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INTRODUCTION TO COHOMOLOGICAL FIELD THEORIES*

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In these lectures, I will give an elementary account of the realization of four dimensional Donaldson theory, and its various cousins (including two dimensional topological gravity) as conventional Lagrangian field theories. I will follow a rather simple nuts and bolts approach, somewhat streamlined compared to [1], but I should note that more sophisticated points of view exist. On the one hand, these theories can be derived [2-4] via BRST-BV gauge fixing of "trivial" underlying gauge invariant Lagrangians. This approach is of substantial conceptual interest, and would be of practical importance as well in more complicated examples than the ones that will be considered here. On the other hand, mathematically, these theories are naturally interpreted in terms of equivariant cohomology [5] and the "equivariant Euler class" [6]. The exposition I will give here has been improved through the influence of comments of S. Axelrod [7]. Also, I recommend introductory lectures by van Baal [8]. Finally, let me note that a superspace formulation that makes regularization and renormalization almost immediate can be found in [9].

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We will be discussing quantum field theories that are associated with the cohomology of various moduli spaces. These are quantum field theories of a rather special and in some ways exceptionally simple kind. From a physical standpoint, one might hastily dismiss them as "trivial." What is perhaps surprising about them is the extent to which they lead to deep results – at least in the case of four dimensional Donaldson theory – and the fact that in at least one important example – two dimensional gravity – what are usually regarded as "physical" quantum field theories are equivalent to special cases of these more primitive "cohomological" theories.

The typical moduli problem that we will consider in constructing quantum field theories has fields, symmetries, and equations. For illustrative purposes, we will consider the moduli problem of Yang-Mills instantons in four dimensions, with some compact gauge group G. The fields are then Yang-Mills fields $A_\mu^a(x)$. The Yang-Mills field is a connection on a G bundle E over a space-time manifold M (which we endow with a Riemannian metric g). The symmetries are gauge symmetries, which infinitesimally take the form $A_\mu^a \to A_\mu^a - D_\mu u^a$, where u^a is a Lie algebra valued zero form. The equations are the self-dual Yang-Mills equations $F = *F$, where $F = dA + A \wedge A$ is the Yang-Mills curvature, and $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. The moduli space $\mathcal{M}$ of interest is the space of all solutions of the self-dual Yang-Mills equations, modulo the action of the symmetry group.

Now, actually, the paradigm of "symmetries, fields, and equations," which we will consider for illustrative purposes, is not universally applicable in moduli problems. On the one hand, the symmetries or the equations may be missing. For instance, "topological sigma models" are associated with the moduli space of holomorphic maps $\Phi : \Sigma \to K$, where Σ is a Riemann surface and K is a Kahler manifold. In this case, the "fields" are the maps $\Phi : \Sigma \to K$, and the "equations" are the instanton equations for a holomorphic map, but there are no "symmetries." Likewise an important moduli problem that can be formulated without "equations" is the problem of describing the moduli space of complex Riemann surfaces of genus g, regarded as the space of all metrics on a smooth genus g surface Σ, modulo diffeomorphisms and Weyl transformations.

Conversely, a moduli problem may be more complicated than suggested by the paradigm of "symmetries, fields, and equations," if for instance the equations are not independent (but are related by "Bianchi identities") or the action of the gauge group is "reducible" (that is, the gauge group does not act freely even generically). In such a case, one would need to use the powerful Batalin-Vilkovisky approach to quantization, and the viewpoint of [2-4] would really pay off. But for the cases that have been important so far, the basic paradigm of "symmetries, fields, and equations" is adequate.

We will also sometimes assume, following [7], that there are metrics on the Lie algebra of the symmetry group, the space of fields, and the space of equations. In the Yang-Mills case, the metric on the Lie algebra is given by

$$|u|^2 = \int d^4x\sqrt{g}\ \mathrm{Tr}\ u^2 \tag{1}$$

if u is a generator of gauge transformations. The metric on the space of fields is

$$|\delta A|^2 = \int d^4x\sqrt{g}\ \mathrm{Tr}\ \delta A_\mu\,\delta A^\mu\ , \tag{2}$$

if δA is a tangent vector to the space of convections. The metric on the space of equations is defined similarly. When such metrics exist, one can make choices of the "antighost multiplets" (which will appear later) that are particularly nice in some ways and are not possible otherwise.

The theory that we will construct has a "ghost number" U, which is related to the dimension of moduli space, as we will see, and which is violated by an anomaly.

We start with *fields* of $U = 0$; in our example these are the Yang-Mills $A_\mu^a(x)$. We also introduce *ghosts* of $U = 1$, with opposite statistics from the fields but otherwise with the same quantum numbers. In our example, these will be an anticommuting one form $\psi_\mu^a(x)$ with values in the Lie algebra. And in $U = 2$, we introduce fields with the quantum numbers of the generators of the symmetry group. In the Yang-Mills case, this means a commuting zero form $\phi^a(x)$ with value in the Lie algebra of the gauge group.

Next one introduces an anticommuting BRST-like symmetry Q, with transformation laws (in the Yang-Mills case)

$$\begin{aligned}
[Q, A_\mu^a] &= \psi_\mu^a \\
\{Q, \psi_\mu^a\} &= -D_\mu\phi^a \\
[Q, \phi^a] &= 0\ .
\end{aligned} \tag{3}$$

It is easy to see that $Q^2 = 0$ up to a gauge transformation. For instance

$$[Q^2, A^a_\mu] = \{Q, [Q, A^a_\mu]\} = -D_\mu \phi^a \tag{4}$$

and the right hand side of (4) is the variation of A^a_μ under an infinitesimal gauge transformation generated by the gauge parameter $u^a = \phi^a$.

The transformation laws (3) can be written abstractly for an arbitrary moduli problem of the class we are considering. We have in general

$$\begin{aligned} [Q, \mathrm{FIELD}] &= \mathrm{GHOST} \\ \{Q, \mathrm{GHOST}\} &= \delta^{\mathrm{GAUGE}}_\phi(\mathrm{FIELD}) \\ [Q, \phi] &= 0\,. \end{aligned} \tag{5}$$

Here $\delta^{\mathrm{GAUGE}}_\phi$ (FIELD) is the transformation of the field under an infinitesimal gauge transformation generated by ϕ.

This formula should be compared with the formulas that arise in Fadde'ev-Popov quantization of gauge theories. In that case one introduces *anticommuting* zero forms $c^a(x)$ of $U = 1$ with values in the Lie algebra. The conventional BRST symmetry $\widetilde{Q}$ of the Fadde'ev-Popov quantization is

$$\begin{aligned} [\widetilde{Q}, A^a_\mu] &= -D_\mu c^a \\ \{\widetilde{Q}, c^a\} &= \frac{1}{2}[c, c]^a\,. \end{aligned} \tag{6}$$

Abstractly, this can be written

$$\begin{aligned} [\widetilde{Q}, \mathit{FIELD}] &= \delta^{\mathrm{GAUGE}}_c(\mathit{FIELD}) \\ \{\widetilde{Q}, c\} &= \frac{1}{2}\, \delta^{\mathrm{GAUGE}}_c\ (\mathrm{FIELD})\,, \end{aligned} \tag{7}$$

in analogy with (5).

Now, both of these multiplets have a natural mathematical interpretation. The conventional BRST symmetry (6) or (7) has to do with the Lie algebra cohomology of the gauge group acting on the space of connections, while (3) or (5) are related in a similar way to the equivariant cohomology of the gauge group acting on the space of connections. We will not develop this point of view, however, and will simply think of the (A_μ, ψ_μ, ϕ) system as a multiplet, with a particular kind of fermionic symmetry, that should be studied by conventional physical methods.

The physical observables in the BRST sense are the cohomology classes of Q (in the space of gauge invariant operators, a natural requirement in any case in gauge theories and necessary here since $Q^2 = 0$ only up to a gauge transformation). Since $[Q, \phi^a(x)] = 0$, and $\phi^a(x)$ (as opposed to its derivative) does not appear on the right hand side of (3), any gauge invariant polynomial in $\phi^a(x)$ defines a physical observable. So we let

$$\mathcal{O}_{k,0}(x) = \mathrm{Tr}\ \phi^k(x)\ . \tag{8}$$

$\mathcal{O}_{k,0}$ has ghost number $U = 2k$. Notice, now, that

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\,\mathcal{O}_{k,0}(x) &= k\ \mathrm{Tr}\ \phi^{k-1}\,D_\mu\phi(x)\\ &= \{Q, -k\ \mathrm{Tr}\ \phi^{k-1}\psi_\mu\}\ .\end{aligned} \tag{9}$$

Thus, although $\mathcal{O}_{k,0}(x)$ is not a BRST commutator, its derivative is. This means that the BRST cohomology class of $\mathcal{O}_{k,0}(x)$ – and thus its correlation functions – are independent of x. This is a first indication that the theory we will construct is generally covariant (topologically invariant). We can rewrite (9) in the form

$$d\ \mathcal{O}_{k,0}(x) = \{Q, \mathcal{O}_{k,1}\} \tag{10}$$

where d is the exterior derivative on M, and $\mathcal{O}_{k,1}$ is the operator valued one form

$$\mathcal{O}_{k,1} = -k\ \mathrm{Tr}\ \phi^{k-1}\psi\ . \tag{11}$$

In addition to implying that correlation functions of $\mathcal{O}_{k,0}(x)$ are independent of x, (10) has the following significance. If C is a circle in M, and

$$W_k(C) = \int_C \mathcal{O}_{k,1}\ , \tag{12}$$

then $W_k(C)$ is BRST invariant, since

$$\{Q, W_k(C)\} = \int_C d\mathcal{O}_{k,0} = 0 . \tag{13}$$

In a similar fashion, one recursively solves the equations

$$\begin{aligned} d\mathcal{O}_{k,1} &= [Q, \mathcal{O}_{k,2}] \\ d\mathcal{O}_{k,2} &= \{Q, \mathcal{O}_{k,3}\} \\ d\mathcal{O}_{k,3} &= [Q, \mathcal{O}_{k,4}] \\ d\mathcal{O}_{k,4} &= 0 . \end{aligned} \tag{14}$$

The first equation in (14) means that $W_k(C)$ is invariant under small deformations of C (and in fact depends only on the homology class of C). This equation also ensures, for any Riemann surface Σ immersed in M, the BRST invariance of

$$W_k(\Sigma) = \int_\Sigma \mathcal{O}_{k,2}(\Sigma) . \tag{15}$$

These observables, which by virtue of the second equation in (14) depend only on the homology class of M, played a key role in Donaldson's work. Likewise, the second and third equations in (14) allow one to define a BRST invariant observable

$$W_k(T) = \int_T \mathcal{O}_{k,3} \tag{16}$$

for every three dimensional homology cycle T in M, and imply that

$$\mathcal{L}_k = \int_M \mathcal{O}_{k,4} \tag{17}$$

is BRST invariant and so is a new interaction term that can be added to any Lagrangian that we may construct.

We now want to add additional multiplets to the A_μ, ψ_μ, ϕ system, and write a Lagrangian possessing the Q symmetry. In doing so, the additional multiplets that we will require have the following trivial structure. We introduce pairs B, β where β has the same quantum numbers as B except that it has opposite statistics and ghost number one greater than that of B. The transformation laws are

$$\begin{aligned} [Q, B] &= \beta \\ [Q, \beta] &= \delta_\phi^{\text{GAUGE}}(B) \ . \end{aligned} \tag{18}$$

This obviously closes in the appropriate way. I will call these multiplets "antighost multiplets," and indeed, they are analogous to the antighost multiplets that are introduced in the BRST approach to conventional Fadde'ev-Popov gauge fixing. Precisely as in that case, there is much freedom in the choice of the antighost multiplets; the only requirement is that they must be chosen so that an appropriate Lagrangian (or, in the conventional case, an appropriate gauge fixing term of the form $\{Q, \Lambda\}$) can be written. With a little practice, one finds that there are many ways that this can be done.

In a case, such as the gauge theory case, in which there are metrics (1), (2) on the various function spaces of interest, a choice of the antighost multiplets that is particularly nice from some points of view can be described as follows. First, we introduce a multiplet (λ, η), where λ has the same quantum numbers as ϕ (and thus, transforms as the Lie algebra of the gauge group) but has $U = -2$, and η is its partner, of $U = -1$. Thus, in the Yang-Mills case, $\lambda^a(x)$ and $\eta^a(x)$ are respectively commuting and anticommuting zero forms with values in the Lie algebra. And we introduce a multiplet (χ, H), where χ has the quantum numbers of the equations but opposite statistics, and H is its partner; χ and H have $U = -1$ and $U = 0$, respectively. Thus, in the Yang-Mills case, if the equation of interest is the instanton equation $F - \widetilde{F} = 0$, which asserts the vanishing of an anti-self-dual two form with values in the Lie algebra, the new multiplet consists of anti-self-dual Lie algebra valued two forms $\chi^a_{\alpha\beta}$ and $H^a_{\alpha\beta}$.

The BRST invariant Lagrangian is now

$$\mathcal{L} = \frac{1}{e^2} \int \{Q, V\} \tag{19}$$

where e is a constant and we make *any choice of V such that $\mathcal{L}$ has a non-degenerate kinetic*

energy for all fields. In the gauge theory case, a very convenient choice is

$$V = \int_M d^4x\sqrt{g}\ \mathrm{Tr}\left[-D_\alpha\lambda\ \psi^\alpha - 2\chi_{\alpha\beta}\ \left(H^{\alpha\beta} - \frac{1}{2}(F^{\alpha\beta} - *F^{\alpha\beta})\right)\right] \tag{20}$$

The various terms can be described abstractly. For instance, the first term is

$$\int d^4x\sqrt{g}\ \mathrm{Tr}(D_\alpha\lambda\ \psi^\alpha) = -(\delta^{\mathrm{GAUGE}}A, \psi) \tag{21}$$

where $\delta_\lambda^{\mathrm{GAUGE}}A_\alpha = -D_\alpha\lambda$ is the variation of A_α under a gauge transformation generated by λ, and (,) is the metric (2) on the space of fields.

Upon computing $\{Q, V\}$ and eliminating the auxiliary field H, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = \frac{1}{e^2}\int_M \sqrt{g}\ \mathrm{Tr}[\frac{1}{8}(F_{\alpha\beta} - *F_{\alpha\beta})^2 + D_\alpha\lambda D^\alpha\phi - D_\alpha\eta\ \psi^\alpha \\ + \lambda[\psi_\alpha, \psi^\alpha] - \chi^{\alpha\beta}(D_\alpha\psi_\beta - D_\beta\psi_\alpha - \epsilon_{\alpha\beta\gamma\delta}D^\gamma\psi^\delta)]\end{aligned} \tag{22}$$

Now, the kinetic energy of the gauge bosons is

$$\mathcal{L}_{gauge} = \frac{1}{4e^2}\int_M \sqrt{g}\ \mathrm{Tr}(F_{\alpha\beta}F^{\alpha\beta})\ -\ \frac{1}{4e^2}\int_M \mathrm{Tr}\ F\wedge F\ . \tag{23}$$

The first term is the conventional Yang-Mills kinetic energy. The second term is the θ term of Yang-Mills theory, which measures the instanton number (but with an imaginary value of the θ angle). This term could be dropped, if one wishes, without spoiling the Q invariance. As for the kinetic energy of the spin zero fields ϕ and λ, this can be interpreted in various ways as the discussion has been somewhat formal up to this point. If one interprets ϕ as a *complex* Lie algebra valued zero form (that is, ϕ takes values in the *complexification* of the Lie algebra), and $\lambda = \bar{\phi}$ as the complex conjugate of ϕ, then the scalar kinetic energy

$$\mathcal{L}_{scalar} = \frac{1}{e^2}\int_M d^4x\ \sqrt{g}\ \mathrm{Tr}\ D_\alpha\bar{\phi}D^\alpha\phi \tag{24}$$

is the conventional kinetic energy of a complex scalar field. The fermion kinetic energy in (22) is of a fairly conventional first order form, except for the fundamental difference that the fermions have integral rather than half-integral spin. This was of course necessary in the construction of the fermionic symmetry Q.

Despite its exotic features, the Lagrangian (22) is obviously rather close to conventional Yang-Mills theories coupled to matter. Indeed, it can easily be seen that, with a minor modification of the choice of V, one obtains precisely a twisted version of conventional $N = 2$ supersymmetric Yang-Mills theory. In particular, this theory is asymptotically free.

Now, what is special about the example that we have just constructed? Mathematically, one could attempt to consider global properties of the solutions of a variety of partial differential equations. But the special case of the self-dual Yang-Mills equations leads to a deep theory, Donaldson theory. Physically, one could construct along the above lines a quantum field theory associated with any moduli problem. But in the special case of the self-dual Yang-Mills equations, one can obtain a representation for this theory that is particularly close to conventional physical theories, and in particular is asymptotically free.

Let us analyze some of the properties of the Lagrangian that we have constructed. The first fundamental property of this theory is topological invariance. A change in the metric g of M will induce a change in the Lagrangian of the form

$$\delta\mathcal{L} = \frac{1}{e^2}\{Q, \delta V\} . \tag{25}$$

In particular, the change in the Lagrangian is a BRST commutator, and is irrelevant in computing the correlation functions of the physical (BRST invariant) observables. The correlation functions of those observables are consequently independent of the metric and so are topological invariants.

Let us state this argument a little more precisely. A path integral

$$\langle \mathcal{O}_1\mathcal{O}_2 \ldots \mathcal{O}_n\rangle = \int \mathcal{D}A \ldots \mathcal{D}\eta \; e^{-\mathcal{L}} \; \mathcal{O}_1\mathcal{O}_2 \ldots \mathcal{O}_n \tag{26}$$

defining a correlation function $\langle \mathcal{O}_1\mathcal{O}_2 \ldots \mathcal{O}_n\rangle$ of physical observables requires a choice of *some* nondegenerate Lagrangian $\mathcal{L}$ in order to be well-defined. Once we pick such a Lagrangian, the correlation functions should be invariant under $\mathcal{L} \to \mathcal{L} + \{Q, \Delta V\}$ for ΔV in some open set in the space of coupling constants. The only restriction on the allowed choice of ΔV is that sufficiently large ΔV might move us to a degenerate Lagrangian (giving an ill-defined theory), or conceivably to a non-degenerate Lagrangian that is in a different "universality class," corresponding to an entirely new quantum field theory. In the case at hand, the Lagrangian that we are considering is nondegenerate for any choice of the background metric on M. As the space of possible metrics is connected, there is therefore no problem with the formal argument showing that correlation functions of BRST invariant observables are independent of the background metric.

A second key property of this theory is that this is a theory in which the semi-classical approximation is exact. To understand this, note that the same argument that shows that the partition function is independent of the metric on M shows that it is independent of the coupling constant e. We can therefore consider the limit of very small e. The leading approximation for small e involves expanding around zeros of the classical action. For the classical action to vanish, first of all the instanton equation $F_{\alpha\beta} - *F_{\alpha\beta} = 0$ must be obeyed. Secondly, ϕ must be covariantly constant, $D_\alpha\phi = 0$. For irreducible instantons, this requires $\phi = 0$, so in good cases in which all of the instantons are irreducible, the space of classical minima is simply the instanton moduli space $\mathcal{M}$, with $\phi = 0$. As a first orientation, we will mainly consider this case. The main difficulties in understanding Donaldson theory, whether by physical or mathematical methods, come from the fact that in general there are reducible instantons for which ϕ can have zero modes.

A third key property of this theory is the ghost number anomaly, and its relation to the zero modes of the instanton moduli problem. Let us recall that of the fermions, ψ has $U = 1$, and η, χ have $U = -1$. According to the index theorem, the net number of $U = 1$ zero modes minus $U = -1$ zero modes is equal to what is called the formal dimension of instanton moduli space (which coincides with the actual dimension in good cases in which the moduli space is smooth and non-singular). To see this explicitly, let us note that if A_α is an instanton, that is, a solution of $F_{\alpha\beta} - *F_{\alpha\beta} = 0$, then the condition that $A_\alpha + \delta A_\alpha$ obeys the same equation to first order in δA_α is

$$D_\alpha \delta A_\beta - D_\beta\, \delta A_\alpha - \epsilon_{\alpha\beta\gamma\delta}\, D^\gamma \delta A^\delta = 0 \qquad (27)$$

To ensure that δA_α is orthogonal to the changes in A_α that would be induced by infinitesimal gauge transformations, one imposes the gauge condition

$$D_\alpha\, \delta A^\alpha = 0\ . \qquad (28)$$

Looking at the form of the Lagrangian, we see that the equations for a ψ_α zero mode are precisely the same as (27) and (28). We see therefore that *the ψ_α zero modes are precisely the tangent vectors to instanton moduli space.* As for η and χ zero modes, it can be shown that they, like ϕ zero modes, are associated with singularities of instanton moduli space. (In fact, an η zero mode must obey $D_\alpha \eta = 0$ which coincides with the condition for a ϕ zero mode.) In getting a first orientation to the theory, we thus wish to consider the case in which only A_α and ψ_α have zero modes.

If $\mathcal{M}$ is the instanton moduli space, then $\widehat{\mathcal{M}}$, the moduli space of the combined (A_α, ψ_α) system, is a supermanifold of a very special kind. Denoting the moduli and supermoduli as m_i and $\widehat{m}_i$, respectively, a function on $\widehat{\mathcal{M}}$ has an expansion

$$\begin{aligned} f(m_1, \ldots, m_n, \widehat{m}_1, \ldots, \widehat{m}_n) = f_0(m_1, \ldots, m_n) & \\ + f_{(1)i}(m_1, \ldots, m_n)\widehat{m}_i & \\ + f_{(2)ij}(m_1, \ldots, m_n)\widehat{m}_i\widehat{m}_j & \\ + \ldots & \end{aligned} \tag{29}$$

where $f_{(k)i_1 \ldots i_k}(m_{i_1} \ldots, m_{i_n})\widehat{m}_{i_1} \ldots \widehat{m}_{i_k}$ can be interpreted as a k form on $\mathcal{M}$. The functions on $\widehat{\mathcal{M}}$ can thus be interpreted as *differential forms* on $\mathcal{M}$.

If we are willing to rename $\widehat{m}_i$ as dm_i, then the BRST formula

$$[Q, m_i] = \widehat{m}_i = dm_i \tag{30}$$

which comes from the underlying formula $[Q, A_\mu] = \psi_\mu$ shows that *on instanton moduli space, Q is the exterior derivative,*

$$Q = \sum_i dm_i \frac{\partial}{\partial m_i} \,. \tag{31}$$

I have described a particular construction of a Lagrangian quantum field theory, but actually theories constructed this way have many equivalent Lagrangian realizations. This should be obvious; there was enormous freedom in the choice of the antighost multiplets. How, then, do we see that different formulations are equivalent? The formulas

$$\begin{aligned} \{Q, \psi_\mu\} &= -D_\mu \phi \\ \{Q, \chi\} &= H = \frac{1}{4}(F - *F) \end{aligned} \tag{32}$$

(the second step follows upon solving the field equation of the auxiliary field H) shows that in the theory that we have constructed *instantons with $D_\mu\phi = 0$ are the Q invariant configurations.* Any theory with this property, and with the property that on instanton moduli space Q reduces to the exterior derivative, will give a Lagrangian realization of Donaldson theory.

In addition to the freedom in the choice of the antighost multiplets, there is another important potential source of freedom in constructing Lagrangians. The theory that we have constructed depended on the choice of a moduli problem. In the case that we considered in detail, this was the problem of Yang-Mills instantons. In that case, the description of the moduli problem in terms of gauge fields and the self-dual Yang-Mills equations was the only apparent one. In other important cases, though, it may happen that the same moduli problem may have several very different-looking descriptions.

A particularly striking instance of this arises in the case of two dimensional gravity. The moduli space of Riemann surfaces of genus g has many different-sounding descriptions. It can be described as the space of all metrics modulo diffeomorphisms and Weyl transformations, as the space of all metrics of $R = -1$ with a given area element modulo area-preserving diffeomorphisms, or as the space of all flat $SL(2, R)$ connections (on a bundle of a particular topological type) modulo gauge transformations and the action of the mapping class group. These three descriptions of moduli space of Riemann surfaces will give rise to three very different-looking constructions of the same quantum field theory. The theory that one obtains by adopting one's favorite point of view about moduli space of Riemann surfaces and then "turning the crank" along the lines sketched above is the two dimensional topological gravity model that has recently turned out to be, apparently, equivalent to the hermitian one matrix model.

One of the striking facts about "physical" two dimensional quantum field theories is that there are many unexpected identities between seemingly different models. Since we are beginning to learn that at least in some instances the physical models can be derived from topological ones, it may well be that some of the relations between physical models can be deduced from more transparent relations among topological models.

Now let us consider the evaluation of a correlation function, such as

$$\langle \mathrm{Tr}\,\phi^{d_1}\cdot \mathrm{Tr}\,\phi^{d_2}\dots \mathrm{Tr}\,\phi^{d_k}\rangle\ . \tag{33}$$

Each operator $\mathrm{Tr}\,\phi^{d_j}$ has ghost number $2d_j$, and will absorb $2d_j$ fermion zero modes. To lowest order of perturbation theory, which we know will give the exact answer, each operator will absorb the requisite number of fermion zero modes independent of the others. Each operator $\mathrm{Tr}\,\phi^{d_j}$ thus becomes a function of $\widehat{\mathcal{M}}$,

$$\mathrm{Tr}\,\phi^{d_j} \to f_{d_j}(m_1,\ldots,m_n;dm_1,\ldots,dm_n) \tag{34}$$

or in other words *a differential form on moduli space*, of dimension $2d_j$. BRST invariance, $[Q,\mathrm{Tr}\,\phi^{d_j}]=0$, implies that each $\mathrm{Tr}\,\phi^{d_j}$ corresponds to a *closed* differential form on moduli space.

So we get a formula

$$\langle \mathrm{Tr}\,\phi^{d_1}\ \mathrm{Tr}\,\phi^{d_2}\ldots\mathrm{Tr}\,\phi^{d_k}\rangle = \int_{\mathcal{M}} f_1\wedge f_2\wedge\ldots\wedge f_k\ , \tag{35}$$

where f_j is the closed form on $\mathcal{M}$ corresponding to $\mathrm{Tr}\,\phi^{d_j}$. If one adds a BRST commutator to one of the operators, $\mathrm{Tr}\,\phi^{d_j}\to\mathrm{Tr}\,\phi^{d_j}+\{Q,\Lambda\}$, then the corresponding differential form transforms as $f_j\to f_j+d\lambda$, where λ is the $2d_j-1$ form on $\mathcal{M}$ determined by the amplitude for Λ to absorb $2d_j-1$ fermion zero modes.

So really, the $\mathrm{Tr}\,\phi^{d_j}$ define *cohomology classes* on moduli space. The correlation functions are given according to (35) by the integral over moduli space of the cup product of these cohomology classes (that is, the integral of the wedge product of the differential forms). Dually, the cohomology classes represented by the $\mathrm{Tr}\,\phi^{d_j}$ are associated with Poincaré dual homology classes, and the correlation functions are the "intersection numbers" of these classes.

Though we have considered only physical methods so far, it is natural at this point to compare to the conventional mathematical viewpoint about the subject. Mathematically, cohomology classes on $\mathcal{M}$ are naturally constructed [10] as characteristic classes of the "universal instanton." What this means is the following. On the product $M\times\mathcal{M}$ of space-time M with moduli space $\mathcal{M}$, we wish to construct a gauge field $\mathcal{A}=(A_\mu,A_i)$ (A_μ and A_i are the components tangent to M and $\mathcal{M}$, respectively) with the following properties. Since $\mathcal{M}$ is the moduli space of instantons on M, every point $p\in\mathcal{M}$ labels an instanton $A_\mu(x^\nu;p)$ on M which is uniquely defined up to a gauge transformation. (Here x^ν are coordinates on M, and the notation $A_\mu(x^\nu;p)$ is meant to emphasize the dependence of the instanton on p.) A universal instanton, if it exists, is simply a connection $\mathcal{A}$ on a suitable G bundle W over $M \times \mathcal{M}$ (finding W is the essential problem) such that on $M \times \{p\}$, for every $p \in \mathcal{M}$, $\mathcal{A}$ coincides up to a gauge transformation with $A_\mu(x^\sigma; p)$.

In general, because of problems associated with the gauge symmetries of instantons, the universal instanton may not exist. However, if we keep away from the singularities of moduli space, then the problem comes only from the center of the gauge group. This would mean that for $G = SU(2)$ the universal instanton would exist as an $SU(2)/\mathbb{Z}_2 = SO(3)$ gauge field, but perhaps not as an $SU(2)$ gauge field. If so, one cannot define the Chern classes of the hypothetical $SU(2)$ universal instanton, but only certain integral multiples thereof. As we are not interested in such questions of integrality, we will proceed as if the universal instanton exists.

If so, the universal instanton $\mathcal{A}$ has a curvature form $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$, and we can define a characteristic class $\mathrm{Tr}\ \mathcal{F}^k$ (that is, $\mathrm{Tr}\ \mathcal{F} \wedge \mathcal{F} \wedge \ldots \wedge \mathcal{F}$, with k factors). Now, $\mathrm{Tr}\ \mathcal{F}^k$ is a closed form of degree $2k$ on $M \times \mathcal{M}$; if we pick $x \in M$, we can restrict it to $\{x\} \times \mathcal{M}$ to get a closed $2k$ form on $\mathcal{M}$ that we will call $\mathrm{Tr}\ \mathcal{F}^k(x)$. This is very reminiscent of the closed $2k$ form on $\mathcal{M}$ coming from the quantum field operator $\mathrm{Tr}\ \phi^k(x)$. It has been precisely demonstrated that the differential forms on $\mathcal{M}$ determined by $\mathrm{Tr}\ \phi^k$ represent the cohomology classes $\mathrm{Tr}\ \mathcal{F}^k$ [3].

I have used $\mathrm{Tr}\ \phi^k$ and $\mathrm{Tr}\ \mathcal{F}^k$ as examples, but more generally any degree r invariant polynomial P on the Lie algebra would lead, in either the physical or mathematical construction, to an observable or characteristic class of dimension $2r$. In the mathematical description we would denote it as $P(\mathcal{F}(x))$ and in the physical description as $P(\phi(x))$. This close parallel between the two constructions is of course no accident. It arises naturally if one thinks of the (A_μ, ψ_μ, ϕ) multiplet as a model for the equivariant cohomology of the gauge group, acting on the space of all connections [5,6].

More general cohomology classes on $\mathcal{M}$ of a similar origin can be described as follows. If Y is an r-dimensional submanifold of M, then we restrict the universal instanton from $M \times \mathcal{M}$ to $Y \times \mathcal{M}$. Then we regard $\mathrm{Tr}\ \mathcal{F}^k$ as an element of $H^{2k}(Y \times \mathcal{M}, \mathbb{R})$, and by integrating along the fibers of the projection $Y \times \mathcal{M} \to \mathcal{M}$, we get an element $W_k(Y)$ in $H^{2k-r}(\mathcal{M}, \mathbb{R})$ (which can be shown to depend only on the homology class of Y). This precisely corresponds to the physical observable of ghost number $2k - r$ that we associated in equations (15)–(17) with the

r dimensional submanifold Y of M.

If we wish to describe two dimensional gravity in a similar vein [12,13], we must replace instanton moduli space by the moduli space of Riemann surfaces. For every way to describe this moduli space in terms of "fields, symmetries, and equations," we will get an alternative Lagrangian realization of the same theory. As I have already noted, there are several quite different-looking possibilities for the formulation of this moduli problem. The low genus correlation functions of the resulting theory can be computed conveniently from their characterization in terms of cohomology of moduli space, as I have explained at length elsewhere [14]. Though the reasons are not yet so well understood as we would like, the resulting theory seems to be equivalent to the hermitian matrix model of two dimensional gravity. At the moment, the matrix model is easily the most powerful method of computation (just as quantum groups are in many respects the most powerful method of computation in another topological field theory, 2+1 dimensional Chern-Simons gauge theory). However, the interpretation of two dimensional gravity in terms of intersection theory on moduli space gives a different vantage point about what the computations mean, embeds two dimensional gravity in a much wider theoretical context that includes, for instance, four dimensional Donaldson theory, and makes obvious the existence of certain types of matter couplings (couplings of topological sigma models, for instance) that are not otherwise known.

If I have given any impression that the story I am explaining is a finished or polished story, this is entirely misleading. The formal construction of Lagrangians is understood well enough. The problem arises when one wishes to understood the physical and mathematical phenomena described by the resulting quantum field theories. I have described formal arguments showing that the quantum field theories in question are theories in which the semiclassical approximation is exact, and I have sketched the topological meaning of the formulas obtained in the semiclassical approximation. The problem is that the semiclassical approximation involves a reduction to a classical moduli space, and if this moduli space is *singular* or *noncompact*, the implementation and interpretation of the semiclassical approximation is not straightforward. Singularities of moduli space mean physically that the Gaussian approximation which is supposed to be the leading term in perturbation theory breaks down because of accidental *zero* modes that appear at the singularities. Noncompactness of moduli space means that convergence of the integrals given by the semiclassical approximation is not automatic; it also means that the topological interpretation of those integrals is not clear. These phenomena may sound at first sight like unlikely nuisances, but actually they are ubiquitous and both physically and mathematically are the main obstacle to understanding Donaldson theory more fully.

To underscore the point, let me note that in Donaldson theory, the space of BRST invariant configurations up to gauge transformation, which is the moduli space we are really interested when we attempt to carry out the semiclassical approximation, is not just the space of instantons. Rather, what arises, as we have seen in the above discussion, is the space of pairs (A_μ, ϕ), where A_μ obeys the instanton equation and ϕ is covariantly constant, $D_\mu\phi = 0$. Now, with a favorable choice of the four manifold and the instanton number, it will happen that the holonomy group of the generic instanton will be the full gauge group, and then the equation $D_\mu\phi = 0$ will force $\phi = 0$. It is in that case that one can forget about ϕ and carry out a semiclassical computation on instanton moduli space. But in general degenerate instantons with a reduced holonomy group will also exist, leading to nonzero solutions of the equation $D_\mu\phi = 0$. Since this equation is linear in ϕ, its solutions are a vector space and always noncompact in a very serious way – one must understand the behavior for $\phi \to \infty$.

One should think of the moduli spaces in Donaldson theory as having two branches. There are the "good" branches, let us call them of type (A), corresponding to irreducible instantons with $\phi = 0$. There are also the "bad" branches, let us call them of type (B), corresponding to $\phi \neq 0$ (and necessarily, therefore, corresponding to degenerate instantons). The contributions of type (A) are more or less understood; this is what I have explained above. The problem in Donaldson theory is that the role of the branches of type (B) is not understood. In addition, there are singularities where the type (A) and type (B) branches intersect (at reducible instantons with $\phi = 0$).* To give a concrete example of a moduli space with a branch of type (B), let the space-time manifold M be a four sphere $\mathbf{S}^4$ and take the instanton number to be zero. The only solution of the self-dual Yang-Mills equation, up to gauge transformation, is then $A_\mu = 0$. With $A_\mu = 0$, the equation $D_\mu\phi = 0$ is then obeyed by $\phi =$constant, giving a branch of type (B).

These notes have been written purely from the path integral point of view. The other im-

* There are other sources of singularities and noncompactness in Donaldson theory, but I have explained the ones that I think are the real obstacles to better understanding. Likewise in discussing the Hamiltonian treatment below, I will describe only the singularities and noncompactness that I regard as the real cause of difficulties.

portant tool in understanding a quantum field theory is of course the Hamiltonian formulation, in which one considers four manifolds of the type $M = Y \times \mathbb{R}^1$, where Y is a three manifold and $\mathbb{R}^1$ represents "time," and by canonical quantization one constructs a "physical Hilbert space" associated with Y. In the context of Donaldson theory the theory of this physical Hilbert space was initiated by A. Floer. Again from the canonical point of view, we have a difficulty analogous to the one that I discussed above. The BRST symmetry can be used to show that the physical Hilbert spaces are naturally independent of the gauge coupling constant e, so we would like to determine them by considering the limit of small e and constructing the leading perturbative approximation. To this aim, we look at the classical Hamiltonian, which contains a term

$$H = \int_Y \left(\frac{1}{4e^2} \operatorname{Tr} F_{ij} F^{ij} + D_i \overline{\phi} D^i \phi \right) \tag{36}$$

(where F_{ij} is the restriction of the curvature to Y). The minima of the classical energy correspond to $F_{ij} = 0$ and $D_i\phi = 0$. Now we have a story analogous to what we have said before. The moduli space of solutions of this pair of equations consists of two branches. A branch of type (A) consists of irreducible flat connections, that is, solutions of $F_{ij} = 0$ with a sufficiently generic holonomy, which necessarily have $\phi = 0$. A branch of type (B) consists of fields with $\phi \neq 0$, in which case the holonomy must be reducible. In addition one has singularities where the two branches intersect (at reducible flat connections with $\phi = 0$). Just as in the path integral approach, the role of the branches of type (A) is clear enough, at least in principle, but the role of the branches of type (B) is not understood. As a simple example in which one encounters a branch of type (B), consider the basic case in which $Y = \mathbf{S}^3$ is the three sphere. Then up to gauge transformation a flat connection is $A_i = 0$, and hence the equation $D_i\phi = 0$ reduces to $\phi =$ constant, giving a branch of type (B).

I have been lecturing about a certain class of quantum field theories. One theory in this class, two dimensional topological gravity, has come to seem "physical" recently, because of its apparent equivalence with matrix models. This indeed has placed the matrix models in a broader and perhaps unexpected context. I hope that other theories in this class will come to be considered "physical" in the relatively near future, and if so this will hopefully lead to progress in understanding the problems I have cited in the last few paragraphs.

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