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String Compactification

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An introduction to compactifications of heterotic strings, IIB, and F-theory.

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1. Heterotic compactification

1.1. Kaluza-Klein Ansatz

Today we are going to focus on the heterotic string. All superstring theories, including the heterotic string, naturally live in ten space-time dimensions. The ten-dimensional fields are as follows:

$$\begin{aligned} \text{gravity multiplet :} \quad & g_{MN}, B_{MN}, \phi, \psi_{M\alpha}, \chi_{\dot\beta} \\ \text{gauge multiplet :} \quad & A_M, \lambda_\alpha \end{aligned} \tag{1.1}$$

We adopted the convention of using capital letters M, N, $P, \ldots$ to denote ten-dimensional space-time indices $0, \ldots, 9$. We use α to denote indices in the ten-dimensional positive chirality spinor representation $\mathbf{16_R}$ (or $\mathbf{16}$ if we want to omit the Majorana condition), and similarly we use $\dot\beta$ for the negative chirality $\mathbf{16'_R}$. The effective ten-dimensional action is of the form

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{|g|}e^{-2\phi}\left[R - \frac{1}{3}H^2 + 4(\nabla\phi)^2 + \frac{\alpha'}{30}\mathrm{Tr}(F^2)\right] + S_{Fermi}(\psi, \lambda, \chi, A, g, B, \phi) \tag{1.2}$$

and the field strength H can be defined at least locally as

$$H = dB + \alpha'\,\omega_L - \frac{\alpha'}{30}\omega_{YM}. \tag{1.3}$$

Here ω_L and ω_{YM} denote the Chern-Simons forms for the spin and Yang-Mills connections, respectively.

The symmetry group of this lagrangian is rather big: we have ten-dimensional super-Poincaré invariance and an $E_8 \times E_8$ or $SO(32)$ gauge group. We are interested in more realistic theories with only four-dimensional Poincaré invariance at large distance scales, and a smaller gauge group. To break the symmetries we will turn on profiles for the ten-dimensional fields.

A good Ansatz is to consider a ten-dimensional background of the form

$$M_{1,3} \times X_6 \tag{1.4}$$

where X_6 is a compact six-dimensional real manifold, and $M_{1,3}$ denotes four-dimensional Minkowski space. As in the original Kaluza-Klein model, at scales large compared to the inverse radius of curvature of X_6, we effectively reduce to a four-dimensional theory.

Which manifold should we take for X_6? The first possibility we might consider is a straightforward generalization of the classic Kaluza-Klein Ansatz: we simply take $X_6 = T^6$. One can do the Kaluza-Klein reduction very explicitly in this case. A toroidal compactification is often referred to as a Narain compactification, and admits an elegant formulation in terms of an even self-dual momentum lattice Γ. For the heterotic string on T^6, the compactification is specified by a Lorentzian lattice $\Gamma^{6,22}$. The space of such lattices (and hence the effective four-dimensional theory) is parametrized by a coset

$$\mathcal{M} = O(6,22;\mathbf{Z})\backslash O(6,22;\mathbf{R})/O(6;\mathbf{R}) \times O(22;\mathbf{R}) \tag{1.5}$$

parametrizing deformations of internal components of g_{MN}, B_{MN} and A_M. Such compactifications have a lot of interesting properties, but for our purposes they are a bit too restrictive. To understand why let us look at the Kaluza-Klein reduction of the ten-dimensional supersymmetry transformations.

Supersymmetry transformation are parametrized by a spinor ϵ_α in the $\mathbf{16_R}$. Under $SO(1,3) \times SO(6)$ a Weyl spinor decomposes as

$$\mathbf{16_R} \;\to\; (\mathbf{2},\mathbf{4}) + (\overline{\mathbf{2}},\overline{\mathbf{4}}) \tag{1.6}$$

Since a torus is simply a quotient of flat space by some translations, spinors which are independent of the internal coordinates will be solutions, so we end up with four independent supersymmetry transformations in four dimensions. So the $N = 1$ supersymmetry in ten dimensions reduces to $N = 4$ supersymmetry in four dimensions. One cannot get a chiral theory from breaking $N \geq 2$ four-dimensional supersymmetry. This is perhaps a little too much of a good thing.

In these lectures we want to focus on compactifications to four dimensions with only $N = 1$ supersymmetry. There is actually a simple way to get an $N = 1$ compactification out of the above: we simply quotient out by a $\mathbf{Z}_3$ symmetry which kills three of the four spinors and leaves one invariant. That is, we take X_6 to be an orbifold $T^6/\mathbf{Z}_3$. This is not the route we are going to take here, although there is a definite connection: one can resolve the fixed points of the $\mathbf{Z}_3$ action to get a smooth manifold, of the general type that we will consider.

1.2. BPS equations

To get a handle on this, we consider the supersymmetry variations of the ten-dimensional fields. The variation of the fermions is given by

$$\begin{aligned} \delta\psi_M &= \nabla_M\epsilon - \frac{1}{4}H_{MAB}\Gamma^{AB}\epsilon + (Fermi)^2 \\ \delta\lambda &= F_{MN}\Gamma^{MN}\epsilon + (Fermi)^2 \end{aligned}$$

$$\delta\chi = \not{\nabla}\phi\epsilon + \frac{1}{24}H_{MNP}\Gamma^{MNP}\epsilon + (Fermi)^2 \tag{1.7}$$

There are similar equations for the bosons, of the form $\delta(Bose) \sim Fermi \times \epsilon$. However to preserve the four-dimensional Poincaré symmetry, the Fermi fields must vanish. Then to get a four-dimensional supersymmetry generator, we have to find a field configuration for g, B, ϕ on X_6 and a spinor ϵ such that $\delta\psi = \delta\lambda = \delta\chi = 0$. We will further assume that H vanishes and ϕ is constant. The equations become

$$\nabla_M\epsilon = 0, \qquad F_{MN}\Gamma^{MN}\epsilon = 0 \tag{1.8}$$

So let us analyze these equations.

We first decompose ϵ according to (1.6) as

$$\epsilon = \epsilon_{4+} \otimes \epsilon_{6+} + \epsilon_{4-} \otimes \epsilon_{6-} \tag{1.9}$$

where $\epsilon_{4-} = \epsilon_{4+}^*$ and $\epsilon_{6-} = \epsilon_{6+}^*$. (It would be more accurate to write $\epsilon_{1,3}$ instead of ϵ_4, but we are saving notation here). Taking M to be an internal index and decomposing into chiral parts, we get the equation

$$\nabla_m\epsilon_{6\pm} = 0. \tag{1.10}$$

This equation means that X_6 must admit a covariantly constant spinor. By Leibnitz' rule, such a spinor will have constant norm, and we can normalize such that the norm is one everywhere. We will assume this in the following.

1.3. Consequences of $\nabla_m\epsilon_{6\pm} = 0$

The existence of a covariantly constant spinor is a strong condition on X_6, and in this subsection we will spell out the consequences. Suppose that we have such a spinor, and let us consider

$$J_m^{\ n} = i\,\epsilon_{6+}^\dagger\Gamma_m^{\ n}\epsilon_{6+} \tag{1.11}$$

where $\Gamma_m^{\ n} = \frac{1}{2}(\Gamma_m\Gamma^n - \Gamma^n\Gamma_m)$. With some algebra, one shows that $J_m^{\ n}$ squares to minus one (i.e. $J_m^{\ n}J_n^{\ p} = -\delta_m^p$). This suggests we should introduce a complex coordinate system z^j, with conjugates $z^{\bar{j}}$, $j = 1, 2, 3$, such that $J_m^{\ n}$ takes the standard form

$$J_j^{\ k} = i\,\delta_j^k, \qquad J_{\bar{j}}^{\ \bar{k}} = -i\,\delta_{\bar{j}}^{\bar{k}} \tag{1.12}$$

But this is generally not possible. Although it can be done at any given point, we may not be able to extend it to an open neighbourhood around that point. Basically if we could construct such a coordinate system, then we would have a set of tangent vector fields that have eigenvalue $+i$ under J and whose Lie bracket preserves that condition. This integrability condition corresponds to the vanishing of a tensor (the anti-holomorphic projection of this Lie bracket), which may be written as the *Nijenhuis tensor*:

$$N_{mn}{}^{p} = J_m{}^q \nabla_q J_n{}^p - J_m{}^q \nabla_n J_q{}^p - J_n{}^q \nabla_q J_m{}^p + J_n{}^q \nabla_m J_q{}^p \tag{1.13}$$

Conversely, the Newlander-Nirenberg theorem states that when this integrability condition is satisfied, then local holomorphic coordinate systems do exist and we have a complex manifold.

Now our $J_m{}^n$ is actually covariantly constant (again by Leibnitz' rule), so we see that in our case the Nijenhuis tensor vanishes and $J_m{}^n$ defines an integrable complex structure. In other words, X_6 must be a *complex manifold*.

It will often be convenient to use complex notation. We will generally use the indices i, j, k when we work in a complex coordinate system. The complexified tangent bundle splits into a holomorphic tangent bundle spanned by $\partial/\partial z^i$ and an anti-holomorphic tangent bundle spanned by $\partial/\partial z^{\bar{i}}$. Similarly the complexified cotangent bundle splits into a holomorphic version spanned by dz^i and an anti-holomorphic version spanned by $dz^{\bar{i}}$. When X is a complex manifold, we will take TX to mean the holomorphic tangent bundle and T^*X to mean the holomorphic cotangent bundle. By the same token, we use $\overline{TX}$ and $\overline{T^*X}$ to denote their anti-holomorphic versions.

We further claim that g_{mn} is a Hermitian metric with respect to the complex structure, in other words that $J_m{}^p J_n{}^q g_{pq} = g_{mn}$. To see this, first note that from (1.11) we get $J_m{}^p g_{pq} = -J_q{}^p g_{pm}$, and hence $J_m{}^p J_n{}^q g_{pq} = -J_q{}^p J_n{}^q g_{pm} = \delta_n{}^p g_{pm} = g_{nm}$. In holomorphic coordinates and using (1.12) this is equivalent to

$$g_{ij} = g_{\bar{i}\bar{j}} = 0 \tag{1.14}$$

i.e. g only has mixed components of the form $g_{i\bar{j}}$ and $g_{\bar{i}j}$. Next we consider the associated two-form

$$J_{mn} = J_m{}^q g_{qn} \tag{1.15}$$

We already noted that anti-symmetry follows from (1.11). In complex notation, this is simply $J_{i\bar{j}} = i g_{i\bar{j}} = -J_{\bar{j}i}$ and $J_{ij} = J_{\bar{i}\bar{j}} = 0$. This two-form is closed, again due to covariance constancy. It is called a Kähler form, and by definition a complex manifold with a Hermitian metric whose Kähler form is closed is called a *Kähler manifold*.

From $\nabla_m \epsilon = 0$ it also follows that

$$[\nabla_m, \nabla_n]\epsilon = \frac{1}{4} R_{mnpq} \Gamma^{pq} \epsilon = 0 \tag{1.16}$$

Contracting with Γ^n and using $\Gamma^n \Gamma^{pq} = \Gamma^{npq} + g^{np}\Gamma^q - g^{nq}\Gamma^p$, we get that $R_{mnpq}\Gamma^{npq}\epsilon - 2R_{mq}\Gamma^q \epsilon = 0$. But $R_{mnpq}\Gamma^{npq} = 0$ by the first Bianchi identity, so $R_{mq}\Gamma^q\epsilon = 0$ and hence $R_{mn} = 0$, and we find that X_6 must be Ricci flat.

On any Kähler manifold, from $\nabla J = 0$ one finds that $J_m^n R^m_{pqr} = J_p^s R^n_{sqr}$. Contracting with g^{qr} and using (1.12) we see that in complex coordinates the only non-vanishing components of the Ricci tensor are of type $(1,1)$. In fact with a bit more calculation one finds that we can locally write

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g) \tag{1.17}$$

Now just as we constructed a two-form from $g_{i\bar{j}}$, so too we construct a two-form from $R_{i\bar{j}}$, the Ricci form $R_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$. It is closed by virtue of the second Bianchi identity, and hence defines a cohomology class, the *first Chern class* of our Kähler manifold:

$$c_1(TX) = \frac{1}{2\pi}[R_{i\bar{j}} dz^i \wedge dz^{\bar{j}}] \ \in \ H^2_{dR}(X_6, \mathbf{R}) \tag{1.18}$$

The first Chern class is usually defined slightly differently, but one can show it is equivalent to the above. Since we found that X_6 must be Ricci flat, $c_1(TX)$ vanishes. A Kähler manifold with $c_1(TX) = 0$ is called a *Calabi-Yau manifold.* So we found that our X_6 should be a Calabi-Yau three-fold. Note that for complex manifolds we typically count the complex rather than the real dimension.

The condition that X_6 is Calabi-Yau is thus clearly necessary to solve the system $\nabla_m \epsilon_{6\pm} = 0$, but could it also be sufficient? It turns out that it is, but before we get to it, we can gain a better understanding by looking at this from the point of view of holonomies.

Recall that given a tangent vector $v \in T_pM$ and a loop $\gamma(t)$ based at p, we can parallel transport v around the loop and get another vector $v' \in T_pM$ by solving the ODE $\nabla_{\dot{\gamma}(t)} v(t) = 0$ along $\gamma(t)$. On a Riemannian manifold they are related as $v' = Rv$ for a rotation $R \in SO(d)$. By considering general loops, these rotations generate a subgroup $H \subset SO(d)$, the holonomy group. In general, if there is no extra structure, then H is all of $SO(d)$. If the loop is very small, enclosing an infinitesimal area element δA^{mn}, then by a local calculation one finds that the holonomy around the loop is $\delta_p{}^q + \delta A^{mn} R_{mnp}{}^q$, so curvature and holonomy are closely related.

We can similarly parallel transport spinors. We have $Spin(6) = SU(4)$, and the positive chirality spinor representation of $SO(6)$ is identified with the fundamental representation of $SU(4)$, usually denoted as $\mathbf{4}$. Now a covariantly constant spinor comes back to itself after parallel transport (since clearly $\nabla_m \epsilon = 0$ implies $\nabla_{\dot{\gamma}(t)} \epsilon = 0$). Using an $SU(4)$ transformation, we can write this spinor in the form

$$(0, 0, 0, *) \ \in \ \mathbf{4} \tag{1.19}$$

Thus the stabilizer, the subgroup of $SU(4)$ that fixes such a spinor, is $SU(3) \subset SU(4)$, acting on the first three components. So the existence of a covariantly constant spinor is the statement that there must exist a metric with $SU(3)$ holonomy on X_6.

On a Kähler manifold the holonomy reduces to $U(3) = SU(3) \times_{Z_3} U(1)$. This is the subgroup of $SO(6)$ that preserves the Kähler form. The $U(1)$ is the determinant of $U(3)$. Now vectors in the cotangent bundle transform as the **3** of $U(3)$, so the top exterior power $\Lambda^3 T^*X$, whose sections are locally of the form

$$dz^1 \wedge dz^2 \wedge dz^3 \tag{1.20}$$

transforms as the determinant. The top exterior power of T^*X appears so often that it has a special name; it is also called the canonical bundle and denoted as K_X. So the $U(1)$ part of the $U(3)$ holonomy on a Kähler manifold comes from parallel transporting $(3,0)$ forms, and by the relation between holonomy and curvature, the holonomy reduces to $SU(3)$ when the curvature of $K_X = \Lambda^3 T^*X$ vanishes.

Indeed, we can use our covariantly constant spinor to write down a global section:

$$\Omega^{3,0}_{ijk} \;=\; \epsilon^T_{6+}\Gamma_{ijk}\epsilon_{6+} \tag{1.21}$$

This is referred to as the 'holomorphic three-form' or sometimes the 'holomorphic volume form'. It is nowhere vanishing, so it explicitly trivializes $\Lambda^3 T^*X$, and hence $c_1(\Lambda^3 T^*X) = -c_1(TX) = 0$, as we derived previously. In fact we can calculate the curvature of K_X. It is given by $R_{mnpq}J_r^{\ q}g^{pr}$, which yields the Ricci form $R_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$, tying everything together.

The conclusion is that we will have solved $\nabla_m \epsilon_{6\pm} = 0$ if we can show that the holonomy is reduced to $SU(3)$, which will be true if we can show that our Kähler manifold admits a Ricci flat metric. Now finding a Ricci flat metric explicitly is extremely hard – effectively we have to solve a complicated non-linear PDE of the form $R_{i\bar{j}} \sim \partial_i \bar{\partial}_{\bar{j}} \log\det g = 0$. Checking the necessary conditions that X is a Kähler manifold with $c_1(TX) = 0$ would clearly be much simpler, but it seems like a weaker condition, a topological condition instead of a differential geometric condition. This is where a famous conjecture of E. Calabi comes in: the claim is that the vanishing of $c_1(TX)$ on a Kähler manifold is also sufficient for the existence of Ricci flat metrics. The conjecture was settled in a Fields Medal-worthy theorem by S.T.Yau:

> *A compact Kähler manifold X with $c_1(TX) = 0$ admits a Ricci flat metric. The metric is uniquely determined by the complex structure and the Kähler class.*

In honour of this theorem, such Ricci flat metrics are called Calabi-Yau metrics.

Let us summarize what we have seen. We were looking for compactifications of the heterotic string to four dimensions with $N = 1$ supersymmetry. Using the Kaluza-Klein Ansatz $M_{1,9} = M_{1,3} \times X_6$, we found that X_6 should admit a covariantly constant spinor. We then further saw that all the following statements about X_6 are equivalent:

1. $\nabla_m \epsilon_{6+} = 0$ (existence of a covariantly constant spinor);

2. X_6 admits a metric of $SU(3)$ holonomy;

3. X_6 is a Kähler manifold with a Ricci flat metric;

4. X_6 is a Calabi-Yau manifold (a Kähler manifold with $c_1(TX) = 0$).

In particular, to construct examples we can use formulation number four, which is the easiest to deal with. In order to satisfy our BPS equations we still have to solve $F_{MN}\Gamma^{MN}\epsilon = 0$. We will set $F = 0$ for now, and get back to this later.

1.4. Examples of Calabi-Yau manifolds

At this stage it is good to look at some examples. Everybody's favourite example of a Kähler manifold is complex projective space: $\mathbf{CP}^n = (\mathbf{C}^n \backslash \{0\})/\mathbf{C}^*$, where the $\mathbf{C}^*$ acts as

$$(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n), \qquad \lambda \in \mathbf{C}^* \tag{1.22}$$

There is a natural metric called the Fubini-Study metric. In the patch $z_0 \neq 0$ it is given by

$$g_{i\bar{j}} \;=\; \partial_i \bar{\partial}_{\bar{j}} \log \left(1 + \sum_{k=1}^{n} |z_k/z_0|^2 \right) \tag{1.23}$$

with analogous expressions in the other patches $z_k \neq 0$. It's straightforward to check that the associated Kähler form is closed. However the metric is not Ricci flat. Indeed, $\Lambda^n T^* \mathbf{CP}^n$ is a non-trivial line bundle, often denoted as $\mathcal{O}_{\mathbf{CP}^n}(-n-1)$. (This notation indicates it has local sections of the form $Q_d(z)/P_{d+n+1}(z)$, where Q and P are polynomials of the indicated degree.)

But submanifolds cut out by algebraic equations are also Kähler manifolds. They are clearly complex manifolds and the pull-back of the Fubini-Study form is still closed. So subvarieties (solutions of polynomial equations) in projective spaces are a natural factory to produce additional Kähler manifolds. By carefully choosing the degree of the equations, we can make examples with $c_1(TX) = 0$.

One of the most famous examples of this kind the quintic hypersurface in $\mathbf{CP}^4$. Its equation is given by

$$P_5 = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 \;=\; 0 \tag{1.24}$$

and deformations thereof. It's not hard to show that $c_1(TX) = 0$, either by a standard Chern class computation or by writing down the holomorphic $(3,0)$ form.[1] More generally, an equation of degree $n+2$ in $\mathbf{CP}^{n+1}$ yields a Calabi-Yau n-fold. For $n = 2$ we get the

[1] Eg. consider the sequence $0 \to TX \to T_{\mathbf{CP}^n}|_X \to N_X \to 0$, and let $c_1(\mathcal{O}_{\mathbf{CP}^n}(1)) = H$ denote the hyperplane class. If X is given by an equation of degree d, then $c_1(N_X) = dH|_X$, and we have $c_1(T_{\mathbf{CP}^n}) = c_1(\mathcal{O}_{\mathbf{CP}^n}(n+1)) = (n+1)H$. It follows that $c_1(TX) = (n+1-d)H|_X$, which vanishes if $d = n+1$.

$K3$ surface (where 'surface' is used in the complex sense; it is a four-manifold). For $n = 1$ we get the elliptic curve (i.e. the two-torus T^2).

By analogy with the Narain story, we would like to understand deformations we can do that preserve the equations we have just studied, i.e. the Calabi-Yau conditions. Such deformations are called moduli and manifest themselves as scalar fields in the four-dimensional effective theory. The $N = 1$ supersymmetry pairs up these scalar fields with chiral fermions into chiral multiplets.

Let's see in how many ways we can deform the equation of a quintic. Naively we can write

$$\begin{pmatrix} 4+5 \\ 5 \end{pmatrix} = 126 \tag{1.25}$$

different monomials of degree five on $\mathbf{CP}^4$. However, adding multiples of P_5 yields the same space, and we further have to mod out by the $PGL(5, \mathbf{C})$ coordinate transformations, of which there are 24. So the equation of a quintic depends on $126 - 1 - 24 = 101$ complex parameters, or 'moduli.'

Counting the deformations of the complex structure this way can be a little hazardous. It happens to work for the quintic, but not in general. It's interesting to do an analogous count for the $K3$ surface, realized as a quartic in $\mathbf{CP}^3$:

$$P_4 = z_0^4 + z_1^4 + z_2^4 + z_3^4 \ = \ 0 \tag{1.26}$$

The same type of count gives us $35 - 1 - 15 = 19$ complex deformations of the quartic. However using the results of the next subsection it is not hard to see that the $K3$ surface actually lives in a 20-dimensional family, as apparently first recognized by Kodaira. The remaining deformation cannot be realized by the embedding.

Apart from this, the quintic has a "breathing mode" inherited from $\mathbf{CP}^4$. This doesn't change the equation of the quintic, in other words it doesn't change the complex structure, but it does change the metric since we are changing volumes. Such a parameter is called a 'Kähler modulus' since we can view it as changing the Kähler form J. In the same way, we can use this two-form not to deform the metric, but to deform the B-field. It is natural to combine the two in a single complex two-form

$$J_{\mathbf{C}} = J + iB \tag{1.27}$$

called the complexified Kähler form, it being understood that deformations of the imaginary part correspond to B-field deformations.

Another famous example of a Calabi-Yau is the Tian-Yau manifold, which is a submanifold of $\mathbf{CP}^3 \times \mathbf{CP}^3$. Using coordinates z_i for the first $\mathbf{CP}^3$ and w_j for the second

$\mathbf{CP}^3$, the Tian-Yau manifold is defined by the equations

$$\sum_{i=0}^{3} z_i w_i = 0, \qquad \sum_{i=0}^{3} z_i^3 = 0, \qquad \sum_{i=0}^{3} w_i^3 = 0 \tag{1.28}$$

and deformations thereof. It has 23 complex structure deformations and 14 Kähler deformations. Only two of the Kähler deformations can be realized in $\mathbf{CP}^3 \times \mathbf{CP}^3$.

1.5. Moduli and Hodge diamond of a Calabi-Yau three-fold

What we saw for the quintic is indicative of the general story. To see this, let us describe some additional properties of Kähler manifolds.

For any manifold we can define the complex of differential forms

$$0 \ \to \ \Omega^0(M, \mathbf{R}) \ \overset{d}{\to} \ \cdots \ \overset{d}{\to} \ \Omega^d(M, \mathbf{R}) \ \to \ 0 \tag{1.29}$$

where $d^2 = 0$. This allows us to define a set of topological invariants called de Rham cohomology classes:

$$H^k_{dR}(M, \mathbf{R}) = \{\text{closed k} - \text{forms/exact k} - \text{forms}\} \tag{1.30}$$

where a form α is closed if $d\alpha = 0$, and exact if $\alpha = d\beta$ for some form β. The dimensions of these cohomology groups are called the Betti numbers, and denoted by b_k.

The next item we want to review is a fundamental result known as the Hodge decomposition. First, by contracting with the ε-tensor and the metric one can define an isomorphism $* : \Omega^k(M, \mathbf{R}) \to \Omega^{d-k}(M, \mathbf{R})$. Explicitly it is given by

$$(*\alpha)_{i_1 \ldots i_{d-k}} \ = \ \frac{1}{k!} \varepsilon_{i_1 \ldots i_{d-k}}{}^{j_1 \ldots j_k} \, \alpha_{j_1 \ldots j_k} \tag{1.31}$$

Using the $*$-operator we can conveniently express the L_2-inner product on forms as

$$\langle \alpha, \beta \rangle \ = \ \int \alpha \wedge *\beta. \tag{1.32}$$

We define the adjoint of d by $\int a \wedge *d\beta = \int d^*\alpha \wedge *\beta$ for any forms α, β. Integrating by parts and using $*^2 = (-1)^{k(n-k)}$ one finds that $d^* = - * d *$. The Hodge Laplacian on differential forms is then defined as

$$\Delta_d \ = \ dd^* + d^*d. \tag{1.33}$$

Since $\langle \alpha, \Delta\alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle d^*\alpha, d^*\alpha \rangle$, a form α is harmonic if and only if $d\alpha = d^*\alpha = 0$. We denote by $\mathcal{H}^k$ the harmonic forms of degree k. Then the Hodge decomposition is given by the following orthogonal decomposition:

$$\Omega^k(M, \mathbf{R}) = \mathcal{H}^k \oplus d\Omega^{k-1}(M, \mathbf{R}) \oplus d^*\Omega^{k+1}(M, \mathbf{R}) \tag{1.34}$$

This decomposition has the following important consequence. Let us consider a general k-form ω and decompose it into its harmonic, exact and co-exact pieces:

$$\omega = \alpha_H + d\beta + d^*\gamma \tag{1.35}$$

Now suppose that we are interested in closed forms. Then $0 = d\omega = dd^*\gamma$. It follows that $\langle d^*\gamma, d^*\gamma \rangle = \langle \gamma, dd^*\gamma \rangle = 0$ and hence $d^*\gamma = 0$. Thus for closed forms we have the simpler decomposition

$$\omega = \alpha_H + d\beta \tag{1.36}$$

Descending to cohomology, we see that *every cohomology class contains a unique harmonic representative*. The spectrum of harmonic forms is completely determined by the cohomology of the underlying manifold. From $d^* = - * d *$ we also see that the $*$-operator commutes with the Laplacian and hence descends to harmonic forms and to cohomology. Thus we have $H^k(M, \mathbf{R}) \simeq H^{d-k}(M, \mathbf{R})$, which is the manifestation of Poincaré duality in this language.

Now we would like to discuss an analogue of this story for complex manifolds. On a complex manifold it is natural to consider complex valued forms and complex valued de Rham cohomology classes. We can then further try to decompose such forms into forms which have a fixed number of p holomorphic and q anti-holomorphic indices. We denote such forms by $\Omega^{p,q}(X, \mathbf{C})$ and refer to (p, q) as the type of the form, which is a refinement of the degree.

The exterior derivative can also be decomposed as

$$d = \partial + \bar{\partial} \tag{1.37}$$

where $\partial : (p, q) \to (p + 1, q)$ and $\bar{\partial} : (p, q) \to (p, q + 1)$. From $d^2 = 0$ and by preservation of degrees one finds that $\bar{\partial}^2 = 0$, so we could use $\bar{\partial}$ to define cohomology groups. For each p we consider the complex

$$0 \;\to\; \Omega^{p,0}(X, \mathbf{C}) \;\overset{\bar{\partial}}{\to}\; \cdots \;\overset{\bar{\partial}}{\to}\; \Omega^{p,n}(X, \mathbf{C}) \;\to\; 0 \tag{1.38}$$

and denote the corresponding cohomology by $H^{p,q}_{\bar{\partial}}(X)$ or simply by $H^{p,q}(X)$. The dimensions of $H^{p,q}$ as a vector spaces are denoted by the Hodge numbers $h^{p,q}$.

On complex forms, the Hodge $*$-operator is defined to include complex conjugation after applying the ordinary $*$-operation. It is therefore sometimes denoted as $\bar{*}$, but we will leave this implicit. Analogous to the de Rham case, we can now define a Hermitian metric on the space of complex forms and define an adjoint $\bar{\partial}^\dagger$, and an associated Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$. There is a Hodge decomposition for the $\bar{\partial}$-complex:

$$\Omega^{p,q}(X, \mathbf{C}) = \mathbf{H}^{p,q} + \bar{\partial}\Omega^{p,q-1}(X, \mathbf{C}) + \bar{\partial}^\dagger\Omega^{p,q+1}(X, \mathbf{C}) \tag{1.39}$$

and as before, every class in $H^{p,q}(X)$ has a unique harmonic representative.

On a Kähler manifold one gets some further nice properties, which imply additional constraints on the $h^{p,q}$. In particular one finds that the d-Laplacian and the $\bar{\partial}$-Laplacian are essentially the same, namely we have $\Delta_d = 2\Delta_{\bar{\partial}}$. It follows that the d-Laplacian commutes with the projection on (p, q) components, and the resulting harmonic forms are those of the $\bar{\partial}$-Laplacian. On a Kähler manifold then we get the decomposition

$$H^k_{dR}(X, \mathbf{C}) = \sum_{p+q=k} H^{p,q}(X) \tag{1.40}$$

By complex conjugation we have $h^{p,q} = h^{q,p}$, and from the $*$-isomorphism we have $h^{p,q} = h^{n-p,n-q}$. By taking powers of the Kähler form one further finds that $h^{p,p} > 0$ for any p.

Let us consider Calabi-Yau three-folds in particular, and try to eliminate the redundancy in the Hodge numbers using the above relations. We can also use that $h^{3,0} = 1$ on a Calabi-Yau three-fold since there is a unique holomorphic three-form. Further, one may show that harmonic $(i, 0)$ forms on a Kähler manifold are covariantly constant. Briefly, one uses a Bochner-Weitzenböck identity

$$\Delta_{\bar{\partial}}\alpha = -\nabla^\dagger\nabla\alpha + R^{\bar{j}}_{\bar{i}}\imath_{\bar{j}}\alpha \tag{1.41}$$

where $\nabla^\dagger\nabla\alpha$ is the Laplacian constructed from the covariant derivative, $\imath_{\bar{j}}$ denotes contraction with $\partial/\partial z^{\bar{j}}$, and $R_{\bar{i}j}$ is the Ricci curvature. Then for harmonic $(i, 0)$ forms we have $0 = \langle\alpha, \nabla^\dagger\nabla\alpha\rangle = \langle\nabla\alpha, \nabla\alpha\rangle$ and so such forms are covariantly constant as promised. (On a Calabi-Yau this even works more generally since the Ricci curvature vanishes). Since harmonic forms of type $(1, 0)$ transform in the $\overline{\mathbf{3}}$ of the $SU(3)$ holonomy, if $h^{1,0}$ were non-zero then we would have a state in the $\overline{\mathbf{3}}$ which would need to be preserved, and so the holonomy would have to be reduced to $SU(2)$ or even smaller. This would mean that there is another covariantly constant spinor and hence an enhanced supersymmetry in the compactified theory. By a Calabi-Yau three-fold we will usually mean a 'proper' Calabi-Yau for which the holonomy group is strictly larger than $SU(2)$. For such 'proper' three-folds then we must have $h^{1,0} = 0$, and by similar reasoning we also find $h^{2,0} = 0$.

Putting all this information together, it follows that the Hodge diamond takes the following form

$$
\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{1.42}
$$

After taking all the constraints into account, there are really only two independent numbers left, $h^{1,1}(X)$ and $h^{2,1}(X)$.

Next we want to study deformations of g and B that preserve the equations we have just studied. We first consider deformations of the metric, δg_{mn}. We can split these into two types of deformations according to the type of the indices, $\delta g_{i\bar{j}}$ and δg_{ij}.

We may view deformations $\delta g_{i\bar{j}}$ as infinitesimal deformations of the Kähler form $J_{i\bar{j}}$. Recall that there exists a unique Calabi-Yau metric for a given Kähler class, so $\delta g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ defines itself a cohomology class of type $(1,1)$. Such deformations are called Kähler moduli and their number is counted by $h^{1,1}(X)$. Again we can complexify this by pairing with deformations of the B-field.

We can view $\delta g_{\bar{i}\bar{j}}$ as deformations of the complex structure, because in the original complex structure it gives a $(0,2)$ component to the deformed metric, so we will have to deform the complex structure to make the metric of type $(1,1)$ again. Alternatively let us consider $\mu_{\bar{i}}^{j} = \delta g_{\bar{i}\bar{k}} g^{\bar{k}j}$, which is a $(0,1)$ form valued in the tangent bundle TX. We can use this to deform the Dolbeault operator

$$
\bar{\partial}_{\bar{i}} \;\to\; \bar{\partial}'_{\bar{i}} = \bar{\partial}_{\bar{i}} + \mu_{\bar{i}}^{j} \partial_j \tag{1.43}
$$

and hence the definition of complex coordinates. In order for this to be consistent we need $\bar{\partial}'^2 = 0$, which for infinitesimal μ gives us $\bar{\partial}\mu = 0$. Furthermore, deformations of the form $\mu_{\bar{i}}^{j} = \bar{\partial}_{\bar{i}} v^j$ can be undone by a holomorphic coordinate transformation, so do not really correspond to complex structure deformations. This means that μ defines a Dolbeault cohomology class in $H^1(TX)$ (defined in the next subsection).

Since I haven't yet defined Dolbeault cohomology of holomorphic bundles, let me map it to something more recognizable. On a Calabi-Yau three-fold we can contract $\mu_{\bar{i}}^{j}$ with the holomorphic $(3,0)$ form, to get the closed form $\omega_{kl\bar{i}} = \mu_{\bar{i}}^{j} \Omega_{jkl}^{3,0}$ of type $(2,1)$. Since $\Omega^{3,0}$ is holomorphic and nowhere vanishing, this gives an isomorphism between $H^1(TX)$ and $H^{2,1}(X)$. Thus the number of such deformations is counted by $h^{2,1}(X)$.

The fact that infinitesimal complex structure deformations of a complex manifold are counted by $H^1(TX)$ is a general result. The fact that on a Calabi-Yau these infinitesimal deformations can be promoted to finite deformations (i.e. there are no obstructions) is not obvious but true, shown independently by Tian and Todorov.

We see that the two types of deformations, complex and Kähler, precisely 'use up' the independent parameters in the Hodge diamond. For the quintic we have $h^{1,1}(X) = 1$, counting the real dimension of the Kähler moduli space, and $h^{2,1}(X) = 101$, counting the complex dimension of the complex structure moduli space. Thus for the metric deformations the moduli space is given by

$$\mathcal{M}_{CY} \;=\; \mathcal{M}_{\rm complex} \times \mathcal{M}_{Kaehler} \qquad (1.44)$$

Adding the B-field deformations has the effect of complexifying $\mathcal{M}_K$.

Note that on a $K3$ surface we can contract a class in $H^1(TX)$ with the holomorphic two-form to get a cohomology class in $H^{1,1}(K3)$. The Hodge diamond for the $K3$ surface turns out to be

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad (1.45)$$

In particular we have $h^{1,1}(K3) = 20$, so we see that the $K3$ surface lives in a 20-dimensional complex family, as promised previously, and not in a 19-dimensional family.

We have arrived at a partial generalization of the Narain story, describing deformations of the metric and B-field. The Narain space also included deformations of the gauge field; we will get to these in the next subsection.

1.6. Vector bundles, stability and Dolbeault cohomology

Now we need to talk about the remaining equation $F_{mn}\Gamma^{mn}\epsilon = 0$. Above we temporarily set $F = 0$, but if $h^{1,0} = 0$ (more precisely if $\pi_1(X_6) = 0$) this means we won't find any non-trivial solutions. So if we want to break the ten-dimensional $E_8 \times E_8$ or $SO(32)$ gauge symmetry, we have to allow for $F \neq 0$. Not only that, setting F to zero is not even consistent with the tadpole cancellation condition, $dH \sim \mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)$.

On a Kähler manifold, the equation $F_{mn}\Gamma^{mn}\epsilon = 0$ can be broken up into two equations:

$$F^{0,2} = 0, \qquad g^{i\bar{j}} F_{i\bar{j}} = 0 \qquad (1.46)$$

On a Calabi-Yau, a quick way to see this is to use $\epsilon_6^T \Gamma^{\bar{i}_1 \ldots \bar{i}_n} \epsilon_6 \sim \Omega^{\bar{i}_1 \ldots \bar{i}_n}$ to get the first equation, and $i\epsilon_6^\dagger \Gamma_{\mu\nu} \epsilon_6 \sim J_{\mu\nu}$ to get the second. On a four-dimensional Kähler manifold these equations say that the field strength is anti-self dual (ASD), so we may think of this as a generalization of the instanton equations. They are called the *Hermitian Yang-Mills equations*. Let us discuss them in turn.

Since $F^{0,2} = 0$, the $(0,1)$ part of the connection is pure gauge

$$A^{0,1} = \Lambda^{-1} \cdot \bar{\partial}\Lambda \tag{1.47}$$

and can be set to zero by a gauge transformation. You must remember here that when we switched to complex coordinates we made our gauge fields complex. Eg. if we started with a $U(N)$ gauge group, then $A^{0,1}$ takes values in $GL(N, \mathbf{C})$, and the gauge transformation Λ used to set $A^{0,1}$ to zero is also valued in $GL(N, \mathbf{C})$. For otherwise we could apply the same argument to $A^{1,0}$ simultaneously and conclude that the bundle is trivial, which is not true in general.

Now let's consider the connection $\widetilde{A}^{0,1}$ on a different patch. On the overlap we would have

$$\widetilde{A}^{0,1} = \Lambda A^{0,1} \Lambda^{-1} - \bar{\partial}\Lambda \cdot \Lambda^{-1} \tag{1.48}$$

We saw that in their own patches, we could set $A^{0,1}$ and $\widetilde{A}^{0,1}$ to zero by a suitable (complexified) gauge transformation. So the condition $F^{0,2} = 0$ says that it is possible to choose transition functions such that $\bar{\partial}\Lambda = 0$, i.e the transition functions are holomorphic. A bundle which admits holomorphic transition functions is a *holomorphic vector bundle.*

By analogy with the Riemannian connection on the tangent bundle we will want the connection to be compatible with a choice of Hermitian metric on our bundle V, in order to reduce the structure group back from the complexified gauge group to the original group (i.e. from $GL(N, \mathbf{C})$ to $U(N)$ in the example given previously). Although not necessary, as we saw above it is frequently convenient to further require the connection to be "compatible with the holomorphic structure," which means that we require that $A^{0,1} = 0$. It's not hard to show that given a complex structure and a Hermitian metric on V, these two conditions determine a unique connection on V (see [2], pg. 73). It is sometimes called the *Chern connection.*

To better understand the second equation, let us first consider a holomorphic subbundle $U \subset V$. The Hermitian metric on V induces a Hermitian metric on U, and one can compute the corresponding curvatures (see [2], pg. 79). Interestingly one finds that

$$F_U \leq F_V|_U \tag{1.49}$$

in the sense that when contracted with holomorphic tangent vectors v^i, the difference $(F_V|_U - F_U)_{i\bar{j}} v^i v^{\bar{j}}$ is positive semi-definite at every point on X. One gets equality only when V splits as a direct sum of holomorphic bundles. In other words in contrast to real geometry, in complex geometry curvature is always non-increasing along holomorphic subbundles.

This is particularly interesting when the connection on V satisfies the Hermitian Yang-Mills equations, since then the 'Kähler trace' $g^{i\bar{j}} F_{U\, i\bar{j}}$ should be negative semi-definite at every point on X, and similarly for then $J \wedge J \wedge F_U \sim g^{i\bar{j}} F_{U\, i\bar{j}} J \wedge J \wedge J$. Taking the trace over gauge indices and integrating, we find that the degree of U, defined as

$$\deg(U) = \frac{1}{2\pi \mathrm{vol}(X)} \int_X J \wedge J \wedge \mathrm{Tr}(F_U) \tag{1.50}$$

is also negative. Since $dJ = 0$ the degree doesn't depend on the full curvature but only on the cohomology class of $\mathrm{Tr}(F_U)$. Up to a factor of 2π this is the first Chern class $c_1(TU)$ of U, a topological quantity.

Using the Chern connection, the equation $J \wedge J \wedge F_V = 0$ is really a complicated non-linear PDE for the Hermitian metric on V. Apart from its dependance on Kähler moduli, the degree of a bundle on the other hand is a topological quantity, and as such requiring the degree of every holomorphic sub-bundle to be negative seems like a much weaker condition than the Hermitian Yang-Mills equations. Nevertheless, somewhat analogous to the problem of finding Calabi-Yau metrics, it turns out that this condition is sufficient. Let us make some definitions.

A stable vector bundle is defined as a vector bundle for which any holomorphic sub-bundle has a smaller slope:

$$U \subset V \;\Rightarrow\; \mu(U) < \mu(V) \tag{1.51}$$

where the slope is defined as

$$\mu_J(V) = \frac{\mathrm{degree}(V)}{\mathrm{rank}(V)} \tag{1.52}$$

A bundle is *poly-stable* if it is a sum of stable bundles of the same slope. Note that for a bundle with a simple gauge group, the degree is always zero.

We've argued that a solution of the Hermitian Yang-Mills equations requires a holomorphic poly-stable bundle, at least for simple gauge groups (in the non-simple case the Hermitian Yang-Mills equations should be slightly generalized by adding a term proportional to the identity matrix). As we hinted, the converse is also true, and this is a difficult theorem of Donaldson-Uhlenbeck-Yau:

> *If a vector bundle V on a compact Kähler manifold is holomorphic and poly-stable, then there exists a unique solution to the Hermitian Yang-Mills equations.*

Of course by unique we mean unique up to conventional gauge transformations.

So why is this a big deal? The beauty of this theorem is that poly-stability is an algebro-geometric condition, so we can use purely algebraic methods to construct lots of examples. In spirit it is similar to Yau's theorem on Calabi-Yau manifolds: it is very hard to write down Ricci flat metrics on a Kähler manifold explicitly, since it corresponds to the solution of a highly non-linear PDE, but a simple criterion (vanishing of the first

Chern class) guaranteed us the existence of a solution of this PDE. In the present case, we can get away with checking the slope of holomorphic sub-bundles.

Furthermore we generally need to know very little about the actual solution; we will see that the massless fields in the effective four-dimensional theory can be studied with quasi-topological methods, much like we did for the moduli of the Calabi-Yau metric.

A canonical example of a stable bundle on a Calabi-Yau is the tangent bundle TX. Rather than use the above result, in this special case we can show it more directly as follows: since $\nabla_m \epsilon = 0$ we also have $0 = [\nabla_m, \nabla_n]\epsilon = \frac{1}{4} R_{mnpq}\Gamma^{pq}\epsilon = \frac{1}{4} R_{pqmn}\Gamma^{pq}\epsilon$, which is precisely $F_{pq}\Gamma^{pq}\epsilon = 0$ for the tangent bundle. Not only that, it also automatically satisfies the tadpole cancelation condition if we take the second E_8 bundle to be trivial, since then $\mathrm{Tr}(F \wedge F) = \mathrm{Tr}(R \wedge R)$ exactly. We will come back to the tangent bundle in a moment.

There are at least two concrete ways to discuss many more examples, monads and spectral cover constructions. They rely on Donaldson-Uhlenbeck-Yau to prove the existence of a solution to the Hermitian Yang-Mills equations. Unfortunately, lack of time precludes us from discussing them.

Now given a solution to the Hermitian Yang-Mills equations, how do we derive the effective four dimensional theory obtained by Kaluza-Klein reduction of the gauge sector? Can we do this without knowing the explicit solution? Given what we have seen earlier, you should expect that this is possible. Massless four-dimensional fields have some kind of harmonic form as internal wave-function, and harmonic forms tend to express quasi-topological information. We will now turn to Dolbeault cohomology and see that this expectation is correct.

Since $\bar\partial^2 = 0$ (or $\bar\partial_A^2 = F^{0,2} = 0$, but we already set $A^{0,1} = 0$ earlier), on holomorphic bundles we can define a complex which is analogous to the de Rham complex:

$$0 \;\to\; \Omega^{0,0}(X_6, V) \;\stackrel{\bar\partial}{\to}\; \cdots \;\stackrel{\bar\partial}{\to}\; \Omega^{0,n}(X_6, V) \;\to\; 0 \tag{1.53}$$

Concretely, these are forms with anti-holomorphic indices and which take values in V. We can then consider the associated cohomology groups

$$H^p(X_6, V) = \{\bar\partial\text{−closed } (0,p) \text{ forms}\}/\{\bar\partial\text{−exact } (0,p) \text{ forms}\} \tag{1.54}$$

which is called Dolbeault cohomology. Taking V to be $\Lambda^q T^* X$ we get back the Hodge numbers we discussed earlier

$$H^{q,p}(X) = H^p(X, \Lambda^q T^* X) \tag{1.55}$$

but Dolbeault cohomology exists more generally; it makes sense for any holomorphic bundle. Note that these Dolbeault cohomology groups are independent of the Hermitian metric on the bundle; they depend only on the holomorphic data.

As we will now see, these bundle valued Dolbeault cohomology groups are just what we need to describe the Kaluza-Klein reduction of the gauge sector. We put a complex structure on the whole 10d space-time (including the uncompactified dimensions) and focus on the $(0,1)$ part $A^{0,1}$, since the $(1,0)$ part may be recovered by Hermitian conjugation. Let us decompose $A^{0,1}$ into its classical profile $A^{0,1}_{cl}$ and the fluctuations $\delta A^{0,1}$ around it. We actually used gauge transformations to set the profile $A^{0,1}_{cl} = 0$ earlier but we temporarily left it for clarity. Now we can decompose the fluctuations into eigenmodes of the internal $\Delta_{\bar{\partial}}$-Laplacian, resulting in an expansion of the form

$$\delta A^{0,1} \;=\; \sum_I A^I_{\bar{\mu}} dx^{\bar{\mu}} \wedge \omega_{(0),I} \;+\; \sum_J \theta^J \, \omega_{(1),J} \tag{1.56}$$

Note that if $\bar{\partial}_A \omega_{(0)} \neq 0$, then $\omega_{(1),I} \equiv \bar{\partial}_A \omega_{(0),I}$ is also an eigenmode of the Laplacian with the same eigen-value. This suggest that such pairs $(\omega_{(0),I}, \omega_{(1),I})$ give rise to a single physical field in four dimensions. Indeed let us now apply a ten-dimensional gauge transformation $A^{0,1} \to A^{0,1} + \bar{\partial}_A \Lambda$ and decompose into eigenmodes, $\Lambda = \lambda^I \omega_{(0),I}$. Then we find

$$A^I_{\bar{\mu}} \to A^I_{\bar{\mu}} + \partial_{\bar{\mu}} \lambda^I, \qquad \theta^I \to \theta^I + \lambda^I \tag{1.57}$$

We learn a couple of things. The ten-dimensional gauge symmetry is spontaneously broken by the compactification, and the pseudo-scalars θ^J describe the longitudinal components of massive Kaluza-Klein gauge bosons A^I_μ. Only gauge transformations of the form $\Lambda = \lambda^I \omega_{(0),I}$ with $\bar{\partial}_A \omega_{(0),I} = 0$ survive as unbroken gauge symmetries. But sections $\omega_{(0),I}$ such that $\bar{\partial}_A \omega_{(0),I} = 0$ are by definition generators of degree zero Dolbeault cohomology. So we conclude that generators of $H^0(X,V)$ precisely count the unbroken four dimensional gauge symmetries. Taking commutators of global holomorphic sections gives a natural Lie algebra structure

$$H^0(X,V) \times H^0(X,V) \;\to\; H^0(X,V) \tag{1.58}$$

which yields the Lie algebra of the four-dimensional gauge group. From our expansion we also see that the A^I_μ corresponding to a generator of $H^0(X,V)$ do not have a longitudinal component, as expected.

Now let's try to deform the profile of the internal components of the gauge field, $A^{0,1}_{cl} \to A^{0,1}_{cl} + \delta A^{0,1}$. In order for this deformation to preserve the condition $F^{0,2} = 0$, we need $\bar{\partial}_A \delta A^{0,1} = 0$. Furthermore deformations of the form $\delta A^{0,1} = \bar{\partial}_A \lambda$ are not honest deformations but (broken) gauge symmetry transformations. Therefore infinitesimal deformations of a holomorphic bundle are described precisely by $H^1(X,V)$. As before deformations give rise to massless bosonic fields in the four-dimensional effective theory, and the unbroken four-dimensional $N = 1$ supersymmetry pairs them up with chiral fermions into chiral multiplets. Again taking commutators there is a natural action

$$H^0(X,V) \times H^1(X,V) \;\to\; H^1(X,V) \tag{1.59}$$

which means that the four-dimensional chiral fields live in certain representations of the unbroken gauge group. Depending on whether they are charged under the unbroken gauge symmetry, we call them either moduli or four-dimensional matter fields.

Let us briefly list some additional things that one can try to understand. We mentioned previously that infinitesimal complex and Kähler moduli on a Calabi-Yau can be promoted to finite deformations. For bundle deformations the situation is more complicated, but it can be encoded in terms of the holomorphic Chern-Simons functional

$$W = \int_{X_6} \Omega^{3,0} \wedge \omega_{YM}(A) \tag{1.60}$$

which gives the superpotential of the effective four-dimensional $N = 1$ theory. Another interesting story is that although the individual Kaluza-Klein modes in (1.56) are hard to understand, a certain combination of them (the holomorphic Ray-Singer torsion) is again quasi-topological and turns out to describe loop corrections to the four-dimensional gauge couplings. Further, since the stability condition depends on Kähler moduli there is an interesting interplay that results in the phenomenon of stability walls in the Kähler moduli space, where solutions of the Hermitian Yang-Mills equations come into or go out of existence. These are slightly more advanced topics that we cannot go into here.

To compare with the Narain story, we have now seen that infinitesimal deformations of g, B and A on X are described by the following cohomology groups:

$$H^1(TX), \qquad H^1(T^*X), \qquad H^1(X,V) \tag{1.61}$$

We can combine this all into a single formula by considering the bundle $TX \oplus T^*X \oplus V$. If you think about how the vertex operators are built from world-volume fields, this perhaps will not seem so surprising.

1.7. The standard embedding

Let us illustrate all this wonderful technology for the case where we embed the tangent bundle in our E_8 bundle. That is we embed the spin connection of the Calabi-Yau metric in the E_8 connection using the maximal subgroup $SU(3) \times E_6 \subset E_8$. The tangent bundle has $SU(3)$ holonomy. Under this subgroup, the adjoint representation of E_8 decomposes as

$$\mathbf{248} = (\mathbf{3}, \mathbf{27}) + (\overline{\mathbf{3}}, \overline{\mathbf{27}}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{8}, \mathbf{1}) \tag{1.62}$$

and therefore

$$\begin{aligned} H^p(X, V_{E_8}) \;\; = \;\; & H^p(TX) \otimes \mathbf{27} + H^p(T^*X) \otimes \overline{\mathbf{27}} \\ & + H^p(\mathcal{O}_X) \otimes \mathbf{78} + H^p(\mathrm{End}_0(TX)) \otimes \mathbf{1} \end{aligned} \tag{1.63}$$

To relate this to (1.62), TX corresponds to the $\mathbf{3}$ representation of the $SU(3)$ holonomy, and T^*X corresponds to the $\overline{\mathbf{3}}$. The trivial line bundle $\mathcal{O}_X$ corresponds to the singlet $\mathbf{1}$ of $SU(3)$. Further, $\mathrm{End}(TX) = T^*X \otimes TX$ corresponds to the $\overline{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}$, and $\mathrm{End}_0(TX)$ corresponds to the projection on the $\mathbf{8}$.

We first consider the four-dimensional gauge fields. According to our previous discussion, they come from $H^p(X, V_{E_8})$ with $p = 0$. Since $H^0(\mathcal{O}_X) = H^{0,0}(X)$ is one-dimensional, we find a gauge field in the $\mathbf{78}$, the adjoint of E_6. We claim that we do not get anything from the other pieces for $p = 0$. Indeed a generator of $H^0(TX)$ corresponds to a global holomorphic section $s : \mathcal{O}_X \to TX$, which embeds $\mathcal{O}_X$ as a holomorphic subbundle of TX. But $\deg(\mathcal{O}_X) = \deg(TX) = 0$, so this would violate the stability condition. Similarly one can show that for any higher rank stable bundle the degree zero Dolbeault cohomology vanishes ('no symmetries'). Therefore as claimed that is all we get from $H^0(X, V_{E_8})$. We conclude that the effective four-dimensional theory has an E_6 gauge group.

Next we consider the four-dimensional matter fields, which come from $H^p(X, V_{E_8})$ with $p = 1$. From $H^1(TX) \otimes \mathbf{27}$ we find $h^1(TX) = h^{2,1}(X)$ chirals charged in the $\mathbf{27}$ of E_6. Similarly there are $h^{1,1}(X) = h^1(T^*X)$ chirals charged in the $\overline{\mathbf{27}}$ of E_6. We further have $H^1(\mathcal{O}_X) = H^{0,1}(X) = 0$. Finally, $H^1(\mathrm{End}_0(TX))$ counts the deformation moduli of the bundle, which preserve the unbroken E_6 gauge group (since they live in the singlet of E_6), but take the connection away from the standard embedding.

Putting this together, we see that the effective four-dimensional theory is an E_6 Grand Unified Theory, plus a hidden sector with whatever descends from the second E_8. The net number of chiral generations is given by

$$N_{\mathbf{27}} - N_{\overline{\mathbf{27}}} \;=\; h^{1,2}(X) - h^{1,1}(X) \;=\; -\chi(X)/2 \tag{1.64}$$

where $\chi(X)$ is the Euler character of X. For the Tian-Yau manifold this gives nine generations, and one can actually mod out this manifold by a discrete symmetry to get precisely three-generations.

We can make the model more realistic. By taking the E_8 connection to be valued in special $SU(6)$ bundles that take the form of an $S(U(5) \times U(1))$-bundle, we break the gauge group precisely to the Standard Model gauge group, $SU(3)_c \times SU(2)_w \times U(1)_Y$. Such configurations can be written down using the general constructions that we mentioned previously.

2. Perturbative IIB

I will spend relatively little time on type IIB, but there are a few things we should discuss. This will be mostly set-up for F-theory, so I will focus on $D7$ and $O7$-planes. There will likely be a more extended discussion in the lectures by H. Ooguri.

2.1. Kaluza-Klein Ansatz

The ten-dimensional fields of the IIB theory are:

$$g_{MN}, B_{MN}, \phi, C^{(0)}, C^{(2)}_{MN}, C^{(4)+}_{MNPQ}, \psi^1_{M\alpha}, \psi^2_{M\alpha}, \chi^1_{\dot\beta}, \chi^2_{\dot\beta} \tag{2.1}$$

The $C^{(i)}$ are anti-symmetric tensor fields, with $C^{(4)}$ satisfying a self-duality condition. There are now two ten-dimensional gravitinos of the same chirality, $\psi^1_{M\alpha}$ and $\psi^2_{M\alpha}$, and similarly two ten-dimensional dilatinos. There are no gauge fields in ten dimensions. In the Einstein frame, the action may be written as

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{|g|}\left[R - \frac{\partial_\mu\bar\tau\partial^\mu\tau}{2\,Im(\tau)^2} - \frac{1}{2}|dC_{(0)}|^2 - \frac{1}{2}\mathcal{M}_{ij}F^i_3F^j_3 - \frac{1}{2}|\tilde{F}_{(5)}|^2\right] + S_{CS} + S_{Fermi} \tag{2.2}$$

where $\tau = ie^{-\phi} + C_{(0)}$ is the axio-dilaton, $F_3 = (dB, dC_{(2)})$ combines the RR and NSNS two-forms, $\tilde{F}_{(5)} = dC_{(4)} - \frac{1}{2}C_{(2)}\wedge dB + \frac{1}{2}B_2\wedge dC_{(2)}$ is the self-dual version of the four-form field strength, and

$$\mathcal{M}_{ij} = \frac{1}{Im(\tau)}\begin{pmatrix} |\tau|^2 & -Re(\tau) \\ -Re(\tau) & 1 \end{pmatrix} \tag{2.3}$$

The Chern-Simons term is given by $S_{CS} = -\frac{\epsilon_{ij}}{4\kappa^2}\int C_4\wedge F^i_3\wedge F^j_3$.

The Kaluza-Klein reduction proceeds much like for the heterotic string. We make an Ansatz

$$M_{1,9} = M_{1,3}\times X_6 \tag{2.4}$$

and set the fluxes to zero. Then we are left with solving $\delta\psi^1_M = \nabla_M\epsilon^1 = 0$ and $\delta\psi^2_M = \nabla_M\epsilon^2 = 0$, where ϵ^1_α and ϵ^2_α parametrize the two ten-dimensional supersymmetries. As before the supersymmetry variations tell us that X_6 should admit a covariantly constant spinor, and so X_6 is Calabi-Yau. However our supersymmetry transformations were parametrized by two ten-dimensional spinors, so compactifying type II strings on a Calabi-Yau yields $N = 2$ supersymmetry in four dimensions.

The natural multiplets for four-dimensional $N = 2$ supersymmetry are the gravity multiplet, the vector multiplet and the hypermultiplet. The bosonic fields in a vector multiplet are (A_μ, φ) where A_μ is a vector field and φ is a complex scalar field. The

$N=2$ *multiplet*	*multiplicity*
gravity	1
vector	$h^{2,1}(X)$
hyper	$h^{1,1}(X)+1$

Table ?: *Spectrum of type IIB compactified on a Calabi-Yau three-fold.*

bosonic fields in a hypermultiplet are a pair of complex scalar fields (φ_1, φ_2). The bosonic fields of the gravity multiplet consist of $(g_{\mu\nu}, A_\mu)$, where $g_{\mu\nu}$ is the metric and A_μ is a vector field called the gravi-photon.

The compactification works out as follows. The only way to get massless four-dimensional vectors is by expanding C_4^+ in harmonic three-forms:

$$C_4^+ \;=\; A_\mu^I dx^\mu \wedge \omega_{3,I} \tag{2.5}$$

From the Hodge diamond we see that there are $2h^{2,1}(X)+2$ such forms, however only half of these are independent due to the self-duality condition. The vector coming from $h^{3,0}(X)$ is the gravi-photon which ends up in the gravity multiplet, the remaining $h^{2,1}$ pair with the complex structure moduli in $h^{2,1}$ vector multiplets.

We further get $4 \times h^{1,1}(X)$ scalars from expanding C_4^+ in four-forms, B_2 and C_2 in two-forms, and Kähler moduli of the metric on X_6. These precisely fill up $h^{1,1}$ hypermultiplets. Two-forms B_2 and C_2 with all indices in the uncompactified directions can be dualized to scalars. They combine with the remaining scalars (ϕ and C_0) into an additional hypermultiplet. We have summarized the spectrum in table ..

There are two ways we could break further to $N=1$ supersymmetry. One is to turn on some of the fluxes. To get a solution, we have to allow a more general warped Ansatz of the form

$$ds^2 = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}g_{mn}dx^m dx^n \tag{2.6}$$

where $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric, g_{mn} is the metric on the six-dimensional internal space, and Z is a non-trivial function on the internal space. The other way to break to $N=1$ supersymmetry is to add $1/2$ BPS branes.

It turns out that if we take g_{mn} to be a Calabi-Yau metric and impose certain conditions on the fluxes, then finding a solution with $N=1$ supersymmetry is reduced to solving

$$-\nabla^2 Z = (2\pi\alpha')^2\rho_{D3} \tag{2.7}$$

Here ∇ refers to the covariant derivative for g_{mn}, and ρ_{D3} is the '$D3$-brane density.' The latter really refers to anything that sources $C_{(4)}$, whether it is explicit $D3$-branes, $H_3 \wedge F_3$ (through the Chern-Simons term S_{CS}), or fluxes and curvature terms on higher dimensional defects. Deriving the effective four-dimensional theory for a warped background can be a difficult business. We are going to think of the warping as a higher order effect and not consider it further, but there are apparently situations where this is not correct and you can miss an important mode, with the mode discovered by Gubser/Herzog/Klebanov in the warped deformed conifold being an example.

2.2. Orientifolds

The perturbative IIB theory has two basic $\mathbf{Z}_2$ symmetries. The first is world-sheet parity, which interchanges the left- and right-movers. This changes the signs of C_0, B_2 and C_4^+, and interchanges the two gravitini (as well as ϵ^1 and ϵ^2). The other basic $\mathbf{Z}_2$ symmetry is the left-moving fermion number $(-1)^{F_L}$, which changes the sign if the left-movers are in the Ramond sector. This therefore changes the sign of C_0, C_2 and C_4^+, as well as the sign of one of the gravitini, say ψ^1_M (and similarly for ϵ^1). We could further combine such involutions with an involution σ of the IIB space-time manifold. By definition, an orientifold is a quotient of the IIB theory that involves the world-sheet parity transformation P.

We are going to consider orientifolds of the following form. We will quotient by

$$(-1)^{F_L} \cdot P \cdot \sigma \tag{2.8}$$

where σ is a holomorphic involution of the Calabi-Yau X_6 that reverses the sign of the holomorphic $(3,0)$ form and has a fixed point locus of complex codimension one. In other words, locally it is of the form $(z_1, z_2, z_3) \to (z_1, z_2, -z_3)$. Such a quotient preserves only one linear combination of the two unbroken spinors ϵ^1 and ϵ^2, and so leads to four-dimensional $N = 1$ supersymmetry. The fixed point locus is called an $O7$-plane. If we denote the holomorphic submanifold that is fixed by S, then we say that the $O7$-plane 'wraps' S.

We can be very explicit. Let $B_3 = X/\mathbf{Z}_2$, which will be a Kähler manifold, and let b_2 be a section of $K_{B_3}^{-2}$, where K_{B_3} is the canonical bundle $K_{B3} \equiv \Lambda^3 T^* X$. Simple examples would be

$$B_3 = \mathbf{CP}^3, \qquad B_3 = \mathbf{CP}^2 \times \mathbf{CP}^1, \qquad B_3 = \mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1. \tag{2.9}$$

but there are many more. For example when $B_3 = \mathbf{CP}^3$, we would have that b_2 is simply a polynomial of degree $2 \times 4 = 8$. Given b_2, we can write an equation for X explicitly as

$$\xi^2 = b_2. \tag{2.10}$$

It has the involution $\xi \to -\xi$ and fixed locus $b_2 = 0$. The Calabi-Yau condition on X is precisely the condition that b_2 lives in $K_{B_3}^{-2}$ (since $d\xi \sim db_2/\sqrt{b_2}$ gives $K_X = \pi^* K_{B_3} + \frac{1}{2}N_{b_2=0}$).

2.3. D-branes

The second type of objects we can add are D-branes. By definition, a D-brane is a defect where a fundamental string can end. By quantizing open strings, we find new degrees of freedom propagating along the D-brane. The low energy world-volume theory on a single D-brane is the dimensional reduction of the ten-dimensional supersymmetric Yang-Mills theory with $U(1)$ gauge group. When N D-branes coincide, the gauge group enhances from $U(1)^N$ to $U(N)$. The extra off-diagonal fields come from the ground states of open strings stretched between the different D-branes. The mass of these modes is proportional to the distance between the branes, so they become massless when the branes coincide.

A common case is for the branes to fill the four uncompactified dimensions (as did the ten-dimensional gauge fields in the heterotic string). Then the branes 'wrap' even dimensional submanifolds of the Calabi-Yau three-fold. To be 1/2 BPS, a D-brane should wrap a holomorphic cycle and the world-volume fields should satisfy a dimensional reduction of the Hermitian Yang-Mills equations that we discussed in the context of the heterotic string.

If we want to preserve the same $N = 1$ supersymmetry that is preserved by (2.8), then we can consider $D7$-branes wrapped on holomorphic four-cycles and $D3$-branes localized at points of X_6. The bosonic world-volume fields on a $D7$-brane are a gauge field A_M and a complex adjoint field Φ. When we wrap the D7-brane on a holomorphic cycle S, these fields may have a non-zero profile along S, satisfying the dimensionally reduced version of the Hermitian Yang-Mills equations. In the abelian case, these equations are simply

$$\bar{\partial}\Phi = 0, \qquad F^{0,2} = 0, \qquad J \wedge F = 0 \tag{2.11}$$

where J is the pull-back of the Kähler form of X_6 to S. When multiple branes are wrapped on the same cycle, we get a non-abelian version of these equations, often referred to as Hitchin's equations.

There is something peculiar going on with the adjoint field Φ, so let us say a few more words about this. We take (z^1, z^2) to be holomorphic coordinates along S, and z^3 for the normal direction. Then to do carry out the dimensional reduction, in the Hermitian Yang-Mills equations we would like to replace gauge fields along the normal direction by adjoint fields:

$$A_{\bar{1}}, A_{\bar{2}}, A_{\bar{3}} \;\to\; A_{\bar{1}}, A_{\bar{2}}, \Phi_{\bar{3}} \tag{2.12}$$

and similarly for their Hermitian conjugates. Restricted to S, the tangent bundle splits holomorphically as $TX|_S = TS \oplus N_S$, where N_S is the normal bundle to S. Similarly the anti-holomorphic cotangent bundle splits as $\overline{TX}|_S = \overline{T^*S} \oplus \overline{N}_S^\vee$, where $N_S^\vee$ is the dual of the normal bundle. The gauge fields (with anti-holomorphic indices) are sections of $\overline{T^*S} \otimes \mathrm{Ad}(P)$, where P is a principal G-bundle and $\mathrm{Ad}(P)$ is its associated adjoint vector bundle. Then the adjoint scalar field should be a section of $\overline{N}_S^\vee \otimes \mathrm{Ad}(P)$. Using the Hermitian metric restricted to S, we can consider $\Phi^3 = \Phi_{\bar{3}} g^{\bar{3}3}$ which takes values in N_S. We say the adjoint field is "valued in" or "twisted by" the normal bundle.

We can go one step further. Using the holomorphic $(3,0)$ form, we can consider $\Phi_{ij} = \Phi_{\bar{3}} g^{\bar{3}3} \Omega_{3ij}$, which is a $(2,0)$ form on S, so we may also consider Φ as a section of $K_S \otimes \mathrm{Ad}(P)$. Similarly, the Hermitian conjugate $\Phi^\dagger$ may get mapped to a $(0,2)$ form on S. Thus the $U(1)$ R-symmetry of the Yang-Mills theory (under which Φ has charge one) is identified with a subgroup of the Lorentz group of the 7-brane. This is the phenomenon of topological twisting, so we found that the world-volume theory of a $D7$ brane wrapped on a non-trivial cycle in a Calabi-Yau is a topologically twisted Yang-Mills theory.

With these identifications, the dimensionally reduced Hermitian Yang-Mills equations can be written as

$$F^{0,2} = 0, \qquad \bar{\partial}\Phi = 0, \qquad J \wedge F + i[\Phi^\dagger, \Phi] = 0 \tag{2.13}$$

(Here we used physicists' conventions in which the gauge generators are Hermitian; in math papers the factor of i would be dropped). The rest of the story is parallel to the heterotic string. The first two equations are equivalent to saying that we need a *Higgs bundle*, i.e. a holomorphic bundle together with a holomorphic section of the twisted adjoint bundle. The last equation is then equivalent to saying that the Higgs bundle is poly-stable.

In the literature you will often find an alternative description of such a configuration as a poly-stable coherent sheaf $\mathcal{L}$. These two descriptions are complementary to each other, with different regimes of validity. Unfortunately I do not have time to go into coherent sheaves here.

In an orientifold, the D-brane configuration should be invariant under the orientifold symmetry. More precisely, the gauge field on the brane transforms as

$$A \;\to\; -\gamma^{-1} \sigma^* A^T \gamma \tag{2.14}$$

where γ is taken to be a constant gauge transformation, which satisfies $\gamma^{-1}\gamma^T = 1$ (up to a further gauge transformation) in order to get an involution. This leads to the following possibilities. A stack of branes that gets mapped to itself, and hence coincides with an O-plane, yields an $SO(N)$ or $USp(N)$ gauge group in the quotient theory, depending on whether $\gamma^T = \pm\gamma$; a pair of stacks that gets interchanged yields a $U(N)$ gauge group in the quotient theory. An O-plane is called an O^--plane if $\gamma^T = \gamma$ and an O^+-plane if $\gamma^T = -\gamma$. Note that unlike for the heterotic string, we do not get exceptional gauge groups in this set-up.

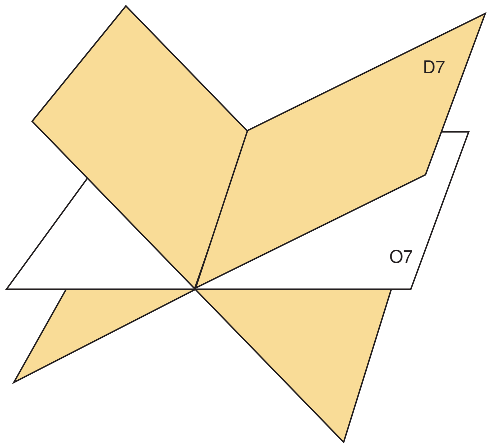

Figure 1: *Tilting a D-brane away from an O-plane by turning on a position dependent profile for the adjoint field of the worldvolume gauge theory.*

We cannot add these defects arbitrarily. In the heterotic case we had the tadpole/generalized Gauss law constraint $dH = \mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)$. In the type II case we get analogous constraints for all the Ramond-Ramond and NS-NS fluxes.

A concrete and generic way to write down a configuration is as follows. A $D7$-brane generates one unit of RR flux for $F_{(1)} = dC_{(0)}$, so we try to cancel this flux by including an $O7^-$-plane, which generates -4 units of RR flux. ($O7^+$-planes source positive RR flux, so they would not be much help). Recall we could write our geometry X with such an $O7$-plane as $\xi^2 = b_2$, where b_2 is a section of $K_{B_3}^{-2}$, $B_3 = X/\mathbf{Z}_2$ (eg. $B_3 = \mathbf{CP}^3$ for simplicity), and $b_2 = 0$ is the $O7$-locus. To cancel the RR charge the $D7$ would then have to be wrapped on a cycle of the form $b_8 = 0$, where b_8 is a section of $K_{B_3}^{-8}$. However consistency imposes a further constraint: a $D7$ brane cannot wrap a generic cycle, because a $D7$-brane should self-intersect (i.e. have singularities of the form $z^2 - w^2 = 0$) when it intersects the $O7$-plane.

To see this, briefly let us consider a coinciding $O7/D7$ system, leading to gauge group $SO(2)$. Now we tilt the $D7$-plane away from the $O7$-plane by turning on a position dependent expectation value for the $SO(2)$-valued field Φ,

$$\Phi(z) \sim z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.15}$$

where z is a complex coordinate along the worldvolume of the $O7$-plane. Then the eigenvalues of Φ are $\pm z$, indicating a pair of $D7$-branes meeting the orientifold plane in a double intersection at $z = 0$, as claimed. This means that our $D7$-brane cannot wrap a

$N=1$ *multiplet*	*multiplicity*
gravity	1
vector	$h^{2,1}_+(X)$
chiral	$h^{1,1}_+(X)+h^{1,1}_-(X)+h^{2,1}_-(X)+1$

Table ?: *Closed string spectrum of type IIB compactified on a Calabi-Yau three-fold, with an orientifold involution of the form* $(-1)^{F_L}P\sigma$.

generic cycle of the form $b_8=0$, but must wrap a cycle of the form

$$\xi^2 b_6 - b_4^2 = 0 \tag{2.16}$$

in order to have double point singularities when the $D7$ and $O7$ intersect (at $\xi=b_4=0$). A more generic expression of the form $b_8=0$ would not have this property. We conclude that we need three sections, b_2, b_4 and b_6 to write down a generic $D7/O7$ configuration. With a bit more effort we can also specify a gauge field configuration. The easiest way to do this is by using the fact that that a quantized field strength $F^{1,1}/2\pi$ of type $(1,1)$ can be dually described by a linear combination of holomorphic curves on the $D7$-brane.

Given a IIB compactification with branes, we would also like to know the effective four-dimensional theory obtained by Kaluza-Klein reduction. The reduction of the closed string fields was already discussed above, with only the slight complication that a subset of the fields will get projected out by the orientifold projection. Briefly, since the involution σ is holomorphic, the cohomology groups $H^{p,q}(X)$ split into eigenspaces with eigenvalue ±1, $H^{p,q}_+(X)$ and $H^{p,q}_-(X)$. Since C_4^+ is even under $(-1)^{F_L}P$, its modes only survive in the quotient theory if its internal wave-function is also even. So for example we only get $h^{2,1}_+(X)$ vector fields, with the remaining $h^{2,1}_+(X)$ of the underlying $N=2$ theory being projected out. The complex structure moduli on the other hand, which where paired with these vector fields under $N=2$ supersymmetry, came from expanding deformations of the holomorphic $(3,0)$-form. Since $\Omega^{3,0}$ is is odd under the involution, only $h^{2,1}_-(X)$ deformations survive in the quotient theory. Continuing in this way, Kähler moduli and modes of C_4^+ with four internal indices give $h^{1,1}_+(X)$ chirals, modes of (B_2, C_2) with two internal indices give $h^{1,1}_-(X)$ chirals, and the axio-dilaton gives one final chiral. This spectrum is summarized in table ..

But how do we reduce the degrees of freedom from the open string sector? It turns out there's a very simple general answer, but because I didn't have time to talk about coherent sheaves I will only mention it in passing.

Recall that in the heterotic string, the massless fields obtained from reduction of the gauge sector were counted by $H^p(X,V)$. For $p=0$ we get four-dimensional gauge fields and for $p = 1$ we get four-dimensional matter fields and moduli. In IIB the 1/2 BPS D-brane configuration can be expressed as a coherent sheaf $\mathcal{L}$, and the deformation theory of sheaves then tell us that the massless fields are then counted by the Ext groups $\mathrm{Ext}^p_X(\mathcal{L}, \mathcal{L})$. For $p = 0$ we get gauge fields and for $p = 1$ we get matter fields and moduli. In an orientifold we keep the odd generators and throw out the even generators.

2.4. S-duality

The type IIB theory also has a strong coupling duality, S-duality which takes $g_s \to 1/g_s$. It is convenient to formulate this in terms of the axio-dilaton

$$\tau \;=\; i\, e^{-\phi} + C_0 \tag{2.17}$$

The axion has a shift symmetry $C_0 \to C_0 + const$ which is non-perturbatively broken to $C_0 \to C_0 + n$, $n \in \mathbf{Z}$. Together these two symmetries act on τ as

$$S : \tau \to 1/\tau, \qquad T : \tau \to \tau + 1 \tag{2.18}$$

and generate an $SL(2, \mathbf{Z})$ duality group. The two-form fields (B_2, C_2) transform as a doublet under $SL(2, \mathbf{Z})$, and the Einstein frame metric is invariant. The whole IIB supergravity action that we wrote in (2.2) is invariant under this duality group.

S-duality takes a fundamental IIB string to a D-string. Since a $D7$-brane is defined as a locus where a fundamental string can end, the S-dual of a $D7$-brane is a defect where a D-string can end. More generally, $SL(2, \mathbf{Z})$ dualities can take a fundamental string to a (p, q) string, a bound state of p fundamental strings and q D-strings. We can define a (p, q) 7-brane as a defect of codimension two where a (p, q) string can end.

This concludes our lightning quick review of perturbative IIB compactification.

3. F-Theory

Heterotic and IIB probably looked quite familiar. They are conventional string theories with a small string coupling. F-theory is a different beast, generalizing some of the structures we have seen previously in the language of geometry. It can reduce to heterotic or IIB in suitable weak coupling limits, but the way in which it does is not obvious.

3.1. Elliptic fibrations

Let us briefly recall some properties of elliptic curves. We have $H_1(T^2, \mathbf{Z}) = \mathbf{Z}^2$, generated by one-cycles a and b such that $a \cap a = b \cap b = 0$ and $a \cap b = 1$. Denote the holomorphic one-form by Ω. Then the complex structure of an elliptic curve can be parametrized as

$$\tau = \frac{\int_b \Omega}{\int_a \Omega} \tag{3.1}$$

The torus has large diffeomorphism or modular transformations which map the one-cycles as $(a, b) \to M(a, b)$, where $M \in SL(2, \mathbf{Z})$. The basic transformations are $(a, b) \to (a, b+a)$ and $(a, b) \to (b, a)$, and the others are generated by these. On the complex structure parameter these act as

$$\tau \to \tau + 1, \qquad \tau \to 1/\tau \tag{3.2}$$

This looks exactly like the S-duality transformations for the axio-dilaton of type IIB. So why not identify the axio-dilaton with the complex structure parameter of an auxiliary torus? This is the basic idea that leads to F-theory [3].

The one-sentence description of F-theory is as follows. F-theory is basically a book-keeping device to describe vacua of IIB string theory with a varying axio-dilaton. We do this by identifying the axio-dilaton τ with the modular parameter of an auxiliary torus, and imagining that this auxiliary torus is real in some sense, by formally attaching this torus at each point in the IIB space-time. In this way we promote the ten-dimensional space-time of the IIB theory to twelve dimensions, and we speak of twelve-dimensional compactifications of F-theory.

You should keep in mind that this is nothing more than a very clever change of variable. Specifying a varying τ directly would get rather complicated due to the branch cuts for τ and the $SL(2, \mathbf{Z})$ monodromies around them. It is much simpler to write down the torus directly, and thereby specify τ implicitly. The T^2 is typically written in Weierstrass form, i.e. described as a cubic equation of the form

$$y^2 = x^3 + fx + g \tag{3.3}$$

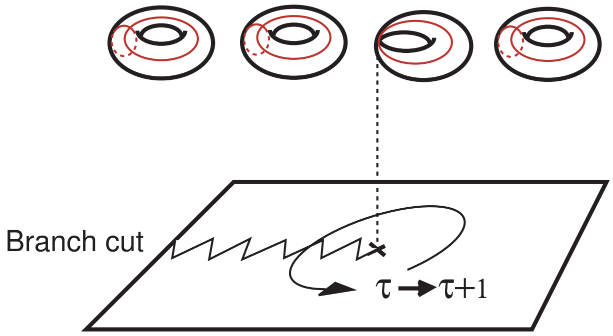

Figure 2: *Elliptic fiber degenerating over the discriminant locus, due to one of its one-cycles pinching to zero size. Singular fibers generate monodromies for τ, and hence are associated to 7-branes.*

which can be done globally on the IIB space-time. Instead of specifying τ, we specify f and g. The area of the torus has no meaning in F-theory and should be taken zero.

Now 7-branes source the axio-dilaton, and supergravity solutions for 7-branes have an axio-dilaton which varies non-trivially over the IIB space-time. The reformulation above is particulary efficient for encoding such solutions, as we discuss next.

As above we label the generating one-cycles of the elliptic fiber as a and b, with $a \cap b = 1, a \cap a = b \cap b = 0$. On a subset of real codimension two on the IIB space-time, the elliptic fiber pinches due to a one-cycle $\gamma = pa + qb$ shrinking to zero. One can calculate that this happens when $\Delta \equiv 4f^3 + 27g^2 = 0$. As we go around this locus, which is called the discriminant locus, the one-cycles undergo a monodromy following the Picard-Lefschetz formula:

$$\delta \to \delta + (\delta \cap \gamma)\gamma \tag{3.4}$$

Let us use our earlier expression $\tau = \int_b \Omega / \int_a \Omega$. With a little algebra, we see that the monodromy acts on τ as

$$\tau \to K_{[p,q]}\tau, \qquad K_{[p,q]} = \begin{pmatrix} 1+pq & p^2 \\ -q^2 & 1-pq \end{pmatrix} \tag{3.5}$$

We claim this identifies the locus with a (p,q) 7-brane, i.e. a type of 7-brane on which a (p,q) string can end. To see this, consider the case of a $(1,0)$ brane. In this case we have $\tau \to \tau + 1$ as we go around a 7-brane, i.e. $C_0 \to C_0 + 1$ and $e^{-\phi}$ invariant. This is precisely the right monodromy for a single $D7$-brane, because it means that the 7-brane sources one unit of RR flux. By applying $SL(2, \mathbf{Z})$ duality transformations, we recover the other cases.

You may start to get suspicious here: we seem to find only 7-branes on which strings can end. But in perturbative IIB we also had $O7$-planes, and strings don't end on $O7$-planes, so what happened to them? It turns out that the $O7$-plane is not an elementary object in F-theory, but is composed of two distinct (p, q) 7-branes. In fact the $SL(2, \mathbf{Z})$ monodromy around an $O7$-plane is given by

$$\begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \tag{3.6}$$

so we see that it can be composed out of a $(1, 1)$ brane and a $(1, -1)$ brane. As we turn on the string coupling, the $O7$-plane splits into its two components.

3.2. The view from M-theory

There is another perspective on F-theory by starting with M-theory, as follows. M-theory on T^2, in the limit that the area goes to zero, is equivalent to type IIB on a circle of radius $R = 1/A$, with axio-dilaton given by the modular parameter τ of the T^2, and A is the area of the T^2. Now let us fiber this duality over a base.

In particular, let us consider an M-theory compactification to three dimensions with $N = 2$ supersymmetry. (This will eventually lift to $N = 1$ supersymmetry in four dimensions). We haven't considered M-theory compactifications so far but the principles are very similar to the other cases we discussed. We make an Ansatz

$$M_{1,10} = M_{1,2} \times Y, \tag{3.7}$$

set the fluxes to zero, and analyze the BPS equations. As usual we have to solve an equation of the form $\delta\psi_M = \nabla_M \epsilon = 0$. The main difference with the cases we discussed previously is that to get $N = 2$ supersymmetry, we need *two* covariantly constant spinors on Y of the same chirality. The positive chirality spinor representation of $SO(8)$ is the $\mathbf{8}_c$, which we therefore want to decompose as $\mathbf{6} + \mathbf{1} + \mathbf{1}$, with the holonomy group preserving the two singlets. This means that Y should have $Spin(6) \simeq SU(4)$ holonomy, and so one finds that Y should be a Calabi-Yau four-fold, henceforth denoted Y_4.

To make use of the M-theory/IIB duality in nine dimensions cited above, we now further require that Y_4 admits an elliptic fibration, i.e. a fibration

$$\pi : Y_4 \to B_3 \tag{3.8}$$

where the fibers are elliptic curves. As usual the subscripts on Y_4 and B_3 denote the complex dimension. If we represent this fibration in Weierstrass form

$$y^2 = x^3 + fx + g \tag{3.9}$$

then in order for the total space Y_4 to be Calabi-Yau, we must have that f is a section of $K_{B_3}^{-4}$ and g is a section of $K_{B_3}^{-6}$. Applying the M-theory/IIB duality fiberwise, we deduce that

$$M - \text{theory on } \mathbf{R}^{1,2} \times Y_4 \qquad \longleftrightarrow \qquad \text{IIB on } \mathbf{R}_{1,2} \times S^1_R \times B_3, \tag{3.10}$$

with a varying axio-dilaton over B_3 on the IIB side. Since $R = 1/A$, we then send $A \to 0$. In the limit that the elliptic fiber shrinks to zero, the S^1 decompactifies and we recover type IIB compactified on $\mathbf{R}_{1,3} \times B_3$, with varying axio-dilaton and $N = 1$ supersymmetry in four dimensions. We say that this corresponds to an F-theory compactification on Y_4 to four dimensions.

3.3. Abelian gauge fields

Since our F-theory compactification contains 7-branes, we expect to see eight-dimensional Yang-Mills theory come out in some way. The way that this happens is quite unlike what we have seen so far from the heterotic string and perturbative IIB, and it will occupy us for the next few subsections.

Although we have some kind of 7-brane defect, the string coupling is not small and we can't go and quantize open strings. Indeed, what we are really doing is describing the 7-brane as a solitonic solution of type IIB supergravity, the 'stringy cosmic string' solution (string here referring to the fact that the soliton is localized in codimension two, which means it's really a 7-brane in ten dimensions). This means that we should expect the world-volume theory to arise from the collective coordinates of the soliton.

In particular, abelian tensor fields on a defect arise as zero modes of the ten-dimensional tensor fields that are roughly localized on the soliton. Since the gauge symmetry we are looking for exists already in eight dimensions, the eight-simensional gauge field must come from zero modes of B_2 and C_2. However B_2 and C_2 are not invariant under the monodromies; they form a doublet under the $SL(2, \mathbf{Z})$ duality group.

In keeping with the philosophy of F-theory, we thus want to reformulate B_2 and C_2 in terms of an object that can be specified globally over the IIB space-time, and is not subject to monodromies. This can be done by encoding the two-form fields in a three-form field:

$$C_{(3)} \sim (C_2 - \tau B_2) \wedge (dx - \tau dy) + c.c. \tag{3.11}$$

where x and y are the two coordinates on the T^2 fiber. Note this is not a completely general three-form, since it has two indices on the IIB space-time and one index on the elliptic fiber. Three-form fields with different numbers of indices in the base and the fiber do not exist in F-theory. This three-form field $C_{(3)}$ is $SL(2, \mathbf{Z})$ invariant and can be defined globally. By compactifying on S^1 and going to M-theory, it corresponds to the usual $C_{(3)}$ field of eleven-dimensional supergravity, except that components with disallowed indices are frozen out in the F-theory limit. The four-form flux of this tensor field is conventionally called the G-flux, $G = dC_{(3)}$.

Now to get a 7-brane gauge field, we need to expand $C_{(3)}$ in terms of real harmonic two-forms, with one index on the base and one index on the fiber (so that the gauge field index lives in the IIB space-time):

$$C_{(3)} = A^I \wedge \omega_{(2),I} \tag{3.12}$$

As a toy example, let us consider F-theory compactified on an elliptically fibered $K3$-surface to eight dimensions. We describe our $K3$ in Weierstrass form as an elliptic fibration over $\mathbf{CP}^1$:

$$y^2 = x^3 + f_8(z)x + f_{12}(z) \tag{3.13}$$

Here f and g are sections of $K_{\mathbf{P}^1}^{-4}$ and $K_{\mathbf{P}^1}^{-6}$, so they correspond to polynomials of degree eight and twelve respectively. Then the discriminant $\Delta = 4f_8^3 + 27g_{12}^2 = 0$ is of degree twenty-four, so a generic elliptic $K3$ has twenty-four singular fibers. From the perspective of IIB this corresponds to a compactification of the form $\mathbf{R}^{1,7} \times \mathbf{CP}^1$ with twenty-four (p, q) 7-branes inserted at special positions on the $\mathbf{CP}^1$. From the Hodge numbers discussed earlier it follows that there are twenty-two harmonic two-forms on a $K3$ surface, but one of these has two indices on the base and one has two indices on the fiber. Thus there are twenty harmonic forms we can expand in, yielding twenty $U(1)$ gauge fields in the eight-dimensional theory.

This is precisely the number of $U(1)$ gauge fields you will find in generic string theory compactifications to eight-dimensions with half-maximal supersymmetry, for example in a heterotic compactification on T^2. Indeed it turns out that there is a duality

$$F - \text{theory on } \mathbf{R}^{1,7} \times \mathbf{CP}^1 \ltimes T^2 \qquad \longleftrightarrow \qquad \text{Heterotic on } \mathbf{R}^{1,7} \times T^2 \tag{3.14}$$

(Here we used the imprecise notation $\mathbf{CP}^1 \ltimes T^2$ to denote an elliptically fibered $K3$.) This is one of the fundamental dualities in the business and has been studied in great detail. Unfortunately lack of time precludes us from discussing it further.

Note that the harmonic forms giving rise to $U(1)$ gauge fields cannot be normalizable in the local supergravity solution for a 7-brane: if so we would get an independent gauge field for each singular fibre, but there are twenty-four singular fibers in an elliptically fibered $K3$ and only twenty gauge field from the 7-branes. Thus in contrast to perturbative IIB, thinking about 7-branes in F-theory as being sharply localized at a codimension two locus (here the discriminant locus) is actually quite misleading: their energy density is quite spread out, much like for other solitonic objects.

Let us say a few more words about F-theory compactifications to four dimensions with $N = 1$ supersymmetry. We saw that if the flux vanishes then F-theory needs to be compactified on an elliptically fibered Calabi-Yau four-fold Y_4. Now suppose we want to consider compactifications with non-zero flux. This situation was analyzed by Becker and Becker. It turns out that the Kaluza-Klein Ansatz needs to be generalized slightly to a warped Ansatz, but when this is done one finds that Y_4 is still a Calabi-Yau four-fold, and the G-flux on Y_4 needs to satisfy the following conditions:

$$\begin{aligned} G &\in H^{2,2}(CY_4) \qquad (\text{F} - \text{term equation}) \\ J \wedge G &= 0 \qquad\qquad\quad\;\; (\text{D} - \text{term equation}) \end{aligned} \tag{3.15}$$

Here J is the Kähler form on the Calabi-Yau four-fold. Note that these conditions are similar to the ASD equations on the internal worldvolume of a 7-brane, $F^{2,0} = 0 = J \wedge F^{1,1}$.

It may seem somewhat strange that we get the simple condition $J \wedge G = 0$, whereas in type IIB and heterotic compactification we found stability conditions, which are even qualitatively dissimilar. It's not hard to see the problem: the equations for the G-flux above were obtained by extrapolating from M-theory. There is a non-renormalization theorem for F-terms but not for D-terms, so the D-terms should be taken with a grain of salt. In fact one should regard even the F-terms with some suspicion; the non-renormalization theorem does not guarantee that after extrapolation the F-terms are expressed in terms of the right degrees of freedom, and ignoring this can easily lead to puzzles.

3.4. Digression on ADE singularities

In order to understand where non-abelian gauge fields come from, we need to make a digression on ADE singularities, their resolutions and their deformations.

Let us consider an isolated singularity of a complex surface. Using complex coordinates x, y, z, the simplest such singularity takes the local form

$$P(x,y,z) = xy + z^2 = 0 \tag{3.16}$$

It is clearly singular as $\partial_x P = \partial_y P = \partial_z P = 0$ at $x = y = z = 0$. This singularity goes under many names in the literature. Algebraic geometers tend to call it an *ordinary double point* singularity.

Now we can think of $P(x,y,z)$ as the determinant of the following matrix:

$$M = \begin{pmatrix} x & z \\ -z & y \end{pmatrix} \tag{3.17}$$

The vanishing of $P = \det(M)$ implies that the matrix M has an eigenvector with eigenvalue zero. We can parametrize the eigenvector as (λ_1, λ_2). Then instead of the surface $P = 0$, we can consider the new surface defined by the pair of equations

$$\begin{pmatrix} x & z \\ -z & y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{3.18}$$

More precisely, this pair of equations defines a surface (and not a three-fold) if we identify $(\lambda_1, \lambda_2) \simeq (c\lambda_1, c\lambda_2)$ for any $c \in \mathbf{C}^*$. In other words, we consider the above as a pair of equations on $\mathbf{C}^3 \times \mathbf{CP}^1$.

When (x,y,z) are not all zero, we can solve for the ratio λ_1/λ_2 in terms of (x,y,z), and so we get back the same surface as before. But when (x,y,z) are all zero, the pair

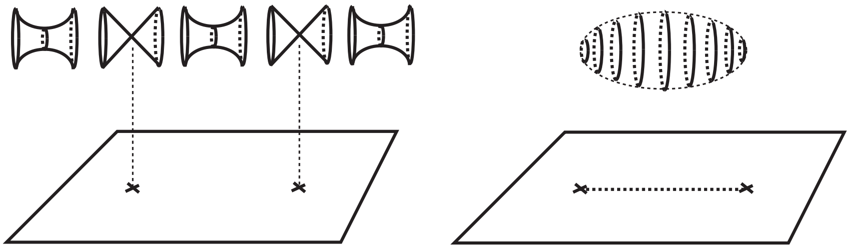

Figure 3: *Pictures for the deformed singularity $xy = P_2(z)$. (A): Fibration of conics over the z-plane, with the conic fiber degenerating over two special points. (B): Fibering the shrinking S^1 of the conic fibers over a path in the z-plane.*

(λ_1, λ_2) is undetermined and parametrizes a $\mathbf{CP}^1$, which we denote by E_1. Furthermore it is easy to check that this new surface is non-singular. The surface defined by (3.18) is said to be the *blow-up* or *resolution* of the singular surface $P = 0$ at $x = y = z = 0$, and the new $\mathbf{CP}^1$ is the *exceptional divisor* of the blow-up.

Note that our eigenvalue equation still implies that $P = \det(M) = 0$, so the process of blowing up is not a complex structure deformation, but rather a Kähler deformation. By varying the Kähler form we can continuously change the size of the exceptional $\mathbf{CP}^1$, making it large or shrinking it back to a singularity.

An alternative way to smooth the singularity is to change the equation to

$$xy + z^2 \ = \ \mu, \qquad \mu \neq 0 \tag{3.19}$$

This is a complex structure deformation, and not a Kähler deformation. In this case the smoothed geometry again contains a topological S^2, but unlike for the resolution it is not a holomorphic submanifold. To see it define $x = u + iv$, $y = u - iv$, in which case the equation becomes $u^2 + v^2 + z^2 = \mu$. Restricting all the variables to be real, this is precisely the equation of a (non-holomorphically embedded) two-sphere. Again as $\mu \to 0$ this two-sphere shrinks to zero size.

It will be useful to formulate the latter construction in a way that will generalize more easily. Let us write the deformation of the singularity as

$$xy \ = \ P_2(z) \tag{3.20}$$

where $P_2(z)$ is a quadratic polynomial with leading term z^2. An equation of the form $xy = t$ is a conic, so we may think of our surface as a conic bundle (a fibration of conics) over the z-plane. As $t \to 0$ the conic degenerates to a pair of lines, given by $x = 0$ and $y = 0$. The conic has a minimal S^1, which can be seen by rewriting $xy = t$ as $u^2 + v^2 = t$

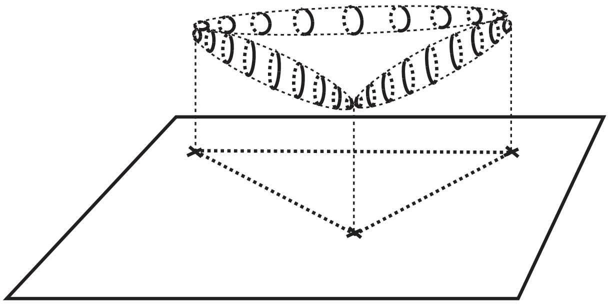

Figure 4: *Picture for the deformed singularity $xy = P_3(z)$. Fibering the shrinking S^1 of the conic fibers over different paths in the z-plane results in topologically distinct two-spheres.*

and taking the variables to be real. Note that the S^1 shrinks to zero as $t \to 0$. There are exactly two points on the z-plane where this happens: the roots z_1^* and z_2^* of the quadratic equation $P_2(z) = 0$. Now we take a path from z_1^* to z_2^* on the z-plane, and construct a manifold by fibering the minimal S^1 of the conic over this path. The resulting manifold is topologically a sphere, see figure .

Let us make the singularity a bit more complicated. We consider

$$Q(x, y, z) \;=\; xy + z^3 \;=\; 0 \tag{3.21}$$

Again as a first step we can consider replacing this by the following pair of equations:

$$\begin{pmatrix} x & z \\ -z^2 & y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \;=\; 0 \tag{3.22}$$

However from the second equation $-z^2\lambda_1 + y\lambda_2 = 0$ we see that our new surface is still singular at $z = y = \lambda_2 = 0$. But we can repeat the procedure, and replace this equation by the pair of equations

$$\begin{pmatrix} \lambda_2 & z \\ z\lambda_1 & y \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \;=\; 0 \tag{3.23}$$

This creates another exceptional $\mathbf{CP}^1$, which we denote E_2, this time parametrized by (μ_1, μ_2). Taking into account that (λ_1, λ_2) and (μ_1, μ_2) each live on a $\mathbf{CP}^1$ and cannot vanish simultaneously, it's not hard to see the surface resulting from our two blow-ups is now non-singular. You can also check that the two exceptional $\mathbf{CP}^1$'s intersect at precisely one point. Indeed when $x = y = z = 0$ the equations simply reduce to $\lambda_2\mu_1 = 0$, which describes two $\mathbf{CP}^1$'s with a simple intersection at $\lambda_2 = \mu_1 = 0$.

We can play a similar game with deforming the singularity. we write the deformed equation as

$$xy = P_3(z) \tag{3.24}$$

Dynkin type	*equation*	restrictions
A_n	$xy + z^{n+1} = 0$	$n \geq 1$
D_n	$x^2 + y^2 z + z^{n-1} = 0$	$n \geq 4$
E_6	$x^2 + y^3 + z^4 = 0$	
E_7	$x^2 + y^3 + yz^3 = 0$	
E_8	$x^2 + y^3 + z^5 = 0$	

Table ?: *Canonical forms of the ADE surface singularities.*

where $P_3(z)$ is a generic degree three polynomial with leading term z^3. Then the conic fibers degenerate when $P_3(z) = 0$, which happens at three special points on the z-plane, say z_1^*, z_2^* and z_3^*. Then by fibering the shrinking S^1's over paths from z_1^* to z_2^*, and from z_2^* to z_3^*, we get two topologically distinct two-spheres with intersection number one. We can make a further two-sphere by fibering the S^1's over a path from z_1^* to z_3^*, but by deforming this path to go through z_2^*, we see that the homology class of this two-sphere is the sum of the homology classes of the other two-spheres.

In general one can make some very nasty surface singularities. But there is a special class of surface singularities, of which the two cases above are the simplest examples, that have the property that the blow-up does not affect the topological type of the canonical bundle. This class is particularly interesting for us, because we are interested in Calabi-Yau spaces, and we want the topological type of the canonical bundle on the resolution to be trivial.

This class of surface singularities goes under many names: Kleinian singularities, DuVal singularities, etc. We will just call them ADE singularities, because they turn out to admit an ADE classification: the homology lattice of S_{ADE}, either after deformation or resolution, is isomorphic to an ADE root lattice:

$$H_2(S_{ADE}, \mathbf{Z}) \ = \ \Gamma_{ADE} \tag{3.25}$$

In particular, picking a set of two-cycles E_i corresponding to fundamental roots, their intersections are given the corresponding Cartan matrix:

$$C_{ij}^{ADE} \ = \ E_i \cdot E_j. \tag{3.26}$$

Up to coordinate transformations, they fit in the list shown in table ??

The appearance of an ADE classification suggests a connection with Lie groups, and hence with gauge theories. Indeed let us consider M-theory compactified on a resolved

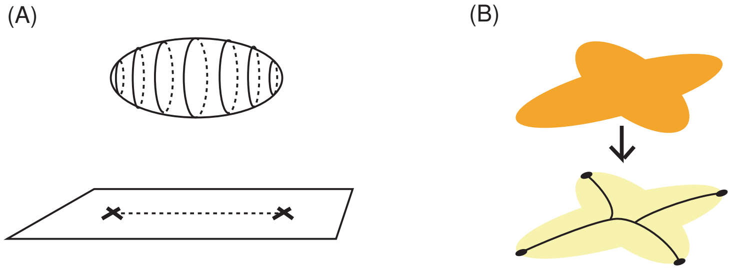

Figure 5: *An open fundamental string in type IIB lifts to a membrane wrapping an exceptional cycle in F-theory. Some W-bosons may correspond to ground states of multi-pronged* (p, q) *strings .*

ADE surface. M-theory has $M2$-branes which we can wrap on the two-cycles. Suppose that an $M2$-brane is wrapped on a cycle $\alpha = \sum n^i E_i$, yielding a particle-like object in the low-energy theory. One can show that when $\alpha \cdot \alpha = -2$, i.e. when α corresponds to a root, then quantizing such a particle yields a (massive) vector multiplet. Furthermore, we also have $H^2(S_{ADE}, \mathbf{Z}) = H_2(S_{ADE}, \mathbf{Z})^* = \Gamma_{ADE}$, so by expanding $C_{(3)}$ in harmonic two-forms we get further $U(1)$ gauge fields, corresponding to the Cartan generators of the ADE gauge group. Almost by definition the non-abelian fields have the right $U(1)$ Cartan charges. In this way we recover a non-abelian gauge theory with ADE gauge group.

The mass of each non-abelian W-boson is proportional to the volume of the cycle that the $M2$-brane is wrapped on. In the limit where we shrink the volume of these cycles to zero size and the geometry becomes singular, the mass of the non-abelian W-bosons goes to zero, and we recover the full unbroken non-abelian gauge symmetry. This is the general mechanism by which string or M-theory relates ADE singularities to ADE gauge theories. Unlike in perturbative IIB, where we could only get classical gauge groups, we also get exceptional gauge symmetries this way.

3.5. Non-abelian gauge fields and the Kodaira classification

Given what we said in the previous subsection, and considering that F-theory compactifications are closely related to M-theory, we would expect that non-abelian gauge fields are closely related to ADE singularities. Let us now discuss how we can see this from the point of view of (p, q) 7-branes and elliptic fibrations.

Let us consider two parallel D7-branes. The non-abelian gauge bosons that enhance the symmetry to $SU(2)$ come from open strings stretched between the two 7-branes. How is this seen in F-theory? Consider the path associated to an open string stretching between the branes. On top of each point of this path we can associate a 1-cycle of the T^2 fiber, which we take to be the $(1, 0)$ cycle. On the left and right ends of the path, this $(1, 0)$ cycle shrinks to zero. Altogether then we reconstruct a topological S^2. Using

ord(f)	*ord (g)*	*ord(Δ)*	*fiber type*	*singularity type*
≥ 0	≥ 0	0	smooth	$-$
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	$-$
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Table 2: *Kodaira classification of singularities of elliptic fibrations, indicating the order of vanishing of* Δ*,* f *and* g*.*

the M-theory perspective, we can wrap an $M2$-brane on this S^2, which turns into the fundamental open string as we go to F-theory. As the we let the 7-branes approach each other, the S^2 shrinks to zero and the Calabi-Yau fourfold develops an A_1 singularity. (The IIb space-time is still perfectly smooth, only when we add the elliptic fibration do we see the singularity). As the S^2 shrinks to zero, the ground states of the wrapped $M2$ brane or open fundamental string become massless, yielding the off-diagonal components of an $SU(2)$ vector multiplet. We get both W^+ and W^- by reversing the orientation of the membrane.

The type of singularities in an elliptic fibration were classified by Kodaira (see table 2). The elliptic fibration may develop an ADE singularity by letting various branes approach each other, and one would naturally expect that by wrapping $M2$ branes on the vanishing cycles one gets an enhanced ADE gauge symmetry.

In perturbative IIB we didn't see exceptional gauge symmetries. How do we understand these more general situations from the IIB space-time? It turns out that the exceptional cycles of a resolved ADE singularity do not necessarily project to open strings with two ends, but may yield so-called multi-pronged strings with multiple ends. This is the key to getting the exceptional groups and can happen when the dilaton cannot be taken small. All the ADE Lie algebras have been reproduced from such (generally multi-pronged) strings. For instance the roots of E_8 can be recovered from a configuration of

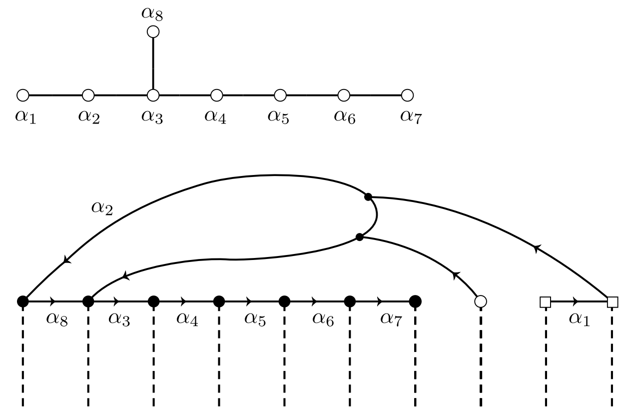

Figure 6: *Representation of the fundamental roots of E_8 from [4]. Here a cross denotes an A-brane, a circle denotes a B-brane, and a box denotes a C-brane.*

seven A-branes, one B brane, and two C-branes, where $A = (1, 0), B = (1, -1), C = (1, 1)$, see figure 6. Configurations for the type D Lie algebras are similar except that they just use one C brane instead of two. A B-brane and C-brane can be combined into an orientifold plane and yield weak coupling limits, but this is not possible for the exceptional cases.

Another difference is that in perturbative IIB we found $U(N)$ gauge symmetry, but in F-theory we find $SU(N)$ gauge symmetry. A similar issue also appears in M-theory. So what happened to the extra $U(1)$ when we turn on the string coupling? It turns out that this extra $U(1)$ is typically already massive in perturbative IIB due to a Stückelberg coupling to Ramond-Ramond fields, with a mass proportional to the string coupling. When the string coupling is finite this mass is of order the Kaluza-Klein scale, so it must correspond to a massive Kaluza-Klein mode.

As an application of this section, let us make a Calabi-Yau four-fold by writing down a fibration of deformed E_8 surfaces over a compact Kähler manifold S. For example we could take S to be $\mathbf{CP}^2$ or $\mathbf{CP}^1 \times \mathbf{CP}^1$, or a del Pezzo surface, i.e. a blow-up of $\mathbf{CP}^2$. We write our fibration over S as the following equation:

$$y^2 = x^3 + a_0 z^5 + a_2 z^3 x + a_3 z^2 y + a_4 z x^2 + a_5 z^5 \tag{3.27}$$

Here a_i is a section of $K_S^i \otimes L$ for some line bundle L on S. The equation has a term linear in y and a term with x^2, so it is not in Weierstrass form, but can be rewritten as such with a few coordinate changes. The terms $y^2 = x^3 + a_0 z^5$ describe an E_8 singularity and the remaining terms are a partial deformation of it, making the fibers less singular. The situation is somewhat analogous to what we saw in the heterotic string: there we started with an E_8 gauge theory, which was broken to a smaller gauge group by turning on a non-trivial profile for the gauge field. Here we also see some kind of breaking of E_8, but it seems less clear how this breaking comes about. You can actually translate this into the language of eight-dimensional gauge theory and see that the breaking of E_8 here is due to a non-trivial Higgs bundle (A, Φ). In particular, the sections a_i with $i > 0$ appearing in equation (3.27) are essentially invariant polynomials in the adjoint field Φ.

If you do the algebra for (3.27) you will see that one generically gets an $SU(5)$ singularity along $x = y = z = 0$, which is a copy of S. So from a IIB space-time perspective we have a bunch of 7-branes with $SU(5)$ gauge symmetry wrapped on S. Inspecting the discriminant, one finds that there are some additional 7-branes which intersect with the 7-branes wrapped on S along two curves, and one can show that charged matter fields propagate along these 7-brane intersections. In particular, one finds matter fields in the $\mathbf{10}$ of $SU(5)$ propagating along the curve $\Sigma_{\mathbf{10}} = \{a_5 = 0\}$ on S, and matter in the $\bar{\mathbf{5}}$ of $SU(5)$ propagating along the curve $\Sigma_{\bar{\mathbf{5}}} = \{a_0 a_5^2 - a_2 a_3 a_5 + a_3^2 a_4 = 0\}$. The geometry (3.27) therefore describes $SU(5)$ Grand Unified Models in F-theory.

3.6. Relation with perturbative IIB and heterotic strings

Finally we would like to briefly describe how the ingredients of perturbative string compactifications that we studied emerge from F-theory.

Let us start with the IIB limit, also known as the Sen limit of an F-theory compactification, which is really the easier one of the known limits. We take our Calabi-Yau four-fold to be an elliptic fibration over a (non-Calabi-Yau) Kähler three-fold B_3. Recall that the elliptic fibration was presented in Weierstrass form as

$$y^2 = x^3 + fx + g \tag{3.28}$$

where f is a section of $K_{B_3}^{-4}$, and g is a section of $K_{B_3}^{-6}$. Now we are going to introduce an alternative parametrization of the Weierstrass form, as follows. We consider three sections b_2, b_4, b_6, where b_i is a section of $K_{B_3}^{-i}$. Then we will write the equation of the elliptic fibers in the form

$$y^2 = \frac{1}{3}s^3 + b_2 s^2 + 2b_4 s + b_6 \tag{3.29}$$

This version has perhaps a bit more redundancy than the Weierstrass form, due to the extra quadratic term s^2 on the right hand side, but it is clearly a perfectly legal way to write a family of elliptic curves. One can relate it explicitly to the Weierstrass form by putting $s = x - b_2$ to eliminate the quadratic term. The coefficients were deliberately denoted by b_2, b_4 and b_6. As we will see shortly, they are closely related to their namesakes in the section on perturbative IIB compactifications.

Next we introduce a parameter t, as follows:

$$y^2 = \frac{1}{3}s^3 + b_2 s^2 + 2b_4 st + b_6 t^2 \tag{3.30}$$

The discriminant takes the form $\Delta = 324\, t^2 b_2^2 (b_2 b_6 - b_4^2) + \mathcal{O}(t^3)$. We claim that the IIB limit corresponds to $t \to 0$, i.e. one can show that the string coupling $e^\phi = Im(\tau)^{-1}$ goes to zero almost everywhere in this limit. So let's see what happens to our geometry when we take this limit.

If we take $t \to 0$ in (3.30), then we are simply left with $y^2 = s^2(b_2+s/3)$, or equivalently $\tilde{y}^2 = b_2 + s/3$ with $\tilde{y} = y/s$. This is just a $\mathbf{CP}^1$, so it might seem that our elliptic fibration has degenerated to a boring $\mathbf{CP}^1$-fibration over B_3. However, there is a slightly different way to take the limit. Let us define $s = tu$, $y = tw$, and then take the limit $t \to 0$. In this case (3.30) becomes

$$w^2 = b_2u^2 + 2b_4u + b_6 \tag{3.31}$$

which describes a fibration of conics over B_3.

The overall picture then is that in the $t \to 0$ limit, our elliptic fibers have split into two pieces, and hence the whole Calabi-Yau four-fold has split into two pieces. It's not hard to see that the intersection of these two pieces is precisely the Calabi-Yau three-fold $\xi^2 = b_2$ that we discussed in the context of IIB orientifolds. Our conics further degenerate to a pair of lines when $b_2b_6 - b_4^2 = 0$, so there is an S^1 shrinking to zero in the fibers there, which should correspond to the $D7$-brane locus as we have seen before. Indeed it has precisely the required form of the locus wrapped by the $D7$-branes discussed in the section on perturbative IIB.

It is somewhat amazing to see what has happened here. We discussed perturbative type IIB compactifications in terms of Calabi-Yau three-folds, orientifold planes and $D7$ branes. Now we see that much of this information can also be captured by Calabi-Yau four-fold with slightly degenerate elliptic fibers. Turning on the string coupling simply corresponds to a smoothing of this degenerate four-fold. To complete the picture we also have to discuss the fluxes, which one can also do quite explicitly.

It is interesting to see how the perturbative IIB spectrum that we discussed previously emerges from the Calabi-Yau four-fold. A precise comparison requires logarithmic cohomology on the degenerate Y_4, but there is a simple intuitive rule that works reasonably well: even cohomology groups $H^{p,q}_+(X)$ on the IIB side lift directly to the four-fold, whereas odd cohomology groups $H^{p,q}_-(X)$ only lift after wedging with a one-form dz on the elliptic fibers. So the $h^{2,1}_-(X)$ complex structure moduli on the IIB side lift to $(3,1)$ forms on Y_4, which do indeed describe complex structure moduli of Y_4. Similarly the $h^{1,1}_+(X)$ Kähler moduli lift to Kähler moduli of Y_4, and the $h^{1,1}_-(X)$ periods of (B_2, C_2) lift to $(2,1)$-forms, which describe periods of the F-theory three-form $C_{(3)}$. The $h^{2,1}_+(X)$ vectors are a little harder to see. The four-form C_4^+ of IIB lifts to $C_{(6)}$ on Y_4 with two indices on the elliptic fiber, where $C_{(6)}$ is the dual of $C_{(3)}$ in M-theory. The $h^{2,1}_+(X)$ lifts to $(3,2)$-forms on Y_4, and expanding $C_{(6)}$ in such forms yields the expected vectors.

One can show that the open string spectrum of perturbative IIB, which we did not discuss in detail, is also reproduced by the four-fold Y_4. Roughly they consist of gauge field deformations on the 7-brane worldvolume described by $(0,1)$-forms, and deformations of the adjoint field described by $(2,0)$-forms. By a $(1,1)$ shift they lift to $(1,2)$-forms and $(3,1)$-forms on Y_4, corresponding to additional deformations of $C_{(3)}$ and complex structure moduli of Y_4 respectively. Additional open string modes are obtained by quantizing wrapped $M2$-branes, as mentioned previously.

The heterotic limit is similar in spirit, but technically more complicated to discuss, essentially due to the more complicated group theory of E_8. Let us consider as a special case a Weierstrass fibration of the form

$$y^2 = x^3 + z^5 + fxz^4 + gz^6 + z^7 \tag{3.32}$$

where we assumed that B_3 admits a $\mathbf{P}^1$-fibration, z is a coordinate on that $\mathbf{P}^1$, and f and g are independent of z. This form can only be achieved in very special cases, because in general all the coefficients are non-trivial sections, but let us ignore that here. The four-fold has an E_8 singularity at $y = x = z = 0$ and another at $z = \infty$. We introduce a parameter t as

$$y^2 = x^3 + tz^5 + fxz^4 + gz^6 + z^7 \tag{3.33}$$

Again as we take $t \to 0$ the Calabi-Yau four-fold splits into two pieces. The first piece is given by taking $t \to 0$ directly in the above equation, and the second is given by defining $(x, y, z) = (t^2\tilde{x}, t^3\tilde{y}, t\tilde{z})$ and then taking $t \to 0$. These two pieces intersect over a Calabi-Yau three-fold, which is identified with the heterotic Calabi-Yau. Since we have an $E_8 \times E_8$ gauge symmetry in this case, the heterotic bundle is trivial.

References

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