Multisymplectic effective General Boundary Field Theory

José A. Zapata

Centro de Ciencias Matemáticas, UNAM, México

zapata@matmor.unam.mx

September 2013

What is presented

$\sphericalangle$ "Multisymplectic Effective General Boundary Field Theory"
M. Arjang and J. A. Zapata (to appear)

A continuation of:

- J. A. Zapata (2004),
  –Loop quantization from a LGT perspective–
- E. Manrique, R. Oeckl, A. Weber and J. A. Zapata (2006),
  –Loop quantization as a continuum limit–
- M. P. Reisenberger (1997, 1994),
  –Classical discrete gauge theory and gravity / S F models–
- R. Oeckl (2003-),
  –General Boundary Field Theory–
- A. P. Veselov (1988),
  –On discrete time hamiltonian systems–
- J. E. Marsden, G. W. Patrick, S. Shkoller (1998),
  –On discrete spacetime multisymplectic field theory–

Closely related to:

- Gambini, Pullin, Di Bartolo, Porto (2002)
  –Consistent discretizations–
- Gambini, Pullin, Di Bartolo, Campiglia (2006)
  –Uniform discretizations–
- Dittrich, Bahr, Hoehn (2009)
  –Variational integrators and improved actions–
- Halvorsen, Sørensen, Christiansen (2011)
  –Noether's theorem for spacetime simplicial gauge theory–
- Kogut, Susskind (1975)
  –Hamiltonian Formulation of Lattice Gauge Theories–

For regularization see

- Calcagni, Oriti, Thurigen (2013)
  –Structure for discrete Hodge star and Laplacian–

Geometry of covariant 1st order filed theory

Continuum histories are local sections
$M \supset U \xrightarrow{\phi} E$ of a bundle with standard fibre $\mathcal{F}$.
Physical motions are selected by Hamilton's principle with action

$$S(\phi) = \int_U \mathcal{L}(j^1\phi) \quad , \quad \text{where } j^1\phi(x) = (x, \phi(x), D\phi(x)).$$

Its variation can be written as

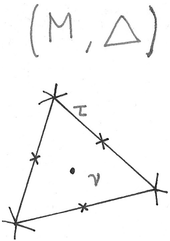
$$dS(\phi)\cdot\delta\phi = -\int_U (j^1\phi)^*(j^1(\delta\phi)\lrcorner\widehat{\Omega}_L) + \int_{\partial U}(j^1\phi)^*(j^1(\delta\phi)\lrcorner\Theta_L)$$

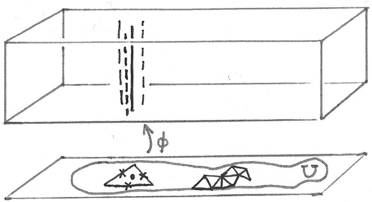
with $\Theta_L, \widehat{\Omega}_L$ differential forms on J^1Y.
If we further define $\Omega_L = -d\Theta_L$ then

** ϕ solution, v, w first var. $\implies$ $\int_{\partial U}(j^1\phi)^*(v\lrcorner w\lrcorner\Omega_L) = 0$ **

* S $\mathcal{G}$-inv., $\xi \in \mathfrak{g}$ gives v_ξ first var. $\implies$ $\int_{\partial U}(j^1\phi)^*(v_\xi\lrcorner\Theta_L) = 0$ *

Decimation for scalar 1st order models





Decimated local record of a history in 1st order format

$$\nu \overset{\tilde{\phi}}{\longmapsto} (\nu, \phi_\nu \in \mathcal{F}, \{\phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$$

A variation $\delta\tilde{\phi}(\nu) = \tilde{v}(\nu) = (v_\nu \in T_{\phi_\nu}\mathcal{F}, \{v_\tau \in T_{\phi_\tau}\mathcal{F}\}_{\tau \subset \partial\nu})$

Scalar field's action and its variation

Our starting point is a variational principle with action

$$S(\phi) = \sum_{\nu \in U_\Delta^n} L(\tilde{\phi}(\nu)) \quad .$$

Its variation yields: (i) eqs. of motion and (ii) geometric structure

$$dS(\phi)[v] = -\sum_{U-\partial U} \tilde{\phi}^*(\tilde{v} \lrcorner \hat{\Omega}_L) + \sum_{\partial U} \tilde{\phi}^*(\tilde{v} \lrcorner \Theta_L)$$

where

$$\Theta_L(\cdot, \tilde{\phi}(\tau_\nu)) = \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau \quad ,$$

$$\hat{\Omega}_L(\cdot, \tilde{\phi}(\nu)) = -\frac{\partial L}{\partial \phi}(\tilde{\phi}(\nu)) d\phi_\nu - \sum_{\tau \in (\partial \nu)^{n-1}} \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau \quad .$$

The multisymplectic formula

We define

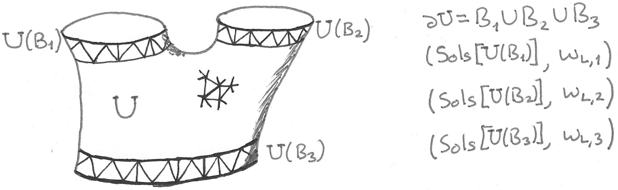
$$\Omega_L(\tilde{v}(\nu), \tilde{w}(\nu), \tilde{\phi}(\tau_\nu)) = -d(\Theta_L|_{\tilde{\phi}(\tau_\nu)})(\tilde{v}(\nu), \tilde{w}(\nu)).$$

The geometrical structure arises when only solutions and first variations are considered. $dS(\phi)[v] = \sum_{\partial U} \tilde{\phi}^*(\tilde{v} \lrcorner \Theta_L)$ implies $0 = -ddS = \sum_{\partial U} \tilde{\phi}^*(\Omega_L)$. More precisely,

$$\sum_{\partial U} \tilde{\phi}^*(\tilde{w} \lrcorner \tilde{v} \lrcorner \Omega_L) = 0$$

for any first variations v, w of any solution ϕ.
This is **the multisymplectic formula**.

Its meaning in GBFT language



$(\mathrm{Sols}[U(\partial U)] = \times_i \mathrm{Sols}[U(B_i)], \mathrm{diag}(\omega_{L,1}, \omega_{L,m}))$ is not trivial.

Solutions in U correlate the spaces of the boundary components.

$$I_U : \mathrm{Sols}[U] \to \mathrm{Sols}[U(\partial U)] = \times_i \mathrm{Sols}[U(B_i)]$$

The multisymplectic formula implies

$$I_U^* \mathrm{diag}(\omega_{L,1}, ..., \omega_{L,m}) = 0 \quad .$$

*** In particular, $U \approx \Sigma \times [0,1] \subset M$ with $\partial\Sigma = \emptyset$ leads to ... ***

Symmetries and conserved quantities

The Lie group $\mathcal{G}$ acts on $\mathcal{F}$ and on histories. In 1st order format

$$(\tilde{g\phi})(\nu) = (\nu, g(\phi_\nu), \{g(\phi_\tau)\}_{\tau \subset \partial \nu})$$

$L(g\tilde{\phi}(\nu)) = L(\tilde{\phi}(\nu)) \quad \forall \quad \nu, \phi, g \implies S$ and Sols$[U]$ are $\mathcal{G}$ inv.

$\xi \in \mathfrak{g}$ induces a first variation v_ξ of any solution ϕ. Then

$$0 = dS(\phi)[v_\xi] = \sum_{\partial \tilde{\phi}(U)} \tilde{v}_\xi \lrcorner \Theta_L .$$

The canonical framework for $\dim M = 1$

$L(\tilde{q}(\nu)) = L(\nu, q^-, q, q^+)$ provides momentum 1-forms

$$(q^-, \frac{\partial L}{\partial q^-}(\tilde{q}(\nu)))dq^- \in T^*Q_{\nu^-},$$

$$(q^+, \frac{\partial L}{\partial q^+}(\tilde{q}(\nu)))dq^+ \in T^*Q_{\nu^+}$$

that can be used to define "Legendre transformations" $f_L^- : Q_{\nu^-} \times Q_\nu \to T^*Q_{\nu^-}$ and $f_L^+ : Q_\nu \times Q_{\nu^+} \to T^*Q_{\nu^+}$.
The relation between the described lagrangian structure and $(T^*Q, \theta = pdq)$ is simply

$$\Theta_L(\cdot, \tilde{q}(\nu)^-) = -(f_L^-)^*\theta \quad , \quad \Theta_L(\cdot, \tilde{q}(\nu)^+) = (f_L^+)^*\theta.$$

Dynamics arises as evolution in the form of canonical transformations with L as their generating function [Veselov].

The canonical framework for $\dim M > 1$

In covariant field theory, J^1Y^* hosts canonical kinematics $(\Theta, \Omega = -d\Theta)$ and dynamics is placed over it.

This canonical str. could be pulled back to the discrete framework.

In a sense, this is what we do.
Another way to see it is that we use

- predetermined codim 1 surfaces
- endowed with a notion of vertical variation (constructed from each face of the surface)

to extract a $1\dim$ system and proceed as in the previous slide.

Example: Particle in euclidean space with a potential

1. $L(\nu, q_\nu^-, q_\nu, q_\nu^+) = \frac{m}{2}(\frac{q_\nu - q_\nu^-}{a})^2 + \frac{m}{2}(\frac{q_\nu^+ - q_\nu}{a})^2 - V(q_\nu)$
2. $dS(q)[v] = \sum_{\nu \in U_{\text{disc}}} \tilde{v}[L](\tilde{q}(\nu)) = \sum_{\nu \in U_{\text{disc}}} \frac{\partial L}{\partial q}(\tilde{q}(\nu)) \cdot v_\nu$
   $+ \sum_{\nu, \nu+1 \in U_{\text{disc}}} \left(\frac{\partial L}{\partial q^+}(\tilde{q}(\nu)) + \frac{\partial L}{\partial q^-}(\tilde{q}(\nu+1)) \right) \cdot v_\nu^+$
   $+ [\frac{\partial L}{\partial q^-}(\tilde{q}(1)) \cdot v_1^- + \frac{\partial L}{\partial q^+}(\tilde{q}(n)) \cdot v_n^+]$
3. $\mapsto \Theta_L(\cdot, \tilde{q}(\nu)^-) = m g_{AB} \frac{(q_\nu - q_\nu^-)^B}{a} dq^{-A}$, $\Theta_L(\cdot, \tilde{q}(\nu)^+)$, $\hat{\Omega}_L(\cdot, \tilde{q}(\nu))$
4. For solutions and first variations:
   (i) ** Conserv. of symplectic structure (indep. of V) **
   $\Omega_L(\cdot, \cdot, \tilde{q}(\nu)^-) \doteq -d(\Theta_L|_{\tilde{q}(\nu)^-}) = \Omega_L(\cdot, \cdot, \tilde{q}(\nu)^+)$
   (ii) ** Symmetries lead to conserved quantities **
   $V = 0 \Rightarrow$ transl. symm. $\Rightarrow 0 = dS(q)[v_\xi] = \tilde{v}_\xi \lrcorner \Theta_L|_{\partial \tilde{q}(U)}$
   $\Rightarrow$ conserv. of $p \Rightarrow \quad q_1 - q_1^- = q_n^+ - q_n$
5. L time indep. $\not\Rightarrow$ energy conserv.: most systems are chaotic

Example: Veselov's top $\quad Q = SO(n)$

1. $L(\nu, g^-, g, g^+) = \frac{1}{2}\operatorname{Tr}(g^{-1}\mathrm{I}g^-)a + \frac{1}{2}\operatorname{Tr}((g^+)^{-1}\mathrm{I}g)a$
2. Veselov: It has a complete set of first integrals in involution.
3. $\tilde{v}_\xi(\nu) = (\xi^-_\nu g^-_\nu, \xi_\nu g_\nu, \xi^+_\nu g^+_\nu)$
4. $\Theta_L(\tilde{v}_\xi, \tilde{g}^-_\nu) \doteq \tilde{v}_\xi[L](\tilde{g}(\nu))|_{\nu^-} = -\frac{1}{2}\operatorname{Tr}(\xi^-_\nu g^-_\nu I g_\nu^T)a$
5. $\Omega_L(\tilde{v}_\xi, \tilde{w}_\eta, \tilde{g}^-_\nu) = -\frac{1}{2}\operatorname{Tr}((\xi_\nu \eta^-_\nu - \eta_\nu \xi^-_\nu) g^-_\nu I g_\nu^T)a$
6. Conserv. of the symplectic str. (q solution, $\tilde{v}_\xi, \tilde{w}_\eta$ first var.)
   $-\frac{1}{2}\operatorname{Tr}((\xi_1\eta^-_1 - \eta_1\xi^-_1)q^-_1 I q_1^T) = -\frac{1}{2}\operatorname{Tr}((\xi_n\eta^-_n - \eta_n\xi^-_n)q^-_n I q_n^T)$
7. Conserv. of ang. momentum (q solution)
   $m(q) = \frac{1}{2}(q^-_\nu I q_\nu^T - q_\nu I q^{-\,T}_\nu) = \frac{1}{2}(q_\nu I q^{+\,T}_\nu - q^+_\nu I q_\nu^T)$

Example: Non linear 2d waves (see Marsden et al)

In a hyper cubical discr. $L = \sum_c L^c$ with $L^{++}(\tilde{\phi}(\nu)) =$

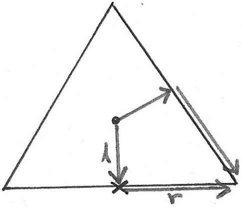
$$\left\{ \frac{1}{2} \left[\left(\frac{\phi_{0+} - \phi_\nu}{h} \right)^2 - \left(\frac{\phi_{1+} - \phi_\nu}{k} \right)^2 \right] + N(\phi_\nu) \right\} hk \quad \text{, etc}$$

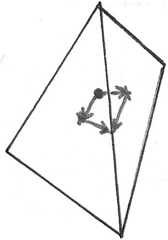
Even with a non linear term, $\sum_{\partial U} \tilde{\phi}^*(\Omega_L) = 0$.
Our framework yields Θ_L, Ω_L (and gluing eqs.) indep. of N, e.g.

$$\Omega_L(\cdot, \cdot, \tilde{\phi}_{1+_\nu}) = -\frac{2h}{k} d\phi_\nu \wedge d\phi_{1+}.$$

Decimation for gauge theories and modified BF theories

Reisenberger's discretization:





Record of a history in 1st order format $\nu \overset{\tilde{A}}{\longmapsto} (\nu, \{h_l\}_\nu, \{k_r\}_\nu)$

Our record of the curvature at ν is $\{g_{\partial s} = h_{l'}^{-1} \circ k_{r'}^{-1} \circ k_r \circ h_l\}_\nu$

A variation in 1st order format $\tilde{v}(\nu) = (\{v_l \in T_{h_l} G\}_\nu, \{v_r \in T_{k_r}\}_\nu)$

Gauge field's action and its variation

$$S(A) = \sum_{\nu \in U^n_\Delta} L(\tilde{A}(\nu)) = \sum_{\nu \in U^n_\Delta} L(\nu, \{g_{\partial s}\})$$

$$dS(A)[v] = -\sum_{U - \partial U} \tilde{A}^*(\tilde{v} \lrcorner \hat{\Omega}_L) + \sum_{\partial U} \tilde{A}^*(\tilde{v} \lrcorner \Theta_L).$$

The resulting geometric structure is, then,

$$\Theta_L(\cdot, \tilde{A}(\tau_\nu)) = \sum_{r \subset \tau} \frac{\partial L}{\partial k_r}(\tilde{A}(\nu)) dk_r \quad ,$$

$$\hat{\Omega}_L(\cdot, \tilde{A}(\nu)) = -\sum_{l \subset \nu} \frac{\partial L}{\partial h_l}(\tilde{A}(\nu)) dh_l - \sum_{r \subset \nu} \frac{\partial L}{\partial k_r}(\tilde{A}(\nu)) dk_r \quad .$$

Internal gauge symmetries

Internal gauge transfs. are local symms. of the action. Then:

- For internal gauge transformations over interior points of U $\implies$ Redundancies of equations of motion.
- For internal gauge transformations over points $C\tau \in \partial U$ $\implies$ Constraints on possible boundary data for solutions.
- For internal gauge transformations over points $C\sigma \in \partial U$ $\implies$ The dynamical content of boundary data can be captured by the reduced variables $k_{r'}^{-1} k_{r1}$.

Example: Lattice gauge theory without fermions

1. $S_{\mathrm{Euc}}(A) = \beta \sum_{\nu \subset U} \sum_{s<\nu} [1 - \frac{1}{n} \operatorname{Re} \operatorname{Tr}(g_{\partial s})]$
2. $\tilde{v}_\xi(\nu) = (\{h_l \xi_l \in T_{h_l} G\}_\nu, \{\xi_r k_r \in T_{k_r}\}_\nu)$
3. $dS(A)[v] = -\sum_{U-\partial U} \tilde{A}^*(\tilde{v} \lrcorner \hat{\Omega}_L) + \sum_{\partial U} \tilde{A}^*(\tilde{v} \lrcorner \Theta_L)$ with

$$\Theta_L(\tilde{v}_\xi, \tilde{A}(\tau_\nu)) = \tilde{v}_\xi L(\tilde{A}(\nu))|_\tau = \frac{-\beta}{n} \operatorname{Re} \sum_{r \subset \tau} \operatorname{Tr}(h_{l'}^{-1} k_{r'}^{-1} \xi_r k_r h_l)$$

$$\hat{\Omega}_L(\tilde{v}, \tilde{A}(\nu)) = \tilde{v}_\xi L(\tilde{A}(\nu)) = \frac{-\beta}{n} \operatorname{Re} \sum_{l \subset \nu} \operatorname{Tr}(h_{l'}^{-1} k_{r'}^{-1} k_r h_l \xi_l) + \frac{-\beta}{n} \operatorname{Re} \sum_{r \subset \partial \nu} \operatorname{Tr}(h_{l'}^{-1} k_{r'}^{-1} \xi_r k_r h_l)$$

4. $0 = \frac{-\beta}{n} \operatorname{Re} \sum_{\partial U} \sum_{r \subset \tau} \operatorname{Tr}(\{\hat{\xi}_r(\eta_l - \eta_{l'}) - \hat{\eta}_r(\xi_l - \xi_{l'})\} g_{\partial s})$
   where $\xi_r = k_r h_l \hat{\xi}_r h_l^{-1} k_r^{-1}$

Example: Reisenberger's model

$$S(A, e, \psi) = \sum_{\nu \subset U} [\sum_{s<\nu} e_{s\,i}\, \mathrm{Tr}(J^i g_{\partial s}) - \frac{1}{60} \sum_{s,\bar{s}<\nu} \psi_\nu^{ij} e_{s\,i} e_{s\,j} sgn(s, \bar{s})]$$

The equations of motion were written by Reisenberger.
Notice that at each ν the boundary dof are purely connection dof.

** $\Theta_L = \Theta_{L,\mathrm{BF}}$ ** Consider $\tilde{v}_\xi(\nu)$ with $\{v_{\xi r} = \xi k_r \in T_{k_r} SU(2)\}$

$$\Theta_L(\tilde{v}_\xi, \tilde{æ}(\tau_\nu)) = \tilde{v}_\xi L(\tilde{æ}(\nu))|_\tau = \sum_{r \subset \tau} e_{s_r\,i}\, \mathrm{Tr}(J^i h_{l'}^{-1} k_{r'}^{-1} \xi_r k_r h_l) \quad ,$$

Thus, the multisymplectic formula, $\sum_{\partial U} \tilde{æ}^* \Omega_L = 0$,
is independent of the constraint term.

Regularization

$$\text{Decimated Histories}_\Delta \xrightarrow{i_\Delta} \text{Histories Continuum}$$

e.g. selecting closest sols. to "the free theory" (if it makes sense)

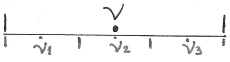
$$S_\Delta = i_\Delta^* S$$

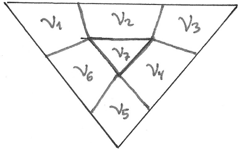
Reisenberger's discretization can be obtained by intersecting a 'prime' simplicial decomposition with its dual.
Having two intersecting discretizations allows for a notion of **Hodge dual for cochains** that sets up a rich structure for regularization; see [Calcagni et al].
We have this structure plus the **simple gluing** already mentioned.

Coarse graining

Hists. at finer scale $\Delta' \geq \Delta$ are coarse grained decimating further:

$$\phi_{\Delta'} \stackrel{\tilde{\pi}}{\longmapsto} \phi_{\Delta}$$





Hists. and vars. in 1st order format are coarse grained likewise.

Notice the difference with directly working on T^*Q or J^1Y^*.

Coarse graining: Correction of a model at Δ

Coarse graning hists. and vars. from $\Delta' > \Delta$ can be used to import **solutions** and **first variations**, but we can do better ...

Measures in the space of histories are also coarse grained.
(In QFT and SFT there is a straight forward interp. of coarse gr.)
They can be corrected by coarse graining from finer scales.
** A probabilistic perfect measure at scale Δ is one that agrees with all its corrections. **
(i) Δ-Effective eqs. of motion are not exactly satisfied, and
(ii) cons. laws are not exactly satisfied either. (See next slide.)

In $\dim(M) = 1$ we have access to Hamilton's function at any scale.
In $\dim(M) > 1$ knowledge of bdary. cond. is only partial;
thus, we have no access to it.

Continuum limit

Fix a history ϕ and a variation $\delta\phi$ from the continuum, and let the measuring (decimation) scale get finer

$$\lim_{\Delta \to M} dS_\Delta(\phi_\Delta) \cdot \delta\phi_\Delta$$

$\mapsto$ Eqs. of motion and geo. str. may converge [Veselov, Marsden et al].
This also gives the error at scale Δ.

– Should we? –

Specify a solution ϕ in the continuum as a sequence $\{\phi_\Delta\}$ of *approx. solutions* to effective eqs. of motion(Δ) s.t.
(i) $\{\phi_\Delta\} \to \phi$, and (ii) dS_Δ converges (see [Gambini et al]),
$\implies$ Symmetries are a limit of "approx. symmetries(Δ)" (discussion Dittrich, Rovelli).