Skein algebras and skein modules beyond semisimplicity

Lukas Woike
Université de Bourgogne

Based on different joint projects with Adrien Brochier (IMJ-PRG) and Lukas Müller (Perimeter Institute)
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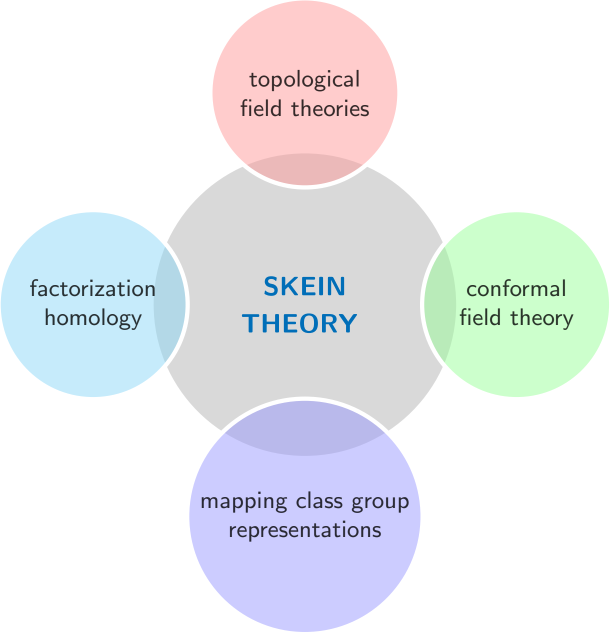


topological
field theories
factorization
homology
SKEIN
THEORY
conformal
field theory
mapping class group
representations

Terminology

- A *finite tensor category* [Etingof-Ostrik] $\mathcal{A}$ over some algebraically closed field k is
  - a linear abelian category $\mathcal{A}$ with finite-dimensional morphism spaces, enough projective objects, finitely many isomorphism classes of simple objects such that every object has finite length,
  - with a monoidal product $\otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$,
  - a rigid duality $-^{\vee}$,
  - and simple unit.
- A *braiding* on a monoidal category is a natural isomorphism $X \otimes Y \longrightarrow Y \otimes X$ subject to the hexagon axioms. A braiding on a finite tensor category is called *non-degenerate* if the only objects that trivially double braid with all other objects are finite direct sums of the monoidal unit.

Terminology

- A *balancing* on a braided monoidal category is a natural isomorphism $\theta_X : X \longrightarrow X$ subject to

$$\theta_{X\otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \ ,$$
$$\theta_I = \mathrm{id}_I \ .$$

- If in presence of duality we have additionally

$$\theta_{X^\vee} = \theta_X^\vee \ ,$$

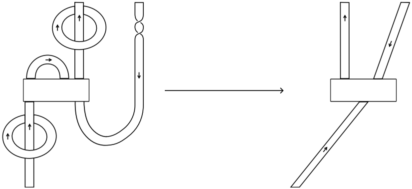
  we call the balancing a *ribbon structure*.
- *Modular category*: finite ribbon category with non-degenerate braiding.

Sources for modular categories

Certain Hopf algebras ($\longrightarrow$ quantum groups) and vertex operator algebras ($\longrightarrow$ two-dimensional conformal field theory).

Category $\mathcal{A}-\mathrm{RibGraphs}(M)$ of ribbon graphs in a three-dimensional manifold M

Let M be an oriented, compact three-dimensional smooth manifold with boundary. Define the category $\mathcal{A}-\mathrm{RibGraphs}(M)$ of Proj $\mathcal{A}$-labeled ribbon graphs in M and replacements in cubes:



Use the Reshetikhin-Turaev graphical calculus to define a functor $\mathcal{A}-\mathrm{RibGraphs}(M) \longrightarrow \mathrm{vect}$ whose colimit we define as $\mathrm{sk}_{\mathcal{A}}(M; B)$ with family B of projective boundary labels $\longrightarrow$ *admissible skein module*, following [Costantino-Geer-Patureau-Mirand 23]. Generalization of the classical construction by Turaev, Kauffman, Przytycki, Hoste, Walker, Masbaum, Roberts, ...

picture: arXiv:2409.17047

Skein categories [Walker (ssi), Brown-Haïoun (nonssi)]

Let $\mathcal{A}$ be a finite ribbon category. For a surface Σ (always compact, oriented, possibly with boundary), define the k-linear *admissible skein category* $\mathrm{skcat}_{\mathcal{A}}(\Sigma)$:

- Objects are embedded intervals in the interior of Σ (at least one in each connected component of Σ) labeled with projective objects of $\mathcal{A}$.
- For two objects B and C, the morphism vector space is $\mathrm{sk}_{\mathcal{A}}(\Sigma \times [0,1]; B^\vee, C)$. Composition is by stacking of cylinders.

Then we get a functor

$$\mathrm{sk}_{\mathcal{A}}(M; -) : \mathrm{skcat}_{\mathcal{A}}(\partial^{\mathrm{oc}} M) \longrightarrow \mathrm{vect} \ ,$$

with $\partial^{\mathrm{oc}} M$ being the parametrized part of the boundary of M.

Connection to factorization homology

Theorem [Cooke 19 (ssi), Brown-Haïoun 24 (nonssi)]

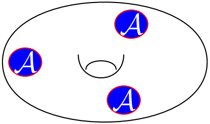
Let $\mathcal{A}$ be a finite ribbon category. For any surface Σ, the skein category $\mathrm{skcat}_{\mathcal{A}}(\Sigma)$, after finite free cocompletion, is equivalent to the factorization homology $\int_\Sigma \mathcal{A}$.

What is *factorization homology*? [Beilinson-Drinfeld, Lurie, Ayala-Francis, . . . ; 2000-]

coefficients: E_2-algebra, e.g. braided category

$$\int_\Sigma \mathcal{A} = \bigoplus_{\substack{\sqcup_n \mathbb{D}^2 \hookrightarrow \Sigma \\ n \geq 0}} \mathcal{A}^{\boxtimes n} \Big/ \sim$$

surface



Excision

Theorem

Let $\mathcal{A}$ be a finite ribbon category. Suppose that M' is obtained by gluing a compact oriented three-manifold M along two oppositely oriented copies of a boundary surface Σ. Then for any $X \in \mathrm{skcat}_{\mathcal{A}}(\partial^{oc} M')$ there is an isomorphism

$$\int^{P \in \mathrm{skcat}_{\mathcal{A}}(\Sigma)} \mathrm{sk}_{\mathcal{A}}(M; X, P, P^{\vee}) \xrightarrow{\simeq} \mathrm{sk}_{\mathcal{A}}(M'; X)$$

The core argument is due to Walker; more recent incarnations of this statement (and its two-dimensional analogue) covering the generality needed here are in Gunningham-Jordan-Safronov, Fuchs-Schweigert-Yang, Müller-Schweigert-W.-Yang, Brown-Haïoun, Runkel-Schweigert-Tham, Müller-W.

Excision

Example: Excision for skein modules and modified traces

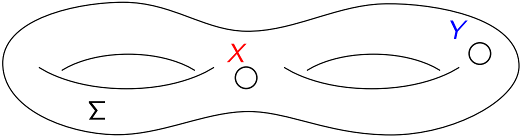
With the above excision result in combination with [Costantino-Geer-Patureau-Mirand 23], we find

$$\int^{P \in \mathrm{Proj}\,\mathcal{A}} \mathcal{A}(I, P) \otimes \mathcal{A}(P, I) \cong \{\text{space of two-sided modified traces}\}^* \,.$$

In case that $\mathcal{A}$ is given by modules over a ribbon Hopf algebra H, the left hand side is $\mathrm{Hom}_H(k, H) \otimes_H k$. This is closely related to the description of modified traces through (co)integrals of H by Beliakova-Blanchet-Gainutdinov, Shibata-Shimizu, Berger-Gainutdinov-Runkel.

Modular functors

Following [Segal 88, Moore-Seiberg 88, Turaev 94, Tillmann 98, Bakalov-Kirillov 01, ...].



$$\mathrm{Map}(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma)) \ ; \quad \text{Example: } \mathrm{Map}(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathrm{SL}(2, \mathbb{Z}) \ .$$

$$(\Sigma; X, Y, \dots) \longmapsto \text{vector space } B(\Sigma; X, Y, \dots) \curvearrowleft \mathrm{Map}(\Sigma)$$

for all surfaces, compatible with the gluing of surfaces. The vector space $B(\Sigma; X, Y, \dots)$ is called *space of conformal blocks*.

Formal definition using modular operads in the sense of Getzler-Kapranov

A *modular functor* is a modular algebra over the modular surface operad (or a certain central extension of it) with values in a symmetric monoidal bicategory of linear categories.

picture: arXiv:2201.07542

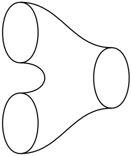
Question: How can we classify genus zero modular functors aka cyclic framed E_2-algebras?

Preliminary observation: The boundary labels form the objects of a linear category, the *circle category*, that we denote by $\mathcal{A}$.

(I will present the situation in which $\mathcal{A}$ is finitely cocomplete and $B(\Sigma, -)$ cocontinuous in the labels. Technically speaking: We work in $\mathsf{Rex}^{\mathsf{f}}$.)

Genus zero modular functors

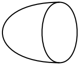
[Wahl 01, Salvatore-Wahl 03]



$\longmapsto \quad \otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$

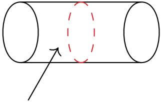
monoidal product

plus braiding $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$



$\longmapsto \quad I \in \mathcal{A}$

monoidal unit



Dehn twist

$\longmapsto \quad \theta : \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

balancing

$$\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y)$$

$$\theta_I = \mathrm{id}_I$$

Genus zero modular functors

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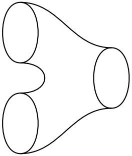

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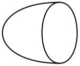

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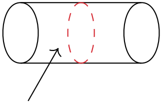
monoidal unit



$\longmapsto \quad \theta : \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

balancing

$$\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y)$$

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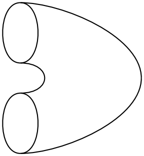
[Müller-W. 20-22]

cyclic structure



ribbon Grothendieck-Verdier duality in the sense of Boyarchenko-Drinfeld

$$D : \mathcal{A} \xrightarrow{\cong} \mathcal{A}^{\mathrm{opp}}$$

$$\mathrm{Hom}_{\mathcal{A}}(- \otimes Y, K) \cong \mathrm{Hom}_{\mathcal{A}}(-, DY)$$

with $K := DI$

$$\theta_{DX} = D\theta_X$$

‘ genus zero modular functors = ribbon Grothendieck-Verdier categories’

Ansular functors

Take a surface Σ with n boundary components and choose a handlebody filling H. If $\mathcal{A}$ is a ribbon Grothendieck-Verdier category, then $\mathcal{A}$ extends uniquely to all handlebodies; it gives us a so-called *ansular functor*.

Theorem [Müller-W. 2022]

Genus zero restriction provides an equivalence between ansular functors in $\mathrm{Rex}^{\mathrm{f}}$ *and ribbon Grothendieck-Verdier categories.*
The ansular functor associated to a ribbon Grothendieck-Verdier category $\mathcal{A}$ *sends a handlebody of genus* g *and* n *disks embedded in its boundary labeled with* $X_1, \ldots, X_n$ *to the hom space*

$$\widehat{\mathcal{A}}(H) \cong \mathcal{A}(X_1 \otimes \cdots \otimes X_n \otimes \mathbb{A}^{\otimes g}, K)^*$$

defined using the canonical end $\mathbb{A} = \otimes \left(\int_{X \in \mathcal{A}} X \boxtimes DX \right)$ *(*D *is the duality functor of* $\mathcal{A}$*).*

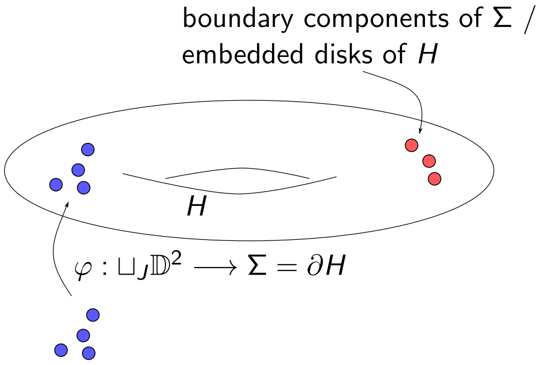
Uses a result of Giansiracusa on the *derived modular envelope* of framed E_2 (a concept due to Costello).
Far-reaching generalization of Lyubashenko's construction, can be applied to very general module categories of vertex operator algebras [Allen-Lentner-Schweigert-Wood 21].

Generalized skein modules [Brochier-W. 22]

Let $\mathcal{A}$ be a ribbon Grothendieck-Verdier category in $\mathsf{Rex}^{\mathsf{f}}$.

- For a handlebody H with $\partial H = \Sigma$ (the n embedded disks of H are converted in boundary components of Σ), consider an embedding $\varphi : \sqcup_J \mathbb{D}^2 \longrightarrow \Sigma$. This endows H with $m := |J|$ more embedded disks in its boundary. We denote this handlebody by H^φ.
- By evaluation of the ansular functor $\widehat{\mathcal{A}}$ associated to $\mathcal{A}$, we get a 1-morphism

$$\mathcal{A}^{\boxtimes m} \xrightarrow{\widehat{\mathcal{A}}(H^\varphi)} \mathcal{A}^{\boxtimes n}$$



- This is natural in φ and hence produces the desired 1-morphism

$$\Phi_{\mathcal{A}}(H) : \int_\Sigma \mathcal{A} = \operatorname*{hocolim}_{\varphi : \sqcup_J \mathbb{D}^2 \longrightarrow \Sigma} \mathcal{A}^{\boxtimes J} \longrightarrow \mathcal{A}^{\boxtimes n} \ .$$

picture: arXiv:2212.11259

How does $\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\boxtimes n}$ correspond to a skein module? Since $\Phi_{\mathcal{A}}(H)(\mathcal{O}_{\Sigma}) \cong \widehat{\mathcal{A}}(H)$ [Brochier-W.] for the quantum structure sheaf $\mathcal{O}_{\Sigma} \in \int_{\Sigma} \mathcal{A}$ of Ben-Zvi-Brochier-Jordan, $\widehat{\mathcal{A}}(H)$ is a module over the *skein algebra*

$$\mathsf{SkAlg}_{\mathcal{A}}(\Sigma) := \mathsf{End}_{\int_{\Sigma} \mathcal{A}}(\mathcal{O}_{\Sigma}) \ .$$

(Agrees with the classical skein algebras in the semisimple case [Cooke 19].)

Comparison

Suppose that $\mathcal{A}$ is a finite ribbon category and H a three-dimensional handlebody. Then we can compare two constructions (on their common domain of definition):

- The value $\widehat{\mathcal{A}}(H)$ of the ansular functor for $\mathcal{A}$, as $\mathrm{Map}(H)$-representation [Müller-W. 22] and module over the skein algebra through the generalized skein module $\Phi_{\mathcal{A}}(H) : \int_\Sigma \mathcal{A} \longrightarrow \mathrm{vect}$ [Brochier-W. 22].
(For this construction, $\mathcal{A}$ does not need to be rigid.)
- The admissible skein module $\mathrm{sk}_{\mathcal{A}}(H)$ of [Costantino-Geer-Patureau-Mirand 23] as $\mathrm{Map}(H)$-representation and module over a generally non-unital skein algebra, but also over the skein algebra defined via factorization homology [Brown-Haïoun 24].
(For this construction, H does not need to be a handlebody.)

Comparison

Recall that the quadruple dual of a finite tensor category is given by

$$-^{\vee\vee\vee\vee} \cong \alpha \otimes - \otimes \alpha^{-1}$$

with the distinguished invertible object α [Etingof-Ostrik-Nikshych 04]. One calls the finite tensor category *unimodular* if $\alpha \cong I$.

Theorem [Müller-W. 24]

For a unimodular finite ribbon category $\mathcal{A}$ and any handlebody H,

$$\widehat{\mathcal{A}}(H) \cong \mathsf{sk}_{\mathcal{A}}(H)$$

as $\mathrm{Map}(H)$*-representations and skein modules.*

This implies that skein modules for three-manifolds have a factorization homology description through the Φ-maps; more precisely,

$$\mathsf{sk}_{\mathcal{A}}(M) \cong \int^{P \in \mathrm{Proj} \int_\Sigma \mathcal{A}} \Phi_{\mathcal{A}}(H'; P^\vee) \otimes \Phi_{\mathcal{A}}(H; P)$$

for a Heegaard splitting $M = H' \cup_\Sigma H$ of a closed three-manifold M.

Strategy

- Prove that both sides form ansular functors: Clear for $\widehat{\mathcal{A}}$. For the skein construction, it boils down to excision.
- The two ansular functors are equivalent if and only if they are equivalent in genus zero by the classification in [Müller-W. 22].
- It remains to determine the underlying cyclic framed E_2-algebra aka ribbon Grothendieck-Verdier category for the skein modular functor. Not very surprisingly, it is $\mathcal{A}$ as balanced braided category, but what is the duality? It is $D = \alpha^{-1} \otimes -$. (All possible ribbon Grothendieck-Verdier dualities relative to the balanced braided structure are given by twists of the rigid duality by an invertible object in the balanced Müger center [Müller-W. 22].) This agrees with $\mathcal{A}$ with the rigid duality if and only if $\mathcal{A}$ is unimodular!

Motivation / Outlook

- Reconcile approaches based on classical skein theory / factorization homology / modular envelope construction.
- Representations of mapping class groups of surfaces.
- Logarithmic conformal field theory.
- Going beyond rigidity.