi_* BU_{(p)} \xrightarrow{\psi^k - 1} \pi_* BU_{(p)} \longrightarrow \pi_{*-1} J_{(p)} \longrightarrow \cdots$$

Now, $\pi_* BU = \mathbb{Z}[\beta]$, where $\beta \in \pi_2 BU$ is the Bott periodicity class. Compute: $\psi^k(\beta) = k\beta$, so if $* = 2n$, this is

$$\cdots \longrightarrow \pi_{2n} J_{(p)} \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{k^n - 1} \mathbb{Z}_{(p)} \longrightarrow \pi_{2n-1} J_{(p)} \longrightarrow \cdots$$

So for $n > 0$, $\pi_{2n} J_{(p)} = 0$, and

$$\pi_{2n-1} J_{(p)} = \mathbb{Z}_{(p)}/(k^n - 1) = \begin{cases} 0, & n \neq (p-1)p^s m \\ \mathbb{Z}/p^{s+1}, & n = (p-1)p^s m \end{cases}$$

Recall that k mod p^2 generates $(\mathbb{Z}/p^2)^\times$. Then:

- $k^n - 1$ is a unit in $\mathbb{Z}_{(p)}$ when $k^n \neq 1 \bmod p \iff (p-1) \nmid n$.
- Further, $k^{(p-1)} \in 1 + p\mathbb{Z}_{(p)}$, so $k^{(p-1)p^s m} \in 1 + p^{s+1}\mathbb{Z}_{(p)}$.

Algebraic K-theory of finite fields

Let $q = p^m$, and define $F\psi^q$ to be the homotopy fibre of $\psi^q - 1 : BU \to BU$.

Quillen used Brauer theory to lift the defining representation of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$ to a virtual complex representation, yielding a map

$$B\,\mathrm{GL}_n(\mathbb{F}_q) \to BU$$

Action of ψ^q on $B\,\mathrm{GL}_n(\mathbb{F}_q)$ is the q-Frobenius so this lifts to $F\psi^q$. In the limit:

Theorem (Quillen)

The map $\Omega^\infty K(\mathbb{F}_q) = B\,\mathrm{GL}_\infty(\mathbb{F}_q)^+ \to F\psi^q$ is an equivalence. Hence

$$K_n(\mathbb{F}_q) = \begin{cases} 0, & n = 2i \\ \mathbb{Z}/(q^i - 1), & n = 2i - 1 \end{cases}$$

Interpretation: Let ℓ be prime, and pick $q = p^m$ so that q mod ℓ^2 is a generator of $(\mathbb{Z}/\ell^2)^\times$. Then from Suslin's theorem:

$$\begin{array}{ccccc} j_\ell^\wedge & \longrightarrow & ku_\ell^\wedge & \xrightarrow{\psi^q - 1} & ku_\ell^\wedge \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\ K(\mathbb{F}_q)_\ell^\wedge & \longrightarrow & K(\overline{\mathbb{F}}_q)_\ell^\wedge & \xrightarrow{\psi^q - 1} & K(\overline{\mathbb{F}}_q)_\ell^\wedge \end{array}$$

Note: This exhibits $K(\mathbb{F}_q)_\ell^\wedge$ as the homotopy fixed points $(K(\overline{\mathbb{F}}_q)_\ell^\wedge)^{h\,\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$.

$K(1)$-local homotopy

Let

$$K_1 := KU/p = \vee_{i=0}^{p-2} \Sigma^{2i} K(1),$$

Pick $k \in \mathbb{Z}$ which generates $(\mathbb{Z}/p^2)^\times$, and define $\mathbb{J}_p$ by the fibre sequence

$$\mathbb{J}_p \longrightarrow KU_p^\wedge \xrightarrow{\psi^k - 1} KU_p^\wedge$$

Theorem

The unit map $e : S^0 \to \mathbb{J}_p$ is an isomorphism in $K(1)_*$, so $\mathbb{J}_p \simeq L_{K(1)} S^0$.

Here $L_{K(1)} S^0$ is the Bousfield localization of S^0 at $K(1)$.

Idea: Compute $K(1)_* KU_p = C(\mathbb{Z}_p^\times, \mathbb{F}_p)$, and the action of ψ^k is by translation by $k \in \mathbb{Z}_p^\times$. Since $\langle k \rangle \leq \mathbb{Z}_p^\times$ is dense, fixed functions are constants = im(e_*).

Conclusion: the localization map $S^0 \to L_{K(1)} S^0$ carries

$$\text{im}(\pi_* J) \cong \pi_*(\mathbb{J}_p), \quad * > 0$$

isomorphically onto $\pi_* L_{K(1)} S^0$ in positive degrees.

Note: This presents $L_{K(1)} S^0$ as the homotopy fixed point spectrum $(KU_p^\wedge)^{h\mathbb{Z}_p^\times}$ for an action of $\mathbb{Z}_p^\times$ by a p-adic extension of the Adams operations.

Morava K and E-theories

Definition

- Let E_n denote the Morava E-theory associated to the Lubin-Tate deformation space of the formal group Γ_n over $\mathbb{F}_{p^n}$ with $[p](x) = x^{p^n}$.
- The Morava stabilizer group is $\mathbb{G}_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathrm{Aut}(\Gamma_n)$.
- The Morava K-theories are $K_n = E_n/\mathfrak{m}$, and $K(n) = K_n^{h\,\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$.

Theorem (Morava, Goerss-Hopkins-Miller, Devinatz-Hopkins, Behrens-Davis)

$\mathbb{G}_n$ acts on E_n in such a way that $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$.

There exists a reduced norm $\det_\pm : \mathbb{G}_n \to \mathbb{Z}_p^\times$ coming from the determinant of the action of $\mathbb{G}_n$ on $\mathrm{End}(\Gamma_n)$. Define

- $S\mathbb{G}_n^\pm := \ker(\det_\pm)$, and
- $R_n := E_n^{hS\mathbb{G}_n^\pm}$: determinantal K-theory, half the sphere, or the Iwasawa extension of $L_{K(n)}S^0$.

Then, for a topological generator $k \in \mathbb{Z}_p^\times$, there is a fibre sequence

$$L_{K(n)}S^0 = (E_n^{hS\mathbb{G}_n^\pm})^{h\mathbb{Z}_p^\times} \longrightarrow R_n \xrightarrow{\psi^k - 1} R_n$$

Higher chromatic analogues

Define $S\langle \det_\pm \rangle = \mathrm{hofib}(\psi^k - k)$. Then $S\langle \det_\pm \rangle \in \mathrm{Pic}_n = \mathrm{Pic}(L_{K(n)}\mathrm{Spectra})$, and

$$(E_n)_* S\langle \det_\pm \rangle \cong (E_n)_*[\det_\pm].$$

When $n = 1$, $S\langle \det_\pm \rangle = L_{K(1)} S^2$.

Theorem (W.)

There exists an essential $\rho_n : S\langle \det_\pm \rangle \to R_n$ which is invertible in $\pi_\star R_n$. Further, the action of $\mathbb{Z}_p^\times$ on the summand

$$\mathbb{Z}_p\{\rho_n^j\} \subseteq [S\langle \det_\pm \rangle^{\otimes j}, R_n]$$

is by j^{th} power of identity character.

Related work of Eric Peterson gives a more algebro-geometric perspective.

Consequently, the same computation for $\pi_* L_{K(1)} S^0$ gives us:

Corollary

There exists a subgroup $\mathbb{Z}/p^{s+1} \subseteq [S\langle \det_\pm \rangle^{\otimes (p-1)p^s m}, L_{K(n)} S^1]$ for m coprime to p.

References

Original formulation:

- G.W. Whitehead:
  On the homotopy groups of spheres and rotation groups.
- M. F. Atiyah: *Thom complexes*.
- J.F. Adams: *On the groups $J(X)$, I – IV*.

Unit spectra:

- J. P. May, F. Quinn, N. Ray, J. Tornehave:
  E_∞ ring spaces and E_∞ ring spectra.
- M. Ando, A. Blumberg, D. Gepner, M. Hopkins, and C. Rezk:
  Units of ring spectra and Thom spectra.
- J. P. May: *What are E_∞ ring spaces good for?*.

Picard spectra:

- M. Ando, A. Blumberg, D. Gepner:
  Parametrized spectra, multiplicative Thom spectra, and the twisted Umkehr map.

References

Adams conjecture and algebraic K-theory of finite fields:

- D. Quillen: *The Adams Conjecture*, and
  On the Cohomology and K-Theory of the General Linear Groups Over a Finite Field.
- D. Sullivan: *Genetics of homotopy theory and the Adams conjecture*.
- E. Friedlander: *Fibrations in etale homotopy theory*.
- W. G. Dwyer: *Quillen's work on the Adams conjecture*.

Image of J space:

- J. P. May, F. Quinn, N. Ray, J. Tornehave:
  E_∞ *ring spaces and* E_∞ *ring spectra*.
- M. Mahowald: *The order of the image of the J-homomorphism*.
- H. R. Miller, D. Ravenel:
  Mark Mahowald's work on the homotopy groups of spheres.

References

Morava K and E-theory:

- J. Morava:
  Noetherian Localisations of Categories of Cobordism Comodules.
- P. Goerss, M. Hopkins:
  Moduli Spaces of Commutative Ring Spectra.
- E. Devinatz, M. Hopkins:
  Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups.
- M. Behrens, D. Davis:
  The homotopy fixed point spectra of profinite Galois extensions.

Determinantal K-theory:

- E. Peterson:
  Annular decomposition of coalgebraic formal variety spectra.
- C. Westerland:
  A higher chromatic analogue of the image of J.