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WALLPAPER GROUPS

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Abstract

In this paper we present the wallpaper groups or plane crystallographic groups. The name wallpaper groups refers to the symmetry group of periodic pattern in two dimensions. We start by presenting the symmetries of the Euclidean plane. Than all 17 types of wallpaper symmetries are described.

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1 Introduction

Intuitively a wallpaper pattern is a design used for making wallpaper. One can make such a pattern by taking a motif and repeating it horizontally and vertically. The example on Figure 1 (left) was made by taking a motif Γ, and translating it horizontally and vertically. There are other ways of repeating a basic motif to get a pattern. For instance, one could repeat the the same motif in an another way to obtain different wallpaper pattern (Figure 1 right). As we will see there are exactly 17 number of ways a motif can be arranged to form a patter.

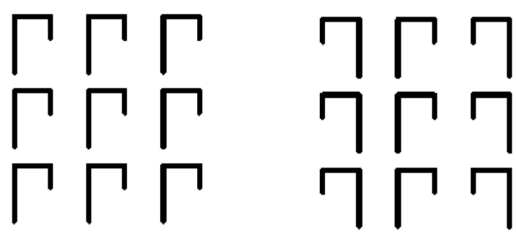


Figure 1: Two different ways of wallpaper pattern construction by repeating the same motif.

Wallpaper patterns occur frequently in architecture and decorative art. A nice example is the Alhambra. This Islamic palace lays in the valleys of Granada, Spain and was built after the Muslims conquered Spain in the 8th century. The palaces interior design exhibits the Islamic necessity to use geometric shapes. This mathematical labyrinth, as it has been referred to, exhibits walls that are decorated using all possible wallpaper patterns. It is not known whether the architects had a mathematical knowledge of all the different wallpaper patterns or simply exhausted all the possibilities.

2 Symmetry in the plane

An *isometry* of $\mathbb{R}^2$ is a distance-preserving transformation of the plane [1]. It is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that for any points P and Q in the plane

$$d(P, Q) = d(f(P), f(Q)), \tag{1}$$

where $d(P, Q)$ is the distance between P and Q. As we will later show there are only four types of isometries of the plane: translations, rotations, reflections and glide reflections.
If vector $\mathbf{v} \in \mathbb{R}^2$, then a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_{\mathbf{v}}(\mathbf{r}) = \mathbf{r} + \mathbf{v}$ for all $\mathbf{r} \in \mathbb{R}^2$ is a **translation** by a vector $\mathbf{v}$. **Rotation** $R_{0,\alpha}$ by an angle α about the origin is in Cartesian coordinates given by

$$R_{0,\alpha}(\mathbf{r}) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2}$$

Rotation by α about a point $P \in \mathbb{R}^2$ can be written as $R_{P,\alpha} = T \circ R_{0,\alpha} \circ T^{-1}$, where T is translation by P. We say that a planer object has rotational symmetry about a point P if it is carried into itself by at least one non-trivial rotation around P. Let l be a line in $\mathbb{R}^2$ going through the origin parallel to a unit vector $\mathbf{w}$, then the **reflection** M_l of vector $\mathbf{r}$ across l is given by

$$M_l(\mathbf{r}) = 2(\mathbf{rw})\mathbf{w} - \mathbf{r}. \tag{3}$$

An object is symmetric with respect to a line l if it is carried into itself by reflection across l. Let M_l be a reflection about a line l passing through the origin. If vector $\mathbf{t} \in \mathbb{R}^2$ is not perpendicular to l, than a composition of reflection and translation $G = T_{\mathbf{t}} \circ M_l$ is a **glide reflection** parallel to l and displaced in perpendicular direction by $\frac{1}{2}\mathbf{t}$.

The composition of two isometries is an isometry [1]. The isometries of the plane form a group under composition of function, the so called Euclidean group E_2. Euclidean group is built from the translational group T the orthogonal group $O_2(\mathbb{R})$ [1]. Every element of $O_2(\mathbb{R})$ can be represented by a 2 x 2 matrix, where any such matrix satisfies the condition $A^T A = I$. Let A be the matrix with respect to the standard basis for an element in $O_2(\mathbb{R})$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4}$$

then the condition $A^T A = I$ gives

$$\begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & d^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

The above relation can be satisfied if there is an angle θ with $a = \cos\theta$, $c = \sin\theta$ and $(b, d) = (-\sin\theta, \cos\theta)$ or $(b, d) = (\sin\theta, -\cos\theta)$. The first choice gives a matrix A with determinant 1 that represents a rotation about the origin by an angle θ. Such matrix is an element of special orthogonal group $SO_2(\mathbb{R})$ a subgroup of $O_2(\mathbb{R})$ whose elements preserve orientation. On the other hand the second choice gives a matrix of determinant -1 that represents a reflection across a line through the origin. Elements of $SO_2(\mathbb{R})$ are rotations and elements of $O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})$ are reflections. Since every element of E_2 is the composition of a translation with an element of $O_2(\mathbb{R})$, it follows that every isometry of the plane is either a translation, a rotation, a reflection or a glide reflection [2]. Every isometry of the plane can be written in a specific notation as an ordered pair $(A, \mathbf{u})$, where $\mathbf{u} \in \mathbb{R}^2$ and $A \in O_2$.

2.1 Composition of isometries

Now we will look at some compositions of isometries, that will later help us construct the wallpaper groups.

<u>Rotation ∘ translation</u>

Composition of rotation $R_{P,\alpha}$ about point P by an angle α and a translation $T_{\mathbf{t}}$ is another rotation $R_{Q,\alpha}$ by α about a point Q that is displaced vertically by $\frac{|\mathbf{t}|}{2}\cot\frac{\alpha}{2}$ along bisector of $\mathbf{t}$ (Figure 2).

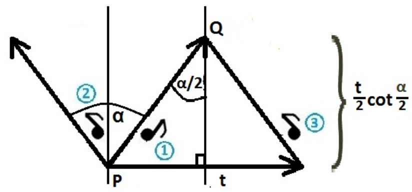


Figure 2: Composition of rotation and translation. Initial object (1) is rotated about P by an angle α to a second object (2), which is than translated by $\mathbf{t}$ to third object (3). The first and third object are directly related with rotation about Q by an angle α.

<u>Reflection ∘ translation</u>

If M_m is a reflection across line m and $T_{\mathbf{t}}$ translation perpendicular to reflection line, than the composition of reflection and translation is a reflection $M_{m'}$ across line m' at the bisector of $\mathbf{t}$. If however the translation is not perpendicular, than the composition is a glide reflection $G_{\mathbf{t}_\parallel}$ parallel to reflection line and displaced by $\frac{\mathbf{t}_\perp}{2}$ in perpendicular direction (Figure 3).

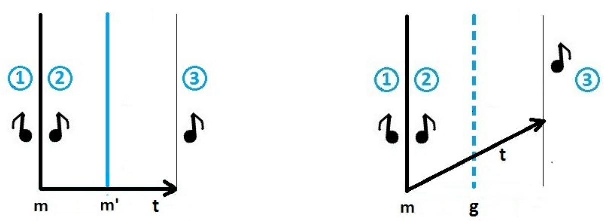


Figure 3: Composition of reflection and translation. Initial object (1) is reflected across line m to a second object (2), which is than translated by $\mathbf{t}$ to third object (3). If the vector of translation is perpendicular to the mirror line m, the first and third object are directly related with reflection across m', otherwise they are related with glide reflection g.

<u>Glide reflection ∘ translation</u>

Composition of a glide reflection G and a perpendicular translation $T_{\mathbf{t}}$ is a glide reflection G' at the bisector of $\mathbf{t}$. Composition of a glide reflection $G_{\mathbf{t}_\parallel}$ and a non-perpendicular translation is a reflection M perpendicular to glide reflection and displaced by $\frac{\mathbf{t}_\perp}{2}$ in perpendicular direction.

<u>Reflection ∘ rotation</u>

If M_m is a reflection across line m and P center of rotation by α that lies on m, than the composition of rotation and reflection is a reflection M'_m across a line m', that is passing through P and is inclined by an angle $\frac{\alpha}{2}$ to line m (Figure 4). If center of rotation does not lie on reflection line, than the composition is a glide reflection.

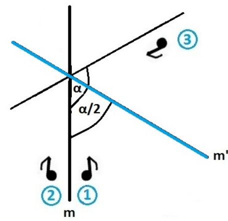


Figure 4: Composition of rotation and reflection. Initial object (1) is reflected across line m to a second object (2), which is than rotated about P by α to third object (3). We can see that the first and third object are related by reflection across line m', which is inclined at an angle $\frac{\alpha}{2}$ relative to line m.

<u>Reflection ∘ reflection</u>
Suppose we have two parallel reflection lines m and m' separated by $\mathbf{t}$. Reflecting an object across reflection line m followed by reflection across the line m' is the same as translating the original object by $2\mathbf{t}$ (Figure 5 left). In the more general case the two reflection lines are inclined to one another by an angle α and we have an intersection point P (Figure 5 right). The composition of two reflections is in this case a rotation $R_{P,\alpha}$ by 2α about point P.

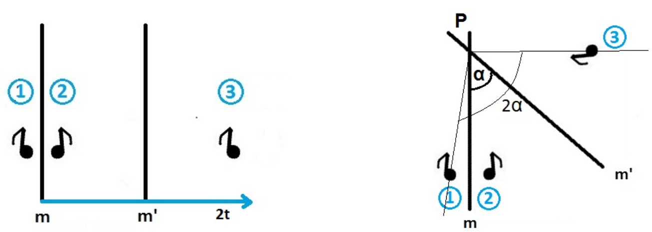


Figure 5: Composition of two parallel (left) and non-parallel (right) reflections. Initial object (1) is reflected across line m to a second object (2), which is than reflected across line m' to third object (3).

2.2 Lattices and point groups

Wallpaper pattern is a two dimensional repeating pattern that fills the whole plane. The symmetry group of a wallpaper pattern is said to be a **wallpaper group**. A subgroup $W \in E_2$ is a wallpaper group if its translation subgroup H is generated by two independent translations and its points group J is finite [2].
A two-dimensional **lattice** L in $\mathbb{R}^2$, is the set of all the points that the origin gets mapped to under the action of H, the translation subgroup of a wallpaper pattern. It is a subgroup of the form $n\mathbf{t}_1 + m\mathbf{t}_2$, where $n, m \in \mathbb{Z}$ for some basis $\mathbf{t}_1, \mathbf{t}_2$ of $\mathbb{R}^2$ [1]. By examining all possible basic parallelograms determined by the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ we will later see that there are only five different types of lattices; parallelogram (oblique), rectangular, centred rectangular, square and hexagonal [1].

Theorem (Crystallographic restriction): The order of a rotation in a wallpaper group can only be 2, 3, 4, or 6.

Proof: Let Q be a lattice point and center of rotation of period n. Transformation operator of the lattice transforms Q into infinitely many other centers of rotation of the same period. Let P'

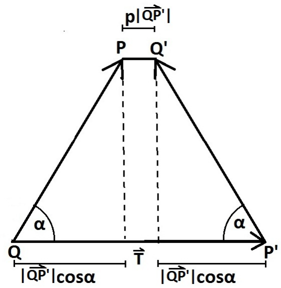


Figure 6: Crystalographic restriction.

be one of these other centers at the least possible distance form Q (Figure 6). A third center of rotation P is obtained by rotation of P' through $\alpha = 2\pi/n$ about Q; and a fourth Q' by rotation of Q by α about P'.
Points generated by rotation must coincide with lattice points, therefore Q' and P are lattice points and $\|Q'P\| = p\|QP'\|$, where $p \in \mathbb{Z}$. Form Figure we can see the following relation

$$\|QP'\| = 2\|QP'\| \cos\alpha + p\|QP'\| \quad \text{or} \quad \cos\alpha = \tfrac{1-p}{2}. \tag{6}$$

By considering each p in turn we can obtain all possible orders of rotation (Table 1).

p	$\cos\alpha$	α	n-fold
4	$-3/2$	/	/
3	-1	180°	2-fold
2	$-1/2$	120°	3-fold
1	0	90°	4-fold
0	1/2	60°	6-fold
-1	1	0°	trivial
-2	3/2	/	/

Table 1: Possible orders of rotations in wallpaper patterns.

Let R be a rotation by $360°/n$ and M a reflection. **Dihedral group** D_n is a group that contains rotations $R, R^2, ..., R^n$ and reflections $M, RM, ..., R^{n-1}M$, where the elements R and M satisfy the relations $R^n = M^2 = I$ and $MRM = R^{-1}$. The subgroup of all rotations in D_n is the **cyclic** subgroup $I, R, ..., R^{n-1}$, denoted by C_n [2].
If J is a point group of a wallpaper group W, then J is isomorphic to one of the following ten groups

$$\left\{ \begin{array}{c} C_1, C_2, C_3, C_4, C_6 \\ D_1, D_2, D_3, D_4, D_6 \end{array} \right\}.$$

For proof see [2]. Since the elements of wallpaper groups are all isometries we will use the notation $(j, \mathbf{t}_j)$ to denote an element of a wallpaper group. For each $j \in J$ there is a vector $\mathbf{t}_j \in \mathbb{R}^2$ with $(j, \mathbf{t}_j) \in J$. Furthermore, $\mathbf{t}_j$ is uniquely determined up to addition by an element of L. We will refer to $\{\mathbf{t}_j\}_{j\in J}$ as point group vectors for wallpaper group W. If $\{\mathbf{t}_1, \mathbf{t}_2\}$ is a basis for the translational lattice, than W is generated by the two translations $(I, \mathbf{t}_1), (I, \mathbf{t}_2)$ together with the elements $\{(j, \mathbf{t}_j); j \in J\}$. This is a finite set of generators. To describe explicitly a wallpaper group W, it is then sufficient to determine all possible vectors $\{\mathbf{t}_j\}$ that satisfy certain restrictions.

3 The Wallpaper groups

Having found all the possible point groups we now move on to classifying all possible wallpaper groups. We will denote wallpaper groups with crystallographic notation. The full name consists of four symbols. The first symbol represents the lattice type; p for primitive and c for centred. This is followed by a digit, n, indicating the highest order of rotational symmetry: 1(none), 2, 3, 4, or 6-fold. The next two symbols is either an m, g, or 1. An m (g) at the place of third symbol means there is a reflection line (glide reflection line) perpendicular to the x-axis while a 1 means there is no line of either type. Finally, the last symbol m (g) represents a reflection line (glide reflection) at an angle α with the x-axis, the angle depending on the largest order of rotation as follows: $\alpha = 90°$ for $n = 1, 2$; $\alpha = 60°$ for $n = 3, 6$; $\alpha = 45°$ for $n = 4$. The short notation drops digits or an m that can be deduced. For example, the group name $p3m1$ represents a group with a 120° rotation, a reflection line perpendicular to the x-axis, and no reflection or glide line at an angle of 60° with the x-axis.

For each wallpaper group we can draw a structure lattice diagram where the lattice is colored in blue, while the present operators are indicated with the well established symbols presented in Table 2.

⬯	center of rotation of order 2 (180°)
△	center of rotation of order 3 (120°)
□	center of rotation of order 4 (90°)
⬡	center of rotation of order 6 (60°)
———	an axis of reflection
- - - - -	an axis of glide reflection

Table 2: Notation used in the structure lattice diagram of wallpaper groups.

3.1 Parallelogram lattices

We choose a basis $\mathbf{t}_1, \mathbf{t}_2$ for the two independent translations, where the x-axis is in direction of $\mathbf{t}_1$. Let lattice point Q be the center of rotation by 180°. With rotation of lattice point P' around Q and Q around P' we obtain the most general situation. On Figure 7 we see that the operation of rotation by 180° does not impose any restriction on the geometry of the lattice. In other words $|\mathbf{t}_1| \neq |\mathbf{t}_1|$ with an arbitrary angle between them.

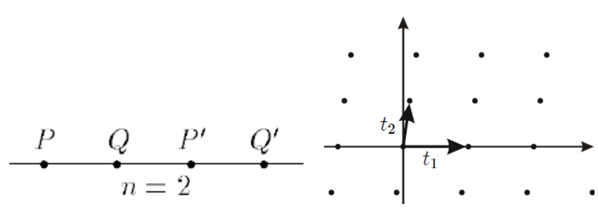


Figure 7: Construction of the lattice in the case of rotation by 180° (left). Parallelogram lattice (right).

Point group C_1

The description of the point group C_1 does not depend on the basis and contains only the identity matrix, therefore

$$C_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{7}$$

The wallpaper group isomorphic to C_1 is the simplest one denoted by **p1**. It is generated by two independent translations and a trivial rotation. An example of group p1 is depicted on Figure 8.

Figure 8: Lattice structure diagram and a pattern example of wallpaper group p1.

Point group $\mathbf{C_2}$

Point group C_2 contains the operation of rotation by 180° and can be written as

$$C_2 = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{8}$$

Suppose we place the center of 2-fold rotation operator at one lattice point. Translation operation produces the rotation operator at all lattice points. The 2-fold rotation centers are therefore present at the corners of parallelogram lattice (Figure 9). Since the composition of 2-fold rotation and translation is another 2-fold rotation there are additional centers located at $\frac{1}{2}\mathbf{t}_1$, $\frac{1}{2}\mathbf{t}_2$ and $\frac{1}{2}(\mathbf{t}_1 + \mathbf{t}_2)$. Obtained wallpaper group is labelled with **p2**.

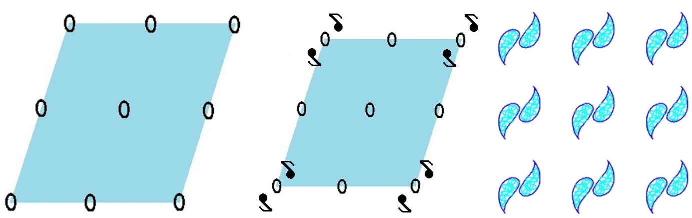


Figure 9: Lattice structure diagram, diagram with the addition of simple motif and a pattern example of wallpaper group p2.

3.2 Rectangular and centerd rectangular (rhombic) lattices

Let M_l be a nontrivial reflection across line l and $\mathbf{t} \in L$ a nonzero vector not parallel to l (Figure 10). Vectors $\mathbf{t} + M(\mathbf{t})$ and $\mathbf{t} - M(\mathbf{t})$ are also elements of L, so L contains nonzero vectors both parallel and perpendicular to the line of reflection l. If $\mathbf{s}_1$ and $\mathbf{s}_2$ are nonzero vectors of minimal length parallel and perpendicular to the reflection line, than for any $\mathbf{t} \in L$ we have $\mathbf{t} + M(\mathbf{t}) = m_t\mathbf{s}_1$ and $\mathbf{t} - M(\mathbf{t}) = n_t\mathbf{s}_2$ for some $m_t, n_t \in \mathbb{Z}$. Solving for $\mathbf{t}$ gives

$$\mathbf{t} = \frac{m_t}{2}\mathbf{s}_1 + \frac{n_t}{2}\mathbf{s}_2. \tag{9}$$

If both integers m_t, n_t are even, than $\{\mathbf{s}_1, \mathbf{s}_2\}$ is the basis for L. The corresponding lattice is rectangular lattice whose basis are two orthogonal not equal in length vectors, one of which is fixed

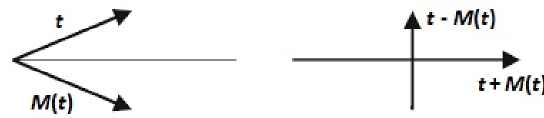


Figure 10: Transformation of nontrivial reflection of vector non parallel to the line of reflection.

by a reflection line (Figure 11). On the other hand, if m_t or n_t is odd for some $\mathbf{t}$, then both have to be odd, otherwise we have a contradiction. If we set $\mathbf{t}_1 = \frac{1}{2}(\mathbf{s}_1 + \mathbf{s}_2)$ and $\mathbf{t}_2 = \frac{1}{2}(\mathbf{s}_1 - \mathbf{s}_2) = f(t_1)$, then

$$\mathbf{t} = \frac{m_t}{2}\mathbf{s}_1 + \frac{n_t}{2}\mathbf{s}_2 = \frac{1}{2}(m_t + n_t)\mathbf{t}_1 + \frac{1}{2}(m_t - n_t)\mathbf{t}_2 = \frac{m'_t}{2}\mathbf{t}_1 + \frac{n'_t}{2}\mathbf{t}_2 \tag{10}$$

with $m'_t, n'_t \in \mathbb{Z}$ and the set $\{\mathbf{t}_1, \mathbf{t}_2\}$ is a basis of vectors of the same length with a reflection that interchanges them. Alternatively we can use larger lattice with additional lattice point where the basis are orthogonal vectors of non equal length. Lattice L is called rhombic or. centred rectangular lattice.

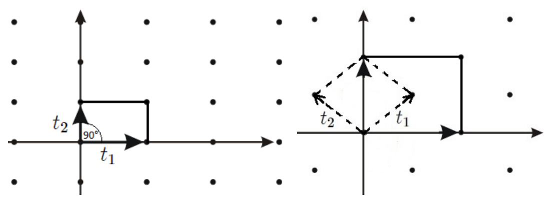


Figure 11: Rectangular (left) and centred rectangular lattice (right).

Point group $\mathbf{D_1}$

Point group D_1 can be combined either with rectangular or centred rectangular lattice. We fist look at the rectangular lattice. The matrix representation of $D_{1,p}$ in above defined basis is

$$D_{1,p} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{11}$$

where p stands for primitive lattice.

Lemma: There are two nonisomorphic wallpaper groups with point group $D_{1,p}$.
Proof: To determinate the point group vectors we have to consider the condition $\mathbf{u} + M(\mathbf{u}) \in L$. Rectangular lattice has the basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ with $M(\mathbf{t}_1) = \mathbf{t}_1$ and $M(\mathbf{t}_2) = -\mathbf{t}_2$. If $\mathbf{u} = \alpha\mathbf{t}_1 + \beta\mathbf{t}_2$ with $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha, \beta < 1$ we have $\mathbf{u} + M(\mathbf{u}) = 2\alpha\mathbf{t}_1$. For this to be an element of L we must have $2\alpha \in \mathbb{Z}$, so $\alpha = 0$ or $\alpha = \frac{1}{2}$, while β is arbitrary. Therefore, there are two wallpaper groups with point group $D_{1,p}$; one corresponds to the choice of $\mathbf{u} = 0$ and the other to $\mathbf{u} = \frac{1}{2}\mathbf{t}_1$.

If wallpaper group W contains a reflection in a horizontal mirror that the wallpaper group is named **pm** that contains group elements

$$\left\{ \mathbf{t}_1, \mathbf{t}_2, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right) \right\}. \tag{12}$$

The reflection lines pass through the lattice points. Another distinct reflections are due to the composition of translation and orthogonal reflection and lie midway between lattice points (Figure 12).

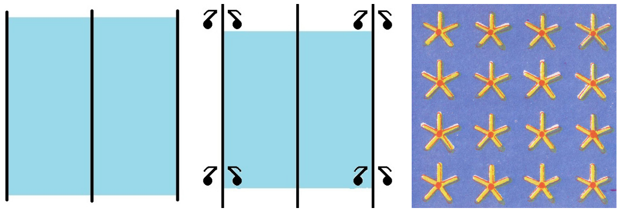


Figure 12: Wallpaper group pm.

If wallpaper group W is isomorphic to point group $D_{1,p}$, yet there are no reflections in W. Than W has to contain a glide reflection whose line is horizontal. Group elements of this wallpaper group denoted **pg** are

$$\left\{\mathbf{t}_1, \mathbf{t}_2, \left(\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \frac{1}{2}\mathbf{t}_1\right)\right\}. \tag{13}$$

By the same argument as before there is also a horizontal glide reflection at $\frac{1}{2}\mathbf{t}_1$ (Figure 13).

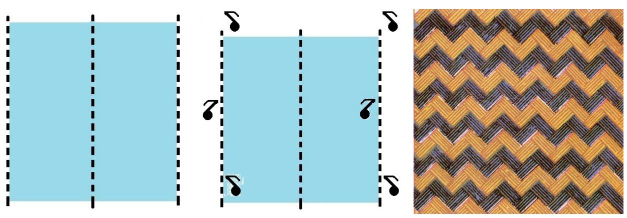


Figure 13: Wallpaper group pg.

In the case of centred rectangular lattice. The matrix representation of $D_{1,c}$ is

$$D_{1,c} = \left\{\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right\}, \tag{14}$$

where c stands for centerd lattice. There is only one way of constructing a wallpaper group with this point group. The group is named **cm** and contains horizontal reflections that go through lattice points and horizontal glide reflections which lie midway between lattice points (Figure 14).

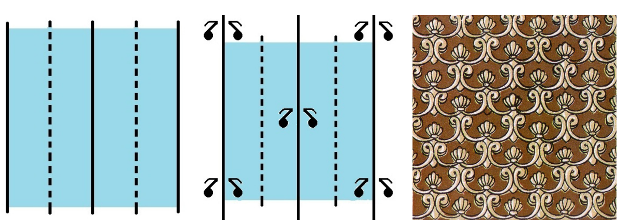


Figure 14: Wallpaper group cm.

Point group $\mathbf{D_2}$

As before there are two possibilities for matrix representation for D_2, corresponding to two different lattices. For the rectangular lattice we have

$$D_{2,p} = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{15}$$

Lemma: There are three nonisomorphic wallpaper groups with point group $D_{2,p}$.
Proof: Let $J = D_{2,p}$. Then L has a basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ with $M(\mathbf{t}_1) = \mathbf{t}_1$ and $M(\mathbf{t}_2) = -\mathbf{t}_2$, while $R(\mathbf{t}) = -\mathbf{t}$ for all $\mathbf{t} \in L$. If $\mathbf{u} = \alpha\mathbf{t}_1 + \beta\mathbf{t}_2$, then condition $R(\mathbf{u}) - \mathbf{u} \in L$ says that $\alpha = 0, \frac{1}{2}$ and $\beta = 0, \frac{1}{2}$. The condition $M(\mathbf{u}) + \mathbf{u} \in L$ gives no further restriction. Therefore, we have four possibilities: $\mathbf{u} = 0, \mathbf{u} = \frac{1}{2}\mathbf{t}_1, \mathbf{u} = \frac{1}{2}\mathbf{t}_2, \mathbf{u} = \frac{1}{2}(\mathbf{t}_1 + \mathbf{t}_2)$. Since groups corresponding to $\mathbf{u} = \frac{1}{2}\mathbf{t}_1, \mathbf{u} = \frac{1}{2}\mathbf{t}_2$ are isomorphic we have only three nonisomorphic wallpaper groups.

If G contains a reflection in a horizontal mirror and a reflection in a vertical mirror than the wallpaper group is named **p2mm** (Figure 15) and generated by

$$\left\{ \mathbf{t}_1, \mathbf{t}_2, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right) \right\}. \tag{16}$$

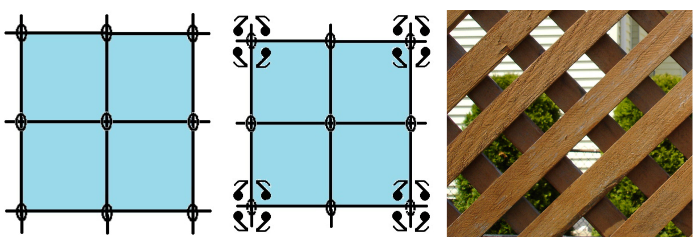


Figure 15: Wallpaper group p2mm.

If G contains a reflection in a horizontal mirror and a glide reflection in a vertical mirror than the wallpaper group is named **p2mg** (Figure 16) and generated by

$$\left\{ \mathbf{t}_1, \mathbf{t}_2, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}\mathbf{t}_1 \right) \right\}. \tag{17}$$

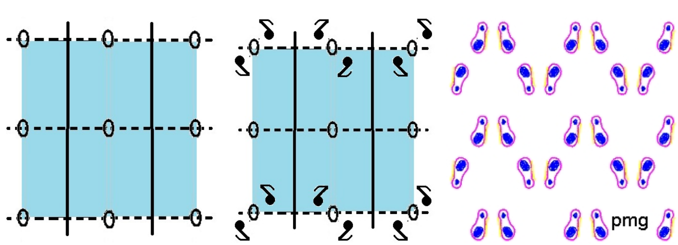


Figure 16: Wallpaper group p2mg.

The third wallpaper group belonging to $D_{2,p}$ is **p2gg** with glide reflections in horizontal and vertical mirror planes (Figure 17) and the generators are

$$\left\{\mathbf{t}_1, \mathbf{t}_2, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}(\mathbf{t}_1 + \mathbf{t}_2)\right)\right\}. \tag{18}$$

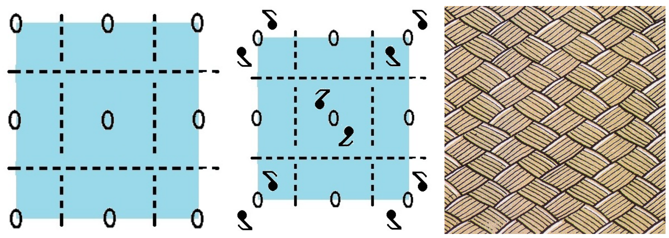


Figure 17: Wallpaper group p2gg.

Matrix representation of D_2 for the centred rectangular lattice is

$$D_{2,c} = \left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}. \tag{19}$$

Wallpaper group corresponding to this point group is named **c2mm** and contains the perpendicular reflections passing through the lattice points and perpendicular glide reflections in between the reflections (Figure 18).

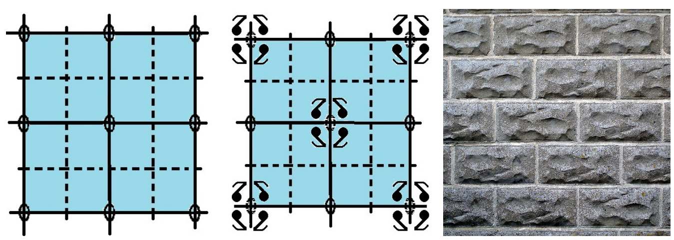


Figure 18: Wallpaper group c2mm.

3.3 Square lattices

Let R be a rotation by 90°. If $\mathbf{t}_1$ is a vector in L of minimal length, then $\{\mathbf{t}_1, R(\mathbf{t}_1)\}$ is a basis for L. The basis vectors are of equal length and the angle between them is equal to 90°. The lattice is called a square lattice (Figure 19).

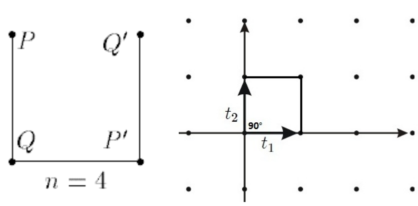


Figure 19: Construction of a square lattice.

Point group C_4
With respect to this basis, we see that if $J = C_4$, then the representation of J by this basis is

$$C_4 = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{20}$$

Figure 20 shows wallpaper group with 4-fold rotational symmetry and no reflections is denoted by **p4**. Adding 4-fold axis of rotation to one lattice point will add a 4-fold rotation at all corners of a square lattice. Additional 4-fold axis lies in the center of square due to composition of 4-fold rotation and translation. Since group C_2 is a subgroup of C_4 we know that 2-fold axis of rotation have to be located at the middle of each lattice edge.

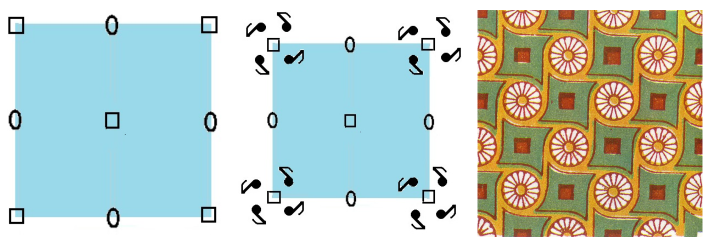


Figure 20: Wallpaper group p4.

Point group D_4
Since D_4 is generated by C_4 and any reflection, using the reflection about the line parallel to $\mathbf{t}_1$, we obtain the representation

$$D_4 = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{21}$$

Lemma: There are two nonisomorphic wallpaper groups with point group D_4.
Proof: Suppose that $J = D_4$, there is a basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ for L with $R(\mathbf{t}_1) = \mathbf{t}_2$ and $R(\mathbf{t}_2) = -\mathbf{t}_1$ and $\mathbf{u} = \alpha\mathbf{t}_1 + \beta\mathbf{t}_2$. The condition $R(\mathbf{u}) - \mathbf{u} \in L$ says $(-\alpha - \beta)\mathbf{t}_1 + (\alpha - \beta)\mathbf{t}_2 \in L$. In other words, $\alpha + \beta \in \mathbb{Z}$ and $\alpha - \beta \in \mathbb{Z}$. Therefore, with $0 \leq \alpha, \beta < 1$, we have the solutions $\alpha = \beta = 0$ and $\alpha = \beta = \frac{1}{2}$. So, either $\mathbf{u} = 0$ or $\mathbf{u} = \frac{1}{2}(\mathbf{t}_1 + \mathbf{t}_2)$.

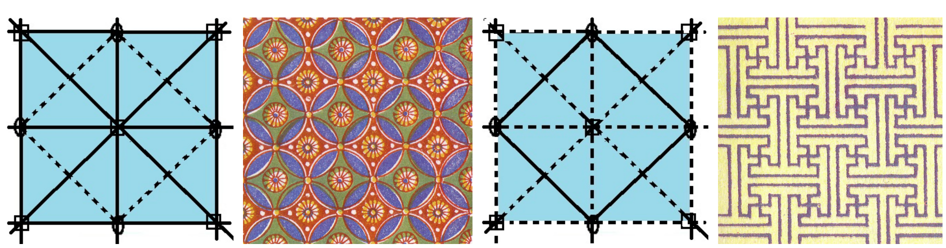


Figure 21: Wallpaper groups p4m (left) and p4g (right).

The wallpaper group **p4m** is generated by

$$\left\{ \mathbf{t}_1, \mathbf{t}_2, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right) \right\}. \tag{22}$$

It contains reflections at angles $0°, 45°, 90°$ and $135°$ with respect to $\mathbf{t}_1$ and glide reflections passing through the 2-fold axes (Figure 21). If we exchange the reflections and glide reflections we obtain wallpaper group **p4g** that is generated by

$$\left\{\mathbf{t}_1, \mathbf{t}_2, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}(\mathbf{t}_1 + \mathbf{t}_2)\right)\right\}. \tag{23}$$

3.4 Hexagonal lattices

Let R be a rotation by $120°$. If $\mathbf{t}_1$ is a vector in L of minimal length, then by setting $\mathbf{t}_2 = R(\mathbf{t}_1)$, the set $\{\mathbf{t}_1, \mathbf{t}_2\}$ is a basis for L. The lattice in this case is called a hexagonal lattice (Figure 22).

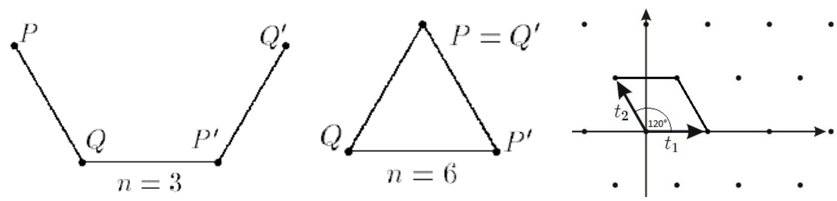


Figure 22: Construction of hexagonal lattice.

The group $\mathbf{C_3}$
In this basis the point group C_3 can be written as

$$C_3 = \left\{\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\right\}. \tag{24}$$

With point group C_3 we obtain the wallpaper group **p3** shown on Figure 23. The centers of 3-fold axis are located at corners of lattice. Two additional centers of 3-fold axis arise due to composition of rotation and translation.

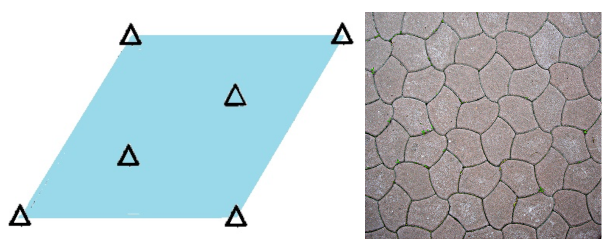


Figure 23: Wallpaper group p3.

The group $\mathbf{C_6}$
Wallpaper group isomorphic to C_6, denoted by **p6** is generated by 6-fold rotation, where

$$C_6 = \left\{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\right\}. \tag{25}$$

To construct the pattern, we use the fact that the point group C_6 contains both C_2 and C_3. Wallpaper group p6 is therefore going to contain all the elements of groups p2 and p3. As we can see on Figure 24 the centers of 6-fold axis are located at corners of hexagonal lattice, while the centres

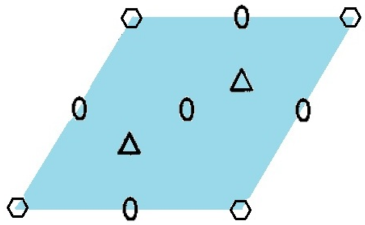


Figure 24: Wallpaper group p6.

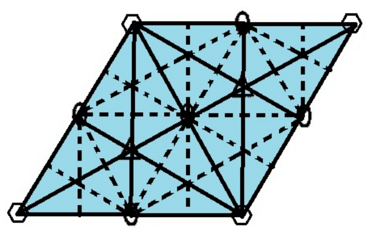
of 3 and 2-fold axis lie at the same positions as in p2 and p3.

The group D_6

If $J = D_6$, then J contains 6 lines of reflection separated by 30°. The group D_6 is generated by C_6 and any reflection; using the reflection that fixes $\mathbf{t}_1$, we have

$$D_6 = \left\{ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\}. \tag{26}$$

The corresponding wallpaper group is named **p6m** (Figure 25). There are additional glide reflections in six distinct directions, whose axes are located halfway between adjacent parallel reflection axes.



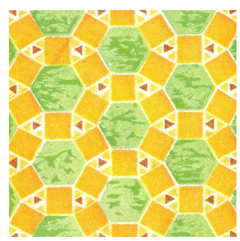


Figure 25: Wallpaper group p6m.

The group D_3

If $J = D_3$, then the point group contains three reflections. The lines of reflection are separated by 60° angles, since if M is a reflection in D_3, then MR is a reflection whose line of reflection makes a 60° angle with that of M. The reflection lines for D_3 must be reflection lines for D_6 since D_3 is a subgroup of D_6. This indicates that D_3 can act in two ways with respect to this basis. The reflection lines can be at angles 30°, 90° and 150° with respect to $\mathbf{t}_1$. The rotation by 120° and the reflection line about the 30° are generating the group $D_{3,l}$, where

$$D_{3,l} = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\}, \tag{27}$$

where l stands for long. The group $D_{3,l}$ contains a reflection about the 150° line, which is the long diagonal of a parallelogram. The wallpaper group associated with $D_{3,l}$ is **p3m1** (Figure 26 left). The centre of every rotation lies on a reflection axis. There are additional glide reflections in three distinct directions, whose axes are located halfway between adjacent parallel reflection axes.

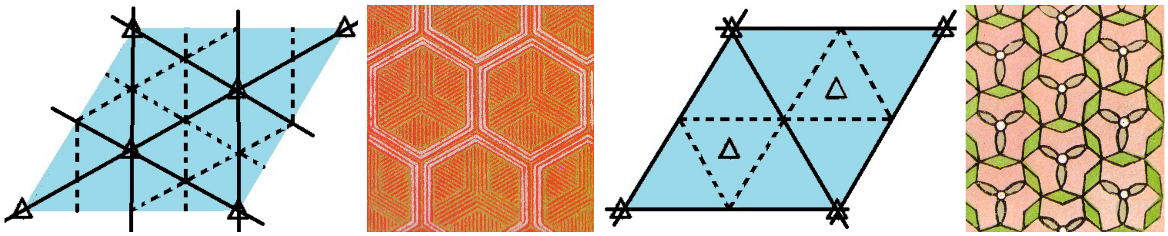


Figure 26: Wallpaper groups p3m1 (left) and p31m (right).

The other possibility are reflection lines at 0°, 60° and 120° with respect to $\mathbf{t}_1$. The rotation by 120° and the reflection line about the 0° are generating the group $D_{3,s}$

$$D_{3,s} = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\}, \tag{28}$$

where s stands for short. The group $D_{3,s}$ contains a reflection about the 60° line, which is the short diagonal of a parallelogram. Isomorphic to $D_{3,s}$ is wallpaper group **p31m** (Figure 26 right). This group has at least one rotation whose centre does not lie on a reflection axis. There are additional glide reflections in three distinct directions, whose axes are located halfway between adjacent parallel reflection axes.

References

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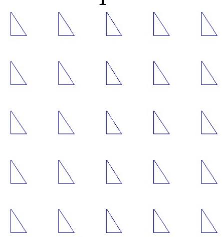
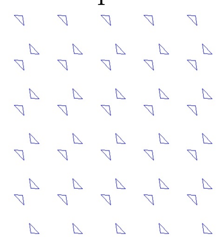
[2] P.J. Morandi, Symmetry Groups: The Classification of Wallpaper Patterns, Mathematics 482/526 New Mexico State University (2007).

A 17 Wallpaper groups

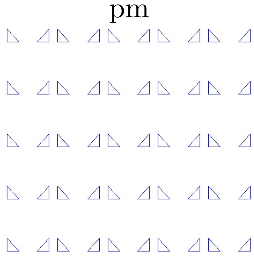
p1



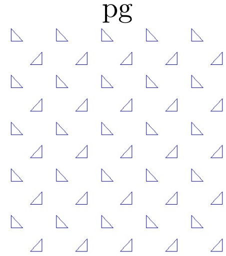
p2



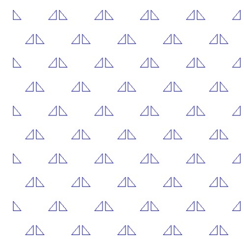
pm



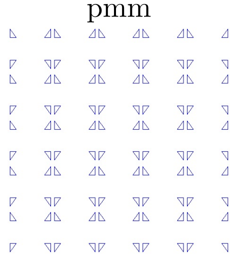
pg



cm



pmm



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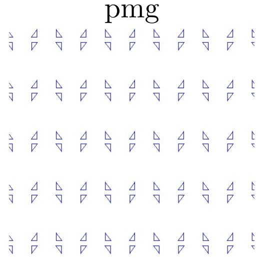

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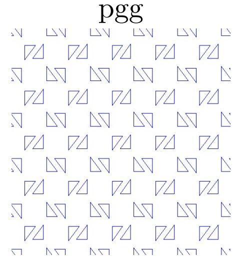
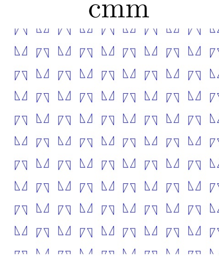
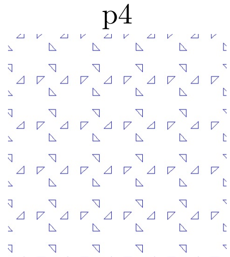


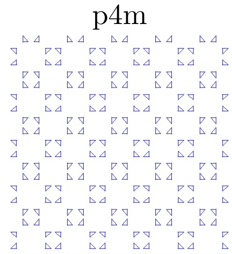
cmm



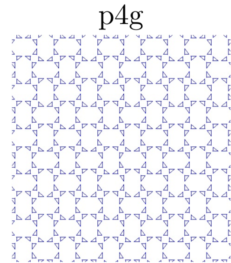
p4



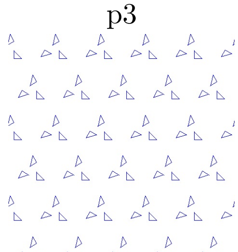
p4m



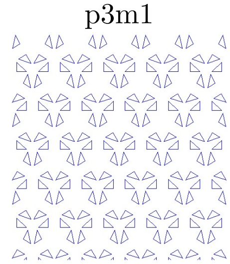
p4g



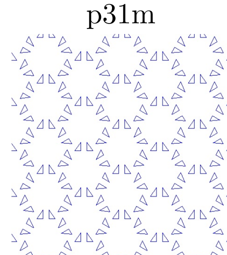
p3



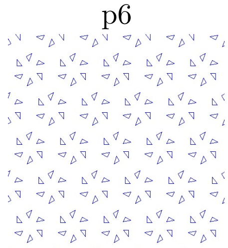
p3m1



p31m



p6



p6m