Christoph Wockel

May 24, 2009

1 Higher dimensional covers

(Joint work with Sven Porst & Chenchang Zhu)

Motivation:
Categorification of structure groups for gauge theories.

Notion of cover: X $(n-1)$-connected space

$$n-cover: \quad a\colon Y \to X \text{ fibration with } \pi_k(a) \text{iso for } k \neq n, \;\; \pi_n(Y) = 0$$

Construction:

$$\text{For characteristic map } X \xrightarrow{f} K(\pi_n, n) \text{ iso on } \pi_n,$$
$$Y = f^*(PK(\pi_n, n))$$

the pullback of the path loop fibration. This is an n-cover of X.

This is unsatisfactory from a group perspective.

Example:

$(n = 1)$	$Spin$	$\to$	SO	$1 -$ cover (simply connected)
$(n = 2)$	?	$\to$	$\Omega Spin$	($\infty -$ dimensional Lie group with $\pi_2 \neq 0$)
$(n = 3)$	?	$\to$	$Spin$	(? $\cong$ "string group")

2 A simple but instructive example: $n = 1$

For G a connected Lie group (or also a topological group), consider the simply connected cover

$$\pi_1 \hookrightarrow \tilde{G} \twoheadrightarrow G$$

This is

- a π_1-principal bundle

- a central extension of G by π_1

Now we know from group cohomology that $\tilde{G}$ is equivalent to $\pi_1 \times_{\theta_1} G$, which is the set $\pi_1 \times G$, endowed with the group multiplication

$$(a, g) \cdot (b, h) = (a + b + \theta_1(g, h), g \cdot h)$$

for some function $\theta_1 : G \times G \to \pi_1$

- associativity requires: $\theta_1(g, h) + \theta_1(gh, k) = \theta_1(g, hk) + \theta_1(h, k)$
- $\theta_1(g, e) = \theta_1(e, g) = 0$ implies that $(0, e)$ is a unit

This defines the group structure $\pi_1 \times_{\theta_1} G$, but how about the smooth structure?

Assume that $\theta_1 \mid_{U \times U}$ is smooth on a unit neighborhood $U \subset G$, then θ_1 gives rise to a Čech cohomology class

$$[\tau\theta_1] \in \check{H}^1(G, \pi_1).$$

Endowing $\pi_1 \times_{\theta_1} G$ with the topology making $\pi_1 \times_{\theta_1} G \stackrel{pr_2}{\to} G$ a π_1-principal bundle with the characteristic class $[\tau\theta_1]$ yields a Lie group topology on $\pi_1 \times_{\theta_1} G$ such that

$$\pi_1 \hookrightarrow \pi_1 \times_{\theta_1} G \twoheadrightarrow G$$

is equivalent to $\tilde{G}$ as a central extension.

3 Construction of θ_1

For each $g \in G$, choose a smooth path α_g, connecting the identity e with g, i.e., a section $\alpha : G \to PG$ of the evaluation map $ev : PG \to G$, where PG is the smooth path space of G (w.l.o.g. we can assume α to be smooth on a unit neighbourhood). Then we can interpret α as a map from G to the group C_1 of singular 1-chains on G, and thus we may take its group differential $\mathtt{d}_{\mathtt{gp}}\,\alpha$. The crucial observation is that $\mathtt{d}_{\mathtt{gp}}\,\alpha$ takes values in the subgroup of 1-cycles Z_1, instead of C_1 (cf. Figure 1). With this we set $\theta_1 := q \circ (\mathtt{d}_{\mathtt{gp}}\,\alpha) : G \times G \to \pi_1$, where $q : Z_1 \to H_1 \cong \pi_1$ is the canonical quotient map. From this it is obvious that θ_1 is a cocycle.

Theorem 1. *$[\theta_1]$ is universal for 2-cocycles f which vanishes on some unit neighborhood, i.e.,*

$$\mathrm{Hom}(\pi_1, A) \to H^1_{\mathrm{gp}}(G, A), \quad \varphi \mapsto [\varphi \circ \theta_1]$$

is bijective for each discrete abelian group A.

- Use standard convering theory for proof. In particular, the path lifting property (or parallel transport).

$$(\mathtt{d}_{\mathrm{gp}}\,\alpha)(g,h)=\alpha_g-\alpha_{gh}+g.\alpha_h=$$

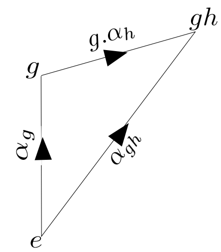

Figure 1: $(\mathtt{d}_{\mathtt{gp}}\,\alpha)(g,h)$ is a closed 1-cycle in G

- $H^n_{\mathrm{gp}}(G,A)$: locally smooth group cohomology

Upshot: The universal locally constant 2-cocycle θ_1 describes simply connected covers! $\rightsquigarrow$ We shall take this as the fundamental property for a generalisation to higher dimensions.

4 Construction of θ_2

Now assume that G is simply connected. Then we find for each $g,h\in G$ a (smooth) map $\beta_{g,h}:\Delta^2\to G$ such that $\partial b_{g,h}=(\mathtt{d}_{\mathtt{gp}}\,\alpha)(g,h)$ (cf. Figure 2).

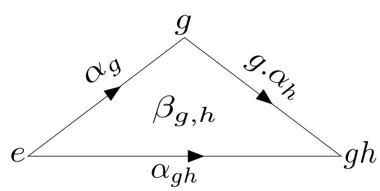

Figure 2: $\partial b_{g,h}=(\mathtt{d}_{\mathtt{gp}}\,\alpha)(g,h)$

As before, we observe that $(\mathtt{d}_{\mathtt{gp}}\,\beta)(g,h,k)$ is a 2-cycle in G (cf. Figure 3) and we set $\theta_2:=q\circ(\mathtt{d}_{\mathtt{gp}}\,\beta):G^3\to\pi_2$. Again, it is obviously true that θ_2 defines a group 3-cocycle. Assuming w.l.o.g. that $\beta_{g,h}$ depends smoothly on g and h on some unit neighbourhood and thus that θ_2 is constant on some unit neighbourhood.

Theorem 2. $[\theta_2]\in H^3_{\mathrm{gp}}(G,\pi_2(G))$ *is universal for locally constant* 3*-cocycles.*

- Proof use path lifting (parallel transport) in 2-bundles.
- Question: To what extend describes θ_2 a 2-connected covering of G?
- algebraically: θ_2 defines and extension of 2-groups

$$B\pi_2\to\mathcal{G}_{\theta_2}\to G$$

$$(\mathtt{d}_{\mathrm{gp}}\,\beta)(g,h,k) = \beta_{g,h} + \beta_{gh,k} - \beta_{g,hk} - g.\beta_{h,k} =$$

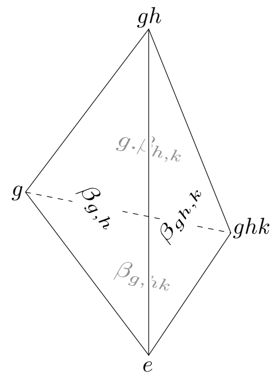

Figure 3: $(\mathtt{d}_{\mathtt{gp}}\,\beta)(g,h,k)$ is a closed 2-cycle in G

- topologically: since θ_2 is constant on a unit neighbourhood, it gives rise to a Čech 2-cocycle $\tau\theta_2$, which leads to a principal $B\pi_2$-2-bundle by the next theorem.

Theorem 3. *Principal $\mathcal{G}$-2-bundles (for $\mathcal{G}$ a strict Lie 2-group) over G are classified by $\check{H}(G,\mathcal{G})$.*

In particular, if $\mathcal{G}$ is $B\pi_2$, then $\check{H}(G,\mathcal{G}) \cong \check{H}^2(G,\pi_2)$ and $[\tau\theta_2] \in \check{H}^2(G,\pi_2)$ gives rise to a principal $B\pi_2$-2-bundle $\mathcal{P}_{\tau\theta_2} \to G$. What would be nice is Lie 2-group structure on $P_{\tau\theta_2}$, but that is too much to ask for! Remedy: invert Morita morphisms of bundles obtain a weak group object in the category of smooth stacks, i.e., a stacky Lie group.

5 What is this good for?

- $\mathcal{G}_{\theta_2}$ provides a generalisation of Lie's Third Theorem to Banach–Lie algebras, (which fails in general when trying to integrate Banach–Lie algebras to Banach–Lie groups).

- Generalisation to higher dimensions in possible, giving a cohomology class $[\theta_n] \in H^{n+1}_{\mathrm{gp}}(G,\pi_n)$, which is universal for locally constant $(n+1)$-group cocycles $\rightsquigarrow$ relation to string group models!

- For G compact, simple and simply connected, the transgression map $\tau : H^4_{\mathrm{gp}}(G,\pi_3) \to \check{H}^3(G,\pi_3)$ may be understood in terms of the Dijkgraaf-Witten correspondence.