Categorical quantum channels

Attacking the quantum version of Birkhoff's theorem with category theory

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For more information on nLab, see http://ncatlab.org/nlab.

Outline

What is a quantum channel?

Definition (Quantum channel)

A *quantum channel* is anything that can be modeled as a **completely positive, trace preserving** map between spaces of operators on Hilbert spaces,

$$\Phi : Mat(n \times n, \mathbb{C}) \to Mat(m \times m, \mathbb{C})$$

- Quantum channels can carry both quantum and classical information.
- Quantum channels have a Kraus decomposition

$$\Phi(A) = \sum_{i \in I} E_i A E_i^\dagger .$$

where the $\{E_i\}$ are the Kraus operators.

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Categories

Definition (Category)

A *category* is a mathematical structure that consists of *Objects*: $A, B, C, \ldots$ and *Arrows*: $f, g, h, \ldots$ (more formally called *morphisms*) where

- each arrow has a *domain* and *codomain*;
- the arrows exhibit **compositeness**, **associativity**, and a **unit** law; and
- for each object there is an associated identity arrow.

Definition (Functor)

A *functor* $F : \mathbf{C} \to \mathbf{D}$ between categories **C** and **D** is a mapping from objects to objects and arrows to arrows such that the functors exhibit **compositeness**, **associativity**, and a **unit** law.

Dagger categories

Definition (Dagger category)

A *dagger category* is a category **C** together with an involutive, identity-on-objects, contravariant functor $\dagger : \mathbf{C} \to \mathbf{C}$, i.e. to every morphism $f : A \to B$ we associate a morphism $f^\dagger : B \to A$, called the *adjoint* of f, such that for all $f : A \to B$ and $g : B \to C$, $\mathrm{id}_A^\dagger = \mathrm{id}_A$, $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$, and $f^{\dagger\dagger} = f$.

Definition (Unitary and self-adjoint maps)

In a dagger category, a morphism $f : A \to B$ is called *unitary* if it is an isomorphism and $f^{-1} = f^\dagger$. A morphism $f : A \to A$ is called *self-adjoint* or *Hermitian* if $f = f^\dagger$.

Note that this formalism encapsulates various quantum processes and protocols including no-cloning, teleportation and so on as detailed in the work of Coecke, Selinger, and others.

Defining QChan

We define the category of all quantum channels **QChan** $\subset$ **Vect**$_{\mathbb{C}}$ (where **Vect**$_{\mathbb{C}}$ is the category of complex vector spaces) whose

- objects are the vector spaces $Mat(n \times n, \mathbb{C})$ for all $n \in \mathbb{N}$;
- morphisms are completely positive and trace-preserving linear maps $\Phi : Mat(n \times n, \mathbb{C}) \to Mat(m \times m, \mathbb{C})$;
- composition of morphisms is, of course, the composition in **Vect**, i.e. the ordinary composition of linear maps.

Any category equipped with some notion of a 'tensor product' is known as a monoidal category.

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Properties of **QChan**

We can form subcategories out of **QChan** fairly easily. For example,

- $\mathbf{QChan}_2$ whose object is the vector space $Mat(2 \times 2, \mathbb{C})$.
- $\mathbf{QChan}_4$ whose object is the vector space $Mat(4 \times 4, \mathbb{C})$.

Notice that the mapping from $\mathbf{QChan}_2$ to $\mathbf{QChan}_4$ is, of course, the normal tensor product which is a functor:

$$- \otimes - : \mathbf{QChan}_2 \to \mathbf{QChan}_4$$

Any category whose morphisms are of the form

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Properties of **QChan**

Call the input to $\mathbf{QChan}_2$ ρ and its output $T(\rho)$. $\mathbf{QChan}_2$ and $\mathbf{QChan}_4$ thus form an arrow category $\mathbf{QChan}^{\rightarrow}$ of **QChan** that has a morphism, $\mathcal{F} = (F, F')$ that is a commutative square

$$\begin{tikzcd} \rho \arrow[r, "F"] \arrow[d, "f"'] & \rho \otimes \rho \arrow[d, "f'"] \\ T \arrow[r, "F'"] & T \otimes T \end{tikzcd}$$

Properties of **QChan**

This can be easily generalized to tensoring over n,

$$\begin{tikzcd}
\rho \arrow[r, "F"] \arrow[d, "f_0"'] & \rho \otimes \rho \arrow[r, "F"] \arrow[d, "f_1"] & \rho \otimes \rho \otimes \rho \arrow[r, "F"] \arrow[d, "f_2"] & \cdots \arrow[r, "F"] & \rho^{\otimes n} \arrow[d, "f_n"] \\
T \arrow[r, "F'"] & T \otimes T \arrow[r, "F'"] & T \otimes T \otimes T \arrow[r, "F'"] & \cdots \arrow[r, "F'"] & T^{\otimes n}
\end{tikzcd}$$

which represents a cumulative hierarchy.

Birkhoff's theorem

Theorem (Birkhoff-von Neumann)

Doubly stochastic matrices of order n are said to form the convex hull of permutation matrices of the same order where the latter are the vertices (extreme points) of the former, i.e. doubly stochastic matrices are convex combinations of permutation matrices.

In the quantum context, doubly stochastic matrices become doubly stochastic channels, i.e. completely positive maps preserving both the trace and the identity. Quantum mechanically we understand the permutations to be the unitarily implemented channels. That is, we expect doubly stochastic quantum channels to be convex combinations of unitary channels.

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Unfortunately, it is known that some quantum channels *violate* this extension of Birkhoff's theorem.

Possible approaches to resolving this issue:

- One approach that has been suggested by A. Winter is based on the idea that large tensor powers of a channel may be easier to represent as a convex hull of permutation matrices, because one need not use only product unitaries in the decomposition.
- A purely categorical approach.
- Some mixture of the two.

Solutions presently exist for all qubit channels and all invertible channels including those that are invertible with environmental assistance.

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A categorical approach to Birkhoff's theorem

Thus we define a (dagger) category of unitarily implemented channels, $\mathbf{QChan}_{\mathbb{U}} \subset \mathbf{QChan}$. The main thrust of a purely categorical approach to this problem is to examine the difference between $\mathbf{QChan}$ and $\mathbf{QChan}_{\mathbb{U}}$ with the aim of identifying some structure-preserving map $\mathbf{QChan} \to \mathbf{QChan}_{\mathbb{U}}$.

A mixed approach to solving this problem would examine whether the cumulative hierarchy of commutative squares in the limit of large n describes a category that is equivalent to $\mathbf{QChan}_{\mathbb{U}}$.

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Identifying $\mathbf{QChan}_{\mathbb{U}}$

Note that $\mathbf{QChan}_{\mathbb{U}}$ consists of channels that are *invertible*.

Definition (Invertible maps)

We say a map $\varepsilon(\rho)$ is invertible if and only if there exists some operator, $\mathcal{D}$, such that

$$\mathcal{D}\left(\varepsilon(\rho)\right) = \rho.$$

Categorically, this means that $\varepsilon(\rho)$ is an isomorphism.

Definition

In any category **C**, an arrow $f : A \to B$ is called an isomorphism if there is an arrow $g : B \to A$ in **C** such that

$$g \circ f = 1_A \qquad \text{and} \qquad f \circ g = 1_B$$

Since inverses are unique, we have that $g = f^{-1}$ and thus A is isomorphic to B.

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Preliminary results

For all subcategories of **QChan** that are monoids of finite-dimensionality:

- Apply Cayley's theorem which states that every group is isomorphic to a permutation group.
- Any representation (e.g. a set of operators) of a finite and compact permutation group may be considered unitary. Note that compact permutation groups are closed subgroups of direct products of finite groups and that their compactness is trivial in this case.
- Thus any quantum channel that preserves the size of the (finite-dimensional) Hilbert space has a unitary representation, even if it is not reversible.

Extending this to non-monoid subcategories of **QChan** may be possible via Yoneda's lemma and will constitute further work. It remains to be seen how non-finite channels can be handled.

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Summary

- We have developed a categorical representation of quantum channels.
- We have shown that all channels whose input and output Hilbert spaces have the same dimension (i.e. those that are monoids) have a unitary representation. This greatly extends the types of channels that can be shown to obey Birkhoff's theorem.
- We propose using Yoneda's lemma to potentially extend our results to all non-monoid subcategories of **QChan** thus achieving a completely general solution.

Thank you!