RTG Geometry–Topology Summer School

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The geometry and topology of braid groups

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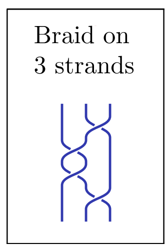
These notes and exercises accompany a 3-part lecture series on the geometry and topology of the braid groups. More advanced exercises are marked with an asterisk.

Lecture 1: Introducing the (pure) braid group

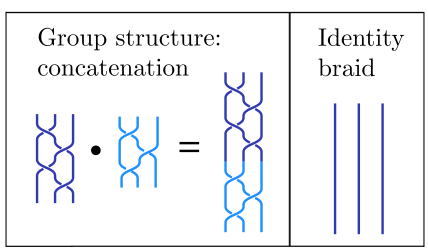
1 Five definitions of the (pure) braid group

1.1 The (pure) braid group via braid diagrams

Our first definition of the braid group is as a group of geometric braid diagrams. Informally, a braid on n strands is (an equivalence class of) pictures like the following, which we view as representing n braided strings in Euclidean 3-space that are anchored at their top and bottom at n distinguished points in the plane.



The strings may move in space but may not double back or pass through each other. These diagrams form a group under concatenation.



We formalize this structure with the following definition.

Definition I. (The (pure) braid group via braid diagrams.) Fix n. Let $p_1, \ldots, p_n$ be n distinguished points in $\mathbb{R}^2$. Let $(f_1, \ldots, f_n)$ be an n–tuple of functions

$$f_i : [0, 1] \longrightarrow \mathbb{R}^2$$

such that

$$f_i(0) = p_i, \qquad f_i(1) = p_j \text{ for some } j = 1, \ldots, n,$$

and such that the n paths

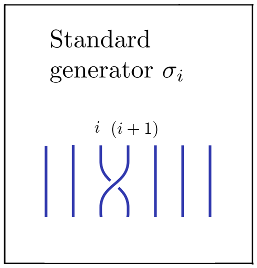
$$\begin{aligned} [0,1] &\longrightarrow \mathbb{R}^2 \times [0,1] \\ t &\longmapsto (f_i(t), t), \end{aligned}$$

called *strands*, have disjoint images. These n strands are called a *braid*. The *braid group* $\boldsymbol{B}_n$ *on* n *strands* is the group of isotopy classes of braids. The product of a braid $(f_1(t), \dots, f_n(t))$ and a braid $(g_1(t), \dots, g_n(t))$ is defined by

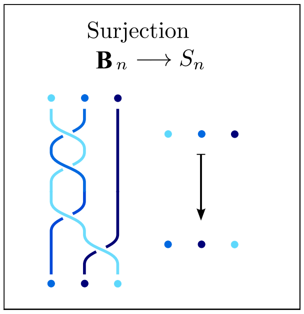
$$(f \bullet g)_i(t) = \begin{cases} f_i(2t), & 0 \leq t \leq \frac{1}{2} \\ g_j(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases} \qquad \text{where } j \text{ is such that } f_i(1) = p_j.$$

Exercise 1. (Inverses in $\mathbf{B}_n$.) Find a geometric rule for constructing the inverse β^{-1} of a braid diagram β.

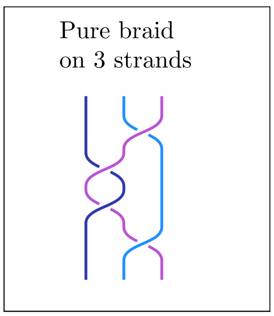
Exercise 2. (Generating $\mathbf{B}_n$.) Verify that every braid in $\mathbf{B}_n$ can be written as a product of the half-twists $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ shown below and their inverses.



Each braid determines a permutation on the points $p_1, \dots, p_n$, giving a surjection $\mathbf{B}_n \twoheadrightarrow S_n$.

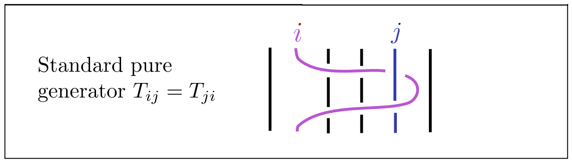


The *pure braid group* $\mathbf{PB}_n$ on n strands is the kernel of the natural surjection $\mathbf{B}_n \to S_n$.



These are the braids for which each path f_i begins and ends at the same point p_i. It is sometimes called the *coloured braid group*, since each strand can be assigned a distinct colour in a way compatible with composition.

Exercise 3. (Generating PB$_n$.) Define the braid $T_{i,j}$ as shown.



Verify that $\mathbf{PB}_n$ is generated by the $\binom{n}{2}$ twists $T_{i,j}$ for $i < j$, $i, j = 1, \ldots, n$.

Our second mode of defining the braid group is by an explicit presentation due to Artin.

1.2 The (pure) braid group via Artin's presentations

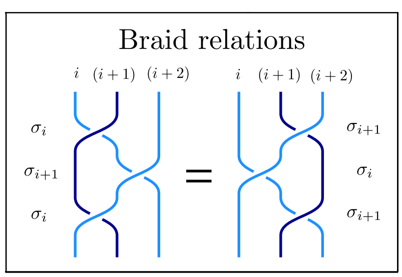
Definition II. (The (pure) braid group via Artin's presentations.) The braid group $\mathbf{B}_n$ on n strands is defined by the presentation

$$\mathbf{B}_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for all } |i-j| > 1 & \end{array} \right\rangle. \tag{1}$$

The pure braid group is defined by the presentation

$$\mathbf{PB}_n = \left\langle \begin{array}{l} T_{i,j} \text{ for } i < j \\ i, j \in \{1, 2, \ldots, n\} \end{array} \;\middle|\; \begin{array}{l} [T_{p,q}, T_{r,s}] = 1 \quad \text{for } p < q < r < s, \\ [T_{p,s}, T_{q,r}] = 1 \text{ for } p < q < r < s, \\ T_{p,r}T_{q,r}T_{p,q} = T_{q,r}T_{p,q}T_{p,r} = T_{p,q}T_{p,r}T_{q,r} \quad \text{for } p < q < r \\ \left[T_{r,s}T_{p,r}T_{r,s}^{-1}, T_{q,s}\right] = 1 \quad \text{for } p < q < r < s \end{array} \right\rangle. \tag{2}$$

Exercise 4. Verify that the generators σ_i for the braid group as defined in Exercise 2 satisfy the relations in Equation 1.



Exercise 5. Verify that the generators $T_{i,j}$ for the pure group as defined in Exercise 3 satisfy the relations in Equation 2.

Exercise 6. Observe that adding the relations $\sigma_i^2 = 1$ to Equation 1 yields a standard presentation for the symmetric group S_n.

(a) Conclude that the twists σ_i^2 form a normal generating set for $\mathbf{PB}_n$.

(b) Express the generators $T_{i,j}$ of $\mathbf{PB}_n$ as conjugates of the elements σ_i^2.

Exercise 7. (Abelianizations of $\mathbf{B}_n$ and $\mathbf{PB}_n$.) Use the presentations of $\mathbf{B}_n$ and $\mathbf{PB}_n$ to do the following.

(a) Show that in the abelianization of $\mathbf{B}_n$, all the generators σ_i are identified to a single nontrivial element. Conclude that

$$\mathbf{B}_n^{ab} \cong \mathbb{Z}.$$

(b) Prove that

$$\mathbf{PB}_n^{ab} \cong \mathbb{Z}^{\binom{n}{2}}$$

is the free abelian group on the images of the $\binom{n}{2}$ generators $T_{i,j}$.

(c) Conclude that

$$H_1(\mathbf{B}_n) \cong \mathbb{Z}, \qquad H^1(\mathbf{B}_n) = \mathrm{Hom}(\mathbf{B}_n, \mathbb{Z}) \cong \mathbb{Z}$$
$$H_1(\mathbf{PB}_n) \cong \mathbb{Z}^{\binom{n}{2}}, \qquad H^1(\mathbf{PB}_n) = \mathrm{Hom}(\mathbf{PB}_n, \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{2}}$$

(d) Show that we can interpret the generator for the cohomology group $H^1(\mathbf{B}_n) = \mathrm{Hom}(\mathbf{B}_n, \mathbb{Z})$ as the function that takes a braid β, viewed as a word in the generators σ_i, to the total exponent of all the generators. Equivalently, it takes a braid β and counts (with sign) the total number of half-twists σ_i in β.

(e) Show that we can interpret the generators for the cohomology group $H^1(\mathbf{PB}_n) = \mathrm{Hom}(\mathbf{PB}_n, \mathbb{Z})$ as elements $T_{i,j}^*$ that take a pure braid β, viewed as a word in the generators $T_{i,j}$, and map β to the total exponent to $T_{i,j}$. Equivalently, it takes a braid β and counts (with sign) the total number of times the i^{th} strand winds clockwise around the j^{th} strand.

Exercise 8. Show that the short exact sequence

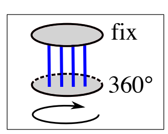
$$1 \longrightarrow \mathbf{PB}_n \longrightarrow \mathbf{B}_n \longrightarrow S_n \longrightarrow 1$$

does **not** split for $n \geq 2$.
Hint: See Exercise 7(a). What is the abelianization of a semi-direct product?
Alternate hint: See Corollary VII.

Exercise 9. (The centre of $\mathbf{B}_n$ and $\mathbf{PB}_n$.) Define the braid

$$z = \Big(\sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\cdots(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)\Big)^2.$$

(a) Show that z defines a "full twist" of all strands, as shown.



(b) Show that the centre $Z(\mathbf{B}_n)$ is an infinite cyclic group generated by z.

(c) Show that $z \in \mathbf{PB}_n$ and $Z(\mathbf{B}_n) = Z(\mathbf{PB}_n)$.

(d) Prove that the inclusion $Z(\mathbf{PB}_n) \hookrightarrow \mathbf{PB}_n$ splits.

Stuck? See Farb–Margalit [FM, Chapter 9].

Exercise* 10. Show that $\mathbf{B}_3 \subseteq \overline{\mathrm{SL}_2(\mathbb{R})}$ is the universal central extension of the modular group

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbf{B}_3 \longrightarrow \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow 1$$

and $\mathrm{PSL}_2(\mathbb{Z}) \cong B_3/Z(B_3)$.

A third viewpoint on the (pure) braid group is as the fundamental group of the configuration space of $\mathbb{C}$.

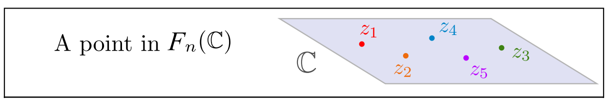
1.3 The (pure) braid group via configuration spaces

Definition III. (The (pure) braid group via configuration spaces.) For a topological space M, define the *ordered configuration space of M on n points* to be the space

$$F_n(M) = \{(m_1, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\},$$

topologized a subspace of M^n. Equivalently, we may view $F_n(M)$ as the space of embeddings

$$\{1, 2, 3, \ldots, n\} \hookrightarrow M.$$



The symmetric group S_n acts on $F_n(M)$ by permuting the coordinates. The quotient space $C_n(M)$ under this action is called the *unordered configuration space of M on n points*. This is the space

$$C_n(M) = \Big\{\{m_1, \ldots, m_n\} \subset M\Big\}$$

of n–element subsets of M. We define the braid group $\mathbf{B}_n$ to be the fundamental group of the unordered configuration space of the plane,

$$\mathbf{B}_n = \pi_1(C_n(\mathbb{C})).$$

We define the pure braid group $\mathbf{PB}_n$ as the fundamental group of the ordered configuration space of the plane,

$$\mathbf{PB}_n = \pi_1(F_n(\mathbb{C})).$$

Exercise 11. Suppose that M is a manifold of (real) dimensional d.

(a) Show that $F_n(M)$ and $C_n(M)$ are manifolds, and compute their dimensions.

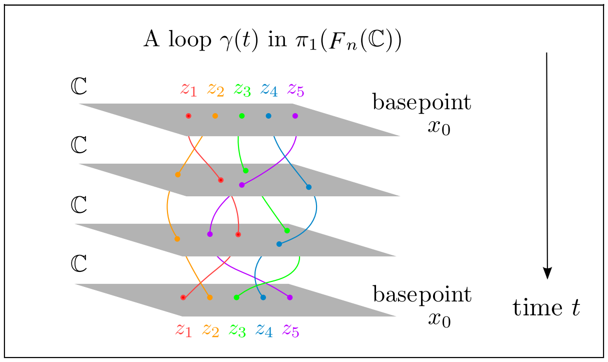
(b) Suppose $d = 1$ and M is the interval $[0, 1]$. Show that $F_n(M)$ is the disjoint union of (contractible) simplices. How many components does it have? What is the space $C_n(M)$?

(c) Suppose that $d \geq 2$ and M is connected. Show that $F_n(M)$ and $C_n(M)$ are connected.

(d) Show that $F_n(\mathbb{C})$ and $C_n(\mathbb{C})$ are complex manifolds.

Exercise 12. Show that the quotient $F_n(M) \to C_n(M)$ is a normal covering space with Deck group S_n.

Exercise 13. Verify that the definition of the (pure) braid group as the fundamental group of configuration space coincides with its definition by braid diagrams given in Definition I.



Interpret the braids σ_i and $T_{i,j}$ as loops in $C_n(M)$ and $F_n(M)$.

Exercise 14. Show by example that even if M and M' are homotopy equivalent, then $F_n(M)$ and $F_n(M')$ need not be homotopy equivalent.

We can identify the configuration space $C_n(\mathbb{C})$ as a space of polynomials indexed by their roots.

1.4 The braid group via polynomials

Definition IV. (The braid group via polynomials.) Let $P_n(\mathbb{F})$ denote the set of monic polynomials with coefficients in a field $\mathbb{F}$, and write P_n for $P_n(\mathbb{C})$. We view P_n as a topological space by identifying

$$\begin{aligned} P_n &\xrightarrow{\cong} \mathbb{C}^n \\ p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n &\longmapsto (a_1, a_2, \ldots, a_n) \end{aligned}$$

Let SP_n denote the subspace of P_n of squarefree polynomials, that is, polynomials having n distinct roots. Then there is a diffeomorphism

$$\begin{aligned} SP_n &\xrightarrow{\cong} C_n(\mathbb{C}) \\ p(x) = (x - z_1) \cdots (x - z_n) &\longmapsto \{z_1, \ldots, z_n\} \end{aligned}$$

We define the braid group as the fundamental group $\pi_1(SP_n)$.

Exercise 15. Prove that a polynomial $p(x)$ is squarefree if and only if $p(x)$ and its derivative $p'(x)$ are coprime.

Exercise 16. (The Viète map.)

(a) Let $s_1, s_2, \ldots s_n$ denote the n elementary symmetric polynomials in n variables, so that a polynomial

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = (x - z_1) \cdots (x - z_n)$$

has coefficients $a_i = s_i(z_1, z_2, \ldots, z_n)$. Verify that the Viète map

$$\begin{aligned} V : \mathbb{C}^n/S_n &\longrightarrow \mathbb{C}^n \\ \{z_1, z_2, \ldots z_n\} &\longmapsto (s_1(z_1, z_2, \ldots, z_n), \ldots, s_n(z_1, z_2, \ldots, z_n)) \\ p(x) = (x - z_1)\cdots(x - z_n) &\longmapsto p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n \end{aligned}$$

is a diffeomorphism between different incarnations of the space P_n.

(b) Verify that the map $SP_n(\mathbb{C}) \to C_n(\mathbb{C})$ is in fact a diffeomorphism.

(c) Consider the restriction of the map V to the space $SP_n = C_n(\mathbb{C})$ of polynomials with distinct roots. Describe the image of SP_n in $\mathbb{C}^n$. *Hint:* discriminant.

(d) Show that the derivative of the composite map $\mathbb{C}^n \to \mathbb{C}^n/S_n \cong \mathbb{C}^n$ is the Vandermonde polynomial. Conclude that the quotient map $\mathbb{C}^n \to \mathbb{C}^n/S_n$ is a local diffeomorphism around a point $\mathbf{z}$ if and only if $\mathbf{z} \in F_n(\mathbb{C})$.

(e) Does the map V define a diffeomorphism $\mathbb{R}^n/S_n \cong \mathbb{R}^n$? What is the preimage of $\mathbb{R}^n$ under V?

We next give a third description of the space $F_n(\mathbb{C})$, realizing it as examples of hyperplane complements.

1.5 The (pure) braid group via hyperplane complements

Definition V. (The (pure) braid group via hyperplane complements.) Let G be a group of linear maps acting on $\mathbb{R}^d$, generated by a finite set of reflections. Let $\{s_i\}$ denote the set of all reflections in G, and let H_i denote the hyperplane fixed by s_i. Then there is a induced action of G on the *hyperplane complement*

$$\mathcal{M}_G = \mathbb{C}^d \backslash \{\text{union of complexified hyperplanes } H_i \otimes_{\mathbb{R}} \mathbb{C}\ \}$$

Let $G \cong S_n$ be the group of $n \times n$ permutation matrices. Then we define the pure braid group $\mathbf{PB}_n$ and the braid group $\mathbf{B}_n$ to be the fundamental groups

$$\mathbf{PB}_n = \pi_1(\mathcal{M}_{S_n}) \qquad \text{and} \qquad \mathbf{B}_n = \pi_1(\mathcal{M}_{S_n}/S_n)$$

More generally, the fundamental group of the quotient $\mathcal{M}_G/G$ is called the *generalized braid group*, and the fundamental group of $\mathcal{M}_G$ is the *pure generalized braid group* associated to G.

Exercise 17. Verify that G stabilizes the set of hyperplanes $\{H_i\}$, and therefore has a well-defined action on the complement of the complexified hyperplanes in $\mathbb{C}^d$.

Exercise 18. Verify that the complexified hyperplanes have real codimension 2 in $\mathbb{C}^d$, and conclude that $\mathcal{M}_G$ is connected.

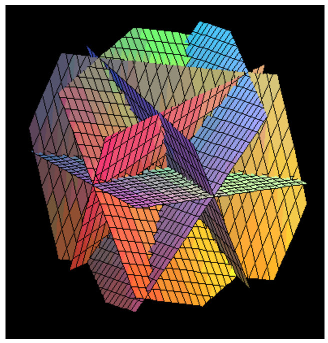
Exercise 19. ($F_n(\mathbb{C})$ is a hyperplane complement.) Let $G \cong S_n$ be the group of $n \times n$ permutation matrices.

(a) Show that G is generated by reflections given by simple transpositions, and the set of all reflections is precisely the set of 2–cycles.

(b) Show that the associated hyperplanes are defined by the equations $z_i - z_j = 0$.

(c) Show that $\mathcal{M}_G$ is precisely the ordered configuration space $F_n(\mathbb{C})$ with its natural S_n–action.

Exercise 20. The action of the permutation matrices on $\mathbb{R}^n$ stabilizes the $(n-1)$–dimensional subspace
$$V \cong \{(x_1, \ldots, x_n) \mid x_1 + x_2 + \cdots + x_n = 0\}.$$
The set of reflecting hyperplanes for the action of S_n on V is the usual hyperplane arrangement associated to the Coxeter presentation of S_n. John Stembridge created the following image of the *real* hyperplane arrangement when $n = 4$.



Let $\mathcal{M}$ denote the corresponding complex hyperplane complement. Show that there is an S_n–equivariant homotopy equivalence $F_n(\mathbb{C}) \to \mathcal{M}$ by projecting along the diagonal
$$z_1 = z_2 = \cdots = z_n.$$

1.6 Other viewpoints on the (pure) braid group

The (pure) braid group arises in a number of other contexts in topology and combinatorics.

Exercise* 21. Show that the (pure) braid group is isomorphic to the (pure) mapping class group of a closed disk. *Stuck? See Farb–Margalit [FM, Chapter 9].*

Exercise* 22. Show that the (pure) braid group embeds in the automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n.

2 The topology of $F_n(\mathbb{C})$

2.1 The fibration $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$

Define a projection map
$$\begin{aligned} \rho_n : F_n(\mathbb{C}) &\longrightarrow F_{n-1}(\mathbb{C}) \\ (z_1, \ldots z_{n-1}, z_n) &\longmapsto (z_1, \ldots, z_{n-1}) \end{aligned}$$

Exercise 23. (The fibrations ρ_n.)

(a) Prove that the map ρ_n is a fibration. Show that the fibre F is homeomorphic to a $(n-1)$–times punctured plane, and hence F is homotopy equivalence to a wedge of 1–spheres $\bigvee^{n-1} S^1$.

(b) Show that the map

$$\begin{aligned}\iota_n : F_{n-1}(\mathbb{C}) &\longrightarrow F_n(\mathbb{C})\\ (z_1, \dots z_{n-1}) &\longmapsto (z_1, \dots, z_{n-1}, \max_i |z_i| + 1)\end{aligned}$$

defines a splitting of the fibration ρ_n.

(c) Describe the maps on $\mathbf{PB}_n$ and $\mathbf{PB}_{n-1}$ induced by ρ_n and ι_n. Interpret these maps as operations on braid diagrams.

(d) Let F_{n-1} denote the free group on $(n-1)$ letters. Show that there is a short exact sequence

$$1 \to F_{n-1} \to \mathbf{PB}_n \to \mathbf{PB}_{n-1} \to 1,$$

and that this sequence is split, so $\mathbf{PB}_n \cong \mathbf{PB}_{n-1} \ltimes F_{n-1}$. Conclude that the pure braid group is an iterated extension of free groups.

(e) What are the generators of F_{n-1} in terms of the generators $T_{i,j}$ of $\mathbf{PB}_n$?

Exercise 24. Show that it is not possible to define a continuous "forget a point" map on the unordered configuration spaces $C_n(\mathbb{R}^2) \to C_{n-1}(\mathbb{R}^2)$.

Exercise* 25. Let $M = S^2$ denote the 2–sphere. Determine whether the projection map $\rho_n : F_n(S^2) \to F_{n-1}(S^2)$ is split. *Stuck? See Chen [Ch].*

These fibrations are valuable tools for studying these configuration spaces. One application is the following result.

2.2 The configuration spaces of $\mathbb{C}$ are $K(\pi, 1)$'s

Theorem VI. *The spaces $F_n(\mathbb{C})$ and $C_n(\mathbb{C})$ are a $K(\boldsymbol{PB}_n, 1)$ and $K(\boldsymbol{B}_n, 1)$ space, respectively. In particular,*

$$H^*(\boldsymbol{PB}_n) = H^*(F_n(\mathbb{C})) \qquad \text{and} \qquad H^*(\boldsymbol{B}_n) = H^*(C_n(\mathbb{C})).$$

The proof of Theorem VI is outlined in the following exercise.

Exercise 26. (The configuration spaces of $\mathbb{C}$ are $K(\pi, 1)$'s.)

(a) Write down the long exact sequence on homotopy groups π_i associated to the fibration $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$.

(b) Compute the homotopy groups of the fibre $F \simeq \bigvee^{n-1} S^1$.

(c) Using induction on n and i, prove that the higher homotopy groups $\pi_i(F_n(\mathbb{C}))$ vanish. Conclude that $F_n(\mathbb{C})$ is a $K(\mathbf{PB}_n, 1)$.

(d) Deduce that the universal cover of $F_n(\mathbb{C})$ is contractible. Conclude that $C_n(\mathbb{C})$ is a $K(\mathbf{B}_n, 1)$.

Exercise 27.

(a) Let $W_n \cong S_n \rtimes \mathbb{Z}/2\mathbb{Z}$ be the group of signed permutation matrices, so W_n is the Weyl group of type B/C. Modify the proof of Theorem VI to prove that $\mathcal{M}_{B_n}$ is a $K(\pi, 1)$ space for the pure generalized braid group in type B/C.

(b)* Do the same for the pure generalized braid group in type D.
Stuck? See Brieskorn [Br, Proposition 2].

Exercise 28.

(a) Let S be a surface. Are the configuration spaces $F_n(S)$ and $B_n(S)$ necessarily $K(\pi, 1)$ spaces? Their fundamental groups are called the *(pure) surface braid groups of S.*

(b) Let M be a manifold of dimension ≥ 3. Are the configuration spaces $F_n(M)$ and $B_n(M)$ $K(\pi, 1)$ spaces?

Corollary VII. *The groups $\boldsymbol{PB}_n$ and $\boldsymbol{B}_n$ are torsion-free.*

The proof is outlined in the following exercise.

Exercise 29. ($\mathbf{PB}_n$ and $\mathbf{B}_n$ are torsion-free.)

(a) Let $G \cong \mathbb{Z}/m\mathbb{Z}$ be a nontrivial finite cyclic group. Compute $H^*(G)$. Conclude that the trivial G–representation $\mathbb{Z}$ does not admit a finite-length free resolution by $\mathbb{Z}[G]$–modules.

(b) Let G be a group, and $H \subseteq G$ a subgroup. Show that any free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$–modules is, under restriction of scalars, a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[H]$–modules.

(c) Deduce that if G contains torsion, then the trivial G–representation $\mathbb{Z}$ does not admit a finite-length free resolution by $\mathbb{Z}[G]$–modules.

(d) Suppose G is a group with CW–complex X a $K(G, 1)$ space. Show that (with an appropriately constructed cell structure) the augmented cellular chain complex on the universal cover $\widetilde{X}$ of X is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$–modules.

(e) Conclude that if G has torsion, then G does not admit a finite-dimensional $K(G, 1)$ space.

(f) Conclude that the braid group and pure braid group are torsion-free.

2.3 Computations for small n

Exercise 30. (Configuration spaces for small n). Prove the following.

(a) $F_1(M) = C_1(M) = M$ for any topological space M.

(b) There is a deformation retract

$$\begin{aligned} F_2(\mathbb{R}^d) &\longrightarrow S^{d-1} \\ (x, y) &\longmapsto \frac{(x-y)}{|x-y|} \end{aligned}$$

In particular $F_2(\mathbb{C}) \simeq S^1$.

(c) There are homeomorphisms

$$F_n(\mathbb{R}^d) \cong \mathbb{R}^d \times F_{n-1}(\mathbb{R}^d \backslash \{0\})$$

$$F_n(\mathbb{C}^\times) \cong (\mathbb{C}^\times) \times F_{n-1}(\mathbb{C}^\times \backslash \{1\})$$

Thus for $n \geq 2$,

$$F_n(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C} \backslash \{0\} \times F_{n-2}(\mathbb{C} \backslash \{0, 1\})$$

Exercise 31. ((Co)homology of configuration spaces for small n). By Exercise 30,

$$F_1(\mathbb{C}) \cong \mathbb{C} \qquad F_2(\mathbb{C}) \simeq S^1 \qquad F_3(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C}\backslash\{0\} \times \mathbb{C}\backslash\{0,1\}$$

Use these results to compute the (co)homology groups of $F_1(\mathbb{C})$, $F_2(\mathbb{C})$, and $F_3(\mathbb{C})$. Explain how to identify the (co)homology classes in degree 1 with the group-theoretic description of the degree 1 (co)homology classes of $\mathbf{B}_n$ and $\mathbf{PB}_n$ from Exercise 7.

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The geometry and topology of braid groups

Jenny Wilson

Lecture 2: The cohomology of the pure braid group

3 The integral cohomology of the pure braid group

3.1 A result of Arnold

The integral cohomology ring of the pure braid group was computed by Arnold in 1969. This section will describe his work.

Viewing $F_n(\mathbb{C}) \subset \mathbb{C}^n$ as a complex manifold, we can define forms

$$\omega_{i,j} := \frac{1}{2\pi I}\left(\frac{dz_i - dz_j}{z_i - z_j}\right), \qquad I \text{ a square root of } -1,\ i \neq j,\ i,j \in \{1,2,\ldots,n\}.$$

We can interpret the form $\omega_{i,j}$ as measuring the "winding number" of a loop around the deleted hyperplane $z_i = z_j$. These forms satisfy the identity

$$\omega_{i,j} \wedge \omega_{j,k} + \omega_{j,k} \wedge \omega_{k,i} + \omega_{k,i} \wedge \omega_{i,j} = 0 \qquad \text{distinct } i,j,k \in \{1,2,\ldots,n\}. \tag{3}$$

The action of S_n on $F_n(\mathbb{C})$ induces an action on these forms by

$$\sigma \cdot \omega_{i,j} = \omega_{\sigma(i),\sigma(j)} \qquad \text{for } \sigma \in S_n.$$

Exercise 32. (Properties of ω_{ij}.)

(a) Show that $\omega_{i,j} = \omega_{j,i}$.

(b) Verify by direct computation that the forms $\omega_{i,j}$ satisfy the relation given in Equation 3.

(c) Show that the cohomology class $\omega_{i,j}$ corresponds to the element $T^*_{i,j} \in H^1(\mathbf{PB}_n)$ as defined in Exercise 7.

Exercise 33. (Inclusions of cohomology.) Because the fibration $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$ is split, show that the induced map on cohomology

$$(\rho_n)^* : H^*(F_{n-1}(\mathbb{C})) \to H^*(F_n(\mathbb{C}))$$

is injective. Show that this map takes $\omega_{i,j} \in H^1(F_{n-1}(\mathbb{C}))$ to $\omega_{i,j} \in H^1(F_n(\mathbb{C}))$.

Theorem VIII (Arnold [A1]). ***(The cohomology of*** $F_n(\mathbb{C})$***).*** *The cohomology algebra $H^*(F_n(\mathbb{C}))$ is the exterior graded algebra generated by the $\binom{n}{2}$ forms $\omega_{i,j}$, which are subject to the $\binom{n}{3}$ relations in Equation 3:*

$$H^*(F_n(\mathbb{C})) \cong \frac{\bigwedge^*_{\mathbb{Z}} \omega_{i,j}}{\langle \omega_{q,r} \wedge \omega_{r,s} + \omega_{r,s} \wedge \omega_{s,q} + \omega_{s,q} \wedge \omega_{q,r} \rangle} \qquad \begin{array}{l} i,j,q,r,s \in \{1,2,\ldots,n\}, \\ i,j \text{ distinct}, \quad q,r,s \text{ distinct}. \end{array}$$

Notably, Arnold proved that an exterior polynomial in the differentials form $\omega_{i,j}$ is cohomologous to zero if and only if it is in fact equal to zero. The cohomology algebra $H^*(F_n(\mathbb{C}))$ is the $\mathbb{Z}$–subalgebra of meromorphic forms on $\mathbb{C}^n$ generated by the elements $\omega_{i,j}$.

Exercise 34. (Additive generators for $H^p(F_n(\mathbb{C}))$.) Assuming Theorem VIII, prove that $H^p(F_n(\mathbb{C}))$ is spanned by exterior monomials of the form

$$\omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \wedge \cdots \wedge \omega_{i_p,j_p}, \qquad \text{where } i_s < j_s, \text{ and } j_1 < j_2 < \cdots < j_p.$$

Exercise 35. (Poincaré polynomial for $F_n(\mathbb{C})$.) Deduce from Theorem VIII that the Poincaré polynomial (the generating function for the Betti numbers) of the pure braid group is

$$p(t) = (1+t)(1+2t)\cdots(1+(n-1)t).$$

Exercise* 36. Assume Theorem VIII. Let $k \geq 2$, and let ω be any rational exterior polynomial of degree k in the differential forms. Prove that its symmetrization $\displaystyle\sum_{\sigma \in S_n} \sigma \cdot \omega$ is zero.

To prove Theorem VIII, we will study the Serre spectral sequence associated to the fibration $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$ we obtain for each n.

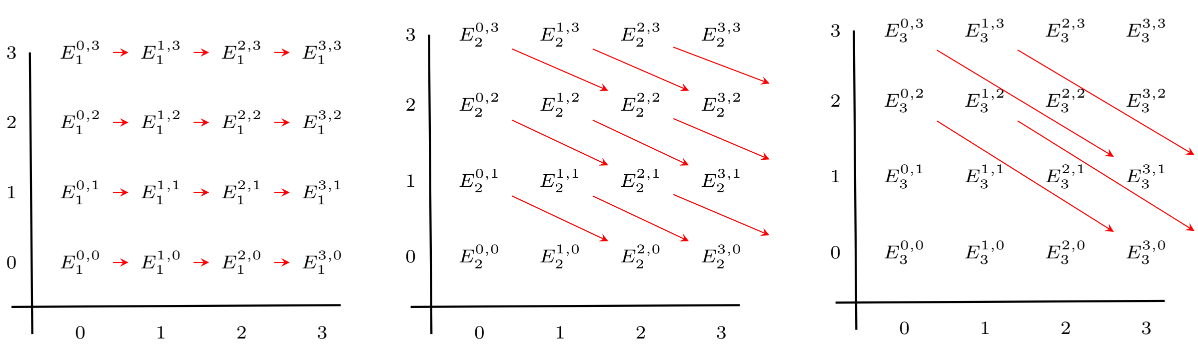
3.2 The structure of a cohomology spectral sequence

Recall that a (cohomology) spectral sequence is a sequence of bigraded abelian groups $E_r = \bigoplus_{p,q} E_r^{p,q}$, called *pages*, for $r = 0, 1, 2, \ldots$. Each page has a differential map $d^r : E_r \to E_r$ satisfying $d_r^2 = 0$, and the page E_{r+1} is the homology of the complex (E_r, d_r), in the sense that

$$E_{r+1}^{p,q} = \frac{\text{kernel of } d_r \text{ at } E_r^{p,q}}{\text{image of } d_r \text{ in } E_r^{p,q}}.$$

In particular $E_{r+1}^{p,q}$ is always a subquotient of $E_r^{p,q}$. For the cohomology Serre spectral sequence, the differentials satisfy

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$



The pages E_1, E_2, and E_3.

The Serre spectral sequence is an example of a *first quadrant spectral sequence*, that is, the groups $E_r^{p,q}$ can be nonzero only when p and q are nonnegative. This implies that, at any fixed point (p,q), for

r sufficiently large, either the domain or the codomain of any differential d_r to or from $E_r^{p,q}$ will be zero. Hence, for r large we find (upon taking homology)

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots$$

We call this stable group $E_\infty^{p,q}$, and call the bigraded abelian group $E_\infty^{*,*}$ the *limit* of the spectral sequence. In general the sequence of groups $\{E_r^{p,q}\}_r$ converges at a page r that depends on (p,q). If there is some r such that $E_r^{p,q} = E_\infty^{p,q}$ for all p and q, then we say that the spectral sequence *collapses* on page E_r.

We now specialize to the Serre spectral sequence.

3.3 The Serre spectral sequence

Let $F \to E \to B$ be a fibration.

> **Exercise 37.** Define the *(algebraic) monodromy* action of the fundamental group $\pi_1(B)$ of the base on the homology and cohomology of the fibre F.
>
> **Exercise 38.** Show that in the case of the fibration $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$, the pure braid group $\mathbf{PB}_{n-1} \cong \pi_1(F_{n-1}(\mathbb{C}))$ acts trivially on the (co)homology of the fibre.

The Serre spectral sequence is a tool that relates the (co)homology of the total space E to the (co)homology of the base B and fibre F.

Theorem IX. ***(The cohomology Serre spectral sequence)***. *Given a fibration $F \to E \to B$ there is an associated (cohomology) spectral sequence $E_*^{p,q}$ with differentials*

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

as follows. The cohomology $H^q(F)$ is a $\mathbb{Z}[\pi_1(B)]$–module, and the E_2 page is the bigraded algebra of cohomology groups with twisted coefficients

$$E_2^{p,q} = H^p(B; H^q(F)).$$

The page E_r has a multiplication

$$E_r^{p,q} \times E_r^{s,t} \longrightarrow E_r^{p+s,q+t}$$

which is, on the E_2 page, $(-1)^{qs}$ times the cup product. The differentials d_r are derivations, satisfying

$$d_r(xy) = (d_r x)y + (-1)^{p+q} x(d_r y).$$

The spectral sequence converges to the cohomology groups

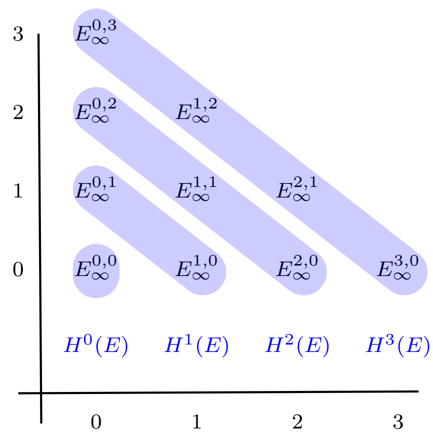
$$H^{p+q}(E)$$

in the sense that there is some filtration of $H^k(E)$

$$0 \subseteq F_k^k \subseteq \cdots \subseteq F_0^k = H^k(E)$$

such that the limiting groups $E_\infty^{p,q}$ are the associated graded pieces

$$E_\infty^{p,q} = F_p^{p+q} / F_{p+1}^{p+q}.$$



The limit of the Serre spectral sequence.

Remark X. (Recovering $H^*(E)$.) In general, knowing the quotient groups $E_\infty^{p,q} = F_p^{p+q}/F_{p+1}^{p+q}$ is not enough to reconstruct the cohomology groups $H^*(E)$; we can only determine these groups "up to extensions". We will see in Exercise 41, however, that in the case of the fibrations $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$ we can completely recover the cohomology groups of the total space $F_n(\mathbb{C})$ from the spectral sequence.

Remark XI. (Trivial monodromy.) Note that if, as in the case with the fibrations ρ_n, the action by $\pi_1(B)$ on the fibre is trivial, then the E_2 page

$$E_2^{p,q} = H^p(B; H^q(F))$$

is cohomology with trivial (ie, non-twisted) coefficients in the abelian group $H^q(F)$.

3.4 The proof of Arnold's result

We will prove Theorem VIII in the following series of exercises. We will see in fact that the cohomology groups of $F_n(\mathbb{C})$ are the same as that of the product of $F_{n-1}(\mathbb{C})$ and the fibre.

Exercise 39. Let $E_*^{p,q}$ be the Serre spectral sequence for the fibration $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$. Show that

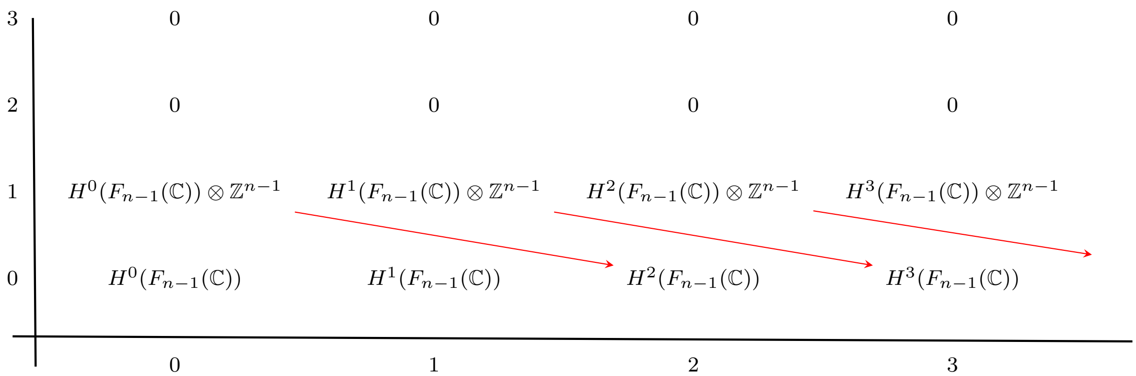
$$E_2^{p,q} = H^p(F_{n-1}(\mathbb{C}); H^q(\vee^{n-1} S^1)) = \begin{cases} H^p(F_{n-1}(\mathbb{C})), & q = 0 \\ H^p(F_{n-1}(\mathbb{C})) \otimes \mathbb{Z}^{n-1}, & q = 1 \\ 0, & q > 1. \end{cases}$$

Exercise 40. ($d_2 = 0$). In this exercise we will show that the spectral sequence collapses on the E_2 page.

(a) Show that the d_2 differentials are the only potentially-nonzero differential d_r for $r \geq 2$, so the spectral sequence must collapse by the E_3 page.

(b) Deduce that

$$E_\infty^{k-1,1} = \ker(d_2) \subseteq E_2^{k-1,1} \qquad \text{and} \qquad E_\infty^{k+1,0} = \frac{E_2^{k+1,0}}{\mathrm{im}(d_2)}.$$



Page E_2 of the Serre spectral sequence for the fibration $\rho_n : F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$.

Conclude that there is an exact sequence

$$0 \longrightarrow E_\infty^{k-1,1} \longrightarrow E_2^{k-1,1} \xrightarrow{d_2} E_2^{k+1,0} \longrightarrow E_\infty^{k+1,0} \longrightarrow 0.$$

(c) The cohomology groups $H^k(F_n(\mathbb{C}))$ are the limit of this spectral sequence in the sense that there are short exact sequences

$$0 \longrightarrow E_\infty^{k,0} \longrightarrow H^k(F_n(\mathbb{C})) \longrightarrow E_\infty^{k-1,1} \longrightarrow 0.$$

Show that we can combine these exact sequences to create a long exact sequence (a variant on the Gysin sequence of a sphere bundle),

$$\cdots \longrightarrow H^k(F_n(\mathbb{C})) \longrightarrow E_2^{k-1,1} \xrightarrow{d_2} E_2^{k+1,0} \longrightarrow H^{k+1}(F_n(\mathbb{C})) \longrightarrow \cdots$$

(d) Verify that the map from $E_2^{k+1,0} = H^{k+1}(F_{n-1}(\mathbb{C}))$ to $H^{k+1}(F_n(\mathbb{C}))$ is the map induced by ρ_n. (This requires a more technical understanding of the construction of the Serre spectral sequence.)

(e) The map ρ_n is split by Exercise 23(b). Conclude that the differential d_2 is zero.

(f) Conclude that the spectral sequence collapses on the E^2 page.

Exercise 41. In this exercise, we will show that the cohomology of the $F_n(\mathbb{C})$ is torsion-free for all n.

(a) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Show that if A and C are free abelian groups, then $B \cong A \oplus C$. In particular, B is free abelian.

(b) Deduce that if the groups $E_2^{p,q}$ of the Serre spectral sequence for the fibration ρ_n : $F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$ are free abelian, then so is its limit $H^*(F_n(\mathbb{C}))$.

(c) Beginning with $F_1(\mathbb{C}) = \mathbb{C}$, use induction on n to show that the groups $H^k(F_n(\mathbb{C}))$ are free abelian (or possibly 0) for all n and all k, and moreover that

$$H^k(F_n(\mathbb{C})) \quad\cong\quad E_2^{k,0} \oplus E_2^{k-1,1} \quad\cong\quad H^k(F_{n-1}(\mathbb{C})) \oplus \left(H^{k-1}(F_{n-1}(\mathbb{C})) \otimes \mathbb{Z}^{n-1}\right)$$

We can view the sequence as relating the cohomology of the product $F_{n-1}(\mathbb{C}) \times \vee^{n-1} S^1$ of base and fibre on the E_2 page to the cohomology of the total space $F_n(\mathbb{C})$ on the E_∞ page.

Exercise 42.

(a) Show that we can identify the $(n-1)$ generators of $H^1(\vee^{n-1}S^1) \cong \mathbb{Z}^{n-1}$ with the $(n-1)$ cohomology classes $\omega_{1,n},\ \omega_{2,n},\ \ldots,\ \omega_{n-1,n}$.

(b) Beginning with $F_1(\mathbb{C}) = \mathbb{C}$, show by induction that $H^k(F_n(\mathbb{C}))$ has an additive basis

$$\omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \wedge \cdots \wedge \omega_{i_p,j_p}, \qquad \text{where } i_s < j_s, \text{ and } j_1 < j_2 < \cdots < j_p.$$

(c) Deduce Arnold's result Theorem VIII.

This argument can be adapted to prove compute the cohomology of the configuration spaces $F_n(\mathbb{R}^d)$ of higher-dimensional Euclidean spaces. A detailed analysis of these cohomology algebras was done by F. Cohen.

Exercise* 43. Compute the cohomology groups $H_*(F_n(\mathbb{R}^d))$.

4 Generalizations of $\mathbf{PB}_n$ and their cohomology

4.1 The cohomology of a hyperplane complement

Many features of the structure of the cohomology of $\mathbb{F}_n(\mathbb{C})$ hold for general complex hyperplane complements. The following results are due to Brieskorn.

Theorem XII (Brieskorn [Br, Théorème 6(i).]). ***(The cohomology of a hyperplane complement $\mathcal{M}_G$).*** *Let G be a finite reflection group. Then the cohomology groups $H^p(\mathcal{M}_G)$ of the complex hyperplane complement $\mathcal{M}_G$ are free abelian, with rank*

$$\textit{rank } H^p(\mathcal{M}_G) = \#\{\ g \in G \mid \textit{length}(g) = p\ \}$$

where the length is taking with respect to the generating set of all reflections in G.

Theorem XIII (Brieskorn [Br, Lemme 5]). ***(Generating the cohomology of $\mathcal{M}$).*** *Let $\mathcal{M}$ be the complement of a finite arrangement of hyperplanes in a complex vector space V. Suppose each hyperplane H_i is determined by a linear form ℓ_i. Then the cohomology algebra of the complex hyperplane complement $\mathcal{M}_G$ is generated by the differential forms*

$$\omega_i := \frac{1}{2\pi I}\left(\frac{d\ell_i}{\ell_i}\right).$$

Moreover, the cohomology algebra is isomorphic to the $\mathbb{Z}$–subsalgebra of meromorphic forms on V generated by the forms ω_i.

Orlik and Solomon [OS] proved that if $\mathcal{M}$ is the complement of a finite arrangement of complex hyperplanes, then the cohomology of $\mathcal{M}$ is completely determined by the combinatorial data of the poset of the hyperplanes' intersections (under inclusion). They give a presentation for the cohomology $H^*(\mathcal{M})$ as an algebra.

4.2 The cohomology of the configuration spaces of a manifold

Let M be a closed orientable real manifold of dimension d. The cohomology of its configuration spaces $H^*(F_n(M))$ and $H^*(C_n(M))$ are subjects of active research, though explicit computations for large n are difficult in most cases. One tool to studying the cohomology groups of the ordered configuration spaces $H^*(F_n(M))$ is a spectral sequence due to Totaro [T], which relates the cohomology of $F_n(M)$ (on the E_∞ page) to (products of) the cohomology groups of M^ℓ and $F_k(\mathbb{R}^d)$ (on the E_2 page). Unfortunately, this spectral sequence typically does not collapse on the E_2 page.

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The geometry and topology of braid groups

Jenny Wilson

Lecture 3: The cohomology of the (pure) braid group, representation stability, and statistics on spaces of polynomials

In this lecture, we will investigate representation-theoretic patterns in the cohomology of the pure braid group, and a striking connection to the combinatorics of polynomials over $\mathbb{F}_q$.

We begin with some foundations on (co)homology with twisted coefficients.

5 Transfer and twisted coefficients

5.1 The transfer map

Definition XIV. (The transfer map). Let X be a connected CW–complex, and let $p : X' \to X$ be an m–sheeted cover for some finite m. We choose CW–structures on X and X' so that each cell in X lifts to m cells in X'. The covering map p induces a map $p_\#$ on cellular chain complexes

$$\begin{aligned} p_\# : C_k(X') &\longrightarrow C_k(X). \\ \sigma &\longmapsto p \circ \sigma \end{aligned}$$

We can also define a map in the other direction

$$\begin{aligned} \tau : C_k(X) &\longrightarrow C_k(X') \\ \sigma &\longmapsto \sum_{\text{lifts } \sigma' \text{ of } \sigma} \sigma' \end{aligned}$$

The map τ induces maps

$$\tau_* : H_k(X) \to H_k(X') \qquad \text{and} \qquad \tau^* : H^k(X') \to H^k(X),$$

called *transfer maps*, or sometimes called "wrong-way maps".

Exercise 44. Explain why the definition of τ requires $m < \infty$.

The following exercise shows that, up to m–torsion, $H^k(X)$ is a subgroup of in $H^k(X')$.

Exercise 45. Show that $p_\# \circ \tau$ is multiplication by m, and therefore $\tau^* \circ p^*$ is multiplication by m. Conclude that the kernel of p^* is contained in the m–torsion subgroup of $H^k(X)$.

The next exercise illuminates the relationship between the rational cohomology of X and that of X'.

Exercise 46. Let $p : X' \to X$ be a normal m–sheeted cover with Deck group Γ, so $X = X'/\Gamma$. Let $\mathbb{F}$ be a field of characteristic 0 or characteristic coprime to m.

(a) Prove that
$$p^* : H^k(X;\mathbb{F}) \longrightarrow H^k(X';\mathbb{F})$$
is injective.

(b) Prove that the image of p^* is the subspace $H^k(X';\mathbb{F})^\Gamma$ of $H^k(X';\mathbb{F})$ that is fixed pointwise by Γ.

Since manifolds have the homotopy type of CW–complexes, we may apply this result to the S_n–covering map $F_n(\mathbb{C}) \to C_n(\mathbb{C})$. We deduce the following corollary.

Corollary XV. ***(Transfer for** $F_n(\mathbb{C}) \to C_n(\mathbb{C})$**).*** *The rational cohomology of* $\boldsymbol{B}_n$ *is isomorphic to the* S_n*–invariants in the cohomology of* $\boldsymbol{PB}_n$*,*

$$H^p(\boldsymbol{B}_n, \mathbb{Q}) \cong H^p(\boldsymbol{PB}_n, \mathbb{Q})^{S_n}.$$

Exercise 47. Use Theorem XV to deduce the following.

(a) Show that $H^1(\mathbf{PB}_n, \mathbb{Q})^{S_n}$ is the 1–dimensional subspace spanned by $\sum_{i<j} \omega_{i,j}$. Interpret this class as a group homomorphism in $\mathrm{Hom}(\mathbf{PB}_n, \mathbb{Z})$.

(b) Use Exercise 36 to conclude that

$$H^p(\mathbf{B}_n;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & p=0 \\ \mathbb{Q}, & p=1 \\ 0 & p>1. \end{cases}$$

(c) What can you say about the groups $H^p(\mathbf{B}_n;\mathbb{F}_q)$ with coefficients in the field of order q?

5.2 (Co)homology with twisted coefficients

Let R be a commutative ring. Let X be a connected CW–complex with fundamental group π and with a universal cover $\widetilde{X}$. Let $C_*(\widetilde{X})$ be the complex of cellular chains on $\widetilde{X}$.

Exercise 48. Explain why in each degree p we can assume the chains $C_p(\widetilde{X})$ form a free $\mathbb{Z}[\pi]$–module.

Recall that we can define the homology $H_*(X;V)$ of X with twisted coefficients in the $\mathbb{Z}[\pi]$–module V as the homology of the chain complex

$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V,$$

and we can define the cohomology $H^*(X;V)$ of X with twisted coefficients V as the homology of the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}), V).$$

Exercise 49. Show that if $V = \mathbb{Z}$ is the trivial $\mathbb{Z}[\pi]$–module, then

$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z} \quad \cong \quad C_*(\widetilde{X})/\pi \quad \cong \quad C_*(X).$$

Conclude that $H_*(X;\mathbb{Z})$ coincides with the usual definition of $H_*(X)$ with integer coefficients. We call the coefficients $\mathbb{Z}$ *trivial coefficients*.

Exercise 50. Suppose that $K \subseteq \pi$ acts trivially on V, so the action of π on V factors through an action of the quotient $Q = \pi/K$. Let X' be the normal cover of X associated to K with Deck group Q.

(a) Show that
$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[Q] \quad \cong \quad C_*(\widetilde{X})/K \quad \cong \quad C_*(X')$$

(b) Show that
$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V \cong C_*(X') \otimes_{\mathbb{Z}[Q]} V \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}), V) \cong \mathrm{Hom}_{\mathbb{Z}[Q]}(C_*(X'), V)$$
so
$$H_*(X; V) \cong H_*(C_*(X') \otimes_{\mathbb{Z}[Q]} V) \qquad \text{and} \qquad H^*(X; V) \cong H_*(\mathrm{Hom}_{\mathbb{Z}[Q]}(C_*(X'), V)).$$

(c) Suppose that Q is a finite group, and V a rational Q–representation. Recall that the group ring $\mathbb{Q}[Q]$ is semisimple. Show that
$$H_*(X; V) \cong H_*(X'; \mathbb{Q}) \otimes_{\mathbb{Q}[Q]} V \qquad \text{and} \qquad H^*(X; V) \cong H^*(X'; \mathbb{Q}) \otimes_{\mathbb{Q}[Q]} V.$$

(d) Let $Q = S_n$. Show that
$$H_*(X; V) \cong \mathrm{Hom}_{\mathbb{Q}[S_n]}(H_*(X'; \mathbb{Q}), V) \qquad \text{and} \qquad H^*(X; V) \cong \mathrm{Hom}_{\mathbb{Q}[S_n]}(H^*(X'), V).$$

(e) Again let $Q = S_n$, and suppose V is an irreducible rational S_n–representation. Recall that S_n–representations are self-dual. Conclude that $H_*(X; V)$ is the isotypic component of V in $H_*(X; \mathbb{Q})$, and similarly for cohomology.

(f) Conclude that, to understand the cohomology of the braid group $\mathbf{B}_n$ with twisted coefficients in a $\mathbb{Q}[S_n]$–module, we must understand the cohomology groups $H^*(\mathbf{PB}_n; \mathbb{Q})$ of the pure braid group as S_n–representations.

6 Stability in the cohomology of braid groups

6.1 The braid group is homologically stable

Arnold [A2] proved that the braid groups are *homologically stable*.

Theorem XVI (Arnold [A2])**.** ***($\boldsymbol{B}_n$ is homologically stable).*** *Fix homological degree k. Then the maps*
$$H_k(\boldsymbol{B}_n) \longrightarrow H_k(\boldsymbol{B}_{n+1})$$
induced by the inclusions $\boldsymbol{B}_n \subseteq \boldsymbol{B}_{n+1}$ are isomorphisms for all $k \geq 2n$.

This result shows that, in a sense, all the degree-k homology $H_k(\mathbf{B}_n)$ of $\mathbf{B}_n$ arises from sub-configurations on $2k$ or fewer points. The analgous statement holds for the cohomology groups.

In contrast, our computations of $H^k(\mathbf{PB}_n)$ shows that these groups fail to stabilize even when $k = 1$; the groups
$$H_1(\mathbf{PB}_n) \cong \mathbb{Z}^{\binom{n}{2}}$$
have ranks growing quadratically in n. Church, Ellenberg, and Farb [CF, CEF1] observed, however, that once we account for the S_n–action on these cohomology groups, some strong stability phenomena emerge.

6.2 The cohomology of the pure braid group is representation stable

Consider the rational cohomology groups $H^k(\mathbf{PB}_n;\mathbb{Q})$. The first sense in which they stabilize is the following "finite generation" result.

Exercise 51. ("Finite generation" for $\{H^k(\mathbf{PB}_n;\mathbb{Q})\}_n$.) Use Arnold's presentation in Theorem VIII to show the following.

(a) Prove that, as an S_n–representation, $H^1(\mathbf{PB}_n;\mathbb{Q})$ is generated by the element $\omega_{1,2}$.

(b) Prove that, as an S_n–representation, $H^k(\mathbf{PB}_n;\mathbb{Q})$ is generated by the image of $H^k(\mathbf{PB}_{2k};\mathbb{Q})$.

We will see moreover that the description of the groups $H^k(\mathbf{PB}_n;\mathbb{Q})$ as S_n–representations stabilizes in a strong sense. Consider first the case of degree $k=1$.

Exercise 52. (Multiplicity stability for $\{H^1(\mathbf{PB}_n;\mathbb{Q})\}_n$.)

(a) Prove that, as an S_n–representation, the decomposition of $H^1(\mathbf{PB}_n;\mathbb{Q})$ into irreducible representations is

$$H^1(\mathbf{PB}_n;\mathbb{Q}) = V_{(n)} \oplus V_{(n-1,1)} \oplus V_{(n-2,2)} \tag{4}$$

for all $n \geq 4$.

(b) Explicitly identify the subrepresentations

$$\left(V_{(n)}\right) \cong \mathbb{Q} \qquad \text{and} \qquad \left(V_{(n)} \oplus V_{(n-1,1)}\right) \cong \mathbb{Q}^n$$

as a vector subspace of $H^1(\mathbf{PB}_n;\mathbb{Q}) = \text{span}_{\mathbb{Q}}(\omega_{i,j})$.

Church–Farb [CF] proved that for every k, the decomposition of $H^k(\mathbf{PB}_n;\mathbb{Q})$ into irreducible S_n–representations is independent of n for all $n \geq 4k$, just as in Equation 4: we can obtain the decomposition for level $n+1$ simply by adding a single box to the top row of each Young diagram in the decomposition at level n. Church and Farb called this phenomenon *multiplicity stability*.

In subsequent work with Ellenberg [CEF1], they proved that moreover the characters of the S_n–representations $H^k(\mathbf{PB}_n;\mathbb{Q})$ satisfy a polynomiality property and are in a sense independent of n. Consider again the case of degree $k=1$.

Exercise 53. (Character polynomials for $\{H^1(\mathbf{PB}_n;\mathbb{Q})\}_n$.) Show that the character χ_1^n of $H^1(\mathbf{PB}_n;\mathbb{Q})$ is given by the formula

$$\chi_1^n(\sigma) = (\#2\text{–cycles in the cycle type of } \sigma) + \binom{\#1\text{–cycles in the cycle type of } \sigma}{2}$$

for all $\sigma \in S_n$ for all n.

Let $X_r : \coprod_{n \geq 0} S_n \to \mathbb{Z}$ denote the class function

$$X_r(\sigma) = \#r\text{–cycles in the cycle type of } \sigma.$$

In this notation,

$$\chi_1^n \quad = \quad X_2 + \binom{X_1}{2} \quad = \quad X_2 + \frac{X_1(X_1 - 1)}{2} \qquad \text{for all } n.$$

Church–Ellenberg–Farb proved that for each fixed k, the characters χ_k^n of $H^k(\mathbf{PB}_n; \mathbb{Q})$ are equal to a unique polynomial $P \in \mathbb{Q}[X_1, X_2, \ldots, X_r, \ldots]$, independent of n, for all n. Such a polynomial is called a *character polynomial*.

Exercise 54. Compute the character polynomial for the representations $\{H^2(\mathbf{PB}_n; \mathbb{Q})\}_n$.

6.3 The cohomology of the pure braid group as a FI–module

The key to Church, Ellenberg, and Farb's results is the following algebraic formalism: they realize the sequence of cohomology groups of $F_n(\mathbb{C})$ as an FI–*module*.

Definition XVII. (The category FI and FI–modules.) Let FI denote the category of finite sets and injective maps. Let R be $\mathbb{Z}$ or $\mathbb{Q}$. Define an FI–*module* V to be a (covariant) functor from FI to the category of R–modules.

The data of an FI–module is a sequence of S_n–representations V_n over R with S_n–equivariant maps $\phi_n : V_n \to V_{n+1}$.

Definition XVIII. (Maps of FI–modules.) A *map of* FI–*modules* $V \to W$ is a natural transformation.

Exercise 55. (The structure of an FI–module.)

(a) Show that this category is equivalent to the category whose objects are the natural numbers n, $(n \geq 0)$, and whose morphisms from m to n are the set of injective maps

$$\{1, 2, 3 \ldots, m\} \to \{1, 2, 3 \ldots, n\}.$$

(b) Explain why an FI–module is a sequence of S_n–representations V_n with S_n–equivariant maps $\phi_n : V_n \to V_{n+1}$, and why the structure is completely determined by the S_n–actions and the maps ϕ_n.

(c) Suppose that $\{W_n\}$ is a sequence of S_n–representations with S_n–equivariant maps $\phi_n : W_n \to W_{n+1}$. Let $\iota_{k,n}$ denote the canonical inclusion

$$\iota_{k,n} : \{1, 2, \ldots, k\} \hookrightarrow \{1, 2, \ldots, n\},$$

and let $G \cong S_{n-k}$ denote its stabilizer under the action of S_n by postcomposition,

$$G = \{ \ \sigma \in S_n \mid \sigma \circ \iota_{k,n} = \iota_{k,n} \ \}.$$

Show that $\{W_n\}$ has the structure of an FI–module, with $(\iota_{n-1,n})_* = \phi_n$ if and only if for all $k < n$, $\sigma \cdot v = v$ for all $\sigma \in G$ and $v \in \text{im}((\iota_{k,n})_*)$.

Exercise 56. (Examples of FI–module.)

(a) Show that the following sequences have the structure of an FI–module over $\mathbb{Z}$.

(i) $V_n = \mathbb{Z}$ trivial S_n–representations, all maps are isomorphisms

(ii) $V_n = \mathbb{Z}^n$ canonical permutation representations, maps $V_n \to V_{n+1}$ the natural inclusions

(iii) $V_n = \bigwedge^k(\mathbb{Z}^n)$, maps $V_n \to V_{n+1}$ the natural inclusions.

(iv) V_n any sequence of S_n–representations, all maps $V_m \to V_n$ with $n > m$ are zero

(v) $V_n = \mathbb{Z}[x_1, \ldots, x_n]$, maps $V_n \to V_{n+1}$ the natural inclusions

(vi) V_n homogenous polynomials in $x_1, \ldots, x_n$ of fixed degree k, maps $V_n \to V_{n+1}$ the natural inclusions

(vii) $V_n = \mathbb{Z}[S_n]$ with action of S_n by conjugation, maps $V_n \to V_{n+1}$ the natural inclusions

(b) Show that the following sequences do **not** have the structure of an FI–module.

(i) $V_n = \mathbb{Z}$ alternating representations, all maps are isomorphisms

(ii) $V_n = \mathbb{Z}[S_n]$ with action of S_n by left multiplication, maps $V_n \to V_{n+1}$ the natural inclusions.

Definition XIX. (Finite generation of FI–modules). An FI–module $V = \{V_n\}$ is *(finitely) generated in degree* $\leq d$ if there is a (finite) set $S \subseteq \coprod_{0 \leq n \leq d} V_n$ such that V is the smallest FI–submodule containing S.

Definition XX. (Representable FI–modules). Define the FI–module $M(d)$ by

$$M(d)_n := R[\mathrm{Hom}_{\mathsf{FI}}(d, n)]$$

and the action of FI–morphisms by post-composition.

The FI–modules $M(d)$ can be thought of as "free" FI–modules.

Exercise 57. (Finite generation of FI–modules.)

(a) Show that an FI–module V is generated in degree $\leq d$ if and only if it admits a surjection

$$\bigoplus_{0 \leq m \leq d} M(m)^{\oplus c_m} \longrightarrow V, \qquad c_m \in \mathbb{Z}_{>0} \cup \{\infty\}.$$

Show moreover that V is finitely generated in degree $\leq d$ if and only if it admits such a surjection with all multiplicites c_m finite.

(b) Determine which of the FI–modules in Exercise 56 are finitely generated.

Exercise 58. Show that $M(d)_n \cong \mathrm{Ind}_{S_{n-d}}^{S_n} R$.

Exercise 59. Explicity describe and compute the decompositions for the rational S_n–representations $M(0)_n$, $M(1)_n$, and $M(2)_n$.

Exercise 60. ($H^k(F_n(\mathbb{C}))$ as an FI–module.)

(a) Show that, for each fixed k, the sequence of S_n–representations $\{H^k(F_n(\mathbb{C}); R)\}_n$ has the structure of an FI–module over R.

(b) Fix k. Show that this FI–module is finitely generated in degree $\leq 2k$. (See Exercise 51).

(c) Show that, for each k, the sequence of S_n–representations $\{H^k(F_n(\mathbb{C}); R)\}_n$ has the structure of an FI^{op}–module.

Church–Ellenberg–Farb proved the following structural results for FI–modules.

Theorem XXI (Church–Ellenberg–Farb [CEF1, CEF2])**.** *Let V be an FI–module over $\mathbb{Q}$ that is finitely generated in degree $\leq d$. Then the following hold.*

- ***(Multiplicity stability).*** *The decomposition of V_n into irreducible representations is independent of n for all n sufficiently large.*
- ***(Polynomial dimension growth).*** *The dimensions $dim_{\mathbb{Q}}(V_n)$ are, for n sufficiently large, equal to the integer points $p(n)$ of a polynomial p of degree $\leq d$.*
- ***(Polynomial characters).*** *For all n sufficiently large, the sequence of characters χ_{V_n} are equal to a character polynomial P that is independent of n;*

$$\chi_n(\sigma) = P(\sigma) \qquad \text{for all } \sigma \in S_n \text{ and all } n \text{ sufficiently large.}$$

- ***(Stable inner products).*** *If Q is any character polynomial, then $\langle \chi_{V_n}, Q \rangle_{S_n}$ is independent of n for all n sufficiently large.*
- ***(Finite presentability).*** *V is* finitely presentable *as an FI–module.*

If V simultaneously has the structure of a module over FI and over FI^{op} in a compatible way, then Church–Ellenberg–Farb call V an *FI#–module.* The cohomology groups $\{H^k(F_n(\mathbb{C}))\}_n$, for example, have this structure. In this case, they obtain the following stronger results.

Theorem XXII (Church–Ellenberg–Farb [CEF1, CEF2])**.** *Let V be an FI♯–module over $\mathbb{Z}$ that is finitely generated as an FI–module in degree $\leq d$. Then the following hold.*

- ***(Multiplicity stability).*** *The decomposition of $\mathbb{Q} \otimes_{\mathbb{Z}} V_n$ into irreducible representations is independent of n for all $n \geq 2d$.*
- ***(Polynomial dimension growth).*** *For all n the ranks $rank_{\mathbb{Z}}(V_n)$ are equal to the integer points $p(n)$ of a polynomial p of degree $\leq d$ that is independent of n.*
- ***(Polynomial characters).*** *The sequence of characters $\chi_{\mathbb{Q} \otimes_{\mathbb{Z}} V_n}$ are equal to a character polynomial P that is independent of n.*
- ***(Stable inner products).*** *If Q is any character polynomial, then $\langle \chi_{\mathbb{Q} \otimes_{\mathbb{Z}} V_n}, Q \rangle_{S_n}$ is independent of n for all $n \geq (d + \deg(Q))$.*
- ***(Structure theorem).*** *For $m = 0, \ldots, d$ there are S_m–representations U_m such that*

$$V_n \cong \bigoplus_{m=0}^{d} Ind_{S_m \times S_{n-m}}^{S_n} U_m \boxtimes \mathbb{Z} \qquad \mathbb{Z} \text{ the trivial } S_{n-m}\text{–representation}$$

and morphisms act by the natural injective maps $V_n \to V_{n'}$.

Each of these consequences can be verified directly by hand in the case of the cohomology groups $H^*(F_n(\mathbb{C}))$, though the results of Church–Ellenberg–Farb allow for a very efficient proof, and also give a conceptual framework for understanding the underlying algebraic structure that drives these stability results. The case of $F_n(\mathbb{C})$ and the pure braid group should be viewed as the test case for a large body of stability results on their various generalizations. Church, Ellenberg, Farb, and others have used this machinery to prove stability results for ordered configuration spaces, hyperplane complements, pure mapping class groups, coinvariant algebras, congruence subgroups of $\mathrm{GL}_n(S)$, ...

7 Polynomials over $\mathbb{F}_q$ and the twisted Grothendieck–Lefschetz fixed point theorem

7.1 The Weil conjectures and étale cohomology

Let V be a nonsingular projective variety defined over $\mathbb{F}_q$ (q prime). A *local zeta function* of V is a particular generating function that encodes point-counts on the $\mathbb{F}_{q^m}$ points $V_{\mathbb{F}_{q^m}}$ of V. In 1949, in the celebrated *Weil conjectures*, Weil anticipated that these generating functions must be rational functions with certain constraints on their roots and poles, and must satisfy certain functional equations in analogy to the Reiman zeta function. These conjectures have been resolved by work of many authors including Dwork, Artin, Grothendieck, and Deligne (1960s-1970s).

To this end, Grothendieck and Artin (building on work of many others) developed the *étale cohomology* of algebraic varieties, an analogue of singular cohomology for topological spaces. These étale cohomology $H^i_{\text{ét}}(V, \mathbb{Q}_\ell)$ satisfy

- $H^i_{\text{ét}}(V, \mathbb{Q}_\ell)$ is finite dimensional vector spaces over the ℓ–adics $\mathbb{Q}_\ell$ for $\ell \neq q$.
- $H^i_{\text{ét}}(V, \mathbb{Q}_\ell) = 0$ for $i < 0$ and for $i > 2\dim(V)$.
- A form of Poincaré duality
- Kunneth theorem
- Actions induced by Frobenius and the Galois groups
- Relationship to the singular cohomology $H^i(V_\mathbb{C}, \mathbb{C})$ in "nice" cases where $V_\mathbb{C}$ is defined
- Lefschetz fixed-point theorem

7.2 The Grothendieck Lefschetz fixed-point theorem

The remaining sections are based on the work of Church–Ellenberg–Farb [CEF2]. Recall the following classical version of the Lefschetz fixed-point theorem:

Theorem XXIII. ***(Lefschetz fixed-point theorem.)*** *Suppose that Y is a compact triangulable manifold, and that $f : Y \to Y$ acts with a finite set Fix(f) of fixed points. Then*

$$\sum_{z \in \mathrm{Fix}(f)} \mathrm{index}(z) = \sum_i (-1)^i \ \mathrm{Trace}\{f \curvearrowright H_i(Y; \mathbb{Q})\}.$$

The index of a fixed point z is a certain (signed) multiplicity of that fixed point. Recall that by Poincaré duality, $H_i(Y;\mathbb{Q}) \cong H^{\dim(Y)-i}(Y;\mathbb{Q})$.

Now let V be a variety over $\mathbb{F}_q$ for some prime power q. Recall that the geometric Frobenius map acts (in an affine chart) on coordinates of V by

$$\begin{aligned} \mathrm{Frob}_q : \overline{\mathbb{F}_q} &\longrightarrow \overline{\mathbb{F}_q} \\ x &\longmapsto x^q \end{aligned}$$

Exercise 61. (Frobenius fixed set.) Consider the map $\mathrm{Frob}_q : \overline{\mathbb{F}_q} \longrightarrow \overline{\mathbb{F}_q}$.

(a) Show that Fix(Frob_q)= $\mathbb{F}_q$.
(b) Show that Fix(Frob_q^m)= $\mathbb{F}_{q^m}$.

There is an analogous fixed-point theorem for étale cohomology:

Theorem XXIV. ***(Grothendieck–Lefschetz fixed-point theorem.)*** *Suppose that X is a smooth projective variety over $\mathbb{F}_q$. Then*

$$|X_{\mathbb{F}_q}| = |\mathrm{Fix}(\mathrm{Frob_q})| = \sum_i (-1)^i \ \mathrm{Trace}\{\mathrm{Frob_q} \curvearrowright H^i_{ét}(X;\mathbb{Q}_\ell)\}.$$

Suppose instead that the smooth variety X over $\mathbb{F}_q$ is not necessarily compact. Then (using Poincaré duality) we can deduce

$$|X_{\mathbb{F}_q}| = |\mathrm{Fix}(\mathrm{Frob_q})| = q^{dim(X)} \sum_i (-1)^i \ \mathrm{Trace}\{\mathrm{Frob_q} \curvearrowright H^i_{ét}(X;\mathbb{Q}_\ell)^\vee\}.$$

7.3 Artin's comparison theorem

Let X be a variety defined over $\mathbb{Z}$, so that both $X_{\mathbb{F}_q}$ and $X_\mathbb{C}$ are defined. Then Artin proved that, if X is sufficiently nice, there is an isomorphism

$$H^i(X_\mathbb{C};\mathbb{Q}_\ell) \xrightarrow{\cong} H^i_{\text{ét}}(X_{/\overline{\mathbb{F}_q}};\mathbb{Q}_\ell).$$

where $H^i(X_\mathbb{C};\mathbb{Q}_\ell)$ denotes the singular cohomology of the topological space $X_\mathbb{C}$.

This isomorphism holds in particular when X is the configuration space C_n.

7.4 The case of $X = C_n$

Exercise 62. (The action of Frobenius on $C_n(\overline{\mathbb{F}_q})$). Show that a polynomial $p \in C_n(\overline{\mathbb{F}_q})$ is fixed by Frobenius exactly when its roots are permuted by the Frobenius map, equivalently, exactly when its coefficients are in $\mathbb{F}_q$. Conclude that the $\mathbb{F}_q$ points of the variety C_n is the space $P_n(\mathbb{F}_q)$ of square-free polynomials with coefficients in $\mathbb{F}_q$. Note that $P_n(\mathbb{F}_q) \neq C_n(\mathbb{F}_q)$ (unless we re-define C_n as the quotient of F_n by S_n in an appropriate scheme-theoretic fashion).

The Frobenius map acts on $H^i_{\text{ét}}(C_n;\mathbb{Q}_\ell)$ by multiplication by q^i. It follows from the fixed-point formula that

$$|P_n(\mathbb{F}_q)| = \sum_{i\geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(C_n;\mathbb{Q}_\ell)^\vee.$$

Exercise 63. ($\mathbb{F}_q$–point counts on $C_n(\overline{\mathbb{F}_q})$).

(a) Use the Artin comparison theorem to deduce that

$$|P_n(\mathbb{F}_q)| = \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}} H^i(C_n(\mathbb{C}); \mathbb{Q}).$$

(b) Use Exercise 47 to conclude that $|P_n(\mathbb{F}_q)| = q^n - q^{n-1}$

(c) Verify this result by a direct count of the number of square-free monic polynomials with coefficients in $\mathbb{F}_q$.

7.5 Twisted Grothendieck–Lefschetz and $X = C_n$

Exercise 64. (The permutation σ_p). Suppose a polynomial $p \in C_n(\overline{\mathbb{F}_q})$ is fixed by Frobenius. Show that this action defines a permutation σ_p on the roots of p. The roots of p are unordered, however, show that this permutation σ_p is well-defined up to conjugacy.

When $X = C_n$, then there is a "twisted" version of the Artin comparison theorem and the Grothendieck–Lefschetz fixed point theorem that holds for cohomology with twisted coefficients. Let V be an S_n–representation with character χ_V. It is possible to define an associated *ℓ-adic sheaf* $\mathcal{V}$ on the variety C_n. Then

$$\begin{aligned} \sum_{p \in P_n(\mathbb{F}_q)} \chi_V(\sigma_p) = \sum_{p \in P_n(\mathbb{F}_q)} \mathrm{Trace}(\mathrm{Frob_q}|_{\mathcal{V}_p}) &= \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(C_n; \mathcal{V})^\vee \\ &= \sum_{i \geq 0} (-1)^i q^{n-i} \dim_{\mathbb{Q}_\ell} H^i(C_n(\mathbb{C}); V) \end{aligned}$$

Exercise 65. (Twisted Grothendieck–Lefschetz for $P_n(\mathbb{F}_q)$). Conclude from Exercise 50 that

$$\sum_{p \in P_n(\mathbb{F}_q)} \chi_V(\sigma_p) = \sum_{i \geq 0} (-1)^i q^{n-i} \langle H^i(F_n(\mathbb{C}); \mathbb{Q}_\ell), V \rangle_{S_n}$$

7.6 Stability of polynomial statistics and its relationship to representation stability for the cohomology of $F_n(\mathbb{C})$

The twisted Grothendieck–Lefschetz formula reveals remarkable relationships between the representation-theoretic stability patterns in the cohomology groups $H^i(F_n(\mathbb{C}); \mathbb{Q})$, and combinatorial stability pattern in certain point-count statistics on the space of polynomials with coefficients in $\mathbb{F}_q$.

Exercise 66. (Polynomial statistics on $P_n(\mathbb{F}_q)$). Recall we defined $X_r : \coprod_n S_n \to \mathbb{Z}$ to be the class function such that $X_r(\sigma)$ is the number of r–cycles in the cycle type of a permutation σ. For a polynomial $p \in P_n(\mathbb{F}_q)$, explain why $X_r(\sigma_p)$ is the number of irreducible r–cycles in the factorization of p over $\mathbb{F}_q$.

We call the point-count statistic on $P_n(\mathbb{F}_q)$ defined by a character polynomial a *polynomial statistic*.

Exercise 67. (The expected number of linear factors of a polynomial over $\mathbb{F}_q$).

(a) Show that the S_n–representation $\mathbb{Q}^n$ has character X_1.

(b) Deduce that the number of linear factors counted over all square-free polynomials with coefficients in $\mathbb{F}_q$ is

$$\sum_{p \in P_n(\mathbb{F}_q)} X_1(\sigma_p) = \sum_{i \geq 0} (-1)^i q^{n-i} \langle H^i(F_n(\mathbb{C}); \mathbb{Q}), \mathbb{Q}^n \rangle_{S_n}$$

(c) Do a direct count to determine this number. Conclude that the "twisted Betti numbers" of the braid group with coefficients in $\mathbb{C}^n$ are

$$\langle H^i(F_n(\mathbb{C}); \mathbb{Q}), \mathbb{Q}^n \rangle_{S_n} = \begin{cases} 1, & i = 0, n-1 \\ 2, & i = 1, 2, \cdots, n-2 \\ 0, & i \geq n. \end{cases}$$

(d)* Verify this conclusion by analyzing the S_n–representations $H^i(F_n(\mathbb{C}))$.

(e) Conclude from these counts that the expected number of linear factors in a random polynomial over $\mathbb{F}_q$ is

$$1 - \frac{1}{q} + \frac{1}{q^2} - \frac{1}{q^3} + \cdots \pm \frac{1}{q^{n-2}}$$

Church–Ellenberg–Farb proved that their representation stability results for the pure braid groups implies that the (normalized) value of any polynomial statistic is stable as $n \to \infty$, and moreover is given by a formula that is in a sense uniform in q.

Theorem XXV (Church–Ellenberg–Farb [CEF2, Theorem 1]). ***(Stability of polynomial staistics.)*** *Let $P \in \mathbb{Q}[X_1, X_2, X_3, \ldots,]$ be a character polynomial, and q a prime power. Then the following two limits exist, and converge to the same value.*

$$\lim_{n \to \infty} q^{-n} \sum_{p \in P_n(\mathbb{F}_q)} P(\sigma_p) \quad = \quad \sum_{i=0}^{\infty} (-1)^i \frac{\lim_{n \to \infty} \langle P, H^i(F_n(\mathbb{C}); \mathbb{Q}) \rangle_{S_n}}{q^i}.$$

Their result illustrates the deep connections between the topology of the configuration space $F_n(\mathbb{C})$, and the combinatorics of the space of square-free polynomials with coefficients in $\mathbb{F}_q$.

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