Analytic aspects of geometric quantization

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Abstract

We give a brief survey of our work (joint with Youliang Tian) on the theme of

"quantization commutes with reduction"

as well as the recent generalization (joint with V. Mathai) on the noncompact setting of proper cocompact actions of non-compact groups on non-compact spaces.

§1. Quantization on compact symplectic manifolds

Let (M, ω) be a closed symplectic manifold. Let J be an almost complex structure on TM such that

$$g^{TM}(v, w) = \omega(v, Jw)$$

defines a Riemmannian metric on TM.

Let E be a Hermitian vector bundle over M admitting a Hermitian connection ∇^E.

Then one can construct a (twisted) **spin**^{c} Dirac operator

$$D^E : \Gamma\left(\Lambda^{0,*}(T^*M) \otimes E\right) \to \Gamma\left(\Lambda^{0,*}(T^*M) \otimes E\right).$$

Remark. When (M, ω, J) is **Kähler** and E is a holomorphic vector bundle over M, one has

$$D^E = \sqrt{2}\left(\overline{\partial}^E + \left(\overline{\partial}^E\right)^*\right).$$

Let $D^E\pm$ be the restriction of D^E:

$$D^E\pm : \Gamma\left(\Lambda^{0,\frac{\text{even}}{\text{odd}}}(T^*M) \otimes E\right) \to$$

$$\Gamma\left(\Lambda^{0,\frac{\mathrm{odd}}{\mathrm{even}}}(T^*M)\otimes E\right).$$

Then

$$D^E = D^E + + D^E -, \qquad \left(D^E +\right)^* = D^E - .$$

Define the **quantization space** of E to be the formal difference

$$Q(E) = \left(\ker D^E +\right) - \left(\operatorname{coker} D^E +\right)$$

$$= \left(\ker D^E +\right) - \left(\ker D^E -\right).$$

It can be viewed as an element in $K(\cdot)$ (point view due to Bott), and does not depend on the choice of J and the metric and connection on E.

Atiyah-Singer index theorem.

$$\dim Q(E) = \operatorname{ind} D^E + = \langle \operatorname{Td}(TM)\operatorname{ch}(E), [M]\rangle$$

$$= \int M \det\left(\frac{e^{\frac{\sqrt{-1}R^{TM}}{2\pi}}}{1-e^{\frac{-\sqrt{-1}R^{TM}}{2\pi}}}\right)\operatorname{Tr}\left[\exp\left(\frac{\sqrt{-1}R^E}{2\pi}\right)\right],$$

where R^{TM} is the curvature of the Levi-Civita connection associated to g^{TM}, $R^E = (\nabla^E)^2$ is the curvature of ∇^E.

Remark. When (M, ω, J) is **Kähler** and E is holomorphic, then

$$Q(E) = H^{0,\mathrm{even}}(M, E) - H^{0,\mathrm{odd}}(M, E).$$

§2. Hamiltonian action and symplectic reduction

Let G be a compact connected Lie group. Let $\mathbf{g}$ be the Lie algebra of G.

Assume G acts on (M, ω) in a Hamiltonian way, and preserves J.

Then there exists a G-equivariant **moment map**

$$\mu : M \to \mathbf{g}^*$$

such that for any $V \in \mathbf{g}$, one has

$$iVM\omega = d\langle \mu, V\rangle,$$

where $VM \in \Gamma(TM)$ denote the vector field on M generated by $V \in \mathbf{g}$.

Clearly, G preserves $\mu^{-1}(0)$.

Definition. The Marsden-Weinstein

symplectic reduction space MG is defined to be

$$MG = \mu^{-1}(0)/G.$$

Basic assumption: $0 \in \mathbf{g}^*$ is a regular value of the moment map $\mu : M \to \mathbf{g}^*$. Then $\mu^{-1}(0)$ is a closed manifold.

For simplicity, also assume that G acts on $\mu^{-1}(0)$ freely, then MG is a closed manifold and carries an induced symplectic form ωG.

Moreover, J induces an almost complex structure JG on TMG such that $\omega G(v, JGw)$ determines a Riemannian metric g^{TMG} on TMG.

Remark. If (M, ω, J) is Kähler, then $(MG, \omega G, JG)$ is also Kähler.

§3. Pre-quantization and symplectic reduction

Let L be an Hermitian line bundle over M carrying an Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2 = \omega.$$

When such an L exists, we call (M, ω) **pre-quantizable**, and call L the pre-quantized line bundle.

We assume the existence of L now.

We make the assumption that the Hamiltonian G action **lifts to an action on** L, which preserves the Hermitian metric and Hermitian connection on L.

Then L descends to a pre-quantized line bundle LG over MG carrying a canonically induced Hermitian metric and Hermitian connection ∇^{LG}.

Remark. When (M, ω, J) is Kähler and L is a holomorphic line bundle over M, then LG is also holomorphic over MG.

§4. Geometric quantization commutes with symplectic reduction

Continue the discussion above.

Then the canonical spin^c Dirac operator D^L commutes with the induced G-action on $\Gamma(\Lambda^{(0,*)}(T^*M))$. Thus, G preserves $\ker D\pm^L$.

Let $(\ker D\pm^L)^G$ denote the G-invariant part in $\ker D\pm^L$.

Define the *reduction* of the quantization space $Q(L)$ of L to be

$$Q(L)^G = \left(\ker D+^L\right)^G - \left(\ker D-^L\right)^G.$$

Also recall that the quantization of LG on MG is defined by

$$Q\left(LG\right) = \left(\ker D+^{LG}\right) - \left(\ker D-^{LG}\right).$$

The Guillemin-Sternberg conjecture (1982):

$$\dim Q(L)^G = \dim Q\left(LG\right). \tag{$*$}$$

Remark. Tautologically, the above means "*Geometric quantization commutes with*

symplectic reduction".

Remark. Guillemin-Sternberg first proved in 1982 that when (M, ω, J) is Kähler and L is holomorphic,

$$\dim H^{(0,0)}(M, L)^G = \dim H^{(0,0)}\left(MG, LG\right)$$

and made the conjecture of generalization. The above formulation (*) of the conjecture was inspired by an observation of Bott.

When G is abelian, (*) was first proved by Guillemin (1995) in a special case, and later in general by Meinrenken (JAMS 1996) and Vergne (DMJ 1996) independently.

The remaining non-abelian case was proved by Meinrenken (Adv. in Math. 1998) by using the technique of *symplectic cut* of Lerman.

There are also approaches of Duistermaart-Guillemin-Meinrenken-Wu (for circle actions) and Jeffrey-Kirwan (for non-abelian group actions with certain extra conditions).

Remark. All of the above use the Atiyah-Bott-Segal-Singer equivariant index theorem in an essential way: first relate $\dim Q(L)^G$ to quantities on the fixed point set of the G-action, and then try to relate the later to quantities on the symplectic quotient (through symplectic cut or through the Jeffrey-Kirwan-Witten non-abelian localization formulas).

Natural question. Whether there is an approach relating $\dim Q(L)^G$ directly to $\dim Q(LG)$?

§5. A direct analytic approach

(with Youliang Tian)

We try to put the problem into the framework of an analytic Morse theory, analogous to what Witten did in the usual (real) case.

Let $\mathbf{g}^*$ be equipped with an AdG-invariant metric. Set

$$\mathcal{H} = |\mu|^2.$$

Let $X^{\mathcal{H}}$ be the associated Hamiltonian
vector field, i.e.,

$$iX^{\mathcal{H}}\omega = d\mathcal{H}.$$

Definition (Tian-Zhang, 1998) For any $T \in \mathbf{R}$, set

$$D^L T = D^L + \frac{\sqrt{-1}T}{2} c\left(X^{\mathcal{H}}\right) :$$

$$\Gamma\left(\Lambda^{0,*}(T^*M) \otimes L\right) \to \Gamma\left(\Lambda^{0,*}(T^*M) \otimes L\right).$$

Remark. If (M, ω, J) is Kähler and L is holomorphic, then one has

$$D^L T = \sqrt{2}\left(e^{\frac{-T\mathcal{H}}{2}}\overline{\partial}^L e^{\frac{T\mathcal{H}}{2}} + e^{\frac{T\mathcal{H}}{2}}\left(\overline{\partial}^L\right)^* e^{\frac{-T\mathcal{H}}{2}}\right).$$

This is an analogue of the Witten deformation in Morse theory, but now in a non-abelian context.

By using this deformation, one can then apply the analytic localization technique of Bismut-Lebeau to complete the proof of the Guillemin-Sternberg conjecture.

There are also many immediate generalizations arising from this analytic approach.

§6. Main idea of proof

Since μ is G-equivariant and the metric on $\mathbf{g}^*$ is AdG-invariant, $\mathcal{H} = |\mu|^2$ is a G-invariant function on M. Thus the associated Hamilton vector field $X^{\mathcal{H}}$ is a G-invariant vector field on M.

(*) For any $T \in \mathbf{R}$,

$$DT^L = D^L + \frac{\sqrt{-1}T}{2} c\left(X^{\mathcal{H}}\right)$$

commutes with the G-action and preserves $\Omega(M, L)^G$, the G-invariant subspace of

$$\Omega(M, L) = \Gamma\left(\Lambda^{0,*}\left(T^*M\right) \otimes L\right).$$

The main point is that when restricted to $\Omega(M, L)^G$, one has

$$\left(DT^L\right)^2 = \left(D^L\right)^2 + TA + 4\pi T\mathcal{H} + \frac{T^2}{4}\left|X^{\mathcal{H}}\right|,$$

where A is a bounded operator.

For simplicity, replace L by L^p, the p-th

tensor power of L. Then one has, when restricted on $\Omega(M, L^p)^G$,

$$\left(DT^{L^p}\right)^2 = \left(DT^{L^p}\right)^2 + TA + 4\pi pT\mathcal{H} + \frac{T^2}{4}\left|X^{\mathcal{H}}\right|,$$

where A is now bounded and not involve p.

Simple observation: If $\mathcal{H}(x) \neq 0$, then when $p > 0$ is large enough,

$$TA + 4\pi pT\mathcal{H} > 0.$$

Proposition. Take any open neighborhood U of $\mu^{-1}(0)$. Then there exists $T0 > 0$, $p0 > 0$, $C > 0$ such that for any $T \geq T0$, $p \geq p0$, $s \in \Omega(M, L^p)^G$ with $\mathrm{Supp}(s) \subset M \setminus U$, one has

$$\left\|DT^{L^p}s\right\|^2 \geq C\left(\|s\|1^2 + T\|s\|0^2\right).$$

This shows that outside U, the restriction of DT^{L^p} on $\Omega(M, L^p)^G$ is "highly invertible". Thus in order to study the kernel of it, one can reduce the problem to U which can be made sufficiently small around $\mu^{-1}(0)$.

One can then apply the Bismut-Lebeau technique in U to prove the quantization formula (at least when $p > 0$ is large enough).

Remark. More refined analysis on $M \setminus U$ works for $p = 1$ in this (compact group action) case.

§7. The non-compact group action case

The original Guillemin-Sternberg conjecture was stated and proved for the case where a compact Lie group acts on a compact manifold.

It is natural to ask the possibility of generalizations to non-compact cases.

Paradan studied the case where G **is compact and** M **is non-compact**, and proved a quantization formula under some extra condition and studied the relations with representation theory of semi-simple groups. A general conjecture in this direction, under the condition that **the moment map is proper**, was proposed by **Vergne** in her

ICM2006 plenary lecture.

In a recent joint work with Xiaonan Ma, we solved this conjecture of Vergne in full generality.

However, in this talk I will present another kind of generalization of the Guillemin-Sternberg conjecture in the noncompact setting.

This is the setting proposed by **Hochs** and **Landsman**, where both G and M are non-compact but M/G is compact.

In what follows, we will outline a generalization of the Guillemin-Sternberg conjecture in the framework of Hochs-Landsman.

This is a joint work with **Varghese Mathai**.

Let G be a locally compact Lie group, and M a locally compact symplectic manifold.

We assume that G acts on M properly and cocompactly, that is M/G is compact and the map

$$G \times M \to M \times M,$$

$$(g, x) \mapsto (x, gx)$$

is proper (i.e. the inverse image of a compact subset is compact).

We make the other assumptions as in the compact case (line bundle L, moment map μ, regularity of μ at $0 \in \mathbf{g}$, G acts on $\mu^{-1}(0)$ freely, etc).

Recall that in the compact case, the quantization formula takes the form

$$\dim Q(L)^G = \dim Q(LG)$$

with

$$Q(L) = \ker D+^L - \ker D-^L .$$

Here since M/G is compact, $MG = \mu^{-1}(0)/G$ is also compact so the right hand side $Q(LG)$ is well defined.

However, since M is now noncompact, $\ker D\pm^L$ might be of infinite dimension.

In their approach, Hochs and Landsman proposed an interpretation of a possible candidate of the left-hand side in the noncompact case by using analytic K-homology in noncommutative geometry.

Our first observation: using the fact that M/G is compact, we can show that even though, $\ker D\pm^L$ might be of infinite dimension, their G-invariant subspaces are of finite dimensions. That is

$$\dim \left(\ker D\pm^L\right)^G < +\infty.$$

Moreover, one can still define naturally a G-invariant index ind$G(D^L+)$, such that for following quantization formula holds in this noncompact setting.

Theorem. (Mathai-Zhang, 2008) There exists $p0 > 0$ such that for any $p \geq p0$,

$$\mathrm{ind}G\left(D^{L^p}+\right) = \dim Q\left(LG^p\right).$$

Moreover, if $\mathbf{g}^*$ admits an AdG-invariant

metric, then one can take $p0 = 1$.

Also, when G is **unimodular**, then indeed,

$$\mathrm{ind}G\left(D^L+\right) = \dim\left(\ker D+^L\right)^G - \dim\left(\ker D-^L\right)^G,$$

which makes the above quantization

formula an explicit extension of the original formulation of the Guillemin-Sternberg conjecture.

§8. The definition of the G-invariant index

Consider the restriction of D^L to the G-invariant subspace

$$D^L : \Omega(M,L)^G \to \Omega(M,L)^G.$$

Since G acts on M properly and the quotient space M/G is compact, their is a compact subset Y of M such that $M = G(Y)$.

It is clear that any section in $\Omega^{0,*}(M,L)^G$ is determined by its restriction on Y. This allows us to reduce the analysis on $\Omega(M,L)^G$ to the analysis near Y.

To be more precise, let $U \subset U'$ be two open neighborhoods of Y in M such that the

closure $\overline{U}$ is compact in U', while the closure $\overline{U}'$ of U' is compact in M.

Let $f : M \to [0,1]$ be a smooth function such that $f|U = 1$, Supp$(f) \subset U'$.

The existence of f is clear.

Let $\|\cdot\|0$ be the standard L^2-norm on $\Omega^{0,*}(M,L)$.

We also fix a (G-invariant) first Sobolev norm $\|\cdot\|1$ on $\Omega^{0,*}(M,L)$.

Let $\mathbf{H}^i f(M,L)^G$, $i = 0$, 1, be the completions of

$$\left\{fs : s \in \Omega^{0,*}(M,L)^G\right\}$$

under the norms $\|\cdot\|i$ respectively.

Let $Pf : L^2(\Omega^{0,*}(M,L)) \to \mathbf{H}^0 f(M,L)^G$ denote the obvious orthogonal projection.

Theorem (Mathai-Zhang 2008). For any G-equivariant first order elliptic differential operator D acting on $\Omega^{0,*}(M,L)$, the

induced operator

$$PfD : \mathbf{H}^1 f(M,L)^G \to \mathbf{H}^0 f(M,L)^G$$

is Fredholm. Moreover,

$$\mathrm{ind}G\left(D+^L\right) := \mathrm{ind}\left(PfD+^L\right)$$

does not depend on the choice of f.

Proof. One has

$$PfD(fs) = fDs + Pf\left([D,f]s\right).$$

Then fD induces a Fredholm operator, while $Pf[D,f]$ induces a compact operator. Q.E.D.

Remark. It is interesting to note that by the above formula, PfD is <u>not</u> a differential operator.

Example If $G = \Gamma$ is discrete and acts on M freely, then one identifies

$$\Omega(M,L)^\Gamma = \Omega\left(M\Gamma, L\Gamma\right)$$

trivially. So the analysis on the left reduces to the analysis on the right. The key point here is that in the left we do not impose any (global on M) L^2-condition.

Remark. If G is unimodular, one can further prove that

$$\mathrm{ind}G\left(D^L+\right) = \dim\left(\ker D+^L\right)^G - \dim\left(\ker D-^L\right)^G.$$

Remark. Bunke showed us that indeed, our G-invariant index $\mathrm{ind}G(D)$ admits a noncommutative KK-theory interpretation.

§9. Proof of the quantization formula

Now in order to study $\mathrm{ind}G(D+^L)$ by our analytic method, another difficulty arises:

Since for noncompact G, $\mathbf{g}^*$ might not

admit an AdG-invariant metric, thus the function $\mathcal{H} = |\mu|^2$ (and then the associated Hamiltonian vector field $X^{\mathcal{H}}$) need not be a G-invariant function (vector) on M.

Consequently,

$$DT^L = D^L + \frac{\sqrt{-1}T}{2} c\left(X^{\mathcal{H}}\right)$$

might **not** commute with G.

To solve this difficulty: take "average" to get a G-equivariant one.

Since G acts properly on M with compact quotient, there exists a smooth, non-negative, compactly supported cut-off function c on X such that

$$\int Gc(gx)^2 dg = 1$$

for any $x \in X$.

Let

$$XG^{\mathcal{H}} = \int Gc(gx)^2 Xg^{\mathcal{H}} dg$$

denote the averaged G-*invariant* vector field on M, where $Xg^{\mathcal{H}}$ denotes the pullback of $X^{\mathcal{H}}$ by g.

For any $T \geq 0$, set

$$D^{L^p}T = D^{L^p} + \frac{\sqrt{-1}T}{2} c\left(XG^{\mathcal{H}}\right),$$

where for later convenience we replace L by L^p.

Then it is <u>G-equivariant</u> (i.e. commutes with the G-action).

One then has, for any $T \geq 0$,

$$\mathrm{ind}G\left(D^{L^p}+\right) = \mathrm{ind}G\left(D^{L^p}+, T\right).$$

Key formula: when restricting on $\Omega(M, L^p)^G$, one has

$$\left(D^{L^p}T\right)^2 = \left(D^{L^p}\right)^2 + TB + 4p\pi T\mathcal{H}G + \frac{T^2}{4}\left|XG^{\mathcal{H}}\right|^2,$$

where $\mathcal{H}G$ is defined by

$$\mathcal{H}G(x) = \int Gc(gx)^2 \mathcal{H}(gx) dg$$

and B is bounded and does not involve p.

Key point: If $\mu(x) \neq 0$, then $\mathcal{H}G(x) > 0$.

When restricted to neighborhoods near Y (recall that Y is compact and $G(Y) = M$), one sees that the techniques in the compact group action case applies, and one can take $p > 0$ large enough to get

$$\mathrm{ind}G\left(D^{L^p}+\right) = \mathrm{ind}\left(D+^{LG^p}\right)$$

which is the quantization formula.

Technical remark. Here $X^{\mathcal{H}}G$ might not be the Hamiltonian vector field associated to $\mathcal{H}G$, thus the refined estimate in the compact group action case does not apply. This explains partly that our quantization formula in the noncompact case need to assume that $p > 0$ is large enough. This fits with a remark of Hochs and Landsman that the quantization formula (for $p = 1$) might not hold for non-unimodular groups.

In particular, if $\mathbf{g}^*$ admits an AdG-invariant metric, then one can take $p = 1$.

Summary: We have two kinds of generalizations of the original Guillemin-Sternberg geometry quantization conjecture to the non-compact settings:

1. The Vergne conjecture for the case where the group is compact and the space is non-compact, with the condition that the moment map is proper: recently solved together with Xiaonan Ma (arXiv:0812.3989);

2. The Hochs-Landsman conjecture for the case where both the group and the space are non-compact, while the action is proper and cocompact: solved up to a power of line bundle together with Mathai (arXiv:0806.3138).

Potential applications: Representation theory for noncompact groups (e.g. semi-simple Lie groups).

Thanks!