SUPER-MODULAR CATEGORIES

A Dissertation

by

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ABSTRACT

A super-modular category is a unitary pre-modular category with Müger center equivalent to the symmetric unitary category of super-vector spaces. Physically, super-modular categories describe universal properties of fermionic topological phases of matter. Mathematically, super-modular categories are important alternatives to modular categories as any unitary pre-modular category is the equivariantization of either a modular or a super-modular category. Unlike the modular case, one does not have a representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$ associated to a super-modular category, but it is possible to obtain a representation of the index 3 θ-subgroup: $\Gamma_\theta < \mathrm{SL}(2,\mathbb{Z})$. We study the image of this representation and conjecture a super-modular analogue of the Ng-Schauenburg Congruence Subgroup Theorem for modular categories, namely that the kernel of the Γ_θ representation is a congruence subgroup. We prove this conjecture for any super-modular category that is a subcategory of a modular category of twice its dimension, i.e., admitting a minimal modular extension.

We also study algebraic methods for classifying super-modular categories by rank. In related work, it was shown that up to fusion rules the only non-split super-modular categories of rank ≤ 6 are $\mathrm{PSU}(2)_{4m+2}$ for $m \in \{0,1,2\}$. We develop super-modular analogs of theorems and techniques previously used in the modular setting. As an application, we classify rank 8 super-modular categories up to Grothendieck equivalence with certain restrictions. In particular, we find three prime super-modular categories of rank 8.

DEDICATION

To my parents.

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NOMENCLATURE

1	The trivial object in $\mathcal{C}$
$\mathcal{C}'$	The Müger center of a braided fusion category $\mathcal{C}$
coev_X	The coevaluation map **1** $\mapsto X \otimes X^*$
d_X	The categorical dimension of X
D^2	The global dimension of a category
D	The positive square root of D^2
ev_X	The evaluation map $X^* \otimes X \mapsto$ **1**
Fib	The Fibonacci modular category
$\mathrm{FPdim}(\mathcal{C})$	The Frobenius-Perron dimension of $\mathcal{C}$
$\mathrm{FPdim}(X)$	The Frobenius-Perron dimension of an object X in $\mathcal{C}$
FSExp($\mathcal{C}$)	The Frobenius-Schur Exponent of $\mathcal{C}$
$\mathrm{Gal}(\mathcal{C})$	$\mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$
$\mathcal{K}_0(\mathcal{C})$	The Grothendieck ring of a fusion category $\mathcal{C}$.
N_{ij}^k	The fusion coefficient $\dim \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_k)$
$\nu_n(X)$	The n^{th} Frobenius-Shur indicator of an object X
$p^{\pm}$	Gauss sums of a pre-modular category
$\mathrm{Rep}(G)$	The representation category of a group G
$\mathfrak{s}$	The generator of the group $\mathrm{SL}(2,\mathbb{Z})$ given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
S	The S-matrix of a pre-modular category
$\hat{S}$	The S-matrix of the fermionic quotient of a super-modular category
Sem	The Semion modular category

sVec	The category of super vector spaces
$\mathfrak{S}_n$	The symmetric group on n letters
$\mathfrak{t}$	The generator of the group $\mathrm{SL}(2,\mathbb{Z})$ given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
T	The T-matrix of a pre-modular category
$\hat{T}$	The T-matrix of the fermionic quotient of a super-modular category
$\mathrm{tr}_{\mathcal{C}}$	The categorical trace in a spherical category
θ_X	The twist of an object X
Vec	The category of vector spaces
Vec$_{G,\omega}$	G-graded vector spaes with the associativity by $\omega \in Z^3(G, \Bbbk^\times)$

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1. INTRODUCTION[1]

Tensor categories, categorical analogs of rings, arise in a variety of areas such as representation theory, noncommutative algebra, operator algebra theory, mathematical physics, and quantum computing. A particularly important class of tensor categories are modular tensor categories, which are braided tensor categories satisfying certain axioms and a non-degeneracy condition. The study of modular tensor categories is related to the representation theory of quantum groups, conformal field theory, link invariants, vertex operator algebras, and topological quantum field theory. In particular, unitary modular categories are algebraic models of anyons in bosonic systems. Non-abelian anyons can be used for topological quantum computing [39], as illustrated in Figure 1.1. Braiding anyons corresponds to acting by a unitary braid group representation on their state space.

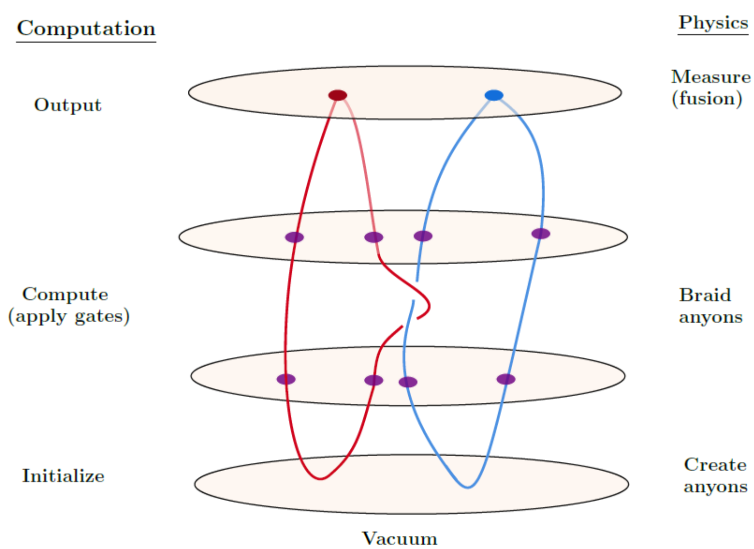


Figure 1.1: TQC

[1]Part of this section is reprinted from "Congruence subgroups and super-modular categories," Parsa Bonderson, Eric C. Rowell, Qing Zhang, Zhenghan Wang, *Pacific Journal of Mathematics*, Vol. 296 (2018), No. 2, 257–270, published by Mathematical Sciences Publishers.

Super-modular categories are the fermionic analogs of modular tensor categories. More precisely, super-modular categories are unitary pre-modular categories with Müger center equivalent to the unitary symmetric fusion category of super-vector spaces (sVec). Physically, super-modular categories are related to the study of fermionic topological phases of matter ([1, 17]). Mathematically, one motivation for pursuing a theory of super-modular categories is that any unitary braided fusion category is the equivariantization of either a modular or super-modular category [63, Theorem 2].

The relationship between super-modular and modular category is not simply one of analogy; quite often, super modular categories can actually be constructed from modular categories. Given any modular category $\mathcal{D}$, the category $\mathrm{sVec} \boxtimes \mathcal{D}$ is super-modular. A super-modular category of this form is called split, and otherwise it is called non-split. An alternative construction involves spin modular categories. A spin modular category $(\mathcal{C}, f)$ is a modular category with a chosen fermion f, which is a simple object f such that $f^{\otimes 2} = \mathbf{1}$ and $\theta_f = -1$. Tensoring with the fermion gives a $\mathbb{Z}_2$-grading: $\mathcal{C} \simeq \mathcal{C}_0 \oplus \mathcal{C}_1$, where $\mathcal{C}_0$ is super-modular. Here we have $2\dim(\mathcal{C}_0) = \dim(\mathcal{C})$. We say that such super-modular category has a minimal modular extension. Conjecturally ([52, Conjecture 5.2], [28, Question 5.15] and [17, Conjecture III.9]), every super-modular category has a minimal modular extension. Moreover, every such minimal modular extension $\mathcal{C}$ is a spin modular category [6].

One of the fundamental invariants of a modular tensor category the following representation of $\mathrm{SL}(2,\mathbb{Z})$. Let $\mathfrak{s}, \mathfrak{t} \in \mathrm{SL}(2,\mathbb{Z})$ be the generators given by $\mathfrak{s} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathfrak{t} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For a modular category, the assignment $\mathfrak{s} \to \frac{S}{D}$, $\mathfrak{t} \to T$ gives a projective representation of $\mathrm{SL}(2,\mathbb{Z})$[4], where the S- and T-matrices are given by the trace of the double braiding and the twist, respecitvely (as defined in Section 2.1.3). The celebrated Congruence Subgroup Theorem of Ng and Schauenburg states that the above representation has kernel a congruence subgroup of level N, where N is the order of the T matrix[55]. In other words, the image factors through $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$, so the group generated by S-matrix and T-matrix is finite.

The definition of the S- and T-matrices makes sense for any premodular category. In the case of super-modular categories, the S- and T-matrices have tensor decompostions: $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}$, where $\hat{S}$ invertible and $\hat{T}$ diagonal ([17]). The assignment of $\mathfrak{s}$ to $\hat{S}$ and $\mathfrak{t}^2$ to $\hat{T}^2$ gives a projective representation $\hat{\rho}$ of the group Γ_θ, where $\Gamma_\theta = \langle \mathfrak{s}, \mathfrak{t}^2 \rangle$ is an index 3 subgroup of $\mathrm{SL}(2,\mathbb{Z})$[17]. Following the analogy with the modular case, it is natural to ask if the kernel of $\hat{\rho}$ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Particularly, one may ask:

Question 1.1. For a super-modular category, do $\hat{S}$ and $\hat{T}^2$ generate a finite group?

We prove the following theorem (Theorem 4.4) for super-modular categories with minimal modular extension.

Theorem 1.2. If a super-modular category has a minimal modular extension, $\hat{\rho}(\Gamma_\theta)$ is a finite group, with kernel a congruence subgroup.

Given their physical motivation, an important problem in the study of super-modular categories is their classification. More broadly, the classification of braided fusion categories (BFCs) stands as a formidable, yet enticing problem. There are many approaches to this problem with varying levels of preciseness and corresponding degrees of difficulty. As examples, one might try to classify by categorical dimension [37, 53, 18, 20, 20, 24, 68], by Witt class [26, 28], by dimension of a generating object [2, 32, 33], or by rank [61, 59]. Each of these approaches has a different motivation and has seen some measure of success. For example, classifying by categorical dimension is related to the problem of classifying groups by their orders, while classifying by the dimension of a generating object is related to the classification of subfactors of finite index and depth. Classification by rank can be motivated physically: for condensed matter systems (e.g. topological phases of matter) modeled by braided fusion categories, the rank of the category corresponds to the number of distinguishable indecomposable particle species [54]. In this dissertation we will be interested in classification of *unitary* BFCs by (low) rank as motivated by this physical interpretation.

Interestingly, the classification of low-rank fusion categories has not progressed very far; it is an open question whether there are finitely many fusion categories of each rank whereas with the braiding assumption rank-finiteness is known [21, 44]. The classification of pivotal fusion categories is complete up to rank 3 [58]. Adding the braiding assumption allows one to go a bit further. For example, there is a complete classification of pre-modular categories of rank at most 5 [15, 23].

Techniques for classifying modular categories are well-established ([61, 22]), and the classification up to rank 6 is nearly complete [25, 43]. Those methods cannot always be applied to general braided fusion categories. For example, a key approach in [22] is to use the representation theory of the modular group $\mathrm{SL}(2,\mathbb{Z})$ to constrain the (modular) S- and (twist) T-matrices, whereas a super-modular category does not provide such representations as the S- matrix has determinant 0.

Some techniques for classifying super-modular categories have been developed recently [16, 19], which lead to a complete classification up to rank 6. There are only 2 such categories: modulo trivial Deligne product constructions and up to fusion rules there are only two examples with rank ≤ 6, and both of them belong to the a family of super-modular categories arising from quantum groups. A particularly useful technique is to formally condense the fermion at the level of fusion rules and modular data to obtain a fermionic quotient, which has naive fusion rules. These can be studied using the concept of a sVec-enriched fusion category [65, 48], but we will not pursue that here. In Section 3 and Section 5 we make progress towards the classification of rank 8 super-modular categories using a stratification by Galois group and some new techniques. We find many non-trivial examples in contrast to lower ranks, and we were unable to give a definitively complete classification–that is, we expect our list to be complete, but do not have an unconstrained proof.

For the following the (standard) notation is explained in Appendix B.

Theorem 1.3. 1. The following are constructions of *prime* rank 8 super-modular categories as centralizers of a distinguished fermion in spin modular categories:

(a) $\mathrm{PSU}(2)_{14} = \langle f\rangle' \subset SU(2)_{14}$ where f is the unique fermion corresponding to highest weight 7ϖ.

(b) $[\mathrm{PSU}(2)_6 \boxtimes \mathrm{PSU}(2)_6]_{\mathbb{Z}_2} = \langle\overline{(f,\mathbf{1})}\rangle' \subset ([\mathrm{SU}(2)_6 \boxtimes \mathrm{SU}(2)_6]_{\mathbb{Z}_2})_0$ where the $\mathbb{Z}_2$-deequivariant-ization in both cases is with respect to the boson (f, f) where f has highest weight 3ϖ, and $\overline{(f, \mathbf{1})}$ is the image of $(f, \mathbf{1})$ under de-equivariantization.

(c) $\langle f \rangle' \subset SO(12)_2$, where f is either of the fermions labelled by $2\varpi_5$ or $2\varpi_6$.

2. Moreover, if we assume that the naive fusion rules $\{\hat{N}_{ij}^k = N_{ij}^k + N_{ij}^{fk}\}_{i,j,k}$ and the simple objects' dimensions d_i are each bounded by 14, then any prime super-modular category of rank 8 has the same fusion rules as one of the above.

A more precise classification with less stringent bounds can be found in Section 5.1.

While we cannot claim this is a complete classification as we have placed bounds in some cases on naive fusion rule multiplicities or dimensions, it is possible that we have listed all possibilities. A counterexample would have large naive fusion multiplicities/dimensions compared to the known examples: the largest naive fusion multiplicity we find among fermionic quotients is 4 while the largest dimension of a simple object is $3 + 2\sqrt{2} \approx 5.8$. There is some precedent for these types of constraints: [42] gives a classification of low rank modular categories with bounded fusion multiplicities and [67] uses numerical techniques to study low rank modular categories with constrained categorical dimension. Although our result is not complete, we provide some new powerful methods for classifying super-modular categories, and illustrate the utility of the existing techniques.

The content of this dissertation is arranged as follows. In Section 2, we collect basic definitions and results on pre-modular and modular categories. We introduce and develop the significant properties of super-modular categories in Section 3. In Section 4 we prove the congruence subgroup theorem for super-modular categories with minimal modular extension. Finally, in Section 5, we determine the rank 8 super-modular categories up to fusion rules with certain restrictions.

2. PRELIMINARIES

In this section, we collect some conventions and preliminary results on pre-modular and modular categories we are interested in, referring the reader to [36, 4, 22, 61] for further details. In this paper, $\Bbbk$ is always assumed to be an algebraically closed field of characteristic zero.

2.1 Pre-modular Categories

2.1.1 Definition of a Pre-Modular Category

Definition 2.1. A **monoidal category** is a category $\mathcal{C}$ equipped with:

(1) a bifunctor: $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product,

(2) an object $\mathbf{1} \in \mathcal{C}$ called the unit object,

(3) a natural isomorphism

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathcal{C}$$

called the associator,

(4) a natural isomorphism

$$\lambda_X : \mathbf{1} \otimes X \to X, \qquad X \in \mathcal{C}$$

called the left unitor,

(5) and a natural isomorphism

$$\rho_X : X \otimes \mathbf{1} \to X, \qquad X \in \mathcal{C}$$

called the right unitor, such that the following diagrams commute for all objects involved:

(1) The triangle diagram

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{Id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{Id}_X \otimes \lambda_Y} & \\ & X \otimes Y & \end{array}$$

(2) The pentagon diagram

$$\begin{array}{ccccc} & & (W \otimes X) \otimes (Y \otimes Z) & & \\ & \nearrow^{\alpha_{W \otimes X,Y,Z}} & & \searrow^{\alpha_{W,X,Y \otimes Z}} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\ \Big\downarrow{\scriptstyle \alpha_{W,X,Y} \otimes \mathrm{Id}_Z} & & & & \Big\uparrow{\scriptstyle \mathrm{Id}_W \otimes \alpha_{X,Y,Z}} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[\alpha_{W,X \otimes Y,Z}]{} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

Examples 2.2. (1) The category of $\Bbbk$-vector spaces is monoidal. The objects of Vec are $\Bbbk$-vector spaces and the morphisms are $\Bbbk$-linear maps. The morphisms α, λ and ρ are defined in an obvious way.

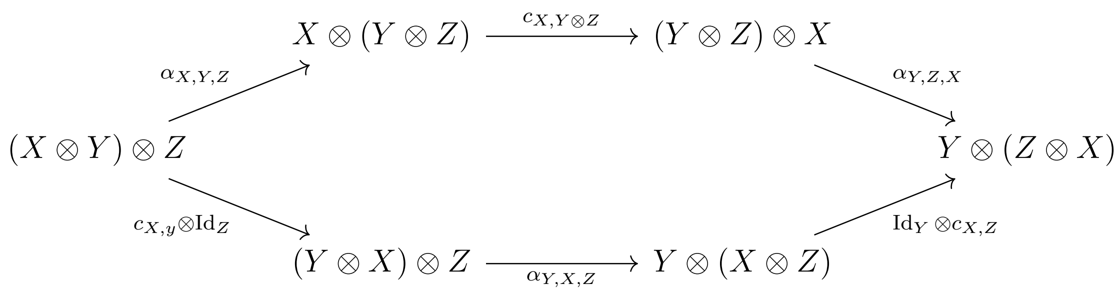
(2) Let G be a group. Then $\mathrm{Rep}(G)$, which is the category of finite dimensional representations of G over $\Bbbk$, is a monoidal category. The objects are the representations of G and the morphisms are the intertwiners. The $\otimes$ is the tensor product of two representations. The unit object is given by the trivial representation.

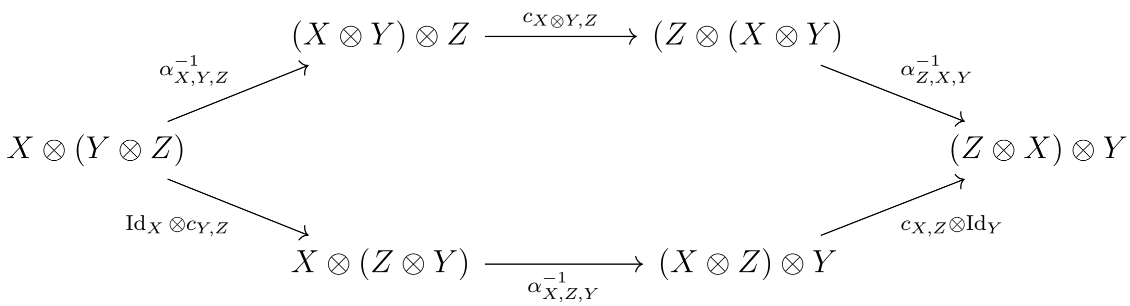
(3) Given a group G, let Vec_G be the category of G-graded vector spaces over $\Bbbk$. The objects are $V = \bigoplus_{g \in G} V_g$ and the morphisms are the grade preserving transformations. The tensor product is given by

$$(V \otimes W)_g = \bigoplus_{jk=g} V_j \otimes V_k$$

The unit $\mathbf{1}$ is given by $\mathbf{1}_e = \Bbbk$ and $\mathbf{1}_g = 0$ otherwise. This gives us a monoidal category with α, λ and ρ being the obvious ones.

Definition 2.3. A **braided monoidal category** is a monoidal category $\mathcal{C}$ with a natural isomorphism $c_{X,Y} : X \otimes Y \simeq Y \otimes X$ called the braiding, such that the following two hexagon diagrams commute





Examples 2.4. (1) The categories Vec and $\mathrm{Rep}(G)$ are braided fusion categories. The braidings are the transpositions of two tensor factors. If G is an abelian group, Vec_G is braided.

(2) The category of super-vector spaces has objects as the $\mathbb{Z}_2$-graded vector spaces. The morphisms are the grade preserving linear transformations. Define the braiding by $c_{X,Y}(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for homogeneous vectors x, y. It is a braided monoidal category and is denoted as sVec. This category plays an important role in our paper.

Definition 2.5. Let $\mathcal{C}$ be a monoidal category and X be an object, a **left-dual** to X is an object X^* with two morphisms

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1},$$
$$\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*,$$

such that the composition

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{Id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{Id}_X \otimes \mathrm{ev}_X} X,$$

is equal to Id_X and the composition

$$X^* \xrightarrow{\mathrm{Id}_X \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{Id}_{X^*}} X^*$$

is equal to Id^*_X. Similarly, an object ${}^*X$ is said to be the **right-dual** if there exist morphisms

$$\mathrm{ev}'_X : X \otimes {}^*X \to \mathbf{1},$$
$$\mathrm{coev}'_X : \mathbf{1} \to {}^*X \otimes X$$

such that the composition

$$X \xrightarrow{\mathrm{Id}_X \otimes \mathrm{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{\alpha^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{Id}_X} X,$$

is equal to Id_X and the composition

$${}^*X \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{Id}_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{\alpha_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathrm{Id}_{{}^*X} \otimes \mathrm{ev}'_X} {}^*X$$

is equal to $\mathrm{Id}_{{}^*X}$. Let X and Y be objects in $\mathcal{C}$ with left-duals X^* and Y^* and $f : X \to Y$ be a morphism. Define the **left-dual morphism** $f^* : Y^* \to X^*$ by

$$Y^* \xrightarrow{\mathrm{Id}_{Y^*} \otimes \mathrm{coev}_X} Y^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{Y^*,X,X^*}} (Y^* \otimes X) \otimes X^*$$

$$\xrightarrow{(\mathrm{Id}_{Y^*} \otimes f) \otimes \mathrm{Id}_{X^*}} (Y^* \otimes Y) \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{Id}_{X^*}} X^*.$$

Similarly, for objects X and Y in $\mathcal{C}$ with right duals ${}^*X$ and ${}^*Y$, the **right-dual morphism** ${}^*f$ is given by

$${}^*Y \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{Id}_{{}^*Y}} ({}^*X \otimes X) \otimes {}^*Y \xrightarrow{\alpha_{{}^*X,X,{}^*Y}} {}^*X \otimes (X \otimes {}^*Y)$$

$$\xrightarrow{\mathrm{Id}_{{}^*X} \otimes (f \otimes \mathrm{Id}_{{}^*Y})} {}^*X \otimes (Y \otimes {}^*Y) \xrightarrow{\mathrm{Id}_{{}^*X} \otimes \mathrm{ev}'_Y} {}^*X.$$

Definition 2.6. A monoidal category $\mathcal{C}$ is called **rigid** if every object in $\mathcal{C}$ has left and right duals.

Definition 2.7. Let $\mathcal{C}$ be a rigid monoidal category, then a **pivotal structure** on $\mathcal{C}$ is an isomorphism of monoidal functors $j_X : X \simeq X^{**}$. A rigid monoidal category with a pivotal structure is called **pivotal**.

Definition 2.8. Let X be an object in a pivital category $\mathcal{C}$ and $\psi \in \mathrm{End}(X)$. We have two trace maps tr_L and tr_R from $\mathrm{End}_{\mathcal{C}}(X)$ to $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ defined by

$$\mathrm{tr}_L(\psi) : \mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{\psi \otimes \mathrm{Id}_X^*} X \otimes X^* \xrightarrow{j_X \otimes \mathrm{Id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\mathrm{ev}_{X^*}} \mathbf{1},$$

$$\mathrm{tr}_R(\psi) : \mathbf{1} \xrightarrow{\mathrm{coev}_{X^*}} X^* \otimes X^{**} \xrightarrow{\mathrm{Id}_{X^*} \otimes j_X^{-1}} X^* \otimes X \xrightarrow{\mathrm{Id}_{X^*} \otimes \psi} X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1}\,.$$

Definition 2.9. A pivotal category is said to be **spherical** if for all objects X and for all morphisms $\psi \in \mathrm{End}(X)$, we have $\mathrm{tr}_L(\psi) = \mathrm{tr}_R(\psi)$. Define the **dimension** of an object X to be $\dim(X) := \mathrm{tr}(\mathrm{Id}_X)$. The **global dimension** is defined by $D^2 = \sum_{j \in \Pi_{\mathcal{C}}} d_j^2$. Note that despite its notation, D^2 need not be a perfect square.

Definition 2.10. Let $\mathcal{C}$ be an abelian category, then an object X is called **simple** if any injection $Y \hookrightarrow X$ is either 0 or an isomorphism.

Remark 2.11. The simple objects in the category Vec over $\Bbbk$ are the 1-dimensional vector spaces, which is isomorphic to $\Bbbk$. For the category $\mathrm{Rep}(G)$, the simple objects are the irreducible representations.

Definition 2.12. An abelian category $\mathcal{C}$ is called **semisimple** if every object in $\mathcal{C}$ is a direct sum of simple objects.

Definition 2.13. Let $\Bbbk$ be a field, then a category $\mathcal{C}$ is said to be $\Bbbk$-linear if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a vector space over $\Bbbk$ for all objects X and Y, and compositions of morphisms are bilinear.

Definition 2.14. A **fusion category** over $\Bbbk$ is a semisimple rigid $\Bbbk$-linear monoidal category with finitely many isomorphism classes of simple objects and finite-dimensional Hom spaces, such that the endomorphisms of the unit object form the ground field $\Bbbk$.

Definition 2.15. A **twist** on a braided rigid monoidal category $\mathcal{C}$ is a natural family of isomorphisms

$$\theta_X : X \to X,$$

such that

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) c_{Y,X} c_{X,Y}\,.$$

We call a twist a ribbon structure if $(\theta_X)^* = \theta_{X^*}$. A **ribbon category** is a braided rigid monoidal category with a ribbon structure.

Definition 2.16. A **pre-modular category** is a spherical braided fusion category.

Remark 2.17. 1. Let $\mathcal{C}$ be a pre-modular category of rank r, we will use the notation $\Pi_{\mathcal{C}} = \{X_i\}_{i=0}^r$ to denote the set of isomorphism classes of simple objects. We label the tensor unit $\mathbf{1} = X_0$. By abuse of notation, $i \in \Pi_{\mathcal{C}}$ means $X_i \in \Pi_{\mathcal{C}}$.

2. Since $\operatorname{End}(X_i) = \Bbbk$, we have $\theta_{X_i} = \theta_i \operatorname{Id}_{X_i}$ and $\dim(X_i) = d_i$ for some θ_i and d_i in $\Bbbk$. d_i is called the **categorical dimension** of X_i.

3. A pre-modular category C is **self-dual** if $X_i = (X_i)^*$ for all $i \in \Pi_{\mathcal{C}}$. $\mathcal{C}$ is self-dual if and only if S-matrix is real.

A conjugation in a monoidal category $\mathcal{C}$ assigns every $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ a morphism $f^\dagger \in \operatorname{Hom}_{\mathcal{C}}(Y, X)$ such that $(f^\dagger)^\dagger = f$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ and $(fg)^\dagger = g^\dagger f^\dagger$. A **Hermitian** ribbon category is a ribbon category $\mathcal{C}$ with a conjugation such that $(c_{X,Y})^\dagger = (c_{Y,X})^{-1}$ and $(\theta_X)^\dagger = (\theta_X)^{-1}$, $(\operatorname{coev}_X)^\dagger = \operatorname{ev}_X c_{X,X^*}(\theta_X \otimes \operatorname{Id}_{X^*})$ and $(\operatorname{ev}_X)^\dagger = (\operatorname{Id}_{X^*} \otimes \theta_X^{-1}) c_{X^*,X}^{-1} \operatorname{coev}_X$. The name Hermitian comes from the fact that $(f, g) \mapsto \operatorname{tr}(fg^\dagger)$ is a non-degenerate Hermitian form on $\operatorname{Hom}(X, Y)$[64, Lemma 5.2.1]. We say that $\mathcal{C}$ is **unitary** if the ground field is $\mathbb{C}$ and $\dagger$ acts on $\mathbb{C}$ by complex conjugation and (f, g) is a positive definite form.

2.1.2 Grothendieck Ring

Given a fusion category $\mathcal{C}$, we will denote by $\mathcal{K}_0(\mathcal{C})$ its Grothendieck semiring(see, e.g. [4, 2.4]). It is a unital based ring of finite rank. The structure coefficients of $\mathcal{K}_0(\mathcal{C})$ are given by

$$X_i \otimes X_j \simeq \bigoplus_k N_{i,j}^k X_k,$$

where $N_{i,j}^k = \dim \operatorname{Hom}(X_k, X_i \otimes X_j)$. We call this formula a **fusion rule**. The coefficients $N_{i,j}^k$ are **fusion coefficients**. The object dual to X_i is denoted by X_i^* or X_{i^*}. The fusion coefficients satisfy

$$N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^*j^*}^{k^*}, \qquad N_{ij}^0 = \delta_{ij^*}. \tag{2.1}$$

Two fusion categories $\mathcal{B}$ and $\mathcal{C}$ are **Grothendieck equivalent** if $\mathcal{K}_0(\mathcal{B}) \simeq \mathcal{K}_0(\mathcal{C})$. The **fusion matrices** N_i are defined by $(N_i)_{k,j} = N_{i,j}^k$. These are integral matrices with non-negative entries. In particular, they satisfy the condition for the Frobenius-Perron theorem. For $X_i \in \Pi_{\mathcal{C}}$, denfine $\mathrm{FPdim}(X_i)$ as the maximal eigenvalue of the corresponding matrix N_i, which is called **Frobenius-Perron dimension** of X_i. In a fusion category, one has $\mathrm{FPdim}(X_i) \geq 1$ for all $i \in \Pi_{\mathcal{C}}$ ([36, Proposition 3.3.4]). The Frobenius-Perron dimension of a fusion category $\mathcal{C}$ is defined to be

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{i \in \Pi_{\mathcal{C}}} \mathrm{FPdim}(X_i)^2.$$

2.1.3 S and T Matrices

Definition 2.18. Let $\mathcal{C}$ be a pre-modular category. For $i, j \in \Pi_{\mathcal{C}}$, define the numbers $\tilde{S}_{ij} \in k$ with entries

$$\tilde{S}_{i,j} := \mathrm{Tr}_{\mathcal{C}}(c_{X_i, X_j^*} c_{X_j^*, X_i}).$$

The S-matrix is a symmetric n-by-n matrix with $n = |\Pi_{\mathcal{C}}|$. The entries of S satisfy $S_{i,j} = S_{i^*,j^*}$ and $S_{0,i} = d_i$. The **global dimension** is defined by $D^2 = \sum_{i \in \Pi_{\mathcal{C}}} d_i^2$. Define the normalized S-matrix by $S := \frac{\tilde{S}}{D}$, where D is the positive square root of the global dimension.

Definition 2.19. A pre-modular category is called **modular** if its S-matrix is non-degenerate.

Let T-matrix be the diagonal matrix such that $T_{i,j} = \delta_{i,j}\theta_i$, where θ_i are roots of unity which have finite order (Vafa's theorem, see [4]) for any pre-modular category. Let $\mathcal{C}$ be pre-modular, we have the following **balancing relation**

$$\theta_i \theta_j \tilde{S}_{ij} = \sum_{k \in \Pi_{\mathcal{C}}} N_{i^*j}^k \theta_k d_k. \tag{2.2}$$

The $\tilde{S}$ and T matrices satisfy (see e.g. [4, Theorem 3.1.7]):

(1) $(\tilde{S}T)^3 = p^+\tilde{S}^2$,

(2) $(\tilde{S}T^{-1})^3 = p^-\tilde{S}^2C$,

(3) $TC = CT, C\tilde{S} = \tilde{S}C$,

where $p^\pm := \sum_{i\in\Pi_\mathcal{C}} \theta_i^\pm d_i^2$ are **Gauss sums** and $C_{i,j} = \delta_{i,j^*}$ is called the **charge conjugation matrix**.

If $\mathcal{C}$ is a pre-modular category and X_i is a simple self-dual object, then we have the **second Frobenius-Schur indicator formula**[15]:

$$\nu_2(X_i) = \frac{1}{D^2}\sum_{j,k} N_{j,k}^i d_j d_k \left(\frac{\theta_j}{\theta_k}\right)^2 - \theta_i \sum_{a\in\Pi'_\mathcal{C}\setminus\{\mathbf{1}\}} d_a \operatorname{Tr}(R_a^{ii}), \tag{2.3}$$

where R_i^{jk} are braiding eigenvalues.

Proposition 2.20. (see e.g., [36, Section 8.13]) Let $\mathcal{C}$ be a modular category, then the entries of S satisfy

$$S_{ij}S_{ik} = d_i \sum_{l\in\Pi_\mathcal{C}} N_{jk}^l S_{il}, \qquad i,j,k \in \Pi_\mathcal{C}. \tag{2.4}$$

Consequently, the maps $\phi_k : i \mapsto \dfrac{S_{ik}}{S_{0k}}$ for each $k \in \Pi_\mathcal{C}$ determine linear characters of $\mathcal{K}_0(\mathcal{C})$.

Remark 2.21. By Proposition 2.20, we know that $\dfrac{S_{ij}}{d_i}$ for any $i, j \in \mathcal{C}$ are eigenvalues of the integer matrix N_i. Therefore the numbers $\dfrac{S_{ij}}{d_i}$ are algebraic integers.

2.1.4 Dimensions

So far, we introduced two notions of dimensions for a fusion category, namely, the FP-dimension and the categorical dimension. Here we introduce more terminologies and results related to dimensions. A fusion category $\mathcal{C}$ is said to be **weakly integral** if $\operatorname{FPdim}(\mathcal{C}) \in \mathbb{Z}$; it is **integral** if $\operatorname{FPdim}(X_i) \in \mathbb{Z}$ for all i; it is **pointed** if $\operatorname{FPdim}(X_i) = 1$ for all i.

Theorem 2.22. [38] Let $\mathcal{C}$ be a fusion category over $\mathbb{C}$. Then

(1) $0 < d_X^2 \leq \mathrm{FPdim}(X)^2$ for all X. So $1 \leq \dim(\mathcal{C}) \leq \mathrm{FPdim}(\mathcal{C})$.

(2) If $\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C})$, then $\mathcal{C}$ admits a unique spherical structure for which $d_X = \mathrm{FPdim}(X) > 0$ for every X. We call such category **pseudo-unitary**.

Remark 2.23. In a pseduo-unitary fusion category $\mathcal{C}$, we have $d_i = \pm\mathrm{FPdim}(X_i)$ for all $i \in \Pi_{\mathcal{C}}$ (See, e.g., [22, Lemma 3.3]). Additionally, if $\mathcal{C}$ is unitary, then $\mathrm{tr}(\mathrm{Id}_X) = \dim(X) > 0$ for all X. Thus in a unitary category, we have $d_X = \mathrm{FPdim}(X)$ for all X.

2.2 Modular Categories

Representation of $\mathrm{SL}(2,\mathbb{Z})$**.** For a modular category, the S and T satisfy (see e.g. [4, Collary 3.1.8]):

$$(ST)^3 = \sqrt{\frac{p^+}{p^-}}S^2, \quad S^2 = C, \quad CT = TC, \quad C^2 = I. \tag{2.5}$$

These imply that from any modular category $\mathcal{C}$ of rank r (i.e. with r isomorphism classes of simple objects) one obtains a projective unitary representation of the modular group $\rho : SL(2,\mathbb{Z}) \to \mathrm{PSU}(r)$ defined on generators by: $\mathfrak{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to S$ and $\mathfrak{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to T$ composed with the canonical projection $\pi_r : \mathrm{U}(r) \to \mathrm{PSU}(r)$. By rescaling the S and T matrices, ρ may be lifted to a linear representation of $SL(2,\mathbb{Z})$, but these lifts are not unique. This representation has topological significance: one identifies the modular group with the mapping class group $\mathrm{Mod}(\Sigma_{1,0})$ of the torus ($\mathfrak{t}$ and $\mathfrak{s}\mathfrak{t}^{-1}\mathfrak{s}^{-1}$ correspond to Dehn twists about the meridian and parallel) and this projective representation is the action of the mapping class group on the Hilbert space associated to the torus by the modular functor obtained from $\mathcal{C}$.

A subgroup $H < \mathrm{SL}(2,\mathbb{Z})$ is called a **congruence subgroup** if H contains a principal congruence subgroup $\Gamma(n) := \{A \in \mathrm{SL}(2,\mathbb{Z}) : A \equiv I \pmod{n}\}$ for some $n \geq 1$. Since $\Gamma(n)$ is the kernel of the reduction modulo n map $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$, any congruence subgroup has finite index. The **level** of a congruence subgroup H is the minimal n so that $\Gamma(n) < H$. More generally, for $G < \mathrm{SL}(2,\mathbb{Z})$ we say $H < G$ is a congruence subgroup if $G \cap \Gamma(n) < H$ with the level of H defined similarly.

The connection between topology and number theory found through the representation above is deepened by the following Congruence Subgroup Theorem:

Theorem 2.24 ([55])**.** Let $\mathcal{C}$ be a modular category of rank r with T matrix of order N. Then the projective representation $\rho : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSU}(r)$ has $\ker(\rho)$ a congruence subgroup of level N.

In particular, the image of ρ factors over $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ and hence is a finite group. This fact has many important consequences: for example, it is related to rank-finiteness [21] and can be used in classification problems [22].

2.2.1 Galois Symmetries for Modular Categories

Let $\mathcal{C}$ be a modular category. The pair (S,T) is called modular data for a given modular category $\mathcal{C}$. The non-degeneracy condition leads to some remarkable properties for modular categores. For example, it can further be shown the fusion coefficients of $\mathcal{C}$ are given by the entries of the S-matrix using the **Verlinda formula** ([4, Theorem 3.1.14]):

$$N_{i,j}^{k} = \frac{1}{D^2}\sum_{m\in\Pi_{\mathcal{C}}} \frac{S_{im}S_{jm}S_{k^*m}}{d_m} \quad \text{for all } i,j,k \in \Pi_{\mathcal{C}}. \tag{2.6}$$

We denote by $\mathbb{Q}(S)$ the smallest field containing all elements of the S-matrix. It is known that $\mathbb{Q}(S)$ is Galois over $\mathbb{Q}$. Let $\mathbb{Q}_N := \mathbb{Q}(\zeta_N)$, where ζ_N is a primitive Nth root of unity. We define $\mathrm{Gal}(\mathcal{C}) = \mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$. Then $\mathrm{Gal}\,\mathcal{C}$ is an abelian subgroup of $\mathcal{S}_r$, the symmetric group on r letters [61]. Recall from Proposition 2.20, the assignments $\phi_k : i \mapsto \dfrac{S_{ik}}{S_{0k}}$ define linear characters for $\mathcal{K}_0(\mathcal{C})$. In the modular setting, since the S-matrix is non-degenerate, $\{\phi_i\}_{i\in\mathcal{C}}$ is the set of all characters of $\mathcal{K}_0(\mathcal{C})$. For any $\sigma \in \mathrm{Gal}(\mathcal{C})$, $\sigma(\phi_k)$ maps i to $\sigma\big(\dfrac{S_{ik}}{S_{0k}}\big)$, which is also a linear character of $\mathcal{K}_0(\mathcal{C})$. Thus, there is a unique $\hat{\sigma} \in S_r$ such that $\sigma(\phi_k) = \phi_{\hat{\sigma}(k)}$ [22]. We will use σ for both the element of the Galois group $\mathrm{Gal}(\mathcal{C})$ and its associated element of S_n. The above argument gives

$$\sigma\Big(\frac{S_{ik}}{S_{0k}}\Big) = \frac{S_{i\sigma(k)}}{S_{0\sigma(k)}}. \tag{2.7}$$

Moreover, we have(see [61], [22])

$$\sigma(S_{j,k}) = \epsilon_{\sigma(k),\sigma} S_{j,\sigma(k)}/d_{\sigma(0)}, \tag{2.8}$$

where $\epsilon_{j,\sigma} = \pm 1$. This action gives the following symmetries:

$$S_{j,k} = \epsilon_{\sigma(j),\sigma}\epsilon_{k,\sigma} S_{\sigma(j),\sigma^{-1}(k)}. \tag{2.9}$$

The smallest field containing T is given by $T = \mathbb{Q}_{\mathrm{FSExp}(\mathcal{C})}$ the cyclotomic field at $\mathrm{FSExp}(\mathcal{C})$ roots of unity. Here $\mathrm{FSExp}(\mathcal{C})$ is called the **Frobenius-Schur Exponent** of $\mathcal{C}$ and is defined to be the minimal integer, n, such that $\nu_n(X_j) = d_j$ for all j. Here ν_n is the n-th Frobenius-Schur indicator and is defined by:

$$\nu_n(X_j) = \frac{1}{D^2}\sum_{i,k=1}^{r} N_{i,k}^{j} d_i d_k \left(\frac{\theta_i}{\theta_k}\right)^n.$$

Note that $\nu_2(k) = 0$ if and only if $X_k \neq X_k^*$ otherwise it is ± 1.

The two field extensions $\mathbb{Q}(T)$ and $\mathbb{Q}(S)$ have the following relation:

Lemma 2.25. [30, Proposition 6.7] If $\mathcal{C}$ is a modular category with modular data (S, T), then $\mathrm{Gal}(\mathbb{Q}(T)/\mathbb{Q}(S))$ is an elementary 2-group.

For a modular category, we have the following generalization of the Cauchy theorem from group theory:

Theorem 2.26. [21, Theorem 3.9] If $\mathcal{C}$ is a modular tensor category, then $\mathrm{FSExp}\,\mathcal{C}$ and $\mathrm{ord}\,T$ have the same prime factors (as ideals).

It was Wang's conjecture that there are finitely many modular categories of a given rank up to equivalence in 2003 ([61]). We call this the rank-finiteness conjecture. A proof of this conjecture was given in [21]:

Theorem 2.27 (Rank finitenes)**.** [21] There are only finitely many modular categories of fixed rank r, up to equivalence.

Therefore the classification of modular categories by rank seems to be a natural and interesting problem. Unitary modular categories were completely classified up to rank 4 in [61]. A classification of modular categories up to monoidal equivalence of rank 5 was obtained in [22].

2.2.2 Low-Rank Modular Categories

We end this section by introducing some low-rank modular categories that will show up later in Section 5. Readers are referred to [61] for more examples and specific details.

Examples 2.28. (1) Semion. The semion modular category has 2 simple objects **1** and s.

Fusion rules: $s \otimes s = \mathbf{1}$.

S-matrix: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

T-matrix: $\mathrm{diag}(1, i)$

(2) Fib. The Fibonacci modular category. There are 2 simple objects **1** and τ. Let $\phi = \frac{1+\sqrt{5}}{2}$ be the Golden ratio.

Fusion rules: $\tau \otimes \tau = \mathbf{1} \oplus \tau$

S-matrix: $\dfrac{1}{\sqrt{1+\phi}}\begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix}$

T-matrix: $\mathrm{diag}(1, e^{\frac{4\pi i}{5}})$

(3) $\mathbb{Z}_4$ modular category has 4 simple objects: **1**, ϵ, σ and σ^*. In particular, it is a non-self dual pointed modular category.

Fusion rules: $\epsilon^{\otimes 2} = \sigma \otimes \sigma^* = \mathbf{1}$, $\sigma^{\otimes 2} = (\sigma^*)^2 = \epsilon$, $\sigma\epsilon = \sigma^*$, $\epsilon^*\sigma = \sigma$

S-matrix: $\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{pmatrix}$

T-matrix: $\mathrm{diag}(1, -1, e^{\frac{\pi i}{4}}, e^{\frac{\pi i}{4}})$

(4) Toric code modular category has simple objects: **1**, e, m and ϵ. It is a self-dual pointed modular category.

Fusion rules: $e^{\otimes 2} = m^{\otimes 2} = \epsilon^{\otimes 2} = \mathbf{1}$, $e \otimes m = \epsilon$, $e \otimes \epsilon = m$, $m \otimes \epsilon = e$

S-matrix: $\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

T-matrix: $\mathrm{diag}(1, 1, 1, -1)$

(5) $(A_1, 7)_{\frac{1}{2}}$ modular category has simple objects: **1**, α, ω and ρ.

Fusion rules: $\alpha^{\otimes 2} = \mathbf{1} \oplus \omega$, $\alpha \otimes \omega = \alpha \oplus \rho$, $\alpha \otimes \rho = \omega \oplus \rho$, $\omega^{\otimes 2} = \mathbf{1} \oplus \omega \oplus \rho$, $\omega \otimes \rho = \alpha \oplus \omega \oplus \rho$, $\rho^{\otimes 2} = \mathbf{1} \oplus \alpha \oplus \omega \oplus \rho$

S-matrix: $\begin{pmatrix} 1 & d^2-1 & 1+d & d \\ d^2-1 & 0 & -d^2+1 & d^2-1 \\ 1+d & -d^2+1 & d & -1 \\ d & d^2-1 & -1 & -d-1 \end{pmatrix}$

T-matrix: $\mathrm{diag}(1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{9}}, e^{\frac{4\pi i}{3}})$

3. SUPER-MODULAR CATEGORIES

In this section, we first introduce the notion of super-modular categories and some of its properties. Most of the results can be found in ([16, 19]) and the references therein. Then we discuss the Galois symmetry for super-modular categories, which is parallel to the modular case in Section 2.2.1.

3.1 Centralizers

Whereas one may always define an S-matrix for any ribbon fusion category $\mathcal{B}$, it may be degenerate. This failure of modularity is encoded it the subcategory of transparent objects called the **Müger center** $\mathcal{C}'$. Here an object X is called **transparent** if all the double braidings with X are trivial: $c_{Y,X}c_{X,Y} = Id_{X\otimes Y}$. Generally, we have the following notion of the centralizer of the braiding.

Definition 3.1. The **Müger centralizer** of a subcategory $\mathcal{D}$ in of a pre-modular category $\mathcal{C}$ is the full fusion subcategory

$$C_{\mathcal{C}}(\mathcal{D}) = \{X \in \mathcal{C} | c_{Y,X}c_{Y,X} = \mathrm{Id}_{X\otimes Y}, \forall Y \in \mathcal{D}\}.$$

The **Müger center** of $\mathcal{C}$ is $\mathcal{C}' = \mathcal{C}_{\mathcal{C}}(\mathcal{C})$.

By a theorem of Bruguières [14] the simple objects in $\mathcal{B}'$ are those X with $\tilde{S}_{X,Y} = d_X d_Y$ for all simple Y, where $d_Y = \dim(Y) = \tilde{S}_{\mathbf{1},Y}$ is the categorical dimension of the object Y. The Müger center is obviously **symmetric**, that is, $c_{Y,X}c_{X,Y} = Id_{X\otimes Y}$ for all $X, Y \in \mathcal{B}'$. Symmetric fusion categories have been classified by Deligne [29], in terms of representations of supergroups. In the case that $\mathcal{B}' \cong \mathrm{Rep}(G)$ (i.e. is Tannakian), the modularization (de-equivariantization) procedure of Bruguières [14] and Müger [50] yields a modular category $\mathcal{B}_G$ of dimension $\dim(\mathcal{B})/|G|$. Otherwise, by taking a maximal Tannakian subcategory $\mathrm{Rep}(G) \subset \mathcal{B}'$ the de-equivariantization $\mathcal{B}_G$ has Müger center $(\mathcal{B}_G)' \cong \mathrm{sVec}$, the symmetric fusion category of super-vector spaces. Generally, a braided fusion category $\mathcal{B}$ with $\mathcal{B}' \cong \mathrm{sVec}$ as symmetric fusion categories is called **slightly degenerate** [31].

The symmetric fusion category sVec has a unique spherical structure compatible with unitarity and has S- and T-matrices: $S_{\mathrm{sVec}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $T_{\mathrm{sVec}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

From this point on we will assume that all our categories are unitary, so that sVec is a unitary symmetric fusion category.

3.2 Definition of a Super-Modular Category

Definition 3.2. A unitary pre-mdoular category $\mathcal{C}$ is called **super-modular** if $\mathcal{C}' \simeq \mathrm{sVec}$.

Remark 3.3. In other terminology, we say $\mathcal{B}$ is super-modular if its Müger center is generated by a **fermion**, that is, an object f with $f^{\otimes 2} \cong \mathbf{1}$ and $\theta_f = -1$.

Super-modular categories (or slight variations) have been studied from several perspectives, see [12, 26, 16, 10, 46] for a few examples. An algebraic motivation for studying these categories is the following: any unitary braided fusion category is the equivariantization [31] of either a modular or super-modular category (see [62, Theorem 2]). Physically, super-modular categories provide a framework for studying fermionic topological phases of matter [16]. Topological motivations include the study of spin 3-manifold invariants ([62, 8, 9]) and $(3+1)$-TQFTs ([66]).

Remark 3.4. We restrict to unitary categories both for mathematical convenience and for their physical significance. On the other hand, there is a non-unitary version sVec^- of sVec: the underlying (non-Tannakian) symmetric fusion category is the same, but with the other possible spherical structure, which leads to negative dimensions. We could define super-modular categories more generally as pre-modular categories $\mathcal{B}$ with Müger center equivalent to either of sVec or sVec^-. However, we do not know of any examples $\mathcal{B}$ with $\mathcal{B}' \cong \mathrm{sVec}^-$ that are not simply of the form $\mathcal{C} \boxtimes \mathrm{sVec}^-$ for some modular category $\mathcal{C}$.

3.3 Spin Modular Cateogories

A **spin modular category** $\mathcal{C}$ is a modular category with a (chosen) fermion. Let $\mathcal{C}$ be a spin modular category, with fermion f, (unnormalized) S-matrix $\tilde{S}$ and T-matrix T. Proposition II.3 of [16] provides a number of useful symmetries of $\tilde{S}$ and T:

1. $\tilde{S}_{f,\alpha} = \epsilon_\alpha d_\alpha$, where $\epsilon_\alpha = \pm 1$ and $\epsilon_f = 1$.

2. $\theta_{\psi\alpha} = -\epsilon_\alpha \theta_\alpha$.

3. $\tilde{S}_{f\alpha,\beta} = \epsilon_\beta \tilde{S}_{\alpha,\beta}$.

Remark 3.5. We have a canonical $\mathbb{Z}/2\mathbb{Z}$-grading $\mathcal{C}_0 \oplus \mathcal{C}_1$ with simple objects $X \in \mathcal{C}_0$ if $\epsilon_X = 1$ and $X \in \mathcal{C}_1$ when $\epsilon_X = -1$. The trivial component $\mathcal{C}_0$ is a super-modular category, since $\mathcal{C}_0' = \langle f \rangle \cong \mathrm{sVec}$.

Further, we have the canonical decomposition $\mathcal{C}_1 = \mathcal{C}_v \oplus \mathcal{C}_\sigma$ as abelian categories (see Section 4.2.1 for details). The following result is useful for classifying spin modular categories.

Lemma 3.6. [19, Lemma 4.2] Given a spin modular category $(\mathcal{C}, f)$ with $\mathcal{C}_0$, $\mathcal{C}_v$ and $\mathcal{C}_\sigma$. Denote their rank as $|\mathcal{C}_0|$, $|\mathcal{C}_v|$ and $|\mathcal{C}_\sigma|$. Then we have

(1) $|\mathcal{C}_0 = |\mathcal{C}_v| + 2|\mathcal{C}_\sigma|$, in particular $|\mathcal{C}| = 2|\mathcal{C}_0| - |\mathcal{C}_\sigma|$.

(2) $\dfrac{3|\mathcal{C}_0|}{2} \leq |\mathcal{C}| \leq 2|\mathcal{C}_0|$.

(3) $|\mathcal{C}_v|$ and $|\mathcal{C}_0|$ are even.

Remark 3.7. Let $\mathcal{B}$ be a ribbon fusion category. A minimal modular extension of $\mathcal{B}$ is a modular category $\mathcal{C}$ such that $\mathcal{B} \subset \mathcal{C}$ and $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{B}') \, \mathrm{FPdim}(\mathcal{B})$. If $\mathcal{B}$ is super-modular, a minimal modular extension of $\mathcal{B}$ is a spin modular category $(\mathcal{C}, f)$, where the fermion f is transparent in $\mathcal{B}$.

3.4 Fermionic Quotient

One interesting feature of super-modular categories $\mathcal{B}$ is that their S- and T-matrices have tensor decompositions:

Theorem 3.8. [16, Theorem 3.9] Let $\mathcal{B}$ be a super-modular category, then $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}$ with $\hat{S}$ symmetric invertible and $\hat{T}$ diagonal.

Note that for the category sVec, we have $\tilde{S}_{\text{sVec}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $T_{\text{sVec}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Consequently, the Gauss sums of a super-modular category is alway 0, i.e., $p^{\pm} := \sum_{i \in \Pi_{\mathcal{B}}} \theta_i^{\pm} d_i^2 = 0$.

If a super-modular $\mathcal{B} \cong \mathcal{C} \boxtimes \mathrm{sVec}$ for some modular $\mathcal{C}$, we call it **split**, otherwise we say it is **non-split**.

By the following proposition, pointed super-modular categories always split.

Proposition 3.9. [31, Corollary A.19.] Let $\mathcal{B}$ be a pointed super-modular category, then $\mathcal{B} \simeq \mathcal{C} \boxtimes \mathrm{sVec}$, where $\mathcal{C}$ is a pointed modular category.

Let f be a fermion in a super-modular category $\mathcal{B}$ with label set $\Pi_{\mathcal{B}}$, by the following lemma, we know that $f \otimes -$ on $\Pi_{\mathcal{B}}$ is fixed-point-free. We will omit the $\otimes$ symbol and denote $f \otimes X$ simply as fX.

Lemma 3.10. [51, Lemma 5.4] Let $\mathcal{B}$ be a super-modular category and f a fermion, then $fX \ncong X$ for any $X \in \Pi_{\mathcal{B}}$.

As a direct consequence of the previous lemma, we have that super-modular categories have even rank.

Lemma 3.11. Let $\mathcal{B}$ be a super-modular category with transparent fermion f. Then $fX \ncong X^*$ for any $X \in \mathcal{B}$.

Proof. By the balancing equation (given in by the third equality) we have that

$$\begin{aligned}-\theta_X d_X &= \theta_X \theta_f d_f d_X \\ &= \theta_X \theta_f S_{f,X} = \sum_Y N_{f,X}^Y d_Y \theta_Y \\ &= d_{fX}\theta_{fX} = d_X \theta_{fX}.\end{aligned}$$

Therefore $\theta_{fX} = -\theta_X$. But since $\theta_{X^*} = \theta_X$, it follows that $fX \ncong X^*$. □

Thus there is a non-canonical partition of the label set $\Pi_{\mathcal{B}} = \Pi_0 \sqcup f\Pi_0$. We can arrange this partition such that $0 \in \Pi_0$. By the previous lemma, we can choice the partition such that if $X \in \Pi_0$, then $X^* \in \Pi_0$. For a rank $2(r+1)$ super-modular $\mathcal{B}$, we have $0, \ldots, r \in \Pi_0$ and $f = f \cdot 0, \ldots, f \cdot r \in f\Pi_0$, where $f \cdot i$ is the label for fX_i, $i = 0, \ldots, r$.

For $i, j, k \in \Pi_0$, we define the **naive fusion rule**

$$\hat{N}_{ij}^k = \dim \mathrm{Hom}(X_i \otimes X_j, X_k) + \dim \mathrm{Hom}(X_i \otimes X_j, f \otimes X_k) = N_{i,j}^k + N_{i,j}^{fk}.$$

and corresponding **naive fusion matrices** $(\hat{N}_i)_{k,j} := \hat{N}_{i,j}^k$. The semisimple commutative algebra they generate will be denoted $\hat{\mathcal{U}}_{\mathcal{B}}$.

Proposition 3.12. [19, Proposition 2.7] Let $\mathcal{B}$ be a super-modular category, then

(a) $\hat{S}$ is symmetric and $\hat{S}\bar{\hat{S}} = \frac{D^2}{2} I$.

(b) $\hat{N}_i \hat{N}_j = \hat{N}_j \hat{N}_i$ for any $i, j \in \Pi_0$.

(c) Let $\{x_i | i \in \Pi_0\}$ denote the basis of $\mathcal{U}_{\mathcal{B}}$. Then the functions $\phi_i(x_j) := \hat{S}_{ij}/\hat{S}_{0i}$ for $0 \leq i \leq r$ form a set of orthogonal characters of the fusion algebra $\mathcal{U}_{\mathcal{B}}$. Thus $\hat{S}$ simultaneously diagonalizes the matrices $\hat{N}_i$.

(d) $\hat{N}_{ij}^k = \dfrac{2}{D^2} \displaystyle\sum_{m \in \Pi_0} \dfrac{\hat{S}_{im}\hat{S}_{jm}\bar{\hat{S}}_{km}}{d_m}$.

Remark 3.13. (1) Let $\hat{\mathcal{N}}$ be the set of $(r+1)\times(r+1)$ matrices $\hat{N}_i = (\hat{N}_{i,j}^k)$ indexed by the label set Π_0. The naive fusion coefficients define a rank $r+1$ unital based ring $\hat{\mathcal{U}}_{\mathcal{B}}$.

(2) $\hat{S}$ is called the **S-matrix of the fermionic quotient**. $\hat{T}$ is the **T-matrix of the fermionic quotient**. $\mathfrak{s} \to \hat{S}$ and $\mathfrak{t}^2 \to \hat{T}^2$ defines a projective representation of Γ_θ, which is an index 3 subgroup of $\mathrm{SL}(2,\mathbb{Z}) \cong <\mathfrak{s}, \mathfrak{t}>$. See Section 4 for details.

Corollary 3.14. Let $\mathcal{B}$ be super-modular and $\hat{N}_{ij}^k$ be its naive fusion rule, where $i, j, k \in \Pi_0$.

$$\hat{N}_{ij}^k = \hat{N}_{ji}^k = \hat{N}_{ik^*}^{j^*} = \hat{N}_{i^*j^*}^{k^*}, \qquad \hat{N}_{ij}^0 = \delta_{ij^*}$$

Proof. The first equation is a direct consequence of Proposition 3.12 (d). $\hat{N}_{ij}^0 = \delta_{ij^*}$ can be derived by combining (a). □

Remark 3.15. One can combine Corollary 3.14 and [4, Equation 2.4.3] to get more relations for the fusion coefficients. For example, we have $N_{ij}^{fk} = N_{ik^*}^{fj^*}$. In fact, the result follows from $\hat{N}_{ij}^k = N_{ij}^k + N_{ij}^{fk} = N_{ik^*}^{j^*} + N_{ij}^{fk} = N_{ik^*}^{j^*} + N_{ik^*}^{fj^*} = \hat{N}_{ik^*}^{j^*}$.

Similar to the proof for modular category case(see, e.g.,[35, Lemma 1.2]), one can derive the following property of the dimensions for super-modular categories.

Lemma 3.16. [68, Corollary 3.4] Let $\mathcal{B}$ be a super-modular category, then $d_i^2 | \frac{D^2}{2}$.

Proof. By Proposition 3.12, we know that $\hat{S}\bar{\hat{S}} = \frac{D^2}{2} I$, hence we have

$$\frac{D^2}{2} = \sum_{j\in\Pi_0} \hat{S}_{ij}\bar{\hat{S}}_{jk} = \sum_{j\in\Pi_0} \hat{S}_{ij}\hat{S}_{jk^*}.$$

The second equation comes from the fact that for pre-modular categories, we have $\bar{S}_{ij} = S_{ij^*}$ since we can embed them into their modular Drifield center. Therefore we have $\sum\limits_{j\in\Pi_0} \frac{\hat{S}_{ij}}{d_j}\frac{\hat{S}_{jk^*}}{d_j} = \frac{D^2/2}{d_i^2}$. The result follows since the left hand side is an algebraic integer. □

The following property of the second Frobenius-Schur indicator can be derived from Equation 2.3 and is useful in section 5:

Lemma 3.17. [19, Lemma 2.8.] Let $\mathcal{B}$ be a super-modular category and X_i a simple object such that $X_i \cong X_i^*$ (self-dual), then

$$\pm 1 = \nu_2(X_i) = \frac{2}{D^2} \sum_{j,k \in \Pi_0} \hat{N}_{j,k}^i d_j d_k \big(\frac{\theta_j}{\theta_k}\big)^2.$$

The following corollary can be derived from the balancing equation for pre-modular categories.

Corollary 3.18. (Balancing equation for super-modular categories) For a super-modular category of rank $2r$, we have:

$$\theta_i \theta_j \hat{S}_{ij} = \sum_{k=0}^{r-1} (N_{i^*j}^k - N_{i^*j}^{fk}) \theta_k d_k.$$

Proof. The balancing equation[4] for a pre-modular category gives us

$$\begin{aligned}
\theta_i \theta_j \hat{S}_{ij} &= \sum_{k=0}^{2r-1} N_{i^*j}^k \theta_k d_k \\
&= \sum_{k=0}^{r-1} N_{i^*j}^k \theta_k d_k + \sum_{k=r}^{2r-1} N_{i^*j}^k \theta_k d_k \\
&= \sum_{k=0}^{r-1} N_{i^*j}^k \theta_k d_k + \sum_{k=r}^{2r-1} N_{i^*j}^{fk} \theta_{fk} d_{fk} \\
&= \sum_{k=0}^{r-1} (N_{i^*j}^k - N_{i^*j}^{fk}) \theta_k d_k.
\end{aligned}$$

□

3.5 Galois Symmetries for Super-modular Categories

In this section, we discuss the Galois symmetry in the fermionic quotient of a super-modular category, which is parallel to the modular setting. We extend results that are well-known for modular categories to this setting.

Let $\mathcal{B}$ be a super-modular category and $\hat{S}$, $\hat{T}$ and $\hat{N}_i$ defined as above. We have the following relation for the entries of $\hat{S}$ and $\hat{N}_i$ [19, Equation 2.3]:

$$\frac{\hat{S}_{ij}\hat{S}_{ik}}{\hat{S}_{0,i}} = \sum_{m\in\Pi_0} \hat{N}_{jk}^{m}\hat{S}_{im} \tag{3.1}$$

This means that $\hat{\lambda}_{ij} := \frac{\hat{S}_{ij}}{\hat{S}_{0j}}$ are eigenvalues of the matrices $\hat{N}_j$ with eigenvectors $(\hat{S}_{im})_{m\in\Pi_0}$. Defining the diagonal matrix $(\hat{\Lambda}_i)_{jk} = \delta_{jk}\frac{\hat{S}_{ij}}{\hat{S}_{0j}}$, then Equation (3.1) can be written as $\hat{N}_i\hat{S} = \hat{S}\hat{\Lambda}_i$ for all $i \in \Pi_0$.

Remark 3.19. Let $\mathbb{Q}(\hat{S})$ be the smallest field containing all elements of the S-matrix. Similarly to the modular setting, $\mathbb{Q}(\hat{S})$ is Galois over $\mathbb{Q}$. Define $\mathrm{Gal}(\mathcal{B}) = \mathrm{Gal}(\mathbb{Q}(\hat{S})/\mathbb{Q})$. Then $\mathrm{Gal}(\mathcal{B})$ is an abelian subgroup of $\mathfrak{S}_r$, where $2r$ is the rank of the corresponding super modular category and $\mathfrak{S}_r$ is the symmetric group on r letters. We will use σ for both the element of the Galois group $\mathrm{Gal}(\mathcal{B})$ and its associated element in $\mathfrak{S}_r$. Indeed, since $\sigma(\frac{\hat{S}_{ik}}{\hat{S}_{0k}})$ is a character of $\hat{\mathcal{U}_{\mathcal{B}}}$ (see Proposition 3.12), the following defines $\sigma(k)$ for $k \in \Pi_0$:

$$\sigma\Big(\frac{\hat{S}_{ik}}{\hat{S}_{0k}}\Big) = \frac{\hat{S}_{i\sigma(k)}}{\hat{S}_{0\sigma(k)}} = \frac{\hat{S}_{i\sigma(k)}}{d_{\sigma(k)}}. \tag{3.2}$$

Lemma 3.20. Let $\hat{S}$ be as above for a super-modular category $\mathcal{B}$.

(i) Let $\sigma \in \mathrm{Gal}(\mathcal{B})$. Then $\sigma(k)^* = \sigma(k^*)$ for all $k \in \Pi_0$.

(ii) The algebraic integers $\hat{S}_{k,\sigma(0)}$ are real numbers.

(iii) We have $\left|\frac{\hat{S}_{k,\sigma(0)}}{d_{\sigma(k)}}\right|^2 = 1$ for all k, σ.

Proof. Let $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be complex conjugation. Now since $\overline{\hat{S}_{ij}} = \hat{S}_{ij^*}$ we have

$$\begin{aligned}\frac{S_{j,k^*}}{d_{k^*}} &= \overline{(\hat{S}_{j,k}/d_k)}\\ &= \tau\left(\hat{S}_{j,k}/d_k\right) = \hat{S}_{j,\tau(k)}/d_{\tau(k)}.\end{aligned}$$

Thus τ sends the normalized k-th column to the $\tau(k)$-th column which is also the k^*-th column. Since $\mathrm{Gal}(\mathcal{B})$ is abelian, we have $\sigma(k)^* = \tau\sigma(k) = \sigma\tau(k) = \sigma(k^*)$.

The second result now follows from the following computation

$$\overline{\hat{S}}_{k,\sigma(0)} = \hat{S}_{k,\sigma(0)^*} = \hat{S}_{k,\sigma(0^*)} = \hat{S}_{k,(0)}.$$

For the third result we compute

$$\begin{aligned}
\sigma\left(D^2\right) &= 2\sum_{j\in\Pi_0}\sigma\left(d_j\right)^2 = 2\sum_{j\in\Pi_0}\sigma\left(d_j\right)\sigma\left(d_j^*\right)\\
&= 2\sum_{j\in\Pi_0}\frac{\hat{S}_{j,\sigma(0)}}{d_{\sigma(0)}}\frac{\hat{S}_{j^*,\sigma(0)}}{d_{\sigma(0)}}\\
&= \frac{2}{d_{\sigma(0)}^2}\sum_{j\in\Pi_0}\hat{S}_{j,\sigma(0)}\left(\hat{S}_{j,\sigma(0)}\right)^* = \frac{D^2}{d_{\sigma(0)}^2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sigma\left(D^2\right) &= 2\sum_{j\in\Pi_0}\sigma\left(\hat{S}_{j,k}\hat{S}_{j,k^*}\right) = 2\sum_{j\in\Pi_0}\sigma\left(\hat{S}_{j,k}\right)\sigma\left(\hat{S}_{j,k^*}\right)\\
&= 2\sum_{j\in\Pi_0}\left(\frac{\hat{S}_{j,\sigma(k)}\hat{S}_{k,\sigma(0)}}{d_{\sigma(0)}d_{\sigma(k)}}\right)\left(\frac{S_{j,\sigma(k^*)}S_{k^*,\sigma(0)}}{d_{\sigma(0)}d_{\sigma(k^*)}}\right)\\
&= \frac{\hat{S}_{k,\sigma(0)}\hat{S}_{k^*,\sigma(0)}}{d_{\sigma(0)}^2 d_{\sigma(k)}d_{\sigma(k^*)}}2\sum_{j\in\Pi_0}\hat{S}_{j,\sigma(k^*)}\hat{S}_{j,\sigma(k)}\\
&= \frac{\hat{S}_{k,\sigma(0)}\hat{S}_{k^*,\sigma(0)}}{d_{\sigma(0)}^2 d_{\sigma(k)}d_{\sigma(k^*)}}D^2.
\end{aligned}$$

Since $d_{\sigma(k^*)} = d_{\sigma(k)^*} = d_{\sigma(k)}$ and $\hat{S}_{k^*,\sigma(0)} = \overline{\hat{S}}_{k,\sigma(0)} = \hat{S}_{k,\sigma(0)}$, the result follows because $D^2/d_{\sigma(0)}^2$ is nonzero. □

We can also derive a result parallel to [22, Equation 2.12] for the S-matrix of the fermionic quotient:

Corollary 3.21. Let $\sigma \in \mathrm{Gal}(\mathcal{B})$ and j, k the indices of simple objects in Π_0. Then

$$\sigma\left(\hat{S}_{j,k}\right) = \pm\frac{\hat{S}_{j,\sigma(k)}}{d_{\sigma(0)}}.$$

Moreover, we have the following symmetries:

$$\hat{S}_{j,k} = \pm\hat{S}_{\sigma(j),\sigma^{-1}(k)}. \tag{3.3}$$

Proof. By Equation (3.2), we have

$$\sigma\left(\hat{S}_{j,k}\right) = \hat{S}_{j,\sigma(k)}\sigma\left(d_k\right)/d_{\sigma(k)},$$
$$\sigma\left(d_k\right) = \hat{S}_{k,\sigma(0)}/d_{\sigma(0)}.$$

In particular,

$$\sigma\left(\hat{S}_{j,k}\right) = \frac{\hat{S}_{j,\sigma(k)}\hat{S}_{k,\sigma(0)}}{d_{\sigma(0)}d_{\sigma(k)}}.$$

So it suffices to show that $\dfrac{S_{k,\sigma(0)}}{d_{\sigma(k)}} = \pm 1$ which follows from Lemma 3.20. For Equation (3.3), we use the symmetry of the $\hat{S}$-matrix and apply $\sigma \circ \sigma^{-1}$ to the first equation. □

Let $(\mathcal{C}, f)$ be a spin modular category, recall that the fermion f gives a grading $\mathcal{C}_0 \oplus \mathcal{C}_1$.

Lemma 3.22. Let $(\mathcal{C}, f)$ be spin modular with (unnormalized) S-matrix S, and $\hat{S}$ the S-matrix for the fermionic quotient. Then $[\mathbb{Q}(S) : \mathbb{Q}(\hat{S})] = 2^n$, for some n.

Proof. Denote by $S^{(0,0)}, S^{(0,1)} = [S^{(1,0)}]^T$ and $S^{(1,1)}$ the 2×2 blocks of the S-matrix S relative to the grading $\mathcal{C}_0 \oplus \mathcal{C}_1$. Suppose that $X_a, X_b \in \mathcal{C}_1$ so that $S_{b,a}$ is an entry in $S^{(1,1)}$. Then, since the normalized ith column $S_{i,a}/d_a$ is a character of the Grothendieck ring $K_0(\mathcal{C})$ for each i, we see that $(S_{b,a})^2 = d_a^2 \sum_j N_{b,a}^j S_{j,a}/d_a$. Since $N_{b,a}^j = 0$ if $X_j \in \mathcal{C}_1$ we find that $(S_{b,a})^2$ lies in the field generated by the entries of $S^{(0,1)}$. In particular, $[\mathbb{Q}(S^{(1,1)}) : \mathbb{Q}(S^{(0,1)})] = 2^k$ for some k, since every entry of $S^{(1,1)}$ satisfies a polynomial equation of degree ≤ 2 over $S^{(0,1)}$.

Now let $S_{b,c}$ be an entry of $S^{(0,1)} = [S^{(1,0)}]^T$, i.e. $X_b \in \mathcal{C}_1$ and $X_c \in \mathcal{C}_0$. A similar argument shows that $(S_{b,c})^2$ lies in the field generated by $S^{(0,0)}$, so that $[\mathbb{Q}(S^{(0,1)}) : \mathbb{Q}(S^{(0,0)})] = 2^\ell$. Since $\mathbb{Q}(\hat{S}) = \mathbb{Q}(S^{(0,0)})$, the result follows.

□

Example 3.23. Consider the Ising modular category with label set $\{\mathbf{1}, \sigma, \psi\}$. It is a spin modular category with fermion ψ. Its S-matrix is

$$\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

The subcategory generated by 1 and ψ is sVec, and we have $[\mathbb{Q}(S) : \mathbb{Q}(S_{\mathrm{sVec}})] = 2$.

Question 3.24. Is there a relationship between the Galois group of the S-matrix of a braided fusion category $\mathcal{B}$ and that of its Drinfeld center $\mathcal{Z}(\mathcal{B})$?

The following lemma can probably be generalized to non-self-dual categories, but we will only use it in the self-dual case:

Lemma 3.25. Suppose that $\mathcal{B}$ is a self-dual super-modular category and z is a label in the fermionic quotient such that $d_z = 1$ and $\hat{S}_{z,z} \neq 1$. Then $\mathcal{B}$ contains a modular pointed subcategory equivalent to $\mathcal{C}(\mathbb{Z}_2, Q)$ (i.e. Sem or $\overline{\mathrm{Sem}}$).

Proof. The hypothesis immediately implies that $\mathcal{B}$ contains an invertible, self-dual simple object Z. Since $S_{Z,Z} = \hat{S}_{z,z} \neq 1$, the object Z is not self-centralizing, hence generates a modular subcategory of dimension 2. □

Question 3.26. Can we drop the self-duality condition in the above, with the same conclusion?

3.5.1 Rank Finiteness

The rank-finiteness property can be extended to categories that do not necessarily admit a spherical structure. It was recently proved that rank-finiteness holds for G-crossed braided fusion categories.

Theorem 3.27. [44, Corollary 4.7.] There are finitely many equivalence classes of G-crossed braided fusion categories of any given rank.

This motives us to pursue a classification of low-rank super-modular categories parallel to [61, 22]. A classification of super-modular categories of rank ≤ 6 is given in [19]. It is shown, for example, that the fusion rules of any non-split super-modular category of rank ≤ 6 are the same as $\mathrm{PSU}(2)_{4k+2}$ for $k = 0, 1$ and 2.

4. CONGRUENCE SUBGROUPS AND SUPER-MODULAR CATEGORIES[1]

Given a super-modular category $\mathcal{B}$, recall that its S and T matrices have tensor decompositions (Theorem 3.8):

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}$$

where $\hat{S}$ is unitary and $\hat{T}$ is a diagonal (unitary) matrix, depending on $r/2-1$ sign choices. Two naive questions motivated by the above are: 1) Do $\hat{S}$ and a choice of $\hat{T}$ provide a (projective) representation of $\mathrm{SL}(2,\mathbb{Z})$? and 2) Is the group generated by $\hat{S}$ and a choice of $\hat{T}$ finite? Of course if $\mathcal{B} = \mathrm{sVec} \boxtimes \mathcal{D}$ for some modular category $\mathcal{D}$ (*split super-modular*) then the answer to both is yes. More generally, as Example 4.2 below illustrates, the answer to both questions is no.

The physical and topological applications of super-modular categories motivate a more refined question as follows. The consideration of fermions on a torus [3] leads to the study of spin structures on the torus $\Sigma_{1,0}$: there are three even spin structures $(A,A),(A,P),(P,A)$ and one odd spin structure (P,P), where A,P denote antiperiodic and periodic boundary conditions. The full mapping class group $\mathrm{Mod}(\Sigma_{1,0}) = \mathrm{SL}(2,\mathbb{Z})$ permutes the even spin structures: $\mathfrak{s}$ interchanges (P,A) and (A,P), and preserves (A,A), whereas $\mathfrak{t}$ interchanges (A,A) and (P,A) and preserves (A,P). Note that both $\mathfrak{s}$ and $\mathfrak{t}^2$ preserve (A,A), so that the index 3 subgroup $\Gamma_\theta := \langle \mathfrak{s}, \mathfrak{t}^2 \rangle < \mathrm{SL}(2,\mathbb{Z})$ is the spin mapping class group of the torus equipped with spin structure (A,A). The spin mapping class group of the torus with spin structure (A,P) or (P,A) is similarly generated by $\mathfrak{s}^2$ and $\mathfrak{t}$, which is projectively isomorphic to $\mathbb{Z}$. On the other hand, Γ_θ is projectively the free product of $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}$ ([60]). Now the matrix $\hat{T}^2$ is unambiguously defined for any super-modular category $\mathcal{B}$, and in [16, Theorem II.7] it is shown that $\mathfrak{s} \to \hat{S}$ and $\mathfrak{t}^2 \to \hat{T}^2$ defines a projective representation $\hat{\rho}$ of Γ_θ. We propose the following:

[1] Part of this section is reprinted from "Congruence subgroups and super-modular categories," Parsa Bonderson, Eric C. Rowell, Qing Zhang, Zhenghan Wang, *Pacific Journal of Mathematics*, Vol. 296 (2018), No. 2, 257–270, published by Mathematical Sciences Publishers.

Conjecture 4.1. Let $\mathcal{B}$ be a super-modular category of rank $2k$ and $\hat{S}$ and $\hat{T}^2$ the corresponding matrices as in equation (4). Then the projective representation $\hat{\rho} : \Gamma_\theta \to \mathrm{PSU}(k)$ given by $\hat{\rho}(\mathfrak{s}) = \pi_k(\hat{S})$ and $\hat{\rho}(\mathfrak{t}^2) = \pi_k(\hat{T}^2)$ has kernel a congruence subgroup.

In particular if this conjecture holds then $\hat{\rho}(\Gamma_\theta)$ is finite. We do not know what to expect the level of $\ker \hat{\rho}$ to be (in terms of, say, the order of $\hat{T}^2$), but we provide some examples below.

An important outstanding conjecture ([27, Question 5.15], [16, Conjecture III.9], see also [49, Conjecture 5.2]) is that every super-modular category $\mathcal{B}$ has a *minimal modular extension*: that is, $\mathcal{B}$ can be embedded in a modular category $\mathcal{C}$ of dimension $\dim(\mathcal{C}) = 2\dim(\mathcal{B})$. One may characterize such $\mathcal{C}$: they are called *spin modular categories* ([5]), see Section 4.2.1 below. Our main result proves Conjecture 4.1 for super-modular categories admitting minimal modular extensions.

4.1 Super-Modular Categories

Equation (4) shows that the S and T matrices of any super-modular category can be expressed as (Kronecker) tensor products: $S = S_{\mathrm{sVec}} \otimes \hat{S}$ and $T = T_{\mathrm{sVec}} \otimes \hat{T}$ with $\hat{S}$ uniquely determined and $\hat{T}$ determined by some sign choices. The projective group generated by $\hat{S}$ and $\hat{T}$ may be infinite for all choices of $\hat{T}$ as the following example illustrates:

Example 4.2. Consider the modular category $\mathrm{SU}(2)_6$. The label set is $I = \{0, 1, 2, 3, 4, 5, 6\}$. The subcategory $\mathrm{PSU}(2)_6$ is generated by 4 simple objects with even labels: $X_0 = \mathbf{1}, X_2, X_4, X_6$. We have $\hat{S} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix}$ and $\hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}$. For either choice of $\hat{T}$ the eigenvalues of $\hat{S}\hat{T}$ are not roots of unity: one checks that they satisfy the irreducible polynomial $x^{16} - x^{12} + \frac{1}{4}x^8 - x^4 + 1$, which has non-abelian Galois group and is not monic over $\mathbb{Z}$.

4.1.1 The θ-Subgroup of $\mathrm{SL}(2, \mathbb{Z})$

The index 3 subgroup $\Gamma_\theta < \mathrm{SL}(2, \mathbb{Z})$ generated by $\mathfrak{s}$ and $\mathfrak{t}^2$ has a uniform description (see e.g. [45]):

$$\Gamma_\theta = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : ac \equiv bd \equiv 0 \pmod 2 \}.$$

The notation Γ_θ comes from the fact that Jacobi's θ series $\theta(z) := \sum_{n=-\infty}^{\infty} e^{n^2 \pi \mathrm{i} z}$ is a modular form of weight $1/2$ on Γ_θ. Moreover,Γ_θ is isomorphic to $\Gamma_0(2)$, the Hecke congruence subgroup of level 2 defined as those matrices in $\mathrm{SL}(2,\mathbb{Z})$ that are upper triangular modulo 2, and $\Gamma(2)$ is a subgroup of both $\Gamma_0(2)$ and Γ_θ. In particular $\Gamma_0(2)$ and Γ_θ are distinct, yet isomorphic, congruence subgroups of level 2. An explicit isomorphism $\vartheta : \Gamma_\theta \to \Gamma_0(2)$ is given by $\vartheta(\mathfrak{g}) = M\mathfrak{g}M^{-1}$ where $M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. This can be verified directly, via:

$$M \begin{pmatrix} a & b \\ c & d \end{pmatrix} M^{-1} = \begin{pmatrix} a+c & \frac{d+b-a-c}{2} \\ 2c & d-c \end{pmatrix}.$$

Observe that $\vartheta(\Gamma(n)) = \Gamma(n)$ for any n, and for n even $\Gamma(n) \lhd \Gamma_\theta$. In particular, we see that $\Gamma_\theta/\Gamma(n) < \mathrm{SL}(2,\mathbb{Z})/\Gamma(n)$ is isomorphic to an index 3 subgroup of $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ that is not normal. Suppose $\varphi : \Gamma_\theta \to H$ has kernel a congruence subgroup, i.e. $\Gamma(n) < \ker(\varphi)$. The congruence level of $\ker(\varphi)$, i.e. the minimal n with $\Gamma(n) < \ker(\varphi)$, is the minimal n so that $\Gamma_\theta/\Gamma(n) \twoheadrightarrow \varphi(\Gamma_\theta)$. The following provides a characterization of such quotients:

Lemma 4.3. Suppose that $n = 2^k q$ with $k \geq 1$ and q odd. Denote by P_k a 2-Sylow subgroup of $\mathrm{SL}(2,\mathbb{Z}/2^k\mathbb{Z})$. Then,

$$\Gamma_\theta/\Gamma(n) \cong P_k \times \mathrm{SL}(2,\mathbb{Z}/q\mathbb{Z}).$$

Proof. By the Chinese Remainder Theorem, non-normal index 3 subgroups of

$$\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z}) \cong \prod_{p|n} \mathrm{SL}(2,\mathbb{Z}/p^{\ell_p}\mathbb{Z})$$

correspond to non-normal index 3 subgroups of $\mathrm{SL}(2,\mathbb{Z}/p^{\ell_p}\mathbb{Z})$ where $n = \prod_{p|n} p^{\ell_p}$ is the prime factorization of n. Any 2-Sylow subgroup of $\mathrm{SL}(2,\mathbb{Z}/2^k\mathbb{Z})$ has index 3 and is not normal (since reduction modulo 2 gives a surjection to $\mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z}) \cong \mathfrak{S}_3$) so it is enough to show that this fails for $\mathrm{SL}(2,\mathbb{Z}/p^k\mathbb{Z})$ with $p > 2$.

In general, if $H < G$ is a non-normal subgroup of index 3 then the (transitive) left action of G on the coset space G/H provides a homomorphism to the symmetric group on 3 letters: $\phi : G \to \mathfrak{S}_3$. If $\phi(G) = \mathfrak{A}_3$ (the alternating group on 3 letters) then we would have $\ker(\phi) = H \lhd G$. Thus $\phi(G) = \mathfrak{S}_3$, so that any such group G must have an irreducible 2 dimensional representation with character values $2, -1, 0$.

By [57, 34] we see that for $p > 2$, the groups $\mathrm{SL}(2, \mathbb{Z}/p^k\mathbb{Z})$ only have 2-dimensional irreducible representations for $p = 3, 5$, and each of these representations factor over the reduction modulo p map $\mathrm{SL}(2, \mathbb{Z}/p^k\mathbb{Z}) \twoheadrightarrow \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$. By inspection neither $\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})$ nor $\mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$ have $\mathfrak{S}_3$ as quotients. □

4.2 Main Results

In this section we prove Conjecture 4.1 for any super-modular category that admits a minimal (spin) modular extension.

4.2.1 Spin Modular Categories

Let $(\mathcal{C}, f)$ be a spin modular category, where f is a chosen fermion. Recall from Remark 3.5, we have a canonical $\mathbb{Z}/2\mathbb{Z}$-grading $\mathcal{C}_0 \oplus \mathcal{C}_1$ with simple objects $X \in \mathcal{C}_0$ if $\epsilon_X = 1$ and $X \in \mathcal{C}_1$ when $\epsilon_X = -1$. The trivial component $\mathcal{C}_0$ is a super-modular category.

Since $\theta_X = -\epsilon_X \theta_{fX}$ it is clear that $fX \ncong X$ for $X \in \mathcal{C}_0$. However, objects in $\mathcal{C}_1$ may be fixed by $- \otimes f$ or not. This provides another canonical decomposition $\mathcal{C}_1 = \mathcal{C}_v \oplus \mathcal{C}_\sigma$ as abelian categories, where a simple object $X \in \mathcal{C}_v \subset \mathcal{C}_1$ if $Xf \ncong X$ and $X \in \mathcal{C}_\sigma \subset \mathcal{C}_1$ if $Xf \cong X$. Finally, using the action of $- \otimes f$ we make a (non-canonical) decomposition of $\mathcal{C}_0 = \breve{\mathcal{C}}_0 \oplus f\breve{\mathcal{C}}_0$ and $\mathcal{C}_v = \breve{\mathcal{C}}_v \oplus f\breve{\mathcal{C}}_v$ so that when $X \in \breve{\mathcal{C}}_0$ we have $Xf \in f\breve{\mathcal{C}}_0$ and similarly for $\mathcal{C}_v$. Notice that for $X \in \mathcal{C}_0$ we have $X^* \ncong f \otimes X$ since $\theta_X = \theta_{X^*}$, so that we may ensure X and X^* are both in $\breve{\mathcal{C}}_0$ or both in $f\breve{\mathcal{C}}_0$. On the other hand, for $Y \in \mathcal{C}_v$ it is possible that $X^* \cong f \otimes X$–for example, this occurs for $SO(2)_1$.

As in [10] we choose an ordered basis $\Pi = \Pi_0 \bigsqcup \psi\Pi_0 \bigsqcup \Pi_v \bigsqcup f\Pi_v \bigsqcup \Pi_\sigma$ for the Grothendieck ring of $\mathcal{C}$ that is compatible with the above partition $\mathcal{C} = \breve{\mathcal{C}}_0 \oplus f\breve{\mathcal{C}}_0 \oplus \breve{\mathcal{C}}_v \oplus f\breve{\mathcal{C}}_v \oplus \mathcal{C}_\sigma$. Using [16, Proposition II.3] we have the block matrix decomposition for the S and T matrices:

$$S = \begin{pmatrix} \frac{1}{2}\hat{S} & \frac{1}{2}\hat{S} & A & A & X \\ \frac{1}{2}\hat{S} & \frac{1}{2}\hat{S} & -A & -A & -X \\ A^T & -A^T & B & -B & 0 \\ A^T & -A^T & -B & B & 0 \\ X^T & -X^T & 0 & 0 & 0 \end{pmatrix} \quad T = \begin{pmatrix} \hat{T} & 0 & 0 & 0 & 0 \\ 0 & -\hat{T} & 0 & 0 & 0 \\ 0 & 0 & \hat{T}_v & 0 & 0 \\ 0 & 0 & 0 & \hat{T}_v & 0 \\ 0 & 0 & 0 & 0 & T_\sigma \end{pmatrix} \tag{4.1}$$

Here B and $\hat{S}$ are symmetric matrices, and each of $\hat{T}, \hat{T}_v$ and T_σ are diagonal matrices.

Now consider the following ordered partitioned basis:

1. $\Pi_0^+ := \{X_i + fX_i : X_i \in \Pi_0\}$,
2. $\Pi_0^- := \{X_i - fX_i : X_i \in \Pi_0\}$,
3. $\Pi_v^+ := \{Y_i + fY_i : Y_i \in \Pi_v\}$,
4. $\Pi_\sigma := \{Z_i \in \Pi_\sigma\}$ and
5. $\Pi_v^- := \{Y_i - fY_i : Y_i \in \Pi_v\}$.

With respect to this partitioned basis, the S and T matrices have the block form:

$$S' = \begin{pmatrix} \hat{S} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2A & X & 0 \\ 0 & 2A^T & 0 & 0 & 0 \\ 0 & 2X^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2B \end{pmatrix} \quad T' = \begin{pmatrix} 0 & \hat{T} & 0 & 0 & 0 \\ \hat{T} & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{T}_v & 0 & 0 \\ 0 & 0 & 0 & T_\sigma & 0 \\ 0 & 0 & 0 & 0 & \hat{T}_v \end{pmatrix}.$$

From this choice of basis one sees that the representation ρ restricted to $\Gamma_\theta = \langle \mathfrak{s}, \mathfrak{t}^2 \rangle$ has 3 invariant (projective) subspaces, spanned by Π_0^+, $\Pi_0^- \cup \Pi_v^+ \cup \Pi_\sigma$ and Π_v^- respectively. In particular we have a surjection $\rho(\Gamma_\theta) \twoheadrightarrow \hat{\rho}(\Gamma_\theta)$, mapping the image of S in $\mathrm{PSU}(|\Pi|)$ to the image of $\hat{S}$ in $\mathrm{PSU}(|\Pi_0^+|)$. We can now prove:

Theorem 4.4. Suppose that $\mathcal{B}$ is a super-modular category with minimal modular extension $\mathcal{C}$ so that $\mathcal{B} = \mathcal{C}_0$. Assume further that the T-matrix of $\mathcal{C}$ has order N. Then $\hat{\rho} : \Gamma_\theta \to \mathrm{PSU}(k)$ has $\ker(\hat{\rho})$ a congruence subgroup of level at most N.

Proof. Let S and T be the S-matrix and T-matrix of $\mathcal{C}$. Consider the projective representation ρ of $\mathrm{SL}(2, \mathbb{Z})$ defined by $\rho(\mathfrak{s}) = S$ and $\rho(\mathfrak{t}) = T$. By Theorem 2.24, $\ker(\rho)$ is a congruence subgroup of level N, i.e. $\Gamma(N) < \ker(\rho)$. Now the restriction of $\rho_{|\Gamma_\theta}$ to Γ_θ has $\ker(\rho_{|\Gamma_\theta}) = \ker(\rho) \cap \Gamma_\theta \supset \Gamma(N) \cap \Gamma_\theta$. However, since $\mathcal{C}$ contains a fermion N is even, so $\Gamma(N) < \Gamma(2) < \Gamma_\theta$ hence $\Gamma(N) \cap \Gamma_\theta = \Gamma(N)$. It follows that $\Gamma(N) < \ker(\rho_{|\Gamma_\theta})$. The discussion above now implies $\Gamma(N) < \ker(\rho_{|\Gamma_\theta}) < \ker(\hat{\rho})$ as we have a surjection $\rho(\Gamma_\theta) \twoheadrightarrow \hat{\rho}(\Gamma_\theta)$. Thus, we have shown that $\ker(\hat{\rho})$ is a congruence subgroup of level at most N, and in particular $\hat{\rho}$ has finite image.

□

4.2.2 Further Questions

The charge conjugation matrix C in the basis above has the form $C'_{i,j} = \pm\delta_{i,j^*}$. Since we have arranged that $X_i \in \Pi_0$ implies $X_i^* \in \Pi_0$, $C'_{i,j} = -1$ can only occur for $i = j \in \Pi_v^-$: if $(W - \psi W)^* = -(W - \psi W)$ for some simple object W, then $W^* = \psi W$. We see that this can only happen if $W \in \mathcal{C}_v$ by comparing twists. Under this change of basis, we have $(S')^2 = \dim(\mathcal{C})C'$ and $(S'T')^3 = \frac{D_+}{D}(S')^2$. It would be interesting to explore the extra relations among the various submatrices of S' and T'.

The 16 spin modular categories of dimension 4 are of the form $\mathrm{SO}(n)_1$ (where $\mathrm{SO}(n)_1 \cong \mathrm{SO}(m)_1$ if and only if $n \cong m \pmod{16}$). For n odd $\mathrm{SO}(n)_1$ has rank 3 whereas for n even $\mathrm{SO}(n)_1$ has rank 4. For example, the Ising modular category corresponds to $n = 1$ and $\mathrm{SO}(2)_1$ has fusion rules like the group $\mathbb{Z}_4$. For any modular category $\mathcal{D}$ and $1 \leq n \leq 16$ the spin modular category $\mathrm{SO}(n)_1 \boxtimes \mathcal{D}$ with fermion $(f, \mathbf{1})$ has either $\mathcal{C}_\sigma = \emptyset$ or $\mathcal{C}_v = \emptyset$. An interesting problem is to classify spin modular categories with either $\mathcal{C}_\sigma = \emptyset$ or $\mathcal{C}_v = \emptyset$, particularly those with no $\boxtimes$-factorization.

4.3 A Case Study

Our result gives an upper bound on the level of $\ker(\hat{\rho})$ for super-modular categories $\mathcal{B}$ with minimal modular extensions $\mathcal{C}$: the level of $\ker(\hat{\rho})$ is at most the order of the T-matrix of $\mathcal{C}$. The actual level can be lower: for a trivial example we consider the super-modular category sVec. In this case $\hat{S} = \hat{T}^2 = I$ so the level $\ker(\hat{\rho})$ is 1, yet the order of the T matrix for its (16) minimal modular extensions can be $2, 4, 8$ or 16. More generally for any split super-modular category $\mathcal{B} = \mathcal{D} \boxtimes \mathrm{sVec} \subset \mathcal{D} \boxtimes \mathrm{SO}(n)_1 = \mathcal{C}$ (with fermion $(\mathbf{1}, \psi)$) the ratio of the levels of the kernels of the $\mathrm{SL}(2,\mathbb{Z})$ (for $\mathcal{C}$) and Γ_θ (for $\mathcal{B}$, i.e. $\mathcal{D}$) representations can be 2^k for $0 \leq k \leq 4$.

To gain further insight we consider a family of non-split super-modular categories obtained from the spin modular category (see [16, Lemma III.7]) $\mathrm{SU}(2)_{4m+2}$. This has modular data:

$$\tilde{S}_{i,j} := \frac{\sin\left(\frac{(i+1)(j+1)\pi}{4m+4}\right)}{\sin(\frac{\pi}{4m+4})}, \quad T_{j,j} := e^{\frac{\pi \mathrm{i}(j^2+2j)}{8m+8}}$$

where $0 \leq i, j \leq 4m+2$. Since T has order $16(m+1)$, Theorem 2.24 implies that the image of the projective representation $\rho : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSU}(4m+3)$ defined via the normalized S-matrix S and T factors over $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ where $N = 16(m+1)$.

The super-modular subcategory $\mathrm{PSU}(2)_{4m+2}$ has simple objects labeled by even i, j. The factorization (4) yields the following:

$$\hat{S}_{i,j} = \frac{\sin\left(\frac{(2i+1)(2j+1)\pi}{4m+4}\right)}{\Xi \sin(\frac{\pi}{4m+4})}, \quad \hat{T}_{j,j} = e^{\frac{\pi \mathrm{i}(j^2+j)}{2m+2}} \tag{4.2}$$

for $0 \leq i, j \leq m$, where $\Xi = \frac{\sqrt{\frac{m+1}{2}}}{\sin(\frac{\pi}{4m+4})}$. In [16] all 16 minimal modular extensions of $\mathrm{PSU}(2)_{4m+2}$ are explicitly constructed and each has T-matrix of order $16(m+1)$ so that the kernel of the corresponding projective $\mathrm{SL}(2,\mathbb{Z})$ representation is a congruence subgroup of level $16(m+1)$. Our computations suggests the following conjecture, with cases verified using Magma software [13] indicated in parentheses. A sample of the results of these computations are found in Table

4.1. The notation $\langle n, k\rangle$ indicates the kth group of order n in the GAP [41] library of small groups. In the last column, we sometimes give a slightly different description than is indicated in part (f) below. We include the groups $\hat{\rho}(\Gamma_\theta)$, $A'_m := [A_m, A_m]$ and $\overline{A}_m := A_m/Z(A_m)$. As $\hat{\rho}$ is not necessarily irreducible, we have $\hat{\rho}(\Gamma_\theta) \twoheadrightarrow \overline{A}_m$. The congruence level of $\ker\hat{\rho}$ is computed using Lemma 4.3.

Conjecture 4.5. Let A_m be the subgroup of $\mathrm{SU}(k)$ generated by $\hat{S}$ and $\hat{T}^2$ associated with $\mathrm{PSU}(2)_{4m+2}$, the quotient $\overline{A}_m := A_m/Z(A_m)$ and the commutator subgroup $A'_m := [A_m, A_m]$. Then

(a) When $m+1 = q$ is odd, $\overline{A}_m = \overline{A}_{q-1} \cong \mathrm{PSL}(2, \mathbb{Z}/q\mathbb{Z})$ (verified for $2 \le m \le 18$).

(b) When $m+1 = 2^n$ we have $|\overline{A}_m| = |\overline{A}_{2^n-1}| = 2^{3n+1}$ (verified for $1 \le n \le 5$).

(c1) If we write $m+1 = 2^n q$ where q is odd, then $\overline{A}_m \cong \overline{A}_{2^n-1} \times \overline{A}_{q-1}$ (verified for $1 \le m \le 14$).

(c2) If we write $m+1 = 2^n q$ where q is odd $|\overline{A}_m| = 2^{3n+1} q^3 \prod_{p|q} \frac{p^2-1}{2p^2}$ (primes p) (verified for $1 \le m \le 21$).

(d) For $5 \le m+1 = p$ prime $A'_{p-1} \cong \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ (verified for $4 \le m \le 12$).

(e) If we write $m+1 = 2^n q$ where q is odd, then $A'_m \cong A'_{2^n-1} \times A'_{q-1}$ (verified for $1 \le m \le 14$).

(f) For $m+1 \not\equiv 0 \pmod 4$, we have $A'_m \lhd \hat{\rho}(\Gamma_\theta)$ and $\hat{\rho}(\Gamma_\theta)$ is an iterated semidirect product of A'_m with cyclic group actions (verified for $1 \le m \le 14$). In general, $\ker(\hat{\rho})$ is a congruence subgroup of level $4(m+1)$ (verified for $1 \le m \le 12$).

Table 4.1: A Sample of $\mathrm{PSU}(2)_{4k+2}$ Results

m	$\lvert\overline{A}_m\rvert$	$\overline{A}_m$	A'_m	$\hat{\rho}(\Gamma_\theta)$
1	2^4	D_{16}	$\mathbb{Z}/8\mathbb{Z}$	$D_{16} = A'_1 \rtimes \mathbb{Z}/2\mathbb{Z}$
2	12	$\mathrm{PSL}(2, \mathbb{Z}/3\mathbb{Z})$	$\mathbf{Q}_8$	$\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$
3	2^7	$\langle 128, 71\rangle$	$\langle 64, 184\rangle$	$\langle 128, 71\rangle$
4	60	$\mathrm{PSL}(2, \mathbb{Z}/5\mathbb{Z})$	$\mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$	$A'_4 \rtimes \mathbb{Z}/2\mathbb{Z}$
5	$2^4 \cdot 12$	$D_{16} \times \mathrm{PSL}(2, \mathbb{Z}/3\mathbb{Z})$	$\mathbb{Z}/8\mathbb{Z} \times \mathbf{Q}_8$	$(\mathbb{Z}/8\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z})) \rtimes \mathbb{Z}/2\mathbb{Z}$
6	168	$\mathrm{PSL}(2, \mathbb{Z}/7\mathbb{Z})$	$\mathrm{SL}(2, \mathbb{Z}/7\mathbb{Z})$	$A'_6 \rtimes \mathbb{Z}/2\mathbb{Z}$
7	2^{10}	$\overline{A}_7$	$\lvert \cdot \rvert = 2^9$	$\overline{A}_7$
8	324	$\mathrm{PSL}(2, \mathbb{Z}/9\mathbb{Z})$	$(\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathbf{Q}_8$	$(A'_8 \rtimes \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$
9	$2^4 \cdot 60$	$D_{16} \times \mathrm{PSL}(2, \mathbb{Z}/5\mathbb{Z})$	$\mathbb{Z}/8\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$	$A'_9 \rtimes \mathbb{Z}/2\mathbb{Z}$
10	660	$\mathrm{PSL}(2, \mathbb{Z}/11\mathbb{Z})$	$\mathrm{SL}(2, \mathbb{Z}/11\mathbb{Z})$	$A'_{10} \rtimes \mathbb{Z}/2\mathbb{Z}$
11	$2^7 \cdot 12$	$\langle 128, 71\rangle \times \mathrm{PSL}(2, \mathbb{Z}/3\mathbb{Z})$	$\langle 64, 184\rangle \times \mathbf{Q}_8$	$\mathrm{SL}(2, \mathbb{Z}/3\mathbb{Z}) \rtimes \langle 128, 71\rangle$
12	1092	$\mathrm{PSL}(2, \mathbb{Z}/13\mathbb{Z})$	$\mathrm{SL}(2, \mathbb{Z}/13\mathbb{Z})$	$\mathrm{SL}(2, \mathbb{Z}/13\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$
13	$2^4 \cdot 168$	$D_{16} \times \mathrm{PSL}(2, \mathbb{Z}/7\mathbb{Z})$	$\mathbb{Z}/8\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z}/7\mathbb{Z})$	$A'_{13} \rtimes \mathbb{Z}/2\mathbb{Z}$
14	720	$\mathrm{PSL}(2, \mathbb{Z}/15\mathbb{Z})$	$\mathbf{Q}_8 \times \mathrm{SL}(2, \mathbb{Z}/5\mathbb{Z})$	$\mathrm{SL}(2, \mathbb{Z}/15\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$

5. CLASSIFICATION OF SUPER-MODULAR CATEGORIES BY RANK

5.1 Main Results

Similarly to modular categories, the Galois group $\mathrm{Gal}(\mathcal{B})$ of a super-modular category $\mathcal{B}$ defined in Section 3.5 is an abelian subgroup of the symmetric group $\mathfrak{S}_r$, where $2r$ is the rank of $\mathcal{B}$ (see Remark 3.19).

In this section, we consider the problem of classifying rank $2r = 8$ super-modular categories. If $\mathcal{B}$ is non-self dual, we can denote the four simple objects in Π_0 as $\mathbf{1}, Y, X, X^*$. The naive fusion rules satisfy the relations in Corollary 3.14 and the argument in [61, Appendix A.2] works for this case. Therefore, we sometimes assume the super-modular categories are self-dual, in which case $\hat{S}$ has real entries.

The abelian subgroups (up to relabeling, but with 0 distinguished) G of $\mathfrak{S}_4$ are listed in the following table:

Table 5.1: Abelian Subgroups of $\mathfrak{S}_4$

$\langle \mathbf{1} \rangle$	$\langle (0) \rangle$
$\mathbb{Z}_2$	$\langle (01) \rangle$, $\langle (23) \rangle$, $\langle (01)(23) \rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle (01)(23), (02)(13) \rangle$, $\langle (01), (23) \rangle$
$\mathbb{Z}_3$	$\langle (012) \rangle$, $\langle (123) \rangle$
$\mathbb{Z}_4$	$\langle (0123) \rangle$

In this section, we determine the possible $\hat{S}$-matrices for super-modular categories, and then derive the fusion rules in Section 5.2. We summarize our results into the following.

Theorem 5.1. Suppose $\mathcal{B}$ is a rank 8 self-dual super-modular category and G is its Galois group as in Table 5.1 then:

- If $G = \langle (23) \rangle$, $\langle (01), (23) \rangle$ or $\langle (123) \rangle$, then $\mathcal{B}$ does not exist.

- If $G = \langle(0)\rangle$, then $\mathcal{B}$ is **pointed**, i.e., of the form $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, Q) \boxtimes \mathrm{sVec}$.
- If $G = \langle(01)\rangle$, then $\mathcal{B}$ is **prime** and **weakly integral** with the same fusion rules as the centralizer of either fermion in $\mathrm{SO}(12)_2$.
- If $G = \langle(01)(23),(02)(13)\rangle$, then $\mathcal{B}$ has the same fusion as $\mathrm{Fib} \boxtimes \mathrm{PSU}(2)_6$.
- If $G = \langle(0123)\rangle$ and $\hat{N}_{ij}^k < 14$, then $\mathcal{B}$ is **prime** and has the same fusion rules as $\mathrm{PSU}(2)_{14}$.
- If $G = \langle(012)\rangle$ and $\hat{N}_{ij}^k < 21$, then $\mathcal{B}$ has the same fusion rules as $\mathrm{PSU}(2)_7 \boxtimes \mathrm{sVec}$.
- If $G = \langle(01)(23)\rangle$ and $d_i \leq 14$ for all i, then the fusion rules of $\mathcal{B}$ are the same as $[\mathrm{PSU}(2)_6 \boxtimes \mathrm{PSU}(2)_6]_{\mathbb{Z}_2}$ and is **prime**, $\mathrm{Fib} \boxtimes \mathrm{Fib} \boxtimes \mathrm{sVec}$, $\mathrm{Sem} \boxtimes \mathrm{Fib} \boxtimes \mathrm{sVec}$ or $\mathrm{Sem} \boxtimes \mathrm{PSU}(2)_6$.

In several cases the proofs in [61] for the classification of rank 4 modular use techniques and results that apply to super-modular categories as well, so we do not repeat the proof here. For many computations the Gröbner basis software in Maple is useful–we used Maple 2018 for our calculations.

5.1.1 $\hat{S}$-Matrices for Rank 8

The naive fusion coefficients $\hat{N}_{ij}^k$ can be computed by the entries of $\hat{S}$ via the Verlinde formula (see Proposition 3.12 (d)). More precisely, to get the $\hat{N}_{ij}^k$'s, it suffices to determine the $\hat{S}$-matrix.

Remark 5.2. We denote by ϕ_n the positive real root of the equation $x^2 - nx - 1 = 0$, where n is an integer, i.e., $\phi_n = \dfrac{n + \sqrt{n^2+4}}{2}$. If an algebraic number ϕ has conjugate $-\frac{1}{\phi}$, then ϕ must be of the form ϕ_n for some $n \in \mathbb{Z}$.

Theorem 5.3. If $\mathcal{B}$ is a rank 8 non-self dual super-modular category, then the corresponding $\hat{S}$-matrix, up to relabeling the simple objects, has the following form:

$$\hat{S} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \pm i & \mp i \\ 1 & -1 & \mp i & \pm i \end{pmatrix}.$$

Proof. The proof in [61, Appendix A.2] carries through, *mutatis mutandis*. □

Remark 5.4. Having dispensed with the non-self-dual case, we assume for the rest of this section that all categories are self-dual. In particular the naive fusion coefficients are cyclically symmetric (see Corollary 3.14), so we will denote $\hat{N}_{ij}^k$ by $n_{i,j,k}$.

Theorem 5.5. There are no rank 8 self-dual super-modular categories with Galois group $G = \langle(23)\rangle$, $\langle(01),(23)\rangle$ or $\langle(123)\rangle$.

Proof. (1) If $G = \langle(23)\rangle$, applying Equation (3.3) with $\sigma = \langle(23)\rangle$, we have the following form for the $\hat{S}$-matrix

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_2 \\ d_1 & s_{11} & s_{12} & \epsilon_1 s_{12} \\ d_2 & s_{12} & s_{22} & s_{23} \\ d_2 & \epsilon_1 s_{12} & s_{23} & \epsilon_2 s_{22} \end{pmatrix}.$$

As 0 and 1 are fixed by G, by Equation (3.2), we know that d_1, d_2, $\frac{s_{11}}{d_1}$, $\frac{s_{12}}{d_1}$, $\frac{s_{12}^2}{d_2^2}$ and $\frac{s_{22}s_{23}}{d_2^2}$ are rationals as they are fixed by the Galois group. Since they are also algebraic integers (see [36, Proposition 8.13.11]), we know these are integers. Consequently, s_{11}, s_{12}, $s_{22}s_{23}$ are also integers.

If $\epsilon_1 = -1$, the orthogonality of the columns of $\hat{S}$ gives

$$d_1(1 + s_{11}) = 0$$

$$d_1 d_2 + s_{11}s_{12} + s_{12}s_{22} - s_{12}s_{23} = 0$$

$$d_1 d_2 - s_{11}s_{12} + s_{12}s_{23} - \epsilon_2 s_{12}s_{22} = 0$$

So we have $s_{11} = -1$. If $\epsilon_2 = 1$, then we have $d_1 d_2 = 0$, which is a contradiction. If $\epsilon_2 = -1$, we have $d_1 d_2 = -s_{12}s_{22}$. Plugging this into the second equation above, we get $s_{12}(1 + s_{23}) = 0$. If $s_{12} = 0$, then $d_1 d_2 = 0$, which is impossible. If $s_{23} = -1$, then s_{22} is an integer. Then all the entries of $\hat{S}$ are integers, which contradicts the assumption that G is $\mathbb{Z}_2$.

If $\epsilon_1 = 1$, the orthogonality of the columns of $\hat{S}$ gives

$$d_2^2 + s_{12}^2 + s_{22}s_{23} + \epsilon_2 s_{22}s_{23} = 0$$

If $\epsilon_2 = -1$, then $d_2^2 + s_{12}^2 = 0$, a contradiction. If $\epsilon_2 = 1$, by applying a Gröebner basis algorithm on Maple, we get $(2s_{22} + s_{11} + 1)(2d_1d_2 + s_{11}s_{12} + 2s_{12}s_{22} - s_{12}) = 0$. One sees that if either factor is 0, we will have trivial G, a contradiction.

(2) Assume $G = \langle(01), (23)\rangle$. Using Equation (3.3), we get

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \pm 1 & \pm d_2 & \pm d_3 \\ d_2 & \pm d_2 & s_{22} & s_{23} \\ d_3 & \pm d_3 & s_{23} & \pm s_{22} \end{pmatrix}.$$

It follows from $\hat{S}^2 = \frac{D^2}{2}I$ that $2d_2^2 + s_{22}^2 + s_{23}^2 = 2d_3^2 + s_{22}^2 + s_{23}^2$. Since d_i's are positive, $d_2 = d_3$. Let

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_2 \\ d_1 & \epsilon_1 & \epsilon_2 d_2 & \epsilon_3 d_2 \\ d_2 & \epsilon_2 d_2 & s_{22} & s_{23} \\ d_2 & \epsilon_3 d_2 & s_{23} & \epsilon_4 s_{22} \end{pmatrix}.$$

This case can be eliminated using orthogonality of the columns of $\hat{S}$. Applying a Gröbner basis algorithm to these equations we find that the only possible sign choice is given by $\epsilon_1 = \epsilon_4 = 1$ and $\epsilon_2 = \epsilon_3 = -1$. We can further deduce that $s_{23} = -1$, $s_{22} = d_1$ and $d_1 = d_2^2$. Therefore, we have

$$\hat{S} = \begin{pmatrix} 1 & d_2^2 & d_2 & d_2 \\ d_2^2 & 1 & -d_2 & -d_2 \\ d_2 & -d_2 & d_2^2 & -1 \\ d_2 & -d_2 & -1 & d_2^2 \end{pmatrix}$$

Notice that $G = \mathrm{Gal}(\mathbb{Q}(d_2)/\mathbb{Q})$. Computing the characteristic polynomial for $\hat{N}_2$, we have

$$p_2(x) = x^4 + (-2d_2 + \frac{2}{d_2})x^3 + (d_2^2 + \frac{1}{d_2^2} - 4)x^2 + (2d_2 - \frac{2}{d_2})x + 1$$

Therefore, $-2d_2 + \frac{2}{d_2}$ must be an integer. In particular, d_2 satisfies a quadratic equation over $\mathbb{Q}$. This means $\mathrm{Gal}(\mathbb{Q}(d_2)/\mathbb{Q})$ is either trivial or $\mathbb{Z}_2$, which contradicts the fact that G is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(3) If $G = \langle(123)\rangle$, then G fixes 0. Therefore $\hat{S}_{i,0} = d_i$ are rational numbers. Since the dimensions d_i's are always algebraic integers, then they must be integers in this case. Moreover, $d_i = \hat{S}_{0,1} = \pm\hat{S}_{0,i+1} = \pm d_{i+1}$. So, by the positivity of the dimensions (i.e., unitarity assumption), we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_1 & d_1 \\ d_1 & s_{11} & \epsilon_1 s_{33} & \epsilon_2 s_{22} \\ d_1 & \epsilon_1 s_{33} & s_{22} & \epsilon_3 s_{11} \\ d_1 & \epsilon_2 s_{22} & \epsilon_3 s_{11} & s_{33} \end{pmatrix}.$$

From Corollary 3.16, we have $d_1^2 | (1 + 3d_1^2)$. We can deduce that $d_1 = 1$. Since d_1 is the largest (in magnitude) eigenvalue of the fusion matrices N_1, N_2 and N_3, we see that the other eigenvalues (which are real numbers) satisfy $\pm\hat{S}_{ii}/d_1 = \pm\hat{S}_{i,i} = \pm 1$. This means the entries of $\hat{S}$ are ± 1's which contradicts the assumption of G being nontrivial.

□

Theorem 5.6. If $G = \langle(0)\rangle$, then the corresponding $\hat{S}$-matrix, up to relabeling the simple objects, is one of the following:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Proof. If G is trivial, then the proof of [61, Theorem 4.1, Case 7] goes through *mutatis mutandis* showing that the corresponding super-modular category is pointed. Thus by Proposition 3.9 the super-modular category splits, so that $\hat{S}$ has the same form as the S-matrix of some rank 4 pointed modular category [61] as in the statement.

□

Theorem 5.7. If $G = \langle (01) \rangle$, then the corresponding $\hat{S}$ is

$$\begin{pmatrix} 1 & 1 & 2 & \sqrt{6} \\ 1 & 1 & 2 & -\sqrt{6} \\ 2 & 2 & -2 & 0 \\ \sqrt{6} & -\sqrt{6} & 0 & 0 \end{pmatrix}.$$

Proof. By Equation (3.3), we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 & \epsilon_2 d_2 & \epsilon_3 d_3 \\ d_2 & \epsilon_2 d_2 & s_{22} & s_{23} \\ d_3 & \epsilon_3 d_3 & s_{23} & s_{33} \end{pmatrix}.$$

We first assume that $\epsilon_1 = 1$. Then we can have $\epsilon_2 \epsilon_3 = -1$ or $\epsilon_2 = \epsilon_3 = -1$.

For the first case, we can assume $\epsilon_2 = 1$, $\epsilon_3 = -1$ and interchange N_2 and N_3 if necessary. Then the orthogonality of $\hat{S}$ gives us $s_{23}(s_{22} + s_{33}) = 0$ and $2d_1 + d_2^2 - d_3^2 = 0$. Assume that $s_{22} + s_{33} = 0$, then since the columns of $\hat{S}$ are of equal length $2d_2^2 + s_{22}^2 = 2d_3^2 + s_{33}^2$. This gives that $d_2 = d_3$, and that $d_1 = 0$, which is a contradiction. So we must have $s_{23} = 0$. Then $\hat{S}$ becomes

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & 1 & d_2 & -d_3 \\ d_2 & d_2 & s_{22} & 0 \\ d_3 & -d_3 & 0 & s_{33} \end{pmatrix}.$$

Since $\sigma = (01)$ is the only non-trivial element of the Galois group, we conclude that $m = \frac{d_2(d_1+1)}{d_1}$, $n = \frac{d_3(d_1-1)}{d_1}$, $t = \frac{s_{22}}{d_2}$, $u = \frac{s_{33}}{d_3}$, $v = \frac{d_2^2}{d_1}$, $w = \frac{(d_1^2+1)}{d_1}$ and $x = \frac{d_3^2}{d_1}$ are integers as coefficients of the minimal polynomials of the $\hat{N}_i$. Notice that m, v, w and x are strictly greater than 0 and $n \geq 0$. Since $d_2 + d_1 d_2 + d_2 s_{22} = 0$, we have $s_{22} < 0$ so $t < 0$. Moreover, we have $t^2 - u^2 \neq 0$. In fact, if $t^2 - u^2 = 0$, then $u^2 + 2 = \frac{s_{33}^2 + 2d_3^2}{d_3^2} = \frac{s_{22}^2 + 2d_2^2}{d_3^2} = \frac{d_2^2}{d_3^2}(\frac{s_{22}^2 + 2d_2^2}{d_2^2}) = \frac{d_2^2}{d_3^2}(t^2+2)$. This implies that $d_2 = d_3$. Using $2d_1 + d_2^2 - d_3^2 = 0$, we have $d_1 = 0$, a contradiction. Thus $t^2 - u^2 \neq 0$ and we have

$$m = -\frac{2t(u^2+2)}{t^2-u^2}, \qquad n = \frac{2u(t^2+2)}{t^2-u^2}, \qquad v = \frac{2(u^2+2)}{t^2-u^2,}$$
$$w = \frac{2(t^2u^2+t^2+u^2)}{t^2-u^2}, \qquad x = \frac{2(t^2+2)}{t^2-u^2}.$$

Since $x > 0$, we have $t^2 - u^2 > 0$. We have $n_{2,2,2} = \frac{t(t^2-u^2-2)}{(t^2-u^2)}$. In order to have $n_{2,2,2} \geq 0$, we must have $t^2 - u^2 - 2 \leq 0$. The only integer solution satisfying all the restrictions here is $t = -1$ and $u = 0$. Then $s_{33} = 0$ and $s_{22} = -d_2$. Thus, we have $d_1 = 1$. The orthogonality condition on the columns of $\hat{S}$ gives that $2d_2 - d_2^2 = 0$. This implies that $d_2 = 2$ and $d_3 = \sqrt{6}$.

If $\epsilon_2 = \epsilon_3 = -1$, we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & 1 & -d_2 & -d_3 \\ d_2 & -d_2 & s_{22} & s_{23} \\ d_3 & -d_3 & s_{23} & s_{33} \end{pmatrix}.$$

Similarly to the previous case, we have $m = \frac{d_3(d_1-1)}{d_2}$, $n = \frac{d_1^2+1}{d_1}$, $t = \frac{d_3^2}{d_1}$, $u = \frac{s_{22}}{d_2}$, $v = \frac{d_2^2}{d_1}$, $w = \frac{s_{33}}{d_3}$, $x = \frac{s_{23}}{d_2}$, $y = \frac{s_{23}}{d_3}$ and $z = \frac{d_2(d_1-1)}{d_1}$ are integers. Here we have $nv - z^2 - 2v = 0$, $t + v - 2 = 0$ and $m^2 + z^2 - 2n + 4 = 0$. Notice that $m^2 + n^2 \neq 0$ since $n \neq 0$. So we have $n = \frac{m^2+z^2}{2} + 2$, $t = \frac{2m^2}{m^2+z^2}$, and $v = \frac{2z^2}{m^2+z^2}$. Since t is an integer, we have $m^2 \geq z^2$.

Similarly, we have $z^2 \geq m^2$. Thus $|m| = |z|$ so $t = v = 1$. This means $d_2 = d_3 = \sqrt{d_1}$. Then $m = d_1 - 1$ and d_1 is an integer. From $|m| = |z|$, we get $d_1 - 1 = \dfrac{d_2(d_1-1)}{d_1}$. If $d_1 = 1$, then we have $d_2 = d_3 = 1$. This would force all the entries of $\hat{S}$ to be integers, which a contradiction to the assumption that the Galois group is $\mathbb{Z}_2$. If $d_1 > 1$, then we have $d_2 = d_1$. Recall that $d_2 = d_3 = \sqrt{d_1}$. This means either $d_2 = d_3 = d_1 = 0$ or $d_2 = d_3 = d_1 = 1$, again a contradiction.

If $\epsilon_1 = -1$, the orthogonality of the columns of $\hat{S}$ gives $\epsilon_2 d_2^2 + \epsilon_3 d_3^2 = 0$. Thus we have $\epsilon_2\epsilon_3 = -1$ and $d_2 = d_3$. But then we have $\sigma(d_2) = \dfrac{d_2}{d_1} = -\dfrac{d_2}{d_1}$ so $d_2 = 0$, a contradiction. □

Theorem 5.8. If $G = \langle (01)(23), (02)(13) \rangle$, then the corresponding $\hat{S}$ has the following form:

$$\begin{pmatrix} 1 & \phi_1\phi_2 & \phi_1 & \phi_2 \\ \phi_1\phi_2 & 1 & -\phi_2 & -\phi_1 \\ \phi_1 & -\phi_2 & -1 & \phi_1\phi_2 \\ \phi_2 & -\phi_1 & \phi_1\phi_2 & -1 \end{pmatrix}.$$

Proof. By Equation (3.3), we have the corresponding $\hat{S}$:

$$\begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 & \epsilon_2 d_3 & \epsilon_3 d_2 \\ d_2 & \epsilon_2 d_3 & \epsilon_4 & \epsilon_5 d_1 \\ d_3 & \epsilon_3 d_2 & \epsilon_5 d_1 & \epsilon_6 \end{pmatrix}.$$

Using orthogonality of the columns of $\hat{S}$ and the fact that $d_i \geq 1$, there are only 2 possibilities for ϵ_i's, namely,

1. $\epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = -1, \epsilon_4 = 1, \epsilon_5 = -1, \epsilon_6 = 1$, or

2. $\epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = -1, \epsilon_4 = -1, \epsilon_5 = 1, \epsilon_6 = -1$.

For the first case, the orthogonality of $\hat{S}$ gives $d_1 = d_2 d_3$, $d_2 = d_1 d_3$ and $d_3 = d_1 d_2$. So we have $d_1 d_2 d_3 = (d_1 d_2 d_3)^2$, we have $d_1 d_2 d_3 = 1$. Since $d_i \geq 1$ for all i, this implies that $d_1 = d_2 = d_3 = 1$. This cannot happen since the corresponding Galois group should be trivial, which is a contradiction to our assumption.

Consider the second case. The orthogonality of $\hat{S}$ gives $d_1 = d_2 d_3$. So we can write the corresponding matrix as

$$\hat{S} = \begin{pmatrix} 1 & d_2 d_3 & d_2 & d_3 \\ d_2 d_3 & 1 & -d_3 & -d_2 \\ d_2 & -d_3 & -1 & d_2 d_3 \\ d_3 & -d_2 & d_2 d_3 & -1 \end{pmatrix}.$$

Notice that Equation (3.2) indicates that d_2 and $-1/d_2$ are conjugates. By Remark 5.2, we know that $d_2 = \phi_m$ for some $m \in \mathbb{Z}$. Similarly, $d_3 = \phi_n$ for some integer n.

Thus we have

$$\hat{S} = \begin{pmatrix} 1 & \phi_m \phi_n & \phi_m & \phi_n \\ \phi_m \phi_n & 1 & -\phi_n & -\phi_m \\ \phi_m & -\phi_n & -1 & \phi_m \phi_n \\ \phi_n & -\phi_m & \phi_m \phi_n & -1 \end{pmatrix}.$$

The corresponding $\hat{N}_i$ matrices have integer entries in terms of m and n. More precisely, we have

$$\hat{N}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & mn & m & n \\ 0 & m & 0 & 1 \\ 0 & n & 1 & 0 \end{pmatrix}, \hat{N}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & m & 0 & 1 \\ 1 & 0 & m & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and } \hat{N}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & n & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & n \end{pmatrix}.$$

Using the formula given in Lemma 3.17 , we calculate the 2nd Frobenius-Schur indicator for the simple object X_2:

$$\nu_2(X_2) = \pm 1 = \frac{2}{D^2}\left(d_2\left(\frac{1}{\theta_2}\right)^2 + md_1^2 + d_1d_3\left(\frac{\theta_1}{\theta_3}\right)^2 + md_2^2 + d_1\theta_2^2 + d_1d_3\left(\frac{\theta_3}{\theta_1}\right)^2\right)$$

from this we obtain

$$\begin{aligned}
\pm\frac{D^2}{2} &= m\left(d_1^2 + d_2^2\right) + d_2\left(\theta_2^2 + \theta_2^{-2}\right) + d_1d_3\left(\left(\frac{\theta_1}{\theta_3}\right)^2 + \left(\frac{\theta_1}{\theta_3}\right)^{-2}\right) \\
&= m\left(d_2^2d_3^2 + d_2^2\right) + 2d_2\operatorname{Re}\left(\theta_2^2\right) + 2d_2d_3^2\operatorname{Re}\left(\frac{\theta_1}{\theta_3}\right)^2 \\
&\leq \frac{D^2}{2} = 1 + d_2^2d_3^2 + d_2^2 + d_3^2 \\
\Rightarrow 0 &\geq md_2^2\left(d_3^2+1\right) + 2d_2\operatorname{Re}\left(\theta_2^2\right) + 2d_2d_3^2\operatorname{Re}\left(\frac{\theta_1}{\theta_3}\right)^2 - 1 - d_2^2d_3^2 - d_2^2 - d_3^2 \\
&= md_2^2\left(d_3^2+1\right) - 2d_2\left(d_3^2+1\right) - d_2^2\left(d_3^2+1\right) - \left(d_3^2+1\right) \\
&= \left(md_2^2 - 2d_2 - d_2^2 - 1\right)\left(d_3^2+1\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 0 &\geq \left(md_2^2 - 2d_2 - d_2^2 - 1\right) \\
&= d_2^2(m-1) - 2d_2 - 1 \\
&= \phi_m^2(m-1) - 2\phi_m - 1 \\
&= (m\phi_m + 1)(m-1) - 2\phi_m - 1 \\
&= (m-2)(\phi_m(m+1)+1).
\end{aligned}$$

Thus m must be $0, 1,$ or 2.

Similarly, we calculate the 2nd Frobenius-Schur indicator for X_3:

$$\nu_2(X_3) = \pm 1 = \frac{2}{D^2}\left(d_3\theta_3^{-2} + nd_1^2 + d_1d_2\left(\frac{\theta_1}{\theta_2}\right)^2 + d_1d_2(\frac{\theta_2}{\theta_1})^2 + nd_3^2 + d_3\theta_3^2\right)$$

$$\begin{aligned}
\Rightarrow \pm\frac{D^2}{2} &= 2d_3\,\mathrm{Re}(\theta_3^2) + n(d_2^2d_3^2 + d_3^2) + 2d_1^2d_3\,\mathrm{Re}\left(\frac{\theta_1}{\theta_2}\right)^2 \\
&\leq \frac{D^2}{2} = 1 + d_2^2d_3^2 + d_2^2 + d_3^2 \\
\Rightarrow 0 \geq& 2d_3\,\mathrm{Re}(\theta_2^2) + nd_3^2(d_2^2+1) + 2d_3d_2^2\,\mathrm{Re}\left(\frac{\theta_1}{\theta_2}\right)^2 - 1 - d_2^2d_3^2 - d_2^2 - d_3^2 \\
&\geq -2d_3 + nd_3^2(d_2^2+1) - 2d_3d_2^2 - d_3^2(d_2^2+1) - (1+d_2^2) \\
&= (nd_3^2 - 2d_3 - d_3^2 - 1)(d_2^2+1) \\
\Rightarrow 0 \geq& (nd_3^2 - 2d_3 - d_3^2 - 1) \\
&= d_3^2(n-1) - 2d_3 - 1 \\
&= \phi_n^2(n-1) - 2\phi_n - 1 \\
&= (n\phi_n + 1)(n-1) - 2\phi_n - 1 \\
&= (n-2)(\phi_n(n+1)+1)
\end{aligned}$$

So n must be $0, 1$, or 2.

Up to symmetry, we can exclude the cases $(m,n) = (0,0), (1,1), (1,0), (2,2)$ since the corresponding Galois groups are not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The possible value for this case, up to symmetry, is $(m,n) = (1,2)$. Notice that $\phi_1 = \dfrac{1+\sqrt{5}}{2}$ and $\phi_2 = 1+\sqrt{2}$.

□

In the last few cases we were unable to complete the classification in general–instead we placed bounds on the $\hat{N}_{ij}^k$'s. Since $N_{ij}^k \leq 2||N_i||_{\max}$, this could also be done in terms of bounds on the N_i's. Sometimes it is easier to work in terms of a bound on the dimensions d_i. Indeed, the proof of [21, Lemma 3.14] goes through with no change, from which we conclude: $\hat{N}_{ij}^k \leq d_i \leq 4||\hat{N}_i||_{\max}$.

Theorem 5.9. If $G = \langle(0123)\rangle$ and $\hat{N}_{ij}^k < 14$, the corresponding $\hat{S}$ is

$$\begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -d_2 & d_3 & 1 \\ d_2 & d_3 & -1 & -d_1 \\ d_3 & 1 & -d_1 & d_2 \end{pmatrix},$$

where $d_1 = 1+\sqrt{2}+\sqrt{2+\sqrt{2}}$, $d_2 = 1+\sqrt{2}+\sqrt{2(2+\sqrt{2})}$, and $d_3 = 1+\sqrt{2+\sqrt{2}}$.

Proof. Applying Equation (3.3) with $\sigma = \langle(0123)\rangle$, we have the following form of $\hat{S}$ matrix

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 d_2 & \epsilon_2 d_3 & \epsilon_3 \\ d_2 & \epsilon_2 d_3 & \epsilon_4 & \epsilon_5 d_1 \\ d_3 & \epsilon_3 & \epsilon_5 d_1 & \epsilon_6 d_2 \end{pmatrix}.$$

Using a Maple's Gröbner basis algorithm, we deduce that $\epsilon_1 = \epsilon_4 = \epsilon_5 = -1$ and $\epsilon_2 = \epsilon_3 = \epsilon_6 = 1$.

So

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -d_2 & d_3 & 1 \\ d_2 & d_3 & -1 & -d_1 \\ d_3 & 1 & -d_1 & d_2 \end{pmatrix}.$$

Let $p_1(x) = x^4 - c_1x^3 + c_2x^2 + c_3x - 1$ be the characteristic polynomial of $\hat{N}_1$. Then $p_3(x) = x^4 - c_3x^3 - c_2x^2 + c_1x - 1$, where $c_i \in \mathbb{Z}$ for $i = 1, 2$ and 3. Notice that $c_1 = \text{Trace}(\hat{N}_1) \geq 0$ and $c_3 = \text{Trace}(\hat{N}_3) \geq 0$ as the $\hat{N}_i$'s are matrices with nonnegative integer entries. Let $p_2(x) = x^4 - b_1x^3 + b_2x^2 + b_3x + 1$ be the characteristic polynomial of $\hat{N}_2$, where

$b_1 = b_3 = d_2 + \frac{d_3}{d_1} - \frac{1}{d_2} - \frac{d_1}{d_3}$ and $b_2 = -2 + \frac{d_1}{d_2d_3} - \frac{d_3}{d_1d_2} - \frac{d_2d_1}{d_3} + \frac{d_2d_3}{d_1}$.

The orthogonality of the rows of $\hat{S}$ gives $d_1 = d_1d_2 - d_2d_3 - d_3$, $d_3 = -d_1 + d_1d_2 - d_2d_3$,

$d_1d_2 = d_3 + d_1 + d_2d_3$ and $d_2d_3 = -d_1 + d_1d_2 - d_3$. So we have $b_2 = -6$ and $b_3 = -b_1$. Thus $p_2(x) = x^4 - b_1x^3 - 6x^2 + b_1x + 1$, where $b_1 = \text{Trace}(\hat{N}_2) \geq 0$.

Notice that $c_1 + c_3 = 2\dfrac{(d_2+1)d_3}{d_2} + 4\dfrac{d_2}{(d_2+1)d_3}$. This gives $c_1 + c_3 \geq 4\sqrt{2}$. Since c_1 and c_3 are integers, we have $c_1 + c_3 \geq 6$. Moreover, we have $4b_1 - c_1^2 + 8c_2 + c_3^2 = 0$.

Let $\Delta = c_1 - c_3$ and $\Sigma = c_1 + c_3$, then $c_2 = \frac{1}{16}[3\Delta\Sigma \pm \sqrt{(32+\Delta^2)(-32+\Sigma^2)}]$ and $b_1 = \frac{1}{8}[-\Delta\Sigma \mp \sqrt{(32+\Delta^2)(-32+\Sigma^2)}]$. Let $P = \frac{16c_2 - 3\Delta\Sigma}{\Delta^2+32} = \pm\sqrt{\frac{\Sigma^2-32}{\Delta^2+32}}$.

We compute the $n_{i,j,k}$'s and we get the following relations:

$$
\begin{aligned}
n_{1,1,1} &= \frac{5c_1 - 3c_3}{8} - \frac{(c_1 - c_3)P}{8} \\
n_{1,1,2} &= 1 - P = 1 + n_{1,2,3} = 2 + n_{2,3,3} \\
n_{1,1,3} &= \frac{c_1 + c_3}{8} - \frac{(c_1 - c_3)P}{8} = n_{1,3,3} = \frac{1}{2}(n_{1,1,1} + n_{3,3,3}) \\
n_{1,2,2} &= \frac{c_1 + c_3}{4} + \frac{(c_1 - c_3)P}{4} = n_{2,2,3} \\
n_{2,2,2} &= \frac{c_1^2 - c_3^2}{4} - 2c_2 + 2P = b_1 + 2P
\end{aligned}
$$

Recall that the fusion coefficients are integral. In particular, since $n_{2,2,2}$ is an integer, we know that c_1 and c_3 are both even. Thus Δ and Σ are divisible by 2. Via a computer search for integer solutions using the above equations, we found there is only one solution when $n_{i,j,k} < 14$, with $c_1 = c_3 = 4$ and $c_2 = 2P = -2$. The corresponding $\hat{S}$ matrix for this case is the one in the statement (and is the same as that of $\text{PSU}(2)_{14}$). □

We can make further progress using more sophisticated number theoretical arguments:

Lemma 5.10. If Σ and Δ are divisible by 4, the corresponding super-modular categories have $c_1 = c_3 = \sqrt{2}(\zeta^{2i-1} - \overline{\zeta}^{2i-1})$, $c_2 = -(\zeta^{2i-1} + \overline{\zeta}^{2i-1})$ and $P = -\frac{1}{2}(\zeta^{2i-1} + \overline{\zeta}^{2i-1})$, where $\zeta = 1 + \sqrt{2}$, $\overline{\zeta} = 1 - \sqrt{2}$ and $i \geq 1$ is an integer.

Proof. Assume that Σ and Δ in the proof above are also divisible by 4. Denote $a = \frac{\Sigma}{4}$, $b = \frac{\Delta}{4}$ and

$c = P$. Then we have the following Diophantine equation

$$a^2 - (b^2 + 2)c^2 = 2.$$

Lemma 5.13 below shows that $b = 0$. Consequently, we have $c_1 = c_3$, and the Diophantine equation becomes $a^2 - 2c^2 = 2$. Since $a = \frac{c_1}{2} \geq 0$ and $c = P = \frac{c_2}{2} \leq -1$ the resulting solutions are

$$a(i) := \frac{1}{\sqrt{2}}(\zeta^{2i-1} - \overline{\zeta}^{2i-1}), \quad c(i) = -\frac{1}{2}(\zeta^{2i-1} + \overline{\zeta}^{2i-1}),$$

where $1 \leq i$ and $\zeta = 1 + \sqrt{2}$ and $\overline{\zeta} = 1 - \sqrt{2}$. This determines all possible fusion rules under these assumptions. The first few are $(a, c) \in \{(2, -1), (10, -7), (58, -41), (338, -239), \ldots\}$.

□

Some cases can be ruled out if we assume the MME conjecture using Lemma 3.22 as follows.

Example 5.11. In the case $(a, c) = (58, -41)$, we find that d_1 is a root of the irreducible polynomial $x^4 - 2 \cdot 58x^3 - 82x^2 + 2 \cdot 58x - 1$. The smallest cyclotomic field in which d_1 resides has degree $464 = 2^4 \cdot 29$ (i.e., the conductor of $\mathbb{Q}(d_1)$ is 464). Now suppose that the corresponding super-modular category $\mathcal{B}$ has a MME $(\mathcal{C}, f)$. Then the order of the T matrix of $\mathcal{C}$ is divisible by 29, so that $7 \mid \varphi(29) \mid [\mathbb{Q}(T) : \mathbb{Q}]$. But Lemma 3.22 and the results of [55] imply that $[\mathbb{Q}(T) : \mathbb{Q}] = 2^m$ for some m (since $[\mathbb{Q}(T) : \mathbb{Q}(S)] = 2^t$). Thus no such category can exist.

Remark 5.12. The $(a, c) = (10, -7)$ case cannot be dealt with in this way since the corresponding conductor is 80.

Lemma 5.13. Assume a, b and c are integers and $a^2 - (b^2 + 2)c^2 = 2$, then $b = 0$.

Proof. Reducing modulo 8 both sides of the equation, there are three cases to consider since a square modulo 8 is 0, 1, or 4.

- If $b^2 \equiv 1 \mod 8$, then we have $a^2 - 2 \equiv 3c^2 \mod 8$. This gives no solutions.
- If $b^2 \equiv 0 \mod 8$, then we have $c \equiv 1 \mod 8$ and $a \equiv 4 \mod 8$.

- If $b^2 \equiv 4 \mod 8$, then we have $c \equiv 1 \mod 8$ and $a \equiv 0 \mod 8$.

Therefore, we must have that a and b are even and c is odd. Moreover, if $4|b$, then $4 \nmid a$ and vice versa.

Now we consider both sides of $a^2 - (b^2+2)c^2 = 2$ modulo 4. This gives us $b^2+2 \equiv 2 \mod 4$. Let $B = b^2 + 2$, and then we need to solve the following Pell-like equation

$$a^2 - Bc^2 = 2$$

As b is even, B is not divisible by 4. So we write $B = m^2 d$, where d is square-free and even and m is odd.

Claim: $d = 2$. Assume otherwise, then we can prove that $a^2 - Bc^2 = 2$ has no solutions by looking at the class group of $\mathbb{Z}[\sqrt{d}]$ via genus theory. In fact, assume $d \neq 2$ and even. Then the equation $a^2 - d(mc)^2 = 2$ can be written as

$$a^2 - dy^2 = 2.$$

If the above equation has no integer solution, then $a^2 - Bc^2 = 2$ has no solution. Now we consider the quadratic number field $K = \mathbb{Q}(\sqrt{d})$. We denote the class group of K by C_K (see [40] Page 45), which is a finite abelian group. Let $V = (\mathbb{Z}/2\mathbb{Z})^g$, where g is the number of distinct prime dividing d. Let $e_i = (0, \ldots, 1, \ldots, 0)$ be the basis of V, where $i = 1, \ldots, g$ and 1 is on the n^{th} position. Let $C_{K,2}$ be the subgroup of C_K consisting of the elements of order 2. For primes $p_1, \ldots, p_g \in \mathbb{Z}$, denote the corresponding prime ideals as $\mathfrak{p}_1, \ldots, \mathfrak{p}_g \in \mathbb{Z}[\sqrt{d}]$. Define the map

$$\phi : V \to C_{K,2}$$
$$e_i \mapsto [\mathfrak{p}_i].$$

This assignment gives a group homomorphism. By Corollary 1 in Section 5 of [40], we know that ϕ is surjective and $\ker(\phi) = \{0, (1, 1, \ldots, 1)\}$. Consequently, $C_{K,2} \simeq (\mathbb{Z}/\mathbb{Z}_2)^{g-1}$. In particular, if $g \geq 2$, then for any prime $p|d$, $\mathfrak{p} = (p, \sqrt{d})$ is not principal.

Now we return to our equation $a^2 - dy^2 = 2$, where $d \neq 2$ and even. Consider the ideal $(a + y\sqrt{d}) \subseteq \mathbb{Z}[\sqrt{d}]$, which has norm 2. We have $(a + y\sqrt{d})(a - y\sqrt{d}) = (2)$. Moreover, we have $(2, \sqrt{d})^2 = (2)$. By the unique factorization, we have $(2, \sqrt{d}) = (a + y\sqrt{d})$. However, if $g \geq 2$, $(2, \sqrt{d})$ is not principal. Consequently, there is no integer solutions for a and y when $d \neq 2$.

Thus we have

$$a^2 - 2m^2c^2 = 2 \qquad\qquad b^2 - 2m^2 = -2.$$

One can further deduce that $4|b$. Let $b = 4\beta$, the second equation gives us $m^2 - 8\beta^2 = 1$. This is a Pell-equation. Notice that $(m, \beta) = (3, 1)$ is the smallest non-trivial solution. Let $z = 3 + 2\sqrt{2}$ and denote its conjugate as $\bar{z}$. The solutions (m, β) of the equation are given by

$$m_n = \frac{z^n + \bar{z}^n}{2} \qquad\qquad \beta_n = \frac{z^n - \bar{z}^n}{4\sqrt{2}},$$

where n is a positive integer. We also have $a^2 - 2y^2 = 2$, which is a Pell-type equation. Notice that $(a, y) = (2, 1)$ is a solution. Let $s = 2 + \sqrt{2}$. By the theorem of K. Mahler [47], the solutions are given by

$$a_k = \frac{s^k + \bar{s}^k}{2\sqrt{2^{(k-1)}}} \qquad\qquad y_k = \frac{s^k - \bar{s}^k}{2\sqrt{2^k}},$$

where k is an odd positive integer. By modifying the indices, we know the solutions of the pair (m_n, y_n) are given by

$$y_n = \frac{(z+1)^{2n+1} - (7-z)^{2n+1}}{2^{3n+2}\sqrt{2}} \qquad\qquad m_n = \frac{z^n + (6-z)^n}{2},$$

where $n \in \mathbb{N}$. Recall that the values of m and y are related by $y = mc$, where m and c are both odd. In particular, $y \geq m$. Now we consider the function given by $f(x) = \dfrac{y_x}{m_x}$. Using standard calculus, we know that f is a monotonic increasing function and $\lim_{x \to \infty} f(x) = 1 + \sqrt{2}$. Therefore, the only possible solution here is $m = 1$. Consequently, we have $b = 0$. $\square$

Remark 5.14. If $n_{i,j,k} < 115$, by a computer search for positive integer values, we find two more solutions with $(\Sigma, \Delta) = (40, 0)$ and $(232, 0)$, which are correspond to $i = 2, 3$ in Lemma 5.10. The first possible solution with $\Sigma \equiv 2 \pmod 4$ has $(\Sigma, \Delta) = (434, 18)$ and $n_{1,1,1} = 115$.

Theorem 5.15. If $G = \langle (012) \rangle$ and $\hat{N}_{ij}^k < 21$, then $\hat{S}$ is

$$\begin{pmatrix} 1 & d & 1+d & d^2-1 \\ d & -(1+d) & -1 & d^2-1 \\ 1+d & -1 & d & -(d^2-1) \\ d^2-1 & d^2-1 & -(d^2-1) & 0 \end{pmatrix},$$

where d is the largest real root of the polynomial $x^3 - 3x - 1 = 0$.

Proof. Applying Equation (3.3) to $\sigma = (012)$, we get

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 d_2 & \epsilon_2 & \epsilon_3 d_3 \\ d_2 & \epsilon_2 & \epsilon_4 d_1 & \epsilon_5 d_3 \\ d_3 & \epsilon_3 d_3 & \epsilon_5 d_3 & s_{33} \end{pmatrix}.$$

A computation using $\hat{S}^2 = \frac{D^2}{2} I$ and $d_i \geq 1$ reduces the sign choices to the following 3 cases:

(1) $\epsilon_3 = \epsilon_4 = -1, \epsilon_1 = \epsilon_5 = 1, \epsilon_2 = -1$,

(2) $\epsilon_3 = \epsilon_4 = 1, \epsilon_1 = \epsilon_5 = -1, \epsilon_2 = -1$, or

(3) $\epsilon_3 = \epsilon_4 = -1, \epsilon_1 = \epsilon_5 = -1, \epsilon_2 = 1$.

In case (3), we find that $d_3^2 + d_1 d_2 - (d_1 + d_2) = 0$. However, since $d_i \geq 1$, we have $d_3^2 + d_1 d_2 \geq 2$ and $-(d_1 + d_2) \leq -2$. So, the equality holds if and only if $d_1 + d_2 = 2 = d_3^2 + d_1 d_2$, which forces $d_1 = d_2 = d_3 = 1$. This is impossible since the Galois group is non-trivial by hypothesis.

Case (1) is equivalent to case (2) by permuting columns/rows 2 and 3 and relabeling $d_1 \leftrightarrow d_2$. So, without loss of generality, we may assume we are in case (2). Let $g(x) = x^3 - c_1 x^2 + c_2 x - c_3$

be an irreducible polynomial for which d_3 is a root. Note that $c_1 = \dfrac{d_3}{d_1d_2}(d_1d_2 + d_2 - d_1)$, $c_2 = \dfrac{d_3^2}{d_1d_2}(d_2 - d_1 - 1)$, and $c_3 = -\dfrac{d_3^3}{d_1d_2}$. The orthogonality of the rows of $\hat{S}$ shows that $c_1 = -c_3$. Moreover, $\dfrac{c_2}{c_3} = -\lambda_{33} \in \mathbb{Z}$. Let $n = \lambda_{33}$ and $c = -c_3 = c_1$, so we have $g(x) = x^3 - cx^2 + ncx + c$. Since the Galois group is $\mathbb{Z}_3$, we have that $\dfrac{dis(g)}{c^2} = c^2(n^2+4) - 2nc(9+2n^2) - 27$ is a square. Take t to be the positive root of this, that is, $t = \dfrac{(d_1-1)(d_1+d_2)(1+d_2)}{d_1d_2}$.

Notice that $c = \dfrac{d_3^3}{d_1d_2} > 0$. Moreover $t > 0$. Computing the fusion rules, we get

$$n_{1,1,1} = \frac{(t - nc - 1)}{2} - \frac{t}{n^2+3} \qquad n_{1,1,2} = n_{1,3,3} = \frac{-cn + 2n^2 + t - 3}{2\,(n^2+3)}$$

$$n_{1,1,3} = \frac{cn^2 + 2c - nt + 3n}{2\,(n^2+3)} \qquad n_{1,2,2} = n_{2,3,3} = \frac{cn - 2n^2 + t + 3}{2\,(n^2+3)}$$

$$n_{1,2,3} = \frac{c - 3n}{n^2+3} \qquad n_{2,2,2} = \frac{1 + nc + t}{2} - \frac{t}{3+n^2}$$

$$n_{2,2,3} = \frac{2c + 3n + cn^2 + nt}{2(3+n^2)} \qquad n_{3,3,3} = \frac{c + n^3}{n^2+3}$$

If we restrict $n_{i,j,k} < 21$, the only integer values of n, t and c that satisfy $t^2 = c^2(n^2+4) - 2nc(9+2n^2) - 27$ and yield $n_{i,j,k} \in \mathbb{Z}$ is $(n,t,c) = (0,3,3)$. The corresponding $\hat{S}$-matrix is the one given in the statement and is the same as that of $\mathrm{PSU}(2)_7$ (see [61]). □

Remark 5.16. Here is an alternative approach that is less computationally intensive, but assumes the minimal modular extension conjecture holds. First notice that c is a divisor of $\dim(\mathcal{C})$, so that if we assume the MME conjecture holds then, by the Cauchy theorem [21], any prime divisor p of c must divide the order N of the T-matrix of any minimal modular extension of the corresponding super-modular category. Now, by Lemma 3.22, we have $\varphi(N) = [\mathbb{Q}(T) : \mathbb{Q}] = 3 \cdot 2^k$ since $|G| = 3$. Thus if $p \mid c$, we also have $\varphi(p) = 2^a 3^b$ where $b \in \{0,1\}$ and at most one prime divisor p can have $3 \mid \varphi(p)$. Thus the prime divisors of c are somewhat uncommon (for example Fermat primes).

For $n = 0$, the discriminant equation above yields the Diophantine equation $(2c)^2 - 27 = t^2$, which has finitely many solutions. The only values of $c > 0$ that correspond to a solution are: 3 and 7. Since $n_{3,3,3} \in \mathbb{Z}$, when $n = 0$ we have $3 \mid c$. So $c = 3$ which, in turn, implies $t = 3$, giving the same solution as above. So in this case we do not need to assume the MME conjecture.

For $n = 1$ the Diophantine discriminant equation $5c^2 - 22c - 27 = t^2$ has infinitely many solutions, with the smallest few c values:

$$c \in \{7, 31, 199, 1351, 9247, 63367, 434311, 2976799, 20403271\}.$$

The method above eliminates all of these values of c except for 7 (notice that $9 \mid \varphi(1351) = 2^7 3^2$). In the case that $c = 7$, we find that $t = 8$ which implies $n_{1,1,1} = -2$, so this cannot occur.

Theorem 5.17. If $G = \langle(01)(23)\rangle$ and $d_i < 14$ for all i, then the corresponding $\hat{S}$ is one of the following:

$$\begin{pmatrix} 1 & \phi_1^2 & \phi_1 & \phi_1 \\ \phi_1^2 & 1 & -\phi_1 & -\phi_1 \\ \phi_1 & -\phi_1 & -1 & \phi_1^2 \\ \phi_1 & -\phi_1 & \phi_1^2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2^2 & \phi_2 & \phi_2 \\ \phi_2^2 & 1 & -\phi_2 & -\phi_2 \\ \phi_2 & -\phi_2 & -1 & \phi_2^2 \\ \phi_2 & -\phi_2 & \phi_2^2 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \phi_1 & 1 & \phi_1 \\ \phi_1 & -1 & \phi_1 & -1 \\ 1 & \phi_1 & -1 & -\phi_1 \\ \phi_1 & -1 & -\phi_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2 & 1 & \phi_2 \\ \phi_2 & -1 & \phi_2 & -1 \\ 1 & \phi_2 & -1 & -\phi_2 \\ \phi_2 & -1 & -\phi_2 & 1 \end{pmatrix}.$$

Proof. Similar as previous cases, we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 & \epsilon_2 d_3 & \epsilon_3 d_2 \\ d_2 & \epsilon_2 d_3 & s_{22} & s_{23} \\ d_3 & \epsilon_3 d_2 & s_{23} & s_{33} \end{pmatrix}.$$

Case (1) $\epsilon_1 = 1$. Using Maple's Gröbner basis algorithm, we deduced that

$$(s_{33}+1)(s_{23}-1)(s_{23}+1) = 0.$$

First, we assume $s_{33}+1 = 0$, then we have $s_{33} = s_{22} = -1$, $\epsilon_2 = \epsilon_3 = -1$, $\epsilon_1 = 1$ and $s_{23} = d_1 = d_2 d_3$. Therefore the corresponding $\hat{S}$ is given by

$$\hat{S} = \begin{pmatrix} 1 & d_2d_3 & d_2 & d_3 \\ d_2d_3 & 1 & -d_3 & -d_2 \\ d_2 & -d_3 & -1 & d_2d_3 \\ d_3 & -d_2 & d_2d_3 & -1 \end{pmatrix}.$$

Notice that this is exactly the same matrix we derived in Theorem 5.8. But here we do not get a contradiction since the Galois group *is* $\mathbb{Z}_2$. Thus the same argument using the 2nd Frobenius-Schur indicator works here. Since the Galois group is $\mathbb{Z}_2$, we have solutions for S-matrix when $(m,n) = (1,1), (1,0), (2,0)$ and $(2,2)$, i.e. $(d_2, d_3) = (\phi_i, \phi_i)$ or $(\phi_i, 1)$ for $i = 1, 2$. The cases $(1,1)$ and $(2,2)$ yield the first two $\hat{S}$-matrices above, while for $(2,0)$ and $(1,0)$ the Galois group $G \neq \langle (01)(23) \rangle$, a contradiction. However, see Case 2 below where these solutions do occur.

If $s_{23} - 1 = 0$, one can show that the corresponding Galois group is trivial.

Now we assume $s_{23}+1 = 0$, then the matrix $\hat{S}$ has the form

$$\hat{S} = \begin{pmatrix} 1 & d_3^2 & d_3 & d_3 \\ d_3^2 & 1 & -d_3 & -d_3 \\ d_3 & -d_3 & d_3^2 & -1 \\ d_3 & -d_3 & -1 & d_3^2 \end{pmatrix}.$$

Notice this is the same matrix as the previous one if $d_2 = d_3$ and permuting the matrices $\hat{N}_2$ and $\hat{N}_3$.

Case (2) $\epsilon_1 = -1$. In this case, the $\hat{S}$ is of the form

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -1 & d_3 & -d_2 \\ d_2 & d_3 & s_{22} & s_{23} \\ d_3 & -d_2 & s_{23} & -s_{22} \end{pmatrix}.$$

Notice that the conjugate of d_1 is $-\dfrac{1}{d_1}$. Moreover, we know that if $d_1 = 1$, then the corresponding Galois group is trivial. Thus the field $\mathbb{Q}(\hat{S}) = \mathbb{Q}(d_1)$, where $d_1 = \phi_n = \dfrac{n + \sqrt{n^2+4}}{2}$ for some n. Now we assume $k\sqrt{P} = \sqrt{n^2+4}$, where k is an integer and P is a square-free integer. Then $d_1 = \dfrac{n + k\xi}{2}$, where $\xi = \sqrt{P}$. Then $\mathbb{Q}(\hat{S}) = \mathbb{Q}(\xi)$. As all the entries of $\hat{S}$ are algebraic integers, we can assume $d_2 = a + b\xi$, $d_3 = c + d\xi$, $s_{22} = e + f\xi$, $s_{23} = g + h\xi$, where a, b, c, d, e, f, g and h are either half integers or integers. Then using Maple's Gröbner basis algorithm to eliminate non-rational variables we obtain 21 Diophantine equations (over $\frac{1}{2}\mathbb{Z}$).

Notice that $\hat{N}_{12}^{3} = -1$ if $d = 0$ or $2h - k = 0$. One Diophatine equation we derive is:

$$2b^2h - b^2k + 2d^2h + d^2k = 0,$$

which can be written as $\dfrac{b^2}{d^2} = -\dfrac{2h+k}{2h-k}$. So we have $(2h-k)(2h+k) \leq 0$, and since $k > 0$, we see that $h \in (-\frac{k}{2}, \frac{k}{2})$. The condition $d_1 < 14$ implies $n \leq 13$ and $k \leq \sqrt{n^2+4}$, and k is determined by n, so we do a brute force search for solutions using parameters (n, h, k). There are 13 cases which pass the non-negative and integral condition of the naive fusion coefficients $\hat{N}_{ij}^{k}$, which are the cases when $n = 1, \ldots, 13$ and $h = -\frac{k}{2}$, for each k corresponding to n. In fact, for these cases, the corresponding $\hat{S}$ matrix has the following form:

$$\begin{pmatrix} 1 & \phi_n & 1 & \phi_n \\ \phi_n & -1 & \phi_n & -1 \\ 1 & \phi_n & -1 & -\phi_n \\ \phi_n & -1 & -\phi_n & 1 \end{pmatrix}.$$

All the cases can be ruled out by Lemma 3.25 except when $(n,k,h) = (1,1,-\frac{1}{2})$ and $(n,k,h) = (2,2,-1)$. For the first case, we have $a = 2d, b = 0, c = d, e = -1, f = 0$, and $g = -\frac{1}{2}$. Then $n_{3,3,3} = 2d - \frac{1}{2d}$, which is non-negative and integral. Thus $d = -\frac{1}{2}$ or $\frac{1}{2}$. Notice that $d_2 = -1$ if $d = -\frac{1}{2}$, which is a contradiction. If $d = \frac{1}{2}$, the corresponding $\hat{S}$-matrix has a modular realization as $\mathrm{Fib} \boxtimes \mathrm{Sem}$. For the second case, we have $n_{2,2,2} = d - \frac{1}{d}$. Thus $d = 1$ and the corresponding S-matrix has a modular realization as $\mathrm{PSU}(2)_6 \boxtimes \mathrm{Sem}$. These are the second pair of $\hat{S}$-matrices.

□

5.2 Fusion Rules

Recall that the naive fusion coefficients are defined as $\hat{N}_{ij}^k = N_{ij}^k + N_{ij}^{fk}$, where $i, j, k \in \Pi_0$. To get the fusion coefficients N_{ij}^k for the corresponding super-modular categories, we need to determine how these $\hat{N}_{ij}^k$ split. Note that for the pointed cases, such as the ones in Theorem 5.3 and Theorem 5.6, the corresponding super-modular categories split by Proposition 3.9. Moreover, the $\hat{S}$ matrices in Theorem 5.6 give the same naive fusion coefficients. From this discussion, we have the following results:

Lemma 5.18. If $\mathcal{B}$ is non-self dual super-modular category of rank 8, then $\mathcal{B}$ has the same fusion rules as $\mathcal{C}(\mathbb{Z}_4, Q) \boxtimes \mathrm{sVec}$ where $\mathcal{C}(\mathbb{Z}_4, Q)$ is a pointed modular category with $\mathbb{Z}_4$ fusion rules.

Lemma 5.19. If $\mathcal{B}$ is a self-dual super-modular category with Galois group $G = \langle (0) \rangle$, then $\mathcal{B}$ has the same fusion rules as $\mathcal{D} \boxtimes \mathrm{sVec}$, where $\mathcal{D}$ is a Toric code modular category.

Lemma 5.20. Let $\mathcal{B}$ be a self-dual super-modular category with $\hat{S}$ of the following form

$$\begin{pmatrix} 1 & 1 & 2 & \sqrt{6} \\ 1 & 1 & 2 & -\sqrt{6} \\ 2 & 2 & -2 & 0 \\ \sqrt{6} & -\sqrt{6} & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{B}$ has the same fusion rules as the centralizer $\langle f \rangle'$ of either fermion f in the modular category $\mathrm{SO}(12)_2$ (see the Appendix).

Proof. $\hat{N}_{11}^1 = \hat{N}_{11}^2 = \hat{N}_{12}^3 = \hat{N}_{22}^3 = \hat{N}_{33}^3 = 0$, $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^2 = 1$ and $\hat{N}_{23}^3 = 2$.

We can assume that $N_{22}^2 = 1$ and $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Similarly, we assume $N_{13}^3 = 1$ and $N_{13}^{f3} = 0$ by interchanging X_3 and fX_3 and X_1 and fX_1 simultaneously, if needed. Using the modified balancing equation on $\hat{S}_{23}$, we get $0 = (N_{23}^3 - N_{23}^{f3})\theta_3\sqrt{6}$. So we have $N_{23}^3 = N_{23}^{f3} = 1$. Now we have:

1. $f^{\otimes 2} = \mathbf{1}$,
2. $X_1^{\otimes 2} = \mathbf{1}$,
3. $X_2^{\otimes 2} = \mathbf{1} \oplus aX_1 \oplus bfX_1 \oplus X_2$,
4. $X_3^{\otimes 2} = \mathbf{1} \oplus X_1 \oplus X_2 \oplus fX_2$,
5. $X_1 \otimes X_2 = aX_2 \oplus bfX_2$,
6. $X_1 \otimes X_3 = X_3$,
7. $X_2 \otimes X_3 = X_3 \oplus fX_3$.

Computing $X_2 \otimes X_2 \otimes X_3$ in two ways gives us: $(2+a)X_3 \oplus (b+1)fX_3 = 2X_3 \oplus 2fX_3$. So we have $a = 0$ and $b = 1$.

□

Lemma 5.21. Let $\mathcal{B}$ be a self-dual super-modular category with

$$\hat{S} = \begin{pmatrix} 1 & \phi_1\phi_2 & \phi_1 & \phi_2 \\ \phi_1\phi_2 & 1 & -\phi_2 & -\phi_1 \\ \phi_1 & -\phi_2 & -1 & \phi_1\phi_2 \\ \phi_2 & -\phi_1 & \phi_1\phi_2 & -1 \end{pmatrix}.$$

Then $\mathcal{B}$ has the same fusion rules as $\mathrm{Fib} \boxtimes \mathrm{PSU}(2)_6$.

Proof. The naive fusion coefficients are: $\hat{N}_{11}^1 = \hat{N}_{33}^3 = 2$, $\hat{N}_{11}^2 = \hat{N}_{12}^3 = \hat{N}_{22}^2 = 1$, $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = 0$. As $\hat{N}_{22}^2 = N_{22}^2 + N_{22}^{f2} = 1$, we assume $N_{22}^2 = 1$ and $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Then we have $X_2^{\otimes 2} = \mathbf{1} \oplus X_2$, so X_2 generates a subcategory $\mathcal{F}$ with fusion rules like those of Fib, which is necessarily modular. Therefore $\mathcal{B} \cong \mathcal{F} \boxtimes \mathcal{D}$ where $\mathcal{D}$ is a super-modular category of rank 4([31, Theorem 3.13]). The classification results in [19] imply that $\mathcal{B}$ has the same fusion rules as $\mathrm{Fib} \boxtimes \mathrm{PSU}(2)_6$.

□

Lemma 5.22. Let $\mathcal{B}$ be a self-dual super-modular category with $\hat{S}$ of the following form

$$\begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -d_2 & d_3 & 1 \\ d_2 & d_3 & -1 & -d_1 \\ d_3 & 1 & -d_1 & d_2 \end{pmatrix},$$

where $d_1 = 1 + \sqrt{2} + \sqrt{2+\sqrt{2}}$, $d_2 = 1 + \sqrt{2} + \sqrt{2(2+\sqrt{2})}$ and $d_3 = 1 + \sqrt{2+\sqrt{2}}$. Then $\mathcal{B}$ has the same fusion rules as $\mathrm{PSU}(2)_{14}$.

Proof. The corresponding naive fusion coefficients are:$\hat{N}_{11}^1 = \hat{N}_{11}^3 = \hat{N}_{12}^3 = \hat{N}_{13}^3 = \hat{N}_{33}^3 = 1$, $\hat{N}_{11}^2 = \hat{N}_{12}^2 = \hat{N}_{22}^2 = \hat{N}_{22}^3 = 2$ and $\hat{N}_{23}^3 = 0$. Since $\hat{N}_{11}^1 = N_{11}^1 + N_{11}^{f1} = 1$, we can assume $N_{11}^1 = 1$ and $N_{11}^{f1} = 0$ by interchanging X_1 and fX_1 if necessary. Similarly, since $\hat{N}_{33}^3 = N_{33}^3 + N_{33}^{f3} = 1$, we can assume $N_{33}^3 = 1$ and $N_{33}^{f3} = 0$. Finally, we may use the X_2 versus fX_2 labeling ambiguity to assume that $N_{13}^2 = 1$. We have:

1. $f^{\otimes 2} = \mathbf{1}$,
2. $X_1^{\otimes 2} = \mathbf{1} \oplus X_1 \oplus aX_2 \oplus bfX_2 \oplus cX_3 \oplus dfX_3$, where $a+b=2$, $c+d=1$,
3. $X_2^{\otimes 2} = \mathbf{1} \oplus gX_1 \oplus hfX_1 \oplus lX_2 \oplus mfX_2 \oplus pX_3 \oplus qfX_3$, where $g+h=2$, $l+m=2$ and $p+q=2$,
4. $X_3^{\otimes 2} = \mathbf{1} \oplus rX_1 \oplus sfX_1 \oplus X_3$, where $r+s=1$,
5. $X_1 \otimes X_2 = aX_1 \oplus bfX_1 \oplus gX_2 \oplus hfX_2 \oplus X_3$,
6. $X_1 \otimes X_3 = cX_1 \oplus dfX_1 \oplus X_2 \oplus rX_3 \oplus sfX_3$,
7. $X_2 \otimes X_3 = X_1 \oplus pX_2 \oplus qfX_2$.

Computing $X_1 \otimes X_3 \otimes X_3$ in two ways and comparing the coefficients of X_1, fX_1, X_2 and fX_2, we have $c+r=2$, $d+s=0$, $ar+bs+1=c+p$ and $br+as=d+q$. Thus we have $c=r=1$, $d=s=0$, $a=p$ and $b=q$. Applying Corollary 3.18 to $\hat{S}_{23}$, we have $|d_1| = |d_1\theta_1 + (p-q)d_2\theta_2| \geq ||(p-q)d_2\theta_2| - d_1|$. If $|p-q|=2$, then $4.26 \approx d_1 \geq |2d_2 - d_1| \approx 5.79$, which is impossible. So we have $p=q=1$. Therefore $a=b=1$. Computing $X_2 \otimes X_3 \otimes X_3$ in two different ways and comparing the coefficients of X_2 and fX_2, we have $g=h=1$. Tensoring $X_2 \otimes X_2 \otimes X_3$ in two ways and comparing the coefficients of X_1 and fX_1, we have $l=1$ and $m=1$. □

Lemma 5.23. Let $\mathcal{B}$ be a self-dual super-modular category with

$$\hat{S} = \begin{pmatrix} 1 & d & 1+d & d^2-1 \\ d & -(1+d) & -1 & d^2-1 \\ 1+d & -1 & d & -(d^2-1) \\ d^2-1 & d^2-1 & -(d^2-1) & 0 \end{pmatrix},$$

where d is the largest real root of $x^3-3x-1=0$. Then $\mathcal{B}$ has the same fusion rules as $\mathrm{PSU}(2)_7 \boxtimes \mathrm{sVec}$.

Proof. We have $\hat{N}_{11}^1 = \hat{N}_{11}^2 = \hat{N}_{13}^3 = 0$ and $\hat{N}_{11}^3 = \hat{N}_{12}^2 = \hat{N}_{12}^3 = \hat{N}_{22}^2 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = \hat{N}_{33}^3 = 1$.

Notice that since $\hat{N}_{22}^2 = N_{22}^2 + N_{22}^{f2} = 1$, we can assume $N_{22}^2 = 1$ and $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Similarly, we can assume $N_{33}^3 = 1$, $N_{33}^{f3} = 0$, $\hat{N}_{22}^1 = 1$ and $\hat{N}_{22}^{f1} = 0$. We have

1. $f^{\otimes 2} = \mathbf{1}$,
2. $X_1^{\otimes 2} = \mathbf{1} \oplus aX_3 \oplus bfX_3$, where $a + b = 1$,
3. $X_2^{\otimes 2} = \mathbf{1} \oplus X_1 \oplus X_2 \oplus gX_3 \oplus hfX_3$, where $g + h = 1$,
4. $X_3^{\otimes 2} = \mathbf{1} \oplus lX_2 \oplus mfX_2 \oplus X_3$, where $l + m = 1$,
5. $X_1 \otimes X_2 = X_2 \oplus pX_3 \oplus qfX_3$, where $p + q = 1$,
6. $X_1 \otimes X_3 = aX_1 \oplus bfX_1 \oplus pX_2 \oplus qfX_2$,
7. $X_2 \otimes X_3 = pX_1 \oplus qfX_1 \oplus gX_2 + hfX_2 \oplus lX_3 \oplus mfX_3$.

Computing $X_1 \otimes X_1 \otimes X_2$ in two different ways and comparing the coefficients of X_2 and fX_2, we have $ag + bh = 1$, $bg + ah = 0$. Thus we have $a = g$ and $b = h$. Similarly, comparing the coefficients of X_3 and fX_3 in $X_1 \otimes X_1 \otimes X_3$ gives us $a = 1$ and $b = 0$. Computing $X_2 \otimes X_2 \otimes X_3$ and comparing the coefficients of X_3 and fX_3, we have $l = 1$ and $m = 0$. Computing $X_1 \otimes X_3 \otimes X_3$ in two different ways and comparing the coefficients for X_2 and fX_2, we have $p = 1$ and $q = 0$. Observing that the simple objects $\mathbf{1}, X_1, X_2$ and X_3 generate a fusion subcategory with the same fusion rules as $\mathrm{PSU}(2)_7$ we obtain the stated result. □

Lemma 5.24. Let $\mathcal{B}$ be a self-dual super-modular category. Suppose that the corresponding $\hat{S}$ has one of the following forms

$$\begin{pmatrix} 1 & \phi_1^2 & \phi_1 & \phi_1 \\ \phi_1^2 & 1 & -\phi_1 & -\phi_1 \\ \phi_1 & -\phi_1 & -1 & \phi_1^2 \\ \phi_1 & -\phi_1 & \phi_1^2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2^2 & \phi_2 & \phi_2 \\ \phi_2^2 & 1 & -\phi_2 & -\phi_2 \\ \phi_2 & -\phi_2 & -1 & \phi_2^2 \\ \phi_2 & -\phi_2 & \phi_2^2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \phi_1 & 1 & \phi_1 \\ \phi_1 & -1 & \phi_1 & -1 \\ 1 & \phi_1 & -1 & -\phi_1 \\ \phi_1 & -1 & -\phi_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2 & 1 & \phi_2 \\ \phi_2 & -1 & \phi_2 & -1 \\ 1 & \phi_2 & -1 & -\phi_2 \\ \phi_2 & -1 & -\phi_2 & 1 \end{pmatrix},$$

then $\mathcal{B}$ has the same fusion rules as $\mathrm{Fib} \boxtimes \mathrm{Fib} \boxtimes \mathrm{sVec}$, $[\mathrm{PSU}(2)_6 \boxtimes \mathrm{PSU}(2)_6]_{\mathbb{Z}_2}$, $\mathrm{Sem} \boxtimes \mathrm{PSU}(2)_6 \boxtimes \mathrm{sVec}$, or $\mathrm{Sem} \boxtimes \mathrm{Fib} \boxtimes \mathrm{sVec}$, respectively.

Proof. Consider the first $\hat{S}$-matrix. We have $\hat{N}_{11}^1 = \hat{N}_{11}^2 = \hat{N}_{11}^3 = \hat{N}_{12}^3 = \hat{N}_{22}^2 = \hat{N}_{33}^3 = 1$ and $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = 0$. Without loss of generality, we may assume $N_{22}^2 = 1$, $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Thus $X_2^{\otimes 2} = \mathbf{1} \oplus X_2$, so X_2 generates a subcategory $\mathcal{F}$ with fusion rules like those of Fib, which is necessarily modular. In particular $\mathcal{B} \cong \mathcal{F} \boxtimes \mathcal{D}$, where $\mathcal{D}$ is a super-modular category of rank 4. The classification results of [19] now imply that $\mathcal{B}$ has the same fusion rules as $\mathrm{Fib} \boxtimes \mathrm{Fib} \boxtimes \mathrm{sVec}$.

For the second $\hat{S}$-matrix, we have that the associated naive fusion coefficients are $\hat{N}_{11}^1 = 4$, $\hat{N}_{11}^2 = \hat{N}_{11}^3 = \hat{N}_{22}^2 = \hat{N}_{33}^3 = 2$, $\hat{N}_{12}^3 = 1$, $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = 0$. We may assume $N_{12}^3 = 1$ and $N_{12}^{f3} = 0$ by interchanging X_3 and fX_3 if necessary. Using Corollary 3.18 on $\hat{S}_{12}$ gives

$$-\theta_1\theta_2\phi_2 = (N_{12}^1 - N_{12}^{f1})\phi_2^2\theta_1 + \phi_2\theta_3.$$

Dividing by ϕ_2, we have

$$-\theta_1\theta_2 = (N_{12}^1 - N_{12}^{f1})\phi_2\theta_1 + \theta_3.$$

Taking absolute value on both sides, we get

$$1 = \left|(N_{12}^1 - N_{12}^{f1})\phi_2\theta_1 + \theta_3\right| \geq \left||(N_{12}^1 - N_{12}^{f1})\phi_2| - 1\right|.$$

So we must have $N_{12}^1 = N_{12}^{f1} = 1$. Similarly, applying Corollary 3.18 to $\hat{S}_{33}$ and $\hat{S}_{13}$ gives

$$-\theta_3^2 = 1 + (N_{33}^3 - N_{33}^{f3})\phi_2\theta_3, \quad -\theta_1\theta_3\phi_2 = (N_{13}^1 - N_{13}^{f1})\phi_2^2 + \phi_2\theta_2$$

and we get $N_{33}^3 = N_{33}^{f3} = 1$ and $N_{11}^3 = N_{11}^{f3} = 1$. A parallel calculation for $\hat{S}_{22}$ yields $N_{22}^2 = N_{22}^{f2} = 1$. By using Corollary 3.18 again for $\hat{S}_{11}$, we get

$$\theta_1^2 = (N_{11}^1 - N_{11}^{f1})\phi_2^2\theta_1 + 1.$$

The potential choices of (N_{11}^1, N_{11}^{f1}) are $(2,2)$, $(4,0)$, $(0,4)$, $(1,3)$ and $(3,1)$, but since $\phi_2^2 > 2$ the only possibility is $(2,2)$. This category has the same fusion rules as $[\mathrm{PSU}(2)_6 \boxtimes \mathrm{PSU}(2)_6]_{\mathbb{Z}_2}$, see the Appendix.

In the last two cases, observe that $\mathcal{B}$ must contain a modular subcategory of the form $\mathcal{C}(\mathbb{Z}_2, Q)$ by Lemma 3.25. Then $\mathcal{B} \cong \mathcal{C}(\mathbb{Z}_2, Q) \boxtimes \mathcal{D}$, where $\mathcal{D}$ is a rank 4 super-modular category. The result now follows from the classification in [19]. □

6. SUMMARY

The following conjecture was first asked in [19] when classifying super-modular categories up to rank 6. Our results provide further supporting evidence.

Conjecture 6.1. If $\mathcal{B}$ is a super-modular category and the corresponding $\hat{S}$ is the S-matrix for some modular category. Then $\mathcal{B}$ is split super-modular.

The twist equation(see [61, Theorem 2.14]) for modular categories is very helpful for classifying low modular categories. One can use this property to figure out the bound for the dimensions, thus could bound all the solutions. We do not have a parallel inequality for the fermionic quotient of super-modular categories so far. By mimicking the proof, we can have some inequalities for the dimensions of super-modular categories. However, these conditions are too weak to use for our classification process. The major obstacle here is that the S-matrix is degenerate.

The Gauss sum mentioned in Section 3.4 is an invariant for pre-modular categories. For super-modular, it is always 0. Higher Gauss sums were recently studied in [56]. It is an interesting question to study the higher Gauss sums for super-modular categories.

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APPENDIX A

MAGMA CODE

For our computational experiments we used the symbolic algebra software Magma [13]. In this appendix we give some basic pseudo-code and some sample Magma code to illustrate how we found the image of $\hat{\rho}(\Gamma_\theta)$ in our case study, so that the interested reader can do similar explorations. Given an integer m, the $(m+1)\times(m+1)$ $\hat{S}$ and $\hat{T}^2$ matrices obtained from $\mathrm{PSU}(2)_{4m+2}$ are given in equation (4.2). In order to use the Magma software we express the entries of $\hat{S}$ and $\hat{T}^2$ in the cyclotomic field $\mathbf{Q}(\omega)$, where ω is an $(8m+8)$-th root of unity. For this we must write $\sin\left(\frac{(2i+1)(2j+1)\pi}{4m+4}\right)$ and $\sqrt{2(m+1)}$ in terms of ω for which we use the result of generalized form of quadratic Gauss sums [7].

Here is the pseudocode to find $\hat{\rho}(\Gamma_\theta)$ for $\mathrm{PSU}(2)_{4m+2}$:

algorithm projective image:

input: integer m

output: $\hat{\rho}(\Gamma_\theta)$

set K to be the cyclotomic field $\mathbf{Q}(\omega)$, where ω is an $(8m+8)$-th root of unity.

set $M = 2(m+1)$

initialize S and T2 to be $(m+1)\times(m+1)$ zero matrices over K.

initialize $\alpha = 0$.

step 1: calculate α

if $M \equiv 0 \pmod 4$ **return** $\alpha = \sum_{n=0}^{M-1} \omega^{4n^2}/(1+\omega^M)$

else Consider $M/2 = m+1 \pmod 4$. Notice there are only two cases: $m+1 \equiv 1$ (mod 4) and $m+1 \equiv 3 \pmod 4$.

if $m+1 \equiv 1 \pmod 4$ **return** $\alpha = \frac{\omega^{m+1}-\omega^{-(m+1)}}{\omega^{2m+2}} \sum_{n=0}^{m} \omega^{8n^2}$

else return $\alpha = \frac{\omega^{m+1}-\omega^{-(m+1)}}{\omega^{2m+2}} \sum_{n=0}^{m} \omega^{8n^2}/(\omega^M)$

return $\alpha = 2/\alpha$.

step 2: define the entries

for $1 \leq i, j \leq m+1$, $S_{i,j} = \alpha \dfrac{\omega^{(2i-1)(2j-1)} - \omega^{-(2i-1)(2j-1)}}{2(\omega^M)}$ and $T2_{j,j} = \omega^{(2(j-1))^2+4(j-1)}$

step 3: find the projective image

set A to be the matrix group generated by S and $T2$ defined above, and ZK the group of scalar matrices over K. The projective image of A is then $A/(ZK \cap A)$.

The following code can be used in Magma [13] to find the $\hat{\rho}(\Gamma_\theta)$ in this case, and slight modifications will give the other headings of Table 4.1:

```
m:=1;
K<w>:= CyclotomicField(8*m+8);
GL:=GeneralLinearGroup(m+1,K);
M:=2*(m+1);
alpha:=0;
if M mod 4 eq 0 then
  for n:=0 to M-1 do
    alpha:=alpha + w^(4*(n^2));
  end for;
  alpha:=alpha/(w^M+1);
else
  if (m+1) mod 4 eq 1 then
    for n:=0 to m do
      alpha:= alpha + w^(8*n^2);
    end for;
```

```
else
  for n:=0 to m do
    alpha:=alpha + w^(8*(n^2));
  end for;
  alpha:=alpha/(w^M);
  end if;
  alpha:=((w^(m + 1) - w^(-(m + 1)))/(w^(2*m + 2)))*alpha;
end if;
alpha:=2/alpha;
S:=ZeroMatrix(K,m+1,m+1);
for i:=1 to m+1 do
   for j:=1 to m+1 do
     S[i,j]:=(w^((2*i-1)*(2*j-1))-w^(-(2*i-1)*(2*j-1)))/(2*(w^M));
     S[i,j]:=S[i,j]*alpha;
   end for;
end for;
T2:=ZeroMatrix(K,m+1,m+1);
for j:=1 to m+1 do
   T2[j,j]:=w^((2*(j-1))^2+4*(j-1));
end for;
A:=MatrixGroup<m+1,K|S,T2>;
ZK:=MatrixGroup<m+1,K|w*IdentityMatrix(K,m+1)>;
F:=(A/(A meet ZK));
```

APPENDIX B

REALIZATIONS OF THE FUSION RULES

Here we record the data for some of the realizations of the super-modular categories that appear in this article, both modular and super-modular, as well as the families of categories in which they reside. We write the T-matrix as an n-tuple with the understanding that these are the diagonal entries.

B.1 Pointed Modular Categories

Pointed braided fusion categories are classified, see [31]. They correspond to pairs (A, Q), where Q is a symmetric quadratic form on A (with values in $U(1)$). The fusion rules of $\mathcal{C}(A, Q)$ are the same as the multiplication in A, and the S- and T-matrices are determined by Q as follows: $S_{a,b} = \frac{Q(a+b)}{Q(a)Q(b)}$ and $\theta_a = Q(a)$. If the symmetric bilinear form given by $S_{a,b}$ is non-degenerate then $\mathcal{C}(A, Q)$ is modular.

For example the semion theory $\mathrm{Sem} = \mathcal{C}(\mathbb{Z}_2, Q)$ that appears in our classification has the following modular data: $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and $T = (1, i)$.

B.2 $\mathbf{PSU(2)}_k$

The rank $k+1$ modular category $\mathrm{SU}(2)_k$ obtained from $U_q\mathfrak{sl}_2$ at $q = e^{\pi i/(2+k)}$ contains the subcategory $\mathrm{PSU}(2)_k$ whose simple objects have even labels ("integer spin" in the physics literature). Denote by ϖ the fundamental weight of type A_1, so that X_ϖ tensor generates $\mathrm{SU}(2)_k$. The object labeled by $\frac{k}{2}\varpi$ is always invertible. When $k \equiv 2 \pmod 4$ the category $\mathrm{PSU}(2)_k$ is super-modular with $f = X_{\frac{k}{2}\varpi}$, when $4 \mid k$, there is a boson $b = X_{\frac{k}{2}\varpi}$ in $\mathrm{PSU}(2)_k$, and when k is odd, $\mathrm{PSU}(2)_k$ is modular, with $X_{\frac{k}{2}\varpi}$ a semion (not in $\mathrm{PSU}(2)_k$.)

The (modular) Fibonacci theory $\mathrm{Fib} = \mathrm{PSU}(2)_3^{rev}$ as well as $\mathrm{PSU}(2)_7$ appear in our classification, and the data can be found in [61].

Some low rank super-modular categories that appear in this article are:

- $\mathrm{PSU}(2)_6$ with data:

$$S = \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } T = (1, i) \otimes (1, -1).$$

- $\mathrm{PSU}(2)_{10}$ with data:

$$S = \begin{pmatrix} 1 & 2+\sqrt{3} & 1+\sqrt{3} \\ 2+\sqrt{3} & 1 & -1-\sqrt{3} \\ 1+\sqrt{3} & -1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } T = (1, -1, e^{\pi i/3}) \otimes (1, -1).$$

- $\mathrm{PSU}(2)_{14}$ with data:

$$S = \begin{pmatrix} 1 & 1+x & 1+\sqrt{2}+x & 1+\sqrt{2}+\sqrt{2}x \\ 1+x & 1+\sqrt{2}+\sqrt{2}x & 1 & -1-\sqrt{2}-x \\ 1+\sqrt{2}+x & 1 & -1-\sqrt{2}-\sqrt{2}x & 1+x \\ 1+\sqrt{2}+\sqrt{2}x & -1-\sqrt{2}-x & 1+x & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where $x = \sqrt{2+\sqrt{2}}$ and $T = (1, e^{i\pi/4}, e^{3i\pi/4}, -i) \otimes (1, -1)$.

The full sequence of super-modular categories $\mathrm{PSU}(2)_{4m+2}$ was studied in [16, 11], where the modular data can be found. If we order the simple objects $[\mathbf{1}, X_1, \ldots, X_{r-1}, fX_{r-1}, \ldots, fX_1, f] = [Y_0, \ldots, Y_{2(r-1)}]$ the fusion rules are completely determined by the rule $Y_1 \otimes Y_k \cong Y_{k-1} \oplus Y_k \oplus Y_{k+1}$ for $0 < k < 2(r-1)$ and the obvious rules involving $Y_{2(r-1)} = f$ and $Y_0 = \mathbf{1}$.

B.3 Other Examples

The following are spin modular categories coming from quantum groups with fermion f so that the subcategory $\langle f \rangle'$ is super-modular, where $r, m \in \mathbb{N}$:

- $\mathrm{SU}(4k+2)_{4m+2}$,

- $\mathrm{SO}(2k+1)_{2m+1}$,

- $Sp(2r)_m$ with $rm = 2 \pmod 4$,

- $\mathrm{SO}(2r)_m$ with $r = 2 \pmod 4$ and $m = 2 \pmod 4$,

- $(E_7)_{4m+2}$.

The pointed sub-category of the rank 13 modular category $\mathrm{SO}(12)_2$ is $\mathrm{sVec} \boxtimes \mathrm{sVec}$ and hence contains two fermions labeled by $2\varpi_5$ and $2\varpi_6$, where ϖ_i are the fundamental weights of type D_6. The centralizer of either of these fermions is super-modular and has modular data:

$$S := \begin{pmatrix} 1 & 1 & 2 & \sqrt{6} \\ 1 & 1 & 2 & -\sqrt{6} \\ 2 & 2 & -2 & 0 \\ \sqrt{6} & -\sqrt{6} & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ and $T = (1, 1, e^{2\pi i/3}, e^{3\pi i/8}) \otimes (1, -1)$. If we label the simple objects of dimension $\sqrt{6}$ by X_3 and fX_3 then the fusion rules are determined by $X_3^{\otimes 2} \cong \mathbf{1} \oplus X_1 \oplus X_2 \oplus fX_2$, $X_1^{\otimes 2} \cong \mathbf{1}$, $X_2^{\otimes 2} \cong \mathbf{1} \oplus fX_1 \oplus X_2$ and $X_2 \otimes X_3 \cong X_3 \oplus fX_3$.

Finally we observe that if $(\mathcal{C}_1, f_1)$ and $(\mathcal{C}_2, f_2)$ are spin modular categories, then $(f_1, f_2) \in \mathcal{C}_1 \boxtimes \mathcal{C}_2$ is a boson and hence can be condensed to obtain a new spin modular category $([\mathcal{C}_1 \boxtimes \mathcal{C}_2]_{\mathbb{Z}_2})_0$, where we de-equivariantize by $\mathrm{Rep}(\mathbb{Z}_2) \cong \langle (f_1, f_2) \rangle$ and then take the trivial component of the corresponding $\mathbb{Z}_2$-grading. For example applying this to $\mathrm{PSU}(2)_6$ we obtain the prime rank 8 example $(\mathrm{PSU}(2)_6 \boxtimes \mathrm{PSU}(2)_6)_{\mathbb{Z}_2}$ with data:

$$S := \begin{pmatrix} 1 & 3+2\sqrt{2} & 1+\sqrt{2} & 1+\sqrt{2} \\ 3+2\sqrt{2} & 1 & -1-\sqrt{2} & -1-\sqrt{2} \\ 1+\sqrt{2} & -1-\sqrt{2} & -1 & 3+2\sqrt{2} \\ 1+\sqrt{2} & -1-\sqrt{2} & 3+2\sqrt{2} & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ and $T = (1, -1, i, i) \otimes (1, -1)$. The fusion rules may be readily determined from those of $\mathrm{PSU}(2)_6$ by condensing the boson $b := (f_1, f_1)$. Notice that $b \otimes X \ncong X$ for any simple X so that there is no ambiguity in labeling the objects in the de-equivariantization. Setting $f := [(f_1, \mathbf{1})] = [(\mathbf{1}, f_1)]$ we have

$$X_1^{\otimes 2} \cong \mathbf{1} \oplus 2(X_1 \oplus fX_1) \oplus X_2 \oplus fX_2 \oplus X_3 \oplus fX_3, \quad X_1 \otimes X_2 \cong X_3 \oplus X_1 \oplus fX_1$$

$$X_1 \otimes X_3 \cong X_2 \oplus X_3 \oplus fX_3, \quad X_2 \otimes X_3 \cong X_1, \text{ and } X_2^{\otimes 2} \cong \mathbf{1} \oplus X_2 \oplus fX_2$$

from which all fusion rules can be recovered.