QMath 16
TU Munich
September 1, 2025

Gap stability and the AKLT model on the hexagonal lattice

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Outline

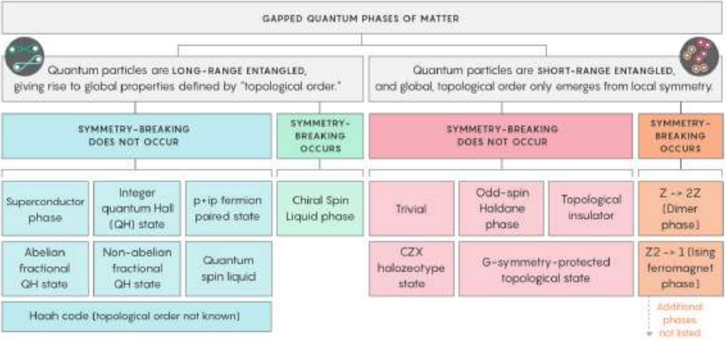
1. Gapped Ground State Phases of Quantum Lattice Models
    1.1 Quantum Spin Systems: Key Definitions and Results
    1.2 Haldane's Conjecture
2. Stability of the Gap and BHM Strategy
3. Main Result: Gap Stability for the Hexagonal AKLT Model

Gapped Ground State Phases

Quantum lattice models are used to investigate and classify phases of quantum matter, including topological phases.

There is a major distinction between gapped and gapless phases, and so determining if a model has a gap is one of the first important questions to settle.

However, proving a nonvanishing gap above the ground state energy is notoriously difficult to determine.



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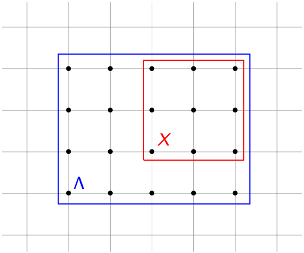
Quantum Spin Systems

A quantum spin system (QSS) is a many-body model defined on a lattice Γ where each vertex $x \in \Gamma$ has only d_x linearly-independent states, i.e. $\mathcal{H}_x = \mathbb{C}^{d_x}$.

For example, $d_x = 2s_x + 1$ if x represents a particle of spin s_x.



These models are initially only well-defined for finite subsets of $\Lambda \subsetneq \Gamma$:

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x, \quad \mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} M_{d_x}(\mathbb{C})$$

For each finite $X \subset \Gamma$ fix an interaction term:

$$\Phi(X)^* = \Phi(X) \in \mathcal{A}_X.$$

The Hamiltonian $H_\Lambda \in \mathcal{A}_\Lambda$ and Heisenberg dynamics $\tau_t^\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ are defined as

$$H_\Lambda = \sum_{X\subseteq\Lambda} \Phi(X), \qquad \tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$$

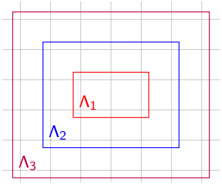
Note: $\mathcal{A}_X \hookrightarrow \mathcal{A}_\Lambda$ via $A \mapsto A \otimes \mathbb{1}_{\Lambda\setminus X}$. The set of possible energies is $\mathrm{spec}(H_\Lambda)$:

$$E_\Lambda^0 < E_\Lambda^1 < E_\Lambda^2 < \dots$$

Ground States and the Spectral Gap

Finite Volume Gap: The ground state space, denoted $\mathcal{G}_\Lambda$, is the eigenspace associated with the ground state energy, E^0_Λ. A QSS is uniformly gapped if there is a sequence $\Lambda_n \uparrow \Gamma$ such that

$$\gamma = \inf_n \mathrm{gap}(H_{\Lambda_n}) > 0, \quad \text{where} \quad \mathrm{gap}(H_{\Lambda_n}) = E^1_{\Lambda_n} - E^0_{\Lambda_n}.$$



Infinite System: The C*-algebra of quasi-local observables is

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}, \quad \mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda \text{ finite}} \mathcal{A}_\Lambda.$$

A weak-* limit $\omega : \mathcal{A}_\Gamma \to \mathbb{C}$ of finite volume ground state functionals $\omega_n(A) = \mathrm{Tr}(\rho_n A)$, $\mathrm{ran}\rho_n \subseteq \mathcal{G}_{\Lambda_n}$, is gapped if the associated GNS Hamiltonian $H_\omega \geq 0$ is gapped, i.e.

$$\mathrm{gap}(H_\omega) := \sup\{\delta > 0 \mid (0,\delta) \cap \mathrm{spec}(H_\omega) = \emptyset\} > 0.$$

Well-known: Under mild assumptions on the interaction $\mathrm{gap}(H_\omega) \geq \inf_n \mathrm{gap}(H_{\Lambda_n})$.

Why is the Gap so Important?

Implications of a nonvanishing ground state gap:

(1) Exponential decay of spin correlations in ground states: [Hastings, Koma '06], [Nachtergaele, Sims '06]

(2) Adiabatic theorems for ground states: [Bachmann, De Roeck, Fraas '17], [Monaco, Teufel '19], [Henheik, Teufel. '22].

(3) The split property for quantum spin chains: [Matsui '10, '13]

(4) Proof of quantization of Hall conductance for interacting electrons on torus: [Hastings, Michalakis '15], [Bachmann, Bols, De Roeck, Fraas '18]

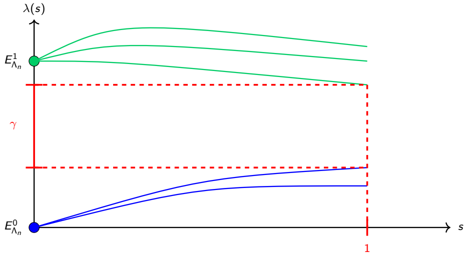
In addition, many properties exhibited by gapped models are stable under small perturbations. [Mariën, Audenaert, Acoleyen, Verstraete '16], [Cha, Naaijkens, Nachergaele '18], [Nachtergaele, Sims, Y. '22]

While the importance of the spectral gap is well known, very few rigorous proving a nonvanishing gap are actually known, especially for multi-dimensional lattices.

Moreover, the spectral gap question is generically undecidable [Cubitt, Pérez-García, Wolf, '15] [Bausch, Cubitt, Lucia, Pérez-García, '18].

Gapped Ground State Phase

A uniform gap is too strong for defining a gapped ground state phase. This can be relaxed to allow for finite volume excited states that converge to infinite volume ground states on Γ.



[Chen, Gu, Wen '10, '11]Two quantum spin interactions Φ_0 and Φ_1 are in the same gapped ground state phase if there exists a path of interactions $\Phi(r)$, $r \in [0,1]$, and sequence of finite volume $\Lambda_n \uparrow \Gamma$ so that

1. $\Phi(X,0) = \Phi_0(X)$ and $\Phi(X,1) = \Phi_1(X)$ for every finite $X \subseteq \Gamma$.
2. $\Phi(X,r)$ is piece-wise differentiable on $(0,1)$ and continuous on $[0,1]$ for each finite $X \subseteq \Gamma$.
3. There exists $\gamma > 0$ and $\epsilon_n(r) \downarrow 0$ as $n \to \infty$ so that

$$\mathrm{spec}(H_{\Lambda_n}(r)) \subseteq [E^0_{\Lambda_n}(r), E^0_{\Lambda_n}(r) + \epsilon_n(r)] \cup [E^0_{\Lambda_n}(r) + \epsilon_n(r) + \gamma, \infty)$$

Example: The Heisenberg Model and Haldane's Conjecture

Fix $s \in \mathbb{N}/2$. For $J = (J_1, J_2) \in \mathbb{R}^2$ with $\|J\| = 1$ define

$$H_N^{(s)}(J) = \sum_{x=1}^{N-1} J_1(\mathbf{S}_x \cdot \mathbf{S}_{x+1}) + J_2(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2, \quad \mathbf{S}_x \cdot \mathbf{S}_{x+1} = \sum_{j=1,2,3} S_x^j \otimes S_{x+1}^j$$

Haldane's Conjecture '83: There exists $\epsilon(s) > 0$ so that if $\|J - (1,0)\| < \epsilon(s)$, then the model $H^{(s)}(J)$ has a unique infinite volume ground state $\omega : \mathcal{A}_{\mathbb{Z}} \to \mathbb{C}$, and

1. $s \in \mathbb{N}$: The model is gapped and ω has exponential decaying correlations.
2. $s \in \frac{\mathbb{N}}{2} \setminus \mathbb{N}$: The model is gapless and ω has power-law decaying correlations.

For $s \in \frac{\mathbb{N}}{2} \setminus \mathbb{N}$, this was not surprising:

1. [Bethe '31], [Lieb, Schultz, Mattis '61] Heisenberg-1/2 model is gapless.
2. [Affleck, Lieb '86] For any s: either non-unique infinite volume ground state, or unique, gapless ground state.

The Heisenberg Model and Haldane's Conjecture

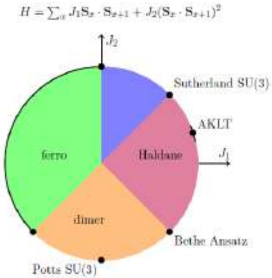


Figure: Phase diagram of antiferromagnetic, $SU(2)$ symmetric, spin-1 chains. Credit: B. Nachtergaele.

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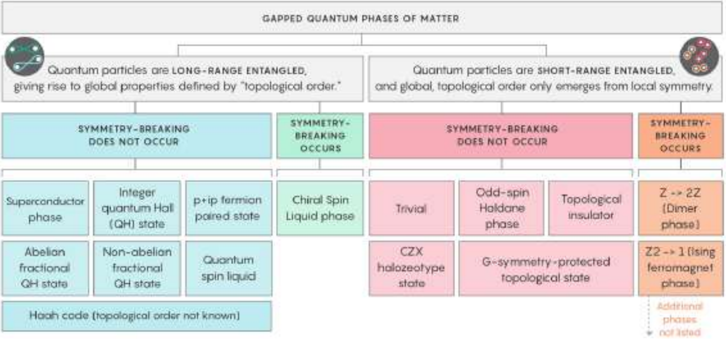
The conjecture for integer s, and in particular $s = 1$, was unexpected. For $s = 1$:

1. [Renard et. al. '87] Experimental evidence.
2. [Affleck, Kennedy, Lieb and Tasaki '87, '88] Proved properties when $J_2/J_1 = 1/3$.
3. [White '92] Numerical evidence for Heisenberg model via DMRG (i.e. $J_2/J_1 = 0$).

Phases of Quantum Matter

Typical strategy for analyzing gapped ground state phases:

1. Prove gap for model H_Λ conjectured to be typical of a phase.
2. Prove the gap is stable under small perturbations: $H_\Lambda(s) = H_\Lambda + sV_\Lambda$ gapped for $|s| << 1$.
3. Refine phase classification by requiring other properties shared, e.g. entanglement structure: [Naaijkens, Ogata '22], symmetry protected phases: [Pollmann, Turner, Berg, Oshikawa '10, '12], [Ogata '18-'21], [Bourne, Schulz-Baldes '20], [Sopenko '21], [Tasaki '25].



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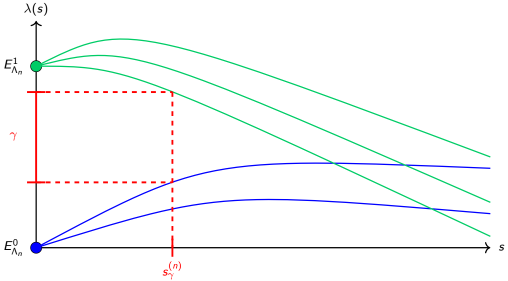
Uniform Gap Stability

We consider gap stability for a sequence of perturbed Hamiltonians

$$H_{\Lambda_n}(s) = H_{\Lambda_n} + sV_{\Lambda_n}, \quad s \in \mathbb{R}$$

$$V_{\Lambda_n} = \sum_{X \subseteq \Lambda} v_X, \qquad \|v_X\| \to 0 \text{ as } \mathrm{diam}(X) \to \infty$$

for which $\gamma_0 := \inf_{n \geq 1} \mathrm{gap}(H_{\Lambda_n}) > 0$.



The spectral gap is stable if for all $0 < \gamma < \gamma_0$,

$$s_\gamma := \inf_{n \geq 1} s^{(n)}_\gamma > 0.$$

Form Bound Implies Gap Estimate

Theorem (Persistence of Gaps): Let H be a densely defined self adjoint operator on a complex Hilbert space $\mathcal{H}$ with domain $\mathcal{D}$ and spectral gap

$$(0,\gamma)\cap \mathrm{spec}(H)=\emptyset.$$

Suppose V is a self-adjoint operator on $\mathcal{H}$ with $\mathcal{D}\subseteq \mathrm{dom}(V)$. If there are constants $\epsilon\geq 0$ and $\beta\in[0,1)$ such that

$$|\langle\psi|V\psi\rangle|\leq \epsilon\|\psi\|^2+\beta\langle\psi|H\psi\rangle \quad \forall\,\psi\in\mathcal{D}$$

then:

$$\mathrm{spec}(H+sV)\cap\big(s\epsilon,\,(1-s\beta)\gamma-s\epsilon\big)=\emptyset.$$

Note: In the case of uniform stability, have H_{Λ_n} and V_{Λ_n}, but want to find ϵ and β independent of Λ_n.

For quantum lattice models, there are various approaches for proving gap stability via form bounds for wide classes of perturbations, including

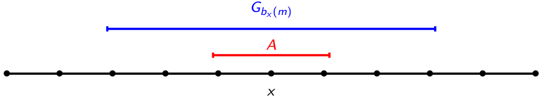
1. Cluster expansion: [Yarotsky, '06], [De Roeck, Salmhofer '19]
2. Quasi-adiabatic continuation: [Bravyi, Hastings, Michalakis, '10], [Michalakis, Zwolak '13], [Nachtergaele, Sims, Y. '22, '23]
3. Lie-Schwinger Diagonalization: [Fröhlich, Pizzo '20] [Del Vecchio, Fröhlich, Pizzo, Rossi '21]

BHM Stability Strategy

Ground State Indistinguishability (LTQO): There is a decay function Ω so that for any observable $A \in \mathcal{A}_{b_x(k)}$ there is a constant $C(A)$ so that for $m \geq k$

$$\|G_{b_x(m)} A G_{b_x(m)} - C(A) G_{b_x(m)}\| \leq \|A\| \Omega(m-k) \to 0, \text{ as } m \to \infty \quad (1)$$

where $G_{b_x(m)}$ is the orthogonal projection onto $\ker(H_{b_x(m)})$.



For a uniformly gapped Hamiltonian, we say the ground states are sufficiently indistinguishable (or satisfy LTQO) if (1) holds and $\sum_{k \geq 0} k^{3\nu/2} \Omega(k) < \infty$, where ν is the spatial dimension.

Theorem: [Bravyi, Hastings, Mickalakis '10], [Michalakis, Zwolak '13], [Nachtergaele, Sims, Y. '22] Let h be a uniformly gapped, frustration-free interaction whose ground states are sufficiently indistinguishable. Then, the spectral gap is stable for any perturbation satisfying

$$\|v_X\| \leq e^{-a\,\mathrm{diam}(X)^\theta} \quad \text{for some } a > 0,\ \theta \in (0,1].$$

BHM Stability Strategy

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$$\|G_{b_x(m)} A G_{b_x(m)} - \omega(A) G_{b_x(m)}\| \leq |b_x(k)| \|A\| \Omega(m-k) \to 0, \quad \text{as} \quad m \to \infty \qquad (1)$$

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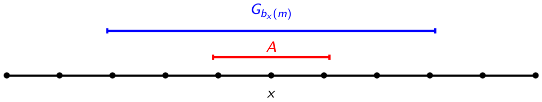

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$$\|v_X\| \leq e^{-a\,\mathrm{diam}(X)^\theta} \quad \text{for some} \quad a > 0,\ \theta \in (0, 1].$$

Results on LTQO

Results utilizing LTQO:

1. [Cubitt, Lucia, Michalakis, Perez-Garcia '15]: Stability of local quantum dissipative systems
2. [Cha, Naaijkens, Nachtergaele '22]: Stability of superselection sectors.
3. [Movassagh, Ouyang '24]: Realizing quantum codes constructed from classical codes as ground states of local Hamiltonians.
4. [Jones, Naaijkens, Penneys, Wallick '23], [Jones, Naaijkens, Penneys '25]: Boundary algebras describing holomorphic dual of bulk topological order.

Gapped models with LTQO:

1. [Nachtergaele, Sims, Y. '22]: Spin chains with matrix product ground states (including AKLT model, XXZ model, PVBS model,. . .)
2. [Bravyi, Hastings, Michalakis '10], [Cui et. al. '19], [Qiu, Wang '20]: Spin models with commuting interactions (e.g. Kitaev's Quantum Double models, String net models, etc.)
3. [Bachmann, Hamza, Nachtergaele, Y. '14]: d-dimensional, single species PVBS models
4. [Lucia, Y. '23], [Lucia, Moon, Y. '24]: AKLT models on sufficiently decorated (hybrid) multi-dimensional lattices and graphs

What about other multi-dimensional models with non-commuting interactions?

A Brief History: Gaps and Gap Stability of the AKLT Model

$$H^{\text{Heis}}_{[1,L]} = \sum_{x=1}^{L-1} \mathbf{S}_x \cdot \mathbf{S}_{x+1}, \qquad \mathcal{H}_{[1,L]} = \bigotimes_{x=1}^{L} \mathbb{C}^3$$

$$H^{\text{AKLT}}_{[1,L]} = \sum_{x=1}^{L-1} \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 = \sum_{x=1}^{L-1} P^{(2)}_{x,x+1}$$

[Affleck, Kennedy, Lieb, Tasaki (AKLT) '88]: Introduced a perturbation of the spin-1 Heisenberg AF chain and showed it belonged to the Haldane phase. Generalized model to other lattices and conjectured that the hexagonal lattice model is gapped.

[Kennedy, Lieb, Tasaki (KLT) '88] Proved the hexagonal model had a unique infinite-volume ground state with exponential decay of correlations.

[Pomata, Wei ' 20], [Lemm, Sandvik, Wang '20]: Provided strong evidence of the gap for the hexagonal model.

[Yarotsky '04], [Nachtergaele, Sims, Y. '21], [Del Vecchio, Fröhlich, Pizzo, Ranallo '23]: Gap stability of the AKLT spin chain using various methods (cluster expansions, LTQO, Lie-Schwinger diagonalization scheme).

[Abdul-Rahman, Lemm, Lucia, Nachtergaele, Y. '20], [Lucia, Y. '23] Spectral gap of decorated AKLT models, including hexagonal model with $d \geq 3$.

[Lucia, Moon, Y. '24] Gap stability for AKLT model on decorated hexagonal lattice with decoration parameters $d \geq 5$. Proved by using cluster expansions to verify LTQO.

Generalized AKLT Models

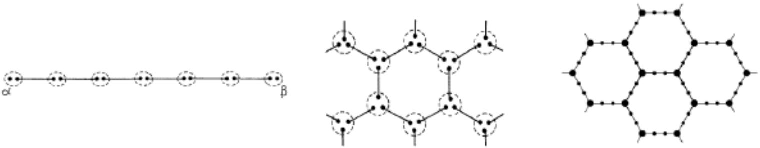


Figure: Valence Bond State description of the AKLT ground states and the 2-decorated hexagonal lattice. Photo Credit: [AKLT, '88], [ALLNY '20]

General AKLT Model [AKLT '88] For any site $x \in \Gamma$, a lattice, let $\mathcal{H}_x = \mathbb{C}^{2s_x+1}$ where $s_x = \deg(x)/2$. Then, for any finite subset $\Lambda \subseteq \Gamma$, define

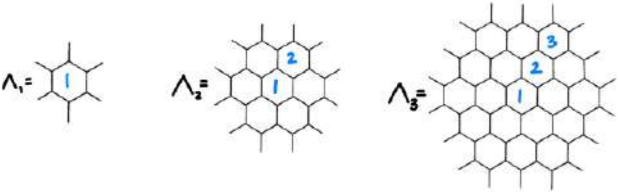
$$H_\Lambda = \sum_{\substack{\text{edge} \\ e=\{x,y\}\in\Lambda}} P_e^{(s_x+s_y)}, \quad \mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x, \quad \mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$$

where $P_e^{(s_v+s_w)}$ projects onto the subspace of total spin-$(s_v + s_w)$ of $\mathcal{H}_v \otimes \mathcal{H}_w$.

Clebsch-Gordon Series: Recall the $(2s_x + 1)$-dimensional irrep of $SU(2)$, denoted $V^{(s_x)}$, acts on $\mathcal{H}_x$ and

$$V^{(s_x)} \otimes V^{(s_y)} \cong V^{(s_x+s_y)} \oplus V^{(s_x+s_y-1)} \oplus \ldots \oplus V^{(|s_x-s_y|)}$$

Ground State Indistinguishability and LTQO for Hexagonal AKLT



Theorem (LTQO) [Jackson, Nachtergaele, Y. in prep] There exists $\epsilon, C > 0$ such that for any $C' > \epsilon^{-1}$, $k \geq 20$, $n \geq k + \max\{20, C' \ln(k)\}$ and $A \in \mathcal{A}_{\Lambda_{k-1}}$, one has

$$\|G_{\Lambda_n} A G_{\Lambda_n} - \omega(A) G_{\Lambda_n}\| \leq C|\partial\Lambda_{k-1}|e^{-2\epsilon(n-k)}$$

This result is a simple consequence of the following indistinguishability result:

Theorem (Indistinguishability) [Jackson, Nachtergaele, Y. in prep] There exists $\epsilon, C > 0$ such that for all $n > k + 20 \geq 40$

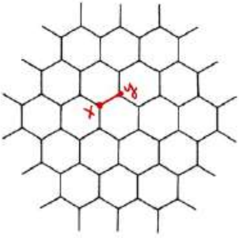
$$|\langle\psi_n|A\psi_n\rangle - \omega(A)| \leq C\|A\|G(n,k)e^{G(n,k)}$$

for all $A \in \mathcal{A}_{\Lambda_{k-1}}$ and normalized $\psi_n \in \mathcal{G}_{\Lambda_n}$ where

$$G(n,k) \propto ke^{-2\epsilon(n-k)}$$

Weyl Representation of the Ground States

As in [KLT '88], we use homogeneous polynomials and the Weyl representations to get a convenient description of the finite volume ground states:



$$\mathcal{H}_x \simeq \mathrm{span}\left\{u_x^k v_x^{3-k} : k = 0,1,2,3\right\} \subseteq L^2(S^2, d\Omega_x)$$

$$u_x = \cos(\theta_x/2)e^{i\phi_x/2}, \quad v_x = \sin(\theta_x/2)e^{-i\phi_x/2}$$

$$\Omega_x = (\sin\theta_x \cos\phi_x, \sin\theta_x \sin\phi_x, \cos\theta_x)$$

$$S_x^3 = \frac{1}{2}(v_x\partial_{v_x} - u_x\partial_{u_x}), \quad S_x^- = u_x\partial_{v_x}, \quad S_x^+ = v_x\partial_{u_x}$$

We note that: $|u_x v_y - v_x u_y|^2 = \frac{1}{2}(1 - \Omega_x \cdot \Omega_y)$

Theorem [KLT '88]: In the Weyl representation,

$$\mathcal{G}_\Lambda = \mathrm{span}\left\{\psi(f) \in \mathcal{H}_\Lambda : \psi(f) = f \cdot \prod_{(x,y)\in\Lambda} (u_x v_y - v_x u_y)\right\}$$

Moreover, for any $A \in \mathcal{A}_{\Lambda'}$, $\Lambda' \subseteq \Lambda$, there is a symbol $A(\mathbf{\Omega})$ depending only on the variables (u_x, v_x) associated to $x \in \Lambda'$ so that for any ground state $\psi(f) \in \mathcal{G}_\Lambda$

$$\langle\psi(f)|A\psi(f)\rangle = C_\Lambda \int d\Omega^\Lambda |f|^2 \prod_{(x,y)\in\Lambda} (1 - \Omega_x \cdot \Omega_y) A(\mathbf{\Omega})$$

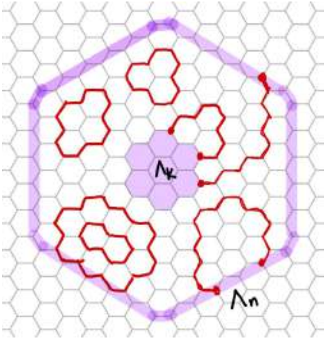
Polymer representation for ground state expectations

Fixing a normalized $\psi_n(f) \in \mathcal{G}_{\Lambda_n}$ and observable $A \in \mathcal{A}_{\Lambda_k}$, $k < n$

$$\langle \psi_n(f) | A \psi_n(f) \rangle = C_{\Lambda_n} \int d\Omega^{\Lambda_n} |f|^2 \prod_{(x,y)\in\Lambda_n} (1 - \Omega_x \cdot \Omega_y) A(\mathbf{\Omega})$$

A hard core polymer representation emerges from integrating $\langle \psi_n(f) | A \psi_n(f) \rangle$ over the sites $x \in \mathring{\Lambda}_n \setminus \Lambda_k$. Namely, using the relations

$$\prod_{(x,y)\in\Lambda_n} (1 - \Omega_x \cdot \Omega_y) = \sum_{\substack{\text{edge induced} \\ G \subseteq \Lambda_n}} \prod_{(x,y)\in G} (-\Omega_x \cdot \Omega_y)$$



$$\int d\Omega_x f(-\Omega_x) = \int d\Omega_x f(\Omega_x)$$

$$\int d\Omega_x (\Omega_y \cdot \Omega_x)(\Omega_x \cdot \Omega_z) = \frac{1}{3} \Omega_y \cdot \Omega_z$$

$$\int d\Omega_x (\Omega_x \cdot \Omega_x) = 1$$

one finds there is a weight function satisfying $|w(\gamma)| \leq 3^{-|\gamma|+1}$ such that

$$\langle \psi_n(f) | A \psi_n(f) \rangle \approx C_{n,k}(f, A) \sum_{\substack{\text{h.c. polymer sets} \\ \{\gamma_1, \ldots, \gamma_\ell\}}} \prod_{i=1}^{\ell} w(\gamma_i).$$

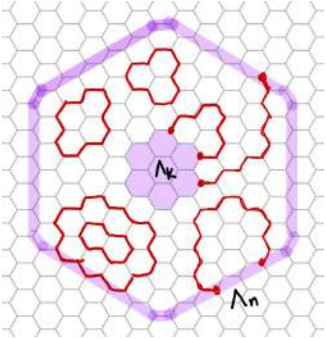
Indistinguishability via Cluster Expansions

$$\langle \psi_n(f) | A \psi_n(f) \rangle \approx C_{n,k}(f, A) \sum_{\substack{\text{h.c. polymer sets} \\ \{\gamma_1, \ldots, \gamma_\ell\}}} \prod_{i=1}^{\ell} w(\gamma_i), \qquad |w(\gamma)| \leq 3^{-|\gamma|+1}$$

Indistinguishability result obtained from identifying a specific bulk ground state ψ_n^{bulk} and bounding

$$|\langle \psi_n(f) | A \psi_n(f) \rangle - \omega(A)| \leq |\langle \psi_n(f) | A \psi_n(f) \rangle - \langle \psi_n^{\text{bulk}} | A \psi_n^{\text{bulk}} \rangle| + |\langle \psi_n^{\text{bulk}} | A \psi_n^{\text{bulk}} \rangle - \omega(A)|$$

Each difference can then be bound in terms of $\|A\|$ and a Renyi-divergence, dependent of quantities such as



$$\log \left(\sum_{\substack{\text{h.c. polymer sets} \\ \{\gamma_1, \ldots, \gamma_\ell\}}} \prod_{i=1}^{\ell} w(\gamma_i) \right).$$

The end result is a consequence of rewriting these logarithms as a sum over clusters of polymers.

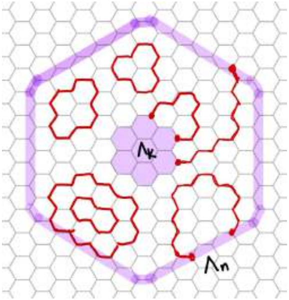
We follow the analogous approach from [KLT '88] to prove the cluster expansion convergence criterion from [Kotecky, Preis '86].

Comparing with the KLT Result

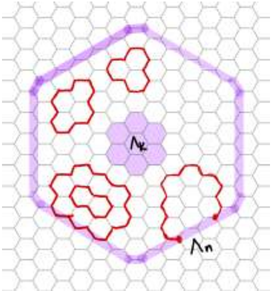
The key difficulty in immediately extending the results of [KLT '88] is that one requires bounds in terms of the operator norm $\|A\|$ for LTQO whereas the bounds from KLT would produce estimates dependent on $\|A(\Omega)\|_\infty$, and these norms are inequivalent:

$$\exists\, A_m \in \mathcal{A}_\Gamma^{\mathrm{loc}} \quad \text{s.t.} \qquad \|A_m\| = 1, \quad \|A_m(\Omega)\|_\infty = \left(\frac{5}{4}\right)^m$$

This stems from how the ground state expectations $\langle \psi_n(f) | A \psi_n(f) \rangle$ were decomposed in KLT, which lead to them to considering a related, but different, polymer set, $\mathcal{P}_{n,k}^{\mathrm{KLT}}$. Since $\mathcal{P}_{n,k}^{\mathrm{JNY}} \not\subseteq \mathcal{P}_{n,k}^{KLT}$, we could not immediately invoke their results.



$\mathcal{P}_{n,k}^{JNY}$



$\mathcal{P}_{n,k}^{KLT}$

Final Comments

- Over the last two decades, significant progress has been made in studying and classifying gapped ground state phases of matter, including topological phases.
- An important endeavor in this study is establishing rigorous gap and gap stability results for models conjectured to be typical of key phases.
- Given that the gap conjecture for the hexagonal AKLT model is true, our result implies the stability of the gap.
- Our result also extends to decorated models on the hexagonal lattice for all $d \geq 1$, improving the result of [Lucia, Moon, Y. '23].

Thank you for your attention!