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K-Theoretic Computation of the Verlinde Ring

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K-Theoretic Computation of the Verlinde Ring

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2018

Acknowledgments

First and foremost, I would like thank my adviser Dan Freed. Dan, you shaped the way I think about math and I am immensely grateful for your continuous support, encouragement, and patience throughout the course of my time at UT.

I thank David Ben-Zvi and Andy Neitzke for being remarkable teachers and for fostering the collaboration between geometry and physics in the department, which I profited from greatly.

I would like to thank my Austin friends: Adrian Clough, Ethan Leeman, Tim Carson, Allison Miller, Robin Neumayer, Soledad Villar, Cornelia Mihaila, Michelle Chu, Sona Akopian, Roberta Guadagni, Andrew Lee, Aaron Royer and Kyle Larson. You made Austin feel like home.

I am grateful to my friends in New York who are always close no matter how far away we live. Especially, I want to thank Liz for helping me through my hardest times.

Finally, I want to thank my family: my parents, Galina and Itsko, and my brother and sister, Vlad and Yelena. Their love and support have made this journey possible and I cannot overstate my gratitude.

K-Theoretic Computation of the Verlinde Ring

by

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The University of Texas at Austin, 2018

Supervisor: Daniel Freed

We compute Verlinde rings of the groups $G = SU(3) \rtimes \mathbb{Z}/2\mathbb{Z}$ and $G = \text{Spin}(8) \rtimes \text{Sym}_3$ at level 1. We use the K-theory formulation developed by Freed, Hopkins and Teleman. More precisely, we compute the twisted equivariant K-theory $K_G^\tau(G)$ where G acts on itself by conjugation. The fusion product, corresponding to the Pontryagin product on the level of K-theory, is partially computed.

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Chapter 1

Introduction

In this thesis we compute the twisted equivariant K-theory groups $K_G^\tau(G)$ for $G = SU(3) \rtimes \mathbb{Z}/2\mathbb{Z}$ and $G = \mathrm{Spin}(8) \rtimes \mathrm{Sym}_3$ with particular K-theory twistings τ. By a theorem of Freed, Hopkins, and Teleman ([FHT11b]), the K-theory group $K_G^\tau(G)$ is isomorphic to the twisted representation group of twisted loop group of G. The interest in this computation lies in the connection of the representations of loop groups to the Chern-Simons topological quantum field theory (TQFT).

A mathematically rigorous definition of a TQFT was given by Atiyah in [Ati88b] after Witten reformulated works of Donaldson and Floer in terms of supersymmetric gauge theory in [Wit88] and found a definition of the Jones polynomial in terms of the Chern-Simons field theory in [Wit89]. Since then, the ideas of TQFTs played integral parts in the study of low dimensional topology (e.g. Seiberg-Witten equation [SW94]), representation theory (e.g. geometric Langlands program [KW07]) and higher category theory (e.g. cobordism hypothesis [Lur09]).

There has recently been a surge of interest in the study of topological phases of matter in condensed matter physics, i.e., the study of materials whose low energy excitations exhibit non-trivial topological properties. It is understood that this non-trivial behavior is described by a TQFT. In condensed matter physics, systems are described by lattice models. Although given a lattice model it is not clear how to determine the TQFT which describes its low energy physics, there are procedures going in the other direction ([Kit03, LW05, WW12]), i.e., given a TQFT one can write down a lattice model whose low energy excitations exhibit the braiding statistics and fusion rules governed by the TQFT. Unlike a fairly abstract definition of a TQFT, lattice models are very concrete descriptions of a physical system and the translation provides a physical intuition for constructions and operations one might write down for TQFTs. It is the literature in this vein that motivated the problem of this thesis. Motivated by the corresponding lattice model description, Barkeshli, Bonderson, Cheng, and Wang described the procedure of gauging a finite symmetry of a 3-2-1 extended TQFT ([BBCW14]). Here, we ask if this procedure applied to the Chern-Simons theory based on a connected compact group results in the Chern-Simons theory based on a non-connected compact group, which it formally should.

A 3-dimensional TQFT (3-2-1 extended, oriented with a p_1-structure) is determined by a modular tensor category $\mathcal{C}$. A modular tensor category is in particular a monoidal, braided, semisimple category with finite number of isomorphism classes of simple objects. The simple objects of $\mathcal{C}$ correspond to particle types of the system and the monoidal product corresponds to fusing particles together. The set of isomorphism classes of $\mathcal{C}$ forms a semiring under direct sum and monoidal product. The Grothendieck group completion of this semiring is called the Verlinde ring.

Chern-Simons theories are examples of *classical* topological field theories. Given a compact Lie group G and a class $l \in H^4(BG, \mathbb{Z})$, called the level, there is a corresponding classical topological (metric-independent) field theory ([Fre95, Fre94, Fre02]). One would then like to quantize this theory, but quantization is a notoriously difficult and non-canonical procedure. In particular, to the best of our knowledge, there does not yet exist a definition that works for all compact Lie groups G. When G is simply-connected, there are several constructions of a modular tensor category $\mathcal{C}_{(G,l)}$ which deserve to be called the quantization of Chern-Simons theory:

(a) The category of representations of the quantum group at a root of unity. (e.g. [Saw06])

(b) The category of representations of affine Lie algebras. (e.g. [HL13]).

(c) Representations of Von Neumann algebras generated by loop group LG on the vacuum representation and Connes fusion product. (e.g. [GF93, Was98])

All three are known to be equivalent to the category of projective positive energy representations of LG at level l as *additive* categories. The categories (a) and (b) are known to be equivalent as modular tensor categories. See [Hen17] for a thorough overview of existing constructions and for a proposal for what the quantum Chern-Simons theory should be as a fully extended theory. It is expected that for any G, the correct category $\mathcal{C}_{G,l}$ should be the category of positive energy representations of LG, but a definition of the modular category structure directly in terms of the loop group is still missing, even for connected groups.

For any compact Lie group G and $l \in H^4(BG, \mathbb{Z})$, Freed, Hopkins, and Teleman ([FHT11b]) prove the isomorphism

$$R^l(LG) \cong K_G^{\tau_l}(G) \tag{1.0.1}$$

of the Grothendieck group of positive energy representations of LG at level l and the twisted equivariant K-theory group $K_G^{\tau_l}(G)$ where G acts on itself by conjugation and τ_l is a twisting constructed from l which we will recall in Section 3.3. Moreover, in [FHT10], the same authors construct the necessary structure on the twisting τ_l so that the push-forward in equivariant K-theory is defined along the multiplication map $\mu : G \times G \to G$, giving rise to a ring structure on $K_G^{\tau}(G)$. When G is simply-connected, the left-hand side of (1.0.1) acquires a ring structure, being the Verlinde ring of one of the modular tensor categories alluded to in the previous paragraph ((a) or (b)). In that case, (1.0.1) is an isomorphism of rings. Hence, the ring $K_G^{\tau_l}(G)$ is a natural candidate for the Verlinde ring of the yet to be defined $\mathcal{C}_{(G,l)}$ for G disconnected.

Let G be a compact group, G_1 be its component of the identity and $F = \pi_0(G)$. Fix $l \in H^4(BG, \mathbb{Z})$ and let $l_1 \in H^4(BG_1, \mathbb{Z})$ be its restriction to BG_1. Formally, this defines an action of F on the quantum Chern-Simons theory of (G_1, l_1). The strongest form a symmetry of a group F on a TQFT is that the theory can be extended to manifolds with a principal F-bundle. In case of Chern-Simons theory based on G_1, the fields of the theory are principal G_1-bundles with connection and the quantization is formally a path integral over the space of fields. Given a 3-manifold X and a principal F-bundle Q, instead of G_1-bundles, we can consider principal G-bundles P with connection and an isomorphism $P \times_G F \cong Q$. When Q is trivial, this recovers the Chern-Simons theory of G_1. The gauging of such symmetry is obtained by summing (with appropriate weights) over all F-bundles. In particular, gauging the F symmetry of the quantum Chern-Simons based on (G_1, l_1) gives the path integral over all G-bundles and hence the Chern-Simons theory based on (G, l).

Although this derivation is formal, a construction of gauging a symmetry of a TQFT provides a candidate for the quantum Chern-Simons theory of (G, l), namely the gauging of the quantum Chern-Simons theory of (G_1, l_1) whose various constructions we discussed above. Such gauging construction is formulated in ([BBCW14]) motivated by the relation to the lattice models. Given a modular tensor category $\mathcal{C}$ and an action of F on $\mathcal{C}$, they write down a category $\mathcal{C}/F$ which corresponds to gauging the action of F. Their construction is based on the work of Kirillov ([Kir04]) on equivariant modular categories and of Etingof, Nikshych, and Ostrik ([ENO05, ENO10]) on fusion categories and group actions on them.

A finite group action of F on a modular tensor category $\mathcal{C}$ is given by an extension of $\mathcal{C}$ to an F-crossed braided tensor category $\mathcal{C}_F^\times$. A more naive notion of an action is a homomorphism $F \to \mathrm{Aut}(\mathcal{C})$ where $\mathrm{Aut}(\mathcal{C})$ is the group of isomorphism classes of auto-equivalences of $\mathcal{C}$. It is shown in ([BBCW14]), that given a naive action of F, there are obstructions to lifting a naive action to a true action and choices of lifts in the case these obstructions vanish. The first obstruction lies in $H^3(BF, \mathcal{A})$ where $\mathcal{A}$ is the group formed by the abelian (quantum dimension 1) simple objects of $\mathcal{C}$. If this obstruction vanishes, there is a $H^2(BF, \mathcal{A})$ torsor of intermediate extensions. For each intermediate extension, there is a second obstruction in $H^4(BF, U(1))$. If the second obstruction vanishes, then there is a $H^3(BF, U(1))$ torsor of extensions to the full group action. This should remind the reader of an extension problem in topology. Indeed, the full action of F on $\mathcal{C}$ is equivalent to a homomorphism $F \to \underline{\mathrm{Aut}}(\mathcal{C})$ where $\underline{\mathrm{Aut}}(\mathcal{C})$ is a certain 3-group of automorphisms of $\mathcal{C}$ [1]. The full action therefore is given by a (homotopy class of a) map of spaces $BF \to B\underline{\mathrm{Aut}}(\mathcal{C})$. The naive action on the other hand corresponds to a map from the 1-skeleton of BF into $B\underline{\mathrm{Aut}}(\mathcal{C})$ which extends to the 2-skeleton. An extension to a full action is a lift of this map to all of BF. The statement about the extension of the group action can be restated as the fact that $\pi_2(B\underline{\mathrm{Aut}}(\mathcal{C})) \cong \mathcal{A}$ and $\pi_3(B\underline{\mathrm{Aut}}(\mathcal{C})) \cong U(1)$.

If we had a definition for $\mathcal{C}_{(G,l)}$ for an arbitrary compact Lie group G, a natural conjecture would be

[1] $\underline{\mathrm{Aut}}(\mathcal{C})$ is the 3-group of invertible module categories of $\mathcal{C}$. It is a sub 3-group of automorphisms of $\mathcal{C}$ when viewed as an object of the 3-category of tensor categories. It is also the looping of the automorphism 4-group of $\mathcal{C}$ when viewed as an object of the 4-category of braided tensor categories ([CGPW16, Section 3.2]).

Conjecture 1. *Let G be a compact Lie group, G_1 be the component of the identity of G and $F = \pi_0(G)$. Let $l_1 \in H^4(BG_1, \mathbb{Z})$ be a level for G_1. Then a choice of $l \in H^4(BG, \mathbb{Z})$ extending l_1 defines an action of F on $\mathcal{C}_{(G_1,l_1)}$ and*

$$\mathcal{C}_{(G_1,l_1)}/F \cong \mathcal{C}_{(G,l)}$$

as modular tensor categories.

Since we only have a definition of $\mathcal{C}_{(G_1,l_1)}$ for G_1 simply connected and we expect $\mathcal{C}_{(G,l)}$ to be equivalent to $Rep^l(LG)$ additively, a more reasonable (and well posed) conjecture for the time being is

Conjecture 2. *Let G, G_1, F, l_1 be as in Conjecture 1. Assume further that G_1 is simply connected. Then a choice of $l \in H^4(BG, \mathbb{Z})$ extending l_1 defines an action of F on $\mathcal{C}_{(G_1,l_1)}$ and*

$$\mathcal{C}_{(G_1,l_1)}/F \cong Rep^l(LG)$$

as abelian categories.

It follows from Serre's spectral sequence of the fibration sequence $BG_1 \to BG \to BF$, that an obstruction to a lift of l_1 to l lies in $H^5(BF, \mathbb{Z}) \cong H^4(BF, U(1))$ and that the set of extensions l is a torsor for $H^4(BF, \mathbb{Z}) \cong H^3(BF, U(1))$. This is precisely the top level obstruction to lifting a naive action of F on a modular tensor category.

Conjecture 3. *Let G, G_1, F, l_1, l be as in Conjecture 2. Then*

$$\mathrm{Ver}(\mathcal{C}_{(G_1,l_1)}/F) \cong K_G^{\tau_l}(G)$$

as rings where the multiplication on $K_G^{\tau_l}(G)$ is the Pontryagin product defined in [FHT10].

The goal of this thesis is to verify the isomorphism of Conjecture 3 in two special cases: $G = SU(3) \rtimes \mathbb{Z}/2$ and $G = \mathrm{Spin}(8) \rtimes \mathrm{Sym}_3$ at $l_1 = 1 \in \mathbb{Z} \cong H^4(BG_1, \mathbb{Z})$. These cases were chosen because the categories $\mathcal{C}_{(G_1,l_1)}/F$ are worked out for them in [BBCW14]. We computed the additive structure of the groups $K_G^{\tau_l}(G)$ in these cases and part of the multiplicative structures. What we were able to compute coincides with results of [BBCW14].

Theorem 1.0.2. *Let $G = SU(3) \rtimes \mathbb{Z}/2$ or $G = Spin(8) \rtimes \mathrm{Sym}_3$. Let $l \in H^4(BG, \mathbb{Z})$ be a class extending $1 \in \mathbb{Z} = H^4(BG_1, \mathbb{Z})$ Then*

$$K_G^{\tau_l}(G) \cong \mathrm{Ver}(\mathcal{C}_{(G_1,l_1)}/F)$$

as abelian groups. Moreover, the restriction of the Pontryagin product on $K_G^{\tau_l}(G)$ to $K_G^{\tau_l}(G) \otimes K_G^{\tau_l}(G_1)$ coincides with the multiplication on $\mathrm{Ver}(\mathcal{C}_{(G_1,l_1)}/F)$.

For the group $G = SU(3) \rtimes \mathbb{Z}/2$, compare the table in Section 8.5 to the computation in [BBCW14, Section X.E]. For $G = \mathrm{Spin}(8) \rtimes \mathrm{Sym}_3$, compare the tables in Section 9.5 to the computation in [BBCW14, Section X.J] where the S-matrix is written down for $\mathcal{C}_{(G_1,l_1)}/F$ [2] or [CGPW16, Appendix B] where the fusion rules are given explicitly. Implicit in the statement of the theorem

[2] The fusion rules can be computed from the S-matrix by the Verlinde formula

is that the structures that we compute are the same for every choice of the extension l of l_1.

The main purpose of this thesis is to compute the K-theory groups appearing in Theorem 1.0.2. In Chapter 2, we review the definition and properties of twisted equivariant K-theory as well as work out some properties of K-theory twistings. In Chapter 3, we review the construction of the twisting τ_l of [FHT10]. It is a part of a 2-dimensional TQFT constructed via a pull-push construction on K-theory similar to the definition of string topology and we review the definition of this TQFT. In Chapter 4, we collect the notation and conventions that deal with compact Lie groups as well as prove some results concerning the Weyl group of semi-direct products $G_1 \rtimes F$ where G_1 is simply-connected. In Chapter 5 we elaborate on the computation of $K^{\tau}_G(G)$ of [FHT11b, part III]. In Chapter 6, we study the properties of the twisting τ_l. In Chapter 7, we study the $R(G)$-module structure on $K^{\tau_l}_G G$, which gives us the part of Pontryagin product we computed. In Chapter 8 and Chapter 9 we compute $K^{\tau_l}_G(G)$ for $G = SU(3) \rtimes \mathbb{Z}/2$ and $G = \mathrm{Spin}(8) \rtimes \mathrm{Sym}_3$ respectively.

Chapter 2

K-Theory Preliminaries

2.1 K-theory of Spaces

In this section, we briefly present the definition of twisted K-theory of Atiyah and Segal ([AS04]). We will be more precise in Section 2.3 when discussing K-theory twistings of stacks as defined by Freed, Hopkins, and Teleman in [FHT11a].

The first definition of K-theory one usually encounters is for compact Hausdorff spaces via the Grothendieck group construction of the commutative monoid of isomorphism classes of complex vector bundles. This functor is represented by Fred($\mathcal{H}$), the space of Fredholm operators on an infinite-dimensional separable Hilbert space with the norm topology. In other words

$$K(X) = [X, \mathrm{Fred}(\mathcal{H})]$$

where $[,]$ denotes the homotopy classes of maps. The simples way to "twist" K-theory is therefore to consider a Fred($\mathcal{H}$)-bundle over a space X and define the twisted K-theory to be the set of homotopy classes of sections. The conjugation action of $U(\mathcal{H})$ on Fred($\mathcal{H}$) descends to an action of the projective unitary group $PU(\mathcal{H})$ and therefore a bundle $P \to X$ of projective Hilbert spaces gives rise to the associated bundle $\mathrm{Fred}(P)$ of Fredholm operators and we may define

$$K^P(X) = \pi_0 \Gamma(\mathrm{Fred}(P))$$

This is the simplest kind of twisted K-theory [1].

Bott periodicy gives rise to a 2-periodic generalized cohomology theory with $K^0(X) = K(X)$. By Brown's representability theorem, it corresponds to a ring spectrum K which Atiyah and Singer ([AS69]) identify with a sequence of spaces of Fredholm operators. Let $\mathrm{Fred}^{(n)}(\mathcal{H}) \subset \mathrm{Fred}(\mathbb{S}_n \otimes \mathcal{H})$ be the subset of odd skew-adjoint operators which graded commute with the Clifford action of Cl_n where $\mathbb{S}_n$ is an irreducible representation of Cl_n and $\mathcal{H}$ is a $\mathbb{Z}/2$-graded Hilbert space. Atiyah and Singer show that when $\mathcal{H}$ has infinite dimensional even and odd components, the space $\mathrm{Fred}^{(n)}(\mathcal{H})$ is homotopy equivalent to $\Omega^n \mathrm{Fred}^{(0)}(\mathcal{H})$. In particular, the spaces $\mathrm{Fred}^{(n)}(\mathcal{H})$ form the K-theory spectrum. The periodicity of K-theory follows from homotopy equivalence $\mathrm{Fred}^{(n)}(\mathcal{H}) \simeq \mathrm{Fred}^{(n+2)}(\mathcal{H})$ which in turn follows from the Morita equivalence $Cl_n \simeq Cl_{n+2}$ and Kuiper's theorem.

We can thus define more general twistings and talk about twisted K-theory in various degrees. The notion of an odd transformation of a $\mathbb{Z}/2$-graded vector space V only uses the decomposition of V as an *unordered* direct sum $V = V^0 \oplus V^1$. A bundle $P \to X$ of projective spaces with two

[1] We implicitly assumed that the bundle P corresponded to a principal PU-bundle with the norm topology. Atiyah and Segal consider more general bundles corresponding to the compact open topology on PU.

infinite-dimensional subbundles P^0, P^1 such that locally $P = \mathbb{P}(\mathcal{H}^0 \oplus \mathcal{H}^1)$ and $P^i = \mathbb{P}(\mathcal{H}^i)$, corresponds to a principal PU^{hom}-bundle where U^{hom} is the group of unitary homogeneous transformations of a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}$ with infinite dimensional even and odd components. The group PU^{hom} acts on $\mathrm{Fred}^{(n)}(\mathcal{H})$ by conjugation and to a bundle P as above we associate the bundle $\mathrm{Fred}^{(n)}(P)$ over X. As n varies, these form a bundle of K-module spectra over X. The P-twisted K-theory of X is then defined by

$$K^{n+P}(X) = \pi_0\Gamma(\mathrm{Fred}^{(n)}(P))$$

There is a general framework ([ABG$^+$14]) for twisted cohomology theories. The ones discussed here are only a subset which appears in geometric examples.

Denote by $Twist'(X)$ the groupoid of twistings of X we just defined, i.e., the groupoid of PU^{hom} principal bundles on X. The set of isomorphism classes in $Twist'(X)$ is given by the set of homotopy classes of maps $X \to BPU^{\mathrm{hom}}$. The classifying space BPU^{hom} is the product of Eilenberg-Maclane spaces [2]

$$BPU^{\mathrm{hom}} = K(\mathbb{Z},3) \times K(\mathbb{Z}/2,1)$$

[2]The group PU^{hom} is $(U \times U)/U(1) \rtimes \mathbb{Z}/2$ with $\mathbb{Z}/2$ acting by permuting the two copies of U. Since U is contractible by Kuiper's theorem, $(U \times U)/U(1) \cong K(\mathbb{Z},3)$ and hence there is a fibration sequence

$$K(\mathbb{Z},3) \longrightarrow BPU^{\mathrm{hom}} \longrightarrow K(\mathbb{Z}/2,1)$$

There is a section induced from a map $\mathbb{Z}/2 \to PU^{\mathrm{hom}}$. The fibration above is therefore associated to a map $K(\mathbb{Z}/2,1) \to BAut_{Top_*}(K(\mathbb{Z},3)) = B(\mathbb{Z}/2)$. Since $\mathbb{Z}/2$ acts trivially on $U(1) \subset U \times U$, this map is trivial.

It follows that the isomorphism classes of twistings of X are given by the set

$$H^1(X, \mathbb{Z}/2) \times H^3(X, \mathbb{Z})$$

and the automorphisms of any twisting are given by

$$H^0(X, \mathbb{Z}/2) \times H^2(X, \mathbb{Z})$$

The set $H^0(X, \mathbb{Z}/2) \times H^2(X, \mathbb{Z})$ classifies $\mathbb{Z}/2$-graded $\mathbb{T}$-principal bundles on X where $\mathbb{T} = U(1)$. Another way to see this bundle is the following. Given an automorphism ϕ of a projective bundle P and a point $x \in X$, the fiber P_x is the projectivization of a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}_x$. Let $L_x^\phi \subset U^{\mathrm{hom}}(\mathcal{H}_x)$ be the $\mathbb{T}$ torsor of lifts of ϕ with the grading 0 if L_x^ϕ belongs to the identity component of $U^{\mathrm{hom}}(\mathcal{H}_x)$ and 1 otherwise. These glue together into a $\mathbb{Z}/2$-graded $\mathbb{T}$-bundle on X whose isomorphism class does not depend on the choices of $\mathcal{H}_x$. The associated $\mathbb{Z}/2$-graded line bundle $L^\phi \times_{\mathbb{T}} \mathbb{C}$ determines a class in $K^0(X)$ and the map $K^P(X) \xrightarrow{\phi_*} K^P(X)$ is the multiplication by this class.

The groupoid $Twist'(X)$ has a symmetric monoidal structure coming from the tensor product of Hilbert spaces. The symmetric structure uses the Koszul sign rule for the tensor product of $\mathbb{Z}/2$-graded vector spaces. As an abelian group, the isomorphism classes of $Twist'(X)$ is the extension of $H^1(X, \mathbb{Z}/2)$ by $H^3(X, \mathbb{Z})$ corresponding to the 2-cocycle $(a, b) \mapsto \beta(a \smile b)$ where β is the Bockstein homomorphism[3].

[3]An element $a \in H^1(X, \mathbb{Z}/2)$ corresponds to a real line bundle L_a with $\omega_1(L_a) = a$ and

There is a relation between the twistings we have thus far defined and $\mathbb{Z}/2$-graded $\mathbb{T}$-gerbes. A low-brow definition of a $\mathbb{Z}/2$-graded $\mathbb{T}$-gerbe on a space X is via the Čech construction, i.e., it is given by an open cover $\{U_i\}$, a $\mathbb{Z}/2$-graded $\mathbb{T}$-bundle L_{ij} over every double intersection $U_{ij} := U_i \cap U_j$, and an isomorphism $L_{ij} \otimes L_{jk} \cong L_{ik}$ over every triple intersection U_{ijk} which satisfies a cocycle condition over quadruple intersections. A twisting P gives such a gerbe via the following construction: let U_i be open sets such that $P\big|_{U_i} \cong \mathbb{P}(\mathcal{H}_i)$ for Hilbert space bundles $\mathcal{H}_i$ over U_i. Let $L_{ij} \subset \hom(\mathcal{H}_i, \mathcal{H}_j)$ be the subset of morphisms lifting the gluing data of P with the grading depending on whether the homomorphisms are even or odd. In [FHT11a], the twistings are defined via the gerbes in the more general setting of twisted K-theory of stacks.

Example 2.1.1. Let $X = S^3$ be the three-sphere and P be a twisting whose isomorphism class is $n \in \mathbb{Z} \cong H^1(X, \mathbb{Z}/2) \times H^3(X, \mathbb{Z})$. Let U_i for $i = 1, 2$ be the open sets $U_i = X \backslash \{p_i\}$ where p_1, p_2 are the oposite poles of X. The twisting P can be trivialized on $U_1 \coprod U_2$ and it corresponds to the $\mathbb{T}$-gerbe given by the complex line bundle $\mathcal{O}(-n)$ over $U_1 \cap U_2 \cong S^2$. The twisted K-theory satisfies

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twisting is given by the complex spinor bundle $\mathbb{S}(L_a)$ by tensoring it with a fixed Hilbert space. For $a, b \in H^1(X, \mathbb{Z}/2)$ the sum of the corresponding twistings is given by the tensor product

$$\mathbb{S}(L_a) \otimes \mathbb{S}(L_b) = \mathbb{S}(L_a \oplus L_b)$$

with $\mathcal{H}$. The projective class of this bundle is

$$\beta(\omega_2(L_a \oplus L_b)) = \beta(a \smile b) \in H^3(X, \mathbb{Z})$$

the Mayer-Vietoris property and there is a long exact sequence

$$\begin{array}{ccccc} K^{0+P}(X) & \longrightarrow & K^{0+P_1}(U_1) \oplus K^{0+P_2}(U_2) & \longrightarrow & K^{0+P_{12}}(U_1 \cap U_2) \\ \uparrow & & & & \downarrow \\ K^{1+P_{12}}(U_1 \cap U_2) & \longleftarrow & K^{1+P_1}(U_1) \oplus K^{1+P_2}(U_2) & \longleftarrow & K^{1+P}(X) \end{array}$$

The restrictions of P to $U_1, U_2, U_1 \cap U_2$ are trivializable and hence the above sequence reduces to

$$\begin{array}{ccccc} K^{0+P}(X) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\alpha} & \mathbb{Z} \oplus \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & K^{1+P}(X) \end{array}$$

We have $K^{0+P_{12}}(U_1 \cap U_2) \cong K^0(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ and we use the basis given by the trivial line bundle $\mathcal{O}(0)$ and the tautological bundle $\mathcal{O}(-1)$. We use the trivialization of P_{12} consistent with the trivialization of P_1. It differs from the trivialization of P_2 by the line $\mathcal{O}(-n)$. The map α above is therefore

$$\begin{pmatrix} 1 & n-1 \\ 0 & -n \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

where we used the relation $\mathcal{O}(-n) = n\mathcal{O}(-1) - (n-1)\mathcal{O}(0)$ in $K^0(S^2)$. We thus get

$$K^{0+P}(X) = \{0\}; \qquad K^{1+P}(X) = \mathbb{Z}/n.$$

One context where twisted cohomology theories arise naturally is the Thom isomorphism. Let $V \to X$ be a real n-dimensional vector bundle and let X^V denote the Thom space of V. Recall that in the case of ordinary cohomology theory, a Thom class $U \in H^n(X^V, \mathbb{Z})$ exists only if V is orientable, in which case we get an isomorphism $H^{\bullet-n}(X,\mathbb{Z}) \cong H^{\bullet}(X^V,\mathbb{Z})$. In a more general case, the Thom isomorphism identifies $H^{\bullet-n}(X,\mathbb{Z}_{or(V)})$ with $H^{\bullet}(X^V,\mathbb{Z})$ where the former is the cohomology with twisted coefficients determined by the $\mathbb{Z}/2$ cover $or(V)$ of orientations of V. The analogous story holds for any generalized cohomology theory given by a spectrum E. Let $\Omega_V(E)$ be the bundle of spectra over X with the fiber over $x \in X$ given by maps of the sphere $\{x\}^{V_x}$ into E, i.e. the fibers are homotopic to $\Sigma^{-n}E$. Globally this bundle might be non-trivial and thus define an E-twisting $-\rho(V)$ of X. The sections of this bundle are manifestly the same as maps $X^V \to E$ and hence

$$E^{\bullet-n-\rho(V)}(X) = E^{\bullet}(X^V)$$

In the case of K-theory, by Atiyah-Singer ([AS04]), the bundle $\Omega_V(K)$ is homotopic to

$$\mathrm{Fred}^{(0)}_{Cl^c(V)}(Cl^c(V)\otimes\mathcal{H}),$$

the space of odd skew-adjoint Fredholm operators which commute with the $Cl^c(V)$ action. We claim that $\mathrm{Fred}_{Cl^c(V)}(Cl^c(V)\otimes\mathcal{H})$ is homotopy equivalent to

$$\mathrm{Fred}^{(0)}_{Cl_n}(\mathbb{S}_n\otimes\mathbb{S}(V)\otimes\mathcal{H}) \cong \mathrm{Fred}^{(n)}(\mathbb{S}(V)\otimes\mathcal{H})$$

To verify it, let M denote Cl_n as an invertible $Cl_n - Cl_n$ bimodule. The orthogonal group $O(n)$ acts projectively on M. We get an associated projective bundle $\mathcal{B}(V)\otimes_{O(n)}\mathbb{P}(M)$ where $\mathcal{B}(V)$ is the bundle of frames of V. The tensor product with this bundle over $Cl^c(V)$ gives the desired equivalence. We leave the details to the reader. We thus get that the twisting $-\rho(V)$ given by

$\mathbb{S}(V) \otimes \mathcal{H}$ satisfies

$$K^{\bullet - n - \rho(V)}(X) \cong K^{\bullet}(X^V)$$

The isomorphism class of $-\rho(V)$ is

$$(w_1(V), \beta(w_2(V)) \in H^1(X, \mathbb{Z}/2) \times H^3(X, \mathbb{Z})$$

When there is no possibility of confusion, we will refer to $\rho(V)$ by V. Note as well that a $Cl^c(V)$-module N on X gives rise to a class in $K^{-n-\rho(V)}(X) = K^0(X^V)$; there is a contractible space of $Cl^c(V)$-linear isomorphisms $Cl^c(V) \otimes \mathcal{H} \cong N \oplus \mathcal{H}'$ where $\mathcal{H}'$ is $\mathbb{Z}/2$-graded with infinite dimensions in both even and odd degrees. The desired class corresponds to the Fredholm operator which vanishes on N and is the identity on $\mathcal{H}$. Perhaps more familiarly, this is the class given by the Atiyah-Bott-Shapiro construction ([ABS64]).

The Thom isomorphism allows us to define the pushforward map. The more general statement of the Thom isomorphism is

$$K^{\bullet - n - \rho(V) + s^*(P)}(X) \cong K^{\bullet + P}(X^V)$$

where P is a twisting on V and s is the 0-section. Given a proper map of manifolds $\phi : X \to Y$ and an isomorphism of twistings

$$P \cong \phi^*(TY) - TX + P_0$$

there is a push-forward map

$$K^P(X) \xrightarrow{\phi_*} K^{P_0 - \dim Y + \dim X}(Y)$$

defined in the following way. There is an embedding $X \xrightarrow{\iota} \mathbb{R}^N \times Y$ such that $\pi_2 \circ \iota = \phi$. The push-forward map is then defined as the following composition

$$\begin{aligned} K^P(X) &\cong K^{\phi^* P_0 - TX + TY}(X) \\ &\cong K^{\phi^* P_0 - \dim X + \dim Y + N}(X^{\nu_\iota}) \\ &\to K^{\phi^* P_0 - \dim X + \dim Y + N}(\Sigma^N Y_+) \\ &\cong K^{P_0 - \dim X + \dim Y}(Y) \end{aligned}$$

where ν_ι is the normal bundle of X in $\mathbb{R}^N \times Y$ and the map

$$K^{\phi^* P_0 - TX + TY + \nu_\iota}(X^{\nu_\iota}) \to K^{\phi^* P_0 + N}(\Sigma^N Y_+)$$

is the Pontryagin-Thom collapse. We have used that the twisting $TY - TX - \nu_\iota$ is trivial. This expression can be made less cumbersome if we include the degree in the data of the twisting, which we will do in a later section.

The Atiyah-Singer index theorem provides an analytic way of computing the pushforward map. Let $\pi : X \to Y$ be a fiber bundle of compact manifolds and let $T(X/Y) \subset TX$ be the kernel of $d\pi$. There is a natural isomorphism of twistings $T(X/Y) = TX - \pi^* TY$. Given a class $\alpha \in K^{\dim X - \dim Y + T(X/Y)}(X)$ represented by a $Cl(T(X/Y))$ module bundle, the Atiyah-Singer index theorem implies that the pushforward $\pi_* \alpha$ equals the index of the Dirac operator acting along the fibers.

2.2 Equivariant K-theory

Let G be a compact Lie group and X be a compact Hausdorff space with a continuous G-action. The equivariant K-theory $K_G(X)$ is the Grothendieck group completion of the monoid of isomorphism classes of G-equivariant complex vector bundles on X. For example, if $X = *$ is a point, then $K_G(*) = R(G)$, the representation ring of G. Segal showed in [Seg68] that in a lot of ways, equivariant K-theory behaves the same way as ordinary K-theory. One defines $K_G^{-n}(X) =: \tilde{K}_G(\Sigma^n X_+)$ where $n \geq 0$ and $\tilde{K}$ denotes reduced K-theory. By periodicity $K_G^{-n}(X) = K_G^{-n-2}(X)$, one extends K_G^n for all $n \in \mathbb{N}$. Just as in non-equivariant case, equivariant K-theory is represented by a space of Fredholm operators. Let $\mathcal{H}$ be a stable G-Hilbert space, i.e. such that every irreducible representation of G occurs infinitely many times in $\mathcal{H}$. Let

$$U_{G-cts}(\mathcal{H}) = \{u \in U(\mathcal{H}) | g \mapsto gug^{-1} \text{ is continuous}\}$$

and let $\mathrm{Fred}_{G-cts}(\mathcal{H})$ be defined analogously. Atiyah and Segal ([AS04, Appendix 3]) prove that $U_{G-cts}(\mathcal{H})$ is equivariantly contractible and that for a G-space X,

$$[X, \mathrm{Fred}_{G-cts}]_G = K_G(X)$$

where $[,]_G$ denotes the set of homotopy classes of G-equivariant maps. Let $\mathcal{H}$ be a $\mathbb{Z}/2$-graded Hilbert space such that $\mathcal{H}^0$ and $\mathcal{H}^1$ are stable G-Hilbert spaces. Defining

$$\mathrm{Fred}^{(n)}_{G-cts}(\mathcal{H}) = \left\{ f \in \mathrm{Fred}_{G-cts}(\mathbb{S}_n \otimes \mathcal{H}) \middle| \begin{array}{l} f \text{ odd, skew-adjoint and} \\ \text{graded commutes with } Cl_n \end{array} \right\},$$

it is shown in [FHT11a, Appendix A.5] that

$$K^n_G(X) = [X, \mathrm{Fred}^n_{G-cts}(\mathcal{H})]_G.$$

Modulo technical difficulties (see [AS04, Section 6]), the equivariant K-twisting of a G-space X is again defined by a G-equivariant projective bundle $P \to X$ which is locally on X the projectivization of a $\mathbb{Z}/2$-graded Hilbert space. It should satisfy the following condition. For every point $x \in X$, there should exist a G_x invariant neighborhood U of x, where $G_x = \mathrm{Stab}_G(x)$, such that $P\big|_U = U \times \mathbb{P}(\mathcal{H})$ where $\mathcal{H}$ is a $\mathbb{Z}/2$-graded Hilbert space with projective action of G_x on $\mathcal{H}$, i.e. with a continuous map $G_x \to PU^{\mathrm{hom}}(\mathcal{H})$. We require $\mathbb{P}(\mathcal{H})$ to be a stable G_x projective bundle in the following sense: the map $G_x \to PU^{\mathrm{hom}}(\mathcal{H})$ defines a $\mathbb{Z}/2$-grading ϵ on G_x by composing with the $\mathbb{Z}/2$-grading of $PU^{\mathrm{hom}}(\mathcal{H})$ and a central extension G_x^τ via pullback

$$\begin{array}{ccc} \mathbb{T} & = & \mathbb{T} \\ \downarrow & & \downarrow \\ G_x^\tau & \longrightarrow & U^{\mathrm{hom}}(\mathcal{H}) \\ \downarrow & & \downarrow \\ G_x & \longrightarrow & PU^{\mathrm{hom}}(\mathcal{H}) \end{array}$$

A (τ, ϵ)-twisted representation of G_x is a representation of G_x^τ on a $\mathbb{Z}/2$-graded vector space such that the central $\mathbb{T}$ acts by its defining representation and $g \in G_x^\tau$ acts by even (odd) homomorphisms if $\epsilon(g) = 0$ ($\epsilon(g) = 1$). The G_x projective bundle $\mathbb{P}(\mathcal{H})$ is called stable if for any (τ, ϵ)-projective representation V, there is a unitary embedding $V \to \mathcal{H}$ of G_x^τ representations. Given such projective bundle P, we form bundles of Fredholm operators $\mathrm{Fred}^{(n)}_{G-cts}(P)$ via the associated bundle construction and define

$$K^{n+P} := \pi_0\Gamma(X, \mathrm{Fred}^{(n)}_{G-cts}(P))^G$$

Consider the simplest case where $X = *$ is a point. Then as above, a twisting gives rise to a central extension

$$1 \to \mathbb{T} \to G^\tau \to G \to 1$$

and a $\mathbb{Z}/2$-grading $\epsilon : G \to \mathbb{Z}/2$. In fact, the category of G-equivariant twistings of a point is equivalent to the category of $\mathbb{Z}/2$-graded central extensions of G as above. The isomorphism classes of the latter are given by the set

$$H^1(BG, \mathbb{Z}/2) \times H^3(BG, \mathbb{Z}).$$

The component in H^1 comes from the grading. The component in H^3 comes from the fact that central extensions are classified by the sheaf cohomology $H^2(BG, \underline{\mathbb{T}})$ where $\underline{\mathbb{T}}$ is the sheaf of continuous functions into $\mathbb{T}$. The isomorphism $H^2(BG, \underline{\mathbb{T}}) \cong H^3(BG, \mathbb{Z})$ is a map in the long exact sequence corresponding to the short exact sequences $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$. It is an isomorphism because the sheaf of continuous maps into $\mathbb{R}$ is fine.

There is an interpretation of the twisted K-theory group $K^{(\tau,\epsilon)}(*)$ in terms of finite dimensional twisted representations. Denote representations of G^τ on which the central $\mathbb{T}$ acts by its defining representation by τ-*twisted representations* of G. Similarly, let (τ, ϵ)-twisted representations of G be the τ-twisted representation of G on a super vector space such that $g \in G^\tau$ preserves the grading if $\epsilon(g) = 0$ and reverses it otherwise. Let $Rep^{(\tau,\epsilon)}(G)$ be the monoid of isomorphism classes of (τ, ϵ)-twisted representations of G and let $Triv^{(\tau,\epsilon)}(G) \subset Rep^{(\tau,\epsilon)}(G)$ be the submonoid generated by super representations for which there exists an odd automorphism super commuting with the action of G^τ. Then

$$K^{(\tau,\epsilon)}(*) = Rep^{(\tau,\epsilon)}(G)/Triv^{(\tau,\epsilon)}(G).$$

This is how twisted K-theory is defined in [FM13, Section 7]. If ϵ is trivial and $V = V^0 \oplus V^1$ is a (τ, ϵ)-twisted representation, then G^τ preserves V^0 and V^1. In that case, $K^{(\tau,0)}$ is the Grothendieck group completion of the monoid of τ-twisted representations of G and $V = V^0 \oplus V^1$ corresponds to the formal difference $V^0 - V^1$.

The isomorphism classes of twistings on a general G-space X are shown by Atiyah and Segal ([AS04, Proposition 6.3]) to coincide with the Borel equivariant cohomology of X. Let $X_G := X \times_G EG$ be the Borel construction where EG is the total space of the universal principal G-bundle. Then

$$H^1(X_G, \mathbb{Z}/2) \times H^3(X_G, \mathbb{Z})$$

is the set of isomorphism classes of G-equivariant twistings on X. As an abelian group, it is again an extension of $H^1(X_G, \mathbb{Z}/2)$ by $H^3(X_G, \mathbb{Z})$ with the 2-cocycle given by $(a, b) \mapsto \beta(a \smile b)$. The π_0 of the automorphisms of a twisting is given by $H^0(X_G, \mathbb{Z}/2) \times H^2(X_G, \mathbb{Z})$.

2.3 K-theory Twistings of Stacks

Let G be a topological group, S be a G-space and $H \lhd G$ be a closed normal subgroup which acts freely on S. Then $K_G(X) \cong K_{G/H}(S/H)$. In fact, the category of G-equivariant vector bundles on S is equivalent to the category of G/H-equivariant vector bundles on S/H. This suggests that K-theory and its twisted variants can be defined on objects which are locally built out of spaces with group actions where a space S with a G action as above is equivalent in an appropriate sense to the space S/H with the action of G/H. Such objects are topological stacks ([Noo05]), but we will refrain from giving a precise definition of those. Instead, we will define topological groupoids, which present topological stacks, as well as the equivalence relation identifying groupoids presenting equivalent stacks. This section mostly presents constructions of [FHT11a].

Definition 2.3.1. A *topological groupoid* X is a groupoid internal to the category of topological spaces. In particular, it consists of a space X_0 of objects, a space X_1 of morphisms and continuous maps

$$\begin{aligned} s, t &: X_1 \to X_0 \\ \circ &: X_1 \underset{X_0}{\times} X_1 \to X_1 \\ \iota &: X_0 \to X_1 \end{aligned}$$

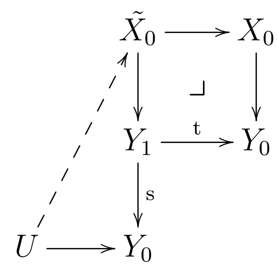
satisfying the standard groupoid relations. Here, s, t are the source and target maps, $\circ$ is the composition, and ι is the map specifying the identity morphisms.

Definition 2.3.2. Let G be a topological group and S be a right G-space. Denote by $S/\!\!/G$ the topological groupoid given by the following data:

$$X_0 = S \qquad X_1 = S \times G$$
$$s(z, g) = z \qquad t(z, g) = z.g$$
$$(z.g_1, g_2) \circ (z, g_1) = (z, g_1 g_2) \qquad \iota(z) = (z, e)$$

where $z \in S$, $g, g_1, g_2 \in G$ and $e \in G$ is the identity element. The groupoids of the form $S/\!\!/G$ are called *quotient groupoids.*

Definition 2.3.3. A *local equivalence* of topological groupoids is a continuous functor $X \to Y$ which is an equivalence of groupoids (when the topology on objects and morphisms is ignored) such that for every object $y \in Y_0$, there exists an open set $U \subset Y_0$ containing y and a lift $U \dashrightarrow \tilde{X}_0$ in the following diagram



Groupoids X, Y are called *weakly equivalent* if there is a zig-zag diagram of local equivalences between X and Y.

A topological groupoid X defines a topological stack by the sheafification of the prestack $U \mapsto U(X)$ where U is a topological space, and $U(X)$ is the groupoid whose objects are continuous maps $U \to X_0$ and whose morphisms are continuous maps $U \to X_1$. A local equivalence of topological groupoids defines an equivalence of the corresponding topological stacks and hence weakly equivalent groupoid represent equivalent topological stacks.

Example 2.3.4. Let S be a topological space and let $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be an open cover of S. There is a groupoid $S_{\mathcal{U}}$ whose space of objects is the disjoint union of the open sets U_i and such for $s_i \in U_i, s_j \in U_j$, the set of morphisms

$S_{\mathcal{U}}(s_i, s_j)$ consists of one element if s_i, s_j correspond to the same point of S and empty otherwise. To summarize

$$(S_{\mathcal{U}})_0 = \coprod_{i \in \mathcal{I}} U_i \qquad\qquad (S_{\mathcal{U}})_1 = \coprod_{i,j \in \mathcal{I} \times \mathcal{I}} U_i \cap U_j$$

The groupoid $S_{\mathcal{U}}$ is locally equivalent to S but it is not equivalent to S as a topological groupoid.

Definition 2.3.5. The *coarse moduli space* $[X]$ of a topological groupoid X is the quotient $X_0/\sim$ where $x_1 \sim x_2$ if there exists a morphism from x_0 to x_1. For $U \subset [X]$ denote by $X\big|_U$ the full subgroupoid of X consisting of objects whose equivalence class lies in U.

Definition 2.3.6. A topological groupoid X is a *local quotient groupoid* if its coarse moduli space $[X]$ admits a countable cover by open sets U such that $X\big|_U$ is weakly equivalent to a global quotient groupoid $S/\!\!/G$ where S is Hausdorff and G is a compact Lie group.

Given a local quotient groupoid X, we will define a Picard 2-groupoid $Twist(X)$ such that for $\tau \in Twist(X)$, an abelian group $K^\tau(X)$ is defined. A map $\phi : X \to Y$ will define the pull-back functor $Twist(Y) \xrightarrow{\phi^*} Twist(X)$ and the pull-back map $K^\tau(Y) \xrightarrow{\phi^*} K^{\phi^*\tau}(X)$. If ϕ is a local equivalence, the the pull-backs are isomorphisms. It is in this sense that the twisted K-theory of topological stacks is defined.

Definition 2.3.7. A $\mathbb{T}$-*bundle* over a groupoid X is a principal $\mathbb{T}$-bundle $\eta \to X_0$ with an isomorphism $s^*\eta \xrightarrow{\sim} t^*\eta$ over X_1 satisfying the obvious associativity condition on $X_1 \underset{X_0}{\times} X_1$.

Definition 2.3.8. A *grading* of a groupoid X is a functor $\epsilon : X \to \mathbb{Z}/2$ into the symmetric monoidal category $\mathbb{Z}/2$ with 2 objects and trivial isotropy groups.

Definition 2.3.9. A $\mathbb{Z}/2$-*graded* $\mathbb{T}$-*bundle* over a groupoid X is a pair (η, ϵ) of a $\mathbb{T}$-bundle and a $\mathbb{Z}/2$-grading of X.

The $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles form a symmetric monoidal category. The space of morphisms $\hom((\eta_1, \epsilon_1), (\eta_2, \epsilon_2))$ is empty unless $\epsilon_1 = \epsilon_2$, in which case it is the space of $\mathbb{T}$-bundle maps $\eta_1 \to \eta_2$. The monoidal product is

$$(\eta_1, \epsilon_1) \otimes (\eta_2, \epsilon_2) = (\eta_1 \otimes \eta_2, \epsilon_1 + \epsilon_2)$$

and the symmetry natural transformation is

$$(a \otimes b) \mapsto (-1)^{\epsilon_1(x)\epsilon_2(x)}(b \otimes a)$$

over a point $x \in X_0$. The category of $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles is equivalent to the category of Hermitian super line bundles. We will sometimes use the two notions interchangeably.

Definition 2.3.10. A *graded* $\mathbb{T}$-*central extension* of a groupoid X is a $\mathbb{Z}/2$-graded $\mathbb{T}$-bundle L over X_1 along with an isomorphism

$$\pi_1^* L \otimes \pi_2^* L \to \circ^* L$$

over $X_1 \underset{X_0}{\times} X_1$ satisfying the obvious cocycle condition.

To spell out the details, a $\mathbb{Z}/2$-graded central extension L assigns a $\mathbb{Z}/2$-graded $\mathbb{T}$-torsor L_f to every morphism $f \in X_1$ and to every pair of composable morphisms f, g it assigns an isomorphism

$$L_g \otimes L_f \to L_{g \circ f}$$

The cocycle condition is that for three composable morphisms f, g, h, the following diagram commutes

$$\begin{array}{ccc} L_h \otimes L_g \otimes L_f & \longrightarrow & L_h \otimes L_{g \circ f} \\ \downarrow & & \downarrow \\ L_{h \circ g} \otimes L_f & \longrightarrow & L_{h \circ g \circ f} \end{array}$$

Equivalently, a $\mathbb{Z}/2$-graded $\mathbb{T}$-central extensions L of X can be given by a grading $\epsilon : X \to * /\!\!/ \mathbb{Z}/2$ and the groupoid $\tilde{X} = (X, L)$ whose space of objects is that of X and whose space of morphisms is a $\mathbb{T}$-bundle over that of X. The structure maps of L define the groupoid structure maps of $\tilde{X}$. There is an obvious morphism $\tilde{X} \to X$ whose fibers are $* /\!\!/ \mathbb{T}$ which is the motivation for the name "$\mathbb{T}$-central extension".

The collection of $\mathbb{Z}/2$-graded $\mathbb{T}$-central extensions of a groupoid X forms a symmetric monoidal 2-groupoid $\mathfrak{Ext}(X)$. The groupoid of morphisms $\mathfrak{Ext}(X)(L^1, L^2)$ consists of $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles η over X_0 equipped with an isomorphism $t^*\eta \otimes L^1 \cong L^2 \otimes s^*\eta$ over X_1, i.e., for a morphism $f \in X(a, b)$ an isomorphism

$$\eta_b \otimes L^1_f \to L^2_f \otimes \eta_a$$

which moreover satisfy the cocycle condition

$$\begin{array}{ccccc} \eta_c \otimes L^1_g \otimes L^1_f & \longrightarrow & L^2_g \otimes \eta_b \otimes L^1_f & \longrightarrow & L^2_g \otimes L^2_f \otimes \eta_a \\ \downarrow & & \circlearrowleft & & \downarrow \\ \eta_c \otimes L^1_{g\circ f} & & \longrightarrow & & L^2_{g\circ f} \otimes \eta_a \end{array}$$

The morphisms $\eta \to \eta'$ are maps of $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles over X_0 satisfying

$$\begin{array}{ccc} \eta_b \otimes L^1_f & \longrightarrow & L^2_f \otimes \eta_a \\ \downarrow & \circlearrowleft & \downarrow \\ \eta'_b \otimes L^1_f & \longrightarrow & L^2_f \otimes \eta'_a \end{array}$$

When $L^1 = L^2 = L$, the structure maps $\eta_b \otimes L_f \to L_f \otimes \eta_a \cong \eta_a \otimes L_f$ give η the structure of a $\mathbb{Z}/2$-graded $\mathbb{T}$-bundle over X. In other words the automorphism groupoid of any $\mathbb{Z}/2$-graded $\mathbb{T}$-central extension is equivalent to the groupoid of $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles over X.

The monoidal structure on $\mathfrak{Ext}(X)$ is the following: the underlying $\mathbb{Z}/2$-graded $\mathbb{T}$-bundle of $L^1 \otimes L^2$ over X_1 is the tensor product of the underlying $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles of L^1 and L^2. The structure maps are

$$\left(L^1_g \otimes L^2_g\right) \otimes \left(L^1_f \otimes L^2_f\right) \to L^1_g \otimes L^1_f \otimes L^2_g \otimes L^2_f \to L^1_{g\circ f} \otimes L^2_{g\circ f}$$

Note that the definition uses the symmetric monoidal structure of $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles $L^2_g \otimes L^1_f \cong L^1_f \otimes L^2_g$ where the Koszul sign rule is used. In particular, if L^1, L^2 are given by trivial $\mathbb{T}$-bundles with trivial structure maps but non-trivial grading, the structure maps of $L^1 \otimes L^2$ might be non-trivial.

We are now ready to define K-Theory twistings of local quotient groupoids. In this presentation we will include the degree of K-theory as part of the twisting.

Definition 2.3.11. For X a local quotient groupoid, let $Twist(X)$ be the groupoid whose objects are triples (P, L, n) where $P \to X$ is a local equivalence, L is a $\mathbb{Z}/2$-graded $\mathbb{T}$-central extensions of P and $n : X \to \mathbb{Z}/2$ is a functor to the Picard groupoid $\mathbb{Z}/2$ consisting of 2 elements and no non-trivial automorphisms. Morphisms between (P_1, L^1, n_1) and (P_2, L^2, n_2) exist only if $n_1 = n_2$, in which case they are given by (P, p, η) where p is a local equivalence

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle p} & & \\ P_1 & \xleftarrow{\pi_1} & P_1 \times_X P_2 & \xrightarrow{\pi_2} & P_2 \end{array}$$

and η is a morphism between $p^*\pi_1^*L^1$ and $p^*\pi_2^*L^2$ on P. Similarly the morphisms between (P, p, η) and (P', p', η') are given by morphisms from η to η' pulled back to some locally equivalent space Q

Example 2.3.12. Let S be a space and let $\mathcal{U}$ be an open cover. Let L_{ij} be a $\mathbb{Z}/2$-graded $\mathbb{T}$-gerbe on S given by $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles over double intersections $U_{i,j}$ along isomorphisms $\lambda_{i,j,k} : L_{i,j} \otimes L_{j,k} \to L_{i,k}$ over triple intersections satisfying the obvious cocycle condition on quadruple intersections. Then L defines a $\mathbb{Z}/2$-graded $\mathbb{T}$-central extension on $S_{\mathcal{U}}$ (defined in Example 2.3.4 and locally equivalent to S). For any locally constant map $n : S \to \mathbb{Z}/2$, we get a twisting $(S_{\mathcal{U}}, L, n)$. It corresponds to the twisting by a $\mathbb{Z}/2$-graded $\mathbb{T}$-gerbe of Section 2.1

Let $|X|$ denote the classifying space of X, i.e., the geometric realization of the nerve of X. It is shown in [FHT11a, Proposition 2.22] that if the restriction maps $H^3(|X|,\mathbb{Z}) \to H^3(X_0,\mathbb{Z})$ and $H^1(|X|,\mathbb{Z}/2) \to H^1(X_0,\mathbb{Z}/2)$ vanish then $Twist(X)$ is equivalent to $\mathfrak{Ext}'(X) := \mathfrak{Ext}(X) \times \hom(X,\mathbb{Z}/2)$. In that case, any twisting is isomorphic to one given by a $\mathbb{Z}/2$-graded central extension of X, i.e., without a need for a locally equivalent groupoid P. In particular, given two elements in $Twist(X)$, we may assume they are given by extension over the same locally equivalent groupoid P.

Definition 2.3.13. The groupoid $Twist(X)$ is symmetric monoidal. The monoidal structure is

$$(P, L^1, n_1) \otimes (P, L^2, n_2) = (P, L^1 \otimes L^2, n_1 + n_2).$$

The symmetry transformation is given by a $\mathbb{Z}/2$-graded bundle η over P_0 which is trivial as a $\mathbb{T}$-bundle, has grading $(-1)^{n_1 n_2}$ and structure maps

$$\eta \otimes L^1_f \otimes L^2_f \to L^1_f \otimes L^2_f \otimes \eta \to L^2_f \otimes L^1_f \otimes \eta$$

given by the symmetric monoidal structure of $\mathbb{Z}/2$-graded $\mathbb{T}$-bundles.

The isomorphism classes of twistings of a groupoid X are given by ([FHT11a, Corollary 2.25])

$$H^0(X,\mathbb{Z}/2) \times H^1(X,\mathbb{Z}/2) \times H^3(X,\mathbb{Z})$$

as a set[4]. As an abelian group, it is an extension of $H^0(X,\mathbb{Z}/2) \oplus H^1(X,\mathbb{Z}/2)$ by $H^3(X,\mathbb{Z})$ with the 2-cocycle given by $\beta(* \smile *)$. The argument is the same as we saw in Section 2.1.

[4]We use the Borel cohomology of topological groupoids throughout, i.e., $H^*(X) := H^*(|X|)$

Example 2.3.14. Let $V \to X$ be a vector bundle on X. Let $\mathcal{B} \to X$ be the bundle of frames of V; it is a principal $O(k)$-bundle. There is a local equivalence $P = \mathcal{B}/\!\!/O(k) \to X$ and a $\mathbb{Z}/2$-graded central extensions $\mathcal{B}/\!\!/\mathrm{Pin}^c(k) \to P$ (grading of the central extension is given by the component in $O(k)$). Let the degree $n : X \to \{0, 1\}$ be the dimension of V mod 2. Denote the resulting twisting by $\rho(V)$. Its equivalence class is given by

$$(\dim V, \omega_1(V), \beta(\omega_2(V))) \in H^0(X, \mathbb{Z}/2) \times H^1(X, \mathbb{Z}/2) \times H^3(X, \mathbb{Z})$$

Consider the category $Twist(*)$. It is a topological symmetric monoidal 2-groupoid with $\pi_0 = \mathbb{Z}/2$, $\pi_1 = \mathbb{Z}/2$, $\pi_2 = \mathbb{T}$. The classifying space of a symmetric monoidal category can be endowed with a structure of an infinite loop space, or equivalently, a connective spectrum, as described by Segal in [Seg74]. More generally, stable homotopy hypothesis says that Picard n-groupoids correspond to stable n-types, i.e., spectra with non-trivial homotopy groups in degrees 0 through n.[5] As an ∞-groupoid, $Twist(*)$ is a Picard 3-groupoid (the homotopy type of its 2-morphisms $\mathbb{T}$ is $B\mathbb{Z}$) and therefore, corresponds to a 3-truncated connective spectrum.

Definition 2.3.15. Let $|Twist|$ be the spectrum corresponding under the stable homotopy hypothesis to $Twist(*)$.

[5]See [JO12, GJO17] for a careful discussion of the cases $n = 1, 2$.

The homotopy groups of $|Twist|$ are

$$\pi_0|Twist| = \mathbb{Z}/2$$

$$\pi_1|Twist| = \mathbb{Z}/2$$

$$\pi_3|Twist| = \mathbb{Z}$$

and the correspondence of the homotopy hypothesis says $Twist(*) = \pi_{\leq\infty}|Twist|$.

The spectrum $|Twist|$ defines a Borel type generalized cohomology theory of stacks by

$$Twist^n(X) = [|X|, \Sigma^n|Twist|]$$

Claim 2.3.16. For X a locally quotient groupoid, the Picard groupoid $Twist(X)$ is equivalent to $\pi_{\leq\infty}Map(|X|, |Twist|)$. In particular, the isomorphism classes in $Twist(X)$ are given by $Twist^0(X)$ and isomorphism classes of automorphisms of any class are given by $Twist^{-1}(X)$.

We will not use the full strength of this claim and therefore will not prove it here. The special cases when X is a space or when X is a quotient groupoid follows from the discussion in Section 2.1. The general case should follow from the fact that the sheaf $Twist$ satisfies the descent condition. It also follows from this that $Twist$ is invariant under homotopy in the sense that if h is a homotopy between $f, g : X \to Y$, then it induces a natural isomorphism between $f^*, g^* : Twist(Y) \to Twist(X)$.

In our computation of $K_G^\tau(G)$, will use the definition of τ in [FHT10] which relies heavily on the structure of $|Twist|$. We therefore analyze its structure. Let ko be connective spectrum associated to the symmetric monoidal category $Vect_{\mathbb{R}}$ of real finite-dimensoinal vector spaces via Segal delooping construction [Seg74] and let $\tilde{ko}$ be its connected cover. The tensor product of vector spaces provides a second monoidal structure on $Vect_{\mathbb{R}}$ and endows ko with a structure of a ring spectrum. Segal shows in [Seg77] that ko is the connective cover of the real K-theory spectrum KO.

Proposition 2.3.17. *The spectrum $|Twist|$ is isomorphic to $\langle \Sigma^{-1}\tilde{ko}\rangle_3$, where $\tilde{ko}$ is the connected cover of KO and $\langle\rangle_n$ denotes the n-th Postnikov section.*

Proof. The homotopy groups of $|Twist|$ agree with those of $\langle \Sigma^{-1}\tilde{ko}\rangle_3$ so it suffices to check that they have the same k-invariants. Consider the Postnikov tower for $|Twist|$

$$\begin{array}{ccccc} \Sigma^3 H\mathbb{Z} & \xrightarrow{\gamma} & |Twist| & & \\ & & \downarrow & & \\ \Sigma^1 H\mathbb{Z}/2 & \xrightarrow{i} & T & \xrightarrow{j} & \Sigma^4 H\mathbb{Z} \\ & & \downarrow & & \\ & & H\mathbb{Z}/2 & \xrightarrow{k} & \Sigma^2 H\mathbb{Z}/2 \end{array}$$

We claim that

$$j \circ i = \beta \circ \mathrm{Sq}^2$$
$$k = \mathrm{Sq}^2$$

where HA is the Eilenberg-Maclane spectrum of an abelian group A. A map $H\mathbb{Z}/2 \to \Sigma^2 H\mathbb{Z}/2$ corresponds to a degree 2 stable cohomology operation on $\mathbb{Z}/2$ cohomology. These are known to coincide with the Steenrod square operations and Sq^2 is the only non-zero operation of degree 2. Any stable operation $H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}$ factors through the Bockstein homomorphism $\Sigma^2 H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}$ and therefore $\beta \circ \mathrm{Sq}^2$ is also the only non-zero map in the specified degrees. It therefore suffices to show that these maps are not zero. The spectrum T corresponds to the 1-Picard groupoid which is 1-truncation of $Twist(*)$. The k-invariant of this Picard groupoid is non-zero. Indeed the symmetry transformation of $1 \otimes 1 \cong 0$, where $1 \in Twist(*)$ corresponds to the trivial twisting in degree 1, is given by the non-identity automorphism of 0. This implies that the map k is non-zero. To see that $j \circ i \neq 0$ note that the pull-back of $|Twist|$ by the map i is the subgroup of twistings of degree 0. As noted in Section 2.1, the group of isomorphism classes of such twistings over a space S is the non-trivial extensions of $H^1(S, \mathbb{Z}/2)$ by $H^3(S, \mathbb{Z})$. This implies that $j \circ i \neq 0$ and hence equal to $\beta(\mathrm{Sq}^2)$. These maps are also non-trivial for $\langle \tilde{ko} \rangle_3$ but we will not prove it here. □

Definition 2.3.18. There are maps of spectra

$$\gamma : \Sigma^3 H\mathbb{Z} \to |Twist|$$
$$\rho : ko \to |Twist|$$
$$P : \tilde{ko} \to \Sigma^1 |Twist|$$

where $\rho(V)$ is as in Example 2.3.14 and P is the projection onto the third Postnikov section. We will write V for $\rho(V)$ when it will not cause confusion.

The spectrum ko classifies virtual real vector bundles and $\tilde{ko}$ classifies the virtual rank 0 bundles. There is a map

$$BO \xrightarrow{\delta} \tilde{ko}$$

which classifies the stable universal bundle over BO. More directly, we have $\Omega^\infty \tilde{ko} \cong BO$ and δ is the adjoint of this isomorphism.

The map γ becomes invertible after taking the smash product with $H\mathbb{Q}$. Consider the composition

$$\tilde{ko} \wedge H\mathbb{Q} \xrightarrow{P} \Sigma^1 |Twist| \wedge H\mathbb{Q} \xrightarrow{\gamma^{-1}} \Sigma^4 H\mathbb{Q}.$$

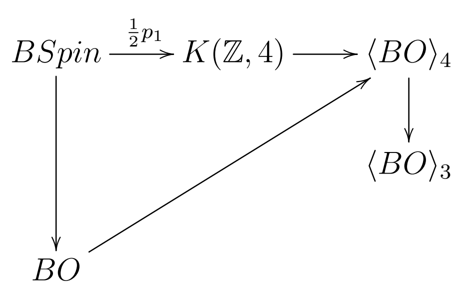
It is the projection to the 4-th Postnikov section of $\tilde{ko} \wedge H\mathbb{Q}$.

Proposition 2.3.19. *The composition*

$$BO \xrightarrow{\delta} \tilde{ko} \wedge H\mathbb{Q} \xrightarrow{P} \Sigma^1 |Twist| \wedge H\mathbb{Q} \xrightarrow{\gamma^{-1}} \Sigma^4 H\mathbb{Q}$$

is the class $\frac{1}{2}p_1$ where $p_1 \in H^4(BO, \mathbb{Z})$ is the first Pontryagin class.

Proof. It suffices to consider the 4-th Postnikov section of BO. Consider the map from $BSpin$, the 4th level of the Whitehead tower for BO, to the 4th Postnikov section $\langle BO \rangle_4$. It factors through the homotopy fiber of the map $\langle BO \rangle_4 \to \langle BO \rangle_3$ and it is well known that the corresponding map is $\frac{1}{2}p_1$.



The result follows. □

Proposition 2.3.20. *The map ρ is the composition*

$$ko \xrightarrow{\eta} \Sigma^{-1}\tilde{ko} \xrightarrow{\Sigma^{-1}P} |Twist|$$

where η is the action of the non-trivial class in $\pi_1(S)$ and S is the sphere spectrum.

Proof. The spectrum $|Twist|$ is a module over ko coming from the action of $Vect_{\mathbb{R}}$ on $Twist(*)$ given by

$$\mathbb{R}^m \otimes (*, L, n) = (*, L^{\otimes m}, n \cdot m)$$

Since both maps in question are ko-module maps, it suffices to check that they coincide on the unit element in ko. This is true since both maps send $\mathbb{R}$ to the non-trivial element.

□

Corollary 2.3.21. *The following maps are homotopic*

$$\rho \circ \pi \simeq \eta \circ P$$

where $\pi : \tilde{ko} \to ko$ is the cover map.

We defined the Borel type cohomology theory $Twist^n(X)$ for any topological groupoid as maps into the spectrum $|Twist|$. The machinery of stable homotopy theory applies to define push-forward in this theory along $|Twist|$-orientable maps. We will not discuss $|Twist|$-orientations and will only encounter push-forwards along stably framed maps.

Proposition 2.3.22. *Let τ be a twisting of $S^1 \times X$ and let $\tau_0 = \tau\big|_{\{*\}\times X}$. The homotopy around the circle induces an isomorphism class of automorphisms of τ_0. It is given by*

$$\pi_{2*}(\tau - \pi_2^*\tau_0) \in Twist^{-1}(X)$$

where $\pi_2 : S^1 \times X \to X$ and the push-forward is with respect to the trivialization of the tangent bundle of S^1.

Proof. The class $\tau - \pi_2^*\tau$ is trivialized over $\{*\} \times X$ and therefore defines a class in

$$\mathrm{Twist}^0\left(S^1 \times X, \{*\} \times X\right) \cong \mathrm{Twist}^0(\Sigma^1 X_+) \cong \mathrm{Twist}^{-1}(X)$$

which is the sought after class. The push-forward of this class along the inclusion $\{0\} \times X \hookrightarrow S^1 \times X$ where $0 \neq *$ with respect to the trivialization of TS^1 is the class $\tau - \pi_2^*\tau_0$. Since the composition of inclusion followed by projection back to $\{0\} \times X$ is the identity map, the result follows by functoriality of the push-forward map. $\square$

2.4 Twisted K-theory of Stacks

Let X be a local quotient groupoid and $\tau \in Twist(X)$. Freed, Hopkins, and Teleman define twisted K-theory group

$$K^\tau(X)$$

in [FHT11a, Section 3]. Their construction is analogous to that outlined in Section 2.1. Here, we simply state the properties that the K-theory functor satisfies. Let $\mathfrak{Twist}$ be the category whose objects are triples (X, A, τ) where X is a local quotient groupoid, A is a subgroupoid and $\tau \in Twist(X)$. A morphism from (X, A, τ_X) to (Y, B, τ_Y) is a continuous functor $f : X \to Y$ which maps A to B and a morphism of twistings $f^*\tau_Y \to \tau_X$. K-theory is a contravariant homotopy functor

$$(X, A, \tau) \mapsto K^\tau(X, A)$$

satisfying

1. There is a natural long exact sequence [6]

$$\begin{array}{ccccc} K^\tau(X,A) & \longrightarrow & K^\tau(X) & \longrightarrow & K^\tau(A) \\ \uparrow & & & & \downarrow \\ K^{\tau+1}(A) & \longleftarrow & K^{\tau+1}(X) & \longleftarrow & K^{\tau+1}(X,A) \end{array}$$

2. For $Z \subset A$ full subgroupoid whose closure belongs to the interior of A, the map

$$K^\tau(X, A) \to K^\tau(X \backslash Z, A \backslash Z)$$

is an isomorphism

3. For $(X, A, \tau) = \coprod_\alpha (X_\alpha, A_\alpha, \tau_\alpha)$, there is a natural isomorphism

$$K^\tau(X, A) \to \prod_\alpha K^{\tau_\alpha}(X_\alpha, A_\alpha)$$

[6] $K^\tau(X) := K^\tau(X, \emptyset)$

4. There is a bilinear pairing

$$K^{\tau_1}(X) \otimes K^{\tau_2}(Y) \to K^{\pi_1^*\tau_1+\pi_2^*\tau_2}(X \times Y)$$

which when pulled back to the diagonal, gives a pairing

$$K^{\tau_1}(X) \otimes K^{\tau_2}(X) \to K^{\tau_1+\tau_2}(X)$$

which is associative and commutative. [7]

5. An automorphism η of a twisting τ corresponds to a $\mathbb{Z}/2$-graded $\mathbb{T}$-bundle L and hence a class in $K^0(X)$. The pullback of $K^\tau(X)$ along (id, η) is multiplication by L.

2.5 Mackey Decomposition

We give a brief treatment following [FHT11b, Sec. 5].[8] Let

$$1 \to T \to N \to W \to 1$$

be a short exact sequence of compact Lie groups. Let Λ be the set if isomorphism classes of irreducible representations of T and fix a representation V_λ for every class $\lambda \in \Lambda$. A representation E of N can then be decomposed as

$$E = \sum_{\lambda \in \Lambda} \mathrm{Hom}_T(V_\lambda, V) \otimes V_\lambda.$$

The vector spaces $\mathrm{Hom}_T(V_\lambda, V)$ form a twisted W-equivariant vector bundle over Λ. There is a family version of this statement:

[7]This uses the symmetric monoidal structure on $Twist(X)$.

[8]We recommend [FM13, Sec. 9] for a thorough discussion of a simple case of this construction.

Lemma 2.5.1 ([FHT11b, Lemma 5.2])**.** *Let N act on a compact Hausdorff space X with T acting trivially and assume T is connected. Let τ be a twisting of the groupoid $X /\!/ N$. Assume moreover that every $x \in X$ has a closed N-equivariant neighborhood equivariantly diffeomorphic to $N \times_{N_x} S_x$ where N_x is the stabilizer of x and S_x is equivariantly contractible. This gives rise to the following:*

1. *a N-equivariant family T_x^τ of central extensions of T by $\mathbb{T}$ parametrized by X.*
2. *a W-equivariant covering space $p : Y \to X$ whose fiber at $x \in X$ is identified with the set of isomorphism classes of irreducible T_x^τ representations where the central $\mathbb{T}$ acts by its defining homomorphism.*
3. *a N-equivariant twisting $\mathbb{P}R$ over Y given by the projective bundle which at point $y \in Y$ is the $T_{p(y)}^\tau$ projective representation of T labeled by y.*
4. *a class $[R] \in K_N^{\mathbb{P}R}(Y)$*
5. *a W-equivariant twisting τ' over Y and an isomorphism of N-equivariant twistings $\tau' \cong p^*\tau - \mathbb{P}R$.*

such that the composition

$$K_W^{\tau'}(Y)_{\mathrm{cpt}} \to K_N^{\tau'}(Y)_{\mathrm{cpt}} \cong K_N^{p^*\tau - \mathbb{P}R}(Y)_{\mathrm{cpt}} \xrightarrow{\otimes R} K_N^{p^*\tau}(Y)_{\mathrm{cpt}} \xrightarrow{p_*} K_N^\tau(X). \tag{2.5.2}$$

is an isomorphism

The case when X is a point is elaborated upon in the example below. The assumption on the neighborhoods ensures that locally, (2.5.2) reduces to the case when X is a point and hence an isomorphism. The general case follows by a Mayer-Vietoris argument.

Example 2.5.3. The simplest example is when X is a point and $\tau = 0$. The group $K^0_N(*)$ is the group completion of the commutative monoid of finite-dimensional representations of N. The main idea of this decomposition is that a representation E of N decomposes under the action of T into its isotypical components which are permuted by W. Let Λ be the set of isomorphism classes of irreducible complex representations of T. It is the space we called Y in Lemma 2.5.1. The group N acts on Λ by conjugating the action and T acts trivially. The action therefore descends to an action of W. Fix an irreducible representation $(V_\lambda, \rho_{V_\lambda})$ for each $\lambda \in \Lambda$ where $\rho_{V_\lambda} : T \to \mathrm{Aut}(V_\lambda)$. For $n \in N$ and a representation (V, ρ_V) of T, let (V^n, ρ_{V^n}) be the representation of T given by

$$\rho_{V^n}(t) = \rho_V(ntn^{-1}).$$

Consider the groupoid P whose set of objects is Λ and whose morphisms from λ to β are pairs $(n, \phi) \in N \times \hom_T(V^n_\lambda, V_\beta)$ where $\hom_T$ is the space of unitary intertwiners. The map of groupoids $P \to \Lambda /\!\!/ N$ defines the desired twisting $\mathbb{P}R$. The class $[R]$ is given by the following tautological vector bundle R over P: for $\lambda \in \Lambda$ we let $R|_\lambda = V_\lambda$ and a morphism $\lambda \xrightarrow{(n,\phi)} \beta$ of P lifts to the map $V_\lambda \to V_\beta$ given by ϕ where we recall that $V_\lambda = V^n_\lambda$ as vector spaces.

The twisting τ' is given by the $\mathbb{T}$-central extension of groupoids $P' \to \Lambda/\!\!/W$ where the morphisms of P' are equivalence classes of pairs (n, ϕ) where $n \in N$ and $\phi \in \hom_T(V_\beta, V_\lambda^n)$ with equivalence given by

$$(n, \phi) \sim (tn, \rho_{V_\lambda}(t) \circ \phi)$$

where $t \in T$. We denote the equivalent class of (n, ϕ) by $[(n, \phi)]$. The isomorphism $\tau' \cong -\mathbb{P}R$ of N-equivariant twistings follows from the fact that $\hom_T(V_\lambda^n, V_\beta)$ is naturally dual to $\hom_T(V_\beta, V_\lambda^n)$.

The sequence of maps (2.5.2) in this case takes the following form: suppose a class in $K_W^{\tau'}(\Lambda)_{\mathrm{cpt}}$ is given by a finite-dimensional compactly supported vector bundle over P' i.e. a collection of vector spaces E_λ for $\lambda \in \Lambda$ which are zero for all but finitely many λ along with maps

$$\lambda_{[(n,\phi)]} : E_\lambda \to E_{\lambda.n}$$

where $[(n, \phi)]$ is a morphism in P'. The image of this class under (2.5.2) is the representation E of N given by

$$E = \bigoplus_{\lambda \in \Lambda} E_\lambda \otimes V_\lambda.$$

The action of N is the following:

$$n : E_\lambda \otimes V_\lambda \xrightarrow{\lambda_{[(n,\phi)]} \otimes \phi^{-1}} E_{\lambda.n} \otimes V_{\lambda.n}$$

where ϕ is any element in $\hom_T(V_{\lambda.n}, V_\lambda^n)$.

The inverse of this construction is more transparent. Given a representation E, we set $E_\lambda = \hom_T(V_\lambda, E)$ and

$$\begin{aligned} \lambda_{[(n,\phi)]} : E_\lambda &\to E_{\lambda.n} \\ f &\mapsto \rho_E(n)^{-1} \circ f \circ \phi. \end{aligned}$$

More generally, if $\tau \in \mathrm{Twist}(\{*\}/\!\!/N)$, then it corresponds to a $\mathbb{T}$-central extension N^τ of N and a $\mathbb{Z}/2$-grading $\epsilon : N \to \mathbb{Z}/2$. The central extension N^τ restricts to a central extension T^τ of T.

$$\begin{array}{ccccccccc} & & \mathbb{T} & & \mathbb{T} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & T^\tau & \longrightarrow & N^\tau & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & T & \longrightarrow & N & \longrightarrow & W & \longrightarrow & 1 \end{array}$$

In this case, Λ^τ is the set of irreducible representations of T^τ such that the central $\mathbb{T}$ acts by its defining homomorphism. The twisting $\mathbb{P}R$ is given by the central extension $P \to \Lambda^\tau/\!\!/N$ where the morphisms from λ to β are equivalence classes of pairs (n, ϕ) where $n \in N$ and $\phi \in \hom_{T^\tau}(V_\lambda^n, V_\beta)$ and the class $R \in K_W^{\mathbb{P}R}(\Lambda^\tau)$ is constructed in exactly the same way as in the untwisted case.

The twisting τ' is given by the central extension $P' \to \Lambda^\tau/\!\!/W$ where the morphisms of P' from λ to β are equivalence classes of pairs (n, ϕ) where $n \in N^\tau$ and $\phi \in \hom_{T^\tau}(V_\beta, V_\lambda^n)$ with equivalence given by

$$(n, \phi) \sim (tn, \rho_{V_\lambda}(t) \circ \phi)$$

for $t \in T^\tau$. The grading ϵ pulls back to give the grading of the twisting τ' since T is assumed connected.

Proposition 2.5.4. *The K-theory group $K_N^\tau(*) = K_W^{\tau'}(\Lambda^\tau)$ is a module for $K_N(*)$. Assume that T is abelian. Then there is a "tensor product" map*

$$\Lambda \times \Lambda^\tau \xrightarrow{\mu} \Lambda^\tau$$

and an isomorphism of twistings $\pi_1^*(\tau_N) + \pi_2^*(\tau') \cong \mu^*(\tau')$ such that the module structure is given by the push-forward along the map μ, i.e., the following diagram commutes.

$$\begin{array}{ccccc}
K_N(*) \otimes K_N^\tau(*) & \longrightarrow & & \longrightarrow & K_N^\tau(*) \\
\| & & & & \| \\
K_W^{\tau_N}(\Lambda) \otimes K_W^{\tau'}(\Lambda^\tau) & \longrightarrow & K_W^{\pi_1^*(\tau_N)+\pi_2^*(\tau')}(\Lambda \times \Lambda^\tau) & \xrightarrow{\mu_*} & K_W^{\tau'}(\Lambda^\tau)
\end{array}$$

Proof. Since we assume that T is abelian, all (τ-twisted) irreducible representations of T are 1-dimensional and hence their tensor product is again irreducible. This determines the map $\mu : \Lambda \times \Lambda^\tau \to \Lambda^\tau$.

Since all irreducible representations of T are 1-dimensional, we can choose an isomorphism $V_\lambda = \mathbb{C}$ for every λ. The spaces of homomorphisms are then trivialized: $\hom_{T^\tau}(V_{\lambda.n}, V_\lambda^n) = \mathbb{T}$. With this trivialization, the twisting $\pi_1^*(\tau_N) + \pi_2^*(\tau')$ is given by the central extension $Q \to \Lambda \times \Lambda^\tau /\!\!/ W$ where the morphisms of Q from (λ, ξ) to $(\lambda.w, \xi.w)$ are the equivalence classes of triples $(n, x_1, x_2) \in N^\tau \times \mathbb{T} \times \mathbb{T}$ with equivalence given by

$$(n, x_1, x_2) \sim (n, x_1 y, y^{-1} x_2)$$

$$(n, x_1, x_2) \sim (tn, \rho_{V_\lambda}(\pi(t)) x_1, \rho_{V_\xi}(t) x_2)$$

for $y \in \mathbb{T}, t \in T^\tau$ and $\pi : T^\tau \to T$. The twisting $\mu^*\tau'$ is given by the central extension Q' whose morphisms from (λ, ξ) to $(\lambda.w, \xi.w)$ are pairs $(n, x) \in N^\tau \times \mathbb{T}$ with equivalence

$$(n, x)\ (tn, \rho_{\mu(\lambda,\xi)}(t)x).$$

The desired isomorphism $\pi_1^*(\tau_N) + \pi_2^*(\tau') \cong \mu^*(\tau')$ is then the map from Q to Q' that maps the equivalence class of (n, x_1, x_2) to the equivalence class of (n, x_1x_2). The map is well defined since $\rho_{V_{\mu(\lambda,\xi)}} = \rho_{V_\lambda}\rho_{V_\mu}$. It is now a matter of tracing definitions to see that the diagram in the statement of the proposition commutes. □

Chapter 3

Twisting τ_l and a 2-D TQFT

In this section, we review the construction of the 2-dimensional oriented topological quantum field theory over $\mathbb{Z}$ defined in [FHT10]. The construction is analogous to string topology but using K-Theory instead of singular homology and the stack of flat principal G-bundles instead of the mapping space.

3.1 Construction of the TQFT

Let Bord_2^{SO} be the symmetric monoidal category whose objects are compact oriented 1-manifolds and whose morphisms are oriented bordisms. The monoidal structure is given by disjoint union. An oriented 2-d TQFT is a symmetric monoidal functor $F : \mathrm{Bord}_2^{SO} \to \mathcal{C}$ for some symmetric monoidal category $\mathcal{C}$ ([Ati88a]). One usually takes $\mathcal{C}$ to be the category of vector spaces over $\mathbb{C}$ with the tensor product as the monoidal structure. Here we take $\mathcal{C} = \mathbb{Z}{-}mod$ with the tensor product.

Fix a compact Lie group G. For M a compact oriented manifold (possibly with boundary), let $\mathcal{M}_M$ denote the stack of flat G connections on M. For X a two dimensional manifold with boundary $Y_0 \coprod Y_1$, i.e. for a morphism

$X : Y_0 \to Y_1$ in the category Bord_2^{SO}, there are restriction maps

$$\begin{array}{ccccc} & & \mathcal{M}_X & & \\ & \overset{s}{\swarrow} & & \overset{t}{\searrow} & \\ \mathcal{M}_{Y_0} & & & & \mathcal{M}_{Y_1} \end{array} . \tag{3.1.1}$$

We would like to define a TQFT by

$$\begin{aligned} Y &\mapsto K^*(\mathcal{M}_Y) \\ (Y_0 \xrightarrow{X} Y_1) &\mapsto t_* \circ s^* : K^*(\mathcal{M}_{Y_0}) \to K^*(\mathcal{M}_{Y_1}), \end{aligned}$$

but the pushforward map $t_* : K^*(\mathcal{M}_X) \to K^*(\mathcal{M}_{Y_1})$ requires a K-orientation of the relative tangent bundle of t, which in general is not K-orientable. This demonstrates the need to introduce a twisting to the definition. The extra data we need is the following: for every one dimensional manifold Y, we need a K-twisting τ_Y of $\mathcal{M}_Y$ and for every bordism $X : Y_0 \to Y_1$ we need an isomorphism of twistings $s^*\tau_{Y_0} \cong t^*\tau_{Y_1} + (T\mathcal{M}_X - T\mathcal{M}_{M_{Y_1}})$. In that case we can define

$$\begin{aligned} F(Y) &= K^{\tau_Y}(\mathcal{M}_Y) \\ F(Y_0 \xrightarrow{X} Y_1) &= t_* \circ s^* : K^{\tau_{Y_0}}(\mathcal{M}_{Y_0}) \to K^{\tau_{Y_1}}(\mathcal{M}_{Y_1}). \end{aligned}$$

In [FHT10], the authors describe the space of such functorial assignments $Y \mapsto \tau_Y$ along with necessary isomorphisms of twistings on $\mathcal{M}_X$ in a way that makes F into a symmetric monoidal functor. We summarize their construction in the next section.

3.2 Universal Orientations

Let $MTSO_n$ denote the Madsen-Tillmann spectrum of SO_n.[1] For E an oriented n-dimensional vector bundle over a compact manifold X, there is a map of spectra $X^{-E} \to MTSO_n$ induced by a classifying map $X \to BSO_n$ of E. There is a natural map $\Sigma^{-1}MTSO_{n-1} \to MTSO_n$ given by stabilization. Consider a family of flat G-connections over a closed oriented 2-manifold X, parametrized by a manifold M. In other words, a G-bundle $P \to M \times X$ with a smoothly varying flat connection over fibers of the projection $\pi : M \times X \to M$. This defines a classifying map $f : M \to \mathcal{M}_X$ of the family. The product of the classifying map $M \times X \to BG$ of P and $X^{-TX} \to MTSO_2$, induces a map

$$M_+ \wedge X^{-TX} \to MTSO_2 \wedge BG$$

where M_+ is the space M with a disjoint base point. There is a map $M_+ \to M_+ \wedge X^{-TX}$ given by embedding X in a sphere and applying Pontryagin-Thom collapse map as in the definition of push-forward maps in generalized cohomology. In [FHT10], a map

$$\sigma_{\text{univ}} : (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG \to |Twist|$$

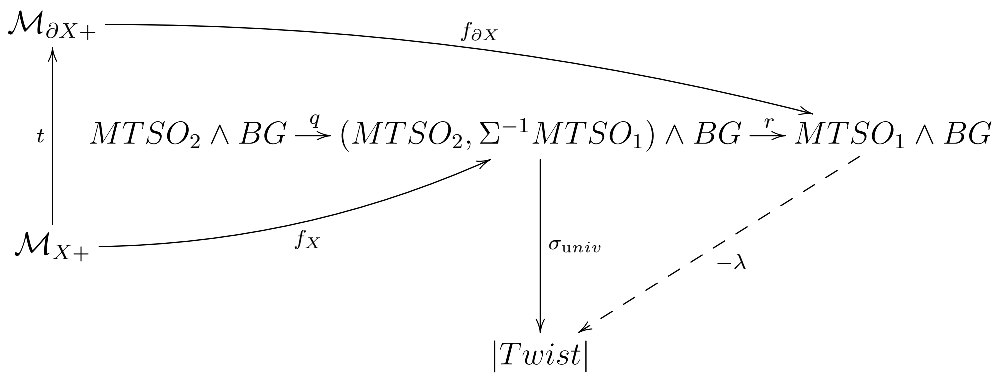
is constructed such that the twisting given by the composition

$$\begin{array}{ccccc} M_+ \wedge X^{-TX} & \longrightarrow & MTSO_2 \wedge BG & \xrightarrow{q} & (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG \\ \uparrow & & & & \downarrow{\scriptstyle \sigma_{\text{univ}}} \\ M_+ & & & & |Twist| \end{array}$$

[1] As a pre-spectrum, it is given by the sequence of Thom spaces $\{Gr_n^+(\mathbb{R}^k)^{Q_{k-n}}\}_{k>n}$ where Gr_n^+ denotes the Grassmannian of oriented n-planes and Q_{k-n} is the quotient of the trivial $\mathbb{R}^k$-bundle by the tautological vector bundle.

is isomorphic to the twisting by the reduced[2] tangent bundle of $\mathcal{M}_X$ pulled back to M along f. If X has non-empty boundary, an analogous construction gives a map $M_+ \to (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG$ which when composed with σ_{univ} is the twisting by the reduced relative tangent bundle of the restriction map $\mathcal{M}_X \xrightarrow{t} \mathcal{M}_{\partial X}$ pulled back to M by f ([FHT10, Lemma 3.19].

Consider the following diagram:



(3.2.1)

The maps $\mathcal{M}_{X+} \xrightarrow{f_X} (MTSO_2, \Sigma^{-1}MTSO_1) \wedge BG$ and $\mathcal{M}_{\partial X+} \xrightarrow{f_{\partial X}} MTSO_1 \wedge BG$ are the classifying maps as described above, written globally on the stacks. The horizontal maps form a cofiber sequence and the diagram consisting of solid arrows commutes.

Definition 3.2.2. A *universal twisting* is a map $\lambda : MTSO_1 \wedge BG \to |Twist|$. A *universal orientation* is a universal twisting λ along with a homotopy from σ_{univ} to $(-\lambda) \circ r$. [3]

[2]For a vector bundle V, the reduced bundle is the difference $V - \underline{\mathbb{R}^{\dim V}}$

[3]The terminology is inconsistent with [FHT10] where they refer to the map λ as the level. We reserve the term level for a class in $H^4(BG, \mathbb{Z})$.

A universal orientation gives us the functorial assignment of K-twistings we wanted. To a 1-manifold Y we assign the twisting

$$\tau_Y := b \dim G + \lambda \circ f_Y$$

of $\mathcal{M}_Y$ where b is the number of connected components in Y.[4] Let X be a 2-manifold with $\partial X = Y$, and $t : \mathcal{M}_X \to \mathcal{M}_Y$ the restriction map. The universal orientation gives rise to an isomorphism

$$t^* \tau_Y \cong -(T\mathcal{M}_X - T\mathcal{M}_Y)$$

where the right hand side is the twisting determined by the virtual vector bundle. Therefore, there is a pushforward map

$$t_* : K^0(\mathcal{M}_X) \to K^{\tau_Y}(\mathcal{M}_Y).$$

In [FHT10], it is also shown that for $Y = Y_0 \bigsqcup Y_1$, if we view X as a bordism $X : Y_0 \to Y_1$, an analogous construction gives rise to a pull-push map

$$K^{\tau_{Y_0}}(\mathcal{M}_{Y_0}) \xrightarrow{s^*} K^{s^*\tau_{Y_0}}(\mathcal{M}_X) \xrightarrow{t_*} K^{\tau_{Y_1}}(\mathcal{M}_{Y_0})$$

where s, t are as in diagram (3.1.1). In summary, a universal orientation allows us to define a TQFT[5] $F : \mathrm{Bord}_2^{SO} \to \mathbb{Z}{-}mod$ where

$$F(Y) = K^{\tau_Y}(\mathcal{M}_Y).$$

[4]The extra degree shift is necessary since $\sigma_{univ} \circ q \circ f_Y$ is the twisting by the reduced relative tangent bundle. The $b \dim G$ term is the dimension of this bundle.

[5]Functoriality is verified in [FHT10].

3.3 Canonical Universal Orientation and Level

It turns out that there is a canonically defined universal orientation h_G for any G which is constructed in [FHT10, Theorem 3.24]. The corresponding universal twisting is given by the map

$$MTSO_1 \wedge BG = \Sigma^{-1}BG \xrightarrow{\mathfrak{g}} \Sigma^{-1}BO \xrightarrow{\delta} \Sigma^{-1}\tilde{ko} \xrightarrow{P} |Twist|$$

where P is as in Definition 2.3.18, and δ classifies the stable universal bundle over BO. We have also used that $MTSO_1$ is isomorphic to the (-1)-suspended sphere spectrum $\Sigma^{-1}S$

Up to homotopy, a universal orientation is equivalent to a null-homotopy of $\sigma_{\text{univ}} \circ q$ in (3.2.1). Therefore, homotopy equivalence classes of universal orientations form a torsor for

$$\mathcal{O}(G) := [\Sigma MTSO_2 \wedge BG, |Twist|].$$

The canonical universal orientation allows us to identify this torsor with $\mathcal{O}(G)$ itself. A class $l \in H^4(BG, \mathbb{Z})$ is called a level and is the input data for a 3-dimensional Chern-Simons theory. At least formally, the dimensional reduction of the Chern-Simons theory for a fixed level l is the 2-dimensional TQFT constructed in the previous section with the universal orientation being the canonical one shifted by the image of l under the map $H^4(BG, \mathbb{Z}) \to \mathcal{O}(G)$ which we now describe.

The spectrum $|Twist|$ is connective and its non-negative homotopy groups are $\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, \ldots$ (it is 3-truncated). The spectrum $\Sigma MTSO_2$ is

(-2)-connected with $\pi_{-1} = \mathbb{Z}$ which implies that $\mathrm{M}ap(\Sigma MTSO_2, |Twist|)$ is 4-truncated with $\pi_4 = \mathbb{Z}$, i.e. its 3-connected cover is the suspended Eilenberg MacLane spectrum $\Sigma^4 H\mathbb{Z}$. We thus get a map

$$H^4(BG, \mathbb{Z}) = [BG, \Sigma^4 H\mathbb{Z}] \to [BG, \mathrm{M}ap(\Sigma MTSO_2, |Twist|)] = \mathcal{O}(G).$$

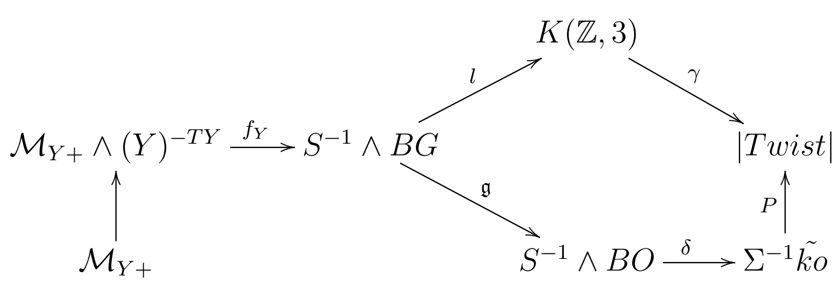
Translating a universal orientation by a class in $H^4(BG, \mathbb{Z})$ via the above map, translates the universal twisting via analogously constructed map

$$H^4(BG, \mathbb{Z}) \to [BG, \mathrm{M}ap(MTSO_1, |Twist|)].$$

Invoking the isomorphism $MTSO_1 \cong \Sigma^{-1}S$, the above map is γ of Definition 2.3.18.

Given a class $l \in H^4(BG, \mathbb{Z})$, the universal twisting corresponding the the canonical universal orientation shifted by l is $\#\pi_0(Y) \cdot \dim G$ plus the sum of the top and bottom rows in the following diagram.

(3.3.1)



We now specialize to $Y = S^1$. The stack $\mathcal{M}_{S^1}$ is represented by the global quotient groupoid $G /\!\!/ G$ where the action is by conjugation. The classifying space of $G /\!\!/ G$ is LBG, the loop space of BG, which could be seen by realizing $G \simeq \Omega BG$ as H-spaces and verifying the homotopy equivalence

$LBG \simeq \Omega BG \times_{\Omega BG} PBG$ where PBG is the contractible space of paths on BG.

The map

$$\Sigma^{-1}(\mathcal{M}_{S^1_+} \wedge S^1_+) = \mathcal{M}_{S^1_+} \wedge (S^1)^{-\mathbb{R}} \xrightarrow{f_{S^1}} S^{-1} \wedge BG$$

is the (-1)-suspension of the evaluation map $LBG \times S^1 \xrightarrow{e} BG$.

Definition 3.3.2. For E any generalized cohomology theory, denote by $\mathrm{tg} : E^*(BG) \to E^{*-1}(LBG)$ the composition $\pi_{1*} \circ \mathrm{ev}^*$ in the following diagram

$$\begin{array}{ccc} LBG \times S^1 & \xrightarrow{\mathrm{ev}} & BG \\ \downarrow{\scriptstyle \pi_1} & & \\ LBG & & \end{array}$$

The push-forward is with respect to the stable framing coming from trivialization of the tangent space of S^1.

For any space X, the push-forward along the projection map $X \times S^1 \xrightarrow{\pi_1} X$ with respect to the trivialization of the tangent bundle of S^1 is the precomposition with the map $X_+ \to X_+ \wedge S^{1-\mathbb{R}} \cong \Sigma^{-1}(X_+ \wedge S^1_+)$. Hence, the twistings in (3.3.1) are given by the transgression of $\gamma \circ l$ and $\mathfrak{g} \circ \delta \circ P$.

Definition 3.3.3. For $l \in H^4(BG, \mathbb{Z})$ denote by $\tau_l \in \mathrm{Twist}(G/\!\!/G)$ the twisting

$$\tau_l := \dim G + \mathrm{tg}(\gamma(l)) + h_G$$

where

$$h_G = \mathrm{tg}(P(\overline{\mathfrak{g}}))$$

and P is defined in Definition 2.3.18.

The twisting τ_l is the universal twisting corresponding the canonical universal orientation shifted by $l \in H^4(BG, \mathbb{Z})$ on $\mathcal{M}_{S^1}$.

Chapter 4

Compact Lie Group Theory

In this section we collect the notation, conventions, and lemmas that have to do with the theory of compact Lie groups.

4.1 Connected Lie Groups

Let G_1 be a connected semi-simple compact Lie group, $\mathfrak{g}$ its Lie algebra, and $T \subset G_1$ a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let $\Pi \subset \mathfrak{t}$ be the kernel of $\exp(2\pi\cdot)$ and $\Lambda \subset \mathfrak{t}^*$ be the $\mathbb{Z}$ dual of Π. The lattice Λ is identified with the group of characters $T \to \mathbb{T}$ via $\exp(t) \mapsto e^{i\langle\lambda,t\rangle}$ for $\lambda \in \Lambda$. The complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ decomposes under the adjoint action of T as a direct sum $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ where $\Delta \subset \Lambda$ is the set of roots.

Let $N_1 := \mathrm{Stab}_{G_1}(T)$ and $W_1 := N_1/T$ be the Weyl group. The hyperplanes $\ker(\alpha)$ for $\alpha \in \Delta$ divide $\mathfrak{t}$ into connected components called Weyl chambers. The group W_1 acts freely and transitively on the set of Weyl chambers. Fix a Weyl chamber and let Δ^+ be the set of roots which have positive values on the fixed chamber. Let $\Phi \subset \Delta^+$ be the set of simple roots, i.e. those which cannot be written as a non-trivial linear combination of elements in Δ^+. The set Φ forms a basis for $\mathfrak{t}^*$.

The Killing form $\langle -, - \rangle$ defines a Weyl invariant symmetric bilinear forms on $\mathfrak{t}$ and $\mathfrak{t}^*$ and hence a map $b : \mathfrak{t}^* \to \mathfrak{t}$. The coroots are defined by $h_\alpha := \frac{2}{\langle \alpha, \alpha \rangle} b(\alpha)$. We have that $h_\alpha \subset \Pi$ and $\Pi / \langle h_\alpha \rangle_{\alpha \in \Delta} \cong \pi_1(G_1)$ where $\langle h_\alpha \rangle_{\alpha \in \Delta}$ is the sublattice spanned by h_α. Define the Cartan matrix by $C_{\alpha\beta} := 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ where $\alpha, \beta \in \Phi$, and the Dynkin diagram by assigning a vertex to each simple root, connecting vertices labeled by α, β by $C_{\alpha\beta} \cdot C_{\beta\alpha}$ edges and orienting the edges in the direction of the shorter root if $C_{\alpha\beta} \cdot C_{\beta\alpha} > 1$. One can reconstruct the Lie algebra up to isomorphism from the Cartan matrix or from the Dynkin diagram. The outer automorphism group of the Lie algebra is isomorphic to the group of automorphisms of the corresponding Dynkin diagram. We will now study particular kind of automorphisms of the Lie algebra coming from the automorphisms of the Dynkin diagram.

For each $\alpha \in \Phi$, choose $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $h_\alpha = -i[X_\alpha, Y_\alpha]$. Let χ be an automorphism of the Dynkin diagram. In particular it acts by a permutation of the simple coroots. Define an automorphism of $\mathfrak{g}_\mathbb{C}$ in the following way:

$$\begin{aligned} h_\alpha &\mapsto h_{\chi(\alpha)} \\ X_\alpha &\mapsto X_{\chi(\alpha)} \\ Y_\alpha &\mapsto Y_{\chi(\alpha)}. \end{aligned}$$

The Lie algebra $\mathfrak{g}_\mathbb{C}$ is generated by X_α, Y_α, the only relations being Serre's relations. In particular, this defines an automorphism of $\mathfrak{g}_\mathbb{C}$. The real subalgebra $\mathfrak{g}$ is generated by $X_\alpha + Y_\alpha, i(X_\alpha - Y_\alpha)$ and hence preserved by χ. It may or may not lift to the Lie group G_1; if G_1 is simply-connected, then the automorphism lifts to the group.

Definition 4.1.1. After having fixed G_1, T, Φ, X_α, an automorphism χ of G_1 is called a *diagram automorphism* if it preserves T and permutes X_α for $\alpha \in \Phi$.

Proposition 4.1.2. *Let G_1 be a simply-connected simple Lie group and $\chi \in \mathrm{Aut}(G_1)$ be a diagram automorphism. If $\langle \chi(\alpha), \alpha \rangle = 0$ for all $\alpha \in \Phi$ such that $\chi(\alpha) \neq \alpha$, then G_1^χ is simple and simply-connected. If $G_1 = SU(2k+1)$ and χ is the order 2 diagram automorphism, then $G_1^\chi \cong SO(2k+1)$.*

Proof. Suppose $\langle \chi(\alpha), \alpha \rangle = 0$ for all $\alpha \in \Phi$ such that $\chi(\alpha) \neq \alpha$. Then a simple computation shows that the coroot vectors of G_1^χ are $\sum_{\alpha \in \mathcal{O}_j} h_\alpha$ where $\mathcal{O}_j \subset \Phi$ are the orbits under χ. Let $\underline{\Pi} \subset \Pi$ be the sublattice fixed by χ. Since G_1 is simply connected, h_α generate Π and it follows that the coroots of G_1^χ generate $\underline{\Pi}$ and hence G_1^χ is simply connected.

The group $G_1 = SU(2k+1)$ corresponds to the Dynkin diagram A_{2k}. The order 2 automorphism χ satisfies $\langle \chi(\alpha), \alpha \rangle = 0$ for all roots except those corresponding to the middle two nodes $\beta, \chi(\beta) \in \Phi$ of A_{2k}. The coroots of G_1^χ are $\sum_{\alpha \in \mathcal{O}_j} h_\alpha$ for $\mathcal{O}_j \neq \{\beta, \chi(\beta)\}$ and $2(h_\beta + h_{\chi(\beta)})$. It follows that the quotient of $\underline{\Pi}$ by the sublattice generated by the coroots is $\mathbb{Z}/2$ and therefore $\pi_1(G_1^\chi) = \mathbb{Z}/2$. Moreover, note that the coroot $2(H_\beta + H_{\chi(\beta)})$ corresponds to a short root and therefore G_1^χ corresponds to the Dynkin diagram B_k. It follows that $G_1^\chi \cong SO(2k+1)$. □

We now collect the results about the geometry of conjugacy classes that we will use later on.

Proposition 4.1.3. *[BtD95, Proposition IV.2.5] The quotient spaces G_1/G_1 and T/N_1 with the action given by conjugation coincide.*

In other words, the groupoids $G_1/\!\!/G_1$ and $T/\!\!/N$ have homeomorphic coarse moduli spaces. Additionally, the inclusion map $T/\!\!/N_1 \xrightarrow{\iota} G_1/\!\!/G_1$ is representable. The fiber over a point $t \in T$ is the quotient G_1^t/N_1^t. We will be considering the push-forward map in K-theory along this map.

Definition 4.1.4. Let G be a compact Lie group. An element $g \in G$ is called *regular* if the connected component of the identity in G^g, the subgroup fixed by conjugation by g, is a torus.

Proposition 4.1.5. *[Bor61, Theorem. 3.4] Let G_1 be a simply-connected Lie group and χ be an automorphism of G_1. Then the fixed subgroup G_1^χ is connected.*

Proposition 4.1.6. *Suppose G_1 is simply connected. Then $t \in T$ is regular if and only if it is not fixed by any element of the Weyl group.*

Proof. The fixed group G_1^t is connected by Proposition 4.1.5 and its Weyl group is isomorphic to the subgroup of W_1 which fixes t. Since a connected group is a torus if and only if its Weyl group is trivial, the result follows. □

4.2 Non-connected Lie Groups

Let G be a compact, but not necessarily connected Lie group whose identity component G_1 is semi-simple, and fix an element $f \in G$. We use the superscript in e.g. G_1^f to indicate the fixed subgroup under the conjugation action of f. Introduce the following notation:

- T - a maximal torus of G_1
- $N = \mathrm{Stab}_G(T)$
- $\Pi = \ker(\exp(2\pi\cdot)) \subset \mathfrak{t}$
- $\Lambda \subset \mathfrak{t}^*$ - the $\mathbb{Z}$-dual of Π,
- $W = N/T$ - the Weyl group of G
- $N_1 = \mathrm{Stab}_{G_1}(T)$
- $W_1 = N_1/T$, the Weyl group of G_1
- $W_{aff} = \Pi \rtimes W$
- $W_{aff_1} = \Pi \rtimes W_1$
- $\underline{T}$ - a maximal torus of G_1^f
- $\underline{\mathfrak{t}}$ - the Lie algebra of $\underline{T}$
- $G(f) = \mathrm{Stab}_G(fG_1)$
- $\underline{N} = \mathrm{Stab}_G(f\underline{T})$
- $\underline{W} = \underline{N}/\underline{T}$
- $\underline{N}_1 = \mathrm{Stab}_{G_1}(f\underline{T})$
- $\underline{W}_1 = \underline{N}_1/\underline{T}$,
- $\underline{\tilde{W}} = \underline{N}^f/\underline{T}$, the Weyl group of G^f.
- $\underline{\tilde{W}}_1 = \underline{N}_1^f/\underline{T}$, the Weyl group of G_1^f.
- $\underline{\Pi} = \ker(\exp(2\pi\cdot)) \subset \underline{\mathfrak{t}}$
- $\underline{\Lambda} \subset \underline{\mathfrak{t}}^*$ - the $\mathbb{Z}$-dual of $\underline{\Pi}$,

- $\underline{W_{aff}} = \{(x, h) \in \underline{\mathfrak{t}} \rtimes \underline{W} \mid h^{-1} \cdot f \cdot \exp(2\pi x) \cdot h = f\}$,

- $\underline{W_{aff}}_1 = \underline{W_{aff}} \cap (\underline{\mathfrak{t}} \times G_1/\underline{T})$.

- $\underline{\tilde{\Pi}} = \underline{W_{aff}} \cap \mathfrak{t} \times (T/\underline{T})$.

Proposition 4.2.1. *[BtD95, Proposition IV.4.7] The quotient spaces $fG_1/G(f)$ and $f\underline{T}/\underline{N}$ coincide.*

Proposition 4.2.2. *Suppose G_1 is simply connected. An element $g \in f\underline{T}$ is regular if and only if it is not fixed by any element in $\underline{W}_1$.*

Proof. The proof is completely analogous to the proof of Proposition 4.1.6 after noting that the fixed subgroup of $\underline{W}_1$ under conjugation by g is the Weyl group of G_1^g. □

Assume from now on that $Ad(f)$ is a diagram automorphism of $\mathfrak{g}$ and that G_1 is simply connected. In that case, we choose $\underline{T}$ to be the identity component of T^f, the fixed subset of T.

Lemma 4.2.3. *If G_1 is simply-connected, then*

$$Z_{G_1}(\underline{T}) = T$$

where $Z_{G_1}(\underline{T})$ is the centralizer of $\underline{T}$ in G_1.

Proof. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ be the Weyl vector. It is not fixed by any element of the Weyl group of G_1 but it is fixed by f. In particular, $b(\rho) \in \underline{\mathfrak{t}}$ where $b : \mathfrak{t}^* \to \mathfrak{t}$ is the map given by the Killing form. It follows that $\underline{\mathfrak{t}}$ is not fixed by any simple reflection, and hence the identity component of $Z_{G_1}(\underline{T})$ is T. Since

$\underline{\mathfrak{t}}$ is not fixed by any element of the Weyl group, $Z_{G_1}(\underline{T})$ is connected, since the Weyl group of $Z_{G_1}(\underline{T})$ is the centralizer of $\underline{\mathfrak{t}}$ in W. □

It follows from the above Lemma that an element normalizing $\underline{T}$ also normalizes T. Hence $\underline{N} \subset N$ and there is a map

$$\underline{\tilde{W}}_1 \subset \underline{W}_1 = \underline{N}_1/\underline{T} \to N_1/T = W_1 \tag{4.2.4}$$

Proposition 4.2.5. *If G_1 is simply-connected, then the map* (4.2.4) *is injective with the image equal to W_1^f.*

Proof. The kernel of the map is $T^f/\underline{T}$ and vanishes by Lemma 4.2.6 below. The remaining statement is proved in [BFM99, Proposition 8.4.3]. □

Lemma 4.2.6. *If G_1 is simply connected, then*

$$T^f = \underline{T}$$

Proof. The statement of the lemma is equivalent to the following statement on the level of the Lie algebra: if $x \in \mathfrak{t}$ such that $(\mathrm{Ad}_f(x) - x) \in \Pi$, then $x \in \Pi + \underline{\mathfrak{t}}$. Suppose we are given such x. Since G_1 is semisimple, the simple coroot vectors h_i form a basis for $\mathfrak{t}$. Let $x = \sum_{i=1}^{k} a_i h_i$. Since we are assuming Ad_f is a diagram automorphism, it permutes the simple coroot vectors, i.e. given by an element $\sigma_f \in \mathrm{Sym}_k$. We therefore have

$$\mathrm{Ad}_f(x) - x = \sum_{i=1}^{k} (a_{\sigma_f(i)} - a_i) h_i.$$

Since G_1 is simply connected, the simple coroot vectors generate the lattice Π. It follows that the coefficients $(a_{\sigma_f(i)} - a_i)$ are integers for all i. The subspace $\underline{\mathfrak{t}}$ consists of elements $\sum_{i=1}^{k} \gamma_i h_i$ such that $\gamma_i = \gamma_j$ if i is in the σ_f orbit of j. The element $x - \sum_{i=1}^{k} \lfloor a_i \rfloor h_i$ therefore belongs to $\underline{\mathfrak{t}}$. □

Corollary 4.2.7. *If G_1 is simply connected, then there is a split short exact sequences*

$$1 \to [T/\underline{T}]^f \to \underline{W}_1 \to W_1^f \to 1.$$

Proof. The image of $\underline{W}_1 \to W_1$ is certainly contained in W_1^f and by Proposition 4.2.5 it is equal to it. The splitting is obtained by the composition $W_1^f \to \underline{\tilde{W}}_1 \hookrightarrow \underline{W}_1$ where the first map is the inverse of the map (4.2.4) □

Corollary 4.2.8. *Let $G = G_1 \rtimes F$ with F acting by diagram automorphisms and G_1 simply connected. Fix $f \in F$. We then have split short exact sequences*

$$1 \to [T/\underline{T}]^f \to \underline{W} \to W_1^f \rtimes H \to 1 \tag{4.2.9}$$

$$1 \to \underline{\tilde{\Pi}} \to \underline{W_{aff}} \to W_1^f \rtimes H \to 1 \tag{4.2.10}$$

where $H = \mathrm{Stab}_F(f)$. Additionally, we have

$$W_1^f \rtimes H \cong \underline{\tilde{W}}$$

Chapter 5

Computation of $K^\tau(G/\!\!/G)$

For G be a compact Lie group and τ a twisting of the groupoid $G/\!\!/G$ where G acts on itself by conjugation, the twisted K-theory $K^\tau(G/\!\!/G)$ has been computed in [FHT11b, part III]. This section is mostly a summary of their computation.

5.1 Local Induction Model

Let G be a compact group, $T \subset G_1$ a maximal torus. Consider the action groupoids $\mathfrak{g}/\!\!/G$ and $\mathfrak{t}/\!\!/N$ with the adjoint action. The fibers of the map $\iota : \mathfrak{t}/\!\!/N \to \mathfrak{g}/\!\!/G$ are compact manifolds and therefore we can push-forward and pull-back K-theory classes along it. More directly, we have $\mathfrak{t}/\!\!/N \cong \mathfrak{t} \times_N G/\!\!/G$ and the map

$$\mathfrak{t} \times_N G/\!\!/G \to \mathfrak{g}/\!\!/G$$

where $[(x, g)] \in \mathfrak{t} \times_N G$ is mapped to $\mathrm{Ad}_{g^{-1}} x \in \mathfrak{g}$. Note that the relative tangent bundle is trivial.

Proposition 5.1.1. *[FHT11b, Proposition 4.13] Let $\tau \in \mathrm{Twist}(\mathfrak{g}/\!\!/G)$ be a twisting. The following composition is the identity map*

$$K^\tau_G(\mathfrak{g})_{\mathrm{cpt}} \xrightarrow{\iota^*} K^\tau_N(\mathfrak{t})_{\mathrm{cpt}} \xrightarrow{\iota_*} K^\tau_G(\mathfrak{g})_{\mathrm{cpt}}$$

Proof. This composition is the multiplication by the class $\iota_*(1) \in K^0_G(\mathfrak{g})$. The restriction of $\iota_*(1)$ to any regular orbit $\mathcal{O} \subset \mathfrak{g}$ is 1 since ι is a diffeomorphism at regular orbits. The intersections of kernels of restriction maps $K^0_G(\mathfrak{g}) \to K^0_G(\mathcal{O})$ for all $\mathcal{O}$ vanishes. It follows that $\iota_*(1) = 1$. □

Claim 5.1.2. Consider the induction map

$$K^\tau_N(\mathfrak{t})_{\mathrm{cpt}} \xrightarrow{\iota_*} K^\tau_G(\mathfrak{g})_{\mathrm{cpt}}.$$

We have $K^\tau_N(\mathfrak{t})_{\mathrm{cpt}} = K^{\tau'-\mathfrak{t}}_W(\Lambda^\tau)_{\mathrm{cpt}}$ by Thom isomorphism. The push-forward of a class supported on a W-orbit $\mathcal{O}$ of Λ^τ vanishes if W_1 does not act freely on $\mathcal{O}$.

By functoriality of the push-forward map, the map in the statement of the claim is given by the push-forward

$$K^{\tau-\mathfrak{t}}_N(*) = K^{\tau-\mathfrak{t}}_G(N\backslash G) \xrightarrow{\pi_*} K^{\tau-\mathfrak{g}}_G(*).$$

This is the Dirac induction map studied in [Sle87, Lan00] and the statement should follow from their work. We will not be using this claim to prove new results.

5.2 Component of the Identity

Let G be a compact Lie group. We use the notation of Section 4.2. Let $\tau \in \mathrm{Twist}(G/\!\!/G)$ where G acts on G by conjugation. For G non-connected, the groupoid $G/\!\!/G$ is the disjoint union of groupoids corresponding to conjugacy classes of the group $\pi_0 G$. The K-theory group $K^\tau_G(G)$ is hence a direct sum of terms corresponding to these conjugacy classes. In this subsection we analyze the summand $K^\tau_G(G_1)$.

We first consider the subgroupoid $T/\!\!/N$ which has the same coarse moduli space as $G_1/\!\!/G$ by Proposition 4.2.1.

The Mackey decomposition construction of Section 2.5 applied to $T/\!\!/N$ gives rise to a W-equivariant covering space $\pi : Y \to T$ and a twisting $\tau' \in \mathrm{Twist}(Y/\!\!/W)$ such that $K^{\tau'}_W(Y)_{\mathrm{cpt}} \cong K^\tau_N(T)$. Denote by Λ^τ the fiber over the identity $e \in T$. It is the set of twisted representations of T, where the twisting is the restriction of τ to $\{e\}/\!\!/T$. In particular it is a W-equivariant torsor for Λ. We have $\pi_1(T) \cong \Pi$ and the theory of covering spaces gives rise to a W-equivariant Λ-affine group action of Π on Λ^τ.

Definition 5.2.1. The holonomy around T defines a W-equivariant homomorphism

$$\kappa : \Pi \to \Lambda$$

Definition 5.2.2. A twisting τ is called *regular* if the corresponding map $\kappa : \Pi \to \Lambda$ is injective.

If τ is a regular twisting, then Y is a union of affine $\mathfrak{t}$ spaces. It is moreover a W equivariant vector bundle over a finite number of points. We locate the zero section of Y when viewed as a vector bundle.[1]

[1] Non-equivariantly, an affine space can be identified with the vector space by choosing a point and declaring it to be the zero vector. When there is a group action, this point needs to be a fixed point.

The group $W_{aff} = \Pi \rtimes W$ acts on Λ^τ with $\pi \in \Pi \subset W_{aff}$ acting by translation by $\kappa(\pi)$. An element $t \in \mathfrak{t}$ defines the following path in T: $s \mapsto \exp(s \cdot t)$ for $s \in [0, 2\pi]$. For $(\lambda, t) \in \Lambda^\tau \times \mathfrak{t}$, denote by $\lambda + t$ the parallel transport of λ along the path defined by t. For $\pi \in \Pi$, we have $\lambda + \pi = \lambda + \kappa(\pi)$.

Proposition 5.2.3. *If τ is regular, then there exists a W_{aff}-equivariant map*

$$\sigma : \Lambda^\tau \to \mathfrak{t}.$$

If G_1 is semi-simple, then σ is unique.

Proof. The space $\Lambda^\tau_{\mathbb{R}} := \Lambda^\tau \otimes \mathbb{R}$ is an affine space for $\mathfrak{t}^*$ with an equivariant W action. Since W is a finite group, there exists a W fixed point λ_0 of $\Lambda^\tau_{\mathbb{R}}$ which identifies it with $\mathfrak{t}^*$ by mapping λ_0 to 0. The inverse of the $\mathbb{R}$-extension $\kappa_{\mathbb{R}} : \mathfrak{t} \to \mathfrak{t}^*$ then defines the desired map σ. If G_1 is semi-simple, then there is a unique W fixed point of $\mathfrak{t}$ and therefore there is a unique W-equivariant identification of $\Lambda^\tau_{\mathbb{R}}$ with $\mathfrak{t}^*$. $\square$

A map σ as in Proposition 5.2.3 determines an injective map

$$\begin{aligned} \sigma' : \Lambda^\tau / \kappa(\Pi) &\to Y \\ \lambda &\mapsto \lambda - \sigma(\lambda) \end{aligned}$$

Viewing this map as the zero section, it expresses Y as a W-equivariant vector bundle over $\Lambda^\tau/\kappa(\Pi)$ with constant fiber $\mathfrak{t}$. By Thom isomorphism we have

$$K^{\tau'}_W(Y)_{\mathrm{cpt}} \cong K^{\sigma'^*\tau' - \mathfrak{t}}_W(\Lambda^\tau/\kappa(\Pi)). \tag{5.2.4}$$

This implies that $K^\tau_N(T)$ is generated by classes pushed-forward from individual N-orbits, namely from $\pi(\mathrm{Im}(\sigma'))$.

Consider the inclusion map

$$w : T /\!\!/ N \to G_1 /\!\!/ G.$$

The map w is the identity on the coarse moduli spaces. The fiber over a point $t \in T \subset G_1$ is G^t/N^t. In particular, the map is a diffeomorphism over regular elements of G. The relative tangent bundle of w is trivial, which allows us to define the push-forward map in K-theory. Explicitly, it is computed by expressing w as a morphism of global quotient groupoids. We have $T /\!\!/ N \cong T \times_N G /\!\!/ G$ where the action of N on T is by conjugation and on G by left multiplication. The action of G on $T \times_N G$ is by right multiplication on G. The map w is then

$$\begin{aligned} w : T \times_N G /\!\!/ G &\to G_1 /\!\!/ G \\ [t, g] &\mapsto g^{-1} t g \end{aligned}$$

We thus have maps

$$\begin{aligned} w^* &: K^\tau_G(G_1) \to K^\tau_N(T) \\ w_* &: K^\tau_N(T) \to K^\tau_G(G_1) \end{aligned}$$

Proposition 5.2.5. *[FHT11b, Theorem 7.9] If τ is a regular twisting then $w_* \circ w^* : K^\tau_G(G_1) \to K^\tau_G(G_1)$ is the identity map.*

Proof. Fix a point $t \in T$ and consider the following diagram:

$$\begin{array}{ccc} \mathfrak{g}^t /\!\!/ G^t & \xrightarrow{x \mapsto t \cdot \exp(x)} & G_1 /\!\!/ G \; . \\ {\scriptstyle \iota}\uparrow & & \uparrow \\ \mathfrak{t} /\!\!/ N^t & \longrightarrow & T /\!\!/ N \end{array}$$

Over a neighborhood of $0 \in \mathfrak{g}^t$, the horizontal maps in the above diagram are local equivalences. The map $\iota^* \circ \iota_*$ is the identity map on K-theory by Proposition 5.1.1. It follows that we can find a good cover of the moduli space of $G_1 /\!\!/ G$ such that restriction of $w_* \circ w^*$ to the open sets and their intersections is the identity map. By the Mayer-Vietoris argument for K-theory with compact supports, $w_* \circ w^*$ is an isomorphism. In particular, w_* is surjective. If τ is a regular twisting, it follows from isomorphism (5.2.4) that $K_N^\tau(T)$ is generated by classes which are push-forwards from individual N orbits. The composition $w_* \circ w^*$ on w_* images of these classes is identity. Since w_* is surjective, $w_* \circ w^*$ is precisely the identity map. $\square$

In particular, w^* is injective and $K_G^\tau(G_1)$ is isomorphic to a summand in $K_N^\tau(T)$. We now identify this summand.

Definition 5.2.6. A weight $\lambda \in \Lambda^\tau$ is *regular* if $\pi \circ \sigma'(\lambda)$ is a regular element of G. Denote the set of regular weights by $\Lambda^\tau_{\text{reg}}$.

Proposition 5.2.7. *[FHT11b, Theorem 7.10] Assume τ is regular and $\sigma : \Lambda^\tau \to \mathfrak{t}$ is a W_{aff}-equivariant map. Then*

$$K_G^\tau(G_1) \cong K_W^{\sigma'^*\tau' - \mathfrak{t}}(\Lambda^\tau_{\text{reg}}/\kappa(\Pi)).$$

Proof. Let $\mathcal{O} \subset \Lambda^\tau/\kappa(\Pi)$ be a W-orbit and $\lambda \in \mathcal{O}$. Under isomorphism (5.2.4) a class supported on $\mathcal{O}$ corresponds to the push-forward of a class $V \in K^\tau_N(\pi \circ \sigma'(\mathcal{O})) \cong K^\tau_{\mathrm{N}^t}(\{t\})$ where $t = \pi \circ \sigma'(\lambda)$. Moreover, V is supported on a single weight. By the local description of w at t described in the proof of Proposition 5.2.5 and by Claim 5.1.2, it follows that the pushforward of this class to $K^\tau_G(G)$ vanishes if t is not a regular element of G. On the other hand if t is a regular element, then w is a diffeomorphism near t and hence w^*w_* is the identity on this class. □

The regular weights in Λ^τ are determined by the action of W_{aff} on Λ^τ.

Proposition 5.2.8. *If G_1 is simply-connected, then $\lambda \in \Lambda^\tau$ is regular if and only if it has trivial stabilizer in W_{aff_1}.*

Proof. For $t \in T$, the centralizer G_1^t of t has T as a maximal torus and $\mathrm{Stab}_{W_1}(t)$ as the Weyl group. The element t is regular if G_1^t has a torus for its component of the identity. Since G_1 is simply connected, by Proposition 4.2.2, G_1^t is connected and hence t is regular if and only if $\mathrm{Stab}_{W_1}(t)$ is trivial. For $x \in \mathfrak{t}$, the group element $\exp(2\pi x)$ has trivial stabilizer in W_1 if and only if x has trivial stabilizer in W_{aff_1}. Since $\Lambda^\tau \xrightarrow{\sigma} \mathfrak{t}$ is an injective W_{aff}-equivariant map, the result follows. □

If $\pi_1(G_1)$ has torsion, then there exists a finite cover $\tilde{G}_1 \to G_1$ where $\tilde{G}_1$ is simply-connected.The preimage $\tilde{T}$ of T is a maximal torus of $\tilde{G}_1$ and the lattice $\tilde{\Pi} = \ker \exp(2\pi \cdot)_{\tilde{T}}$ is a sublattice of Π. An element $g \in G_1$ is regular if and only if any of its preimages in $\tilde{G}_1$ is regular. Therefore $\exp(2\pi t)$ is regular if and only if it has a trivial stabilizer in $\widetilde{W_{aff_1}} := \tilde{\Pi} \rtimes W_1$.

We have that if G_1 is simply connected, then

$$K_G^\tau(G_1) = K_{\pi_0(G)}^{\sigma'^*\tau-\mathfrak{t}}(\Lambda_{\text{reg}}^\tau / W_{aff_1}). \tag{5.2.9}$$

since $W/W_1 \cong \pi_0(G)$ and the following groupoids are equivalent

$$\left(\Lambda_{\text{reg}}^\tau / W_{aff_1}\right) /\!\!/ (W/W_1) \cong \left(\Lambda_{\text{reg}}^\tau / \kappa(\Pi)\right) /\!\!/ W.$$

When G is connected, the right hand side of (5.2.9) is the K-theory of a finite number of points. If $\deg \tau = \dim G \mod 2$ then the degree of $\sigma'^*\tau - \mathfrak{t}$ is zero and

$$K_G^\tau(G) \cong \mathbb{Z}^{\oplus \#\left(\Lambda_{\text{reg}}^\tau / W_{aff}\right)}.$$

If $\deg \tau \neq \dim G \mod 2$, then the K-theory vanishes.

We now discuss how to find κ given an isomorphism class of τ. Let T be a torus and τ a twisting of $T /\!\!/ T$. The isomorphism class of τ lies in

$$H_T^0(T, \mathbb{Z}/2) \times H_T^1(T, \mathbb{Z}/2) \times H_T^3(T, \mathbb{Z})$$

and we have

$$H_T^3(T, \mathbb{Z}) \cong H^2(BT, \mathbb{Z}) \otimes H^1(T, \mathbb{Z}) \oplus H^3(T, \mathbb{Z}).$$

The twistings that live in $H^3(T, \mathbb{Z})$ and $H_T^1(T, \mathbb{Z}/2) \cong H^1(T, \mathbb{Z}/2)$ are pulled back from a twisting on $T /\!\!/ \{*\}$ and do not affect the map κ. We have a natural isomorphism

$$H^2(BT, \mathbb{Z}) \otimes H^1(T, \mathbb{Z}) \cong \Lambda \otimes \Lambda.$$

Proposition 5.2.10. *For τ a twisting of $T/\!\!/T$, the map κ is given by the component of the isomorphism class of τ in $H^2(BT,\mathbb{Z})\otimes H^1(T,\mathbb{Z})$ via the map*

$$H^2(BT,\mathbb{Z})\otimes H^1(T,\mathbb{Z})\cong\Lambda\otimes\Lambda\cong\Lambda\otimes\Pi^*\cong\hom(\Pi,\Lambda).$$

Note that it is important which copy of Λ we contract with Π. Here, we contract the copy corresponding to $H^1(T,\mathbb{Z})$.

Proof. An element $\pi\in\Pi$ corresponds to a loop in T and the value of $\kappa(\pi)\in\Lambda$ is the character of the line bundle over the groupoid $\{e\}/\!\!/T$ corresponding to the automorphism of $\tau\big|_{\{e\}}$ determined by this loop. Consider $\overline{\tau}$, the pullback of τ along the map $S^1/\!\!/T\to T/\!\!/T$ determined by π. By Proposition 2.3.22, the desired automorphism is $\pi_*(\overline{\tau}-\pi^*(\tau\big|_{\{e\}}))$. Since we only care about the character of this line and not its $\mathbb{Z}/2$ grading, we can restrict τ to a class in $H^3_T(T,\mathbb{Z})$ and compute the push-forward in cohomology. For a class α in $H^3_T(T,\mathbb{Z})\cong H^2(BT,\mathbb{Z})\otimes H^1(T,\mathbb{Z})\oplus H^3(T,\mathbb{Z})$, its pullback to $S^1/\!\!/T$ followed by push-forward to $\{e\}/\!\!/T$ is precisely contraction of its component in $H^2(BT,\mathbb{Z})\otimes H^1(T,\mathbb{Z})$ with π.

□

Example 5.2.11. Let us consider the group $G=SU(2)$. From the discussion underneath (5.2.9), it suffices to determine the action of W_{aff} on Λ^τ to compute $K^\tau_G(G)$.

The isomorphism classes of twistings of $G/\!\!/G$ are classified by

$$H^0_G(G,\mathbb{Z}/2)\times H^1_G(G,\mathbb{Z}/2)\times H^3_G(G,\mathbb{Z})\cong\mathbb{Z}/2\times\{0\}\times\mathbb{Z}.$$

To see that $H^3_G(G,\mathbb{Z}) \cong \mathbb{Z}$, one can consider the Serre spectral sequence for the fibration

$$\begin{array}{ccc} G & \longrightarrow & G \times_G EG\,. \\ & & \downarrow \\ & & BG \end{array}$$

Since G is connected and simply connected, many of the low degree terms vanish and $H^3_G(G,\mathbb{Z})$ is the kernel of the fourth page differential

$$d_4 : \mathbb{Z} \cong H^3(G,\mathbb{Z}) \to H^4(BG,\mathbb{Z}) \cong \mathbb{Z}.$$

Since the fibration has a splitting given by the fixed point $e \in G$ for the conjugation action, every non-zero class of $H^4(BG,\mathbb{Z})$ lifts to a non-zero class in $H^4_G(G,\mathbb{Z})$. This implies that d_4 is the zero map.

We can restrict a generator of $H^3_G(G,\mathbb{Z})$ to

$$H^3_T(T,\mathbb{Z}) \cong H^2(BT,\mathbb{Z}) \otimes H^1(T,\mathbb{Z}) \cong \Lambda \otimes \Lambda$$

where we have used that $H^3(T,\mathbb{Z}) \cong \{0\}$. A generator of $H^3_G(G,\mathbb{Z})$ maps to twice a generator of $\Lambda \otimes \Lambda$. This can be verified explicitly using the Cartan model for equivariant cohomology.[2] Moreover, $\Lambda \otimes \Lambda$ has a natural choice of a generator: $\lambda \otimes \lambda$ where λ is any generator of Λ. We trivialize $H^3_G(G,\mathbb{Z})$ by picking the generator that is mapped to twice the distinguished generator of $\Lambda \otimes \Lambda$. An isomorphism class of τ is given by an integer $k \in H^3_G(G,\mathbb{Z}) \cong \mathbb{Z}$ and degree $n \in \mathbb{Z}/2\mathbb{Z}$.

[2]See e.g. [AB84]

The integer k specifies the action of Π on Λ^τ: a generator π shifts Λ^τ by the character $\kappa(\pi)$ specified uniquely by $\langle \kappa(\pi), \pi \rangle = 2k$. To compute the rank of K-theory in question, it remains to determine the action of $W = \{\pm 1\}$ on Λ^τ. The action on $\Lambda^\tau \otimes \mathbb{R}$ fixes an element x. There are two possibilities: either $x \in \Lambda^\tau$ or $x \in \frac{1}{2}\Lambda^\tau$. In the former case, sending x to $0 \in \Lambda$ gives an isomorphism between Λ and Λ^τ as W_{aff} spaces. Proposition 5.2.12 below implies that this is in fact the case. We see that there are $k-1$ free orbits of Λ^τ under the action of W_{aff} and therefore

$$K_G^{(1,0,k)}(G) \cong \mathbb{Z}^{\oplus k-1}.$$

Proposition 5.2.12 ([FHT11a, Lemma 4.20])**.** *Let G be a compact simply-connected group and τ a twisting of $G/\!\!/G$. Restricting the twisting to $T/\!\!/N$, the construction of Section 2.5 defines an action of W on Λ^τ. This action has a fixed point.*

Proof. The twisting τ restricted to $\{e\}/\!\!/G$ determines a graded central extension which restricts to one on T.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & G^\tau & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & T^\tau & \longrightarrow & T & \longrightarrow & 1 \end{array}$$

The set Λ^τ is the set of splittings of $\mathbb{T} \to T^\tau$ and the action of W is by conjugation by elements in G^τ. Since G is simply connected, there is a splitting $G^\tau \to \mathbb{T}$. Restricting this splitting to $T^\tau \subset G^\tau$ gives the desired element in Λ^τ. $\square$

5.3 Other Components

We only consider the case where $G = G_1 \rtimes F$ with F acting by diagram automorphisms and G_1 simply connected. Fix an element $f \in F \subset G$ and let $G(f)$ be the stabilizer of the component fG_1. We use the notation of Section 4.2. We analyze the group $K^{\tau}_{G(f)}(fG_1)$. The full K-theory of $G/\!\!/G$ is then the direct sum of these K-theory groups for f belonging to different conjugacy classes of $\pi_0(G)$.

We first consider the subgroupoid $f\underline{T}/\!\!/\underline{N}$ which has the same coarse moduli space as $fG_1/\!\!/G(f)$ by Proposition 4.2.1. The Mackey decomposition construction of Section 2.5 applied to $f\underline{T}/\!\!/\underline{N}$ gives rise to a $\underline{W}$-equivariant covering space $\pi : \underline{Y} \to f\underline{T}$ and a twisting $\tau' \in \mathrm{Twist}(\underline{Y}/\!\!/\underline{W})$ such that $K^{\tau'}_{\underline{W}}(Y)_{\mathrm{cpt}} \cong K^{\tau}_{\underline{N}}(f\underline{T})$. Denote by $\underline{\Lambda}^{\tau}$ the fiber over f. It is a $\underline{\tilde{W}}$-equivariant $\underline{\Lambda}$ torsor. The entire group $\underline{W}$ does not act on $\underline{\Lambda}^{\tau}$ since $\underline{W}$ does not fix f.

Definition 5.3.1. The holonomy around $\underline{T}$ defines a $\underline{\tilde{W}}$ equivariant map

$$\kappa : \underline{\Pi} \to \underline{\Lambda}$$

Definition 5.3.2. A twisting τ is called *regular* on fG_1 if the holonomy map $\sigma : \underline{\Pi} \to \underline{\Lambda}$ is injective.

If τ is regular on fG_1, then $\underline{Y}$ is a $\underline{W}$-equivariant vector bundle over a finite number of points that we now locate. Define the following group

$$\underline{W_{aff}} = \{(x, h) \in \underline{\mathfrak{t}} \rtimes \underline{W} \mid h^{-1} \cdot f \cdot \exp(2\pi x) \cdot h = f\}$$

We have that $\underline{W_{aff}}$ acts on $\underline{\Lambda}$. To describe the action, first note that $\underline{\mathfrak{t}}$ acts on $\underline{Y}$ by parallel transport: for $y \in \underline{Y}, t \in \underline{T}$, denote the parallel transport of y along the path $\pi(y) \cdot \exp(s \cdot t)$ for $s \in [0, 2\pi]$ by $\lambda + t$. This action is equivariant with respect to the $\underline{W}$ action on $\underline{Y}$ and $\underline{\mathfrak{t}}$. We thus get an action of $\underline{\mathfrak{t}} \rtimes \underline{W}$ on $\underline{Y}$ which moreover maps fibers to fibers of the projection $\underline{Y} \to f\underline{T}$. The group $\underline{W_{aff}}$ is the stabilizer of the fiber over f. Explicitly, the action is given by

$$\lambda.(t, h) = (\lambda + t).h$$

Define also the following subgroup

$$\underline{\tilde{\Pi}} = \underline{W_{aff}} \cap (\underline{\mathfrak{t}} \times (T/\underline{T}))$$

The kernel of the projection $\pi_1 : \underline{\tilde{\Pi}} \to \mathfrak{t}$ is $T^f/\underline{T}$, which is the trivial group by Lemma 4.2.6. We will identify $\underline{\tilde{\Pi}}$ with $\pi_1(\underline{\tilde{\Pi}}) \subset \underline{\mathfrak{t}}$ which is a lattice containing $\underline{\Pi}$. For $\xi \in \underline{\Lambda}^\tau, \underline{\lambda} \in \Lambda, (t, h) \in \underline{W_{aff}}$, we have

$$(\xi + \lambda).(t, h) = \xi.(t, h) + \lambda.h$$

and therefore the action of $\underline{\tilde{\Pi}}$ on $\underline{\Lambda}^\tau$ is equivariant with respect to the $\underline{\Lambda}$-action on $\underline{\Lambda}^\tau$. It is therefore given by a homomorphism $\underline{\tilde{\Pi}} \to \underline{\Lambda}$ which restricts to κ on $\underline{\Pi}$. It should not cause confusion if we call this homomorphism κ as well.

$$\kappa : \underline{\tilde{\Pi}} \to \underline{\Lambda}$$

Proposition 5.3.3. *If τ is regular on fG_1, then there exists a $\underline{W_{aff}}$-equivariant map*

$$\sigma : \underline{\Lambda}^\tau \to \underline{\mathfrak{t}}.$$

If G_1^f is semi-simple, then σ is unique.

Proof. The space $\underline{\Lambda}^\tau_{\mathbb{R}} := \underline{\Lambda}^\tau \otimes \mathbb{R}$ is an affine space for $\underline{\mathfrak{t}}^*$ with an equivariant $\underline{\tilde{W}}$ action. Since $\underline{\tilde{W}}$ is a finite group, there exists a $\underline{\tilde{W}}$ fixed point of $\underline{\Lambda}^\tau_{\mathbb{R}}$ which identifies it with $\underline{\mathfrak{t}}^*$. Let σ be the inverse of the $\mathbb{R}$-extension $\kappa_{\mathbb{R}} : \underline{\mathfrak{t}} \to \underline{\mathfrak{t}}^*$. It is manifestly equivariant with respect to $\underline{\Pi} \rtimes \underline{\tilde{W}}$. Since the action of $\underline{\tilde{\Pi}}$ is by translation by κ, it is also equivariant with respect to $\underline{\tilde{\Pi}}$. Since $\underline{\tilde{\Pi}}$ and $\underline{\tilde{W}}$ generate $\underline{W_{aff}}$, the result follows. □

A map σ determines an injective map

$$\sigma' : \underline{\Lambda}^\tau/\kappa(\underline{\Pi}) \to \underline{Y}$$
$$\lambda \mapsto \lambda - \sigma(\lambda)$$

The subgroup $\underline{\Pi} \subset \underline{W_{aff}}$ acts on $\underline{\Lambda}^\tau$ by the homomorphism κ and the quotient group $\underline{W_{aff}}/\underline{\Pi}$ is isomorphic to $\underline{W}$. In particular, this defines a $\underline{W}$ action on $\underline{\Lambda}^\tau/\kappa(\underline{\Pi})$ and the map σ' is $\underline{W}$-equivariant. This map expresses $\underline{Y}$ as a $\underline{W}$-equivariant vector bundle over $\underline{\Lambda}^\tau/\kappa(\underline{\Pi})$ with constant fiber $\underline{\mathfrak{t}}$. By Thom isomorphism, we again get

$$K_{\underline{W}}^{\tau'-\underline{\mathfrak{t}}}(\underline{Y}) \cong K_{\underline{W}}^{\sigma'^*\tau'-\underline{\mathfrak{t}}}(\underline{\Lambda}^\tau/\kappa(\underline{\Pi}))$$

Let

$$w : f\underline{T}/\!\!/\underline{N} \to fG_1/\!\!/G(f)$$

be the inclusion map. Locally, on a neighborhood of $f' \in f\underline{T}$ the map w has the form of the local restriction map we considered in Section 5.1.

$$\begin{array}{ccc} \mathfrak{g}^{f'} /\!\!/ G^{f'} & \xrightarrow{x \mapsto f' \cdot \exp(x)} & fG_1 /\!\!/ G(f) \\ {\scriptstyle \iota}\uparrow & & \uparrow \\ \underline{\mathfrak{t}} /\!\!/ N(G^{f'}) & \longrightarrow & f\underline{T} /\!\!/ \underline{N} \end{array}$$

where $N(G^{f'})$ is the normalizer of $\underline{T}$ in $G^{f'}$.

Proposition 5.3.4. *[FHT11b, Theorem 7.9] If τ is a regular twisting on fG_1, then $w_* \circ w^* : K^{\tau}_{G(f)}(fG_1) \to K^{\tau}_{G(f)}(fG_1)$ is the identity map.*

Proof. The proof is analogous to the proof of Proposition 5.2.5 □

Definition 5.3.5. A weight $\lambda \in \underline{\Lambda}^{\tau}$ is *regular* if $\pi \circ \sigma'(\lambda)$ is a regular element of G. Denote the set of regular weights by $\underline{\Lambda}^{\tau}_{\text{reg}}$.

Proposition 5.3.6. *[FHT11b, Theorem 7.10] Assume τ is a regular twisting on fG_1 and fix $\sigma : \underline{\Lambda}^{\tau} \to \underline{\mathfrak{t}}$, a $\underline{W_{aff}}$-equivariant map. Then*

$$K^{\tau}_{G(f)}(fG_1) \cong K^{\sigma'^{*}\tau' - \underline{\mathfrak{t}}}_{\underline{W}}(\underline{\Lambda}^{\tau}_{\text{reg}}/\kappa(\underline{\Pi})).$$

Proof. The proof is analogous to the proof of Proposition 5.2.7 □

Proposition 5.3.7. *A weight $\xi \in \underline{\Lambda}^{\tau}$ is regular if and only if it has trivial stabilizer in $\underline{W_{aff}}_1$. In particular, we have*

$$K^{\tau}_{G(f)}(fG_1) \cong K^{\sigma'^{*}\tau' - \underline{\mathfrak{t}}}_{H}(\underline{\Lambda}^{\tau}_{\text{reg}}/\underline{W_{aff}}_1).$$

where $H := \text{Stab}_{\pi_0(G)}(fG_1)$.

Proof. The proof is analogous to the proof of Proposition 5.2.8. □

Proposition 5.3.7 identifies K-theory classes in $K_G^\tau(fG_1)$ which are pushforwards from regular orbits in fG_1. In the cases we will be considering those classes can be pushed forward from $\{f\}/\!\!/G^f$ instead. We now fix a condition on the action of $\underline{W_{aff}}$ on Λ^τ which will ensure that that is the case.

Condition 5.3.8. For every $\underline{W_{aff}}$-orbit $\tilde{\mathcal{O}}_i \subset \underline{\Lambda}^\tau_{\text{reg}}$ there is an element $\beta_i \in \tilde{\mathcal{O}}_i$ such that

$$H_i := \text{Stab}_{\underline{W_{aff}}}(\beta_i) \subset H \subset \underline{\tilde{\Pi}} \rtimes W_1^f \rtimes H$$

Let $\mathcal{O}_i \subset \underline{\Lambda}^\tau_{\text{reg}}$ be the $\underline{\tilde{W}}$ orbit of β_i. Then $\bigcup_i \mathcal{O}_i \times \underline{\mathfrak{t}}/\!\!/\underline{\tilde{W}}$ is equivalent to $\underline{Y}_{\text{reg}}/\!\!/\underline{W}$ with the equivalence map given by $(\lambda, t) \mapsto \lambda + t$. The stabilizer condition ensures that this map is injective.

Proposition 5.3.9. *If Condition 5.3.8 is satisfied, we have*

$$K^\tau_{G(f)}(fG_1) \cong \bigoplus_i K^{\tau'-\underline{\mathfrak{t}}}_{\underline{\tilde{W}}}(\mathcal{O}_i) \cong \bigoplus_i K^{\tau'-\underline{\mathfrak{t}}}_{H_i}(\{\beta_i\})$$

We now discuss the action of $\underline{W_{aff}}$ on $\underline{\Lambda}^\tau$.

Proposition 5.3.10. *If the isomorphism class of τ restricted to $\{f\}/\!\!/G_1^f$ has trivial component in $H^3_{G_1^f}(*, \mathbb{Z})$, then there is a unique $\underline{W_{aff}}$ equivariant isomorphism $\underline{\Lambda} \cong \underline{\Lambda}^\tau$ where the action of $\underline{W_{aff}} \subset \underline{\mathfrak{t}} \times \underline{W}$ on $\underline{\Lambda}$ is given by $\lambda.(t, w) = \kappa(t).w + \lambda.w$.*

Proof. The twisting τ restricted to $\{f\}/\!\!/G_1^f$ determines a central extension

$$1 \to \mathbb{T} \to \tilde{G}_1^f \to G_1^f \to 1 \tag{5.3.11}$$

which restricts to a central extension

$$1 \to \mathbb{T} \to \underline{\tilde{T}} \to \underline{T} \to 1$$

whose set of splittings is $\underline{\Lambda}^\tau$. The Weyl group of G_1^f is W_1^f and its action on $\underline{\Lambda}^\tau$ is given by conjugation. The condition on τ in the statement of the proposition is equivalent to sequence (5.3.11) splitting. Precomposing a splitting $\tilde{G}_1^f \to \mathbb{T}$ with the inclusion $\underline{\tilde{T}} \hookrightarrow \tilde{G}_1^f$ defines a twisted weight λ_0 which is fixed by W_1^f. Since G_1^f is semisimple, the element $0 \in \underline{\Lambda}$ is the unique element fixed by W_1^f. This implies that λ_0 is the unique element of $\underline{\Lambda}^\tau$ fixed by W_1^f and hence is also fixed by H. The desired map is the one sending $0 \in \underline{\Lambda}$ to $\lambda_0 \in \underline{\Lambda}^\tau$. $\square$

Chapter 6

Computation of the Twisting

In this section we compute the aspects of the twisting necessary to compute $K^{\tau}_{G}(G)$ for the twisting

$$\tau_l = \dim \mathfrak{g} + \mathrm{tg}(l) + h_G$$

defined in Definition 3.3.3. Here, $\mathrm{tg}(l)$ is the transgressed level and $h_G = \mathrm{tg}(P(\bar{\mathfrak{g}})$ is the twisting corresponding to the canonical universal orientation. The level we will take in our computations will be a class which pulls back to $l_0 = 1 \in \mathbb{Z} \cong H^4(BG_1, \mathbb{Z})$ along the map $BG_1 \to BG$. We show that such class exists in Section 6.1. We compute the holonomy map κ in Section 6.2. In Section 6.3, we compute the restriction of h_G to $\{f\}/\!\!/G^f$.

6.1 Extension of a level from $H^4(BG_1)$ to $H^4(BG)$

For any compact Lie group G, there is a fibration

$$\begin{array}{ccc} BG_1 & \xrightarrow{i} & BG \\ & & \downarrow{\scriptstyle \pi} \\ & & B\pi_0(G) \end{array} \tag{6.1.1}$$

induced from the short exact sequence

$$1 \to G_1 \to G \to \pi_0(G) \to 1. \tag{6.1.2}$$

Given a level $l_0 \in H^4(BG_1)$ of G_1, an extension of l_0 to G is a class $l \in H^4(BG, \mathbb{Z})$ such that $i^* l = l_0$.

Proposition 6.1.3. *Suppose G_1 is simple simply-connected and the sequence (6.1.2) splits. Then any class $l_0 \in H^4(BG_1, \mathbb{Z})$ extends to a class $l \in H^4(BG, \mathbb{Z})$. The set of such extensions is a torsor for $H^4(B\pi_0(G), \mathbb{Z})$.*

Here, $H^4(B\pi_0(G), \mathbb{Z})$ acts via pullback along $\pi : BG \to B\pi_0(G)$.

Proof. The statement follows from the Serre spectral sequence of the fibration (6.1.1). Since G_1 is simply connected it follows that

$$H^k(BG_1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ \{0\} & k = 1, 2, 3 \end{cases}$$

and $H^4(BG_1, \mathbb{Z})$ is invariant under automorphisms of G_1. We thus have that the term $E_2^{0,4}$ in the Serre spectral sequence is $H^4(BG_1, \mathbb{Z})$ and an element $l_0 \in H^4(BG_1, \mathbb{Z})$ extends to $H^4(BG, \mathbb{Z})$ if and only if it is in the kernel of the differential map $d_5 : H^4(BG_1, \mathbb{Z}) \to H^5(B\pi_0(G), \mathbb{Z})$. Since we assume that the sequence (6.1.1) splits, every non-zero class in $H^5(B\pi_0(G), \mathbb{Z})$ lifts to a non-zero class in $H^5(BG, \mathbb{Z})$ and therefore the differential d_5 vanishes. Since $H^4(B\pi_0(G), \mathbb{Z})$ does not receive any non-trivial maps in the spectral sequence, it follows that the set of extensions is a torsor for $H^4(B\pi_0(G), \mathbb{Z})$. □

6.2 Restriction to maximal tori

In this section we compute κ of Definition 5.3.1. Let us first consider the case where G is simple and simply connected. Fix a level $l \in H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$

and consider the twisting $\tau = \tau_l$.

Proposition 6.2.1. *Let $l \in \mathbb{Z} = H^4(BG_1, \mathbb{Z})$. Then $\kappa : \mathfrak{t} \to \mathfrak{t}^*$ is a W-equivariant symmetric map such that*

$$\langle \kappa(h_\alpha), h_\alpha \rangle = 2(l + h_G^\vee)$$

where $h_G^\vee$ is the dual Coxeter number of G and h_α is a long simple coroot.

Proof. By Proposition 5.2.10, the holonomy map $\kappa : \Pi \to \Lambda$ is determined by the restriction of the component in H^3 of τ to $T/\!/T$. The cohomology group $H^3_T(T, \mathbb{Z})$ is torsion free, and therefore we don't lose any information by working with

$$\tau_{\mathbb{Q}} = \tau \otimes \mathbb{Q} \in Twist(G/\!/G) \otimes \mathbb{Q} \cong H^3_G(G, \mathbb{Q})$$

instead. By Proposition 2.3.19 the composition

$$BG \xrightarrow{\mathfrak{g}} BO \xrightarrow{\delta} \tilde{ko} \wedge H\mathbb{Q} \xrightarrow{P} \Sigma^1 |Twist| \wedge H\mathbb{Q} \xrightarrow{\gamma^{-1}} \Sigma^4 H\mathbb{Q}$$

equals $\frac{1}{2}p_1(\mathfrak{g})$. For a simple simply-connected group G, we have $p_1(\mathfrak{g}) = 2h_G^\vee \in H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$.[1] It follows that

$$\tau_{l\mathbb{Q}} = \mathrm{tg}(l + h_G^\vee)$$

is the transgression of $l + h_G^\vee \in H^4(BG, \mathbb{Q}) \cong \mathbb{Q}$ defined in Definition 3.3.2.

[1]The cohomology of $H^4(BSO(k), \mathbb{R})$ can be identified with the space of ad-invariant bilinear forms on $\mathfrak{so}(k)$. The first Pontryagin class corresponds to the bilinear form $(a, b) \mapsto tr(ab)$. The pullback of p_1 to $H^4(BG, \mathbb{R})$ is then the Killing form of $\mathfrak{g}$ which is $2h_G^\vee$ times the image of a generator of $H^4(BG, \mathbb{Z})$ in $H^4(BG, \mathbb{R})$.

The transgression map commutes with pull back to subgroups. Consider first $T \subset G$, a maximal torus. We have $H^1(T, \mathbb{Q}) = H^2(BT, \mathbb{Q}) \cong \mathfrak{t}^*$ and $H^4(BT, \mathbb{Q}) = \mathrm{Sym}^2(H^2(BT, \mathbb{Q}))$. Since the conjugation action of T on T is trivial, we have $LBT = B(T /\!\!/ T) = BT \times T$. The transgression map therefore has the following form

$$\begin{array}{ccc} BT \times T \times S^1 & \xrightarrow{e} & BT \\ \Big\downarrow{\scriptstyle \pi} & & \\ BT \times T & & \end{array}$$

where e on $T \times S^1$ component is given by homotopy equivalence $T \cong \Omega BT$, and on BT by the identity map. Implicitly this used an H-space structure on BT. For $a \in \mathfrak{t}^*$, let $\alpha_a \in H^1(T, \mathbb{Q}), \beta_a \in H^2(BT, \mathbb{Q})$ correspond to a and let $\eta \in H^1(S^1, \mathbb{Q})$ be the dual to the fundamental class of S^1. We have

$$e^*(\alpha_a) = \alpha_a + \eta\beta_a$$

It follows that

$$\begin{aligned} e^*(\alpha_a\alpha_b) &= (\alpha_a + \eta\beta_a)(\alpha_b + \eta\beta_b) \\ &= \alpha_a\alpha_b + \eta\left(\beta_a\alpha_b + \beta_b\alpha_a\right) \end{aligned}$$

In particular

$$\mathrm{tg}(\alpha_a\alpha_b) = \beta_a\alpha_b + \beta_b\alpha_a \in H^3(T \times BT, \mathbb{Q})$$

belongs to the component

$$H^1(BT, \mathbb{Q}) \otimes H^2(BT, \mathbb{Q}) \cong \mathfrak{t}^{*\otimes 2}$$

and corresponds to a symmetric bilinear form. This shows that κ is symmetric map.

A choice of a root α of G determines a homomorphism $\rho : SU(2) \to G$ which sends the coroot h of $SU(2)$ to h_α. If α is a long root, this determines an isomorphism $\rho^* : H^3(G, \mathbb{Z}) \to H^3(SU(2), \mathbb{Z})$ ([BS58, Prop III.10.2]) and hence also an isomorphism $H^4(BG, \mathbb{Z}) \to H^4(BSU(2), \mathbb{Z})$. The map κ for $SU(2)$ is computed in Example 5.2.11. We thus have

$$\langle \kappa^{\tau_l}(h_\alpha), h_\alpha \rangle = \langle \kappa^{\rho^* \tau_l}(h), h \rangle = 2(l + h_G^\vee)$$

where we denoted by κ^τ the holonomy map constructed from the twisting τ.

□

Consider now the case $G = G_1 \rtimes F$ where G_1 is simple and simply connected and $l \in H^4(BG, \mathbb{Z})$. We again consider the twisting τ_l. For $f \in F$, let the holonomy map of Definition 5.3.1 be denoted by $\kappa_f : \underline{\Pi} \to \underline{\Lambda}$. Proposition 6.2.1 computes κ_{id} in terms of the pullback of l to $H^4(BG_1, \mathbb{Z}) \cong \mathbb{Z}$. The holonomy maps at other $f \in F$ turn out to be the restriction of κ_{id} to $\underline{\Pi}$. In particular, it only depends on the restriction of l to $H^4(BG_1, \mathbb{Z})$.

Proposition 6.2.2. *Let $f \in F$ and $\iota : \underline{\mathfrak{t}} \to \mathfrak{t}$ be the inclusion map. Then*

$$\kappa_f = \iota^* \circ \kappa_{\mathrm{id}} \circ \iota$$

Proof. The holonomy map depends only on

$$\tau_{\mathbb{Q}} \in \mathrm{Twist}(G /\!\!/ G) \otimes \mathbb{Q} \cong H^3_G(G, \mathbb{Q})$$

which is the transgression of a class $\omega \in H^4(BG, \mathbb{Q})$. The exact expression of ω will not be important for this argument. Since transgression commutes with restriction to a subgroup we can assume without loss of generality that $G = \underline{T} \times \mathbb{Z}/k\mathbb{Z}$. In this case $BG = B\underline{T} \times B(\mathbb{Z}/k\mathbb{Z})$ and $LBG = LB\underline{T} \times LB(\mathbb{Z}/k\mathbb{Z})$. Consider the following diagram

$$\begin{array}{ccccc} LB\underline{T} \times S^1 & \xrightarrow{(\mathrm{id},\gamma_{\mathrm{id}},\mathrm{id})} & LB\underline{T} \times LB(\mathbb{Z}/k\mathbb{Z}) \times S^1 & \xrightarrow{e} & B\underline{T} \times B(\mathbb{Z}/k\mathbb{Z}) \\ & \nearrow^{(\mathrm{id},\gamma_f,\mathrm{id})} & & & \downarrow{\scriptstyle \pi_1} \\ LB\underline{T} \times S^1 & & & & B\underline{T} \end{array}$$

where $\gamma_g \in LB(\mathbb{Z}/k\mathbb{Z})$ is any element in the component of $LB(\mathbb{Z}/k\mathbb{Z})$ corresponding to $[g] \in \pi_0(G) = \mathbb{Z}/k\mathbb{Z} = \pi_0 LB(\mathbb{Z}/k\mathbb{Z})$.

The restrictions of $\tau_{\mathbb{Q}}$ to $\underline{T}/\!\!/\underline{T}$ and $f\underline{T}/\!\!/\underline{T}$ are the pushforwards along S^1 of the pullbacks of ω to the "top" and "bottom" copies of $LB\underline{T} \times S^1$ in the diagram respectively. Clearly the two classes are the same if ω is pulled pack from $B\underline{T}$ by π_1. Since π_1^* is an isomorphism on cohomology with rational coefficients, the result follows.

□

6.3 Twisting h_G

We analyze the twisting h_G restricted to the subgroupoid $\{f\}/\!\!/G^f$. Recall that h_G is the pushforward along S^1 of the following composition

$$LBG \times S^1 \xrightarrow{e} BG \xrightarrow{\mathfrak{g}} BO \xrightarrow{\delta} ko_1 \xrightarrow{P} \Sigma^1 |Twist|$$

Let us first consider $f = \mathrm{id}$. The subset $\{e\}/\!\!/G \subset G/\!\!/G$ corresponds on the level of classifying spaces to the subset of constant loops $BG \subset LBG$. It follows that the restriction of h_G to $\{e\}/\!\!/G$ is the pushforward along S^1 of the composition

$$BG \times S^1 \xrightarrow{\pi_1} BG \xrightarrow{\mathfrak{g}} BO \xrightarrow{\delta} ko_1 \xrightarrow{P} \Sigma^1|Twist|$$

In particular, the class that is being pushed forward is a pullback of a class in BG. This pull-push procedure can be performed in any cohomoloy theory, and therefore is given by multiplication by an element in $\pi_1(S)$ where S is the sphere spectrum.

Lemma 6.3.1. *Let E be any generalized cohomology theory and X be any space. Let $\pi : X \times S^1 \to X$ be the projection map. Then the map*

$$E^*(X) \to E^{*-1}(X)$$
$$\omega \mapsto \pi_* \circ \pi^*\omega$$

is given by the action of the non-trivial element $\eta \in \pi_1(S)$. The push-forward is with respect to stable framing of S^1 induced by the trivialization of the tangent bundle of S^1.

Proof. Recall the construction of the stable framing corresponding to the trivialization of the tangent bundle. Let M be a k-manifold and let $\xi : TM \to \underline{\mathbb{R}^k}$ be a trivialization of TM. Fix an embedding $\chi : M \to \mathbb{R}^N$. We have an isomorphism $TM \oplus \nu_\chi \cong \mathbb{R}^N$ where ν_χ is the normal bundle of the embedding.

The isomorphism ξ affords an isomorphism

$$\mathbb{R}^k \oplus \nu_\chi \cong \mathbb{R}^N$$

The left-hand side of this expression is the normal bundle of the embedding $M \xrightarrow{\xi} \mathbb{R}^N \hookrightarrow \mathbb{R}^{N+k}$ and hence we get a stable normal framing of M.

Let $M = S^1$ and χ be the embedding of S^1 into the unit circle in $\mathbb{R}^2$. Let ξ be the embedding $S^1 \hookrightarrow \mathbb{R}^3$. The above construction gives rise to the trivialization of the normal bundle of ξ which "rotates" once as you go around the circle. The Pontryagin-Thom collapse defines a map $S^3 \to (S^1)^{\nu_\xi}$ into the Thom space of ν_ξ. The trivialization of ν_ξ gives rise to a map

$$S^3 \to \Sigma^2 S^1_+ \tag{6.3.2}$$

Now consider E given by the sphere spectrum and $\omega \in E^0(S^0)$ corresponding to the identity map $S^0 \to S^0$. The pullback $\pi^*\omega$ is given by the map $S^1_+ \to S^0$ which maps S^1 to the non-base point of S^0. The push-forward $\pi_*\pi^*\omega \in E^{-1}(*) \cong E^2(S^3)$ is the composition

$$S^3 \to \Sigma^2 S^1_+ \to \Sigma^2 S^0 \cong S^2$$

where the first map is (6.3.2). This is the Hopf map. Indeed, preimage of two points forms a Hopf link. Since every spectrum is a module for the sphere spectrum, the result follows.

□

Corollary 6.3.3. *The restriction of the twisting h_G to $\{e\}/\!\!/G$ is $\mathfrak{g} - \dim \mathfrak{g}$.*

Proof. By Lemma 6.3.1 and the discussion preceding it, the restriction of h_G is $\eta \circ P$ applied to the virtual rank 0 bundle given by $BG \xrightarrow{\mathfrak{g}} BO \xrightarrow{\delta} \tilde{ko}$. By Corollary 2.3.21, it is also the composition

$$BG \xrightarrow{\mathfrak{g}} BO \xrightarrow{\delta} \tilde{ko} \xrightarrow{\pi} ko \xrightarrow{\rho} |Twist|$$

which is precisely the twisting $\mathfrak{g} - \dim \mathfrak{g}$. $\square$

On $\{f\}/\!\!/G^f$, h_G is obtained by the push-forward of

$$BG^f \times S^1 \to BG \xrightarrow{\mathfrak{g}} BO \xrightarrow{\delta} \tilde{ko} \xrightarrow{P} \Sigma^1 Bwist.$$

The composition $BG^f \times S^1 \to BG \to BO(\dim \mathfrak{g})$ classifies the vector bundle over $BG^f \times S^1$ given by the vector bundle $\mathfrak{g}$ over BG^f by restricting the action of G to G^f and with the holonomy f around S^1.

Lemma 6.3.4. *Let M be a space and V be a finite-dimensional $\mathbb{R}$-vector bundle over M. Let ϕ be an automorphism of V which covers the identity map on M. This gives rise to a vector bundle (V, ϕ) on $M \times S^1$ obtained by gluing the ends of $V \times [0,1]$ on the boundary by ϕ. Assume further that the fixed subspace V^ϕ is a subvector bundle of V. Then the push-forward in ko-theory*

$$\pi_*[(V, \phi)] = \eta \cdot [V^\phi] \in ko^{-1}(M)$$

where η is the nontrivial class in π_1 of the sphere spectrum.

Proof. We first compute the push-forward in KO theory. By the index theorem, the push-forward of a vector bundle W on $M \times S^1$ is obtained by the index of the Dirac operator. Introducing a connection ∇ along the fibers, we get a Dirac operator acting on sections of $W \otimes Cl_1$ where Cl_1 is viewed as $Cl^c(TS^1) - Cl_1$ bimodule via the trivialization of TS^1. Point by point, the kernel of the Dirac operator is a finite-dimensional vector space and has a remaining right Cl_1 action. If the kernels form a vector bundle over M, then this vector bundle with right Cl_1 action represents the push-forward of W in $KO^{-1}(M)$ via the Atiyah-Bott-Shapiro construction.

In our case, there is a natural connection on the fibers of (V, ϕ) with the holonomy ϕ. The kernel of the Dirac operator consists of flat sections and is therefore $V^\phi \otimes Cl_1$. In particular, it is the product of the class $V^\phi \in KO^0(M)$ and $Cl_1 \in KO^{-1}(M)$. By Lemma 6.3.1, the class Cl_1 is the action of η on the class $1 \in KO^0(M)$. It follows that the push-forward in KO of (V, ϕ) is $\eta \cdot V^\phi$.

The homotopy fiber of the connective covering map $ko \to KO$ has homotopy groups in degrees -2 and below. Since a topological space has homotopy groups only in non-negative degrees, if a class in $KO^i(M)$ for $i < 2$ lifts to a class in $ko^i(M)$, then the lift is unique. The push-forward in ko theory of (V, ϕ) is a lift of the push-forward of this class in KO. The class $\eta \cdot [V^\phi]$ is the unique such lift. □

Proposition 6.3.5. *The restriction of the twisting h_G to $\{f\}/\!/G^f$ is isomorphic to $\mathfrak{g}^f - \dim \mathfrak{g}$.*

Proof. By Lemma 6.3.4, the push-forward of $((\mathfrak{g}, f) - \dim \mathfrak{g}) \in ko^0(BG^f \times S^1)$ along S^1 is $\eta \cdot (\mathfrak{g}^f - \dim \mathfrak{g})$. The statement now follows from Proposition 2.3.20. □

Chapter 7

$R(G)$-Module Structure

Let X be the "pair of pants" bordism from $S^1 \coprod S^1$ to S^1. The stack of flat G connections on X is represented by the groupoid $G \times G /\!\!/ G$ where G acts by conjugation on both copies of G. The construction in Section 3.1 applied to X gives rise to the map

$$K_G^\tau(G) \otimes K_G^\tau(G) \to K_G^{\pi_1^*\tau + \pi_2^*\tau}(G \times G) \xrightarrow{\mu_*} K_G^\tau(G)$$

where μ_* is the pushforward along the multiplication map of G. In particular, the construction in Section 3.1 gives an isomorphism of twistings

$$\mu^*\tau \cong \pi_1^*\tau + \pi_2^*\tau + \mathfrak{g}. \tag{7.0.1}$$

Here, we used that $T(G \times G)$ is the equivariantly trivial vector bundle $\mathfrak{g} \oplus \mathfrak{g}$ and hence there is an isomorphism of virtual vector bundles

$$T(G \times G) - \mu^* T(G) \cong \mathfrak{g}.$$

Restricting 7.0.1 to $\{e\} \times \{e\} \subset G \times G$ we get

$$\tau\big|_e \cong \tau\big|_e + \tau\big|_e + \mathfrak{g}$$

and hence

$$\tau\big|_e \cong \mathfrak{g}.$$

We verified this isomorphism independently in Corollary 6.3.3. This allows us to define the pushforward map

$$\iota_* : K^0_G(\{e\}) \to K^\tau_G(G)$$

where $\iota : \{e\} \to G$ is the inclusion of the identity element. The following diagram commutes

$$\begin{array}{ccccc}
K^0_G(\{e\}) \otimes K^\tau_G(G) & \longrightarrow & K^{\pi_2^*\tau}_G(\{e\} \times G) & \xrightarrow{\sim} & K^\tau_G(G)\,. \\
\downarrow{\scriptstyle \iota_* \otimes \mathrm{id}} & & {\scriptstyle (\iota \times \mathrm{id})_*}\downarrow & & \| \\
K^\tau_G(G) \otimes K^\tau_G(G) & \longrightarrow & K^{\pi_1^*\tau + \pi_2^*\tau}_G(G \times G) & \xrightarrow{\mu_*} & K^\tau_G(G)
\end{array}$$

The top row is the action of $K^0_G(*)$ on G-equivariant K-theory coming from the ring structure of K. In this section we compute the pushforward map ι_* and the $K^0_G(*)$ action on $K^\tau_G(G)$. As such, we will obtain partial Pontryagin product structure on $K^\tau_G(G)$.

In all our situations, the twisting will satisfy Condition 5.3.8 and therefore all classes $K^\tau_G(G)$ will be push-forwards from $\{f\}/\!\!/G^f$. We have $\tau\big|_{\{e\}} \cong \mathfrak{g}$ and for most cases, we will have $\tau\big|_{\{f\}/\!\!/G^f} \cong \mathfrak{g}^f$. We start by studying equivariant twistings over a point.

7.1 Twistings of $\{*\}/\!\!/N$

Let $G = G_1 \rtimes F$ where F acts by diagram automorphisms and G_1 is simple and simply connected. Applying Mackey decomposition construction of Section 2.5 to $T \lhd N$ and the trivial twisting gives rise to $\tau_N \in \mathrm{Twist}(\Lambda/\!\!/W)$

and an isomorphism

$$K_N(*) \cong K_W^{\tau_N}(\Lambda)_{\mathrm{cpt}}$$

Proposition 7.1.1. *The twisting τ_N is trivializable.*

What is more important than abstract trivializability is the particular trivialization that we exhibit in the proof. It will be used throughout.

Proof. We trivialize τ_N by constructing a τ_N twisted line bundle over Λ. By construction, such a line bundle is equivalent to a representation of N with a 1-dimensional weight-space for every weight. For every W_1 orbit $\mathcal{O}' \subset \Lambda$, there is a unique G_1 irreducible representation $V_{\mathcal{O}'}$ with a maximal weight in this orbit. Let $\mathcal{O}$ be a W orbit in Λ which is a union $\bigcup_i \mathcal{O}'_i$ of W_1-orbits. We produce an action of G on $V := \bigoplus_i V_{\mathcal{O}'_i}$. The action of G_1 is already specified so it suffices to give an action of F on V. Let $\lambda_i \subset \mathcal{O}'_i$ be the dominant weight in the orbit $\mathcal{O}'_i$ and trivialize the weight spaces $V_{\lambda_i} \subset V_{\mathcal{O}'_i}$. For $f \in F$, $f(\lambda_i)$ is a dominant weight since f acts by diagram automorphisms. We define the action of f on $V_{\mathcal{O}'_i}$ to be the one that sends $1 \in V_{\lambda_i}$ to $1 \in V_{f(\lambda_i)}$. The action on the rest of the space $V_{\mathcal{O}'_i}$ follows since $W_{\mathcal{O}'_i}$ is generated by V_{λ_i} as a representation of G_1. It is straight-forward to check that this defines a representation of G. The direct sum $\bigoplus_{\lambda \in \mathcal{O}} V_\lambda$ is a representation of N with 1-dimensional weight spaces. Taking the direct sum of these representations for every W orbit $\mathcal{O}$ defines the desired representation.

□

Consider now the twisting

$$\nu := \mathfrak{g} - \mathfrak{t}$$

It is of interest because when Condition 5.3.8 is satisfied, the summand of $K_N^\tau(T)$ corresponding to $K_G^\tau(G_1)$ consists of classes which are pushforwards of classes in $K_N^\nu(\{e\})$. The Mackey decomposition gives rise to an isomorphism

$$K_N^\nu(*) = K_W^{\nu'}(\Lambda^\nu)_{\text{cpt}}$$

We will construct a trivialization of ν' on a subset of Λ^ν by constructing ν-twisted representations of N with 1-dimensional weight spaces.

We have the following decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{t} \oplus P$$

where

$$P = \bigoplus_{\alpha \in \Delta^+} P_\alpha$$

and P_α is the real subspace of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. The twisting ν is therefore given by the N-equivariant vector space P. The corresponding central extension of N is the following pullback

$$\begin{array}{ccc} N^\nu & \longrightarrow & Pin^c(P) \\ \downarrow & & \downarrow \\ N & \longrightarrow & O(P) \end{array}$$

with the $\mathbb{Z}/2\mathbb{Z}$ grading given by composition $N \to O(P) \xrightarrow{\det} \{\pm 1\}$. There is a splitting on the level of Lie algebras $\mathfrak{t} \to \mathfrak{t}^\nu$ induced from the splitting of Lie algebras

$$\mathfrak{so}(P) \to \mathfrak{spin}^c(P) \cong \mathfrak{so}(P) \otimes \mathbb{C}$$

given by sending $\mathfrak{so}(P)$ to the real subspace of $\mathfrak{so}(P) \otimes \mathbb{C}$. This lets us identify Λ^ν with a subset of $\mathfrak{t}^*$. It is a translate of Λ. The spinor representation $\mathbb{S}(P)$, i.e., the irreducible representation of $Cl^c((P)$, is a ν-twisted $\mathbb{Z}/2\mathbb{Z}$ graded representation of N. As a T^ν representation, it is the tensor product of $\mathbb{S}(P_\alpha)$ for $\alpha \in \Delta^+$. The character of $\mathbb{S}(P_\alpha)$ is $e^{\alpha/2} - e^{-\alpha/2}$ and therefore the character of $\mathbb{S}(P)$ is the Weyl denominator

$$\prod_{\alpha \in \Delta^+} e^{\alpha/2} - e^{-\alpha/2}.$$

By the Weyl character formula, the character of $\mathbb{S}(P)$ is also

$$\sum_{w \in W_1} \epsilon(w) e^{w(\rho)}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ is the Weyl vector and $\epsilon(w) \in \{\pm 1\}$ is the determinant of the action of w on $\mathfrak{t}$. A standard fact in Lie theory is that ρ is the sum of the fundamental weights. In particular, if a dominant weight λ does not lie of a wall of the Weyl chamber, then it pairs positively with every positive coroot and hence $\lambda - \rho$ is also dominant.

Definition 7.1.2. Let $\Lambda^\nu_{\mathrm{reg}} \subset \Lambda^\nu$ be the subset of weights on which W_1 acts freely.

Proposition 7.1.3. *The twisting ν' is trivializable on $\Lambda^\nu_{\mathrm{reg}}$.*

Proof. For $\mathcal{O}$ a W-orbit of a dominant $\lambda \in \Lambda$, $V_{\mathcal{O}} \otimes \mathbb{S}(P)$ is a $\mathbb{Z}/2\mathbb{Z}$ graded ν twisted representation of N where $V_{\mathcal{O}}$ is as in the proof of Proposition 7.1.1. By the Weyl character formula, it has 1-dimensional weight spaces, supported on $\mathcal{O}'$, the W orbit of $\lambda + \rho$. Denote this representation by $\tilde{V}_{\mathcal{O}'}$. Union of the orbits $\mathcal{O}'$ as $\mathcal{O}$ vary is precisely $\Lambda^{\nu}_{\text{reg}}$. Direct sum of these representations gives rise to a ν' twisted $\mathbb{Z}/2\mathbb{Z}$-graded line bundle over $\Lambda^{\nu}_{\text{reg}}$. □

Definition 7.1.4. For $\xi \in \Lambda^{\nu}_{\text{reg}}$, let

$$\epsilon(\xi) = \epsilon(w) \in \{\pm 1\}$$

where w is such that $\xi.w$ is dominant.

Proposition 7.1.5. *The isomorphism of twistings*

$$\pi_1^*(\tau_N) + \pi_2^*(\nu') \cong \mu^*(\nu')$$

over $\Lambda \times \Lambda^{\nu}_{\text{reg}} \cap \mu^{-1}\Lambda^{\nu}_{\text{reg}}$ of Proposition 2.5.4 with respect to trivializations of Proposition 7.1.1 and Proposition 7.1.3 is given by the trivial line bundle with the grading on (λ, ξ) being $\epsilon(\xi) \cdot \epsilon(\xi + \lambda)$.

Proof. If L_1, L_2 are the τ_N, ν' twisted line bundles trivializing τ_N and ν', then the line bundle in question is $\pi_1^* L_1 \otimes \pi_2^* L_2 \otimes \mu^* L_2^{-1}$ with their corresponding gradings. Note that L_1 is positively graded everywhere and the grading of L_2 is given by ϵ. The result follows. □

7.2 Pushforward from $K_N^\nu(*)$

Recall that $\tau\big|_{\{e\}/\!\!/G} \cong \mathfrak{g}$ and we have the push-forward map $K_G^0(*) \to K_G^\tau(G)$.

Remark 7.2.1. It follows from the properties of the push-forward map that the composition

$$K_G(*) \xrightarrow{\iota_*} K_G^\tau(G_1) \xrightarrow{\text{res}} K_N^\tau(T)$$

equals the composition

$$K_G(*) \xrightarrow{\iota_*} K_G^{\mathfrak{g}}(\mathfrak{g})_{\text{cpt}} \xrightarrow{\text{res}} K_N^{\mathfrak{g}}(\mathfrak{t})_{\text{cpt}} \xrightarrow{\pi_*} K_N^{\mathfrak{g}-\mathfrak{t}}(*) \qquad (7.2.2)$$

followed by the push-forward $K_N^{\mathfrak{g}-\mathfrak{t}}(*) \to K_N^\tau(T)$.

Proposition 7.2.3. *The composition (7.2.2) is given by the tensor product with $\mathbb{S}(P)$. For $\mathcal{O}$ a W orbit of Λ, it maps the G-representation $V_{\mathcal{O}}$ defined in the proof of Proposition 7.1.1 to the class $1 \in K_W^0(\mathcal{O}') \cong K_W^{\nu'}(\mathcal{O}') \subset K_W^{\nu'}(\Lambda^\nu)$ where $\mathcal{O}'$ is the W orbit of $\xi + \rho$ for any dominant weight in $\mathcal{O}$.*

Proof. Let $G^{\mathfrak{g}}$ be the graded extension of G given by the pullback

$$\begin{array}{ccc} \mathbb{T} & = & \mathbb{T} \\ \downarrow & & \downarrow \\ G^{\mathfrak{g}} & \longrightarrow & \text{Pin}^c(\mathfrak{g}) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\text{Ad}} & O(\mathfrak{g}) \end{array} .$$

There is an action of G on $Cl(\mathfrak{g})$ induced from the adjoint action on $\mathfrak{g}$. The Atiyah-Bott-Shapiro construction associates to a $Cl(\mathfrak{g})$-module with an action of $G^{\mathfrak{g}}$ which is equivariant with respect to the $Cl(\mathfrak{g})$-action, a class in $K_G^{\mathfrak{g}}(\mathfrak{g})$. With this construction, the map $K_G(*) \xrightarrow{\iota_*} K_G^{\mathfrak{g}}(\mathfrak{g})$ is given by

$$V \mapsto V \otimes \mathbb{S}(\mathfrak{g}).$$

Similarly, the map $K_N^*(*) \xrightarrow{\iota_*} K_N^{*+\mathfrak{t}}(\mathfrak{t})$ is given by tensoring with $\mathbb{S}(\mathfrak{t})$. The result then follows from the fact that $\mathfrak{g} = \mathfrak{t} \oplus P$. □

We now assume that the twisting τ satisfies Condition 5.3.8 and we make the choices of $\beta_i \in \Lambda^\tau_{\text{reg}}$. By Proposition 5.3.9, we have

$$K_G^\tau(G_1) \cong K_W^{\tau'}(Y_{\text{reg}})_{\text{cpt}} \cong \bigoplus_i K_W^{\tau'}(Y_i)_{\text{cpt}} \cong \bigoplus_i K_{H_i}^{\tau'-\mathfrak{t}}(\{\beta_i\})$$

where $Y_{\text{reg}} \subset Y$ is the subset containing $\Lambda^\tau_{\text{reg}}$ and Y_i is the union of component containing the W orbit of β_i. The last equality is by Thom isomorphism, or equivalently, via the push-forward map $W(\beta_i) \hookrightarrow Y_i$.

Fix an element $\beta_i \in \{\beta_i\}$ and an element $\pi \in \Pi$. Let $\mathcal{O}$ be the W orbit of β_i, and let $\mathcal{O}'$ be the W orbit of $\beta_i + \kappa(\pi)$. The orbit $\mathcal{O}'$ belongs to the same component of Y_{reg} as $\mathcal{O}$ and therefore the image of $K_W^{\nu'}(\mathcal{O}')$ under the push-forward map $K_W^{\nu'}(\Lambda^\tau_{\text{reg}})_{\text{cpt}} \to K_W^{\tau'}(Y_{\text{reg}})_{\text{cpt}}$ belongs to the summand isomorphic to $K_W^{\nu'}(\mathcal{O})$. We thus get a map

$$r : K_W^{\nu'}(\mathcal{O}') \to K_W^{\nu'}(\mathcal{O}).$$

Proposition 7.2.4. *The map r is the push-forward along the W-equivariant map*

$$s : \mathcal{O}' \to \mathcal{O}$$

which maps $\beta_i + \kappa(\pi)$ to β_i.

Proof. The map r is the push-forward along the inclusion $\mathcal{O}' \hookrightarrow Y_i$ followed by the push-forward along the projection $Y_i \to \mathcal{O}$. The push-forward map is invariant under homotopy. There is a homotopy $h : I \times \mathcal{O}' \to Y_i$ between the inclusion map $\mathcal{O}' \hookrightarrow Y_i$ and the map $\mathcal{O}' \to \mathcal{O}$ defined in the statement of the proposition. The image of $\beta_i + \kappa(\pi)$ under this homotopy is the lift to Y of the loop determined by π. This homotopy defines an isomorphism of twistings $\nu' \cong s^*\nu'$ with respect to which the push-forward $s : K_W^{\nu'}(\mathcal{O}') \to K_W^{\nu'}(\mathcal{O})$ is defined. □

Definition 7.2.5. Let δ be the $\mathbb{Z}/2\mathbb{Z}$ graded line bundle corresponding to the automorphism of twistings $\nu' \cong s^*\nu'$ over $\mathcal{O}' /\!\!/ W$ defined in the proof of Proposition 7.2.4.

The line bundle δ is well defined since both ν' and $s^*\nu'$ are trivialized. Let

$$H' = \mathrm{Stab}_W(\beta_i + \kappa(\pi)).$$

Since $\beta_i + \kappa(\pi)$ is in the same connected component of Y as β_i, the group H' is a subgroup of $H_i = \mathrm{Stab}_W(\beta_i)$.

The groupoid $\{\beta_i + \kappa(\pi)\} /\!\!/ H'$ is equivalent to $\mathcal{O}' /\!\!/ W$. We will compute the isomorphism class of the restriction of δ to this smaller groupoid. Consider the restriction of τ to the subgroupoid $T /\!\!/ H'$. The loop in T determined by π is preserved by H' and therefore defines an automorphism of the restriction of $\tau - \mathfrak{t}$ to $\{e\} /\!\!/ H'$ which is a graded line bundle we denote by $L_{\pi,H'}^{\tau-\mathfrak{t}}$. It depends only on $\tau - \mathfrak{t}$ and the group H', and not the other choices we have made in this section.

Lemma 7.2.6. *The line bundle δ is given by the graded line bundle*

$$\epsilon(\beta_i)\epsilon(\beta_i + \kappa(\pi))L^{\tau-\mathfrak{t}}_{\pi,H'}$$

In particular, if $L^{\tau-\mathfrak{t}}_{\pi,H'}$ is trivial, then δ is given by the trivial line bundle with the grading $\epsilon(\beta_i) \cdot \epsilon(\beta_i + \kappa(\pi))$

Proof. The result is obtained by restricting to the subgroupoid $T/\!\!/T \rtimes H'$ and observing that ν' is trivialized by a line bundle whose grading at ξ is given by $\epsilon(\xi)$. □

Proposition 7.2.7. *If $G_1^{H'}$ is simply connected, then the line bundle $L^{\tau-\mathfrak{t}}_{\pi,H'}$ is trivial.*

Proof. The twisting $\mathfrak{t}$ is pulled back from a point and therefore $L^{\tau-\mathfrak{t}}_{\pi,H'} = L^{\tau}_{\pi,H'}$. The twisting τ on the other hand extends to a twisting over $G_1^{H'}/\!\!/H'$. Since $G_1^{H'}$ is simply connected, the loop determined by π is contractible in it. □

Condition 7.2.8. The line bundle $L^{\tau-\mathfrak{t}}_{\pi,H'}$ is trivial for all choices of β_i and π.

We compute the push-forward map $K^{\nu}_W(\Lambda^{\nu}_{\mathrm{reg}})_{\mathrm{cpt}} \to K^{\tau'}_W(Y_{\mathrm{reg}})_{\mathrm{cpt}}$ when τ satisfies Condition 7.2.8. Since we trivialized all the twistings in sight, it can now be phrased in terms of pushforward of non-twisted K-theory of discrete groupoids.

Definition 7.2.9. For X a discrete groupoid, let $Z\mathbb{C}[X]$ be the center of the groupoid algebra of X. It is a subset of functions on morphisms of X. We have

$$K^0(X)_{\text{cpt}} \otimes \mathbb{C} \cong Z\mathbb{C}[X].$$

Moreover, since $K^0(X)_{\text{cpt}}$ is torsion-free, $Z\mathbb{C}[X]$ contains all the information of $K^0(X)$. In the case of a global quotient, for $\chi \in Z\mathbb{C}[S/\!\!/H]$, we denote by χ_h the function

$$\begin{aligned} \chi_h : S &\to \mathbb{C} \\ s &\mapsto \chi(s \xrightarrow{h} s.h) \end{aligned}$$

The push-forward and pull-back in K^0 along a map $X \to Y$ of discrete groupoids corresponds to the analogous operations on $Z\mathbb{C}[X]$ viewed as a set of functions. The non-trivial K-orientations are given by line bundles over X which correspond to elements in $K^0(X)$. Push-forward with respect to such orientation correspond to precomposing with multiplication with corresponding classes in $Z\mathbb{C}[X]$.

Corollary 7.2.10. *Consider the diagram*

$$\begin{array}{ccccc} K_W(\Lambda^{\nu}_{\text{reg}})_{\text{cpt}} & \xrightarrow{\iota_*} & K^{\tau}_W(Y_{\text{reg}})_{\text{cpt}} & \xrightarrow{\sim} & K_W(\bigcup \mathcal{O}_i) \\ \downarrow & & & & \downarrow \\ P : Z\mathbb{C}[\Lambda^{\nu}_{\text{reg}}/\!\!/W] & & \longrightarrow & & Z\mathbb{C}[\bigcup \mathcal{O}_i/\!\!/W] \end{array}$$

Fix $\beta_i \in \{\beta_i\}$ *and let* $h \in H_i = \text{Stab}_W(\beta_i)$. *Assuming Condition 7.2.8, we have*

$$P(\chi)_h(\beta_i) = \epsilon(\beta_i) \sum_{\pi \in \Pi} \epsilon(\beta_i + \kappa(\pi))\chi_h\left((\beta_i + \kappa(\pi))\right)$$

where $\epsilon(\beta_i) \in \{\pm 1\}$ *is defined in Definition 7.1.4.*

Corollary 7.2.11. *Let* $\iota : \Lambda^{\nu}_{\mathrm{reg}} \to Y_{\mathrm{reg}}$*,* $\beta_i \in \{\beta_i\}$*,* $h \in H_i$*,* $A \in K_N(*)$ *and* $B \in K^{\nu'}_W(\Lambda^{\nu})_{\mathrm{cpt}}$*. Assume further that* $\epsilon(\beta_i) = 1$*. Then the restriction of the* $K_N(*)$ *action map* $K_N(*) \otimes K^{\tau}_N(T) \to K^{\tau}_N(T)$ *to the direct summand in* $K^{\tau}_N(T)$ *corresponding to* $K^{\tau'}_W(Y_{\mathrm{reg}})_{\mathrm{cpt}}$ *is given on the level of character maps by*

$$\chi(A \otimes \iota_*(B))_h(\beta_i) = \sum_{\pi\in\Pi} \sum_{\substack{\lambda_1\in\Lambda;\ \lambda_2\in\Lambda^{\nu}_{\mathrm{reg}} \\ \lambda_1+\lambda_2=\beta_i+\kappa(\pi)}} \epsilon(\lambda_2)\chi(A)_h(\lambda_1)\cdot\chi(B)_h(\lambda_2)$$

7.3 Pushforward from $K^{\nu}_{N^f}(\{f\})$

For other components of $G/\!\!/G$ the story is very similar and we will be brief.

Condition 7.3.1. The restriction of $\mathrm{tg}(l)$ to $\{f\}/\!\!/N^f$ vanishes. In this case, 6.3.5 implies that $\tau_l|_{\{f\}/\!\!/N^f} \cong \mathfrak{g}^f$.

We also assume that Condition 5.3.8 is satisfied. By Proposition 5.3.9, we have

$$K^{\tau}_{G(f)}(fG_1) \cong K^{\tau'}_{\underline{W}}(\underline{Y}_{\mathrm{reg}})_{\mathrm{cpt}} \cong \bigoplus_i K^{\tau'}_{\underline{W}}(\underline{Y}_i)_{\mathrm{cpt}} \cong \bigoplus_i K^{\tau'-\mathfrak{t}}_{\underline{\tilde{W}}}(\mathcal{O}_i) \cong \bigoplus_i K^{\tau'-\mathfrak{t}}_{H_i}(\beta_i)$$

Fix an element $\beta_i \in \{\beta_i\}$ and an element $\pi \in \underline{\tilde{\Pi}}$. Let $\mathcal{O}$ be the $\underline{\tilde{W}}$ orbit of β_i, and let $\mathcal{O}'$ be the $\underline{\tilde{W}}$ orbit of $\beta_i+\kappa(\pi)$. It belongs to the same component of the groupoid $\underline{Y}_{\mathrm{reg}}/\!\!/\underline{W}$ as $\mathcal{O}$ [1] and therefore the image of $K^{\tau'}_W(\mathcal{O}')$ in $K^{\tau'}_{\underline{\tilde{W}}}(\underline{Y}_{\mathrm{reg}})_{\mathrm{cpt}}$

[1]It might be in a different component of the space $\underline{Y}_{\mathrm{reg}}$ but related by a morphism in $\underline{W}$

belongs to the summand $K^{\tau'}_{\underline{\tilde{W}}}(\mathcal{O})$. We get a map

$$r : K^{\tau'}_{\underline{\tilde{W}}}(\mathcal{O}') \to K^{\tau'}_{\underline{\tilde{W}}}(\mathcal{O}).$$

which is again the the push-forward along the $\underline{\tilde{W}}$-equivariant map

$$s : \mathcal{O}' \to \mathcal{O}$$

which maps $\beta_i + \kappa(\pi)$ to β_i.

Assuming $\tau\big|_{\{f\}/\!\!/G^f} \cong \mathfrak{g}^f$, we trivialize $\tau' - \mathfrak{t} = \nu'$ on $\underline{\Lambda}^{\tau}_{\mathrm{reg}}$ as in Proposition 7.1.3. The K-orientation of s is then again given by a line bundle δ. Let

$$H' = \mathrm{Stab}_{\underline{\tilde{W}}}(\beta_i + \kappa(\pi)).$$

The groupoid $\{\beta_i + \kappa(\pi)\}/\!\!/H'$ is equivalent to $\mathcal{O}'/\!\!/\underline{\tilde{W}}$.

There is a line bundle $L^{\tau-\mathfrak{t}}_{\pi,H'}$ over $\{f\}/\!\!/H'$ which is constructed analogously to the one in the case of the component of the identity. If $\pi \in \underline{\Pi}$ then the construction is exactly analogous. For a general π, recall that $\underline{\tilde{\Pi}} \subset \underline{\mathfrak{t}} \times T/\underline{T}$ and choose $t \in T$ such that $\pi = (x, [t])$ for $x \in \underline{\mathfrak{t}}$. The group H' fixes $x \in \mathfrak{t}$ and acts on $\langle \underline{T}, t\rangle$, the subgroup generated by $\underline{T}$ and t. We can pullback the twisting τ by the map of groupoids

$$x\mathbb{R}/\left(\underline{\Pi} \cap x\mathbb{R}\right)/\!\!/\langle \underline{T}, t\rangle \rtimes H' \xrightarrow{f\exp} f\underline{T}/\!\!/\underline{N}$$

The action of t on the groupoid on the left is by translation by x. In particular, the groupoid is locally equivalent to $x\mathbb{R}/x\mathbb{Z}/\!\!/\underline{T} \rtimes H'$ which is a circle acted on trivially by $\underline{T} \rtimes H'$. The pullback of τ defines an automorphism of the restriction of τ to a base-point and hence an equivalence class of a line bundle over $*/\!\!/\underline{T} \rtimes H'$. The character for the action of $\underline{T}$ is $\kappa(\pi)$ and $L^{\tau-\mathfrak{t}}_{\pi,H'}$ is the representation restricted to $H' \subset \underline{T} \rtimes H'$.

We again have $\delta = \epsilon(\beta_i)\epsilon(\beta_i + \pi)L^{\tau-\mathfrak{t}}_{\pi,H'}$ and under the condition that $L^{\tau-\mathfrak{t}}_{\pi,H'}$ is trivial we have the following computation of the $K_{\underline{N}}(*)$ action on the summand isomorphic to $K^{\tau}_{G(f)}(fG_1)$ in $K^{\tau}_{\underline{N}}(f\underline{T})$.

Corollary 7.3.2. *Let* $\iota : \underline{\Lambda}^{\tau}_{\text{reg}} \to \underline{Y}_{\text{reg}}$, $\beta_i \in \{\beta_i\}$, $h \in H_i$, $A \in K_{\underline{N}}(*)$ *and* $B \in K^{\nu'}_{\underline{\tilde{W}}}(\underline{\Lambda}^{\tau})_{\text{cpt}}$. *Assume further that* $\epsilon(\beta_i) = 1$. *Then the restriction of the* $K_{\underline{N}}(*)$ *action map* $K_{\underline{N}}(*) \otimes K^{\tau}_{\underline{N}}(f\underline{T}) \to K^{\tau}_{\underline{N}}(f\underline{T})$ *to the direct summand in* $K^{\tau}_{\underline{N}}(f\underline{T})$ *corresponding to* $K^{\tau'}_{\underline{W}}(\underline{Y}_{\text{reg}})_{\text{cpt}}$ *is given on the level of character maps by*

$$\chi(A \otimes \iota_*(B))_h(\beta_i) = \sum_{\pi \in \underline{\tilde{\Pi}}} \sum_{\substack{\lambda_1 \in \underline{\Lambda};\ \lambda_2 \in \underline{\Lambda}^{\nu}_{\text{reg}} \\ \lambda_1 + \lambda_2 = \beta_i + \kappa(\pi)}} \epsilon(\lambda_2)\chi(A)_h(\lambda_1) \cdot \chi(B)_h(\lambda_2)$$

The $K_G(*)$ *action is obtained by first restricting a representation to* $\underline{N}$.

Chapter 8

$G = SU(3) \rtimes \mathbb{Z}/2\mathbb{Z}$

8.1 Lie Theory

Let $G_1 = SU(3) \subset GL_3(\mathbb{C})$. The Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ consists of skew-hermitian matrices. Fix the maximal torus

$$T = \left\{ \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix} \,\middle|\, \alpha, \beta, \gamma \in \mathbb{T};\ \alpha\beta\gamma = 1 \right\}$$

and let $\mathfrak{t}$ be its Lie algebra. Choose the following basis of $\mathfrak{t}$

$$\eta_1 = i \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}; \quad \eta_2 = i \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

and let $\{\xi_1, \xi_2\}$ be its dual basis. The lattices Π, Λ are generated by $\{\eta_1, \eta_2\}$, $\{\xi_1, \xi_2\}$ respectively. The Weyl group W_1 is the symmetric group Sym_3 which acts by permuting the three basis vectors in $\mathbb{C}^3$ the $GL_3(\mathbb{C})$ acts on.

There are six roots

$$\pm(\xi_1 - \xi_2), \pm(2\xi_1 + \xi_2), \pm(\xi_1 + 2\xi_2)$$

with the corresponding coroots

$$\pm(\eta_1 - \eta_2), \pm\eta_1, \pm\eta_2.$$

We fix the following set of simple roots

$$\alpha_1 = (\xi_1 - \xi_2), \qquad \alpha_2 = (\xi_1 + 2\xi_2)$$

and consider an outer automorphism that interchanges the coresponding co-roots. One such automorphism is given by complex conjugation followed by conjugation by $\left(\begin{smallmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)$. We denote it by γ and let $G = G_1 \rtimes_\gamma \mathbb{Z}/2$. The action of γ on $\mathfrak{t}$ is given by

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

in the basis $\{\eta_1, \eta_2\}$.

There are two connected components of $G/\!\!/G$ corresponding to $f = e$ and $f = \gamma$. We analyze the group $\underline{W_{aff}}$ for the second case. We have

- $\underline{\Pi} = \langle \eta_1 \rangle$,
- $\underline{\Lambda} = \langle \xi_1 \rangle$,
- $[T/\underline{T}]^f = \{\pm 1\}$ with non-trivial element given by $\exp \frac{1}{2}\eta_2$,
- $\underline{\tilde{\Pi}} = \langle \frac{1}{2}\eta_1 \rangle$,
- $W_1^f = \{\mathrm{id}, (1,3)\} \subset \mathrm{Sym}_3 = W_1$. It acts on $\underline{\mathfrak{t}}$ by multiplication by -1.

8.2 Twisting τ (level 1)

Let $l \in H^4(BG, \mathbb{Z})$ be a class that pulls back to $1 \in \mathbb{Z} \cong H^4(BG_1, \mathbb{Z})$ and let τ be the twisting corresponding to level l, $\tau = \tau_l = \dim g + \mathrm{tg}(l) + h_G$.

By Proposition 6.2.1, the map $\kappa : \mathfrak{t} \to \mathfrak{t}^*$ is

$$\kappa = \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$$

in the basis $\{\eta_i\}$ and $\{\xi_i\}$. Indeed, this map is symmetric, Weyl invariant and satisfies $\langle \kappa(h_\alpha), h_\alpha \rangle = 2(1 + h_G^\vee) = 8$ for any coroot h_α.

The restriction of the twisting τ_l to $\{f\}/\!\!/G^f$ when $f = \gamma$ is non-trivial and therefore the analysis of Section 7.3 does not apply directly. In this case though, we will not need the full strength of that computation and simply knowing the action of $\underline{W_{aff}}$ on $\underline{\Lambda}^\tau$ will suffice. We will show that the component in $H^3_{G^f}(\{f\}, \mathbb{Z})$ of τ vanishes and therefore, by Proposition 5.3.10 the action of $\underline{W_{aff}}$ on $\underline{\Lambda}^\tau$ is isomorphic to that on $\underline{\Lambda}$.

Proposition 8.2.1. *The component of τ in $H^3_{G^f}(\{f\}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ vanishes.*

Recall that we have $\tau = \tau_l = \dim \mathfrak{g} + \mathrm{tg}(l) + h_G$. We analyze the summands $\mathrm{tg}(l)$ and h_G separately.

Lemma 8.2.2. *The restriction of $\mathrm{tg}(l) \in H^3(G/\!\!/G, \mathbb{Z})$ to $H^3(\{f\}/\!\!/G_1^f, \mathbb{Z}) \cong \mathbb{Z}/2$ is non-zero iff l restricts to an odd integer in $H^4(G_1, \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. Here $l \in H^4(BG, \mathbb{Z})$ and $\mathrm{tg}(l)$ is given by $\pi_* e^* l$.

$$\begin{array}{ccc} LBG \times S^1 & \xrightarrow{e} & BG \\ \big\downarrow{\scriptstyle \pi} & & \\ LBG & & \end{array}$$

where LBG, the loop space of G, is the classifying space of $G/\!\!/G$. Restricting to $\{f\}/\!\!/G_1^f$ corresponds to restriction to

$$BG_1^f \cong \{\gamma_f\} \times_{\Omega BG_1^f} PBG \subset \Omega BG \times_{\Omega BG} PBG \cong LBG$$

where $\gamma_f \in \Omega BG \cong G$ is a loop corresponding to the element f. Hence we have the following diagram

$$\begin{array}{ccccc} BG_1^f \times S^1 & \longrightarrow & LBG \times S^1 & \xrightarrow{e} & BG \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ BG_1^f & \longrightarrow & LBG & & \end{array}$$

where the composition in the top row is given by

$$BG_1^f \times S^1 \cong B\left(G_1^f \times \mathbb{Z}\right) \to B(G)$$

induced by the homomorphism

$$\begin{array}{rcl} G_1^f \times \mathbb{Z} & \to & G \\ (x,n) & \mapsto & xf^n \end{array}$$

We have $H^4(BG,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, and we are interested in the transgression of a class $l \in H^4(BG,\mathbb{Z})$ pulling back to $1 \in H^4(BG_1,\mathbb{Z})$[1]. We have $H^3(BG_1^f,\mathbb{Z}) = \mathbb{Z}/2$ and the reduction mod-2, $H^3(BG_1^f,\mathbb{Z}) \to H^3(BG_1^f,\mathbb{Z}/2)$ is injective. The image is the the class $w_3 \in H^3(BSO(3),\mathbb{Z}/2)$.[2] We can therefore reduce the

[1]There are two such classes and they differ by a pullback of a class in $H^4(B\mathbb{Z}/2,\mathbb{Z})$, which transgresses to 0.

[2]The class in $H^3(BSO(3),\mathbb{Z})$ is $\beta(w_2)$ and its mod 2 reduction is w_3.

class l mod 2 to the class $\tilde{l} \in H^4(BG, \mathbb{Z}/2)$ and compute transgression in mod 2 cohomology.

Consider the map

$$\theta : B\left(SU(3) \rtimes \mathbb{Z}/2\right) \to B\left(O(6)\right)$$

given by the action of $SU(3)$ and complex conjugation on the underlying real vector space $\mathbb{C}^3_{\mathbb{R}}$. We claim that

$$\theta^*(w_4) = \tilde{l}.$$

It suffices to show that the pullback of $\theta^*(w_4)$ to $H^4(BSU(3), \mathbb{Z}/2)$ is non-trivial. The non-trivial class in $H^4(BSU(3), \mathbb{Z}/2)$ is the reduction mod 2 of c_2 and the claim follows from the fact that $w_4(V_{\mathbb{R}}) = c_2(V) \mod 2$ for a complex vector bundle V.

We thus get the diagram

$$\begin{array}{ccccc} BSO(3) \times S^1 & \longrightarrow & B\left(SU(3) \rtimes \mathbb{Z}/2\right) & \xrightarrow{\theta} & BO(6) \\ \downarrow & & & & \\ BSO(3) & & & & \end{array}$$

along which we want to compute the transgression of $w_4 \in H^4(BO(6), \mathbb{Z}/2)$. Let $\sigma \in H^1(S^1, \mathbb{Z}/2)$, and $w_1, w_2, w_3 \in H^*(BSO(3), \mathbb{Z}/2)$.[3] We want to compute w_4 of the vector bundle over $BSO(3) \times S^1$ specified by the top row map above and pick out the component $w_3 \otimes \sigma$. The vector bundle is $V \oplus V \otimes L$

[3]Of course $w_1 = 0$ on $BSO(3)$, but we include it right now so that the formulas below are cleaner.

where V is the tautological vector bundle on $BSO(3)$ and L is the Mobius line bundle on S^1. By Whitney sum formula and the splitting principal, for any n-dimensional vector bundle W and a line bundle $L^{'}$, the total Stiefel-Whitney class of $W \otimes L^{'}$ is

$$\begin{aligned} w(W \otimes L') = w(W) + w_1(L') \left(n + (n-1)w_1(W) + (n-2)w_2(W) + \dots\right) \\ + w_1(L')^2 \left(\binom{n}{2} + \binom{n-1}{2} w_1(W) + \dots\right) + \dots \end{aligned}$$

In particular, for our V, L, we get

$$w(V \otimes L) = \dots + \sigma(3 + 2w_1 + w_2) + \dots$$

where $\dots$ denote the terms not containing σ and

$$w(V \oplus V \otimes L) = \dots + (1 + w_1 + w_2 + w_3) \cdot \sigma(3 + 2w_1 + w_2) + \dots.$$

We get that

$$w_4(V \oplus V \otimes L) = w_3\sigma + \dots$$

and its push-forward along $BSO(3) \times S^1 \to BSO(3)$ is non-zero. □

Lemma 8.2.3. *The component of the restriction of h_G to $\{f\}/\!\!/G_1^f$ in $H^3(BG_1^f, \mathbb{Z}) \cong \mathbb{Z}/2$ is the non-zero element.*

Proof. By Proposition 6.3.5, the restriction of h_G to $\{f\}/\!\!/G_1^f$ is given by the equivariant vector bundle $\mathfrak{g}^f - \dim \mathfrak{g}$. The component in H^3 of the twisting determined by a vector bundle V is $\beta(\omega_2(V))$ where β is the Bockstein homomorphism and ω_2 is the second Stiefel-Whitney class. We have $G_1^f \cong SO(3)$

and the adjoint action on its Lie algebra is isomorphic to the defining representation of $SO(3)$. The result follows by noting that the non-trivial class in $H^3(BSO(3),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is $\beta(\omega_2)$. □

Proof of Proposition 8.2.1. The proposition follows from the two lemmas above. □

8.3 $K^\tau_G(G)$

8.3.0.1 Component of the identity

Let $f = \mathrm{id} \in G$. We first understand the structure of $\Lambda_{\mathrm{reg}}/W_{aff}(f)_1$ where $W_{aff}(f)_1 = \Pi \rtimes W_1$ and $\Lambda_{\mathrm{reg}} \subset \Lambda$ are the elements with trivial stabilizer in $W_{aff}(f)_1$. Let P be the fundamental domain of the action of Π on Λ via κ given by the parallelogram spanned by $\left(\begin{smallmatrix}8\\4\end{smallmatrix}\right), \left(\begin{smallmatrix}4\\8\end{smallmatrix}\right) \in \Lambda$. For a fixed element $w \in W_1$, an element $\lambda \in \Lambda$ is fixed by an element $(\pi, w) \in W_{aff}(f)_1$ for some $\pi \in \Pi$ if $(w - \mathrm{id})(\lambda) \in \kappa(\Pi)$. We thus compute

$$U(w) := (w - \mathrm{id})^{-1}(\kappa(\Pi)) \cap P$$

for $w \in W_1$:

- $U(1,2)$: $\{(x,x)|\ 0 \le x \le 12\}$
- $U(1,3)$: $\{(4,x)|\ 2 \le x \le 8\} \cup \{(8,x)|\ 4 \le x \le 10\}$
- $U(2,3)$: $\{(x,4)|\ 2 \le x \le 8\} \cup \{(x,8)|\ 4 \le x \le 10\}$
- $U(1,2,3)$: $\{(4x,4y)|\ 0 \le x,y \le 3\}$

- $U(1,3,2)$: $\{(4x,4y)|\ 0 \le x, y \le 3\}$

There are 18 elements in P not belonging to the above sets forming 3 free orbits of W_1. The subgroup $F = \mathbb{Z}/2 \subset W$ permutes two of these orbits and fixes one of them. The action satisfies Condition 5.3.8 and we pick representatives of W_{aff} orbits

$$\beta_1 = (2\xi_1 + \xi_2) \quad \beta_2 = (3\xi_1 + 2\xi_2).$$

The corresponding stabilizers are

$$H_1 = \langle(\mathrm{id}, \mathrm{id}, \gamma)\rangle \in W_{aff}(f); \qquad H_2 = \{0\}$$

We thus get

$$K^\tau_G(G_1) \cong K^0_{H_1}(\{\beta_1\}) \oplus K^0_{H_2}(\{\beta_2\}) = \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}.$$

8.3.0.2 Component of the transposition

Let $f = (\mathrm{id}, \chi) \in G$. We have $\underline{\tilde{\Pi}} = \langle \frac{1}{2}\eta_1\rangle$ and $\kappa(\frac{1}{2}\eta_1) = 4\xi_1 \in \underline{\Lambda} = \langle \xi_1 \rangle$. There is one free $W_{aff}(f)_1 = \underline{\tilde{\Pi}} \rtimes W^f_1$ orbit $\tilde{\mathcal{O}}$ represented by $\beta_3 = \xi_1$. We have $H_3 := \mathbb{Z}/2$ and Condition 5.3.8 is again satisfied and

$$K^\tau_G(fG_1) \cong K^0_{H_3}(\{\beta_3\}) = \mathbb{Z}^{\oplus 2}.$$

8.4 $R(G)$ module structure

We fix the following basis of $K^\tau_G(G)$:

$$(1, \mathbb{I}), (1, \sigma), (2, \mathbb{I}), (3, \mathbb{I}), (3, \sigma)$$

where $(i, V) \in K^0_{H_i}(\beta_i)$ is given by representation V. The fundamental weights of G_1 are

$$w_1 = \xi_1 \quad w_2 = \xi_2 + \xi_1.$$

The irreducible representations of G_1 are labeled by pairs of non-negative integers (m_1, m_2) corresponding to the highest-weight representation $V_{(m_1,m_2)}$ with highest weight $m_1 w_1 + m_2 w_2$. Let $X = \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ be the set of isomorphism classes of irreducible representations of G_1. The representations of G are given by $\mathbb{Z}/2$ equivariant vector bundles over X as in Example 2.5.3. The action of $\mathbb{Z}/2$ on X is by exchanging the two factors. The irreducible representations are therefore given by

$$W_{(m_1,m_2)} := V_{(m_1,m_2)} \oplus V_{(m_2,m_1)}$$

with $m_1 < m_2$,

$$W_{(m,m),\mathbb{I}} := V_{(m,m)}$$

with $\mathbb{Z}/2$ acting trivially, and

$$W_{(m,m),\sigma} := V_{(m,m)}$$

with $\mathbb{Z}/2$ acting by -1.

We first analyze the map $\iota_* : R(G) \to K^\tau_G(G)$. We use Proposition 7.2.3. The Weyl vector of G_1 is

$$\rho = 2\xi_1 + \xi_2 = w_1 + w_2$$

We have

$$\iota_* W_{(0,0),\mathbb{I}} = (1, \mathbb{I}),$$

$$\iota_* W_{(0,0),\sigma} = (1,\sigma),$$

$$\iota_* W_{(0,1)} = (2,\mathbb{I}).$$

We determine the action of $W_{(0,0),\mathbb{I}}, W_{(0,0),\sigma}, W_{(0,1)}$ on

$$\{(1,\mathbb{I}),(1,\sigma),(2,\mathbb{I}),(3,\mathbb{I}),(3,\sigma)\}.$$

The following are the character maps, i.e. elements in $Z\mathbb{C}[\Lambda /\!\!/ W]$, of the classes in $K_G(*)$ that we will use. We will denote by an element $\lambda \in \Lambda$ the function on Λ which equals 1 on λ and 0 everywhere else. We will write (n,m) for $n\xi_1 + m\xi_2$.

$$\begin{aligned}
\chi(W_{(0,0),\sigma})_e &= (0,0) \\
\chi(W_{(0,0),\sigma})_\gamma &= -(0,0) \\
\chi(W_{(0,1)})_e &= (0,-1)+(-1,0)+(1,1)+(-1,-1)+(0,1)+(1,0) \\
\chi(W_{(0,1)})_\gamma &= 0
\end{aligned}$$

We used the Kostant multiplicity formula to determine the weight decomposition of $W_{(0,1)}$. The elements in $K_G^\tau(G)$ correspond to character maps in $Z\mathbb{C}[\Lambda^\tau_{\text{reg}} /\!\!/ W]$:

$$\begin{aligned}
\chi((1,\mathbb{I}))_e = \chi((1,\mathbb{I}))_\gamma &= \sum_{w\in W_1} w((2,1)) \\
&= (1,2)+(-1,-2)+(2,1)+(1,-1)+(-2,-1)+(-1,1) \\
\chi((1,\sigma))_e = -\chi((1,\mathbb{I}))_\gamma &= \sum_{w\in W_1} w((2,1))
\end{aligned}$$

$$
\begin{aligned}
\chi((2,\mathbb{I}))_e &= \sum_{w\in W} w((2,3)) \\
&= (2,3)+(-1,-3)+(3,2)+(1,-2)+(-3,-1)+(-2,1) \\
&+ (1,3)+(-2,-3)+(3,1)+(2,-1)+(-3,-2)+(-3,-3) \\
\chi((2,\mathbb{I}))_\gamma &= 0
\end{aligned}
$$

The rest is a direct computation using Corollary 7.2.11. For example, to compute $W_{(0,1)} \otimes (2,\mathbb{I})$ we compute the corresponding character:

$$
\begin{aligned}
\chi(W_{(0,1)} \otimes (2,\mathbb{I}))_e(\beta_1) &= 2 \\
\chi(W_{(0,1)} \otimes (2,\mathbb{I}))_\gamma(\beta_1) &= 0 \\
\chi(W_{(0,1)} \otimes (2,\mathbb{I}))_e(\beta_2) &= 1 \\
\chi(W_{(0,1)} \otimes (2,\mathbb{I}))_\gamma(\beta_2) &= 0
\end{aligned}
$$

where for example the contributions to $\chi(W_{(0,1)} \otimes (2,\mathbb{I}))_e(\beta_1)$ come from the following pairs of weights which add to $\beta_1 = (2,1)$:

$$
\begin{aligned}
(-1,-1),(3,2) \\
(-1,0),(3,1)
\end{aligned}
$$

and the contribution to $\chi(W_{(0,1)} \otimes (2,\mathbb{I}))_e(\beta_2)$ comes from the pair $(0,1),(3,1)$. This implies

$$
W_{(0,1)} \otimes (2,\mathbb{I}) = (1,\mathbb{I}) + (1,\sigma) + (2,\mathbb{I}).
$$

Analogously, we compute the other products

	$(1,\mathbb{I})$	$(1,\sigma)$	$(2,\mathbb{I})$
$W_{(0,0),\mathbb{I}}$	$(1,\mathbb{I})$	$(1,\sigma)$	$(2,\mathbb{I})$
$W_{(0,0),\sigma}$	$(1,\sigma)$	$(1,\mathbb{I})$	$(2,\mathbb{I})$
$W_{(0,1)}$	$(2,\mathbb{I})$	$(2,\mathbb{I})$	$(1,\mathbb{I})+(1,\sigma)+(2,\mathbb{I})$

For the other component, $\{f\}/\!\!/G^f$, we don't have $\tau|_{\{f\}/\!\!/G^f} \cong \mathfrak{g}^f$, but the W_{aff} action on $\underline{\Lambda}^\tau$ is enough in this case to compute the $R(G)$ structure because there is only one free $\underline{W_{aff}}$ orbit and the translations of β_3 by characters of any of $W_{(0,0),\mathbb{I}}, W_{(0,0),\sigma}, W_{(0,1)}$ does not land in a $\kappa(\tilde{\Pi})$ shift of β_3. This is evident from the character maps of the restrictions of these classes to $\underline{N}^f$:

$$\chi(\underline{W_{(0,0),\sigma}})_e = (0)$$

$$\chi(\underline{W_{(0,0),\sigma}})_\gamma = -(0)$$

$$\chi(\underline{W_{(0,1)}})_e = 2(-1) + 2(0) + 2(1)$$

$$\chi(\underline{W_{(0,1)}})_\gamma = 0.$$

where the underline denotes restriction to $\underline{N}^f$. The action should be clear from these considerations:

	$(3,\mathbb{I})$	$(3,\sigma)$
$W_{(0,0),\mathbb{I}}$	$(3,\mathbb{I})$	$(3,\sigma)$
$W_{(0,0),\sigma}$	$(3,\sigma)$	$(3,\mathbb{I}))$
$W_{(0,1)}$	$(3,\mathbb{I})+(3,\sigma)$	$(3,\mathbb{I})+(3,\sigma)$

8.5 Multiplication Table

	$(1,\mathbb{I})$	$(1,\sigma)$	$(2,\mathbb{I})$	$(3,\mathbb{I})$	$(3,\sigma)$
$(1,\mathbb{I})$	$(1,\mathbb{I})$	$(1,\sigma)$	$(2,\mathbb{I})$	$(3,\mathbb{I})$	$(3,\sigma)$
$(1,\sigma)$	$(1,\sigma)$	$(1,\mathbb{I})$	$(2,\mathbb{I})$	$(3,\sigma)$	$(3,\mathbb{I}))$
$(2,\mathbb{I})$	$(2,\mathbb{I})$	$(2,\mathbb{I})$	$(1,\mathbb{I})+(1,\sigma)+(2,\mathbb{I})$	$(3,\mathbb{I})+(3,\sigma)$	$(3,\mathbb{I})+(3,\sigma)$

Chapter 9

$G = \mathbf{Spin}(8) \rtimes \mathrm{Sym}_3$

9.1 Lie Theory

We first consider the group $\overline{G} = \mathrm{SO}(8)$ and its Lie theory. We fix four orthogonal planes $P_1, P_2, P_3, P_4 \subset \mathbb{R}^8$ and let $\overline{T}$ be the maximal torus consisting of rotations in the planes P_i. The period lattice $\overline{\Pi}$ of $\overline{T}$ is generated by elements $\eta_1, \ldots, \eta_4 \in \bar{\mathfrak{t}}$ which exponentiate to rotations by 2π in planes $P_1, \ldots, P_4$ respectively. Let $\xi_1, \ldots, \xi_4$ be the dual basis of $\overline{\Lambda} = \hom(\overline{\Pi}, \mathbb{Z}) \subset \bar{t}^*$. Elements of $\overline{G}$ which normalize $\overline{T}$ have to preserve the set of planes $P_1, \ldots, P_4$. It follows that up to the action of $\overline{T}$ an element in the normalizer of $\overline{T}$ permutes the planes P_i and acts by reflection over a line on even number of planes P_i (since it has to preserve the overall orientation). If we let $R \subset (\mathbb{Z}/2)^{\times 4}$ be the subgroup consisting of even number of -1's, then we can realize the Weyl group of $\overline{G}$ as

$$W_1 = R \rtimes \mathrm{Sym}_4.$$

The action of W_1 on $\bar{\mathfrak{t}}$ is the following: the subgroup Sym_4 permutes the basis elements $\{\eta_i\}$ and the subgroup R negates an even number of basis elements. The action on $\overline{\Lambda}$ is the same with $\{\eta_i\}$ replaced by $\{\xi_i\}$.

We now let $G_1 = \mathrm{Spin}(8)$ and T the pullback of $\overline{T}$ by the projection

$G_1 \to \overline{G}$. The Lie algebras of $\overline{T}$ and T are naturally identified. The period lattice Π of T is the sublattice

$$\Pi = \{\sum_{i=1}^{4} a_i\eta_i |\ a_i \in \mathbb{Z},\ \ \sum_{i=1}^{4} a_i \in 2\mathbb{Z}\}.$$

The weight lattice Λ of T is larger than the weight lattice of $\overline{T}$:

$$\Lambda = \left\{ \sum_{i=1}^{4} c_i\xi_i |\ c_i \in \frac{1}{2}\mathbb{Z},\ \ \begin{cases} c_i \in \mathbb{Z} & \text{for all i} \\ c_i \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z} & \text{for all i} \end{cases} \right\}.$$

The Weyl group of G_1 is the same as the Weyl group of $\overline{G}$ with the same action on $\mathfrak{t}$ and $\mathfrak{t}^*$.

The roots of G_1 are elements $\{\pm\xi_i \pm \xi_j\}$ where $i, j \in \{1, \ldots, 4\}$ and $i \neq j$. The corresponding coroots are $\{\pm\eta_i \pm \eta_j\}$. We fix a set of simple roots $\alpha_1, \ldots, \alpha_4$:

$$\alpha_1 = \xi_1 - \xi_2; \quad \alpha_2 = \xi_2 - \xi_3; \quad \alpha_3 = \xi_3 - \xi_4; \quad \alpha_4 = \xi_3 + \xi_4.$$

The outer automorphism group of a simply connected simple group is isomorphic to the automorphism group of the corresponding Dynkin diagram. In the case of the group Spin(8), the Dynkin diagram is D_4 whose group of automorphisms is the symmetric group Sym_3 which acts by permuting the outer nodes of D_4. With the choice of simple roots as above, the correspondence with the Dynkin diagram is given by associating the inner node of D_4 to the simple root α_2 and the outer nodes to the roots $\alpha_1, \alpha_3, \alpha_4$. The identification of the set $\{\alpha_1, \alpha_3, \alpha_4\}$ with the set $\{1, 2, 3\}$ in that order defines an action of Sym_3 on the Dynkin diagram and therefore by diagram automorphisms on $\mathfrak{g}$.

The action restricted to $\mathfrak{t}$ is given by a homomorphism $\rho : \mathrm{Sym}_3 \to \mathrm{GL}(\mathfrak{t})$. In the basis $\{\eta_i\}$, this homomorphism is given by the following matrices

$$\rho((1,2)) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \quad \rho((1,3)) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\rho((2,3)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \rho((1,2,3) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

$$\rho((1,3,2)) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

The above choices define an action of Sym_3 on $\mathrm{Spin}(8)$. Let G be the semi-direct product $\mathrm{Spin}(8) \rtimes \mathrm{Sym}_3$. We compute here the structures necessary for the calculation of $K^\tau_G(G)$. We use the same notation as in Section 5.3. The group $\pi_0(G) = \mathrm{Sym}_3$ has three conjugacy classes: $[\mathrm{id}], [(2,3)], [(1,2,3)]$. The groupoid $G/\!\!/G$ is therefore the union of three groupoids of the form $fG_1/\!\!/G(f)$ with $f = \mathrm{id}, (2,3), (1,2,3)$.

We treat each case separately.

1. Case $f = (2,3)$

   - $\underline{\mathfrak{t}} = \langle \eta_1, \eta_2, \eta_3 \rangle$
   - $\underline{\Pi} = \{\sum_{i=1}^3 a_i\eta_i |\ a_i \in \mathbb{Z},\ \ \sum_{i=1}^3 a_i \in 2\mathbb{Z}\}$,
   - $\underline{\Lambda} = \left\{ \sum_{i=1}^3 c_i\xi_i |\ c_i \in \tfrac{1}{2}\mathbb{Z},\ \ \begin{cases} c_i \in \mathbb{Z} & \text{for all i} \\ c_i \in \tfrac{1}{2}\mathbb{Z}\backslash\mathbb{Z} & \text{for all i} \end{cases} \right\}$

- $[T/\underline{T}]^f = \{\pm 1\}$ with non-trivial element given by $\exp \frac{1}{2}\eta_4$,
- $\underline{\tilde{\Pi}} = \{\sum_{i=1}^{3} a_i\eta_i |\ a_i \in \mathbb{Z}\}$
- $W_1^f = R \rtimes \mathrm{Sym}_3 \subset W_1 = R \rtimes \mathrm{Sym}_4$ where $\mathrm{Sym}_3 \subset \mathrm{Sym}_4$ is the subgroup fixing the last entry.
- $Z_{\pi_0(G)}(fG_1) = \{\mathrm{id}, (2,3)\} \subset \mathrm{Sym}_3$.

2. Case $f = (1,2,3)$

- $\underline{\mathfrak{t}} = \langle \eta_1 + \eta_3, \eta_2 - \eta_3 \rangle$,
- $\underline{\Pi} = \{a_1(\eta_1 + \eta_3) + a_2(\eta_2 - \eta_3) |\ a_i \in \mathbb{Z}\}$,
- $\underline{\Lambda} = \left\{c_1\frac{1}{3}(\xi_1 - \xi_2 + 2\xi_3) + c_2\frac{1}{3}(2\xi_1 + \xi_2 + \xi_3) |\ c_i \in \mathbb{Z}\right\}$. For convenience, define the following basis of $\underline{\Lambda}$

$$z_1 := \frac{1}{3}\left(\xi_1 - \xi_2 + 2\xi_3\right), \quad z_2 := \frac{1}{3}\left(2\xi_1 + \xi_2 + \xi_3\right)$$

- $[T/\underline{T}]^f = \mathbb{Z}/3\mathbb{Z}$ with a generator given by $\exp(\frac{1}{3}\eta_4)$,
- $$\begin{aligned}\underline{\tilde{\Pi}} &= \{a_1\frac{1}{3}(\eta_1 - \eta_2 + 2\eta_3) + a_2(\eta_2 - \eta_3) |\ a_i \in \mathbb{Z}\} \\ &= \{a_1\frac{1}{3}(\eta_1 - \eta_2 + 2\eta_3) + a_2\frac{1}{3}(2\eta_1 + \eta_2 + \eta_3) |\ a_i \in \mathbb{Z}\}\end{aligned}$$
- W_1^f has 12 elements:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The action of W_1^f on $\underline{\mathfrak{t}}^*$ with respect to the basis $\{z_1, z_2\}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \tag{9.1.1}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- $Z_{\pi_0(G)}(fG_1) = \{\mathrm{id}, (1,2,3), (1,3,2)\} \subset \mathrm{Sym}_3$.

9.2 Twisting τ (level 1)

We specialize to the twisting τ_l corresponding to a level $l \in H^4(BG, \mathbb{Z})$ which pulls back to $1 \in \mathbb{Z} \cong H^4(BG_1, \mathbb{Z})$. By Proposition 6.2.1, we have

$$\kappa = (1+6) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in the bases $\{\eta_i, \xi_i\}$ since $h_G^\vee = 6$.

By Proposition 4.1.2, the fixed subgroups G^f are simply connected. In particular that implies that $H^3_{G^f}(\{f\}, \mathbb{Z}) = \{0\}$ for all choices of f. Therefore Condition 7.3.1 is satisfied and the analysis of Section 7.3 applies.

9.3 $K^\tau_G(G)$

9.3.0.1 Component of the Identity

We would like to first understand the action of W_1 on $\Lambda/\kappa(\Pi)$. First, note that it acts by homomorphisms since W_1 acts on Λ by homomorphisms. Next note that W_1 preserves the subgroup $J = \{\sum_{i=1}^4 c_i\xi_i |\ c_i \in \mathbb{Z}\}/\kappa(\Pi)$. It also fixes the point $s := \frac{7}{2}(\sum_i \xi_i)$ and therefore the two J-cosets are isomorphic as W_1-sets: the isomorphism is obtained by sending $0 \in J$ to $s \in sJ$. Next observe that $J = (\mathbb{Z}/7)^{\bigoplus 4} \oplus \mathbb{Z}/2$ with the four elements of order 7 being $2\xi_i$ for $i = 1, \ldots 4$ and the element of order 2 being $d := 7\xi_1$. Note that W_1 fixes d and preserves the subgroup $(\mathbb{Z}/7)^{\bigoplus 4} := J' \subset J$. On J', $W_1 = R \rtimes \mathrm{Sym}_4$ acts manifestly by permuting the entries and negating even number of elements. It follows that $(c_1, c_2, c_3, c_4) \in J'$ is not fixed by any non-zero element of W_1 if and only if $c_i \neq \pm c_j$ for $i \neq j$. There is only one free W_1 orbit in J' generated by $(0, 1, 2, 3) \in J'$.

In summary, there are 4 free W_1 orbits in $\Lambda/\kappa(\Pi)$

$$\mathcal{O}_1 = \langle 2\xi_2 + 4\xi_3 + 6\xi_4\rangle \qquad \mathcal{O}_2 = \langle 7\xi_1 + 2\xi_2 + 4\xi_3 + 6\xi_4\rangle$$

$$\mathcal{O}_2' = \langle 2\xi_2 + 4\xi_3 + 6\xi_4 + \frac{7}{2}(\sum_i \xi_i)\rangle \qquad \mathcal{O}_2'' = \langle 7\xi_1 + 2\xi_2 + 4\xi_3 + 6\xi_4 + \frac{7}{2}(\sum_i \xi_i)\rangle$$

The group Sym_3 acts on the orbits by preserving $\mathcal{O}_1$ and permuting $\mathcal{O}_2, \mathcal{O}_2', \mathcal{O}_2''$. We fix elements $\beta_1 \in \mathcal{O}_1$ and $\beta_2 \in \mathcal{O}_2$ such that $\mathrm{Stab}_{W_{aff}}(\beta_i) \subset \{0\} \rtimes \{e\} \rtimes \mathrm{Sym}_3$

$$\beta_1 := 3\xi_1 + 2\xi_2 + \xi_3 \qquad \beta_2 := 10\xi_1 + 2\xi_2 + \xi_3.$$

The element β_1 is chosen to coincide with the Weyl vector. We have

$$H_1 = \mathrm{Sym}_3 \subset \{0\} \rtimes W_1 \rtimes \mathrm{Sym}_3$$

$$H_2 = \mathbb{Z}/2 = \langle (2,3) \rangle \subset \mathrm{Sym}_3 \subset \{0\} \rtimes W_1 \rtimes \mathrm{Sym}_3.$$

The stabilizers H_i satisfy Condition 5.3.8 and therefore

$$K_G^\tau(G_1) \cong K^0_{H_1}(\{\beta_1\}) \oplus K^0_{H_2}(\{\beta_2\}) = \mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}^{\oplus 2}$$

9.3.0.2 Component of a 2-cycle

We first understand the action of W_1^f on $\underline{\Lambda}/\kappa(\underline{\tilde{\Pi}})$. We have that W_1^f preserves the subgroup $J = \{\sum_{i=1}^3 c_i\xi_i | \ c_i \in \mathbb{Z}\}/\kappa(\underline{\tilde{\Pi}})$ and fixes the element $\frac{7}{2}(\sum_i \xi_i)$. There is only one free W_1^f orbit in J. All in all we have two orbits generated by

$$\beta_3 := \xi_1 + 2\xi_2 + 3\xi_3 \qquad \beta_4 := \frac{7}{2}\left(\sum_{i=1}^{3} \xi_i\right) + \xi_1 + 2\xi_2 + 3\xi_3.$$

The stabilizers are

$$H_4 = H_3 = \langle (2,3) \rangle$$

and they satisfy Condition 5.3.8. We have

$$K_G^\tau(fG_1) \cong K^0_{H_3}(\{\beta_3\}) \oplus K^0_{H_4}(\{\beta_4\}) = \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}^{\oplus 2}$$

9.3.0.3 Component of a 3-cycle

We again first understand the action of W_1^f on $\underline{\Lambda}/\kappa(\tilde{\underline{\Pi}})$. We have that $\underline{\Lambda}/\kappa(\tilde{\underline{\Pi}}) \cong \langle z_1, z_2 \rangle = \mathbb{Z}/7 \otimes \mathbb{Z}/7$. The action of W_1^f is as in 9.1.1. Note that $\underline{\Lambda}/\kappa(\tilde{\underline{\Pi}})$ is a 2-dimensional vector space over $\mathbb{F}_7$ and that W_1^f acts linearly. In particular, for each non-identity element $w \in W_1^f$, the set of points in $\underline{\Lambda}/\kappa(\tilde{\underline{\Pi}})$ which are fixed by w is a linear subspace of dimension 0 or 1. Moreover, the only point in the intersection of two distinct subspaces of dimension 1 in $\underline{\Lambda}/\kappa(\tilde{\underline{\Pi}})$ is the zero vector. We now list the non-identity elements of W_1^f with eigenvalue 1 and specify the corresponding eigenspaces

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$
$$\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In particular, the union of the fixed subspaces has order 37. There are 49 elements in total in $\underline{\Lambda}/\kappa(\tilde{\underline{\Pi}})$, hence there is one free W_1^f orbit. This orbit is for example generated by the element

$$\beta_5 = z_1 + 2z_2 = \frac{1}{3}(\xi_1 - \xi_2 + 2\xi_3) + \frac{2}{3}(2\xi_1 + \xi_2 + \xi_3) = \frac{5}{3}\xi_1 + \frac{1}{3}\xi_2 + \frac{4}{3}\xi_3.$$

For the stabilizer, we have

$$H_5 = \langle (1,2,3) \rangle \subset \mathrm{Sym}_3 \subset \{0\} \rtimes W_1 \rtimes \mathrm{Sym}_3$$

which satisfies Condition 5.3.8 and therefore

$$K_G^\tau(fG_1) \cong K_{H_5}^0(\{\beta_5\}) = \mathbb{Z}^{\oplus 3}.$$

9.4 $R(G)$ module structure

We fix the following basis of $K^\tau_G(G)$:

$$\{(1,\mathbb{I}),(1,\sigma),(1,\mathrm{s}t),(2,\mathbb{I}),(2,\sigma),(3,\mathbb{I}),(3,\sigma),(4,\mathbb{I}),(4\sigma),(5,\mathbb{I}),(5,\omega),(5,\omega^2)\}$$

where $(i,V)\in K^0_{H_i}(\beta_i)$ given by representation V of H_i, σ is the sign representation, st is the standard irreducible representation of Sym_3 and ω is the 1-dimensional representation of $\langle(1,2,3)\rangle\cong\mathbb{Z}/3$ where $(1,2,3)$ acts by $e^{\frac{2}{3}\pi i}$.

The fundamental weights of G_1 are

$$w_1=\xi_1 \quad w_2=\xi_1+\xi_2 \quad w_3=\frac{1}{2}(\xi_1+\xi_2+\xi_3-\xi_4) \quad w_4=\frac{1}{2}(\xi_1+\xi_2+\xi_3+\xi_4).$$

The group Sym_3 acts on the weight vectors by fixing w_2 and permuting w_1,w_3,w_4. A representation of G is given by a tuple $n=(n_1,n_2,n_3,n_4)\in\mathbb{Z}^{\times 4}_{\geq 0}$ and a representation V of $\mathrm{Stab}_{\mathrm{Sym}_3}(n)$. We label such representation by $W_{n,V}$.

We first analyze the map $\iota_*:R(G)\to K^\tau_G(G_1)$. The map is surjective and we will find preimages of the generators we fixed above. The Weyl vector of G_1 is

$$\rho=3\xi_1+2\xi_2+\xi_1=w_1+w_2+w_3+w_4.$$

We have

$$\iota_*W_{(0),\mathbb{I}}=(1,\mathbb{I})$$

$$\iota_*W_{(0),\sigma}=(1,\sigma)$$

$$\iota_*W_{(0),\mathrm{s}t}=(1,\mathrm{s}t)$$

$$\iota_*W_{(7,0,0,0),\mathbb{I}}=(2,\mathbb{I})$$

$$\iota_* W_{(7,0,0,0),\sigma} = (2, \sigma).$$

The twisting τ satisfies Condition 7.2.8 by Proposition 7.2.7 and therefore Corollary 7.3.2 gives an algorithm for computing the $R(G)$ action. The algorithm is the same as described in more detail for $SU(3) \rtimes \mathbb{Z}/2$. We performed the computation using the program Sage which is particularly adept at performing computations in representation rings of Lie groups. We simply record the answer here.

9.5 Multiplication Table

	$(1, \mathbb{I})$	$(1, \sigma)$	$(1, \mathrm{s}t)$
$(1, \mathbb{I})$	$(1, \mathbb{I})$	$(1, \sigma)$	$(1, \mathrm{s}t)$
$(1, \sigma)$	$(1, \sigma)$	$(1, \mathbb{I})$	$(1, \mathrm{s}t)$
$(1, \mathrm{s}t)$	$(1, \mathrm{s}t)$	$(1, \mathrm{s}t)$	$(1, \mathbb{I}) + (1, \sigma) + (1, \mathrm{s}t)$
$(2, \mathbb{I})$	$(2, \mathbb{I})$	$(2, \sigma)$	$(2, \sigma) + (2, \mathbb{I})$
$(2, \sigma)$	$(2, \sigma)$	$(2, \mathbb{I})$	$(2, \sigma) + (2, \mathbb{I})$
$(3, \mathbb{I})$	$(3, \mathbb{I})$	$(3, \sigma)$	$(3, \mathbb{I}) + (3, \sigma)$
$(3, \sigma)$	$(3, \sigma)$	$(3, \mathbb{I})$	$(3, \mathbb{I}) + (3, \sigma)$
$(4, \mathbb{I})$	$(4, \mathbb{I})$	$(4, \sigma)$	$(4, \mathbb{I}) + (4, \sigma)$
$(4, \sigma)$	$(4, \sigma)$	$(4, \mathbb{I})$	$(4, \mathbb{I}) + (4, \sigma)$
$(5, \mathbb{I})$	$(5, \mathbb{I})$	$(5, \mathbb{I})$	$(5, w) + (5, w^2)$
$(5, w)$	$(5, w)$	$(5, w)$	$(5, w^2) + (5, \mathbb{I})$
$(5, w^2)$	$(5, w^2)$	$(5, w^2)$	$(5, \mathbb{I}) + (5, w)$

	$(2,\mathbb{I})$	$(2,\sigma)$
$(1,\mathbb{I})$	$(2,\mathbb{I})$	$(2,\sigma)$
$(1,\sigma)$	$(2,\sigma)$	$(2,\mathbb{I})$
$(1,\mathrm{s}t)$	$(2,\sigma)+(2,\mathbb{I})$	$(2,\sigma)+(2,\mathbb{I})$
$(2,\mathbb{I})$	$(1,\mathbb{I})+(1,\mathrm{s}t)+(2,\mathbb{I})+(2,\sigma)$	$(1,\sigma)+(1,\mathrm{s}t)+(2,\mathbb{I})+(2,\sigma)$
$(2,\sigma)$	$(1,\sigma)+(1,\mathrm{s}t)+(2,\mathbb{I})+(2,\sigma)$	$(1,\mathbb{I})+(1,\mathrm{s}t)+(2,\mathbb{I})+(2,\sigma)$
$(3,\mathbb{I})$	$(3,\mathbb{I})+(4,\mathbb{I})+(4,\sigma)$	$(3,\sigma)+(4,\mathbb{I})+(4,\sigma)$
$(3,\sigma)$	$(3,\sigma)+(4,\mathbb{I})+(4,\sigma)$	$(3,\mathbb{I})+(4,\mathbb{I})+(4,\sigma)$
$(4,\mathbb{I})$	$(3,\mathbb{I})+(3,\sigma)+(4,\mathbb{I})$	$(3,\mathbb{I})+(3,\sigma)+(4,\sigma)$
$(4,\sigma)$	$(3,\mathbb{I})+(3,\sigma)+(4,\sigma)$	$(3,\mathbb{I})+(3,\sigma)+(4,\mathbb{I})$
$(5,\mathbb{I})$	$(5,\mathbb{I})+(5,w)+(5,w^2)$	$(5,\mathbb{I})+(5,w)+(5,w^2)$
$(5,w)$	$(5,\mathbb{I})+(5,w)+(5,w^2)$	$(5,\mathbb{I})+(5,w)+(5,w^2)$
$(5,w^2)$	$(5,\mathbb{I})+(5,w)+(5,w^2)$	$(5,\mathbb{I})+(5,w)+(5,w^2)$

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