Self-dual Hall modules

A Dissertation Presented

by

Matthew Bruce Young

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2013

Stony Brook University
The Graduate School

Matthew Bruce Young

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Michael Movshev - Dissertation Advisor
Associate Professor, Department of Mathematics

Oleg Viro - Chairperson of Defense
Professor, Department of Mathematics

Alexander Kirillov, Jr.
Associate Professor, Department of Mathematics

Anton Khoroshkin
Research Assistant Professor, Simons Center for Geometry and Physics

This dissertation is accepted by the Graduate School.

Charles Taber
Interim Dean of the Graduate School

Abstract of the Dissertation

Self-dual Hall Modules

by

Matthew Bruce Young

Doctor of Philosophy

in

Mathematics

Stony Brook University

2013

In the past twenty years Hall algebras have played an important role in many areas of mathematics and physics, including the theory of quantum groups and string theory. In its original setting the Hall algebra is constructed from a finitary exact category, the multiplication encoding the extension structure of the category. In this dissertation we introduce the Hall module of a finitary exact category with duality. The duality structure allows objects to carry sesquilinear forms and the action of the Hall algebra provides a categorical generalization of parabolic induction for the classical groups preserving a sesquilinear form. After developing the basic theory of Hall modules we study in detail Hall modules arising from the representation theory of a quiver with involution. In this setting we find a connection with twisted versions of quantum groups. We also study arithmetic properties of the Hall module structure constants.

To all my teachers.

Contents

Acknowledgements

First and foremost I would like to thank my advisor, Michael Movshev. Throughout my time as his student he has consistently been very generous with both his time and ideas. I consider myself fortunate to have had him as my advisor.

I would like to thank Professors Oleg Viro, Alexander Kirillov Jr. and Anton Khoroshkin for serving on my dissertation committee and providing valuable comments and feedback.

For all of their help along the way, I would also like to thank Professors Michael Anderson, Ivette Fuentes, Alexander Kirillov Jr., William Linch III, Robert Mann, Abdol-Reza Mansouri, Tony Phillips, Martin Roček, Albert Schwarz, Greg Smith and Jason Starr.

I would like to thank fellow students Ken Knox, Cheng Hao, Somnath Basu, Jeff Calder, Marcelo Disconzi, Carl Hammarsten, Savan Kharel, Jimmy Mathews, Paul McGrath, Alex Venditti and Joe Walsh for all the discussions we've had about mathematics and physics. I also thank Ryan Zade for helpful writing tips.

I have always enjoyed the full support of my family, Nancy, Bruce, Kelly, Chris and Kristin, and without them none of this would have been possible.

Finally, I would like to thank Sabrina for her companionship and support over the past six years.

My studies were funded by the Natural Sciences and Engineering Research Council of Canada, Renaissance Technologies and Stony Brook University. I gratefully acknowledge each for their support.

Chapter 1

Introduction

1.1 Hall algebras in mathematics and physics

The first incarnation of what is currently known as a Hall algebra was introduced in 1901 by Steinitz [64]. The construction was rediscovered and studied in greater detail by Hall [24] a half century later. For a fixed prime p, the vector space with basis labeled by isomorphism classes of finite abelian p-groups was given an associative product by counting certain short exact sequences of finite abelian p-groups. This algebra, now termed the classical Hall algebra, was called the algebra of partitions by Hall due its basis being in bijection with the set of all partitions. The classical Hall algebra is isomorphic to the algebra of symmetric functions and was used by Green [19] in his work describing characters of finite general linear groups. From this point of view, multiplication in the classical Hall algebra is given by parabolic induction of characters. A thorough account of the combinatorial and representation theoretic aspects the classical Hall algebra can be found in [47], [71].

Ringel extended the work of Steinitz and Hall to construct from an arbitrary abelian category $\mathcal{A}$ with finite Hom and Ext^1 sets an associative algebra $\mathcal{H}_{\mathcal{A}}$, again called the Hall algebra. The multiplication encodes the first order extension structure of $\mathcal{A}$. When $\mathcal{A}$ is the category of finite abelian p-groups the classical Hall algebra is recovered. Ringel showed [57] that when $\mathcal{A} = \mathrm{Rep}_{\mathbb{F}_q}(Q)$ is the category of representations of a quiver over a finite field the Hall algebra $\mathcal{H}_Q$ contains a subalgebra isomorphic to the negative part of the quantum derived Kac-Moody algebra $U_v(\mathfrak{g}_Q)$ attached to Q, specialized at $\sqrt{q}^{-1}$. Later, Green [20] endowed $\mathcal{H}_{\mathcal{A}}$ with a natural coproduct which for hereditary $\mathcal{A}$ makes $\mathcal{H}_{\mathcal{A}}$ into a (braided) bialgebra. In particular, the Hopf algebra structure of $U^{\leq 0}_{\sqrt{q}^{-1}}(\mathfrak{g}_Q)$ is naturally realized by $\mathcal{H}_Q$, thereby clarifying the relationship between Hall algebras and quantum groups. More recently, Bridgeland [7] realized the full quantum group using the Hall algebra of $\mathbb{Z}_2$-graded complexes of quiver representations.

A second important class of examples of Hall algebras occurs when $\mathcal{A}$ is the (hereditary) category of coherent sheaves on a smooth projective curve X defined over $\mathbb{F}_q$. In this case the Hall algebra is closely related to automorphic forms for the groups GL_n, $n \geq 0$, over the function field of X. When $X = \mathbb{P}^1$, using the (quadratic) functional equations for Eisenstein series Kapranov showed [33] (see also [2]) that the Hall algebra contains a subalgebra isomorphic to the non-standard negative part of the quantum affine algebra $U_{\sqrt{q}^{-1}}(\hat{\mathfrak{sl}}_2)$. The Hall algebra of an elliptic curve is related to Cherednik's double affine Hecke algebra [60], while Hall algebras of higher genus curves remain an active area of research.

The Hall algebra approach to quantum groups paved the way to more geometric realizations of interesting algebraic structures. In [42, 43, 44], Lusztig used induction and restriction operators on categories of perverse sheaves on the moduli stack of $\overline{\mathbb{F}_q}$-representations of Q to construct $U_v^-(\mathfrak{g}_Q)$ and its canonical basis. In a similar way, Ringel's served as motivation for Nakajima's construction [49] of modules of quantum affine algebras using the homology groups of Nakajima quiver varieties.

While Hall algebras have proven to be very interesting from a purely mathematical point of view, recently their role in physics has been emphasized. In quantum field theories and string theories with extended supersymmetry the Bogomol'nyi-Prasad-Sommerfield (BPS) states form a subspace of the Hilbert space of all states about which exact statements can often be made [50]. BPS states are quantum states that are invariant under only a fraction of the supersymmetries of the theory. As such BPS states form short, or of lower expected dimension, representations of the supersymmetry algebra. This implies that the subspace of BPS states is invariant under certain deformations of the theory. For example, in Type IIA string theory compactified on a three dimensional Calabi-Yau variety an appropriate index of BPS states (which arise from holomorphic cycles) is invariant under deformations of the complex structure of the Calabi-Yau. From a physical point of view the invariance of BPS indices yields valuable information about strong/weak coupling duality of the theory. BPS states are also of interest to mathematicians since these states often admit interesting geometric interpretations, examples including cohomology classes of moduli spaces of sheaves, holomorphic curves and special Lagrangian submanifolds.

One important property of the space of BPS states is that it is expected to form an algebra [25]. The product encodes the ways in which two BPS states can form a third BPS state as a bound state. From this description it is natural to regard BPS algebras as a kind of generalization of the Hall algebra. For example, the algebra of BPS states in Type IIA theory on the crepant resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_N$ is related to a complex version of the Hall algebra of the affine quiver associated to the singularity through the McKay correspondence [16]. See [17], [9] for related mathematical treatments. Recently, general mathematical models for BPS algebras have been suggested, most notably the motivic Hall algebras of Joyce [30] and Kontsevich-Soibelman [37] (see also [38], [6] for reviews) and the cohomological Hall algebra of Kontsevich-Soibelman [39]. These algebras are constructed from the category of coherent sheaves on a Calabi-Yau threefold and, often conjecturally, from a large class of three dimensional Calabi-Yau categories.

The Hall algebras of Joyce and Kontsevich-Soibelman are central to the theory of generalized Donaldson-Thomas (DT) invariants, which roughly speaking count semistable objects of the category under consideration. Physically, DT invariants encode the BPS spectrum of the theory. Under certain smoothness assumptions DT invariants are defined using the intersection theory of the moduli stack of semistable objects. However, without these assumptions the definition of DT invariants uses the Hall algebra in an essential way. The Hall algebra also provides an effective tool to study the basic properties of DT invariants, such as their integrality and behaviour under change of stability condition, known as wall-crossing. As a concrete example, using identities in the motivic Hall algebra Bridgeland proved the Donaldson-Thomas/Pandharipande-Thomas correspondence for smooth projective Calabi-Yau threefolds [5].

1.2 Dissertation work

We briefly summarize the contents of each chapter.

In Chapter 2 we review the basic theory of Hall algebras of abelian and exact categories that will be needed throughout the dissertation. We state Green's theorem asserting the compatibility of the algebra and coalgebra structures of the Hall algebra of a hereditary abelian category.

Following the previously described character theoretic interpretation of the classical Hall algebra, multiplication in $\mathcal{H}_{\mathcal{A}}$ can be viewed as a categorical version of parabolic induction for general linear groups. It is therefore natural to seek a modification of the Hall algebra construction that gives a categorical generalization of parabolic induction for the classical groups preserving a non-degenerate sesquilinear form. This is one of the primary goals of this dissertation and is the focus of Chapter 3. To formulate the problem we work in the setting of exact categories with duality. We therefore assume that along with a finitary exact category $\mathcal{A}$ we are given the additional data of a duality structure, that is, an exact contravariant functor $S : \mathcal{A} \to \mathcal{A}$ whose square is compatibly isomorphic to the identity. A pair (M, ψ) consisting of an object $M \in \mathcal{A}$ and a symmetric isomorphism $\psi : M \xrightarrow{\sim} S(M)$ is called a self-dual object. The morphism ψ can be thought of as a non-degenerate form on M. As is the case in linear orthogonal or symplectic geometry, given an isotropic subobject $U \subset M$ the reduced object $U^{\perp}/U$ inherits a canonical self-dual structure. Denote by $\mathcal{M}_{\mathcal{A}}$ the vector space with basis labelled by the set of isometry classes of self-dual objects. Our first result defines the Hall module of $(\mathcal{A}, S)$, which is the central object of this dissertation.

Theorem 1.2.1 (Theorem 3.2.3). *The formula*

$$[U] \star [M] = \sum_{N} G_{U,M}^{N} [N]$$

gives $\mathcal{M}_{\mathcal{A}}$ the structure of a left $\mathcal{H}_{\mathcal{A}}$-module, where $U \in \mathcal{A}$, M, N are self-dual objects of $(\mathcal{A}, S)$ and $G^N_{U,M}$ is the number of isotropic subobjects of N isomorphic to U and with reduction isometric to M.

There is also a version of Theorem 1.2.1 giving $\mathcal{M}_{\mathcal{A}}$ the structure of a left $\mathcal{H}_{\mathcal{A}}$-comodule. The Hall module carries a canonically defined non-degenerate symmetric bilinear form for which the module and comodule structure maps are adjoint. This form plays a central role in structure of the Hall module. The Hall module is naturally graded by the Grothendieck-Witt group of $\mathcal{A}$ and decomposes as a direct sum of modules labeled by the Witt group of $\mathcal{A}$; see Proposition 3.3.1. We use the Grothendieck-Witt grading to twist the action in Theorem 1.2.1 to obtain a module over the Ringel's twisted Hall algebra. This twist is essential for the connection to quantum groups discussed later in the dissertation. The definition of the twist uses a homologically defined function $\mathcal{E}$ that plays the role of the Euler form for exact categories with duality and is therefore of independent interest; see Theorem 4.3.3 and Proposition 4.3.5.

The main source of examples of Hall modules studied in this dissertation is constructed in Chapter 4. We show that when a quiver is given a contravariant involution σ there are a number of naturally defined duality structures on $\text{Rep}_k(Q)$. In particular cases the self-dual objects recover the orthogonal and symplectic quiver representations of Derksen-Weyman [11]. In all cases, the total space of the representation carries a hermitian, orthogonal or symplectic form and the structure maps satisfy certain symmetry conditions. By the work of Ringel and Green and the general theory of Hall modules, to each duality structure on $\text{Rep}_{\mathbb{F}_q}(Q)$ there is a corresponding representation $\mathcal{M}_Q$ of $U^-_{\sqrt{q}^{-1}}(\mathfrak{g}_Q)$. To better understand $\mathcal{M}_Q$ we partially describe the compatibility of its module and comodule structures.

Theorem 1.2.2 (Theorem 4.4.1)**.** *The operators of induction and restriction along the simple representations associated to the nodes of Q give $\mathcal{M}_Q$ the structure of a module over $B_\sigma(\mathfrak{g}_Q)$, the reduced σ-analogue of the quantum Kac-Moody algebra attached to Q, specialized at $\sqrt{q}^{-1}$.*

The proof of Theorem 1.2.2 is based on a series of weighted counts of diagrams of pairs of isotropic subrepresentations. Using Theorem 1.2.2, $\mathcal{M}_Q$ can be written entirely in terms of irreducible highest weight $B_\sigma(\mathfrak{g}_Q)$-modules $V_\sigma(\lambda)$. We call a homogeneous (with respect to the Grothendieck-Witt grading) element $\xi \in \mathcal{M}_Q$ cuspidal if it is annihilated by all simple restriction operators.

Theorem 1.2.3 (Theorem 4.4.7)**.** *The $B_\sigma(\mathfrak{g}_Q)$-submodule generated by a homogeneous cuspidal $\xi \in \mathcal{M}_Q$ is isomorphic to $V_\sigma(\lambda_\xi)$, the weight λ_ξ being written explicitly in terms of the Euler form and the function $\mathcal{E}$. Moreover, if $\mathcal{C}_Q$ is a homogeneous basis for the submodule of cuspidals, then there is a direct sum decomposition of $B_\sigma(\mathfrak{g}_Q)$-modules*

$$\mathcal{M}_Q \simeq \bigoplus_{\xi \in \mathcal{C}_Q} V_\sigma(\lambda_\xi).$$

In particular, the trivial self-dual representation is cuspidal and so generates an irreducible direct summand of $\mathcal{M}_Q$. Theorem 1.2.3 can therefore be viewed as a generalization of a result of Enomoto [14], which states that the submodule generated by the trivial orthogonal representation is isomorphic to $V_\sigma(0)$. Enomoto's approach uses entirely different methods than those employed here, working over an algebraically closed field in Lusztig's geometric setting of perverse sheaves. Our approach is more basic, relying on the combinatorics of representations defined over finite fields.

From a representation theoretic point of view, the simplest quivers are those having only finitely many indecomposable representations over any field. These are the so-called finite type quivers. As proved by Gabriel [18], the connected finite type quivers are orientations of ADE Dynkin diagrams and their indecomposables are in bijection with the positive roots of the root system of the corresponding simple Lie algebra. In Chapter 5 we show that the quivers with involution having only finitely many indecomposable self-dual representations are precisely the finite type quivers admitting an involution. This gives a slight generalization of previous results of Derksen-Weyman [11]. The two basic building blocks of finite type quivers with involution are type A quivers and disjoint unions of ADE quivers. In these cases, we explicitly classify the indecomposable self-dual $\mathbb{F}_q$-representations. Unlike ordinary quiver representations, the self-dual indecomposables over $\mathbb{F}_q$ differ from those over algebraically closed fields (as studied in [11]) because of forms over finite fields. We can phrase our classification to give a version of Gabriel's theorem for self-dual $\mathbb{F}_q$-representations.

Theorem 1.2.4 (Theorem 5.1.3)**.** *Fix a finite ground field of odd characteristic.*

1. The self-dual indecomposables of a disjoint union $Q \sqcup Q^{op}$, with Q finite type, are in bijection with the positive roots of the root system Δ_Q.

2. If the graph underlying Q is of Dynkin type A, then the self-dual indecomposable representations are in bijection with the positive roots of a (possibly non-reduced) root system whose short roots are decorated with data labeling inequivalent forms over finite fields.

Only when Q is finite type is $\mathcal{H}_Q$ finitely generated over the subalgebra $U^-_{\sqrt{q}^{-1}}(\mathfrak{g}_Q)$, in which case it is cyclic. Motivated by this we expect $\mathcal{M}_Q$ to have only finitely many cuspidals precisely when Q is finite type. When all self-dual representations are hyperbolic this expectation is true and there is a simple description of $\mathcal{M}_Q$.

Theorem 1.2.5 (Theorem 5.2.1)**.** *If Q is finite type and all self-dual representations are hyperbolic, then $\mathcal{M}_Q$ is generated by the trivial representation. Equivalently, the only cuspidal is the trivial self-dual representation.*

When there are non-hyperbolic self-dual representations, based on Theorem 1.2.4 and evidence from the case of equioriented type A quivers, we conjecture (Conjecture

5.2.5) that the cuspidals are given by explicit alternating sums of inequivalent forms of self-dual indecomposables. We outline a strategy to prove this conjecture using the character theory of $B_\sigma(\mathfrak{g})$-modules.

In Chapter 6 we study the dependence of the Hall module structure constants $G^N_{U,M}$ on the underlying finite field $\mathbb{F}_q$. For finite type quivers, the Hall polynomials [56] specialize to the Hall algebra structure constants at each prime power q, showing that the Hall numbers depend in a uniform way on the ground field. The polynomality of the structure constants of $\mathcal{M}_Q$ is more subtle. We introduce the notion of a self-dual Hall semi-polynomial, which consists of a pair of polynomials, one of which specializes to the structure constants of $\mathcal{M}_Q$ for all $q \equiv 1 \pmod 4$ and the other for all $q \equiv 3 \pmod 4$

Theorem 1.2.6 (Theorem 6.2.1)**.** *Let Q be an equioriented type A quiver or a finite type disjoint union quiver. If the indecomposable self-dual representations of Q have no forms, then self-dual Hall polynomials with integer coefficients exist. Otherwise, self-dual Hall semi-polynomials with half integer coefficients exist.*

We first reduce Theorem 1.2.6 to showing the existence of self-dual Hall semi-polynomials for the action of simple induction operators. We then prove this simpler statement by a careful case by case analysis. We conjecture that Theorem 1.2.6 holds without the equioriented assumption, and hence for all finite type quivers. As an application we prove in Theorem 6.3.1 that the Hall module is insensitive to the existence of self-dual Hall semi-polynomials instead of polynomials in the sense that the two generic Hall modules are isomorphic as modules for the generic Hall algebra.

1.3 Physical motivation: Orientifold and open BPS states

Compactifying string theory in ten dimensions on a Calabi-Yau threefold X produces a four dimensional field theory with unitary gauge groups and matter in bifundamental representations. More realistic physical models require also orthogonal and symplectic gauge groups together with matter in symmetric and skew-symmetric representations. One method to obtain these modifications is the orientifold projection. For a review see [51].

Quite generally, orientifolds are defined using a parity operator P, which to a first approximation is the composition of two operators. Denote by Ω the worldsheet parity operator, which reverses the orientation of strings. Given a holomorphic involution τ of X, the parity operator is schematically $P = \tau\Omega$. If an open string stretches from the D-brane B_1 to the D-brane B_2, then upon parity reversal the open string will stretch from $P(B_2)$ to $P(B_1)$. Since open strings are precisely the morphisms in the category $\mathcal{D}$ of topological D-branes, we see that the parity operator defines a contravariant functor on $\mathcal{D}$. More careful analysis shows that in fact $(\mathcal{D}, P)$ is a triangulated category with duality [27], [12]. The self-dual brane configurations are precisely the brane configurations invariant with respect to the orientifold and therefore survive the orientifold projection. The theory is unchanged away from the fixed points of the orientifold action, but near the fixed points orthogonal and symplectic gauge groups appear.

Several categorical models for orientifolds have been introduced in the physics literature. In the large volume limit of Type IIB theory compactified on a Calabi-Yau X, $\mathcal{D} = D^b(Coh_X)$ and $P = \mathbf{L}\tau^* \circ \mathbf{R}\mathsf{Hom}(-, \mathcal{O}_X)$ [12]. At the Landau-Ginzburg point $\mathcal{D}$ is the triangulated category of matrix factorizations of a superpotential W and P is the composition of the graded transpose and the pullback along τ [27]. At the orbifold point, where $\mathcal{D}$ is described through the representation theory of a quiver, the duality structure for orientifolds is of the type considered in this dissertation [13], [68].

Just as BPS states in oriented string theories are expected to form an algebra, so too are BPS states in unoriented string theories. Since the latter can be obtained using the orientifold projection, it suggests that a Hall-type algebra should be associated to certain categories with duality. We expect that the Hall modules constructed in this dissertation are a shadow of this structure. From this perspective, the module structure would arise from a morphism from the BPS algebra to the orientifold BPS algebra, most likely defined using the hyperbolic functor.

One application of this algebraic structure should be to orientifold, or real, DT invariants. In [40] candidates for orientifold DT invariants of local toric Calabi-Yau varieties were defined at a number of points in the Kähler moduli space directly using equivariant torus localization. However, no fundamental definition is given from which the above examples can be derived as special cases. In view of the relationship between Joyce and Kontsevich-Soibelman's Hall algebras and DT invariants, the orientifold BPS algebra or module may provide a fundamental approach to the study of orientifold DT invariants.

So far we have discussed only applications to closed BPS states. Another important class are open BPS states, which are those BPS states present only in theories with defects. One such example are maps from a bordered Riemann surface into a Calabi-Yau threefold with boundary in a Lagrangian submanifold, as appear in open Gromov-Witten theory. In this example the open BPS states are related to homological knot invariants [22], [21]. The algebraic structure of the space of open BPS states is that of a module over the (closed) BPS algebra [23]. This motivates the general study of representations of Hall or BPS algebras that are again of Hall-type. The Hall modules described in this dissertation are but a single example of this structure. The most accessible example in which to study the open BPS module may be for D0/D2/D6 branes in a toric Calabi-Yau, in which case the Hall module may admit a construction in terms of the framed representation theory of quivers [48]. The idea of constructing a module in this was has also appeared in [65].

Finally, we remark that the open and orientifold BPS invariants are likely related. This follows from the belief that there should be an orientifold Donaldson-Thomas/Gromov-Witten correspondence [40] and the fact that the orientifold Gromov-Witten invariants can sometimes be expressed in terms of open Gromov-Witten invariants where the Lagrangian submanifold is the fixed point locus of the involution. It would be very interesting to understand this relationship at the level of Hall modules.

Chapter 2

The Hall algebra of an exact category

In this chapter we review the basic properties of the Hall algebra of an exact category. Full details and proofs can be found in [59], [28].

We begin by recalling the definition of exact categories in the sense of Quillen [53]. We refer the reader to [8] for a thorough account of exact categories.

A kernel-cokernel pair (i, π) in an additive category $\mathcal{A}$ is a diagram

$$U \xrightarrow{i} X \xrightarrow{\pi} V$$

with i a kernel of π and π a cokernel of i. Fix a class $\mathcal{F}$ of kernel-cokernel pairs in $\mathcal{A}$. Elements of $\mathcal{F}$ will be called short exact sequences. We say that a morphism i is an admissible monic, and write $\overset{i}{\rightarrowtail}$, if it appears in a pair $(i, \pi) \in \mathcal{F}$. Admissible epics are similarly defined and denoted by $\overset{\pi}{\twoheadrightarrow}$.

Definition ([53])**.** *The pair $(\mathcal{A}, \mathcal{F})$ is an exact category if $\mathcal{F}$ is closed under isomorphisms and satisfies the following axioms:*

- *For any $U \in \mathcal{A}$, the identity $U \xrightarrow{1_U} U$ is both an admissible monic and an admissible epic.*
- *The class of admissible monics is closed under composition.*
- *The class of admissible epics is closed under composition.*
- *The push out of an admissible monic exists and gives an admissible monic:*

$$\begin{array}{ccc} U & \rightarrowtail & X \\ \downarrow & & \downarrow \\ Y & \rightarrowtail & V \end{array}$$

- *The pullback of an admissible epic exists and gives an admissible epic:*

$$\begin{array}{ccc} U & \dashrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & V \end{array}$$

We often refer to $\mathcal{A}$ as an exact category, instead of $(\mathcal{A}, \mathcal{F})$, if no confusion will result. Any abelian category is in a canonical way an exact category, taking $\mathcal{F}$ to be the class of all short exact sequences (arising from the abelian structure of $\mathcal{A}$); this is setting of most examples considered in this dissertation. Similarly, any extension closed full subcategory of an abelian category has a canonically induced exact structure.

We assume for the remainder of the dissertation that the category $\mathcal{A}$ is essentially small. In this case there is a set $Iso(\mathcal{A})$ of isomorphism classes of objects of $\mathcal{A}$. The Grothendieck group $K(\mathcal{A})$ is the abelian group generated by the symbols $|X|$, $X \in Iso(\mathcal{A})$, modulo the relation $|X| = |U| + |V|$ whenever $U \rightarrowtail X \twoheadrightarrow V$ is a short exact sequence.

The set of all short exact sequences in $\mathcal{A}$ of the form $U \rightarrowtail X \twoheadrightarrow V$ is written $\underline{\mathcal{F}}^X_{U,V}$. If for all $U, V \in \mathcal{A}$ the set $\operatorname{Hom}(U, V)$ is finite and $\underline{\mathcal{F}}^X_{U,V}$ is non-empty for only finitely many $X \in Iso(\mathcal{A})$, the category $\mathcal{A}$ is called finitary. In this case the Hall numbers are defined as the cardinalities $F^X_{U,V} = |\underline{F}^X_{U,V}|$, where

$$\underline{F}^X_{U,V} = \{\tilde{U} \subset X \mid \tilde{U} \simeq U,\ X/\tilde{U} \simeq V\}.$$

In $\underline{F}^X_{U,V}$ all subobjects $\tilde{U}$ are assumed to be admissible. Setting $a(U) = |Aut(U)|$, the Hall numbers are related to the cardinality of $\underline{\mathcal{F}}^X_{U,V}$ through the equation

$$F^X_{U,V} = \frac{|\underline{\mathcal{F}}^X_{U,V}|}{a(U)a(V)}.$$

All results below can therefore also be stated in terms of $|\underline{\mathcal{F}}^X_{U,V}|$, as in [28].

Let R be an integral domain containing $\mathbb{Q}$ and let $\mathcal{H}_{\mathcal{A}}$ be the free R-module with basis $Iso(\mathcal{A})$:

$$\mathcal{H}_{\mathcal{A}} = \bigoplus_{U \in Iso(\mathcal{A})} R[U].$$

Fix a bilinear form $c : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ and a unit $\nu \in R^\times$.

Theorem 2.0.1 ([56], [28]; [20])**.** *1. $\mathcal{H}_{\mathcal{A}}$ is a unital associative algebra when given the product*

$$[U][V] = \nu^{c(V,U)} \sum_{X \in Iso(\mathcal{A})} F^X_{U,V}[X]$$

and unit $[0]$, the class of the zero object of $\mathcal{A}$.

2. $\mathcal{H}_{\mathcal{A}}$ is a topological counital coassociative coalgebra when given the coproduct

$$\Delta[X] = \sum_{U,V \in Iso(\mathcal{A})} \nu^{c(V,U)} \frac{a(U)a(V)}{a(X)} F^X_{U,V} [U] \otimes [V]$$

and counit $\epsilon[X] = \delta_{X,0}$

We remark that the Hall algebra as defined above uses the opposite multiplication and comultiplication as compared to [56].

Both the algebra and coalgebra structures respect the natural $K(\mathcal{A})$-grading

$$\mathcal{H}_{\mathcal{A}} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{H}_{\mathcal{A}}(\alpha), \qquad \mathcal{H}_{\mathcal{A}}(\alpha) = \bigoplus_{|U|=\alpha} R[U].$$

In general, the coproduct Δ takes values in the completion $\mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}}$ consisting of all formal linear combinations $\sum_{U,V} c_{U,V} [U] \otimes [V]$. The meaning of the word topological in Theorem 2.0.1 is, first, that the coassociativity compositions

$$(\Delta \otimes 1) \circ \Delta, \quad (1 \otimes \Delta) \circ \Delta : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}}$$

are well-defined, and second, that they are in fact equal. See [59, §1.4] and the proof of Theorem 3.2.3 below for details. If every object of $\mathcal{A}$ has only finitely many subobjects then Δ is a coproduct in the ordinary sense; this will be the case for most examples in this paper.

The algebra and coalgebra structures of $\mathcal{H}_{\mathcal{A}}$ are compatible only under stronger assumptions on $\mathcal{A}$. Suppose that $\mathcal{A}$ is abelian, of finite homological dimension and, for simplicity, $\mathbb{F}_q$-linear.[1] The Euler form is defined as

$$\langle U, V \rangle = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{F}_q} \mathrm{Ext}^i(U, V)$$

and descends to a bilinear form on $K(\mathcal{A})$. The symmetrization of $\langle \cdot, \cdot \rangle$ is written $(\cdot, \cdot)$. The Ringel-Hall algebra of $\mathcal{A}$ is the Hall algebra $\mathcal{H}_{\mathcal{A}}$ with the choices $\nu = \sqrt{q}^{-1}$, $R = \mathbb{Q}[\nu, \nu^{-1}]$ and $c = -\langle \cdot, \cdot \rangle$, $\mathcal{H}_{\mathcal{A}}$.

We recall that an abelian category is said to be hereditary if its homological dimension is at most one.

Theorem 2.0.2 (Green's theorem [20])**.** *Let $\mathcal{H}_{\mathcal{A}}$ be the Ringel-Hall algebra of a hereditary abelian category $\mathcal{A}$. Equip $\mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}}$ with the algebra structure given on homogeneous elements by*

$$(x \otimes y) \cdot (z \otimes w) = \nu^{-(y,z)} xz \otimes yw, \quad x, y, z, w \in \mathcal{H}_{\mathcal{A}}.$$

[1] Without the linearity assumption multiplicative Euler forms should be used.

Then $\Delta : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{A} \hat{\otimes} \mathcal{H}_\mathcal{A}$ *is an algebra homomorphism.*

The (co)algebra structures of $\mathcal{H}_\mathcal{A}$ can be extended to $\mathcal{H}_\mathcal{A} \otimes_R R[K(\mathcal{A})]$ in such a way that Theorem 2.0.2 becomes the statement that the extended Hall algebra $\mathcal{H}_\mathcal{A} \otimes_R R[K(\mathcal{A})]$ is a topological bialgebra.

Finally, Green [20] defined an R-valued non-degenerate symmetric bilinear form on $\mathcal{H}_\mathcal{A}$ by $([U],[V])_\mathcal{H} = \dfrac{\delta_{U,V}}{a(U)}$. This form satisfies the important property

$$(x \otimes y, \Delta z)_{\mathcal{H}\otimes\mathcal{H}} = (xy, z)_\mathcal{H}, \quad x, y, z \in \mathcal{H}_\mathcal{A}$$

where $(x \otimes y, x' \otimes y')_{\mathcal{H}\otimes\mathcal{H}} = (x, x')_\mathcal{H}(y, y')_\mathcal{H}$.

Example. If $\mathcal{A}, \mathcal{A}'$ are finitary exact categories, then the product category $\mathcal{A} \times \mathcal{A}'$ is also finitary exact and its Hall algebra satisfies $\mathcal{H}_{\mathcal{A}\times\mathcal{A}'} \simeq \mathcal{H}_\mathcal{A} \otimes_R \mathcal{H}_{\mathcal{A}'}$. So, from the point of view of Hall algebras is suffices to restrict attention to connected categories. ◁

Example. The classical Hall algebra $\mathcal{H}_{cl}$ arises when $\mathcal{A}$ is the (hereditary) category of finite length modules over a discrete valuation ring with finite residue field $\mathbb{F}_q$. Alternatively, $\mathcal{H}_{cl}$ can be identified with the ring of unipotent characters of $GL_n(\mathbb{F}_q)$, $n \geq 0$, with multiplication given by parabolic induction [19], [71]. Moreover, $\mathcal{H}_{cl}$ is isomorphic to the bialgebra of symmetric functions. The structure constants of $\mathcal{H}_{cl}$ are the classical Hall polynomials and the Green bilinear form recovers the Hall inner product on symmetric functions. ◁

We close this chapter by discussing the two primary sources of finitary hereditary abelian categories. By Theorem 2.0.2 Hall algebras of such categories are twisted bialgebras. Since the first example is the focus of Chapter 4, we discuss it here only briefly and refer the reader to Chapter 4 for details and references.

Example. The abelian category of $\mathbb{F}_q$-representations of a quiver Q is finitary and hereditary. The corresponding twisted bialgebra is denoted by $\mathcal{H}_Q$. In particular, the Hall algebra of nilpotent representations of the quiver consisting of a single loop recovers $\mathcal{H}_{cl}$ from the previous example. The algebra of $\mathcal{H}_Q$ contains a subalgebra isomorphic to the quantum Kac-Moody algebra $U^-_{\sqrt{q}^{-1}}(\mathfrak{g}_Q)$. The entirety of $\mathcal{H}_Q$ is itself isomorphic to a specialized quantum Borcherds algebra, whose identification remains a difficult problem. ◁

Example. Let X be a smooth projective variety of dimension n defined over $\mathbb{F}_q$. The abelian category of coherent sheaves on X, written Coh_X, is finitary and of homological dimension n. Theorem 2.0.2 therefore only applies when X is a curve and we henceforth restrict to this case. The category of vector bundles $Vect_X$ is a full exact subcategory of Coh_X. The submodule $\mathcal{H}_{Vect_X} \subset \mathcal{H}_{Coh_X}$ spanned by elements labeled by $Iso(Vect_X)$ is a subalgebra (but not a subcoalgebra). Identifying the moduli stack of rank r vector bundles on X with the double coset

$$GL_r(\mathbb{O})\backslash GL_r(\mathbb{A})/GL_r(\mathbb{F}_q(X))$$

shows that elements of $\mathcal{H}_{Vect_X}$ may be interpreted as unramified automorphic forms for GL defined over $\mathbb{F}_q(X)$. Here $\mathbb{O}$, $\mathbb{A}$ and $\mathbb{F}_q(X)$ are the integer adèles, adèles and function field of X, respectively. Multiplication in $\mathcal{H}_{Vect_X}$ corresponds to the parabolic Eisenstein series map. Incorporating the action of torsion sheaves recovers Hecke operators on automorphic forms. This interpretation allows geometric properties of automorphic forms to be translated into algebraic properties of $\mathcal{H}_{Coh_X}$. For example, the quadratic identities satisfied by cusp eigenforms become quadratic relations in $\mathcal{H}_{Coh_X}$ [33]. In genus zero these relations fully determine $\mathcal{H}_{Coh_{\mathbb{P}^1}}$, showing that it is isomorphic to the semidirect product of the Hall algebra of torsion sheaves (which is a tensor product of classical Hall algebras) with a non-standard negative part of the quantum affine algebra $U_{\sqrt{q}^{-1}}(\hat{\mathfrak{sl}}_2)$ [33]. In higher genus these relations no longer determine $\mathcal{H}_{Coh_X}$ and the situation is more complicated. A number of natural subalgebras of $\mathcal{H}_{Coh_X}$, which play the role of $U^-_{\sqrt{q}^{-1}}(\hat{\mathfrak{sl}}_2)$ for higher genus curves, have been studied and shown to be related to Cherednik's double affine Hecke algebra [60] and geometric Langlands duality [61]. Recently, the entire Hall algebra of an arbitrary curve was described in terms of Rankin-Selberg L-functions of cusp eigenforms for $GL_r(\mathbb{F}_q(X))$, $r \geq 0$ [34]. When the dimension of X is greater than one very little is known about $\mathcal{H}_{Coh_X}$. $\triangleleft$

For detailed examples of Hall algebras of hereditary abelian categories the reader is referred to [59]. See also [3] where similarities between Hall algebras of a large class of exact categories and quantum nilpotent groups are emphasized.

Chapter 3

The Hall module of an exact category with duality

After reviewing some basic properties of exact categories with duality we show that from such a category there corresponds a module over the Hall algebra, called the Hall module. This is the central object of this dissertation. We then establish a number of structural properties of Hall modules that will be used in later chapters.

3.1 Exact categories with duality

A general reference for exact categories with duality is [1].

Definition. *An exact category with duality is a triple $(\mathcal{A}, S, \Theta)$ consisting of an exact category $\mathcal{A}$, an exact functor $S : \mathcal{A}^{op} \to \mathcal{A}$ and an isomorphism of functors $\Theta : 1_{\mathcal{A}} \to S^2$ satisfying $S(\Theta_U)\Theta_{S(U)} = 1_{S(U)}$ for all $U \in \mathcal{A}$.*

We sometimes refer to $\mathcal{A}$, instead of $(\mathcal{A}, S, \Theta)$, as an exact category with duality if S and Θ are clear from the context.

Definition. *A self-dual structure on $N \in \mathcal{A}$ is an isomorphism $N \xrightarrow{\psi} S(N)$ satisfying $S(\psi)\Theta_N = \psi$. The pair (N, ψ), or just N if no confusion will arise, is called a self-dual object.*

We will write $N \in \mathcal{A}_S$ to indicate that N is a self-dual object. An isometry of self-dual objects $(N, \psi) \to (N', \psi')$ is an isomorphism $N \xrightarrow{\Phi} N'$ satisfying $S(\Phi)\psi'\Phi = \psi$. In this case we write $N \simeq_S N'$. The set of isometry classes of self-dual objects $Iso(\mathcal{A}_S)$ is an abelian monoid under the operation of orthogonal direct sum, defined by $(N, \psi) \oplus (N', \psi') = (N \oplus N', \psi \oplus \psi')$.

Example. Let ι be an involutive field automorphism of k with fixed subfield k_0. Define a duality structure on $Vect_k$ as follows. For $V \in Vect_k$ define $S(V) = \overline{V}$

to be the vector space of additive maps $V \xrightarrow{f} k$ satisfying $f(cv) = \iota(c)f(v)$ for all $v \in V, c \in k$. On morphisms, S sends $V \xrightarrow{\phi} V'$ to $S(V') \xrightarrow{\phi^{\vee}} S(V)$. The map $\overline{ev} : V \to S^2(V)$ given by

$$\overline{\mathrm{ev}(v)}(f) = \iota(f(v)), \quad f \in \overline{V},$$

is a k-linear isomorphism. For each $s \in \{\pm 1\}$, the triple $(Vect_k, S, s \cdot \overline{ev})$ is a k_0-linear abelian category with duality. When ι is the identity self-dual structures are simply orthogonal ($s = 1$) or symplectic ($s = -1$) forms. If ι is non-trivial and $s = 1$ then a self-dual structure is a hermitian form. ◁

Example. Generalizing the previous example consider $Vect_X$, the exact category of vector bundles $Vect_X$ on a scheme X. Given a line bundle $\mathcal{L} \to X$ define a duality functor by $S(\mathcal{V}) = \mathcal{V}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{L}$ with $\mathcal{V}^{\vee} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X)$. Let $\Theta_{\mathcal{V}} : \mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\vee\vee}$ be the standard evaluation isomorphism. Then $(Vect_X, S, \pm\Theta)$ is an exact category with duality [36], the self-dual objects interpreted as $\mathcal{L}$-valued orthogonal or symplectic vector bundles on X. ◁

Definition. *An isotropic subobject of (N, ψ) is an admissible monic $U \overset{i}{\rightarrowtail} N$ such that $S(i)\psi i = 0$ and the canonically induced morphism $U \hookrightarrow U^{\perp}$ is admissible. Here $U^{\perp}$ is the kernel of $S(i)\psi$.*

We will use the notation $U \overset{\perp}{\subset} N$ to indicate that U is an isotropic subobject.

Definition. *1. A self-dual object is called hyperbolic it is isomorphic to the self-dual object*

$$H(U) = \left(U \oplus S(U), \psi = \begin{pmatrix} 0 & 1_{S(U)} \\ \Theta_U & 0 \end{pmatrix} \right),$$

for some $U \in \mathcal{A}$.

2. A self-dual object N is called metabolic if it contains a Lagrangian, i.e. an isotopic subobject $U \subset N$ such that $U^{\perp} = U$.

In particular, every hyperbolic self-dual object is metabolic.

The following result provides a categorical generalization of linear orthogonal and symplectic reduction along an isotropic subspace.

Proposition 3.1.1 ([52, Proposition 5.2])**.** *Let U be an isotropic subobject of (N, ψ). There exists a self-dual structure $\tilde{\psi}$ on $N//U := U^{\perp}/U$, unique up to isometry, making the following diagram commute:*

$$\begin{array}{ccc} U^{\perp} & \xrightarrow{\pi} & N//U \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle S(\pi)\tilde{\psi}} \\ N & \xrightarrow[S(k)\psi]{} & S(U^{\perp}) \end{array} .$$

Motivated by Proposition 3.1.1, we introduce a notion of short exact sequences for exact categories with duality. This will be used in counting arguments in Chapter 4

Definition. *Given $U \in \mathcal{A}$, $M, N \in \mathcal{A}_S$, let $\underline{\mathcal{G}}_{U,M}^{N}$ be the set of all equivalence classes of exact commutative diagrams of the form*

$$\begin{array}{ccccc} U & \overset{j}{\rightarrowtail} & E & \overset{\pi}{\twoheadrightarrow} & M \\ \| & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle S(\pi)\psi_M} \\ U & \overset{i}{\rightarrowtail} & N & \xrightarrow{S(k)\psi_N} & S(E) \\ & & \downarrow{\scriptstyle S(i)\psi_N} & & \downarrow{\scriptstyle S(j)} \\ & & S(U) & = & S(U) \end{array} \tag{3.1}$$

Two such diagrams are equivalent if there exists an isomorphism $E \xrightarrow{\sim} E'$ making all appropriate diagrams commute.

The equivalence relation in the previous definition is imposed so that the orthogonal $U^{\perp}$ can be thought of as a subobject of N. Elements of $\underline{\mathcal{G}}_{U,M}^{N}$ are called self-dual exact sequences and are denoted by $U \overset{i}{\rightarrowtail} N \overset{\pi}{\dashrightarrow\!\!\!\!\!\rightarrow} M$.

3.2 Hall modules

Let $\mathcal{A}$ be a finitary exact category with duality. For $U \in \mathcal{A}$ and $M, N \in \mathcal{A}_S$ define the finite set

$$\underline{G}_{U,M}^{N} = \{\tilde{U} \overset{\perp}{\subset} N \mid \tilde{U} \simeq U, \;\; N/\!/\tilde{U} \simeq_S M\}.$$

The cardinalities $G_{U,M}^{N} = |\underline{G}_{U,M}^{N}|$ are called self-dual Hall numbers. Denote by $Aut_S(M)$ the group of isometries of M and put $a_S(M) = |Aut_S(M)|$. Also put $\mathcal{G}_{U,M}^{N} = |\underline{\mathcal{G}}_{U,M}^{N}|$.

Lemma 3.2.1. *The identity $G_{U,M}^{N} = \dfrac{\mathcal{G}_{U,M}^{N}}{a(U)a_S(M)}$ holds.*

Proof. Denoting a diagram as in (3.1) by $(E; i, j, k, \pi)$, the group $Aut(U) \times Aut_S(M)$ acts on $\underline{\mathcal{G}}_{U,M}^{N}$ by

$$(g, h) \cdot (E; i, j, k, \pi) = (E; ig^{-1}, jg^{-1}, k, h\pi), \quad (g, h) \in Aut(U) \times Aut_S(M).$$

If (g,h) fixes the equivalence class of $(E;i,j,k,\pi)$, then $g=1_U$, $kr=k$ and $g\pi=\pi r$ for some $r\in Aut(E)$. The first equation implies $r=1_E$ while the second implies $g=1_M$, showing that the action is free.

The map $\underline{\mathcal{G}}_{U,M}^N \to \underline{G}_{U,M}^N$ assigning to $(E;i,j,k,\pi)$ the image of i is $Aut(U)\times Aut_S(M)$-invariant and descends to a bijection $\underline{\mathcal{G}}_{U,M}^N/Aut(U)\times Aut_S(M) \xrightarrow{\sim} \underline{G}_{U,M}^N$, proving the lemma. □

Self-dual Hall numbers obey the following finite support condition that can be viewed as a self-dual analogue of Ext^1-finiteness of $\mathcal{A}$.

Lemma 3.2.2. *For fixed $U\in\mathcal{A}$, $M\in\mathcal{A}_S$, the set $\underline{\mathcal{G}}_{U,M}^N$ is non-empty for only finitely many $N\in Iso(\mathcal{A}_S)$.*

Proof. If $\underline{\mathcal{G}}_{U,M}^N$ is non-empty, then N fits into the diagram (3.1). By Ext^1-finiteness only finitely many isomorphism types of E, and hence N, may appear. By Hom-finiteness, N admits at most finitely many self-dual structures and the statement follows. □

Let $\mathcal{M}_{\mathcal{A}}$ be the free R-module with basis indexed by the set of isometry classes of self-dual objects:

$$\mathcal{M}_{\mathcal{A}} = \bigoplus_{M\in Iso(\mathcal{A}_S)} R[M].$$

The next theorem defines the Hall module associated to $(\mathcal{A},S,\Theta)$. To begin we take $c=0$ in Theorem 2.0.1 to consider untwisted Hall algebras.

Theorem 3.2.3. *1. The formula*

$$[U]\star[M] = \sum_{N\in Iso(\mathcal{A}_S)} G_{U,M}^N [N]$$

gives $\mathcal{M}_{\mathcal{A}}$ the structure of a left $\mathcal{H}_{\mathcal{A}}$-module.

2. The formula

$$\rho[N] = \sum_{U\in Iso(\mathcal{A})}\sum_{M\in Iso(\mathcal{A}_S)} \frac{a(U)a_S(M)}{a_S(N)} G_{U,M}^N [U]\otimes[M].$$

gives $\mathcal{M}_{\mathcal{A}}$ the structure of a topological left $\mathcal{H}_{\mathcal{A}}$-comodule

Proof. The associativity of the action $\star$ is equivalent to the identity

$$\sum_{W\in Iso(\mathcal{A})} F_{U,V}^W G_{W,M}^N = \sum_{P\in Iso(\mathcal{A}_S)} G_{U,P}^N G_{V,M}^P, \quad U,V\in\mathcal{A}, \quad M,N\in\mathcal{A}_S. \tag{3.2}$$

Interpreting equation (3.2) in terms of isotropic filtrations, we find that it is equivalent to the statement that, for fixed $U \overset{\perp}{\subset} N$, the assignment $V \mapsto V/U$ gives a bijection

$$\{V \overset{\perp}{\subset} N \mid U \subset V\} \longleftrightarrow \{\tilde{V} \overset{\perp}{\subset} N//U\} \tag{3.3}$$

such that $(N//U)//(V/U) \simeq_S N//V$. To establish the bijection (3.3) we first observe that the assignment $V \mapsto V/U$ gives a bijection

$$\{V \subset U^{\perp} \mid U \subset V\} \leftrightarrow \{\tilde{V} \subset N//U\}.$$

Using the commutative diagram

$$\begin{array}{ccccccc}
N & \twoheadrightarrow & S(U^{\perp}) & \twoheadrightarrow & S(V^{\perp}) & \twoheadrightarrow & S(V) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
U^{\perp} & \twoheadrightarrow & N//U & \twoheadrightarrow & S(V^{\perp}/U) & \twoheadrightarrow & S(V/U) \\
\uparrow & & \uparrow & & \uparrow & & \\
V^{\perp} & \twoheadrightarrow & V^{\perp}/U & \twoheadrightarrow & N//V & & \\
\uparrow & & \uparrow & & & & \\
V & \twoheadrightarrow & V/U & & & &
\end{array}$$

we conclude that $V/U \subset N//U$ is isotropic if and only if $V \subset N$ is isotropic. The bijection (3.3) therefore holds.

Next, we claim that $(V/U)^{\perp} \subset N//U$ is naturally identified with $V^{\perp}/U$. Indeed, the previous diagram implies $V^{\perp}/U \rightarrowtail (V/U)^{\perp}$. This map is surjective since the canonical map $V^{\perp} \to (V/U)^{\perp}$ is surjective. Hence

$$(N//U)//(V/U) \simeq N//V.$$

That this isomorphism is an isometry can be seen by considering the three nested central squares in the diagram, presenting $N//U$, $N//V$ and $(N//V)//(V/U)$ as self-dual reductions as in Proposition 3.1.1. This proves the first statement of the theorem.

Turning to the second statement, we must first show that the compositions

$$(1 \otimes \rho) \circ \rho, \quad (\Delta \otimes 1) \circ \rho : \mathcal{M}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{M}_{\mathcal{A}}$$

are well-defined and equal. The completions again consist of all formal linear combinations. For any $\xi \in \mathcal{M}_{\mathcal{A}}$, the terms in $\rho(\xi)$ contributing to the coefficient of

$[U_1] \otimes [U_2] \otimes [M]$ in $(1 \otimes \rho) \circ \rho(\xi)$ are of the form $[U_1] \otimes [N]$ with $N//U_2 \simeq_S M$. By Lemma 3.2.2 the number of such terms is finite. Similarly, the number of terms contributing to $[U_1] \otimes [U_2] \otimes [M]$ in $(\Delta \otimes 1) \circ \rho(\xi)$ is finite. Hence, the compositions are indeed well defined. A direct calculation now shows that the equality $(1 \otimes \rho) \circ \rho = (\Delta \otimes 1) \circ \rho$ is equivalent to equation (3.2), which holds from the first part of the theorem. □

The following analogue of Green's bilinear form will play an important role in Chapters 4 and 5 when we discuss the structure of $\mathcal{M}_{\mathcal{A}}$ in specific examples.

Lemma 3.2.4. *The R-valued symmetric bilinear form on $\mathcal{M}_{\mathcal{A}}$ given by*

$$([M], [N])_{\mathcal{M}} = \frac{\delta_{M,N}}{a_S(M)}$$

is non-degenerate and satisfies

$$(x \otimes \xi, \rho(\zeta))_{\mathcal{H} \otimes \mathcal{M}} = (x \star \xi, \zeta)_{\mathcal{M}}, \quad x \in \mathcal{H}_{\mathcal{A}},\ \xi, \zeta \in \mathcal{M}_{\mathcal{A}}$$

where $(x \otimes \xi, x' \otimes \xi')_{\mathcal{H} \otimes \mathcal{M}} = (x, x')_{\mathcal{H}} (\xi, \xi')_{\mathcal{M}}$.

Proof. Non-degeneracy is clear. It suffices to check the second statement on a basis. The definition gives

$$([U] \otimes [M], \rho[N])_{\mathcal{H} \otimes \mathcal{M}} = \sum_{\substack{V \in Iso(\mathcal{A}) \\ P \in Iso(\mathcal{A}_S)}} \frac{a(V) a_S(P)}{a_S(N)} G^N_{V,P} ([U] \otimes [M], [V] \otimes [P])_{\mathcal{H} \otimes \mathcal{M}}$$

which simplifies to $\frac{G^N_{U,M}}{a_S(N)}$. On the other hand,

$$([U] \star [M], [N])_{\mathcal{M}} = \sum_{Q \in Iso(\mathcal{A}_S)} G^Q_{U,M} ([Q], [N])_{\mathcal{M}} = \frac{G^N_{U,M}}{a_S(N)}.$$

□

3.3 Gradings, twists and functorial properties

We continue to denote by $\mathcal{A}$ a finitary exact category with duality.

We recall the definitions of the Grothendieck-Witt and Witt groups of an exact category with duality. To ease notation, we omit the dependence on the duality from the notation.

Definition. *1. The Grothendieck-Witt group $GW(\mathcal{A})$ is the Grothendieck group of $Iso(\mathcal{A}_S)$ modulo the relation $|N| = |H(U)|$ if N contains U as a Lagrangian subobject.*

2. The Witt group $W(\mathcal{A})$ is the abelian monoid $Iso(\mathcal{A}_S)$ modulo the submonoid of metabolic objects.

There is a group homomorphism $K(\mathcal{A}) \xrightarrow{H} GW(\mathcal{A})$ defined at the level of objects by $U \mapsto H(U)$. Whenever $U \subset N$ is isotropic we have in $GW(\mathcal{A})$ the identity (see [52])

$$|N| = |N/\!/U| + |H(U)|. \tag{3.4}$$

The Hall module decomposes as an R-module

$$\mathcal{M}_{\mathcal{A}} = \bigoplus_{\gamma \in GW(\mathcal{A})} \mathcal{M}_{\mathcal{A}}(\gamma)$$

with $\mathcal{M}_{\mathcal{A}}(\gamma)$ the R-submodule spanned by self-dual objects of class γ. Using equation (3.4) we find for all $\alpha \in K(\mathcal{A})$ and $\gamma, \delta \in GW(\mathcal{A})$

$$\mathcal{H}_{\mathcal{A}}(\alpha) \star \mathcal{M}_{\mathcal{A}}(\gamma) \subset \mathcal{M}_{\mathcal{A}}(H(\alpha) + \gamma), \qquad \rho(\mathcal{M}_{\mathcal{A}}(\delta)) \subset \bigoplus_{H(\alpha)+\gamma=\delta} \mathcal{H}_{\mathcal{A}}(\alpha) \hat{\otimes} \mathcal{M}_{\mathcal{A}}(\gamma).$$

We summarize.

Proposition 3.3.1. *The group homomorphism H makes $\mathcal{M}_{\mathcal{A}}$ is a $GW(\mathcal{A})$-graded $\mathcal{H}_{\mathcal{A}}$-module. Moreover, $\mathcal{M}_{\mathcal{A}}$ decomposes as a direct sum of $\mathcal{H}_{\mathcal{A}}$-modules indexed by $W(\mathcal{A})$.*

Proof. By a slight abuse of notation, the first statement is simply a restatement of the previous two inclusions. The second statement follows from the first and from the exact sequence of abelian groups [52]

$$K(\mathcal{A}) \xrightarrow{H} GW(\mathcal{A}) \to W(\mathcal{A}) \to 0. \tag{3.5}$$

□

Remark. Proposition 3.3.1 can be sharpened as follows. The $K(\mathcal{A})$-grading of $\mathcal{H}_{\mathcal{A}}$ can be refined to a grading by $\Gamma_{\mathcal{A}}$, the Grothendieck monoid [3]. Define the Grothendieck-Witt monoid $\Gamma^S_{\mathcal{A}}$ as $Iso_S(\mathcal{A})$ modulo the relation $|M| = |M'|$ if both $\mathcal{G}^M_{U,N}$ and $\mathcal{G}^{M'}_{U,N}$ are non-empty for some $U \in \mathcal{A}$, $N \in \mathcal{A}_S$. The hyperbolic functor defines a monoid homomorphism $\Gamma_{\mathcal{A}} \to \Gamma^S_{\mathcal{A}}$ and induces a $\Gamma^S_{\mathcal{A}}$-grading of $\mathcal{M}_{\mathcal{A}}$. This refined may play an important role in determining a minimal generating set of $\mathcal{M}_{\mathcal{A}}$. We use a different approach to this problem below where the $GW(\mathcal{A})$-grading suffices.

We now discuss how to extend Theorem 3.2.3 to include non-trivial twists c of the Hall algebra. Let $\tilde{c} : GW(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be a function satisfying, for all

$\alpha, \beta \in K(\mathcal{A})$ and $\gamma \in GW(\mathcal{A})$,

$$c(\alpha, \beta) + \tilde{c}(\gamma, \alpha + \beta) = \tilde{c}(\gamma, \alpha) + \tilde{c}(\gamma + H(\alpha), \beta). \tag{3.6}$$

This relation guarantees that the twisted actions

$$[U] \star [M] = \nu^{\tilde{c}(M,U)} \sum_{N \in Iso(\mathcal{A}_S)} G^N_{U,M} [N]$$

and

$$\rho[N] = \sum_{U \in Iso(\mathcal{A})} \sum_{M \in Iso(\mathcal{A}_S)} \nu^{\tilde{c}(M,U)} \frac{a(U) a_S(M)}{a_S(N)} G^N_{U,M} [U] \otimes [M]$$

define on $\mathcal{M}_{\mathcal{A}}$ the structure of a (co)module for the c-twisted Hall algebra. We suppose that $\mathcal{A}$ is abelian and focus on the case $c = -\langle \cdot, \cdot \rangle$. For simplicity we take $\mathcal{A}$ and S to be k-linear. For each $p \in \{\pm 1\}$ define

$$\mathrm{Ext}^i(S(U), U)^{pS} = \left\{ \xi \in \mathrm{Ext}^i(S(U), U) \mid S(\xi) = p \Theta^{-1}_{U*} \xi \right\}.$$

Proposition 3.3.2. *Suppose that char$(k) \neq 2$ and that the abelian category $\mathcal{A}$ has finite homological dimension. Then the function $\mathcal{E} : Iso(\mathcal{A}) \to \mathbb{Z}$ given by*

$$\mathcal{E}(U) = \sum_{i \geq 0} (-1)^i \dim_k \mathit{Ext}^i(S(U), U)^{(-1)^{i+1} S}$$

descends to a function on $K(\mathcal{A})$.

Proof. Given a short exact sequence $0 \to U \to W \to V \to 0$ in $\mathcal{A}$ there is a corresponding dual exact sequence $0 \to S(V) \to S(W) \to S(U) \to 0$. Applying the bifunctor $\mathrm{Hom}(\cdot, \cdot)$ to the previous two sequences gives six long exact sequences fitting into the diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Hom(S(U),U) & \longrightarrow & Hom(S(U),W) & \longrightarrow & Hom(S(U),V) \\
 & & \downarrow & & -\downarrow & & \downarrow \\
0 & \longrightarrow & Hom(S(W),U) & \longrightarrow & Hom(S(W),W) & \longrightarrow & Hom(S(W),V) \\
 & & \downarrow & & -\downarrow & & \downarrow \\
0 & \longrightarrow & Hom(S(V),U) & \longrightarrow & Hom(S(V),W) & \longrightarrow & Hom(S(V),V) \\
 & & \downarrow & & -\downarrow & & \downarrow \\
 & \longrightarrow & Ext^1(S(U),U) & \longrightarrow & Ext^1(S(U),W) & \longrightarrow & Ext^1(S(U),V) \\
 & & \downarrow & & -\downarrow & & \downarrow \\
 & \longrightarrow & Ext^1(S(W),U) & \longrightarrow & Ext^1(S(W),W) & \longrightarrow & Ext^1(S(W),V) \\
 & & \downarrow & & -\downarrow & & \downarrow \\
 & \longrightarrow & Ext^1(S(V),U) & \longrightarrow & Ext^1(S(V),W) & \longrightarrow & Ext^1(S(V),V) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

The minus signs, indicating that negatives of the canonical maps are taken, are included to ensure that each square of the diagram anti-commutes. We can then form the total complex by summing across the diagonals of the previous diagram. The first three terms of this complex are given by

$$
0 \to \mathrm{Hom}(S(U),U) \to \begin{array}{c} \mathrm{Hom}(S(U),W) \\ \oplus \\ \mathrm{Hom}(S(W),U) \end{array} \to \begin{array}{c} \mathrm{Hom}(S(U),V) \\ \oplus \\ \mathrm{Hom}(S(W),W) \\ \oplus \\ \mathrm{Hom}(S(V),U) \end{array} \to \cdots
$$

There is an action of $\mathbb{Z}_2$ on the total complex, commuting with all differentials, defined as follows. The generator of $\mathbb{Z}_2$ acts by $(-1)^{i+1}S$ on $\mathrm{Ext}^i(S(U),U)$, $\mathrm{Ext}^i(S(V),V)$ and $\mathrm{Ext}^i(S(A),B) \oplus \mathrm{Ext}^i(S(B),A)$ with $A \neq B$ and acts by $(-1)^i S$ on $\mathrm{Ext}^i(S(W),W)$. Viewed as a virtual representation of $\mathbb{Z}_2$, the character of the total complex is zero. This implies the following relation between virtual dimensions of spaces of $\mathbb{Z}_2$-invariants

$$
0 = \mathcal{E}(U) - \langle S(U),W\rangle + \langle S(U),V\rangle + (\langle S(W),W\rangle - \mathcal{E}(W)) - \langle S(W),V\rangle + \mathcal{E}(V).
$$

The term $\langle S(W),W\rangle - \mathcal{E}(W)$ appears instead of $\mathcal{E}(W)$ because of the extra sign in the action of $\mathbb{Z}_2$ on $\mathrm{Ext}^i(S(W),W)$. Using additivity of the Euler form, the previous equality is rewritten as

$$
\mathcal{E}(W) = \mathcal{E}(U) + \mathcal{E}(V) + \langle S(U),V\rangle.
$$

The right hand side of this equation is easily seen to coincide with $\mathcal{E}(U \oplus V)$, proving the proposition. □

Taking $c = -\langle \cdot, \cdot \rangle$ and $\tilde{c}(M, U) = -\langle M, U \rangle - \mathcal{E}(U)$ we easily verify that equation (3.6) holds. With these choices, $\mathcal{M}_{\mathcal{A}}$ will be called the Ringel-Hall module.

We give an interpretation of $\mathcal{E}$ for hereditary abelian categories in terms of Lagrangians; see Proposition 4.3.5 below for an interpretation in terms of self-dual quiver representations.

Proposition 3.3.3. *1. The stabilizer of a Lagrangian* $U \overset{i}{\rightarrowtail} (N, \psi)$ *under the action of* $Aut_S(N)$ *is isomorphic to* $Hom(S(U), U)^{-S}$.

2. Suppose $\mathcal{A}$ *is linear over a field whose characteristic is not two. There is a canonical bijection between* $Ext^1(S(U), U)^S$ *and the set of all Lagrangian exact sequences*

$$0 \to U \xrightarrow{i} N \xrightarrow{S(i)\psi} S(U) \to 0 \tag{3.7}$$

modulo equivalence coming from isometries of the middle term.

Proof. Complete the Lagrangian $U \overset{i}{\hookrightarrow} N$ to the short exact sequence (3.7). An element $\phi \in Aut_S(N)$ stabilizing (3.7) can be written $\phi = 1_N + i\beta S(i)\psi$ for a unique $\beta \in \mathrm{Hom}(S(U), U)$. Then ϕ is an isometry if and only if $\beta \in \mathrm{Hom}(S(U), U)^{-S}$. The converse is similar.

As for the second statement, it is clear that any equivalence class of Lagrangian exact sequences (3.7) gives a class in $\mathrm{Ext}^1(S(U), U)^S$. In the other direction, let $\xi \in \mathrm{Ext}^1(S(U), U)^S$ be represented by the exact sequence

$$0 \to U \xrightarrow{i} N \xrightarrow{\pi} S(U) \to 0. \tag{3.8}$$

By assumption there exists $\psi_0 \in \mathrm{Hom}(N, S(N))$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{i} & N & \xrightarrow{\pi} & S(U) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \Theta_U} & & \Big\downarrow{\scriptstyle \psi_0} & & \Big\| & & \\ 0 & \longrightarrow & S^2(U) & \underset{S(\pi)}{\longrightarrow} & S(N) & \underset{S(i)}{\longrightarrow} & S(U) & \longrightarrow & 0 \end{array} \tag{3.9}$$

Define $\Psi_0 \in \mathrm{Hom}(N, S(N))$ as the difference $\Psi_0 = \psi_0 - S(\psi_0)\Theta_N$. Since $\Psi_0 i = 0$ and $S(i)\Psi_0 = 0$ there exists a unique $\beta \in \mathrm{Hom}(S(U), U)^{-S}$ for which $\Psi_0 = S(\pi)\Theta_U \beta \pi$. Then $\psi = \psi_0(1_N - \frac{1}{2} i\beta\pi)$ is a self-dual structure on N stabilizing (3.8).

To prove uniqueness of the isometry type of N suppose that ψ_1 and ψ_2 are two self-dual structures on N making the diagram (3.9) commute. Arguing as above we find $\psi_1 - \psi_2 = S(\pi)\Theta_U \beta \pi$ for a unique $\beta \in \mathrm{Hom}(S(U), U)^S$. Then $1_N + \frac{1}{2} i\beta\pi$ is the required isometry from ψ_1 to ψ_2. □

Remark. Propositions 3.3.2 and 3.3.3 and their proofs remain valid for extension closed full subcategories of abelian categories, such as $Vect_X \subset Coh_X$.

To any finitary exact category $\mathcal{A}$ there corresponds a finitary exact category with duality $H\mathcal{A}$, the hyperbolic category. The underlying exact category of $H\mathcal{A}$ is $\mathcal{A}\times\mathcal{A}^{op}$ and the duality structure is defined by $S(A,B) = (B,A)$ and $\Theta = \mathrm{Id}$. As the name suggests, all self-dual objects are hyperbolic. Let $\mathcal{H}_{\mathcal{A}}^{op-cop}$ be the (co)algebra obtained from $\mathcal{H}_{\mathcal{A}}$ by taking the opposite multiplication and comultiplication. There are canonical isomorphisms $\mathcal{H}_{\mathcal{A}^{op}} \simeq \mathcal{H}_{\mathcal{A}}^{op-cop}$ and $\mathcal{H}_{H\mathcal{A}} \simeq \mathcal{H}_{\mathcal{A}} \otimes_R \mathcal{H}_{\mathcal{A}}^{op-cop}$. The next proposition shows that in a certain sense the Hall algebra appears as a special case of the Hall module.

Proposition 3.3.4. *The assignment $[X] \mapsto [H(X)]$ extends to an isomorphism $\mathcal{H}_{\mathcal{A}} \xrightarrow{\sim} \mathcal{M}_{H\mathcal{A}}$ of $K(\mathcal{A})$-graded left $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}}^{op-cop}$-(co)modules preserving Green forms.*

Proof. The above assignment induces an R-module isomorphism $\mathcal{H}_{\mathcal{A}} \xrightarrow{\sim} \mathcal{M}_{H\mathcal{A}}$. A subobject of $H(X)$, $X \in \mathcal{A}$, is necessarily of the form $U_1 \oplus S(U_2)$ for some $U_1, U_2 \in \mathcal{A}$, and is isotropic if and only if $S(U_2) \subset S(X/U_1)$. Summing over the possible isomorphism types of X/U_1 we find

$$G^{H(X)}_{U_1\oplus S(U_2),H(Y)} = \sum_{W\in Iso(\mathcal{A})} F^X_{U_1,W} F^{S(W)}_{S(U_2),S(Y)}.$$

By definition we have $F^{S(W)}_{S(U_2),S(Y)} = F^W_{Y,U_2}$ so that

$$G^{H(X)}_{U_1\oplus S(U_2),H(Y)} = \sum_{W\in Iso(\mathcal{A})} F^X_{U_1,W} F^W_{Y,U_2}. \tag{3.10}$$

Equation (3.10) shows that $G^{H(X)}_{U_1\oplus S(U_2),H(Y)}$ is the coefficient of $[X]$ in $[U_1][Y][U_2]$, all multiplication being in $\mathcal{H}_{\mathcal{A}}$, proving that $\mathcal{M}_{H\mathcal{A}} \simeq \mathcal{H}_{\mathcal{A}}$ as $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}}^{op}$-modules. Using

$$Aut(U_1 \oplus S(U_2)) = Aut(U_1) \times Aut(U_2), \quad Aut_S(H(X)) = Aut(X),$$

a similar argument establishes the isomorphism of comodules.

That the isomorphism respects gradings follows from the fact that the restriction of the hyperbolic functor to $\mathcal{A} \subset H\mathcal{A}$ induces an isomorphism $K(\mathcal{A}) \xrightarrow{\sim} GW(H\mathcal{A})$. Finally, that the Green forms are preserved follows from the previous description of $Aut_S(H(X))$. □

Example. Let $\mathcal{H}_{Vect_X}$ and $\mathcal{M}_{Vect_X}\mathcal{M}_{Vect_X}$ be the Hall algebra and module associated to $Vect_X$, X a curve, with duality determined by a line bundle $\mathcal{L}$. Following the interpretation of $\mathcal{H}_{Vect_X}$ in terms of automorphic forms, $\mathcal{M}_{Vect_X}$ is identified with the space of $\mathcal{L}$-twisted automorphic forms for symplectic or orthogonal groups. For curves, the Witt group of $(Vect_X, \mathcal{L}, \Theta)$ is finite [29] and therefore provides a finite decomposition of $\mathcal{M}_{Vect_X}$. As the duality S does not extend to Coh_X there is not a direct Hall module interpretation of Hecke operators on $\mathcal{M}_{Vect_X}$. ◁

We end this section with some comments on the functorial properties of the assignment $(\mathcal{A}, S, \Theta) \mapsto \mathcal{M}_{\mathcal{A}}$.

Definition ([1])**.** *A form functor from* $(\mathcal{A}, S, \Theta)$ *to* $(\mathcal{B}, T, \Xi)$ *is a pair* (Φ, η) *consisting of an exact functor* $\Phi : \mathcal{A} \to \mathcal{B}$ *and a natural transformation* $\eta : \Phi \circ S \xrightarrow{\sim} T \circ \Phi$ *making the following diagram commute:*

$$\begin{array}{ccc} \Phi(M) & \xrightarrow{\Phi(\Theta_M)} & \Phi(S^2(M)) \\ {\scriptstyle \Xi_{\Phi(M)}}\downarrow & & \downarrow{\scriptstyle \eta_{S(M)}} \\ T^2(\Phi(M)) & \xrightarrow[T(\eta_M)]{} & T(\Phi(S(M))) \end{array}$$

It is straightforward to verify that a form functor (Φ, η) defines a set map $\mathcal{A}_S \to \mathcal{B}_T$ by $(M, \psi) \mapsto (\Phi(M), \eta_M \Phi(\psi))$.

If $\mathcal{A}$ and $\mathcal{B}$ are finitary, define R-module homomorphisms by

$$\begin{array}{ccc} \mathcal{H}_{\mathcal{A}} & \xrightarrow{\Phi_*} & \mathcal{H}_{\mathcal{B}} \\ [U] & \mapsto & [\Phi(U)] \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{A}} & \xrightarrow{(\Phi,\eta)_*} & \mathcal{M}_{\mathcal{B}} \\ [M] & \mapsto & [\Phi(M)] \end{array}$$

The following statement is motivated by a functoriality property for Hall algebras [59, §1.8].

Proposition 3.3.5. *Let* (Φ, η) *be a form functor such that* Φ *induces isomorphisms*

$$Ext^i(U, V) \xrightarrow{\sim} Ext^i(\Phi(U), \Phi(V))$$

for all $i \geq 0$*. Then* $(\Phi, \eta)_*$ *is a module embedding over* Φ_**. If, moreover,* $\Phi(\mathcal{A})$ *is essentially stable under taking subobjects, then* $(\Phi, \eta)_*$ *is a comodule embedding over* Φ_**.*

Proof. The conditions on Φ ensure that Φ_* is a (co-)algebra embedding [59]. To show that no additional assumptions are needed to ensure the module theoretic statements we use the following facts. First, a subobject $U \subset N$ is isotropic if and only if $\Phi(U) \subset \Phi(N)$ is isotropic. Second, the induced maps

$$\Phi : \text{Ext}^i(S(U), U)^{\pm S} \to \text{Ext}^i(T(\Phi(U)), \Phi(U))^{\pm T}$$

are isomorphisms, implying that Φ preserves $\mathcal{E}$. Finally, if $M, M' \in \mathcal{A}_S$ are self-dual objects such that $\Phi(M) \simeq_S \Phi(M')$, then $M \simeq_T M'$. To see this fix an isometry

$\Phi(M) \xrightarrow{\sim} \Phi(M')$, which is necessarily of the form $\Phi(g)$ for some $g : M \to M'$. Then

$$T(\Phi(g))\eta_{M'}\Phi(\psi_{M'})\Phi(g) = \eta_M\Phi(\psi_M)$$

which can be rewritten as

$$\eta_M\Phi(S(g)\psi_{M'}g) = \eta_M\Phi(\psi_m),$$

showing that g is an isometry. In particular, the set map $(\Phi, \eta)_*$ is injective.

With these preliminaries, a direct calculation shows that Φ_* is a module homomorphism if and only if $G^N_{U,M} = G^{\Phi(N)}_{\Phi(U),\Phi(M)}$. This equality follows from the first fact above and the isometry $\Phi(N//U) \simeq_T \Phi(N)//\Phi(U)$.

Finally, Φ_* is a comodule homomorphism if and only if

$$G^{\Phi(N)}_{A,R} = \sum_{\Phi(U)\simeq A, \Phi(M)\simeq_T R} G^N_{U,M}.$$

By assumption there exists a (necessarily unique) U such that $\Phi(U) \simeq A$. In this case $R \simeq_T \Phi(M)$ and the equality follows from the first part of the proof. □

The assumptions on Φ in Proposition 3.3.5 are quite strong but suffice for our applications. An example where Φ_* fails to be a coalgebra morphism is the inclusion $\Phi : Vect_X \hookrightarrow Coh_X$. Explicitly, the object $\mathcal{O}_{\mathbb{P}^1} \in Coh_{\mathbb{P}^1}$ has many non-trivial subobjects, namely any $\mathcal{O}_{\mathbb{P}^1}(m)$ with $m < 0$, while $\mathcal{O}_{\mathbb{P}^1} \in Vect_{\mathbb{P}^1}$ is simple.

Chapter 4

Hall modules from quivers with involution

In this chapter we study Hall modules arising from the representation theory of a quiver with involution. Before introducing these objects we recall some preliminary results about quantum groups and their relationship with Hall algebras of quivers.

4.1 Quantum Kac-Moody algebras

Let A be a symmetric generalized Cartan matrix. That is, $A = (a_{ij})_{i,j=1}^{n}$ satisfies $a_{ij} \in \mathbb{Z}$ and

$$a_{ii} = 2, \quad a_{ij} = a_{ji} \quad \text{and} \quad a_{ij} \leq 0 \text{ if } i \neq j.$$

Let $(\pi, \pi^{\vee}, \mathfrak{h})$ be a realization of A. Then $\mathfrak{h}$ is a complex vector space of dimension $n + \operatorname{corank} A$ and π, $\pi^{\vee}$ are linearly independent sets

$$\pi^{\vee} = \{h_1, \dots, h_n\} \subset \mathfrak{h}, \quad \pi = \{\epsilon_1, \dots, \epsilon_n\} \subset \mathfrak{h}^{\vee}$$

satisfying $\epsilon_i(h_j) = a_{ij}$. Denote by $\mathfrak{g} = \mathfrak{g}(A)$ the symmetric Kac-Moody algebra attached to A [31]. The Cartan form on $\mathfrak{h}^{\vee}$ is written $(\cdot, \cdot)$. We denote by $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ the corresponding derived algebra, whose Cartan subalgebra is $\mathfrak{h}' = \bigoplus_{i=1}^{n} \mathbb{C} h_i \subset \mathfrak{h}$ and whose root lattice is $\Phi = \bigoplus_{i=1}^{n} \mathbb{Z} \epsilon_i^{\vee}$.

Let v be an indeterminate and let $\mathbb{Q}(v)$ be the field of rational functions in v. Define the quantum combinatorial symbols by

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]_v! = \prod_{i=1}^{n} [i]_v, \quad \begin{bmatrix} n \\ k \end{bmatrix}_v = \frac{[n]_v!}{[k]_v! [n-k]_v!}.$$

Note that each of these expressions is an element of $\mathbb{Z}[v]$.

Definition. *The quantum Kac-Moody algebra* $U_v(\mathfrak{g})$ *is the* $\mathbb{Q}(v)$*-algebra generated by the symbols* $E_i, F_i, i = 1, \dots, n$ *and* v^h, $h \in \mathfrak{h}$, *subject to the relations*

1. $v^h v^{h'} = v^{h+h'}$
2. $v^h E_i v^{-h} = v^{\epsilon_i(h)} E_i, \qquad v^h F_i v^{-h} = v^{-\epsilon_i(h)} F_i$
3. $[E_i, F_j] = \delta_{ij} \dfrac{v^{h_i} - v^{-h_i}}{v - v^{-1}}$
4. $\displaystyle\sum_{l=0}^{1-(\epsilon_i,\epsilon_j)} (-1)^l \begin{bmatrix} 1-(\epsilon_i,\epsilon_j) \\ l \end{bmatrix}_v E_i^l E_j E_i^{1-(\epsilon_i,\epsilon_j)-l} = 0, \quad i \neq j.$
5. $\displaystyle\sum_{l=0}^{1-(\epsilon_i,\epsilon_j)} (-1)^l \begin{bmatrix} 1-(\epsilon_i,\epsilon_j) \\ l \end{bmatrix}_v F_i^l F_j F_i^{1-(\epsilon_i,\epsilon_j)-l} = 0, \quad i \neq j$

The final two relations in the previous definition are called the quantum Serre relations. The quantum group of the derived algebra $\mathfrak{g}'$, denoted $U_v(\mathfrak{g}')$, is the $\mathbb{Q}(v)$-subalgebra of $U_v(\mathfrak{g})$ generated by $E_i, F_i, T_i^{\pm 1} = v^{\pm h_i}$. The algebra $U_v(\mathfrak{g})$ is a Hopf algebra with coproduct defined by

$$\Delta(v^h) = v^h \otimes v^h, \quad \Delta(E_i) = E_i \otimes 1 + v^{h_i} \otimes E_i, \;\; \Delta(F_i) = 1 \otimes F_i + F_i \otimes v^{-h_i}$$

and antipode given by

$$S(v^h) = v^{-h}, \;\; S(E_i) = v^{-h_i} E_i, \;\; S(F_i) = -v^{h_i} F_i.$$

Then $U_v(\mathfrak{g}')$ is a Hopf subalgebra. Denote by $U_v^-(\mathfrak{g})$ (resp. $U_v^{\leq 0}(\mathfrak{g})$) the subalgebras of $U_v(\mathfrak{g})$ generated by F_i (resp. F_i, v^h). Note that $U_v^{\leq 0}(\mathfrak{b})$ is a Hopf subalgebra, but $U_v^-(\mathfrak{g})$ is not. Similar statements hold for the derived quantum group.

If $\nu \in \mathbb{C}^\times$ is not a root of unity, the specialized quantum group $U_\nu(\mathfrak{g})$ is the $\mathbb{Q}[\nu, \nu^{-1}]$-algebra with the same generators and relations as $U_v(\mathfrak{g})$ but with v replaced with ν. The algebras $U_\nu(\mathfrak{g}')$, $U_\nu^-(\mathfrak{g})$ are defined analogously.

The reader is referred to [46] for a detailed discussion of quantum groups.

4.2 The Hall algebra of a quiver

Fix a ground field k. A quiver Q consists of a finite set of nodes Q_0, a finite set of arrows Q_1 along with head and tail maps $h, t : Q_1 \to Q_0$. An arrow α from the node i to the node j, that is, $t(\alpha) = i$ and $h(\alpha) = j$, is written $i \xrightarrow{\alpha} j$. A cyclic in Q is an oriented path which starts and ends at the same node. A loop is cycle of length one. A representation of Q is a finite dimensional Q_0-graded vector space $V = \bigoplus_{i \in Q_0} V_i$ together with a linear map $V_i \xrightarrow{v_\alpha} V_j$ for each $i \xrightarrow{\alpha} j$. The dimension vector of V is

$\mathbf{dim}\, V = (\dim V_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$. The abelian group $\mathbb{Z}^{Q_0}$ has a natural basis $\{\epsilon_i\}_{i \in Q_0}$ consisting of unit vectors supported at $i \in Q_0$. To each node $i \in Q_0$ there is an associated simple representation S_i with dimension vector ϵ_i and all structure maps zero. The category of finite dimensional k-representations of Q is an abelian category and will be denoted by $\mathrm{Rep}_k(Q)$. It is hereditary and, if k is a finite field, finitary. For further discussion of the representation theory of quivers the reader is referred to [10].

Suppose that Q has no loops. Denote by $\mathcal{H}_Q$ the Ringel-Hall algebra of $\mathrm{Rep}_{\mathbb{F}_q}(Q)$. The composition subalgebra $\mathbf{C}_Q$ is defined to be the subalgebra of $\mathcal{H}_Q$ generated by the simple representations $[S_i]$. Since Q has no loops, the matrix of the symmetrized Euler form of Q is a generalized Cartan matrix. Write $\mathfrak{g}_Q$ for the corresponding symmetric Kac-Moody algebra. The following fundamental theorem provides a connection between the representation theory of quivers and the theory of quantum groups.

Theorem 4.2.1 ([57], [20])**.** *Let Q be a quiver without loops. The composition subalgebra $\mathbf{C}_Q$ is isomorphic to $U_\nu^-(\mathfrak{g}_Q)$, the negative part of the quantum Kac-Moody algebra associated to $\mathfrak{g}_Q$ specialized at $\sqrt{q}^{-1}$.*

In the isomorphism of Theorem 4.2.1 the Chevalley generator F_i is mapped to $[S_i]$. That the simple representations satisfy the quantum Serre relations is then verified by direct calculation. The proof of injectivity of $U_\nu^-(\mathfrak{g}_Q) \to \mathcal{H}_Q$ relies on the verification that the Green bilinear form pulls back to Lusztig's non-degenerate bilinear form.

Theorem 4.2.1 can be used to realize the Hopf algebra $U_\nu^{\leq 0}(\mathfrak{g}'_Q)$ inside the extended Hall algebra of Q [69]. Theorem 4.2.1 has been generalized to quivers with loops, the (suitably modified) composition subalgebra being the negative part of a quantum generalized Kac-Moody (or Borcherds) algebra [32]. In another direction, $\mathcal{H}_Q$ itself is the negative part of a quantum generalized Kac-Moody algebra, which in general depends on the finite field $\mathbb{F}_q$ [63].

4.3 Quivers with involution

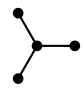
We study natural duality structures on $\mathrm{Rep}_k(Q)$ constructed from a contravariant involution of Q. We assume throughout that the characteristic of the ground field k is not two.

Definition. *An involution of Q is a pair of involutions $Q_i \xrightarrow{\sigma} Q_i$, $i = 0, 1$, such that for all $\alpha \in Q_1$, $h(\sigma(\alpha)) = \sigma(t(\alpha))$ and if $\sigma(t(\alpha)) = h(\alpha)$ then $\sigma(\alpha) = \alpha$.*

Diagrammatically, the first condition reads

$$\underset{i}{\bullet} \xrightarrow{\alpha} \underset{j}{\bullet} \quad \Longleftrightarrow \quad \underset{\sigma(j)}{\bullet} \xrightarrow{\sigma(\alpha)} \underset{\sigma(i)}{\bullet}$$

While the second assumption on σ can be dropped, nothing new is gained by doing so. Not every quiver has an involution. For example, no orientation of the graph



admits an involution.

Let (Q, σ) be a quiver with involution. Fix an involutive field automorphism $k \xrightarrow{\iota} k$ with fixed field $k_0 \subset k$, a sign $s \in \{\pm 1\}$ and a σ-invariant function $\tau : Q_1 \to \{\pm 1\}$. Define an exact k_0-linear contravariant functor $S : \mathrm{Rep}_k(Q) \to \mathrm{Rep}_k(Q)$ as follows. For a representation (M, m) put $S(M)_i = \overline{M}_{\sigma(i)}$ and define the structure map $S(m)_\alpha = \tau_\alpha m^\vee_{\sigma(\alpha)}$. Similarly, given a morphism $M \xrightarrow{\phi} M'$ the components of

$$S(M') \xrightarrow{S(\phi)} S(M)$$

are $S(\phi)_i = \phi^\vee_{\sigma(i)}$. Exactness of the functor S follows from its exactness at the level of Q_0-graded vector spaces. Define $\Theta : 1_{\mathrm{Rep}(Q)} \to S^2$ by $\Theta_M = s \cdot \overline{\mathrm{ev}_M}$. The triple $(\mathrm{Rep}_k(Q), S, \Theta)$ is then a k_0-linear abelian category with duality.

A self-dual representation (M, ψ) has the following geometric interpretation. Define a non-degenerate form on the total space of M by $\langle v, w \rangle = \psi(v)(w)$. The form is linear in the first variable, ι-linear in the second variable and satisfies $\langle v, w \rangle = s\iota(\langle w, v \rangle)$. Moreover, the subspaces M_i and M_j are orthogonal unless $i = \sigma(j)$. The isomorphism $M \xrightarrow{\psi} S(M)$ requires that the structure maps of M satisfy

$$\langle m_\alpha v, w \rangle - \tau_\alpha \langle v, m_{\sigma(\alpha)} w \rangle = 0, \qquad v \in M_{t(\alpha)}, \ w \in M_{\sigma(h(\alpha))}.$$

If ι is the identity then $\langle \cdot, \cdot \rangle$ is s-symmetric. In this case and with the choice $\tau = -1$, the self-dual objects are precisely the orthogonal and symplectic quiver representations originally introduced by Derksen-Weyman [11] and we will refer to them as such. If $k_0 \subset k$ is a separable quadratic extension and ι is the unique non-trivial k_0-linear automorphism of k, then $\langle \cdot, \cdot \rangle$ is a hermitian form. For example, if the prime power q is a perfect square, the extension $\mathbb{F}_{\sqrt{q}} \subset \mathbb{F}_q$ is quadratic and the involution is given by $\iota(x) = x^{\sqrt{q}}$. In this case the self-dual representations with the choice $(s, \tau) = (1, -1)$ will be referred to as unitary representations. It will be convenient to also refer to this as the $s = 0$ case.

Self-dual representations can also be described in terms of collections of matrices as follows. Define a partition $Q_0 = Q_0^\sigma \sqcup Q_0^+ \sqcup Q_0^-$ where Q_0^σ consists of nodes fixed by σ and Q_0^+ consists of a single node from each two-point σ-orbit. There is a similar decomposition of Q_1. We impose the compatibility condition $\mathrm{supp}(Q_1^+) \subset Q_0^+ \sqcup Q_0^\sigma$. For concreteness, fix k to be a finite field. The non-degenerate form on the total space of a d-dimensional self-dual representation is nearly unique up to isometry. The only ambiguity is in the orthogonal case, where the Witt type of each node $i \in Q_0^\sigma$ must be given. Fix a model for this form. The affine scheme of orthogonal, unitary or symplectic representations of dimension vector d is then

$$R_d^{\mathfrak{g}} = \bigoplus_{i \xrightarrow{\alpha} j \in Q_1^+} \operatorname{Hom}_k(k^{d_i}, k^{d_j}) \oplus \bigoplus_{i \xrightarrow{\alpha} \sigma(i) \in Q_1^{\sigma}} \begin{cases} \Lambda^2 k^{d_i}, & \text{if } s = 1 \\ \mathfrak{sherm}(d_i), & \text{if } s = 0 \\ \operatorname{Sym}^2 k^{d_i} & \text{if } s = -1 \end{cases}$$

Here $\mathfrak{sherm}(d_i)$ is the k_0-vector space of $d_i \times d_i$-skew-hermitian matrices. The reductive algebraic group

$$G_d^{\mathfrak{g}} = \prod_{i \in Q_0^{\sigma}} G_{d_i} \times \prod_{i \in Q_0^+} GL_{d_i}.$$

acts on $R_d^{\mathfrak{g}}$, the quotient being the moduli space of d-dimensional self-dual representations. Here G_{d_i} is O_{d_i}, Sp_{d_i} or U_{d_i} as appropriate.

We will sometimes refer to a symplectic representation of (Q, σ) simply as a representation of $Q^{\mathfrak{sp}}$ and write $\mathcal{M}_Q^{\mathfrak{sp}}$ for the associated Hall module.

In the unitary case, $\operatorname{Rep}_{\mathbb{F}_q}(Q)$ is regarded as a $\mathbb{F}_{\sqrt{q}}$-linear category and the corresponding Ringel-Hall algebra is defined over $\mathbb{Q}[\nu_0, \nu_0^{-1}]$ with $\nu_0 = \sqrt[4]{q}^{-1}$. Then $\mathcal{H}_Q \simeq \mathcal{H}_Q(\mathbb{F}_q) \otimes_{\mathbb{Q}} \mathbb{Q}[\nu_0, \nu_0^{-1}]$, where $\mathcal{H}_Q(\mathbb{F}_q)$ is the Ringel-Hall algebra of $\operatorname{Rep}_{\mathbb{F}_q}(Q)$ viewed as a $\mathbb{F}_q$-linear category. For a more uniform discussion in what follows, we write the $\mathcal{H}_Q$-module structure of $\mathcal{M}_Q^{\mathfrak{u}}$ in terms of $\nu = \sqrt{q}^{-1}$ instead of ν_0. We also rescale the definition of $\mathcal{E}$ in Proposition 3.3.2 by a factor of $\frac{1}{2}$. While $\mathcal{E}$ is then only half-integral valued, expressions of the form $\nu^{-\mathcal{E}(U)}$ remain well-defined.

Example. The quiver $A_2 = \bullet \!\rightarrow\! \bullet$ has a unique involution, swapping the nodes and fixing the arrow. An orthogonal representation consists of a vector space V and a skew-symmetric linear map $V \xrightarrow{v} V^{\vee}$. The form on $V \oplus V^{\vee}$ is induced from the canonical pairing $V \times V^{\vee} \to k$. Isometry classes of (n, n)-dimensional orthogonal representations are parameterized by $\Lambda^2 k^n / GL_n(k)$. $\triangleleft$

Example. The Jordan quiver $\circlearrowleft$ has a unique involution, fixing the node and arrow. A symplectic representation consists of a symplectic vector space M and $m \in \mathfrak{sp}(M)$. The moduli space of $2n$-dimensional symplectic representations is $\mathfrak{sp}_{2n}/Sp_{2n}$. $\triangleleft$

Example. For any quiver Q let Q^{op} be the quiver obtained by reversing the orientations of all arrows Q. Then $Q^{\sqcup} = Q \sqcup Q^{op}$ has a canonical involution that sends a node or arrow of Q to the corresponding node or arrow of Q^{op}. The following proposition describes the associated Hall modules. $\triangleleft$

Proposition 4.3.1. *For all* (τ, s), $\mathcal{M}_{Q^{\sqcup}} \simeq \mathcal{M}_{HRep(Q)} \simeq \mathcal{H}_Q(k_0)$ *as* $\mathcal{H}_Q(k_0) \otimes_{\mathbb{Q}} \mathcal{H}_Q(k_0)^{op}$*-(co)modules.*

Proof. Let S be the duality functor on $\operatorname{Rep}_k(Q^{\sqcup})$ determined by (s, τ) and S_H the duality functor on $H\operatorname{Rep}_k(Q)$. Define $F : H\operatorname{Rep}_k(Q) \to (\operatorname{Rep}_k(Q^{\sqcup}), S, \Theta)$ by $F(A, B) =$

$A \oplus S(B)$ and define $\eta : F \circ S_H \to S \circ F$ as $\begin{pmatrix} 0 & 1 \\ \Theta & 0 \end{pmatrix}$. The pair (F, η) is a form functor and induces an equivalence of k_0-linear categories with duality. The proposition now follows by functoriality; see Proposition 3.3.5. □

For arbitrary Q the Grothendieck group $K(\mathrm{Rep}_k(Q))$ is the free abelian group on the set $\mathfrak{S}$ of simple objects of $\mathrm{Rep}_k(Q)$. The duality S induces an involution on $\mathfrak{S}$ which gives a non-canonical decomposition $\mathfrak{S} = \mathfrak{S}^+ \sqcup \mathfrak{S}^S \sqcup \mathfrak{S}^-$. Here $\mathfrak{S}^S$ consists of simples fixed by S and S restricts to a bijection from $\mathfrak{S}^+$ to $\mathfrak{S}^-$.

Proposition 4.3.2. *There are group isomorphisms*

$$GW(Rep_k(Q)) \simeq \bigoplus_{U \in \mathfrak{S}^+} \mathbb{Z}|H(U)| \oplus \bigoplus_{U \in \mathfrak{S}^S} GW(\mathcal{A}_U)$$

and

$$W(Rep_k(Q)) \simeq \bigoplus_{U \in \mathfrak{S}^S} W(\mathcal{A}_U)$$

where $\mathcal{A}_U$ is the semisimple abelian category with duality generated by U.

Proof. Let N be a self-dual representation. If $U \overset{i}{\hookrightarrow} N$ is a simple subrepresentation, by Schur's lemma the composition $S(i)\psi i$ is either zero or an isomorphism. If it is zero then U is isotropic and $|N| = |H(U)| + |N//U|$ in $GW(\mathrm{Rep}_k(Q))$. If instead $S(i)\psi i$ is an isomorphism then it is a self-dual structure on U and there is a corresponding orthogonal decomposition $N \simeq_S U \oplus \tilde{N}$, implying $|N| = |U| + |\tilde{N}|$. Since every representation has a finite composition series we can repeatedly apply the above procedure, establishing the claimed description of the Grothendieck-Witt group. The isomorphism of Witt groups now follows from the exact sequence (3.5). □

The primary interest of this paper is $k = \mathbb{F}_q$, q odd and Q acyclic. In this case $\mathfrak{S} = \{S_i\}_{i \in Q_0}$ and the categories $\mathcal{A}_U$ in Proposition 4.3.2 are equivalent to $Vect_k$ with dualities determined by s. Explicitly, $GW(k) := GW(Vect_k)$ is isomorphic to $\mathbb{Z}$ if $s = -1, 0$, with generators $H(S_0)$ and S_0, respectively. If $s = 1$ then $GW(k)$ is isomorphic to $\mathbb{Z}^2$ with generators the two non-isometric orthogonal structures on $S_0 = k$. The corresponding Witt groups are

$$W(k) \simeq \begin{cases} pt, & \text{if } s = -1 \\ \mathbb{Z}_2, & \text{if } s = 0 \\ \mathbb{Z}_4, & \text{if } s = 1 \text{ and } char(k) \equiv 3 \pmod 4 \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } s = 1 \text{ and } char(k) \equiv 1 \pmod 4 \end{cases}$$

In particular, the Grothendieck-Witt class of N is essentially its dimension vector, with additional decorations at σ-fixed vertices in the orthogonal case.

We now study the function $\mathcal{E}$ of Proposition 3.3.2 in more detail for quivers. Given representations (V, v) and (W, w), define

$$A^0(V, W) = \bigoplus_{i \in Q_0} \mathrm{Hom}_k(V_i, W_i), \qquad A^1(V, W) = \bigoplus_{i \xrightarrow{\alpha} j \in Q_1} \mathrm{Hom}_k(V_i, W_j).$$

There is a differential $A^0(V, W) \xrightarrow{\delta} A^1(V, W)$ given by $\delta\{f_i\}_i = \{w_\alpha f_i - f_j v_\alpha\}_\alpha$. The complex $A^\bullet(V, W)$ fits into the exact sequence

$$0 \to \mathrm{Hom}(V, W) \to A^0(V, W) \xrightarrow{\delta} A^1(V, W) \to \mathrm{Ext}^1(V, W) \to 0. \tag{4.1}$$

An immediate consequence is that the Euler form depends only on the dimension vectors of its arguments and is given explicitly by

$$\langle d, d' \rangle = \sum_{i \in Q_0} d_i d'_i - \sum_{i \xrightarrow{\alpha} j} d_i d'_j, \qquad d, d' \in \mathbb{Z}^{Q_0}.$$

The following identity plays an important role in the next chapter but is also of independent interest. Take $k = \mathbb{F}_q$.

Theorem 4.3.3. *For all representations U and self-dual representations M the identity*

$$\sum_N \frac{\mathcal{G}^N_{U,M}}{a_S(N)} = q^{-\langle M, U \rangle - \mathcal{E}(U)}$$

hods, the sum being over isometry classes of self-dual representations.

Proof. The functor S defines an involution on the complex $A^\bullet(S(U), U)$ through the composition

$$A^i(S(U), U) \xrightarrow{S} A^i(S(U), S^2(U)) \xrightarrow{\Theta^{-1}_{U*}} A^i(S(U), U),$$

or in components

$$S\{f_i\}_i = \{\Theta_i^{-1} f^\vee_{\sigma(i)}\}_i, \quad S\{e_\alpha\}_\alpha = \{\tau_\alpha \Theta_j^{-1} e^\vee_{\sigma(\alpha)}\}_\alpha.$$

Then $\delta(S\{f_i\}_i) = \{u_\alpha \Theta_i^{-1} f^\vee_{\sigma(i)} - \tau_\alpha \Theta_j^{-1} f^\vee_{\sigma(j)} u^\vee_{\sigma(\alpha)}\}_\alpha$ while

$$\begin{aligned} S(\delta\{f_i\}_i) &= \{\Theta_j^{-1} \tau_\alpha (u_{\sigma(\alpha)} f_{\sigma(j)} - \tau_{\sigma(\alpha)} f_{\sigma(j)} u^\vee_\alpha)^\vee\}_\alpha \\ &= \{-\Theta_j^{-1} u^{\vee\vee}_\alpha f^\vee_{\sigma(i)} + \tau_\alpha \Theta_j^{-1} f^\vee_{\sigma(j)} u^\vee_{\sigma(\alpha)}\}_\alpha. \end{aligned}$$

Comparing these two expressions and using $\Theta_j^{-1} u_\alpha^{\vee\vee} = u_\alpha \Theta_i^{-1}$, we conclude $\delta(S\{f_i\}) = -S(\delta\{f_i\})$. The subcomplex of (anti-)fixed points $A^0(S(U), U)^{-S} \xrightarrow{\delta} A^1(S(U), U)^S$, denoted $B^\bullet(U)$, fits into the exact sequence

$$0 \to \operatorname{Hom}(S(U), U)^{-S} \to B^0(U) \xrightarrow{\delta} B^1(U) \to \operatorname{Ext}^1(S(U), U)^S \to 0. \tag{4.2}$$

Note that each of the terms in (4.2) are k_0-vector spaces.

Let $\underline{N}$ be the Q_0-graded vector space with sesquilinear form underlying the split self-dual extension $U \oplus M \oplus S(U)$. The degree zero term of the direct sum $C^\bullet(M, U) = A^\bullet(M, U) \oplus B^\bullet(U)$ is naturally interpreted as the Lie algebra of the unipotent radical $\mathcal{U}$ of the parabolic subgroup of $Aut_S(\underline{N})$ stabilizing the isotropic subspace $\underline{U} \subset \underline{N}$. To each $c = \{d_\alpha, e_\alpha\}_\alpha \in C^1(M, U)$ there corresponds a self-dual representation N_c with structure maps, in the natural basis of $\underline{N}$,

$$n_\alpha = \begin{pmatrix} u_\alpha & d_\alpha & e_\alpha - d_\alpha \psi_j^{-1} d^\vee_{\sigma(\alpha)} \\ 0 & m_\alpha & \psi_j^{-1} d^\vee_{\sigma(\alpha)} \\ 0 & 0 & \tau_\alpha u^\vee_{\sigma(\alpha)} \end{pmatrix}.$$

The group $\mathcal{U}$ acts on $C^1(M, U)$ by conjugating the associated structure maps $\{n_\alpha\}_\alpha$. Then $g \in \mathcal{U}$ fixes $c \in C^1(M, U)$ if and only if $g \in Aut_S(N_c)$, in which case it is immediate that g preserves the natural self-dual exact sequence structure of N_c. We have defined a set map

$$\bigsqcup_N \underline{\mathcal{G}}^N_{U,M} / Aut_S(N) \leftarrow C^1(M, U)/\mathcal{U}.$$

We show that this is a bijection by constructing an inverse. Given a self-dual exact sequence with middle term N, pick an isometry $N \simeq \underline{N}$ that is also an equivalence of self-dual extensions of vector spaces. In the natural basis of $\underline{N}$, the difference between the structure maps of N and $H(U) \oplus M$ is an element of $C^1(M, U)$ and defines the inverse map. Moreover, the stabilizer of $c \in C^1(M, U)$ under the action of $\mathcal{U}$ has cardinality equal to that of the stabilizer of N_c, viewed as a self-dual exact sequence, under the action of $Aut_S(N_c)$. Applying Burnside's lemma gives

$$\sum_N \frac{\mathcal{G}^N_{U,M}}{a_S(N)} = \frac{|C^1(M, U)|}{|\mathcal{U}|}$$

As $|\mathcal{U}| = |C^0(M, U)|$, this sum is equal to $|k_0|^{-\chi(C^\bullet(M,U))}$ and the theorem now follows from the exact sequences (4.1) and (4.2). □

Remark. The analogue of Theorem 4.3.3, which holds for arbitrary hereditary finitary abelian categories, states [55]

$$\sum_{X \in Iso(\mathcal{A})} \frac{\mathcal{F}^X_{U,V}}{a(X)} = q^{-\langle V, U \rangle}.$$

Its proof relies on the simple fact that the stabilizer of a short exact sequence $0 \to U \to X \to V \to 0$ under the action of $Aut(X)$ is canonically isomorphic to $\mathrm{Hom}(V, U)$. The situation is necessarily more subtle in the self-dual situation as the stabilizer of a self-dual exact sequence $0 \to U \to N \dashrightarrow M \to 0$ under the action of $Aut_S(N)$ is not determined solely by M and U. The additional data needed to fix the cardinality of the stabilizer is the extension class ξ associated to $0 \to U \to U^{\perp} \to M \to 0$. Explicitly, the stabilizer of the above self-dual exact sequence has cardinality $|\ker \delta^S_{\xi}||\mathrm{Hom}(S(U), U)^{-S}|$ where $\delta^S_{\xi} : \mathrm{Hom}(M, U) \to \mathrm{Ext}^1(S(U), U)^S$ is defined by

$$\delta^S_{\xi}\beta = \beta_*\xi + \Theta^{-1}_{U*}S(\beta_*\xi) = \delta_{\xi}\beta + \Theta^{-1}_{U*}S(\delta_{\xi}\beta).$$

Here δ_{ξ} is the connecting homomorphism in the long exact sequence obtained by applying $\mathrm{Hom}(-, U)$ to ξ.

Corollary 4.3.4. *When $\mathcal{A} = Rep_k(Q)$, $\mathcal{E}(U)$ depends only on $u = \mathbf{dim}\, U$ and is given explicitly by*

$$\mathcal{E}(U) = \sum_{i \in Q_0^{\sigma}} \frac{u_i(u_i - s)}{2} + \sum_{i \in Q_0^+} u_{\sigma(i)}u_i - \sum_{(\sigma(i) \xrightarrow{\alpha} i) \in Q_1^{\sigma}} \frac{u_i(u_i + \tau_{\alpha}s)}{2} - \sum_{(i \xrightarrow{\alpha} j) \in Q_1^+} u_{\sigma(i)}u_j.$$

Proof. This follows by direct calculation using the exact sequence (4.2). □

The function $\mathcal{E}$ has the following interpretation that should be compared with the fact that the dimension of the moduli stack of d-dimensional representations of Q is $-\langle d, d\rangle$. Let $\mathfrak{M}^{\mathfrak{g}}_d$ be the global quotient stack associated to the action of $G^{\mathfrak{g}}_d$ on $R^{\mathfrak{g}}_d$.

Proposition 4.3.5. *The dimension of $\mathfrak{M}^{\mathfrak{g}}_d$ is $-\mathcal{E}(d)$.*

Proof. Verifying that $\dim R^{\mathfrak{g}}_d - \dim G^{\mathfrak{g}}_d$ is equal to $-\mathcal{E}(d)$ gives a simple proof of the proposition. We give here a more conceptual proof.

The scheme $R^{\mathfrak{g}}_d$ represents the functor $\mathsf{f.g.Alg}_k \to \mathsf{Set}$ given by

$$A \mapsto \mathrm{Hom}_{kQ_0}(kQ, \mathrm{End}_A(A \otimes \underline{N}))^S.$$

Here kQ is the path algebra of the quiver and kQ_0 is the subalgebra generated by the nodes of Q. The functor S acts on the path algebra by σ and on $\mathrm{End}_A(A \otimes \underline{N})$ by its action on $\underline{N}$. Writing $\mathbb{D} = k[\epsilon]/\langle \epsilon^2 \rangle$ for the algebra of dual numbers, a Zariski tangent vector to $R^{\mathfrak{g}}_d$ at N is an element $v \in \mathrm{Hom}_{kQ_0}(kQ, \mathrm{End}_k(\underline{N}))$ such that $N + \epsilon v \in \mathrm{Hom}_{kQ_0}(kQ, \mathrm{End}_{\mathbb{D}}(\mathbb{D} \otimes \underline{N}))^S$. This requires for all $\alpha, \alpha' \in Q_1$

$$n_{\alpha\alpha'} + \epsilon v_{\alpha\alpha'} = n_{\alpha}n_{\alpha'} + \epsilon(v_{\alpha}n_{\alpha'} + n_{\alpha}v_{\alpha'})$$

which is equivalent to requiring $v \in \mathrm{Der}_{kQ_0}(kQ, \mathrm{End}(N))$. The self-duality of $N + \epsilon v$ requires that v be self-dual. This gives $T_N R^{\mathfrak{g}}_d = \mathrm{Der}_{kQ_0}(kQ, \mathrm{End}(N))^S$. Next, we describe the tangent space to the G_d^σ-orbit through N. For any $x \in Lie(G_d^\sigma) = \mathrm{End}_{kQ_0}(\underline{N})^{-S}$ we have

$$(1_N + \epsilon x)N(1_N - \epsilon x)_{|\alpha} = n_\alpha + \epsilon(x_j n_\alpha - n_\alpha x_i).$$

The coefficient of ϵ, which is a tangent vector to the orbit, is an element of the inner derivations $\mathrm{InnDer}_{kQ_0}(kQ, \mathrm{End}(N))^S$. Combining the previous two results we conclude $T_N \mathfrak{M}_d^{\mathfrak{g}} = \mathrm{Ext}^1(N, N)^S$.

To complete the proof we use that the infinitesimal isometries of N are $\mathrm{End}(N)^{-S}$, which can be verified as in the previous calculation. □

4.4 Quantum groups and Hall modules

In view of the realization of quantum groups via Hall algebras it is natural to expect a connection between Hall modules and representations of quantum groups. The goal of the remainder of this chapter is to investigate such a connection. We assume that Q has no loops.

Theorems 3.2.3 and 4.2.1 show at once that $\mathcal{M}_Q$ is a representation of $U_\nu^-(\mathfrak{g}_Q)$. Using [63] we can similarly conclude that $\mathcal{M}_Q$ is a representation of a much larger quantum group, namely $\mathcal{H}_Q$ itself. However, without a better understanding of the latter quantum group it seems unlikely that this structure can be used to say much about $\mathcal{M}_Q$. In order to further constrain $\mathcal{M}_Q$ we instead focus on incorporating the comodule structure. The most obvious guess is that $\mathcal{M}_Q$ is a Hopf module, i.e. the comodule structure map

$$\rho : \mathcal{M}_Q \to \mathcal{H}_Q \otimes \mathcal{M}_Q$$

is a homomorphism of modules, possibly with $\mathcal{H}_Q \otimes \mathcal{M}_Q$ having a twisted $\mathcal{H}_Q \otimes \mathcal{H}_Q$-module structure; compare with Theorem 2.0.2. However, already for A_1 this is seen to not be the case. We therefore seek a modification of the Hopf module condition to describe the compatibility of the module and comodule structures of $\mathcal{M}_Q$. To this end, we describe in Theorem 4.4.1 below the relationship between multiplication and comultiplication along the simple representations S_i.

To state the result we require a twisted version of quantum groups. To motivate this, given a symmetric Kac-Moody algebra $\mathfrak{g}$ define linear operators E_i' and E_i^* on $U_v^-(\mathfrak{g})$ according to

$$[E_i, F_j] = \delta_{ij} \frac{E_i^* F_j T_i - T_i^{-1} E_i' F_j}{v - v^{-1}}.$$

The operators E_i' satisfy the quantum Serre relations and also the relation

$$E_i' F_j = v^{-(\epsilon_i, \epsilon_j)} F_j E_i' + \delta_{ij}.$$

The subalgebra of $\mathrm{End}_{\mathbb{Q}(v)}(U_v^-(\mathfrak{g}))$ generated by E_i' and F_i is called the reduced analogue of $U_v(\mathfrak{g})$ and is denoted by $B(\mathfrak{g})$ [35].

To modify $B(\mathfrak{g})$ in the self-dual case, suppose that we are given a $(\cdot,\cdot)$-preserving involution σ of the set of simple roots.

Definition ([15])**.** *The reduced σ-analogue $B_\sigma(\mathfrak{g})$ is the $\mathbb{Q}(v)$-algebra generated by symbols E_i, F_i, T_i, T_i^{-1}, $i = 1, \ldots, n$, subject to the relations*

1. $T_i T_j = T_j T_i, \quad T_i T_i^{-1} = 1, \quad T_i = T_{\sigma(i)}$ (4.3)
2. $T_i E_j = v^{(\epsilon_j + \epsilon_{\sigma(j)}, \epsilon_i)} E_j T_i, \qquad T_i F_j = v^{-(\epsilon_j + \epsilon_{\sigma(j)}, \epsilon_i)} F_j T_i$ (4.4)
3. $E_i F_j = v^{-(\epsilon_i, \epsilon_j)} F_j E_i + \delta_{i,j} + \delta_{i,\sigma(j)} T_i$ (4.5)
4. $\displaystyle\sum_{l=0}^{1-(\epsilon_i,\epsilon_j)} (-1)^l \begin{bmatrix} 1-(\epsilon_i,\epsilon_j) \\ l \end{bmatrix}_v E_i^l E_j E_i^{1-(\epsilon_i,\epsilon_j)-l} = 0, \quad i \neq j.$ (4.6)
5. $\displaystyle\sum_{l=0}^{1-(\epsilon_i,\epsilon_j)} (-1)^l \begin{bmatrix} 1-(\epsilon_i,\epsilon_j) \\ l \end{bmatrix}_v F_i^l F_j F_i^{1-(\epsilon_i,\epsilon_j)-l} = 0, \quad i \neq j$ (4.7)

For generic $t \in \mathbb{C}^\times$ the specialization $B_\sigma(\mathfrak{g})_t$ is the $\mathbb{Q}[t, t^{-1}]$-algebra with the same generators and relations as $B_\sigma(\mathfrak{g})$, but with v replaced by t.

Returning to the quiver setting, for each $i \in Q_0$ define the induction operator F_i by

$$F_i[M] = [S_i] \star [M]$$

and define the restriction operator E_i as the projection of the comodule structure map onto the subspace $[S_i] \otimes \mathcal{M}_Q \subset \mathcal{H}_Q \otimes \mathcal{M}_Q$. That is,

$$\rho([M]) = [S_i] \otimes E_i[M] + (\rho[M])'$$

where $(\rho[M])'$ is a linear combination of terms of the form $[U] \otimes [N]$ with $U \not\simeq S_i$. The action of T_i is given by

$$T_i[M] = \nu^{-(M,\epsilon_i) - \mathcal{E}(\epsilon_i) - \mathcal{E}(\epsilon_{\sigma(i)})}[M].$$

Theorem 4.4.1. *The operators E_i, F_i, T_i, $i = 1, \ldots, n$, give $\mathcal{M}_Q$ the structure of a $B_\sigma(\mathfrak{g}_Q)_\nu \otimes_{\mathbb{Q}} \mathbb{Q}[\nu_0, \nu_0^{-1}]$-module.*

Beginning of Proof. We first verify the relations (4.3), (4.4), (4.6) and (4.7). The first two parts of relation (4.3) are trivial and $T_i = T_{\sigma(i)}$ because $(d, \epsilon_i) = (d, \epsilon_{\sigma(i)})$ whenever $d = \sigma(d)$. Relation (4.4) follows from the explicit description of the action of T_i and the fact that F_i (resp. E_i) increases (resp. decreases) the dimension vector by $\epsilon_i + \epsilon_{\sigma(i)}$. The Serre relation (4.7) follows from Theorems 3.2.3 and 4.2.1. Interpreting Lemma 3.2.4 in terms of induction and restriction operators we find

$$(F_i \xi, \zeta)_{\mathcal{M}} = \frac{1}{\nu^{-2} - 1} (\xi, E_i \zeta)_{\mathcal{M}}, \quad \xi, \zeta \in \mathcal{M}_Q.$$

Using this, the relation (4.6) now follows from the non-degeneracy of $(\cdot,\cdot)_{\mathcal{M}}$ and the Serre relations for F_i. To complete the proof it remains to verify relations (4.5). Since this is quite involved, we break the verification into a number of parts. $\square$

Using Lemma 3.2.1 to rewrite the action of the Hall algebra in terms of the numbers $\mathcal{G}^N_{U,M}$, we find that relation (4.5) holds if and only if for all $i,j \in Q_0$ and self-dual representations N, Y, the following identity holds:

$$\sum_X \frac{\mathcal{G}^X_{S_i,N}\mathcal{G}^X_{S_j,Y}}{a_S(X)} = \frac{|\mathrm{Ext}^1(S_{\sigma(j)},S_i)|}{|\mathrm{Hom}(S_{\sigma(j)},S_i)|}\sum_Z \frac{\mathcal{G}^Y_{S_i,Z}\mathcal{G}^N_{S_j,Z}}{a_S(Z)} + \delta_{i,\sigma(j)}\delta_{N,Y}a(S_i)a_S(N)$$
$$+\, \delta_{i,j}\delta_{N,Y}a(S_i)a_S(N)\frac{|\mathrm{Ext}^1(N,S_i)||\mathrm{Ext}^1(S_{\sigma(i)},S_i)^S|}{|\mathrm{Hom}(N,S_i)||\mathrm{Hom}(S_{\sigma(i)},S_i)^{-S}|} \qquad (4.8)$$

Unless otherwise noted, all sums in what follows will be over isometry classes of self-dual representations. We will establish relation (4.5) in this formulation.

Given self-dual representations X, Y, N, let $C_X(i,j;N,Y)$ be the set of crosses of self-dual exact sequences

$$\begin{array}{ccccc} & & S_j & & \\ & & \big\downarrow{\scriptstyle b} & & \\ S_i & \xrightarrow{a} & X & \overset{\pi}{\dashrightarrow} & N \\ & & \big\downarrow{\scriptstyle \rho} & & \\ & & Y & & \end{array} \qquad (4.9)$$

The group $Aut_S(X)$ acts on $C_X(i,j;N,Y)$ with orbit space $\tilde{C}_X(i,j;N,Y)$. Burnside's lemma gives for the sum on the left-hand side of equation (4.8)

$$\sum_X \frac{\mathcal{G}^X_{S_i,N}\mathcal{G}^X_{S_j,Y}}{a_S(X)} = \sum_X \frac{|C_X(i,j;N,Y)|}{a_S(X)} = \sum_X \sum_{\mathcal{C}\in\tilde{C}_X(i,j;N,Y)} \frac{1}{|\mathrm{Stab}_{Aut_S(X)}\mathcal{C}|}.$$

Similarly, for a self-dual representation Z let $D_Z(i,j;N,Y)$ be the set of all corners of self-dual exact sequences

$$\begin{array}{ccccc} & & & & S_j \\ & & & & \big\downarrow{\scriptstyle \tilde{b}} \\ & & & & N \\ & & & & \big\downarrow{\scriptstyle \tilde{\rho}} \\ S_i & \xrightarrow[\tilde{a}]{} & Y & \underset{\tilde{\pi}}{\dashrightarrow} & Z \end{array} \qquad (4.10)$$

The group $Aut_S(Z)$ acts freely on $D_Z(i,j;N,Y)$ with orbit space $\tilde{D}_Z(i,j;N,Y)$. The sum on the right-hand side of equation (4.8) can then be written as

$$\sum_{Z} \frac{\mathcal{G}^{Y}_{S_i,Z}\mathcal{G}^{N}_{S_j,Z}}{a_S(Z)} = \sum_{Z} \frac{|D_Z(i,j;N,Y)|}{a_S(Z)} = \sum_{Z} |\tilde{D}_Z(i,j;N,Y)|.$$

If in the notation of the cross (4.9) the subrepresentation $\mathrm{Im}(a \oplus b) \subset X$ is isotropic, then the cross descends to a corner on $Z = X//\mathrm{Im}(a \oplus b)$ as follows. The map $S_j \overset{\tilde{b}}{\rightarrowtail} N$ is induced by $E_i \overset{\pi}{\twoheadrightarrow} N$, where $E_i = \mathrm{Im}(a)^{\perp}$, the map $S_i \overset{\tilde{a}}{\rightarrowtail} Y$ being defined similarly. Picking an orthogonal $E' \subset X$ for $Im(a \oplus b)$, define $\tilde{\rho}$ and $\tilde{\pi}$ by the pushout diagram

$$\begin{array}{ccccc}
 & & S_j & = & S_j \\
 & & {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle \tilde{b}} \\
S_i & \overset{a'}{\rightarrowtail} & E' & \overset{\pi'}{\twoheadrightarrow} & \tilde{E}_j \\
\| & & {\scriptstyle \rho'}\downarrow & & \downarrow{\scriptstyle \tilde{\rho}} \\
S_i & \underset{\tilde{a}}{\rightarrowtail} & \tilde{E}_i & \underset{\tilde{\pi}}{\twoheadrightarrow} & Z
\end{array} \tag{4.11}$$

where π' and ρ' are the canonical maps induced by $E_i \overset{\pi}{\twoheadrightarrow} N$ and $E_j \overset{\rho}{\twoheadrightarrow} Y$, respectively.

Lemma 4.4.2. *If the cross (4.9) does not descend to a corner, then $N \simeq_S Y$.*

Proof. The cross fails to descend if and only if $\mathrm{Im}(a \oplus b)$ is not a two dimensional isotropic subrepresentation, which occurs in precisely two cases. The first is if $\mathrm{Im}(a) = \mathrm{Im}(b)$, in which case clearly $N \simeq_S Y$. The second is if $\mathrm{Im}(a \oplus b)$ is non-degenerate and hence necessarily isometric to $H(S_i)$. In this case $X \simeq_S H(S_i) \oplus N \simeq_S H(S_i) \oplus Y$ and we again have $N \simeq_S Y$. □

The strategy to establish equation (4.8) is as follows. We will show that the sum on the right hand side counts crosses that descend to corners, with appropriate multiplicity, while the two remaining terms count crosses that fail to descend for each of the two reasons indicated in the proof of Lemma 4.4.2. Since the left hand side of equation (4.8) counts all crosses, the identity will follow. From this point of view, the spirit of the proof is similar to [20, Theorem 2].

Lemma 4.4.3. *There are exactly $a(S_j)a_S(N)$ crosses in $\tilde{C}_X(i,j;N,Y)$ in which $Im(a \oplus b)$ is non-degenerate.*

Proof. Suppose we are given a cross in $C_X(i,j;N,N)$ with $\mathrm{Im}(a \oplus b)$ non-degenerate. Then $X \simeq_S H(S_i) \oplus N$. Acting by the subgroups $Aut_S(H(S_i))$ and $Aut_S(N)$ of $Aut_S(X)$ we may take a to be the standard inclusion and π to be the projection onto N. The set of all pairs (b, ρ) completing the cross forms a torsor for $Aut(S_i) \times Aut_S(N)$ and different choices for (b, ρ) give different classes in $\tilde{C}_X(i,j;N,Y)$. □

Proposition 4.4.4. *Let $\mathcal{C} \in C_X(i,j;X,N)$.*

1. *If $\mathcal{C}$ descends to a corner, then $Stab_{Aut_S(X)}(\mathcal{C}) \simeq Hom(S_{\sigma(i)}, S_j)$.*
2. *If $Im(a \oplus b)$ is non-degenerate, then $Stab_{Aut_S(X)}(\mathcal{C}) = \{1\}$.*

Proof. We make two preliminary observations. Suppose $\phi \in Aut_S(X)$ fixes $(E_i; a, l, k, \pi) \in \underline{\mathcal{G}}^X_{S_i,N}$. Then there exists $r \in Aut(E_i)$ such that

$$rl = l, \quad \pi r^{-1} = \pi, \quad kr^{-1} = \phi k.$$

The first two equations state that r stabilizes $S_i \overset{l}{\rightarrowtail} E_i \overset{\pi}{\twoheadrightarrow} N$ and hence is uniquely determined by an element of $\mathrm{Hom}(N, S_i)$. The third equation states that E_i is ϕ-stable and $\phi_{|E_i} = r$, so that $\phi_{|E_i}$ is also determined by an element of $\mathrm{Hom}(N, S_i)$. If moreover, $\phi_{|E_i} = 1_{E_i}$, then ϕ also stabilizes $E_i \overset{k}{\rightarrowtail} N \overset{S(a)\psi}{\twoheadrightarrow} S(S_i)$. In this case ϕ is uniquely determined by an element of $\mathrm{Hom}(S_{\sigma(i)}, S_i)^{-S}$

We prove the first statement; the second is similar. Suppose that $\mathcal{C}$ descends to a corner and let $\phi \in Stab_{Aut_S(X)}(\mathcal{C})$. The restrictions of $\phi_{|E_i}$ and $\phi_{|E_j}$ are determined uniquely by elements of $\mathrm{Hom}(N, S_i)$ and $\mathrm{Hom}(Y, S_j)$, respectively. As $\mathrm{Im}(a) \cap \mathrm{Im}(b) = 0$, the restriction of ϕ to $E_i \cap E_j$ is the identity and ϕ is determined uniquely by an element of $\mathrm{Hom}(S_{\sigma(i)} \oplus S_{\sigma(j)}, S_i \oplus S_j)^{-S}$. Compatibility with $\phi_{|E_i}$ and $\phi_{|E_j}$ requires that the component of ϕ from $\mathrm{Hom}(S_{\sigma(i)}, S_i)$ and $\mathrm{Hom}(S_{\sigma(j)}, S_j)$ vanish, leaving only a factor determined by $\mathrm{Hom}(S_{\sigma(i)}, S_j)$. Reversing this argument, any of $\mathrm{Hom}(S_{\sigma(i)}, S_j)$ gives rise to an isometry of X stabilizing $\mathcal{C}$. ∎

We need a final result before being able to establish equation (4.8).

Proposition 4.4.5. *There are $|Ext^1(S_{\sigma(i)}, S_j)|$ elements of $\bigsqcup_X \tilde{C}_X(i,j;N,Y)$ that descend to each $\mathcal{D} \in \bigsqcup_Z \tilde{D}_Z(i,j;N,Y)$.*

Proof. The first part of the proof is as in [20]. To avoid clutter write $U = S_i$ and $V = S_j$. Start with the corner (4.10). Let E' be the pullback of $\tilde{\pi}$ and $\tilde{\rho}$ in diagram (4.11) and let $\xi \in \mathrm{Ext}^1(N, U)$ map to $(a', \pi') \in \mathrm{Ext}^1(\tilde{E}_V, U)$ in the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(S(V), U) \to \mathrm{Hom}(N, U) \to \mathrm{Hom}(\tilde{E}_V, U) \to \\ \mathrm{Ext}^1(S(V), U) \to \mathrm{Ext}^1(N, U) \to \mathrm{Ext}^1(\tilde{E}_V, U) \to 0 \end{aligned}$$

The set of morphisms τ making the diagram

$$\begin{array}{ccccc} U & \overset{a'}{\rightarrowtail} & E' & \overset{\pi'}{\twoheadrightarrow} & \tilde{E}_V \\ \| & & \downarrow \tau & & \downarrow \tilde{k} \\ U & \rightarrowtail & E_U & \overset{\pi}{\twoheadrightarrow} & N \\ & & \downarrow & & \downarrow S(\tilde{b})\psi_N \\ & & S(V) & = & S(V) \end{array}$$

commute is a torsor for $\mathrm{Hom}(\tilde{E}_V, U)$. The middle row is a representative of ξ. The map $U \rightarrowtail E_U$ is determined by commutativity. The group $Aut(E_U)$ acts transitively on the set of choices for π, with stabilizer $\mathrm{Hom}(N, U)$. From the long exact sequence, we find that there are

$$a(E_U)\frac{|\mathrm{Ext}^1(S(V), U)_{E_U}|}{|\mathrm{Hom}(S(V), U)|}$$

diagrams as above with central term E_U. Because of the equivalence in the definition of self-dual exact sequences, we are only interested in such diagrams up to the action of $Aut(E_U)$. Each diagram has stabilizer $\mathrm{Hom}(S(V), U)$ under $Aut(E_U)$. Summing over isomorphism types of E_U, we find that there are $|\mathrm{Ext}^1(S(V), U)|$ equivalence classes of diagrams, keeping the outside maps fixed.

Fix a diagram as above. We show by constraining the possible structure maps that there is a unique way of extending this diagram to a cross inducing the original corner. Let $\epsilon_U \in A^1(N, U)$ determine E_U and let X, the central term in the cross, be determined by

$$(x_1, x_2) \in A^1(S(U), E_U) = A^1(S(U), N) \oplus A^1(S(U), U).$$

If the canonical bilinear form is to make X self-dual, then x_1 must be the transpose of ϵ_U. Moreover, the element x_2 is uniquely determined by the requirement that the canonical map $V^\perp \to Y$ be an isometry. $\square$

Completion of the proof of Theorem 4.4.1. Write

$$C_X(i, j; N, Y) = C_X^{(1)}(i, j; N, Y) \bigsqcup C_X^{(2)}(i, j; N, Y)$$

where $C_X^{(1)}(i, j; N, Y)$ is the set of crosses that descend to corners. By Proposition 4.4.4

$$\sum_X \frac{|C_X(i, j; N, Y)|}{a_S(X)} = \sum_X \frac{|\tilde{C}_X^{(1)}(i, j; N, Y)|}{|\mathrm{Hom}(S_{\sigma(i)}, S_j)|} + \sum_X \frac{|C_X^{(2)}(i, j; N, Y)|}{a_S(X)}.$$

Proposition 4.4.5 shows

$$\sum_X \frac{|\tilde{C}_X^{(1)}(i, j; N, Y)|}{|Hom(S_{\sigma(i)}, S_j)|} = \frac{|\mathrm{Ext}^1(S_{\sigma(j)}, S_i)|}{|\mathrm{Hom}(S_{\sigma(j)}, S_i)|} \sum_Z |\tilde{D}_Z(i, j; N, Y)|.$$

The number of crosses in $C_X(i, j; N, Y)$ that fail to descend to corners because $\mathrm{Im}(a) = \mathrm{Im}(b)$ is equal to $a(S_i) a_S(N) \sum_X \mathcal{G}_{S_i, N}^X$. Applying Lemma 4.4.3 and Proposition 4.4.4 gives

$$\sum_X \frac{|C_X^{(2)}(i, j; N, Y)|}{a_S(X)} = \delta_{N,Y} \delta_{i,j} a(S_i) a_S(N) \sum_X \frac{\mathcal{G}_{S_i, N}^X}{a_S(X)} + \delta_{N,Y} \delta_{i,\sigma(j)} a(S_i) a_S(N)$$

Using Theorem 4.3.3 to evaluate the sum on the right hand side now completes the proof. □

In order to say more about the $B_\sigma(\mathfrak{g}_Q)_\nu$-module structure of $\mathcal{M}_Q^{\mathfrak{g}}$ we recall a characterization of certain highest weight $B_\sigma(\mathfrak{g})$-modules.

Proposition 4.4.6 ([15, Proposition 2.11])**.** *Let $\lambda \in Hom(\Phi, \mathbb{Z})$ be a σ-invariant integral weight. Then there exists a $B_\sigma(\mathfrak{g})$-module $V_\sigma(\lambda)$ generated by a non-zero vector ϕ_λ such that $T_i\phi_\lambda = v^{\lambda(\epsilon_i)}\phi_\lambda$, $E_i\phi_\lambda = 0$ for all $i = 1, \dots, n$ and*

$$\{x \in V_\sigma(\lambda) \mid E_i x = 0, \ \ \forall i = 1, \dots, n\} = \mathbb{Q}(v)\phi_\lambda.$$

Moreover, $V_\sigma(\lambda)$ is irreducible and unique up to isomorphism.

We require two straightforward modifications of Proposition 4.4.6. The first is an extension to σ-invariant half-integral weights $\lambda \in \mathrm{Hom}(\Phi, \frac{1}{2}\mathbb{Z})$ that will arise from unitary representations. In this case we have a representation of $B_\sigma(\mathfrak{g}) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v^{\frac{1}{2}})$. The second is a version of Proposition 4.4.6 (or its aforementioned extension) for the specializations $B_\sigma(\mathfrak{g})_t$ to generic $t \in \mathbb{C}$, the corresponding modules being denoted by $V_\sigma(\lambda)_t$. The proof of Proposition 4.4.6 given in [15] carries over directly in both cases.

Definition. *An non-zero element $\xi \in \mathcal{M}_Q^{\mathfrak{g}}$ is cuspidal if $E_i\xi = 0$ for all $i \in Q_0$.*

Example. If U is a simple representation admitting a self-dual structure ψ, then $[(U, \psi)] \in \mathcal{M}_Q^{\mathfrak{g}}$ is cuspidal. The submodule generated by $[(U, \psi)]$ lies in the direct summand of $\mathcal{M}_Q^{\mathfrak{g}}$ labelled by the Witt class of ψ; see Proposition 3.3.1. ◁

If ξ is cuspidal, then its components of homogeneous dimension vector are also cuspidal. We may therefore without loss of generality take cuspidals to be of homogeneous dimension. Let $\mathcal{C}_Q^{\mathfrak{g}}$ be a homogeneous orthogonal basis for the R-submodule of cuspidals of $\mathcal{M}_Q^{\mathfrak{g}}$. To any σ-invariant dimension vector $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ we associate a σ-invariant weight $\lambda_d \in \mathrm{Hom}(\Phi, \frac{1}{2}\mathbb{Z})$ by defining λ_d on a basis of Φ

$$\lambda_d(\epsilon_i) = -(d, \epsilon_i) - \mathcal{E}(\epsilon_i) - \mathcal{E}(\epsilon_{\sigma(i)})$$

and extending linearly. Note that $T_i[M] = v^{\lambda_{\mathbf{dim} M}(\epsilon_i)}[M]$. The weight λ_d is independent of the orientation of Q. Given $\xi \in \mathcal{C}_Q^{\mathfrak{g}}$, we will write λ_ξ for $\lambda_{\mathbf{dim}\,\xi}$.

Theorem 4.4.7. *There is a direct sum decomposition of $B_\sigma(\mathfrak{g}_Q)_\nu \otimes_{\mathbb{Q}} \mathbb{Q}[\nu_0, \nu_0^{-1}]$-modules*

$$\mathcal{M}_Q^{\mathfrak{g}} = \bigoplus_{\xi \in \mathcal{C}_Q^{\mathfrak{g}}} V_\sigma(\lambda_\xi)_\nu.$$

Proof. We first prove that the submodule $\langle \xi \rangle \subset \mathcal{M}_Q^{\mathfrak{g}}$ generated by $\xi \in \mathcal{C}_Q^{\mathfrak{g}}$ is isomorphic to $V_\sigma(\lambda_\xi)_\nu$. Suppose that $x \in \langle \xi \rangle$ is non-zero with $E_i x = 0$ for all $i \in Q_0$. If $x = \sum_{i \in Q_0} F_i y_i$ for some $y_i \in \langle \xi \rangle$, then

$$(x, x)_{\mathcal{M}} = \sum_{i \in Q_0} (x, F_i y_i)_{\mathcal{M}} = \frac{1}{\nu^{-2} - 1} \sum_{i \in Q_0} (E_i x, y_i)_{\mathcal{M}} = 0.$$

However, writing x in the natural basis of $\mathcal{M}_Q^{\mathfrak{g}}$ as $x = \sum_M c_M [M]$ we have

$$(x, x)_{\mathcal{M}} = \sum_M \frac{c_M^2}{a_S(M)} > 0,$$

a contradiction. Therefore x must be a scalar multiple of ξ. From Proposition 4.4.6 we conclude $\langle \xi \rangle \simeq V_\sigma(\lambda_\xi)_\nu$.

Next, suppose that ξ_1 and ξ_2 are two distinct cuspidals. An arbitrary element of $\langle \xi_1 \rangle$ is of the form $p(F_i)\xi_1$ for some non-commutative polynomial p. Then $(p(F_i)\xi_1, \xi_2)_{\mathcal{M}}$ is zero, as follows from the adjointness of E_i and F_i and the fact that ξ_2 is cuspidal. This shows that $\langle \xi_1 \rangle$ and ξ_2 are orthogonal. Arguing in the same way we find $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ are orthogonal. Hence we have $\bigoplus_{\xi \in \mathcal{C}_Q^{\mathfrak{g}}} V_\sigma(\lambda_\xi)_\nu \hookrightarrow \mathcal{M}_Q^{\mathfrak{g}}$.

To prove that this inclusion is an isomorphism note that the restriction of $(\cdot, \cdot)_{\mathcal{M}}$ to $\langle \xi \rangle$, and hence to $\bigoplus_{\xi \in \mathcal{C}_Q^{\mathfrak{g}}} V_\sigma(\lambda_\xi)_\nu$, is non-degenerate. Let $0 \neq \zeta \in \mathcal{M}_Q^{\mathfrak{g}}$ be orthogonal to $\bigoplus_{\xi \in \mathcal{C}_Q^{\mathfrak{g}}} V_\sigma(\lambda_\xi)_\nu$ and of minimal dimension with this property. In particular, ζ is not cuspidal. Hence there exists $i \in Q_0$ so that $E_i \zeta$ is non-zero. By the minimality assumption on ζ, $E_i \zeta \in \bigoplus_{\xi \in \mathcal{C}_Q^{\mathfrak{g}}} V_\sigma(\lambda_\xi)_\nu$. Since $F_i E_i \zeta \in \bigoplus_{\xi \in \mathcal{C}_Q^{\mathfrak{g}}} V_\sigma(\lambda_\xi)_\nu$ it follows that

$$(E_i \zeta, E_i \zeta)_{\mathcal{M}} = (\nu^{-2} - 1)(\zeta, F_i E_i \zeta)_{\mathcal{M}} = 0,$$

contradicting $E_i \zeta \neq 0$. This completes the proof. □

For all quivers the trivial self-dual representation $[0]$ is cuspidal. We define the composition submodule of $\mathcal{M}_Q^{\mathfrak{g}}$ by $\mathbf{M}_Q^{\mathfrak{g}} = \langle [0] \rangle$. The proof of Theorem 4.4.7 shows $\mathbf{M}_Q^{\mathfrak{g}} \simeq V_\sigma(\lambda_{[0]})_\nu$. In particular, the isomorphism type of $\mathbf{M}_Q^{\mathfrak{g}}$ is independent of the orientation of Q.

For orthogonal representations we have $\mathcal{E}(\epsilon_i) = 0$ for all $i \in Q_0$, from which we conclude $\lambda_{[0]} = 0$ and $\mathbf{M}_Q^{\mathfrak{o}} \simeq V_\sigma(0)_\nu$. A geometric version of this result was previously obtained by Enomoto [14, Theorem 5.12]. In *loc. cit.*, by studying induction and restriction operators on the Grothendieck group ${}^\sigma K(Q)$ of a certain category of perverse sheaves on the moduli stack of orthogonal representations of Q (with $Q_0^\sigma = \emptyset$), Enomoto established an isomorphism of $B_\sigma(\mathfrak{g}_Q)$-modules ${}^\sigma K(Q) \simeq V_\sigma(0)$. He also showed that the simple perverse sheaves in ${}^\sigma K(Q)$ give a lower global basis of $V_\sigma(0)$, giving an orthogonal analogue of Lusztig's construction of the lower global basis of $U_v^-(\mathfrak{g}_Q)$ [42]. See also [45] where orthogonal representations of $Q = A_\infty$ are treated using perverse sheaves. In [67] Enomoto's approach was generalized to construct lower global bases of $V_\sigma(\lambda)$ for more general λ.

In another direction, van Leeuwen [66] studied the Hall module of unipotent characters of the orthogonal, symplectic and unitary groups over finite fields, following the approach of Green [19] and Zelevinsky [71]. In the language of quivers, this can be interpreted as the study of self-dual Hall modules of nilpotent representations of the Jordan quiver. van Leeuwen obtained a complete description of the relationship of the module and comodule structures. Precisely, it is shown that there is a ring homomorphism $\Psi : \mathcal{H}_Q \to \mathcal{H}_Q \otimes \mathcal{H}_Q$ such that $\rho([U] \star [M]) = \Psi([U]) \star \rho([M])$. If Ψ were the coproduct (and therefore first order in the Hall numbers), this would simply be the Hopf module condition. Instead, Ψ is third order in the Hall numbers. The relation (4.5) recovers a particular component of this Ψ-twisted Hopf module structure.[1] It would be very interesting to generalize this result to arbitrary (Q, σ).

[1] While Theorem 4.4.1 is stated for loopless quivers, the verification of relation (4.5) above holds without this assumption.

Chapter 5

Finite type Hall modules

In this chapter we restrict attention to finite type quivers with involution. In this case we completely classify self-dual representations. The classification is then used to obtain a more detailed description of the corresponding Hall modules.

5.1 Classification of self-dual representations over finite fields

Recall that a quiver Q is called finite type if it has only finitely many indecomposable representations over any algebraically closed ground field. A connected finite type quiver is an orientation of an ADE Dynkin diagram, as shown by Gabriel [18]. Moreover, in this case the indecomposable representations are in bijection with the set of positive roots $\Delta_Q^{>0}$ of the simple Lie algebra $\mathfrak{g}_Q$ [18].

Definition ([11])**.** *A quiver with involution (Q, σ) is called finite type if it has only finitely many isometry classes of indecomposable self-dual representations over any algebraically closed field whose characteristic is not two.*

We will need the following basic result.

Lemma 5.1.1.

1. *The representation underlying a self-dual indecomposable is either indecomposable or of the form $I \oplus S(I)$ for some indecomposable I.*
2. *Let Q be finite type and suppose that the indecomposable I does not admit a self-dual structure. Then, up to isometry, the hyperbolic form is the unique self-dual structure on $I \oplus S(I)$.*

Proof. The first statement is given in [11, Proposition 2.7]. In *loc. cit.* the authors work over an algebraically closed field but this result and its proof are valid without this assumption.

Before proving the second statement, recall [10] that when Q is finite type there exists a total order $\preceq$ on the indecomposable representations such that $\mathrm{Hom}(I, J) =$

$\mathrm{Ext}^1(J,I)=0$ if $J \prec I$. Such an order is independent of the ground field and will be called an Auslander-Reiten order. Moreover, $\mathrm{Ext}^1(I,I)=0$ and $\mathrm{End}(I)=k$ for any indecomposable I.

As for the second statement, write a self-dual structure ψ on $I \oplus S(I)$ as

$$I \oplus S(I) \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} S(I) \oplus S^2(I).$$

This requires $S(a)\Theta_I = a$. If $I \simeq S(I)$, then $\mathrm{Hom}(I, S(I)) \simeq \mathrm{End}(I) = k$. In this case $a=0$; otherwise a is a self-dual structure on I, a contradiction. Similarly $d=0$. It is now straightforward to see that ψ is isometric to $H(I)$. If instead $I \not\simeq S(I)$, we may without loss of generality assume $S(I) \prec I$. In this case $a=0$ and acting by $Aut(I)$ we may also take $b = 1_{S(I)}$ and $c = \Theta_I$. Then $\begin{pmatrix} 1 & -\frac{1}{2}d \\ 0 & 1 \end{pmatrix}$ is an isometry from ψ to $H(I)$. □

Remark. When k is algebraically closed the second part of Lemma 5.1.1 is true for arbitrary Q [11].

We now give a slight generalization of a result of Derksen-Weyman [11, Theorem 3.1], which describes the pairs (Q,σ) that are finite type when considering only orthogonal and symplectic representations. We find that including unitary representations case does not affect the classification.

Theorem 5.1.2. *(Q,σ) is finite type if and only if Q is finite type.*

Proof. Suppose that Q is finite type. If I is an indecomposable that does not admit a self-dual structure, the second part of Lemma 5.1.1 implies that $H(I)$ is the unique self-dual structure on $I \oplus S(I)$. If I does admit a self-dual structure, then the isometry classes of self-dual structures on I are in bijection with the isometry classes of orthogonal or hermitian forms on k, according to the choice of duality functor. Since the latter are finite in number (see for example [62]), so too are the isometry classes of self-dual structures on I. Using the first part of Lemma 5.1.1 we conclude that (Q,σ) is finite type.

Conversely, if Q is not finite type let $\{I_\beta\}_\beta$ be an infinite set of pairwise non-isomorphic indecomposables such that $I_\beta \not\simeq S(I_\gamma)$ if $\beta \neq \gamma$. Then $\{I_\beta, H(I_\beta)\}_\beta$ contains an infinite set of pairwise non-isometric indecomposable self-dual representations. Hence (Q,σ) is not finite type. □

If (Q,σ) is finite type with Q connected, then Q is necessarily of Dynkin type A, as orientations of type DE Dynkin diagrams do not admit involutions. If Q is not connected but (Q,σ) is not a disjoint union of quivers with involution, then $Q = Q'^{\sqcup}$ with Q' an orientation of an ADE Dynkin diagram. All other finite type quivers with involution can be obtained as disjoint unions of the previous two types, so we do not consider these in what follows.

We now show that for the purposes of studying Hall modules of finite type quivers it suffices to restrict attention to orthogonal, symplectic and unitary representations, i.e. $\tau \equiv -1$. We saw this in Chapter 4 for disjoint union quivers. Let Q be of type A with τ, τ' two σ-invariant functions on Q_1 determining duality functors S, S', respectively. For simplicity we take ι to be the identity; non-trivial ι is dealt with similarly. Assume that τ and τ' differ at a single $\alpha_* \in Q_1^+$; the general case can be obtained inductively from this case. Define an autoequivalence F of $\text{Rep}_k(Q)$ by sending (U, u) to (U, u'), where $u_\alpha = u'_\alpha$ if $\alpha \neq \alpha_*$ and $u'_{\alpha_*} = -u_{\alpha_*}$. Then $F_* = 1_{\mathcal{H}_Q}$. The pair (F, id) is a form functor and by Proposition 3.3.5 induces an isomorphism $\mathcal{M}_Q^{\tau} \xrightarrow{\sim} \mathcal{M}_Q^{\tau'}$. If instead τ and τ' differ at $\alpha_* \in Q_1^{\sigma}$ and define F as above but take $S' = -S$. Then $(F, \text{id})_*$ is an isomorphism $\mathcal{M}_Q^{\tau,S} \xrightarrow{\sim} \mathcal{M}_Q^{\tau',S'}$. Since F preserves dimension vectors, and hence $\langle \cdot, \cdot \rangle$ and $\mathcal{E}$, the above isomorphisms extend to Ringel-Hall modules.

Before describing the self-dual indecomposables of type A quivers we recall the ordinary indecomposables. For Q of type A_{2n} or A_{2n+1}, label the nodes $-n, \dots, n$, omitting 0 for A_{2n}, with i and $i+1$ adjacent. Denote by $I_{i,j}$, $i \leq j$, the representation with dimension vector $\epsilon_i + \cdots + \epsilon_j$ and all intermediate structure maps the identity. Over any field, the collection $\{I_{i,j}\}_{-n \leq i \leq j \leq n}$ is a complete set of isomorphism classes of indecomposable representations.

The indecomposable orthogonal and symplectic representations of finite type quivers, over an algebraically closed field k whose characteristic is not two, were classified in [11] and shown to have a partial interpretation in terms of root systems; see Theorem 5.1.3 below. We require a version of this result for finite fields and for unitary representations. Write $\Delta_Q^{\mathfrak{g}}$ and $\overline{\Delta_Q^{\mathfrak{g}}}$ for the self-dual indecomposables over $\mathbb{F}_q$ and $\overline{\mathbb{F}_q}$, respectively.

To describe the self-dual indecomposables we proceed as in the proof of Theorem 5.1.2. The only non-trivial task is to identify the self-dual structures on indecomposable representations, which can be done directly. For $A_{2n}^{\mathfrak{o}}$ and $A_{2n+1}^{\mathfrak{sp}}$ indecomposables do not admit self-dual structures. Hence in these cases the self-dual indecomposables are simply the hyperbolics $\{H(I_{i,j})\}$. For $A_{2n}^{\mathfrak{sp}}$ and $A_{2n+1}^{\mathfrak{o}}$ the indecomposables $I_{-i,i}$ admit exactly two self-dual structures, corresponding to the two non-isometric orthogonal forms on k. For $A_n^{\mathfrak{u}}$ the indecomposables $I_{-i,i}$ admit a unique unitary, corresponding to the unique hermitian form on k.

We denote by R_i^c a self-dual representation with underlying representation isomorphic to $I_{-i,i}$. The superscript c labels the Witt type of the the induced sesquilinear form on $k_i \subset R_i^c$ obtained by composing all structure maps of R_i^c. The label c may be omitted when it is trivial.

Example. The indecomposables of $A_3^{\mathfrak{o}}$ are

$$H(S_1) : k \to 0 \to k, \quad H(I_{0,1}) : k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 0 & -1 \end{pmatrix}} k$$

and

$$R_0^c : 0 \to k \to 0, \quad R_1^c : k \xrightarrow{1} k \xrightarrow{-c} k$$

where in the final two examples the orthogonal form of the central node has Witt index c, identified with either 1 or a fixed element of $k^2 \backslash k$. ◁

Given a root system Δ, let Δ^2 and Δ^δ be the decorated root systems obtained from Δ by giving each short root of Δ multiplicity two. In Δ^2 the doubled roots are viewed as independent, whereas those in Δ^δ are viewed as being split from a single short root. In the iteration $\Delta^{2,\delta}$ each short root has multiplicity three; two viewed as split from a single short root and the third viewed as independent. With this notation we can now give a self-dual analogue of Gabriel's theorem for finite fields, refining the algebraically closed version of [11, Propositions 3.6, 3.8].

Theorem 5.1.3. *1. The indecomposable self-dual $\mathbb{F}_q$-representations of $Q^{\sqcup}$ are in bijection with $\Delta_Q^{>0}$.*

2. The indecomposable self-dual $\mathbb{F}_q$-representations of A_{2n+1} are in bijection with

$$\Delta_{A_{2n+1}}^{\mathfrak{sp}} \simeq C_{n+1}^{>0}, \quad \Delta_{A_{2n+1}}^{\mathfrak{o}} \simeq B_{n+1}^{>0,\delta}, \quad \Delta_{A_{2n+1}}^{\mathfrak{u}} \simeq B_{n+1}^{>0}$$

3. The indecomposable self-dual $\mathbb{F}_q$-representations of A_{2n} are in bijection with

$$\Delta_{A_{2n}}^{\mathfrak{o}} \simeq BC_n^{>0}, \quad \Delta_{A_{2n}}^{\mathfrak{sp}} \simeq B_n^{>0,2,\delta}, \quad \Delta_{A_{2n}}^{\mathfrak{u}} \simeq B_n^{>0,2}.$$

Proof. The self-dual indecomposables of $Q^{\sqcup}$ are bijection with the indecomposables of Q. By Gabriel's theorem, the latter are in bijection with $\Delta_Q^{>0}$. This establishes the first statement.

We describe the bijections for $A_{2n}^{\mathfrak{g}}$; the discussion for $A_{2n+1}^{\mathfrak{g}}$ is similar. To fix notation, set $B_n^{>0} = \{\epsilon_i \pm \epsilon_j, \epsilon_i \mid 0 \leq i \leq j \leq n-1\}$ and $BC_n^{>0} = B_n^{>0} \bigsqcup \{2\epsilon_i\}_{i=0}^{n-1}$. For $A_{2n}^{\mathfrak{o}}$ we assign $2\epsilon_{n-i}$ to the indecomposable $H(I_{-i,i})$. Consistency then requires

$$H(I_{i,j}) \mapsto \epsilon_{n-j} - \epsilon_{n-i+1}, \quad 1 \leq i \leq j \leq n.$$

The convention $\epsilon_n = 0$ is used, so that in particular $H(I_{1,j}) \mapsto \epsilon_{n-j}$. Similarly

$$H(I_{-i,j}) \mapsto \epsilon_{n-i} + \epsilon_{n-j}, \quad 1 \leq i < n, 1 \leq j \leq n$$

and we obtain the bijection with $BC_n^{>0}$.

For $A_{2n}^{\mathfrak{sp}}$ the only difference is that we begin by assigning ϵ_{n-i}^c to R_i^c. The assignments for hyperbolics are unchanged. In this case R_i^c and $H(I_{1,i})$ have the same image, giving the short roots multiplicity three.

Finally, for $A_{2n}^{\mathfrak{u}}$ we assign ϵ_{n-i} to R_i and proceed as above. □

For algebraically closed ground fields Theorem 5.1.3 is modified by omitting the δ construction, since the indecomposables $I_{-i,i}$ do not have non-trivial forms. This is the result of [11]. We will write $\overline{\Delta}_Q^{\mathfrak{g}}$ for the corresponding set of self-dual indecomposables.

A weak version of the Krull-Schmidt theorem holds for self-dual representations over $\mathbb{F}_q$. Namely, any self-dual representation decomposes into an orthogonal direct sum of self-dual indecomposables. However, this decomposition is in general non-unique; the Witt type of each isotypic component $R_i^{\oplus m,c}$ is determined but the Witt types of the individual summands are not. In any case, combining this with Theorem 5.1.3 we find that isometry classes of representations of $Q^{\mathfrak{g}}$ are in bijection with a subset of the set of all functions

$$\overline{\Delta}_Q^{\mathfrak{g}} \to \mathbb{Z}_{\geq 0} \times L.$$

When $Q^{\mathfrak{g}}$ has no forms, L is trivial, there are no restrictions on the functions and uniqueness holds in the Krull-Schmidt theorem. When $Q^{\mathfrak{g}}$ has forms, $L = W$ and the W-label of $r \in \overline{\Delta}_Q^{\mathfrak{g}}$ is required to be the identity if the self-dual representation associated to r has no forms and to be of appropriate dimension otherwise.

5.2 Applications to Hall modules

Hall algebras of finite type quivers are particularly simple: the embedding

$$U_{\nu}^{-}(\mathfrak{g}_Q) \hookrightarrow \mathcal{H}_Q$$

of Theorem 4.2.1 is an isomorphism, as follows from Gabriel's theorem and the quantum Poincaré-Birkhoff-Witt theorem. For all other quivers the Hall algebra is much larger than $U_{\nu}^{-}(\mathfrak{g}_Q)$ and lacks an explicit description.

With this in mind we move towards describing the entire Hall module of a finite type quiver. We approach this problem using Theorem 4.4.7. From this point of view, a complete description of the Hall module is equivalent to the classification of cuspidal elements.

We begin with two important examples.

Example. The decomposition of the Hall module $\mathcal{M}_{A_1}^{\mathfrak{g}}$ into irreducible $B_{\sigma}(\mathfrak{sl}_2)$-modules agrees with the Witt decomposition of Proposition 3.3.1. Explicitly,

$$\mathcal{M}_{A_1}^{\mathfrak{sp}} = V_{\sigma}(-2), \qquad \mathcal{M}_{A_1}^{\mathfrak{u}} = V_{\sigma}(-1) \oplus V_{\sigma}(-2)$$

and

$$\mathcal{M}^{\mathfrak{o}}_{A_1} = V_\sigma(0) \oplus V_\sigma(-1)^{\oplus 2} \oplus V_\sigma(-2).$$

Here we have identified the T-weights with $\mathbb{Z}$. ◁

Example. The first non-trivial example is $\mathcal{M}^{\mathfrak{o}}_{A_2}$. We show that $\mathcal{M}^{\mathfrak{o}}_{A_2}$ is generated by $[0]$. For simplicity we work with the untwisted Hall module and the orientation $-1 \to 1$. Let $T = H(S_1)$ and $R = H(I_{-1,1})$. We claim that for all $k + l = t + 2r$ with $k \leq t$ we have

$$G^{T^{\oplus t} \oplus R^{\oplus r}}_{S_{-1}^{\oplus k} \oplus S_1^{\oplus l}, 0} = \begin{bmatrix} t \\ k \end{bmatrix}$$

where

$$[n] = \frac{q^n - 1}{q - 1}, \quad [n]! = \prod_{i=1}^{n} [i], \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

To see this, note that if $S_{-1}^{\oplus k} \oplus S_1^{\oplus l}$ is embedded as a Lagrangian in $T^{\oplus t} \oplus R^{\oplus r}$ then $S_{-1}^{\oplus k} \subset T^{\oplus t}$, since $R^{\oplus r}$ has no kernel. This gives $\begin{bmatrix} t \\ k \end{bmatrix}$ choices for the image of $S_{-1}^{\oplus k}$. The image of $S_1^{\oplus l}$ is determined by the condition that it be orthogonal to the image of $S_{-1}^{\oplus k}$, establishing the formula.

Since $A_2^{\mathfrak{o}}$ has no forms, an arbitrary representation is of the form $T^{\oplus t} \oplus R^{\oplus r}$, $t, r \geq 0$. We prove by induction on r that $[T^{\oplus t} \oplus R^{\oplus r}] \in \mathbf{M}^{\mathfrak{o}}_{A_2}$. The case $r = 0$ follows from $[S_{-1}^{\oplus t}] \star [0] = [T^{\oplus t}]$. Assuming the statement holds for $r < n$, using the Lagrangian Hall number above we find

$$[S_{-1}^{\oplus k} \oplus S_1^{\oplus 2n}] \star [0] = [T^{\oplus k} \oplus R^{\oplus n}] + \sum_{i=1}^{n} \begin{bmatrix} k + 2i \\ k \end{bmatrix} [T^{\oplus k + 2i} \oplus R^{\oplus n - i}].$$

By the inductive hypothesis each term in the sum lies in $\mathbf{M}^{\mathfrak{o}}_{A_2}$, and hence so too does $[T^{\oplus k} \oplus R^{\oplus n}]$. ◁

This example generalizes as follows.

Theorem 5.2.1. *If (Q, σ) is finite type and $Q^{\mathfrak{g}}$ has only hyperbolic representations, then $\mathcal{M}^{\mathfrak{g}}_Q = \mathbf{M}^{\mathfrak{g}}_Q \simeq V_\sigma(\lambda_{[0]})_\nu$.*

Proof. Consider a hyperbolic representation $H(U)$. Writing U as a direct sum of indecomposables we have

$$H(U) \simeq_S \bigoplus_{i=1}^{l} H(I_i)^{\oplus m_i}$$

for positive integers m_i and indecomposables I_i such that $I_i \not\simeq I_j$ and $I_i \not\simeq S(I_j)$ for $i \neq j$. Since Q is finite type, relabeling if necessary we may suppose that $S(I_i) \preceq I_i \prec I_{i+1} \prec \cdots \prec I_l$ for $i = 1, \ldots, l$. This implies $\mathrm{Ext}^1(S(I_i), I_j) = 0$ for all $i \leq j$, which by duality gives $Ext^1(S(I_i), I_j) = 0$ for all $i \geq j$. Hence $\mathrm{Ext}^1(S(U), U) = 0$ and we find

$$[U] \star [0] = \nu^{\mathcal{E}(U)} G_{U,0}^{H(U)} [H(U)],$$

showing $[H(U)] \in \mathbf{M}_Q^{\mathfrak{g}}$. The equality $\mathcal{M}_Q^{\mathfrak{g}} = \mathbf{M}_Q^{\mathfrak{g}}$ now follows from the fact that the Hall algebra of a finite type quiver is generated by simple representations. The isomorphism $\mathcal{M}_Q^{\mathfrak{g}} \simeq V_\sigma(\lambda_{[0]})_\nu$ follows from Theorem 4.4.7. □

Remark. The previous example showed slightly more than Theorem 5.2.1, namely, that the integral form $\mathcal{M}_{A_2,\mathbb{Z}}^{\mathfrak{o}}$ (defined by taking $R = \mathbb{Z}[\nu, \nu^{-1}]$) is generated by [0]. A similar result likely extends to $\mathcal{M}_{A_{2n}}^{\mathfrak{o}}$.

The Hall modules for $A_{2n+1}^{\mathfrak{o}}$, $A_{2n}^{\mathfrak{sp}}$ and $A_n^{\mathfrak{u}}$ are not covered by Theorem 5.2.1. Before describing their (conjectural) decompositions into irreducibles we study an illustrative example.

Example. We claim that $\mathcal{M}_{A_2}^{\mathfrak{sp}}$ is generated by [0] and $[R^+]-[R^-]$. We first show that $\mathbf{M}_{A_2}^{\mathfrak{sp}}$ has a basis consisting of all hyperbolics and all elements of the form $[H(U) \oplus R^+] + [H(U) \oplus R^-]$ and $[H(U) \oplus R^{\oplus 2,-}]$. That $\mathbf{M}_{A_2}^{\mathfrak{sp}}$ contains all hyperbolics follows from the proof of Theorem 5.2.1. Mirroring the calculations in the example of $\mathcal{M}_{A_2}^{\mathfrak{o}}$ above, we find

$$[S_{-1}^{\oplus k} \oplus S_1^{\oplus r}] \star [0] = [T^{\oplus k} \oplus R^{\oplus r,+}] + [T^{\oplus k} \oplus R^{\oplus r,-}] + \sum_c \sum_{i=1}^{r} \begin{bmatrix} k+i \\ k \end{bmatrix} [T^{\oplus k+i} \oplus R^{\oplus r-i,c}]$$

Proceeding by induction on the number r of symmetric summands we find that $\mathbf{M}_{A_2}^{\mathfrak{sp}}$ contains all elements of the form $[H(U) \oplus R^+] + [H(U) \oplus R^-]$. We also have

$$[S_{-1}^{\oplus k} \oplus I^{\oplus n}] \star [0] = |OGr^+(n, 2n)| [T^{\oplus k} \oplus R^{\oplus 2n,+}] = \prod_{i=0}^{n-1} (q^i + 1) [T^{\oplus k} \oplus R^{\oplus 2n,+}]$$

where $OGr^+(n, 2n)$ denotes the Lagrangian Grassmannian of the $2n$-dimensional hyperbolic orthogonal space. It follows that $\mathbf{M}_{A_2}^{\mathfrak{sp}}$ also contains all elements of the form $[H(U) \oplus R^{\oplus 2,-}]$.

Let $U = S_{-1}^{\oplus k} \oplus I^{\oplus i}$. Note that $Ext^1(S(U), U) = 0$. We have

$$\begin{aligned} [U] \star ([R^+] - [R^-]) &= |OGr(i, 2i+1)| \left([H(U) \oplus R^+] - [H(U) \oplus R^-] \right) \\ &= \prod_{j=1}^{i} (q^j + 1) \left([H(U) \oplus R^+] - [H(U) \oplus R^-] \right). \end{aligned}$$

Combining this calculation with those for $\mathbf{M}_{A_2}^{\mathfrak{sp}}$, a basis of $\langle [R^+] - [R^-] \rangle$ consists of the elements $[H(U) \oplus R^+] - [H(U) \oplus R^-]$, with U an arbitrary representation of A_2. We conclude that [0] and $[R^+] - [R^-]$ are the sole cuspidals of $A_2^{\mathfrak{sp}}$. In particular, $\mathcal{M}_{A_2}^{\mathfrak{sp}} = V_\sigma(1)_\nu \oplus V_\sigma(0)_\nu$. ◁

We now generalize this example to $\mathcal{M}^{\mathfrak{sp}}_{A_{2n}}$. Let $W = W(A^{\mathfrak{o}}_1)$ and write $w^+_1, w^-_1 \in W$ for the classes of the one dimensional orthogonal spaces with descriminant one and a non-square, respectively. Set $\xi_0 = [0]$ and for each $1 \leq i \leq n$ define

$$\xi_i = \sum_{\underline{w} \in \{w^+_1, w^-_1\}^i} a_{\underline{w}}[R^{\underline{w}}] \in \mathcal{M}^{\mathfrak{sp}}_{A_{2n}}.$$

Here $R^{\underline{w}} = \bigoplus_{j=1}^i R_j^{w_j}$ and $a_{\underline{w}} = \prod_{j \,\mathrm{odd}} w_j$, with $w^{\pm}_1$ being with $\pm 1 \in \mathbb{Z}$. Let $\lambda^{\mathfrak{sp}}_i$ be the T-weight of λ_{ξ_i}. Explicitly, $\lambda^{\mathfrak{sp}}_n = 0$ and

$$\lambda^{\mathfrak{sp}}_i = \epsilon^{\vee}_{-(i+1)} + \epsilon^{\vee}_{i+1}, \quad i = 0, \dots, n-1$$

where $\epsilon^{\vee}_i$ is dual to $\epsilon_i \in \mathbb{Z}^{Q_0}$. We say that $Q = A_n$ is equioriented if its arrows are $i \to i+1$.

Proposition 5.2.2. *If $Q = A_{2n}$ equioriented, there is an inclusion of $B_\sigma(\mathfrak{sl}_{2n+1})_\nu$-modules*

$$\bigoplus_{i=0}^n V_\sigma(\lambda^{\mathfrak{sp}}_i)_\nu \subset \mathcal{M}^{\mathfrak{sp}}_Q.$$

Proof. For $j > 0$ we have

$$R^{\pm}_j // S_j \simeq_S R^{\pm \eta(-1)}_{j-1} \tag{5.1}$$

with $\eta : \mathbb{F}_q \to \mathbb{Z}$ is the quadratic character. For $w \in \{w^{\pm}_1\}^i$, the orientation assumption implies $R^{\underline{w}}$ contains at most one (necessarily isotropic) subrepresentation isomorphic to S_j if $1 \leq j \leq i$ and none otherwise. Using this and equation (5.1) it is straightforward to verify that the choice of coefficients $a_{\underline{w}}$ ensures that ξ_i is cuspidal. The proposition now follows from Theorem 4.4.7. □

The case $Q = A^{\mathfrak{o}}_{2n+1}$ requires a slight refinement because of its non-trivial Witt group. For each $b \in W$ let $W_b = \{\underline{w} \in \{w^{\pm 1}_1\}^{i+1} \mid \sum_{j=0}^i w_j = b\}$. For each $0 \leq i \leq n$ this gives a partition $\bigsqcup_{b \in W} W^i_b = \{w^{\pm 1}_1\}^{i+1}$. Define

$$\xi^b_i = \sum_{\underline{w} \in W^i_b} a_{\underline{w}}[R^{\underline{w}}] \in \mathcal{M}^{\mathfrak{o}}_{A_{2n+1}}$$

with $R^{\underline{w}} = \bigoplus_{j=0}^i R_j^{w_j}$ and $a_{\underline{w}} = \prod_{j \,\mathrm{odd}} w_j$. Let $\lambda^{\mathfrak{o}}_-$ and $\lambda^{\mathfrak{o}}_i$ be the T-weights of $[R_0^{\oplus 2,-}]$ and ξ^b_i (which is independent of b), respectively. Explicitly,

$$\lambda^{\mathfrak{o}}_- = 2\epsilon^{\vee}_{-1} - 4\epsilon^{\vee}_0 + 2\epsilon^{\vee}_1, \quad \lambda^{\mathfrak{o}}_i = \epsilon^{\vee}_{-(i+1)} - 2\epsilon^{\vee}_0 + \epsilon^{\vee}_{i+1}, \quad i = 0, \dots, n-1$$

and $\lambda^{\mathfrak{o}}_n = 2\epsilon^{\vee}_0$. If $n = 0$ the terms involving $\epsilon^{\vee}_{\pm 1}$ are omitted.

Proposition 5.2.3. *If $Q = A_{2n+1}$ equioriented, there is an inclusion of $B_\sigma(\mathfrak{sl}_{2n+2})_\nu$-modules*

$$V_\sigma(0)_\nu \oplus V_\sigma(\lambda_-^{\mathfrak{o}})_\nu \oplus \bigoplus_{i=0}^{n} V_\sigma(\lambda_i^{\mathfrak{o}})_\nu^{\oplus 2} \subset \mathcal{M}_Q^{\mathfrak{o}}$$

Proof. The submodule $V_\sigma(0)_\nu$ is the composition submodule while $V_\sigma(\lambda_-^{\mathfrak{o}})_\nu$ is generated by $R_0^{2,-}$, which is obviously cuspidal. The summands $V_\sigma(\lambda_i^{\mathfrak{o}})_\nu$ are generated by ξ_i^b. The remainder of the proof is as in Proposition 5.2.2. □

Finally, for unitary representations the statement is much simpler.

Proposition 5.2.4. *For all σ-compatible orientations there are inclusions of $B_\sigma(\mathfrak{g}_Q)_\nu$-modules*

1. $V_\sigma\left(\frac{1}{2}(\epsilon_1^\vee + \epsilon_{-1}^\vee)\right)_\nu \subset \mathcal{M}_{A_{2n}}^{\mathfrak{u}}$
2. $V_\sigma(-\epsilon_0^\vee)_\nu \oplus V_\sigma\left(-3\epsilon_0^\vee + (\epsilon_1^\vee + \epsilon_{-1}^\vee)\right)_\nu \subset \mathcal{M}_{A_{2n+1}}^{\mathfrak{u}}$.

Proof. In each case the first summand is the composition submodule, while for A_{2n+1} the second summand is generated by $[R_0]$. □

We believe that the previous discussion in fact gives a full description of the finite type Hall modules.

Conjecture 5.2.5. *The inclusions in Propositions 5.2.2, 5.2.3 and 5.2.4 are isomorphisms and hold for all σ-compatible orientations.*

One strategy to prove Conjecture 5.2.5 is to develop the character theory for $B_\sigma(\mathfrak{g})$-modules. Precisely, given a $B_\sigma(\mathfrak{g})$-module L with finite dimensional T-weight spaces, define its character by

$$\mathrm{ch}(L) = \sum_{\lambda \in \Phi^\vee} (\dim L_\lambda) e^\lambda$$

where $L_\lambda = \{x \in L \mid T_i x = v^{\lambda(\epsilon_i)} x, \quad i = 1, \dots, n\}$. We first compute the character of the entire Hall module. The form in which we write the character is motivated by Lemma 5.2.7 below.

Proposition 5.2.6. *If (Q, σ) is finite type, then the character of the Hall module is given by*

$$ch(\mathcal{M}_Q^{\mathfrak{g}}) = \frac{e^{\lambda_{[0]}} \prod_{\alpha \in \Delta^{>0,+}} (1 - e^{-(\alpha + \sigma(\alpha))}) \prod_{\alpha \in \Delta^{>0,\sigma}} (1 + e^{-\alpha})^m}{\prod_{\alpha \in \Delta^{>0}} (1 - e^{-(\alpha + \sigma(\alpha))})}$$

where Δ is the root system associated to Q and

$$m = \begin{cases} 2, & \text{for } A_{2n}^{\mathfrak{sp}}, \quad A_{2n+1}^{\mathfrak{o}} \\ 1, & \text{for } A_n^{\mathfrak{u}} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Partition the set of positive roots as $\Delta^{>0} = \Delta^{>0,+} \sqcup \Delta^{>0,\sigma} \sqcup \Delta^{>0,-}$. Here $\Delta^{>0,\sigma}$ is the set of positive roots fixed by σ and $\Delta^{>0,+}$ is a choice of one point from each two-point orbit of σ on $\Delta^{>0}$. First, suppose that all self-dual representations of $Q^{\mathfrak{g}}$ are hyperbolic. In this case the Krull-Schmidt theorem holds and from Theorem 5.1.3 we have

$$\text{ch}(\mathcal{M}_Q^{\mathfrak{g}}) = \frac{e^{\lambda_{[0]}}}{\prod_{\alpha \in \Delta^{>0,+} \sqcup \Delta^{>0,\sigma}} (1 - e^{-(\alpha+\sigma(\alpha))})}.$$

This proves the proposition when $m = 0$. For $A_{2n}^{\mathfrak{sp}}$ and $A_{2n+1}^{\mathfrak{o}}$ the symmetric indecomposables, i.e. those representations corresponding to elements of $\Delta^{>0,\sigma}$, each admit exactly two distinct self-dual structures. Hence each symmetric indecomposable $\alpha \in \Delta^{>0,\sigma}$ contributes to the character a factor of

$$1 + 2e^{-\alpha} + 2e^{-2\alpha} + \cdots = \frac{1+e^{-\alpha}}{1-e^{-\alpha}} = \frac{(1+e^{-\alpha})^2}{1-e^{-2\alpha}},$$

establishing the case $m = 2$. For $A_n^{\mathfrak{u}}$ the symmetric indecomposables admit a unique unitary structure and therefore contribute to a factor of $\frac{1}{1-e^{-\alpha}} = \frac{1+e^{-\alpha}}{1-e^{-2\alpha}}$, which is the case $m = 1$. □

The problem is then to compute the characters $\text{ch}(V_\sigma(\lambda))$. Consider the subalgebra $\mathbb{Q}(v)[T_1, \dots, T_n] \subset B_\sigma(\mathfrak{g})$ generated by $T_1, \dots, T_n$ and let $\mathbb{Q}(v)_\lambda$ be the rank one $\mathbb{Q}(v)[T_1, \dots, T_n]$-module with generator ϕ'_λ and module structure $T_i \cdot \phi'_\lambda = v^{\lambda\ (\epsilon_i)} \phi'_\lambda$. Define a highest weight λ Verma-type module for $B_\sigma(\mathfrak{g})$ by

$$M(\lambda) = B_\sigma(\mathfrak{g}) \otimes_{B_\sigma^{\leq 0}(\mathfrak{g})} \mathbb{Q}(v)_\lambda.$$

Lemma 5.2.7. *For an arbitrary acyclic quiver in with involution*

$$ch(M(\lambda)) = e^\lambda \prod_{\alpha \in \Delta^{>0}} \left(1 - e^{-(\alpha+\sigma(\alpha))}\right)^{-mult(\alpha)}.$$

Proof. This follows at once from the Poincaré-Birkhoff-Witt theorem for $U_v^-(\mathfrak{g})$ and the isomorphism $B_\sigma^-(\mathfrak{g}) \simeq U_v^-(\mathfrak{g})$. □

We believe that the irreducible representation $V_\sigma(\lambda)$ has a resolution of the following form, analogous to the BGG resolution of a highest weight $U_q(\mathfrak{g})$-module [4], [26]:

$$0 \to M(w_0 \star \lambda) \to \cdots \to \bigoplus_{\ell(w)=1} M(w \star \lambda) \to M(\lambda) \to V_\sigma(\lambda) \to 0. \tag{5.2}$$

The surjection $M(\lambda) \twoheadrightarrow V_\sigma(\lambda)$ is canonically defined. One corollary of the resolution (5.2) would be a Weyl character-type formula for $\text{ch}(V_\sigma(\lambda))$. The proof of Conjecture 5.2.5 would then be reduced to the verification of an algebraic identity. We give a number of examples providing some hints as to the likely structure of (5.2).

Example. If $Q^{\mathfrak{g}}$ has only hyperbolic representations and is finite type, by Theorem 5.2.6 the numerator of $\mathrm{ch}(\mathcal{M}_Q^{\mathfrak{g}})$ is the product over the root subsystem $\widetilde{\Delta^{\mathfrak{g}}} \subset \Delta^{\mathfrak{g}}$ obtained by removing from $\Delta^{\mathfrak{g}}$ all indecomposable symmetric roots. From this observation, the Weyl denominator identity for $\widetilde{\Delta^{\mathfrak{g}}}$ and Theorem 5.2.1 we find

$$\mathrm{ch}(V_\sigma(\lambda_{[0]})) = e^{\lambda_{[0]}} \sum_{w \in \tilde{W}} (-1)^{\ell(w)} \mathrm{ch}(M(w \cdot 0)).$$

Here $\tilde{W}$ denotes the Weyl group of $\widetilde{\Delta^{\mathfrak{g}}}$. Explicitly, $\widetilde{\Delta^{\mathfrak{g}}}$ is $\Delta_Q^{>0}$, $B_n^{>0}$ and $D_n^{>0}$ for $Q^{\sqcup}$, $A_{2n}^{\mathfrak{o}}$ and $A_{2n+1}^{\mathfrak{sp}}$, respectively. ◁

Example. The case of $B_\sigma(\mathfrak{sl}_2)$ is easily dealt with: $M(\lambda) = V_\sigma(\lambda)$ and the resolution (5.2) collapses to

$$0 \to M(\lambda) \to V_\sigma(\lambda) \to 0.$$

Note that in this case $\widetilde{\Delta^{\mathfrak{g}}}$ is empty. ◁

Example. For $B_\sigma(\mathfrak{sl}_3)$, the resolution (5.2) should be

$$0 \to M(-\lambda + 1) \to M(\lambda) \to V_\sigma(\lambda) \to 0.$$

Here $\lambda \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. This can be proved directly for $\lambda = 0, \frac{1}{2}, 1$, which is sufficient to prove Conjecture 5.2.5 for $A_2^{\mathfrak{g}}$. This resolution resembles the BGG resolution for $B_1 = A_1$, but with the half-sum of the positive roots replaced by the quarter sum. This must be accounted for in the $\star$ action in (5.2). ◁

To end this chapter we speculate on some general properties of the set of cuspidals $\mathcal{C}_Q^{\mathfrak{g}}$. There is a decomposition $\mathcal{C}_Q^{\mathfrak{g}} = \bigsqcup_{d \in \mathbb{Z}^{Q_0}} \mathcal{C}_Q^{\mathfrak{g}}(d)$ according the homogenous dimension vector of cuspidals. Note that each set $\mathcal{C}_Q^{\mathfrak{g}}(d)$ is finite.

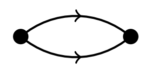
Conjecture 5.2.8. *1. (Q, σ) is finite type if and only if $\mathcal{C}_Q^{\mathfrak{g}}$ is finite.*

2. The cardinality of $\mathcal{C}_Q^{\mathfrak{g}}(d)$ is the specialization at q of a polynomial that depends only on the underlying graph of Q with involution.

Note that Conjecture 5.2.5 is closely related to the finite type part of Conjecture 5.2.8. We give some evidence for Conjecture 5.2.8 in non-finite type cases.

Example. Any element $\xi \in \mathcal{M}_{Q^{\sqcup}}^{\mathfrak{g}}$ is of the form $x + S(x)$ for a unique $x \in \mathcal{H}_Q$. Then ξ is cuspidal if and only if $e_i x = 0$ for all $i \in Q_0$, where e_i denotes the operator of restriction along $[S_i]$ in $\mathcal{H}_Q$. A basis for the R-submodule of such x is a minimal generating set of $\mathcal{H}_Q$, viewed as a $\mathbf{C}_Q$-module. If Q is not finite type then $\mathcal{H}_Q$ is not a finitely generated $\mathbf{C}_Q$-module. This can be seen by comparing the Hilbert series of $\mathcal{H}_Q$ and $\mathbf{C}_Q$, viewed as $\mathbb{Z}^{Q_0}$-graded R-modules. The first part of Conjecture 5.2.8 therefore holds for disjoint union quivers. ◁

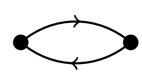
Example. The Kronecker quiver



has a unique involution, swapping the nodes and fixing the arrows. Label a $(1,1)$-dimensional representation by $(\mu, \lambda) \in \mathbb{F}_q^2$, where μ corresponds to the upper arrow. There are $2q+3$ isometry classes of $(1,1)$-dimensional symplectic representations, with representatives $(0,0), (0,1), (0,\delta), (1,\lambda), (\delta,\lambda)$ where $\delta \in \mathbb{F}_q \backslash \mathbb{F}_q^2$ is fixed and $\lambda \in \mathbb{F}_q$ is arbitrary. Then $F_{-1}[0] = [(0,0)]$ and $F_1[0]$ is the sum of all isometry classes of $(1,1)$-dimensional representations. The $(1,1)$-cuspidals can therefore be described as a basis for the $(2q+1)$-dimensional hyperplane in $\mathit{span}\{[(0,1)], [(0,\delta)], [(1,\lambda)], [(\delta,\lambda)]\}_{\lambda \in \mathbb{F}_q}$ orthogonal to the vector

$$[(0,1)] + [(0,\delta)] + \sum_{\lambda \in \mathbb{F}_q} ([(1,\lambda)] + [(\delta,\lambda)]).$$

If we consider instead the equioriented affine Dynkin diagram $\tilde{A}_1$



with the same involution, we may take for representatives of the $(1,1)$-cuspidals $[(1,0)] - [(\delta,0)]$ and $[(0,1)] - [(0,\delta)]$ together with $\{[(1,\lambda)], [(\delta,\lambda)]\}_{\lambda \neq 0}$. There are again $2q+1$ cuspidals. $\triangleleft$

Example. If U is a simple representation that is not isomorphic to S_i for some $i \in Q_0$, then $\{H(U)^{\oplus m}\}_{m \in \mathbb{N}}$ is an infinite set of cuspidals. $\triangleleft$

Chapter 6

Self-dual Hall polynomials

In this final chapter we study the dependence of self-dual Hall numbers of finite type quivers on the finite field $\mathbb{F}_q$. We prove existence of universal polynomials specializing to self-dual Hall numbers for equioriented type A quivers without forms. The situation is more complicated when forms are present. In this case we prove that universal polynomials exist only once the residue of q modulo four is fixed.

6.1 Definitions and reduction to simple case

We begin by recalling the definition of Hall polynomials [56]. Fix a finite type quiver Q with associated root system Δ. By Gabriel's theorem, isomorphism classes of representations of Q are labelled, independently of the ground field, by functions $\Delta^{>0} \rightarrow \mathbb{Z}_{\geq 0}$. Write $U(\alpha)$ for the representation assigned to $\Delta^{>0} \xrightarrow{\alpha} \mathbb{Z}_{\geq 0}$. For each triple of such functions α, β, γ there exists a polynomial $f^{\beta}_{\alpha,\gamma} \in \mathbb{Z}[\mathfrak{q}]$ such that

$$f^{\beta}_{\alpha,\gamma}(q) = F^{U(\beta)}_{U(\alpha),U(\gamma)}$$

for all prime powers q. In this equation the Hall numbers are defined with respect to the ground field $\mathbb{F}_q$. The polynomials $f^{\beta}_{\alpha,\gamma}$ are called Hall polynomials and arise in many areas of representation theory. For example, Hall polynomials are closely related to the Poincaré-Birkhoff-Witt basis for Lusztig's integral form of $U_v^-(\mathfrak{g}_Q)$. From a different perspective, Hall polynomials and their generalizations to polynomials counting rational points of quiver Grassmannians play an important role in the theory of cluster algebras.

When Q admits an involution, the discussion after Theorem 5.1.3 defined a bijection between the isometry classes of representations of $Q^{\mathfrak{g}}$ and certain functions $\overline{\Delta}^{\mathfrak{g}} \rightarrow \mathbb{Z}_{\geq 0} \times L$. Moreover, this bijection was independent of the (finite) ground field. This allows us to make the following definition.

Definition. *Fix functions $\alpha : \Delta^{>0} \rightarrow \mathbb{Z}_{\geq 0}$ and $\chi, \omega : \overline{\Delta}^{\mathfrak{g}} \rightarrow \mathbb{Z}_{\geq 0} \times L$ determining a representation $U(\alpha)$ and self-dual representations $M(\chi), M(\omega)$. A polynomial $g^{\omega}_{\alpha,\chi} \in$*

$\mathbb{Q}[q]$ *satisfying*

$$g^{\omega}_{\alpha,\chi}(q) = G^{N(\omega)}_{U(\alpha),N(\chi)}$$

for all odd prime powers q is called a self-dual Hall polynomial. In the above equality the self-dual Hall number is defined for $\mathbb{F}_q$-representations.

We need two slight modifications of the previous definition. In the unitary case only finite fields $\mathbb{F}_{q^2}$ should be considered and the polynomials $g^{\omega}_{\alpha,\chi}$ are specialized to q as opposed to q^2. If instead there exists a pair of polynomials $g^{\omega}_{\alpha,\chi}, g'^{\omega}_{\alpha,\chi}$ specializing to self-dual Hall numbers for all prime powers $q \equiv 1, 3 \pmod 4$, respectively, we refer to the pair $\{g^{\omega}_{\alpha,\chi}, g'^{\omega}_{\alpha,\chi}\}$ as self-dual Hall semi-polynomials.

Example. Self-dual Hall polynomials with integer coefficients exist for $A_1^{\mathfrak{g}}$ and are equal to the polynomials counting $\mathbb{F}_q$-rational points of appropriate isotropic Grassmannians. The existence of these polynomials can be deduced from a Schubert cell decomposition, or alternatively, by a direct counting argument. ◁

Example. The existence of Hall polynomials of a finite type quiver Q together with equation (3.10) show that self-dual Hall polynomials exist for $Q^{\sqcup}$ and have integer coefficients. ◁

Example. Symplectic Hall polynomials do not exist for $A_{2n}^{\mathfrak{sp}}$, $n \geq 2$. Taking the equioriented case for simplicity, we have

$$G^{R_2^+}_{S_2,R_1^+} = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} := \left\{ \begin{array}{ll} 1, & \eta(-1) = 1 \\ 0, & \eta(-1) = -1 \end{array} \right.$$

We have introduced the notation $\{\cdot\}$ as a shorthand for the rightmost expression. However in this example self-dual Hall semi-polynomials exist

$$g^{R_2^+}_{S_2,R_1^+} = 1, \quad g'^{R_2^+}_{S_2,R_1^+} = 0.$$

◁

Motivated by these examples we make the following conjecture.

Conjecture 6.1.1. *Let (Q, σ) be finite type.*

1. If $Q^{\mathfrak{g}}$ has no forms, then self-dual Hall polynomials exist and have coefficients in $\mathbb{Z}$.

2. If $Q^{\mathfrak{g}}$ has forms, then self-dual Hall semi-polynomials exist and have coefficients in $\mathbb{Z}[\frac{1}{2}]$.

Having established Conjecture 6.1.1 for disjoint union quivers above, we take Q to be of type A. We first reduce Conjecture 6.1.1 to showing that the self-dual Hall numbers of the form $G^{N}_{S_i,M}$ are (semi-)polynomial in q. We will refer to this as the simple case. Let R be $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{2}]$ as in Conjecture 6.1.1.

Proposition 6.1.2. *Let (Q, σ) be of Dynkin type A. If simple self-dual Hall (semi-) polynomials exist, then self-dual Hall (semi- polynomials exist.*

Proof. The proof uses the reduction method of [56] and ultimately relies on Theorem 3.2.3 and the fact that the Hall algebra of a finite type quiver is generated by simples. We will prove the proposition for self-dual Hall polynomials; the argument for semi-polynomials is the same.

We are given that $g^N_{S_i,M} \in R[q]$ exists for all M, N with the desired properties. Assuming $g^N_{S_i^{\oplus d},M} \in R[q]$ exists, define

$$g^N_{S_i^{\oplus(d+1)},M} = \frac{1}{[d+1]} \sum_P g^N_{S_i,P} g^P_{S_i^{\oplus d},M} \in \mathbb{Q}(q). \tag{6.1}$$

Since $[S_i][S_i^{\oplus d}] = [d+1][S_i^{\oplus(d+1)}]$, equation (3.2) shows that $g^N_{S_i^{\oplus(d+1)},M}$ specializes to $G^N_{S_i^{\oplus(d+1)},M}$ for each odd prime power q. This ensures $g^N_{S_i^{\oplus d},M} \in \mathbb{Q}[q]$; see for example [54, Proposition 6.1]. From equation (6.1), we have $[d+1]g^N_{S_i^{\oplus(d+1)},M} \in R[q]$ which in turn implies $g^N_{S_i^{\oplus(d+1)},M} \in R[q]$. Hence self-dual Hall polynomials exist when the ordinary representation is an isotypic direct sum of simples.

We now proceed by induction on $\dim U$. If $U = 0$ put $g^N_{U,M} = \delta^N_M$. Suppose self-dual Hall polynomials exist for representations with dimension less than $\dim U$. Writing U interms of indecomposables as $U = \bigoplus_{i=1}^r I_i^{\oplus m_i}$ with $I_1 \prec \cdots \prec I_r$, $m_i \geq 1$. If $r \geq 2$, put $U' = I_1^{\oplus m_1}$, $U'' = \bigoplus_{i=2}^r I^{\oplus m_i}$ and define

$$g^N_{U,M} = \sum_P g^N_{U',P} g^P_{U'',M} \in R[q].$$

Specializing to q, equation (3.2) shows

$$g^N_{U,M}(q) = \sum_V F^V_{U',U''} G^N_{V,M}.$$

But $F^V_{U',U''} = \delta^V_U$ by definition of U' and U'' and it follows that $g^N_{U,M}(q) = G^N_{U,M}$.

Finally, if $U = I^{\oplus m}$ write $\mathbf{dim}\, U = \sum_{i \in Q_0} d_i \epsilon_i$ with $S_i \prec S_j$ if $i < j$. Then

$$[U] = [S_1^{\oplus d_1}] \cdots [S_r^{\oplus d_r}] - \sum_V [V] \in \mathcal{H}_Q,$$

the sum being over classes of representations V, $V \not\simeq U$, with dimension vector $\mathbf{dim}\, U$. Then

$$g^N_{U,M} = \sum_{N_1,\ldots,N_r} g^N_{S_1^{\oplus d_1},N_1} \cdots g^{N_r}_{S_r^{\oplus d_r},M} - \sum_V g^N_{V,M} \in R[q]$$

specializes at q to $G^N_{U,M}$. □

6.2 Equioriented self-dual Hall polynomials

In this section we use Proposition 6.1.2 to prove Conjecture 6.1.1 for equioriented type A quivers.

Theorem 6.2.1. *Conjecture 6.1.1 holds for equioriented type A quivers.*

Proof. Fix $i \in Q_0$. An arbitrary self-dual representation N can be uniquely decomposed as $N = N_1 \oplus N_2$ where N_1 contains all isotypic summands of N into which S_i can be isotropically embedded. Then $G^N_{S_i,M}$ is non-zero only if $M = M' \oplus N_2$, in which case $G^N_{S_i,M} = G^{N_1}_{S_i,M'}$. It therefore suffices to assume that $N = N_1$.

We first establish two special cases of the theorem and then combine them to prove the general case. Consider a self-dual representation of the form $H(U)$ with

$$U = \bigoplus_{j \leq i, j \neq -i} I_{j,i}^{\oplus m_j}, \quad m_j \geq 0.$$

This is the most general self-dual representation without symmetric summands and containing S_i as an isotropic subrepresentation. If $S_i \simeq V \subset H(U)$ is isotropic, then necessarily $V \subset U$ and $H(U)//V \simeq_S H(U/V)$. This gives

$$G^{H(U)}_{S_i,H(X)} = F^U_{S_i,X}. \tag{6.2}$$

Next suppose that $N = R_i^{\oplus m,c}$. The structure maps of N define a bilinear form B on N_i. This leads to a partition of the space of lines in N_i

$$\mathbb{P}N_i \simeq \mathbb{P}^{m-1} = \mathcal{Q}_+ \sqcup \mathcal{Q}_0 \sqcup \mathcal{Q}_-$$

with $\mathcal{Q}_b = \{\ell \in \mathbb{P}N_i \mid \eta(B(\ell,\ell)) = b\}$. If needed we will write $\mathcal{Q}^c_b$ to indicate the type $c \in W$ of B. The partition determines the reduction type $N//V$ as follows:

$$R_i^{\oplus m,c}//V \simeq_S \begin{cases} R_i^{\oplus m-2,c} \oplus H(I_{-(i-1),i}), & \text{if } V \in \mathcal{Q}_0 \\ R_i^{\oplus m-1,c-b} \oplus R_i^b//S_i, & \text{if } V \in \mathcal{Q}_b, \quad b = \pm 1. \end{cases} \tag{6.3}$$

To see this, let $v \in V$ and write $\overline{v} \in N_{-i}$ for the unique element with image v under the structure maps of N. If $V \in \mathcal{Q}_0$, then $\langle \overline{v}, v \rangle = 0$ and the subrepresentation generated by $\overline{v}$, written $\langle \overline{v} \rangle$, is isotropic and isomorphic to $I_{-i,i}$. Hence $N \simeq_S H(\langle \overline{v} \rangle) \oplus R_i^{\oplus m-2,c}$ with $V \subset H(\langle \overline{v} \rangle)$ and the first equality follows. Similarly, if $V \in \mathcal{Q}_b$ with $b = \pm 1$, then $\langle \overline{v} \rangle$ is isometric to R_i^b. Hence $N \simeq_S R_i^{\oplus m-1,c-b} \oplus R_i^b$ with $V \subset R_i^b$ and the second equality follows.

Next we show that equation (6.3) gives polynomial self-dual Hall numbers. Identify $w_i^{\pm} \in W$ with $\pm 1 \in \mathbb{Z}$. Consider first the orthogonal and symplectic cases. Using

[41, Theorems 6.24, 6.27] we compute

$$|\mathcal{Q}^c_0| = \begin{cases} [m-1], & \text{if } m \text{ is odd} \\ [m-1] + cq^{\frac{m-2}{2}}, & \text{if } m \text{ is even} \end{cases}$$

and

$$|\mathcal{Q}^c_\pm| = \begin{cases} \frac{1}{2}(q^{m-1} \pm cq^{\frac{m-1}{2}}), & \text{if } m \text{ is odd} \\ \frac{1}{2}(q^{m-1} - cq^{\frac{m-2}{2}}), & \text{if } m \text{ is even.} \end{cases}$$

Applying equation (6.3) we get

$$G^{R_i^{\oplus m,c}}_{S_i, R_i^{\oplus m-2,c} \oplus H(I_{-(i-1),i})} = |\mathcal{Q}^c_0|$$

which is polynomial in q with integer coefficients. If $Q^{\mathfrak{g}}$ has no forms the sets $\mathcal{Q}^c_\pm$ are empty. Otherwise, we must deal with Hall numbers counting quotients of the second type in equation (6.3).

When m is odd, we find

$$G^{R_i^{\oplus m,c}}_{S_i, R_i^{\oplus m-1, w_2^+} \oplus R^c_{i-1}} = \begin{Bmatrix} |\mathcal{Q}^c_c| \\ 0 \end{Bmatrix}, \quad G^{R_i^{\oplus m,c}}_{S_i, R_i^{\oplus m-1, w_2^+} \oplus R^{\overline{c}}_{i-1}} = \begin{Bmatrix} 0 \\ |\mathcal{Q}^c_c| \end{Bmatrix}$$

and

$$G^{R_i^{\oplus m,c}}_{S_i, R_i^{\oplus m-1, w_2^-} \oplus R^{\overline{c}}_{i-1}} = \begin{Bmatrix} |\mathcal{Q}^c_{\overline{c}}| \\ 0 \end{Bmatrix}, \quad G^{R_i^{\oplus m,c}}_{S_i, R_i^{\oplus m-1, w_2^-} \oplus R^c_{i-1}} = \begin{Bmatrix} 0 \\ |\mathcal{Q}^c_{\overline{c}}| \end{Bmatrix}.$$

In these calculations we have used equation (5.1) to determine the dependence of the Hall numbers on $\eta(-1)$. From these expressions it is obvious that once the residue of q modulo four is fixed, the self-dual Hall numbers of the second type in equation (6.3) are polynomial in q with coefficients in $\mathbb{Z}[\frac{1}{2}]$.

When m is even $|\mathcal{Q}^c_\pm|$ is independent of $\pm$ and the self-dual Hall numbers (6.3) for m even are polynomial in q with coefficients in $\mathbb{Z}[\frac{1}{2}]$, with no condition on the residue of q. For example,

$$G^{R_i^{\oplus m, w_2^+}}_{S_i, R_i^{\oplus m-1,c} \oplus R^c_{i-1}} = \begin{Bmatrix} |\mathcal{Q}^{w_2^+}_c| \\ |\mathcal{Q}^{w_2^+}_{\overline{c}}| \end{Bmatrix} = \frac{1}{2}(q^{m-1} - q^{\frac{m-2}{2}}).$$

In the unitary case equation (6.3) simplifies to

$$R_i^{\oplus m} /\!/ V \simeq_S \begin{cases} R_i^{\oplus m-2} \oplus H(I_{-(i-1),i}), & \text{if } W \in \mathcal{Q}_0 \\ R_i^{\oplus m-1} \oplus R_{i-1}, & \text{otherwise.} \end{cases} \tag{6.4}$$

The variety $\mathcal{Q}_0$ is set of all isotropic lines in $\mathbb{F}^m_{q^2}$ with its standard hermitian form.

We have

$$|\mathcal{Q}_0| = \begin{cases} \left[\dfrac{m}{2}\right](q^{m-1}+1), & \text{if } m \text{ is even} \\ \left[\dfrac{m-1}{2}\right](q^{m}+1), & \text{if } m \text{ is odd.} \end{cases}$$

Hence the cardinalities of $\mathcal{Q}_0$ and its complement are polynomial in q with integer coefficients.

We now reduce the polynomality of $G^N_{S_i,M}$ for arbitrary N to the previous two cases. Let $R_i^{\oplus m,c}$, $m \geq 1$, be the symmetric summand of N and suppose that N contains at least one summand of the form $H(I_{j,i})$ with $-i < j$. Write a basis vector of $V \subset N$ as $v = v_1 + v_R$ with v_R the component of v in $R_i^{\oplus m,c}$. There exists $\phi \in \mathrm{Hom}(I_{j,i}, R_i^{\oplus m,c})$ such that $\phi(v_1) = -v_R$. Then

$$1_N + \phi - S(\phi)\psi - \frac{1}{2}S(\phi)\psi\phi \in Aut_S(N)$$

maps v to v_1, showing $N//V \simeq_S N//\langle v_1 \rangle$. If instead all non-symmetric summands of N have $j < -i$ in the previous notation above, we may apply a similar argument to reduce v to v_R, in which case $N//V \simeq_S N//\langle v_R \rangle$.

Summarizing, we see that an arbitrary simple self-dual Hall polynomial $G^N_{S_i,M}$ can be written as a power of q times one of the two types of simple self-dual Hall polynomials described at the beginning of the proof. Since the latter have the desired polynomality properties this completes the proof. □

The proof of Theorem 6.2.1 shows that self-dual Hall polynomials with integer coefficients exist for $A_2^{\mathfrak{sp}}$ (this could also be checked directly). The quivers $A_1^{\mathfrak{o}}$ and $A_2^{\mathfrak{sp}}$ are exceptional in the sense that they are the only finite type quivers with forms for which self-dual Hall polynomials exist.

The cardinalities $|\mathcal{Q}_{\pm}|$ are responsible for both the semi-polynomiality and non-integrality of self-dual Hall numbers. This is related to the fact that $Q_{\pm}$ do no have natural scheme structures. However, the cardinality of $\mathcal{Q}_+ \sqcup \mathcal{Q}_-$ is polynomial in q with integer coefficients, as can be verified from the proof of Theorem 6.2.1. Indeed, this must be the case as the counting polynomial of a variety defined over a finite field (like $\mathcal{Q}_+ \sqcup \mathcal{Q}_-$) must have integer coefficients [54].

Another example in which to investigate self-dual Hall polynomials is the Hall module of nilpotent representations of the Jordan quiver. Partial results in this direction were obtained for symplectic representations [70], where it was shown that Lagrangian Hall numbers $G^N_{U,0}$ are polynomial in q.

6.3 Generic Hall modules

We briefly discuss an application of self-dual Hall polynomials to Hall modules.

The existence of Hall polynomials $f^{\gamma}_{\alpha,\beta} \in \mathbb{Z}[q]$ for a finite type quiver Q allows a direct definition of the generic Hall algebra [58]

$$\underline{\mathcal{H}}_Q = \bigoplus_{\alpha:\Delta^{>0}\to\mathbb{Z}_{\geq 0}} \mathbb{Z}[v,v^{-1}]u_{\alpha}$$

with multiplication given by

$$u_{\alpha}u_{\beta} = v^{-\langle \beta,\alpha\rangle}\sum_{\gamma} f^{\gamma}_{\alpha,\beta}(v^{-2})u_{\gamma}.$$

Here v is an indeterminate satisfying $v^{-2} = q$. Theorem 4.2.1 is upgraded to the statement that $\underline{\mathcal{H}}_Q$ is isomorphic to $U^{-}_v(\mathfrak{g}_Q)_{\mathbb{Z}}$, Lusztig's integral form of the quantum group [57]. In particular, $\underline{\mathcal{H}}_Q$ is a coalgebra whose structure constants specialize at q^{-1} to the coalgebra structure constants of $\mathcal{H}_Q$.

In situations where self-dual Hall polynomials exist we can follow the above approach and define the generic Hall module by

$$\underline{\mathcal{M}}^{\mathfrak{g}}_Q = \bigoplus_{\omega:\overline{\Delta}^{\mathfrak{g}}\to\mathbb{Z}_{\geq 0}\times L} R[v,v^{-1}]\xi_{\omega}$$

with $\underline{\mathcal{H}}_Q$-module structure

$$u_{\alpha}\star\xi_{\chi} = v^{-\langle\chi,\alpha\rangle-\mathcal{E}(\alpha)}\sum_{\omega} g^{\omega}_{\alpha,\chi}(v^{-2})\xi_{\omega}.$$

There are two generic Hall modules $\underline{\mathcal{M}}_Q$, $\underline{\mathcal{M}}'_Q$ with structure constants g, g' when $Q^{\mathfrak{g}}$ has forms. We would also like to give $\underline{\mathcal{M}}^{\mathfrak{g}}_Q$ a comodule structure. However, to do this a version of Conjecture 6.1.1 is needed for the comodule structure constants. In the following basic examples the comodule structure can be defined directly and we obtain non-specialized a result from the previous chapter.

Example. The generic Hall module $\underline{\mathcal{M}}_{Q^{\sqcup}}$ is well-defined when Q is finite type. Moreover, a simple argument using the existence of Hall polynomials for comultiplication in $\mathcal{H}_Q$ and equation (3.10) shows that the comodule structure constants of $\mathcal{M}_{Q^{\sqcup}}$ are polynomial in v. Combining this with Theorem 5.2.1 we conclude $\underline{\mathcal{M}}_{Q^{\sqcup}} \simeq V_{\sigma}(0)$. ◁

Example. The comodule structure constants of $\underline{\mathcal{M}}^{\mathfrak{sp}}_{A_1}$ count the reciprocal of the number of $\mathbb{F}_q$-rational points of unipotent radicals of maximal parabolic subgroups of the symplectic group and so are polynomial in v^2. Hence $\underline{\mathcal{M}}^{\mathfrak{sp}}_{A_1}$ is a $B_{\sigma}(\mathfrak{sl}_2)$-module and $\underline{\mathcal{M}}^{\mathfrak{sp}}_{A_1} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v) \simeq V_{\sigma}(-2)$. Similar statements hold for $\underline{\mathcal{M}}^{\mathfrak{o}}_{A_1}$ and $\underline{\mathcal{M}}^{\mathfrak{u}}_{A_1}$. ◁

Finally, we show that in an appropriate sense the generic Hall module is insensitive to the splitting of self-dual Hall polynomials when $Q^{\mathfrak{g}}$ has forms. Define an automorphism $\varphi \in Aut(W)$ by $w^{\pm}_1 = w^{\mp}_1$. Extend this to an automorphism of $Iso(Q^{\mathfrak{g}})$, again denoted by φ. Define an R-linear map $\Psi : \underline{\mathcal{M}}_Q \to \underline{\mathcal{M}}'_Q$ by $[M] \mapsto [\varphi^{\lceil \frac{m}{2} \rceil}(M)]$ where $m = \dim M$ and $\lceil \cdot \rceil$ is the ceiling function.

Theorem 6.3.1. *If Q is an equioriented type A Dynkin quiver then Ψ is a $\underline{\mathcal{H}}_Q$-module isomorphism.*

Proof. We note two symmetries of self-dual Hall semi-polynomials. First, we claim that $g^N_{U,M} = g^{\varphi(N)}_{U,\varphi(M)}$ with an analogous identity holding for $g'^N_{U,M}$. When U is simple the identity is checked directly using the explicit self-dual Hall semi-polynomials computed in the proof of Theorem 6.2.1. The rest of the proof can then be carried out by induction on the Auslander-Reiten quiver, as in Proposition 6.1.2. Second, self-dual Hall polynomials for different residue classes of q in $\mathbb{Z}_4$ are related through the identity $g^N_{U,M} = g'^N_{U,\varphi^u(M)}$. This can be proved in the same way as the first identity.

Denote the action of $\underline{\mathcal{H}}_Q$ on $\underline{\mathcal{M}}_Q$ and $\underline{\mathcal{M}}'_Q$ by $\star$ and $\star'$. Since Ψ preserves dimension vectors it suffices to omit twists. Using the two identities above we verify that Ψ is a homomorphism:

$$\begin{aligned}
\Psi([U] \star [M]) &= \sum_N g^N_{U,M} [\varphi^{\lceil \frac{m}{2} \rceil + u}(N)] \\
&= \sum_N g'^N_{U,\varphi^u(M)} [\varphi^{\lceil \frac{m}{2} \rceil + u}(N)] \\
&= \sum_N g'^{\varphi^{\lceil \frac{m}{2} \rceil + u}(N)}_{U,\varphi^u(M)} [N] \\
&= \sum_N g'^N_{U,\varphi^{\lceil \frac{m}{2} \rceil}(M)} [N] = [U] \star' \Psi([M])
\end{aligned}$$

□

The isomorphism Ψ acts rather simply on the cuspidals described before Conjecture 5.2.5. For $\mathcal{M}^{\mathfrak{sp}}_{A_{2n}}$ the map Ψ acts by ± 1 while for $\mathcal{M}^{\mathfrak{o}}_{A_{2n+1}}$ it acts by the identity or swaps the cuspidal ξ^b_i with $\xi^{b'}_i$. The simple form of Ψ is not surprising in view of Conjecture 5.2.5 and the belief that this result should extend to the generic Hall module.

Bibliography

[1] P. Balmer. Witt groups. In *Handbook of K-theory. Vol. 1, 2*, pages 539–576. Springer, Berlin, 2005.

[2] P. Baumann and C. Kassel. The Hall algebra of the category of coherent sheaves on the projective line. *J. Reine Angew. Math.*, 533:207–233, 2001.

[3] A. Berenstein and J. Greenstein. Primitively generated Hall algebras. arXiv:1209.2770, 2012.

[4] I. Bernšteĭn, I. Gel′fand, and S. Gel′fand. Differential operators on the base affine space and a study of $\mathfrak{g}$-modules. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 21–64. Halsted, New York, 1975.

[5] T. Bridgeland. Hall algebras and curve-counting invariants. *J. Amer. Math. Soc.*, 24(4):969–998, 2011.

[6] T. Bridgeland. An introduction to motivic Hall algebras. *Adv. Math.*, 229(1):102–138, 2012.

[7] T. Bridgeland. Quantum groups via Hall algebras of complexes. *Ann. of Math. (2)*, 177(2):739–759, 2013.

[8] T. Bühler. Exact categories. *Expo. Math.*, 28(1):1–69, 2010.

[9] T. Cramer. Kac's conjecture and the algebra of BPS states. arXiv:1212.5832, 2012.

[10] W. Crawley-Boevey. Lecture on representations of quivers. Available at www1.maths.leeds.ac.uk/~pmtwc/, 1992.

[11] H. Derksen and J. Weyman. Generalized quivers associated to reductive groups. *Colloq. Math.*, 94(2):151–173, 2002.

[12] D.-E. Diaconescu, A. Garcia-Raboso, R. Karp, and K. Sinha. D-brane superpotentials in Calabi-Yau orientifolds. *Adv. Theor. Math. Phys.*, 11(3):471–516, 2007.

[13] M. Douglas and G. Moore. *D*-branes, quivers and *ALE* instantons. hep-th/9603167, 1996.

[14] N. Enomoto. A quiver construction of symmetric crystals. *Int. Math. Res. Not.*, 12:2200–2247, 2009.

[15] N. Enomoto and M. Kashiwara. Symmetric crystals for $\mathfrak{gl}_\infty$. *Publ. Res. Inst. Math. Sci.*, 44(3):837–891, 2008.

[16] B. Fiol and M. Mariño. BPS states and algebras from quivers. *J. High Energy Phys.*, 7:Paper 31, 40, 2000.

[17] I. Frenkel, A. Malkin, and M. Vybornov. Quiver varieties, affine Lie algebras, algebras of BPS states, and semicanonical basis. In *Algebraic combinatorics and quantum groups*, pages 11–29. World Sci. Publ., River Edge, NJ, 2003.

[18] P. Gabriel. Unzerlegbare Darstellungen. I. *Manuscripta Math.*, 6:71–103; correction, ibid. 6 (1972), 309, 1972.

[19] J. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.

[20] J. Green. Hall algebras, hereditary algebras and quantum groups. *Invent. Math.*, 120(2):361–377, 1995.

[21] S. Gukov. Gauge theory and knot homologies. *Fortschr. Phys.*, 55(5-7):473–490, 2007.

[22] S. Gukov, A. Schwarz, and C. Vafa. Khovanov-Rozansky homology and topological strings. *Lett. Math. Phys.*, 74(1):53–74, 2005.

[23] S. Gukov and M. Stosić. Homological algebra of knots and BPS states. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 125–172. Amer. Math. Soc., Providence, RI, 2011.

[24] P. Hall. The algebra of partitions. *Proceedings of the 4th Canadian mathematical congress, Banff*, pages 147–159i, 1959.

[25] J. Harvey and G. Moore. On the algebras of BPS states. *Comm. Math. Phys.*, 197(3):489–519, 1998.

[26] I. Heckenberger and S. Kolb. On the Bernstein-Gelfand-Gelfand resolution for Kac-Moody algebras and quantized enveloping algebras. *Transform. Groups*, 12(4):647–655, 2007.

[27] K. Hori and J. Walcher. D-brane categories for orientifolds—the Landau-Ginzburg case. *J. High Energy Phys.*, 4:030, 36, 2008.

[28] A. Hubery. From triangulated categories to Lie algebras: a theorem of Peng and Xiao. In *Trends in representation theory of algebras and related topics*, volume 406 of *Contemp. Math.*, pages 51–66. Amer. Math. Soc., Providence, RI, 2006.

[29] J. Jacobson. Finiteness theorems for the shifted Witt and higher Grothendieck-Witt groups of arithmetic schemes. *J. Algebra*, 351:254–278, 2012.

[30] D. Joyce. Configurations in abelian categories. II. Ringel-Hall algebras. *Adv. Math.*, 210(2):635–706, 2007.

[31] V. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[32] S.-J. Kang and O. Schiffmann. Canonical bases for quantum generalized Kac-Moody algebras. *Adv. Math.*, 200(2):455–478, 2006.

[33] M. Kapranov. Eisenstein series and quantum affine algebras. *J. Math. Sci. (New York)*, 84(5):1311–1360, 1997.

[34] M. Kapranov, O. Schiffmann, and E. Vasserot. The Hall algebra of a curve. arxiv:1201:6185, 2012.

[35] M. Kashiwara. On crystal bases of the Q-analogue of universal enveloping algebras. *Duke Math. J.*, 63(2):465–516, 1991.

[36] M. Knebusch. Symmetric bilinear forms over algebraic varieties. In *Conference on Quadratic Forms—1976 (Proc. Conf., Queen's Univ., Kingston, Ont., 1976)*, pages 103–283. Queen's Papers in Pure and Appl. Math., No. 46. Queen's Univ., Kingston, Ont., 1977.

[37] M. Kontsevich and Y. Soibelman. Stability structures, motivic Donaldston-Thomas invariants and cluster transformations. arXiv:0811.2435, 2008.

[38] M. Kontsevich and Y. Soibelman. Motivic Donaldson-Thomas invariants: summary of results. In *Mirror symmetry and tropical geometry*, volume 527 of *Contemp. Math.*, pages 55–89. Amer. Math. Soc., Providence, RI, 2010.

[39] M. Kontsevich and Y. Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. *Commun. Number Theory Phys.*, 5(2):231–352, 2011.

[40] D. Krefl. Wall crossing phenomenology of orientifolds. arXiv:1001.5031, 2010.

[41] R. Lidl and H. Niederreiter. *Finite fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1997. With a foreword by P. M. Cohn.

[42] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):447–498, 1990.

[43] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, 4(2):365–421, 1991.

[44] G. Lusztig. Canonical bases and Hall algebras. In *Representation theories and algebraic geometry (Montreal, PQ, 1997)*, volume 514 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 365–399. Kluwer Acad. Publ., Dordrecht, 1998.

[45] G. Lusztig. Graded Lie algebras and intersection cohomology. In *Representation theory of algebraic groups and quantum groups*, volume 284 of *Progr. Math.*, pages 191–224. Birkhäuser/Springer, New York, 2010.

[46] G. Lusztig. *Introduction to quantum groups*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[47] I. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[48] K. Nagao and M. Yamazaki. The non-commutative topological vertex and wall-crossing phenomena. *Adv. Theor. Math. Phys.*, 14(4):1147–1181, 2010.

[49] H. Nakajima. Quiver varieties and finite-dimensional representations of quantum affine algebras. *J. Amer. Math. Soc.*, 14(1):145–238, 2001.

[50] D. Olive and E. Witten. Supersymmetry algebras that include topological charges. *Phys. Lett. B*, 88(1):97–101, 1978.

[51] J. Polchinski. *String theory. Vol. I and II*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2005.

[52] H.-G. Quebbemann, W. Scharlau, and M. Schulte. Quadratic and Hermitian forms in additive and abelian categories. *J. Algebra*, 59(2):264–289, 1979.

[53] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[54] M. Reineke. Counting rational points of quiver moduli. *Int. Math. Res. Not.*, pages Art. ID 70456, 19, 2006.

[55] C. Riedtmann. Lie algebras generated by indecomposables. *J. Algebra*, 170(2):526–546, 1994.

[56] C. Ringel. Hall algebras. In *Topics in algebra, Part 1 (Warsaw, 1988)*, volume 26 of *Banach Center Publ.*, pages 433–447. PWN, Warsaw, 1990.

[57] C. Ringel. Hall algebras and quantum groups. *Invent. Math.*, 101(3):583–591, 1990.

[58] C. Ringel. From representations of quivers via Hall and Loewy algebras to quantum groups. In *Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 381–401, Providence, RI, 1992. Amer. Math. Soc.

[59] O. Schiffmann. Lectures on Hall algebras. arXiv:math/0611617v2, 2006.

[60] O. Schiffmann and E. Vasserot. The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials. *Compos. Math.*, 147(1):188–234, 2011.

[61] O. Schiffmann and E. Vasserot. Hall algebras of curves, commuting varieties and Langlands duality. *Math. Ann.*, 353(4):1399–1451, 2012.

[62] J.-P. Serre. *Galois cohomology.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[63] B. Sevenhant and M. Van den Bergh. A relation between a conjecture of Kac and the structure of the Hall algebra. *J. Pure Appl. Algebra*, 160(2-3):319–332, 2001.

[64] E. Steinitz. Zur Theorie der Abel'schen Gruppen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 9:80–85, 1901.

[65] B. Szendrői. Nekrasov's partition function and refined Donaldson-Thomas theory: the rank one case. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 8:16pp, 2012.

[66] M. van Leeuwen. An application of Hopf-algebra techniques to representations of finite classical groups. *J. Algebra*, 140(1):210–246, 1991.

[67] M. Varagnolo and E. Vasserot. Canonical bases and affine Hecke algebras of type B. *Invent. Math.*, 185(3):593–693, 2011.

[68] M. Wijnholt. Geometry of particle physics. *Adv. Theor. Math. Phys.*, 13(4):947–990, 2009.

[69] J. Xiao. Drinfeld double and Ringel-Green theory of Hall algebras. *J. Algebra*, 190(1):100–144, 1997.

[70] E. Zabric. Hall polynomials for symplectic groups. II. *J. Algebra*, 181(3):663–696, 1996.

[71] A. Zelevinsky. *Representations of finite classical groups*, volume 869 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.