A New Tangential structure for type IIA.

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Physical motivations

- Swampland program.
- Cobordism conjecture $\Omega^{QG}_{*} = 0$ no global symmetries.

What we want is Ω^{IIA}_{*} ⇝ defects needed to trivialize this group.

We to approximate IIA tangential structure.

- Goal: Introduce a tangential structure that incorporates $\underbrace{F_2}_{\text{quatization}}$ & W_7 anomaly of DMW.

String^h structure.

Mathematical motivations

- Study twists of spectra that are relevant for physics. $\longrightarrow$ allow for computations

- $MTSpin \to ko$ $\qquad$ $\underbrace{MTString \xrightarrow{\frac{1}{2}p_1 = 0} tmf}_{\text{String theory \& elliptic coh.}}$

  $MTSpin^c \to ku$

Why care? B.c. it plays well with the Adams S.S.

$$H^*(ko; \mathbb{Z}/2) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$$

$$H^*(tmf; \mathbb{Z}/2) = \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbb{Z}/2.$$

$\exists$ another change of rings $\qquad H^*(tmf_1(3); \mathbb{Z}/2) = \mathcal{A} \otimes_{\mathcal{E}(2)} \mathbb{Z}/2$

$$\boxed{??} \longrightarrow tmf_1(3)$$

Devalpurkar showed that $\exists$ an orientation

$$\mathrm{MTString}^h \longrightarrow \underbrace{\mathrm{tmf}_1(3)}_{\text{defined on } \Gamma_1(3)}$$

$\exists$ $\mathrm{tmf}_1(n)$ defined on $\Gamma_1(n)$.

Results: - We will relate $\mathrm{MTString}^h$ to manifolds with trivial W_7.

Thm: $\mathrm{MTString}^h[\tfrac{1}{n}] \longrightarrow \mathrm{tmf}_1(n)$.

- Compute homotopy groups of $\mathrm{MTString}^h$ which have applications to questions regarding anomaly cancellation of IIA on certain compactifications.

Slogan: String^h is the spin^c analogue of String.

Review on Spin^c:

A spin^c structure on an oriented vector bundle $V \to X$ is equivalently:

- A trivialization of $\beta_{\mathbb{Z}}(w_2(V))$

$$\beta_{\mathbb{Z}} : H^2(\,;\mathbb{Z}/2) \to H^3(-;\mathbb{Z})$$

- A class $c_1 \in H^2(X;\mathbb{Z})$ and an identification

$$c_1 \bmod 2 = w_2(V)$$

- The data of a complex line bundle L and a Spin structure on $V \oplus L$. (twisted spin structure)

Let $V \to X$ be a spin^c vector bundle, and det line bundle L.

$$\lambda = \frac{p_1}{2}.$$

$$\lambda^c(V) = \lambda(V \oplus L)$$

$$\hookrightarrow \frac{p_1(V) + c_1^2(L)}{2}$$

String^h: A string^h structure on a Spin^c vector bundle $V \to X$ is equiv to:

- A trivialization of $\square_{ku}(\lambda^c(V)) \in ku^7(X)$.

$\square_{ku}$ is the connecting map: $\Sigma^2 ku \xrightarrow{\beta} ku \xrightarrow{\tau} H\mathbb{Z}$.

- A class $c_2^{ku}(V) \in ku^4(X)$ and an identification

$$\tau_0(c_2^{ku}(V)) = \lambda^c(V).$$

(twisted String) For $V \to X$ a virtual vector bundle, then a twisted string structure on $E \to M$ with the data of a map $f: M \to X$ is a string structure on $E \oplus f^* V$.

- A $(\underset{X}{BU}, \underset{V}{S})$-twisted String structure. $S \to BU$ is the tautological bundle.

$$
\begin{tikzcd}
BString \arrow[r] \arrow[d] & BSpin \arrow[r] \arrow[d] & K(\mathbb{Z},4) \arrow[d] \\
\underline{BString^h} \arrow[r] & \underline{BSpin^c} \arrow[r] & \Sigma^7 ku
\end{tikzcd}
$$

$MTString^h$ is complex oriented, and from the last defn.

$$MTString^h \simeq MTString \wedge MU$$

$$
\begin{tikzcd}
MTString^h \arrow[r] & MTString \wedge MU \arrow[r, "\underset{\sim}{\text{AHR}}\ \sigma \wedge M(n)"] & tmf \wedge Tmf_1(n) \arrow[dll, bend left, "A(n) \wedge id"] \\
Tmf_1(n) \wedge Tmf_1(n) \arrow[r, "\mu"] & Tmf_1(n) \arrow[r] & tmf_1(n).
\end{tikzcd}
$$

Absmeier.

Hill - Lawson

$\boxtimes$.

DMW anomaly was studied in the context of the M-theory partition function.

In a particular topological sector:

$$Z \sim \sum_{\alpha \in H^4(X;\mathbb{Z})} (-1)^{f(\alpha)} e^{-\int G_\alpha \wedge G_\alpha}.$$

The W_7 anomaly appears from looking at $\alpha \to \alpha + \underbrace{\gamma}_{\text{torsion valued}}$ gauge transformations.

- DMW show if $W_7(X) \neq 0$, then the above partition function vanishes.

<u>DMW structure</u>: $\mathrm{Spin}^c\langle W_7 \rangle$ structure.

Thm: A string structure implies $W_7 = 0$.

Pf: Uses the defn that involves a trivialization of a K-theory class.

Observe the following commuting square:

$$\begin{tikzcd}
\lambda^c \in H^4(X;\mathbb{Z}) \arrow[r, "\beta_{ku}"] \arrow[d, "id"] & ku^7(X) \arrow[d] \\
\lambda^c \in H^4(X;\mathbb{Z}) \arrow[r, "\beta_{\mathbb{Z}} Sq^2"'] & H^7(X;\mathbb{Z}).
\end{tikzcd}$$

$\Rightarrow \beta_{ku}(\lambda^c) = 0$ by assumption.

$\beta_{\mathbb{Z}} Sq^2(\lambda^c) = W_7 = 0$.

first consider λ^c mod 2.

$\boxed{x}$

Now you can ask DMW structure $\longrightarrow$ String^h ?

- For spin^c manifolds in dim 5 or below, the lift to String^h is unique.

- In dim $6 \leq d \leq 9$. Spin^c & DMW are not the same! But still every DMW structure lifts to a String^h.

- In dim 10, we don't know.

<u>Application</u>: $\Omega_*^{\mathrm{Spin}^c\langle w_? \rangle}(BG)$ G is compact simple, simply connected Lie group.

$$\Omega_*^{\mathrm{String}^h}(x) = \mathbb{Z}[X_2, X_4, X_6, X_8, Y_8, X_{10}, Y_{12}, \dots]$$

these groups in $* = 10$ or below have <u>no torsion</u>.

$\longrightarrow$ No global anomalies.

Remark: $\exists$ String^c structure.

This is given by a trivialization of $\lambda^c = 0$.

Rather than $\Pi_{ku}(\lambda^c) = 0$.

A String^c structure on X, leads to a spin^c structure on LX. $\rightsquigarrow$ parallel of Witten's work for String.

Problem is that it is too strong to be a tangential structure for $\mathrm{II}A$. $\lambda^c = 0 = \lambda(TM \otimes L)$

Future: Add in NS sector $\longrightarrow$ twisted String^h.

Give some insights to the S-duality problem in type $\mathrm{II}B$ on $M \times S^1$ by using String^h.