AN INTRODUCTION TO THE DERIVED CATEGORY

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ABSTRACT. The derived category is the proper framework for hyperhomology. The treatment of derived functors on the level of the derived category yields simple identities that underlie most of the familiar spectral sequences. By pursuing a formal analogy to algebraic topology, we obtain Verdier's axioms of a triangulated category which are an attempt to capture the essential structure of the homotopy category of chain complexes and the derived category. We outline the construction of the derived category as a localization of the homotopy category and show how this leads to its triangulated structure. Finally, we briefly discuss derived functors.

CONTENTS

1. INTRODUCTION

Our standpoint is that homological algebra is the theory of chain complexes and short exact sequences.

Let $\mathscr{A}$ be an abelian category. A chain complex is a diagram

$$\cdots \xrightarrow{d_A^{n-2}} A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \xrightarrow{d_A^{n+1}} \cdots$$

such that $d_A^{n+1} d_A^n = 0$ for all $n \in \mathbb{Z}$. A chain map is a morphism of such diagrams. We denote the category of chain complexes by $\mathbf{Ch}(\mathscr{A})$.

EXERCISE 1.1. Let $\mathscr{A}$ be an abelian category. Prove that $\mathbf{Ch}(\mathscr{A})$ is abelian. A sequence of chain maps is short exact if and only if it is short exact in each degree. Prove that the functor $\mathscr{A} \to \mathbf{Ch}(\mathscr{A})$ that arises from considering an object of $\mathscr{A}$ as a complex concentrated in degree zero is fully faithful (bijective on Hom-sets) and exact.

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It is nice to know that $\mathbf{Ch}(\mathscr{A})$ is abelian, however this knowledge does not help us in understanding the notions of chain homotopy equivalence and quasi-isomorphisms. In order to do this, we need to introduce two further categories:

(i) The *homotopy category* $\mathbf{K}(\mathscr{A})$. Objects are chain complexes over $\mathscr{A}$ and morphisms are the homotopy classes of chain maps.
(ii) The *derived category* $\mathbf{D}(\mathscr{A})$ obtained from $\mathbf{K}(\mathscr{A})$ by formally inverting the quasi-isomorphisms.

It turns out that neither the homotopy category nor the derived category are abelian except in trivial cases (see [Ver96, Chapitre II, Proposition 1.3.6, p.108]), so there is a loss of structure. However, the mapping cone construction gives rise to the structure of *triangulated categories* on both $\mathbf{K}(\mathscr{A})$ and $\mathbf{D}(\mathscr{A})$, i.e., there is a class of diagrams, called "distinguished triangles", which serves as substitute for short exact sequences.

Now let us be given an additive functor $F : \mathscr{A} \to \mathscr{B}$ between abelian categories. By applying F degreewise on chain complexes and chain maps, we obtain a functor $\mathbf{Ch}\, F : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$. Because F is additive, it preserves chain homotopies, hence it descends to a functor $\mathbf{K}\, F : \mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$. Because the triangulated structure on $\mathbf{K}\, F$ is defined in "additive terms", this functor is compatible with the triangulations, i.e., it is a *triangle functor*. However, applying F degreewise yields a well-defined functor $\mathbf{D}\, F : \mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$ on the level of the derived categories if and only if F is exact, and in this case, this will be a triangle functor. Otherwise $\mathbf{K}\, F$ will fail to map quasi-isomorphisms to quasi-isomorphisms. The left and right derived functors will then appear as "best approximations" to $\mathbf{K}\, F$ among the triangle functors $\mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$.

2. The Mapping Cone

The category $\mathbf{Ch}(\mathscr{A})$ has rather more structure than just being abelian. First, there is the suspension functor

$$\Sigma : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{A})$$

defined on complexes by $(\Sigma A)^n = A^{n+1}$ and $d^n_{\Sigma A} = -d^{n+1}_A$ and on chain maps by $(\Sigma f)^n = f^{n+1}$. The suspension functor is additive and invertible. Second, there is the mapping cone construction, which we will treat in detail in this section.

Given a chain map $A \xrightarrow{f} B$, we define the mapping cone to be the complex $\operatorname{cone}(f)^n = A^{n+1} \oplus B^n$ with differential

$$d^n_f = \begin{bmatrix} -d^{n+1}_A & 0 \\ f^{n+1} & d^n_B \end{bmatrix}.$$

Observe that $d^{n+1}_f d^n_f = 0$ because f is a chain map.

Exercise 2.1. For a complex A, we define $\operatorname{cone}(A)$ to be the cone of 1_A. Prove that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}\downarrow & \text{PO} & \downarrow{\scriptstyle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \\ \operatorname{cone}(A) & \xrightarrow[\left[\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right]]{} & \operatorname{cone}(f) \end{array}$$

is a push-out in $\mathbf{Ch}(\mathscr{A})$. Compare this to the definition given in basic algebraic topology $Cf = CX \cup_f Y$.

EXAMPLES 2.2.

(i) The cone of the zero morphism $A \xrightarrow{0} B$ is $\operatorname{cone}(0) = \Sigma A \oplus B$. In particular, if $B = 0$ then $\operatorname{cone}(0) = \Sigma A$, and if $A = 0$ then $\operatorname{cone}(0) = B$.

(ii) The cone of the identity morphism $A \xrightarrow{1_A} A$ is contractible; a contracting homotopy is given by $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ as witnessed by the verification

$$\begin{bmatrix} -d_A^n & 0 \\ 1_{A^n} & d_A^{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1_{A^n} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1_{A^{n+1}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -d_A^{n+1} & 0 \\ 1_{A^{n+1}} & d_A^n \end{bmatrix} = \begin{bmatrix} 1_{A^{n+1}} & 0 \\ 0 & 1_{A^n} \end{bmatrix} = 1_{\operatorname{cone}(A)^n}.$$

(iii) Let $A \xrightarrow{f} B$ be a morphism of $\mathscr{A}$. Its mapping cone is the complex

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0 \longrightarrow \cdots,$$

where A sits in degree -1.

(iv) Let $A \xrightarrow{\varepsilon} E^\bullet$ be a resolution. Its augmented complex is nothing but $\operatorname{cone}(\varepsilon)$.

EXERCISE 2.3 (The Homotopy Invariant).

(i) Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of chain complexes which splits in each degree. Construct a chain map $C \to \Sigma A$. The construction will depend on some choices. Prove that the homotopy equivalence class of this chain map is well-defined, it is called the *homotopy invariant* of the degreewise split sequence.

(ii) Let $A \xrightarrow{f} B$ be a chain map. Determine the homotopy invariant of the sequence

$$0 \to B \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \operatorname{cone}(f) \xrightarrow{[\,1\ 0\,]} \Sigma A \to 0.$$

(iii) Prove that the connecting morphism of the long exact homology sequence associated to the sequence in (ii) is $H^*(\Sigma f)$.

The mapping cone construction gives rise to a diagram

$$A \xrightarrow{f} B \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \operatorname{cone}(f) \xrightarrow{[\,1\ 0\,]} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \cdots$$

reminiscent of the *cofiber sequence* in topology.

EXERCISE 2.4. Prove that the composition of any two consecutive maps in the above cofiber sequence is homotopic to zero.

The cofiber sequence is functorial in f: Given a morphism of chain maps

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array}$$

there is the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & \operatorname{cone}(f) & \xrightarrow{[\,1\ 0\,]} & \Sigma A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} \Sigma a & 0 \\ 0 & b \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \Sigma a} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & \operatorname{cone}(f') & \xrightarrow{[\,1\ 0\,]} & \Sigma A'. \end{array}$$

EXERCISE 2.5. Let $A \xrightarrow{f} B$ be a chain map. Check that f is homotopic to zero if and only if it extends to a chain map $\operatorname{cone}(1_A) \xrightarrow{[\,s\ f\,]} B$.

From algebraic topology, one expects that homotopic maps have homotopic mapping cones. This is true with the familiar proviso that there is no distinguished homotopy equivalence. More precisely, let $f - g = ds + sd$ be homotopic chain maps $A \to B$. The choice of an explicit chain homotopy s yields an isomorphism

$$\operatorname{cone}(f) \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right]} \operatorname{cone}(g).$$

More generally, consider a homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle a} & \simeq & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array}$$

that is, there exists s such that $f'a - bf = d_{B'}s + sd_A$. The *choice* of such an s yields the following diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & \operatorname{cone}(f) & \xrightarrow{[1\ 0]} & \Sigma A \\ \downarrow{\scriptstyle a} & \simeq & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} a & 0 \\ s & b \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \Sigma a} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & \operatorname{cone}(f') & \xrightarrow{[1\ 0]} & \Sigma A' \end{array}$$

in which the first square is homotopy commutative and the other two squares are commutative.

3. The Homotopy Category

We denote by $\mathbf{K}(\mathscr{A})$ the *homotopy category* of chain complexes over the additive category $\mathscr{A}$. Its objects are the chain complexes and its morphisms are the homotopy equivalence classes of chain maps.

Exercise 3.1.

(i) Prove that $\mathbf{K}(\mathscr{A})$ is an additive category.
(ii) There is a functor $\mathscr{A} \to \mathbf{K}(\mathscr{A})$ that is obtained by considering an object of $\mathscr{A}$ as a complex concentrated in degree zero. Prove that this functor is fully faithful.

The suspension automorphism $\Sigma : \mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{A})$ is defined as in $\mathbf{Ch}(\mathscr{A})$.

Exercise 3.2. Let A, B be chain complexes over $\mathscr{A}$. Define a complex of abelian groups by

$$\hom(A,B)^i = \prod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathscr{A}}(A^n, B^{n+i})$$

with differential

$$d^i : (f^n)_{n\in\mathbb{Z}} \mapsto \left(d_B^{n+i} f^n - (-1)^i f^{n+1} d_A^n\right)_{n\in\mathbb{Z}}.$$

Prove that $\operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(A, \Sigma^i B) \cong H^i(\hom(A,B))$.

Exercise 3.3 ([Kel90, 2.3 a)]). Let A be a chain complex and define IA to be the complex $(IA)^n = A^n \oplus A^{n+1}$ with differential $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. There is a chain map $i_A : A \to IA$ given by $\left[\begin{smallmatrix} 1 \\ d_A^n \end{smallmatrix}\right]$ in each degree.

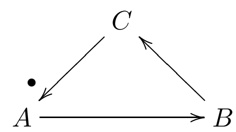
(i) Prove that $f : A \to B$ is homotopic to zero if and only if it factors over i_A.
(ii) Prove that $f : A \to B$ admits a retraction in $\mathbf{K}(\mathscr{A})$ if and only if the chain map $\left[\begin{smallmatrix} f \\ i_A \end{smallmatrix}\right] : A \to IA$ admits a retraction in $\mathbf{Ch}(\mathscr{A})$.

(iii) Conclude that A is null-homotopic if and only if it is a retract (in $\mathbf{Ch}(\mathscr{A})$) of a direct sum of complexes of the form $\cdots \to 0 \to B \xrightarrow{1} B \to 0 \to \cdots$.

Definition 3.4 (Triangles). A triangle in $\mathbf{K}(\mathscr{A})$ is a diagram of the form

$$A \to B \to C \to \Sigma A.$$

Triangles are often depicted diagrammatically as



where the bullet indicates that the morphism is of degree one.

A *morphism of triangles* is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma a \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

in $\mathbf{K}(\mathscr{A})$.

A triangle is called *distinguished* if it is isomorphic in $\mathbf{K}(\mathscr{A})$ to a *strict triangle*, i.e., a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \operatorname{cone}(f) \xrightarrow{[1\ 0]} \Sigma A.$$

Remark 3.5. Strictly speaking, the expression $\operatorname{cone}(f)$ does not make sense on the level of $\mathbf{K}(\mathscr{A})$ since the mapping cones of homotopic maps are not canonically isomorphic in general.

Lemma 3.6 (Rotation). *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a distinguished triangle. Then the* rotated *triangles*

$$B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$$

and

$$\Sigma^{-1} C \xrightarrow{-\Sigma^{-1} h} A \xrightarrow{f} B \xrightarrow{h} C$$

are distinguished as well.

Proof. We only prove the first statement and leave the second statement to the reader as an exercise. We may assume that the initial triangle is represented by

$$A \xrightarrow{f} B \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \operatorname{cone}(f) \xrightarrow{[1\ 0]} \Sigma A$$

in $\mathbf{Ch}(\mathscr{A})$. We have to prove that the triangle

$$B \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \operatorname{cone}(f) \xrightarrow{[1\ 0]} \Sigma A \xrightarrow{-\Sigma f} \Sigma B.$$

is isomorphic in $\mathbf{K}(\mathscr{A})$ to a strict triangle. The cone over $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ is the complex

$$B^{n+1} \oplus A^{n+1} \oplus B^n \qquad \text{with differential} \qquad \begin{bmatrix} -d_B^{n+1} & 0 & 0 \\ 0 & -d_A^{n+1} & 0 \\ 1 & f^{n+1} & d_B^n \end{bmatrix}.$$

In the diagram

$$\begin{array}{ccccccc}
B & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \operatorname{cone}(f) & \xrightarrow{[1\ 0]} & \Sigma A & \xrightarrow{-\Sigma f} & \Sigma B \\
\| & & \| & & \downarrow{\scriptsize\begin{bmatrix}-\Sigma f\\1\\0\end{bmatrix}} & & \| \\
B & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \operatorname{cone}(f) & \xrightarrow{\left[\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right]} & \operatorname{cone}(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) & \xrightarrow{[1\ 0\ 0]} & \Sigma B
\end{array}$$

the two outer squares are commutative while the middle square is commutative up to homotopy since

$$\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix} - \begin{bmatrix}-\Sigma f&0\\1&0\\0&0\end{bmatrix} = \begin{bmatrix}\Sigma f&0\\0&0\\0&1\end{bmatrix} = \begin{bmatrix}-d_B^n&0&0\\0&-d_A^n&0\\1&f^n&d_B^{n-1}\end{bmatrix}\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix} + \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}-d_A^{n+1}&0\\f^{n+1}&d_B^n\end{bmatrix}.$$

Similarly, in the following diagram

$$\begin{array}{ccccccc}
B & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \operatorname{cone}(f) & \xrightarrow{\left[\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right]} & \operatorname{cone}(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) & \xrightarrow{[1\ 0\ 0]} & \Sigma B \\
\| & & \| & & \downarrow{\scriptsize [0\ 1\ 0]} & & \| \\
B & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \operatorname{cone}(f) & \xrightarrow{[1\ 0]} & \Sigma A & \xrightarrow{-\Sigma f} & \Sigma B
\end{array}$$

the left and the middle square are commutative while the right hand square is commutative up to homotopy since

$$\begin{bmatrix}1&\Sigma f&0\end{bmatrix} = (-d_B^n)\begin{bmatrix}0&0&1\end{bmatrix} + \begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}-d_B^{n+1}&0&0\\0&-d_A^{n+1}&0\\1&f^{n+1}&d_B^n\end{bmatrix}.$$

The last two diagrams exhibit an isomorphism of triangles since

$$1_{\Sigma A} = \begin{bmatrix}0&1&0\end{bmatrix}\begin{bmatrix}-f\\1\\0\end{bmatrix}$$

and

$$\begin{aligned}\begin{bmatrix}1&\Sigma f&0\\0&0&0\\0&0&1\end{bmatrix} &= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}-\Sigma f\\1\\0\end{bmatrix}\begin{bmatrix}0&1&0\end{bmatrix} \\ &= \begin{bmatrix}-d_B^n&0&0\\0&-d_A^n&0\\1&f^n&d_B^{n-1}\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} + \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}-d_B^{n+1}&0&0\\0&-d_A^{n+1}&0\\1&f^{n+1}&d_B^n\end{bmatrix}\end{aligned}$$

so we are done. □

Definition 3.7 (Verdier). A *triangulated category* $(\mathbf{K}, \Sigma, \Delta)$ is a triple consisting of an additive category $\mathbf{K}$, an auto-equivalence $\Sigma : \mathbf{K} \to \mathbf{K}$ and a class of *distinguished* Σ-triangles satisfying the axioms below:

[**TR 1**] (i) The class Δ is closed under isomorphisms of triangles.
(ii) For every object $A \in \mathbf{K}$ the triangle

$$A \xrightarrow{1_A} A \to 0 \to \Sigma A$$

is distinguished.
(iii) For every morphism $A \xrightarrow{f} B$ there exists a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.$$

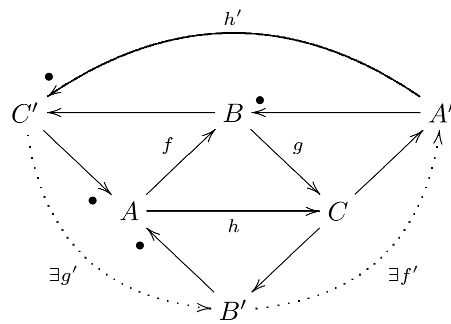
[**TR 2**] (Rotation Axiom) The triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is distinguished if and only if the triangle $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$ is distinguished.

[**TR 3**] (Morphism Axiom) If the rows of the following diagram are distinguished triangles and the left hand square is commutative

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \vdots{\scriptstyle \exists c} & & \downarrow{\scriptstyle \Sigma a} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A', \end{array}$$

then there exists a morphism c making the whole diagram commutative.

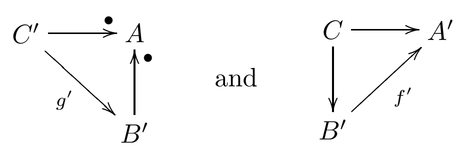
[**TR 4**] (Octahedral Axiom) Given two composeable morphisms f and g, embed f, g and $h = gf$ into distinguished triangles according to [**TR 1**] (iii).



Let h' be the composite $A' \to \Sigma B \to \Sigma C'$. There exist morphisms f' and g' such that

(i) The triangle $C' \xrightarrow{g'} B' \xrightarrow{f'} A' \xrightarrow{h'} \Sigma C'$ is distinguished.

(ii) The triangles



are commutative.

(iii) The two morphisms $B \to B'$ (via C and C') coincide as well as the two morphisms $B' \to \Sigma B$ (via ΣA and A').

Remark 3.8. The octahedral axiom is admittedly hard to digest. In the context of the homotopy category it simply expresses how the cone of the composition of two morphisms is linked to the cone of the components, namely via a distinguished triangle fitting into a nice diagram. The name stems from the fact that the big diagram may be re-arranged so as to obtain an octahedron, four faces of which are commutative, the other four are distinguished triangles.

We will only make use of the octahedral axiom in our discussion of Verdier localization, so the reader is advised to ignore it at the moment. However, for more advanced applications of triangulated categories such as abstract truncation, gluing, tilting theory, Brown representability, *etc.*, the octahedron turns out to be the flesh and bone of the axiomatics. It seems fair to say that every serious application of triangulated categories involves at least one octahedron.

Theorem 3.9 (Verdier). *Let $\mathscr{A}$ be an additive category. The homotopy category* $\mathbf{K}(\mathscr{A})$ *is triangulated.*

Proof. Our discussion of the mapping cone and the definition of distinguished triangles take care of axioms [**TR 1**], [**TR 2**] and [**TR 3**]. The proof of the octahedral axiom is a straightforward but rather tedious verification which we leave to the reader. ☐

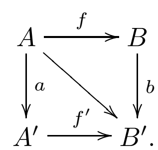
Exercise 3.10 (Triangulated Subcategories). Let $\mathscr{T} \subset \mathbf{K}$ be a full additive subcategory. Prove that $\mathscr{T}$ inherits the structure of a triangulated category from $\mathbf{K}$ if it is closed under Σ and Σ^{-1} and if for every morphism $S \to T$ of $\mathscr{T}$ there exists a distinguished triangle $S \to T \to C \to \Sigma S$ with $C \in \mathscr{T}$.

Exercise 3.11 (Duality). Let $(\mathbf{K}, \Sigma, \Delta)$ be a triangulated category, define a triangulated structure on $\mathbf{K}^{\mathrm{op}}$.

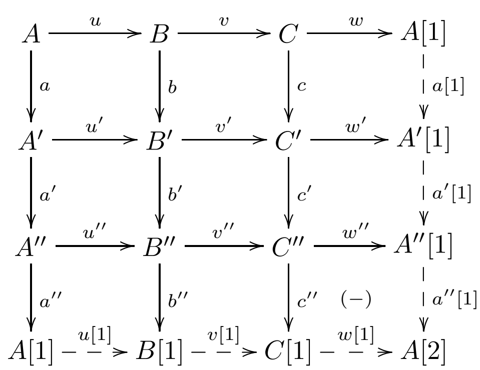
Exercise 3.12 (Verdier's Exercise).

(i) Prove that [**TR 1**] and the octahedral axiom [**TR 4**] imply the morphism axiom [**TR 3**].

Hint: Build octahedra over the two commutative triangles in the diagram



(ii) Every commutative square as in (i) can be embedded into a diagram



in which all unlabeled squares commute, the square labeled $(-)$ is sign commutative and all rows and columns with solid arrows are distinguished triangles.

Hint: Proceed as in (i) to obtain the top three squares and the three left-hand squares. The morphism c is constructed as a composition of two morphisms, use this to build a third octahedron and to complete the diagram. You will have to rotate one triangle, and this is the reason for the sign.

Remark 3.13. There are various reasons not to drop [**TR 3**] from the axiomatics. First, there is the argument concerning historical tradition. Second, [**TR 3**] is usually neither hard to prove nor to apply, whereas [**TR 4**] is considerably more subtle in both respects. See Balmer's MathSciNet review of [May01] for a more serious discussion.

4. Elementary Properties of Triangulated Categories

In this section, $\mathbf{K}$ will be a triangulated category and we will discuss the most important consequences of axioms [**TR 1**], [**TR 2**] and [**TR 3**]. The reader is advised to bear in mind the slogan *"distinguished triangles should be thought of as the triangulated analog of exact sequences in abelian categories"* throughout this section.

PROPOSITION 4.1. *The composition of two consecutive maps in a distinguished triangle is zero.*

PROOF. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma C$ be a distinguished triangle. By [**TR 1**] and [**TR 3**] there is a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{1_A} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\
\downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle 1_{\Sigma A}} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A
\end{array}$$

from which we conclude that $gf = 0$. That $hg = 0$ and $(\Sigma f)h = 0$ now follows from [**TR 2**]. □

EXERCISE 4.2. Prove that one may change the sign of any two morphisms in a distinguished triangle.

Caution: This is *not* true for the sign of a single morphism, see [Ive86, Example 4.21, p. 32].

PROPOSITION 4.3. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a distinguished triangle. The morphism f is a* weak kernel *of g and h is a* weak cokernel *of h. In detail:*

(i) There exists a factorization $x = fx'$

$$\begin{array}{ccccccc}
 & & X & & & & \\
 & {\scriptstyle \exists x'}\swarrow & \downarrow{\scriptstyle x} & \searrow{\scriptstyle 0} & & & \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A
\end{array}$$

for every $X \xrightarrow{x} B$ such that $gx = 0$. Moreover, $gf = 0$.

(ii) There is a factorization $y = y'h$

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
 & & & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle y} & \swarrow{\scriptstyle \exists y'} & \\
 & & & & Y & &
\end{array}$$

for every $C \xrightarrow{y} Y$ such that $yg = 0$. Moreover, $hg = 0$.

PROOF. We only prove (ii). By [**TR 1**] and [**TR 2**] we have a commutative diagram whose rows are distinguished triangles

$$\begin{array}{ccccccc}
B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A & \xrightarrow{-\Sigma f} & \Sigma B \\
\downarrow & & \downarrow{\scriptstyle y} & & \vdots{\scriptstyle \exists y'} & & \downarrow \\
0 & \longrightarrow & Y & \xrightarrow{1_Y} & Y & \longrightarrow & 0
\end{array}$$

and the existence of y' follows from [**TR 3**]. □

COROLLARY 4.4. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a distinguished triangle and suppose*

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 0} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A
\end{array}$$

is a morphism of triangles. Then $c^2 = 0$.

Proof. By assumption $cg = 0$, so by the weak cokernel property there exists $\Sigma A \xrightarrow{c'} C$ such that $c = c'h$, so $c^2 = c'hc = 0$. □

Corollary 4.5 (Five Lemma). *Let*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma a} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

be a morphism of distinguished triangles. If two out of a, b, c are isomorphisms then so is the third.

Proof. By [**TR 2**] it suffices to consider the case that a and b are isomorphisms. We leave it to the reader to reduce the question to the following statement: If

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \end{array}$$

is an endomorphism of a distinguished triangle then c is an isomorphism. By the previous corollary we know that $(c-1)^2 = 0$, hence $c = 1+(c-1)$ is an isomorphism with inverse $1-(c-1)$. □

Corollary 4.6. *If $A \xrightarrow{f} B$ is a morphism, any two distinguished triangles of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ are isomorphic via a* non-unique *isomorphism.* □

Corollary 4.7. *A morphism $A \xrightarrow{f} B$ is an isomorphism if and only if the triangle $A \xrightarrow{f} B \to 0 \to \Sigma A$ is distinguished.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \Sigma f} \\ B & \xrightarrow{1} & B & \longrightarrow & 0 & \longrightarrow & \Sigma B. \end{array}$$

If f is an isomorphism, the top row is a distinguished triangle because it is isomorphic to the distinguished triangle in the bottom row. Conversely, if both rows in the diagram are distinguished triangles, we conclude by the five lemma that f is an isomorphism. □

Definition 4.8. Let $\mathscr{A}$ be an abelian category. A functor $F : \mathbf{K} \to \mathscr{A}$ is called *homological* if for each distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact (at $F(B)$).

Remark 4.9. Let $F : \mathbf{K} \to \mathscr{A}$ be a homological functor. It is customary to write $F^n(A) := F(\Sigma^n A)$ for $n \in \mathbb{Z}$. It follows from [**TR 2**] that a homological functor gives rise to a long exact sequence

$$\cdots \to F^{n-1}(C) \to F^n(A) \to F^n(B) \to F^n(C) \to F^{n+1}(A) \to \cdots$$

for each distinguished triangle $A \to B \to C \to \Sigma A$.

EXERCISE 4.10. Prove that homology in degree zero $H^0 : \mathbf{K}(\mathscr{A}) \to \mathscr{A}$ is a homological functor.

EXERCISE 4.11. The represented functor $\mathrm{Hom}_{\mathbf{K}}(X, -) : \mathbf{K} \to \mathbf{Ab}$ is homological for every $X \in \mathbf{K}$.

EXERCISE 4.12 ([BBD82, 1.1.9]). Consider the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \vdots\exists a & & \downarrow b & & \vdots\exists c & & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

and assume that the rows are distinguished triangles. The following four conditions are equivalent:

(i) $g'bf = 0$.
(ii) There exists a morphism $a : A \to A'$ making the left hand square commutative.
(iii) There exists a morphism $c : C \to C'$ making the right hand square commutative.
(iv) The morphism b fits into a morphism of triangles (a, b, c).

If in addition to these conditions $\mathrm{Hom}^{-1}_{\mathbf{K}}(A, C') = 0$ then a and c are unique.

Hint: To see the uniqueness claim, apply the homological functors $\mathrm{Hom}_{\mathbf{K}}(A, -)$ and $\mathrm{Hom}_{\mathbf{K}}(-, C')$.

EXERCISE 4.13 ([BBD82, 1.1.10]). Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a distinguished triangle with the property that $\mathrm{Hom}^{-1}_{\mathbf{K}}(A, C) = 0$. Conclude from the previous exercise that:

(i) Every other distinguished triangle $A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A$ containing f is isomorphic to the original triangle via a *unique* isomorphism of triangles of the form $(1, 1, c)$.
(ii) The morphism $C \xrightarrow{h} \Sigma A$ is the only morphism $x \in \mathrm{Hom}_{\mathbf{K}}(C, \Sigma A)$ such that the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{x} \Sigma A$ is distinguished.

5. Triangle Functors

Let $(\mathbf{K}, \Sigma, \Delta)$ and $(\mathbf{K}', \Sigma', \Delta')$ be triangulated categories. A *triangle functor* $\mathbf{K} \to \mathbf{K}'$ is a pair (F, α) consisting of an additive functor $F : \mathbf{K} \to \mathbf{K}'$ and a natural transformation $\alpha : F\Sigma \Rightarrow \Sigma' F$ such that for each distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathbf{K}$ the triangle

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{(\alpha_A)F(h)} \Sigma' F(A), \qquad F(C) \xrightarrow{F(h)} F(\Sigma A) \xrightarrow{\alpha_A} \Sigma' F(A)$$

is distinguished in $\mathbf{K}'$. Since there is usually an "obvious" choice for the natural transformation α, it is often suppressed notationally.

REMARK 5.1. Let (F, α) be a triangle functor. Consider the distinguished triangle

$$X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{-1} \Sigma X$$

and apply (F, α) to obtain the distinguished triangle

$$FX \longrightarrow 0 \longrightarrow F\Sigma X \xrightarrow{-\alpha_X} \Sigma' FX.$$

Therefore the five lemma for triangulated categories implies that α is an isomorphism of functors.

Exercise 5.2. Let $(F,\alpha):\mathbf{K}\to\mathbf{K}'$ be a triangle functor and let

$$\mathscr{T}=\{A\in\mathbf{K}\,:\,F(A)\cong 0\}$$

be the kernel of F. Prove that $\mathscr{T}$ is a triangulated subcategory of $\mathbf{K}$. Moreover, $\mathscr{T}$ is *thick* in the sense that for all objects $A,B\in\mathbf{K}$ whose sum $A\oplus B$ is in $\mathscr{T}$ already $A,B\in\mathscr{T}$.

Remark 5.3. Verdier's localization theorem gives a converse to the previous exercise: every thick triangulated subcategory of a triangulated category is the kernel of some triangle functor. We will later provide more details.

The point of the following two results is that "the" adjoint of a triangle functor is itself a triangle functor.

Lemma 5.4 ([KV87, 1.6]). *Let $(R,\rho):\mathbf{K}\to\mathbf{K}'$ and $(L,\lambda):\mathbf{K}'\to\mathbf{K}$ be triangle functors such that L is left adjoint to R. Let $\varphi:LR\Rightarrow\mathrm{id}_{\mathbf{K}}$ and $\psi:\mathrm{id}_{\mathbf{K}'}\Rightarrow RL$ be the adjunction morphisms. The following four conditions are equivalent:*

(i) $\lambda=(\varphi\Sigma L)(L\varrho^{-1}L)(L\Sigma'\psi)$;
(ii) $\varrho^{-1}=(R\Sigma\varphi)(R\lambda R)(\psi\Sigma'R)$;
(iii) $\varphi\Sigma=(\Sigma\varphi)(\lambda R)(L\varrho)$;
(iv) $\Sigma'\psi=(\varrho L)(R\lambda)(\psi\Sigma')$.

If these conditions are satisfied, the triangle functors (L,λ) and (R,ϱ) are called an adjoint pair of triangle functors.

Proposition 5.5 ([KV87, 1.6]). *Let $(R,\varrho):\mathbf{K}\to\mathbf{K}'$ be a triangle functor and suppose that it has a left adjoint $L\dashv R$. Let*

$$\psi:\mathrm{id}_{\mathbf{K}'}\Rightarrow RL \qquad \text{and} \qquad \varphi:LR\Rightarrow\mathrm{id}_{\mathbf{K}}$$

be the adjunction morphisms. For $X\in\mathbf{K}'$ put

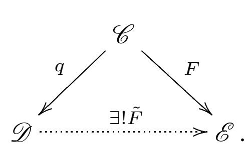
$$\lambda_X=\varphi_{\Sigma LX}\circ L\varrho^{-1}_{LX}\circ L\Sigma'\psi_X.$$

Then $\lambda:L\Sigma'\Rightarrow\Sigma L$ is a natural transformation such that (L,λ) is a triangle functor and (L,λ) and (R,ϱ) are an adjoint pair of triangle functors.

6. Localization of Categories

Definition 6.1. Let $\mathscr{C}$ be an arbitrary category and let $\mathscr{W}$ be a class of morphisms in $\mathscr{C}$. A *localization* of $\mathscr{C}$ with respect to $\mathscr{W}$ is a pair $(\mathscr{D},q)$ consisting of a category $\mathscr{D}$ and a functor $q:\mathscr{C}\to\mathscr{D}$ satisfying:

(i) The functor q transforms the morphisms in $\mathscr{W}$ to isomorphisms in $\mathscr{D}$.
(ii) If $F:\mathscr{C}\to\mathscr{E}$ is any functor transforming the morphisms in $\mathscr{W}$ to isomorphisms then F factors uniquely over $\mathscr{D}$:



If it exists, the category $\mathscr{D}$ is unique up to equivalence of categories and by abuse of notation any such category is denoted by $\mathscr{C}[\mathscr{W}^{-1}]$.

Theorem 6.2 (Gabriel-Zisman [GZ67, Chapter 1]). *For every category $\mathscr{C}$ and every class $\mathscr{W}$ of morphisms in $\mathscr{C}$ there exists a (possibly large) localization $\mathscr{C}[\mathscr{W}^{-1}]$.*

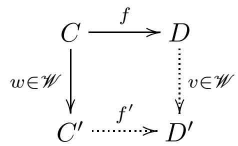
The proof of the Gabriel-Zisman theorem does not yield a very tractable description of $\mathscr{C}[\mathscr{W}^{-1}]$. However, under suitable hypotheses on $\mathscr{W}$, one can give quite a simple construction.

Definition 6.3. The class of morphisms $\mathscr{W}$ admits a *calculus of left fractions* if

[**F 1**] For every object $C \in \mathscr{C}$, the identity 1_C belongs to $\mathscr{W}$.

[**F 2**] The composition of two elements of $\mathscr{W}$ is again an element of $\mathscr{W}$.

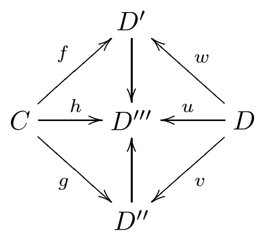
[**F 3**] (Left Ore Condition) Every diagram $C' \xleftarrow{w} C \xrightarrow{f} D$ with $w \in \mathscr{W}$ can be completed to a commutative square



with $v \in \mathscr{W}$.

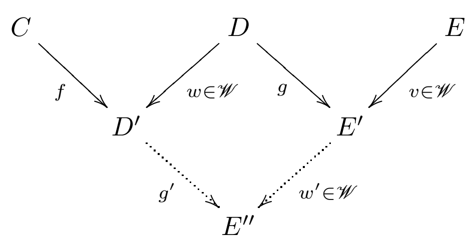
[**F 4**] (Left Cancellation) If f, g are morphisms in $\mathscr{C}$ and there exists $w \in \mathscr{W}$ such that $fw = gw$ then there exists $v \in \mathscr{W}$ such that $vf = vg$.

The idea is that the diagram $C \xrightarrow{f} D' \xleftarrow{w} D$ is a fraction representing the morphism $w^{-1} \circ f : C \to D$ in $\mathscr{C}[\mathscr{W}^{-1}]$. More precisely, consider the pairs of morphisms (w, f) with the same target and $w \in \mathscr{W}$. Two such pairs (w, f) and (v, g) are equivalent if there exists a commutative diagram



with $u \in \mathscr{W}$. Thinking of fractions this just means that the fractions (w, f) and (v, g) can be expanded to the same fraction. Reflexivity and symmetry of this relation are obvious, to prove transitivity one uses axioms [**F 2**], [**F 3**] and [**F 4**].

Denote the equivalence class of (w, f) by $(w \setminus f)$. The equivalence classes $(w \setminus f)$ and $(v \setminus g)$ can be composed as follows: [**F 3**] guarantees the existence of a diagram



and [**F 2**] shows that $w'v \in \mathscr{W}$. The composition is then given by

$$(v \setminus g) \circ (w \setminus f) = (w'v \setminus g'f)$$

and it is not hard to check that this composition is well-defined, associative and by [**F 1**] the fractions $(1 \setminus 1)$ are two-sided neutral elements.

The localization $\mathscr{C}[\mathscr{W}^{-1}]$ can now be described as the category with the same objects as $\mathscr{C}$ and whose morphisms are the equivalence classes $(w \setminus f)$ with the composition described above. The localization functor $q : \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ is defined as $q(f) = (1 \setminus f)$. The inverse of $q(w)$ is $(w \setminus 1)$, so the images of the morphisms in $\mathscr{W}$

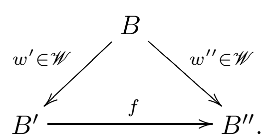
under q are indeed invertible. If the functor $F : \mathscr{C} \to \mathscr{E}$ transforms elements of $\mathscr{W}$ to isomorphisms then a factorization is obtained by setting $\tilde{F}((w \setminus f)) = F(w)^{-1}F(f)$ and one easily convinces oneself that this is the only way to get a factorization of F over q.

Remark 6.4. Dualization yields the notion of admitting a *calculus of right fractions* and hence a description of $\mathscr{C}[\mathscr{W}^{-1}]$ using right fractions. If $\mathscr{W}$ admits both a calculus of left fractions and a calculus of right fractions, the universal property of the localization yields that the two descriptions of $\mathscr{C}[\mathscr{W}^{-1}]$ via left and right fractions are canonically isomorphic.

Remark 6.5. Despite the simplicity of the construction, the localization $\mathscr{C}[\mathscr{W}^{-1}]$ is usually quite hard to identify explicitly. It is often non-trivial to decide whether $\mathscr{C}[\mathscr{W}^{-1}]$ is (equivalent to) the zero category or not.

Remark 6.6 (Set-Theoretic Caveat). We deliberately glossed over a set-theoretic difficulty: if $\mathscr{C}$ is not small, the equivalence relation in the definition of the morphisms of $\mathscr{C}[\mathscr{W}^{-1}]$ may have proper classes as equivalence classes, and there is no *a priori* reason why the class of equivalence classes should form a set. In particular, the "category" $\mathscr{C}[\mathscr{W}^{-1}]$ is not a category *stricto sensu*, because $\mathrm{Hom}_{\mathscr{C}[\mathscr{W}^{-1}]}(C, D)$ need not be a set.

Remark 6.7 (Gabriel-Zisman). Let $\mathscr{W}$ admit a calculus of left fractions and let B be an object of $\mathscr{C}$. Consider the "comma category" $\mathscr{W}_B$ whose objects are the diagrams $B \xrightarrow{w'} B'$ with $w' \in \mathscr{W}$ and whose morphisms are the commutative diagrams of the form



By the Ore condition this category is filtered and one can identify

$$\mathrm{Hom}_{\mathscr{C}[\mathscr{W}^{-1}]}(A, B) = \varinjlim_{\mathscr{W}_B} \mathrm{Hom}_{\mathscr{C}}(A, B')$$

(exercise). In order to ensure that $\mathrm{Hom}_{\mathscr{C}[\mathscr{W}^{-1}]}(A, B)$ is a set, it is sufficient to require that each $\mathscr{W}_B$ has a small cofinal subcategory. More explicitly, $\mathscr{W}$ is said to be *locally small (on the right)* if for each object $B \in \mathscr{C}$ there is a *set* $\mathscr{S}_B$ of morphisms in $\mathscr{W}$ (objects of $\mathscr{W}_B$) such that whenever there is a morphism $B \to B'$ in $\mathscr{W}$ there exists a morphism $B' \to B''$ such that the composite $B \to B' \to B''$ is in $\mathscr{S}_B$. It then follows that

$$\mathrm{Hom}_{\mathscr{C}[\mathscr{W}^{-1}]}(A, B) = \varinjlim_{\mathscr{S}_B} \mathrm{Hom}_{\mathscr{C}}(A, B'),$$

and the latter is manifestly a set.

Exercise 6.8. Let $\mathscr{C}$ be an additive category and let $\mathscr{W}$ admit a calculus of left fractions. Prove that the localization $\mathscr{C}[\mathscr{W}^{-1}]$ is additive and that the quotient functor $q : \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ is additive.

Hint: To see that $\mathrm{Hom}_{\mathscr{C}[\mathscr{W}^{-1}]}(A, B)$ is an abelian group, use its representation as a colimit given in the previous remark. Then prove the general facts that q preserves initial and terminal objects and finite products and coproducts (whenever they exist in $\mathscr{C}$).

7. Verdier Localization

Let $\mathbf{K}$ be a triangulated category and let $\mathscr{W}$ be a class of morphisms in $\mathbf{K}$ admitting a calculus of left fractions. We want to ensure that the localization $\mathbf{K}[\mathscr{W}^{-1}]$ has a natural triangulated structure. To this end, we impose two further conditions on the class $\mathscr{W}$:

[**F 5**] $\Sigma^{\pm 1}\mathscr{W} \subset \mathscr{W}$.

[**F 6**] If in the situation of the morphism axiom [**TR 3**] for triangulated categories $a, b \in \mathscr{W}$ then the morphism c can be chosen to be in $\mathscr{W}$ as well:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle a\in\mathscr{W}} & & \downarrow{\scriptstyle b\in\mathscr{W}} & & \vdots{\scriptstyle \exists c\in\mathscr{W}} & & \downarrow{\scriptstyle \Sigma a\in\mathscr{W}} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A'. \end{array}$$

Let $\mathscr{W}$ satisfy axioms [**F 1**] to [**F 6**] and construct the category $\mathbf{K}[\mathscr{W}^{-1}]$ via left fractions as described in the previous section. It not difficult to see that $\mathbf{K}[\mathscr{W}^{-1}]$ is additive. Axiom [**F 5**] ensures that the suspension Σ descends to an auto-equivalence of $\mathbf{K}[\mathscr{W}^{-1}]$, which we still denote by Σ. Let $q : \mathbf{K} \to \mathbf{K}[\mathscr{W}^{-1}]$ be the quotient functor. By construction, $q\Sigma = \Sigma q$. Declare a triangle in $\mathbf{K}[\mathscr{W}^{-1}]$ to be distinguished if it is isomorphic to the image under q of some distinguished triangle in $\mathbf{K}$. With this structure we have:

Theorem 7.1 (Verdier Localization). *Let $\mathbf{K}$ be a triangulated category and let $\mathscr{W}$ be a class of morphisms satisfying axioms [**F 1**] to [**F 6**]. The category $\mathbf{K}[\mathscr{W}^{-1}]$ is triangulated and the quotient functor $q : \mathbf{K} \to \mathbf{K}[\mathscr{W}^{-1}]$ is a triangle functor. Every triangle functor $\mathbf{K} \to \mathbf{K}'$ transforming morphisms in $\mathscr{W}$ to isomorphisms in $\mathbf{K}'$ factors uniquely over a triangle functor $\mathbf{K}[\mathscr{W}^{-1}] \to \mathbf{K}'$.*

Of course, the question arises how one can find classes $\mathscr{W}$ satisfying the hypotheses of Verdier's localization theorem. The octahedral axiom furnishes a definitive answer to this question:

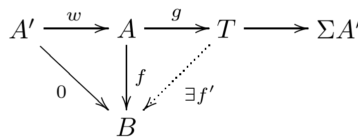
Proposition 7.2. *Let $\mathscr{T}$ be a triangulated subcategory of $\mathbf{K}$. Let $\mathscr{W}$ be the class of morphisms w for which there exists a distinguished triangle $A \xrightarrow{w} B \xrightarrow{g} T \xrightarrow{h} \Sigma A$ with $T \in \mathscr{T}$. Then $\mathscr{W}$ satisfies axioms [**F 1**] to [**F 6**] and their duals.*

Sketch of the Proof. Axioms [**F 1**] and [**F 5**] are obvious. To prove [**F 2**], we use the octahedral axiom. If $A \xrightarrow{w} B \xrightarrow{w'} C$ are two morphisms of $\mathscr{W}$, then we have to show that $w'w \in \mathscr{W}$ as well. There are distinguished triangles $A \xrightarrow{w} B \xrightarrow{g} T \xrightarrow{h} \Sigma A$ and $B \xrightarrow{w'} C \xrightarrow{g'} T' \xrightarrow{h'} \Sigma B$ with T and T' both in $\mathscr{T}$. Since $\mathscr{T}$ is triangulated, there is a distinguished triangle $T' \xrightarrow{(\Sigma g)h'} \Sigma T \xrightarrow{g''} \Sigma T'' \xrightarrow{h''} \Sigma T'$ with $\Sigma T'' \in \mathscr{T}$. Rotate these three triangles appropriately and build the octahedron over the commutative triangle

$$\begin{array}{ccccc} & & \Sigma B & & \\ & {\scriptstyle h'}\nearrow & & \searrow{\scriptstyle \Sigma g} & \\ T' & & \xrightarrow{(\Sigma g)h'} & & \Sigma T. \end{array}$$

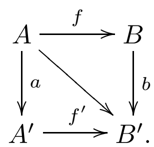
The distinguished triangle furnished by the octahedral axiom shows $\Sigma(w'w)$ to be in $\mathscr{W}$ and we conclude by [**F 5**].

To prove [**F 4**], it suffices to show that for a morphism f in $\mathbf{K}$ for which there exists $w \in \mathscr{W}$ such that $fw = 0$ then there exists $v \in \mathscr{W}$ such that $vf = 0$. Choose a distinguished triangle with base w and third object $T \in \mathscr{T}$ and apply the weak cokernel property:



Build a triangle $T \xrightarrow{f'} B \xrightarrow{v} C \to \Sigma T$. By rotation, we see that $v \in \mathscr{W}$. Moreover, $vf = vf'g = 0$ since the composition of two consecutive morphisms in a distinguished triangle is zero.

Axiom [**F 6**] is easy: Consider the diagram



Applying the octahedral axiom to each of the two commutative triangles, one obtains two morphisms in $\mathscr{W}$ and composing these one finds the required morphism $c \in \mathscr{W}$.

We leave it to the reader as an exercise to prove that [**F 1**], [**F 5**] and [**F 6**] imply [**F 3**] and to convince himself that all the arguments are subject to dualization. □

REMARK 7.3. Let $\mathscr{T} \subset \mathbf{K}$ be a triangulated subcategory and let $\mathscr{W}$ the associated class of morphisms as described in the previous proposition. Let $q : \mathbf{K} \to \mathbf{K}[\mathscr{W}^{-1}]$ be the quotient functor. Since q is a triangle functor and since it sends the morphisms in $\mathscr{W}$ to isomorphisms, it must annihilate $\mathscr{T}$. It is therefore customary to write $\mathbf{K}[\mathscr{W}^{-1}] = \mathbf{K}/\mathscr{T}$ and refer to it as the *Verdier quotient* of $\mathbf{K}$ by $\mathscr{T}$. One can prove that the kernel of q is the *thick closure* of $\mathscr{T}$, i.e.,

$$\operatorname{Ker} q = \{A \in \mathbf{K} \,:\, A \oplus B \cong T \in \mathscr{T} \text{ for some } B \in \mathbf{K}\},$$

see e.g. [Nee01, Chapter 2]. In case $\mathscr{T}$ is already thick, it follows that q transforms *precisely* the morphisms in $\mathscr{W}$ to isomorphisms.

EXERCISE 7.4. Let $\mathscr{A}$ be an abelian category and let $\mathbf{Ac}(\mathscr{A}) \subset \mathbf{K}(\mathscr{A})$ be the class of acyclic complexes. Prove that $\mathbf{Ac}(\mathscr{A})$ is a thick triangulated subcategory and that the morphisms in the associated class $\mathscr{W}$ are precisely the quasi-isomorphisms (in other words: a chain map is a quasi-isomorphism if and only if its cone is acyclic).

Combining the previous exercise with the previous remark, we see that a chain map becomes invertible in the derived category $\mathbf{D}(\mathscr{A}) = \mathbf{K}(\mathscr{A})/\mathbf{Ac}(\mathscr{A})$ if and only if it is a quasi-isomorphism.

The previous exercise is a special case of:

EXERCISE 7.5. Let $F : \mathbf{K} \to \mathscr{A}$ be a homological functor. Prove that

$$\operatorname{Ker} F = \{A \in \mathscr{K} \,:\, F^i(A) = 0 \text{ for all } i \in \mathbb{Z}\}$$

is a thick triangulated subcategory of $\mathbf{K}$.

8. The Derived Category of an Abelian Category

DEFINITION 8.1. The *derived category* $\mathbf{D}(\mathscr{A})$ is the localization $\mathbf{K}(\mathscr{A})[\mathscr{Q}^{-1}]$ of the homotopy category with respect to the class $\mathscr{Q}$ of quasi-isomorphisms. Equivalently, $\mathbf{D}(\mathscr{A})$ is the Verdier quotient $\mathbf{K}(\mathscr{A})/\mathbf{Ac}(\mathscr{A})$.

Verdier's localization theorem gives:

Theorem 8.2 (Verdier). *The derived category of an abelian category is triangulated. A chain map becomes invertible in the derived category if and only if it is a quasi-isomorphism. A triangle functor $F : \mathbf{K}(\mathscr{A}) \to \mathbf{K}'$ factors over a triangle functor $\mathbf{D}(\mathscr{A}) \to \mathbf{K}'$ if and only if F sends quasi-isomorphisms to isomorphisms in $\mathbf{K}'$.*

Example 8.3. Let $0 \to A' \xrightarrow{m} A \xrightarrow{e} A'' \to 0$ be a short exact sequence in $\mathscr{A}$. Consider the diagram

$$\begin{array}{ccccccc} A' & \xrightarrow{m} & A & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \operatorname{cone}(m) & \xrightarrow{[\,0\ 1\,]} & \Sigma A' \\ \| & & \| & & \big\downarrow{\scriptstyle [\,0\ e\,] \in \mathscr{Q}} & & \\ A' & \xrightarrow{m} & A & \xrightarrow{e} & A'' & & \end{array}$$

and observe that $[\,0\ e\,] \in \mathscr{Q}$. Thus we get a distinguished triangle

$$A' \xrightarrow{m} A \xrightarrow{e} A'' \xrightarrow{[\,0\ 1\,][\,0\ e\,]^{-1}} \Sigma A'$$

in $\mathbf{D}(\mathscr{A})$ and obtain the result: every short exact sequence in $\mathscr{A}$ naturally embeds into a distinguished triangle of $\mathbf{D}(\mathscr{A})$.

Exercise 8.4. Generalize the previous example to short exact sequences in $\mathbf{Ch}(\mathscr{A})$.

Example 8.5. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. It induces a triangle functor $\mathbf{K}\,F : \mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ which sends quasi-isomorphisms in $\mathbf{K}(\mathscr{A})$ to quasi-isomorphisms in $\mathbf{K}(\mathscr{B})$ if and only if F is exact. Thus, we obtain a factorization $\mathbf{D}\,F : \mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$ if and only if F is exact.

Exercise 8.6. Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor. Describe $\mathbf{D}\,F$ explicitly.

9. Boundedness Conditions

There is a whole arsenal of abelian subcategories of $\mathbf{Ch}(\mathscr{A})$, obtained by imposing boundedness conditions. We only introduce three of them:

Definition 9.1. A complex is *left (right) bounded*, if $A^n = 0$ for all sufficiently small (large) $n \in \mathbb{Z}$. A complex is *bounded* if it is both left bounded and right bounded. The full subcategories of $\mathbf{Ch}(\mathscr{A})$ consisting of left bounded, right bounded, and bounded complexes are denoted by $\mathbf{Ch}^+(\mathscr{A})$, $\mathbf{Ch}^-(\mathscr{A})$ and $\mathbf{Ch}^b(\mathscr{A})$, respectively.

Exercise 9.2. Prove that $\mathbf{Ch}^*(\mathscr{A})$ is abelian for $* \in \{+, -, b\}$.

Exercise 9.3. For $* \in \{+, -, b\}$ let $\mathbf{K}^*(\mathscr{A})$ be the essential image of $\mathbf{Ch}^*(\mathscr{A})$ in $\mathbf{K}(\mathscr{A})$ under the canonical quotient functor. Prove that $\mathbf{K}^*(\mathscr{A})$ is a triangulated subcategory of $\mathbf{K}(\mathscr{A})$.

Exercise 9.4. For $* \in \{+, -, b\}$ let $\mathbf{D}^*(\mathscr{A})$ be the essential image of $\mathbf{K}^*(\mathscr{A})$ in $\mathbf{D}(\mathscr{A})$ under the canonical quotient functor. Prove that $\mathbf{D}^*(\mathscr{A})$ is a triangulated subcategory of $\mathbf{D}(\mathscr{A})$.

Theorem 9.5. *Let $\mathscr{A}$ be an abelian category and assume that there are enough injectives in $\mathscr{A}$. Let $\mathscr{I} \subset \mathscr{A}$ be the full subcategory of injective objects. The composition i*

$$\mathbf{K}^+(\mathscr{I}) \to \mathbf{K}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{A})$$

is an equivalence of triangulated categories.

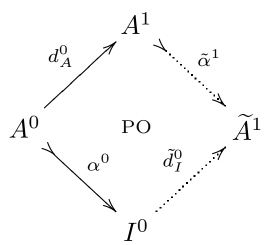
Remark 9.6. Similarly, if $\mathscr{A}$ has enough projectives, there is an equivalence of categories $\mathbf{K}^-(\mathscr{P}) \cong \mathbf{D}^-(\mathscr{A})$, where $\mathscr{P} \subset \mathscr{A}$ denotes the full subcategory of projectives.

Proof. Since the composition of two triangle functors is again a triangle functor, we need only check that i is fully faithful and essentially surjective, this is the content of the following lemmas. □

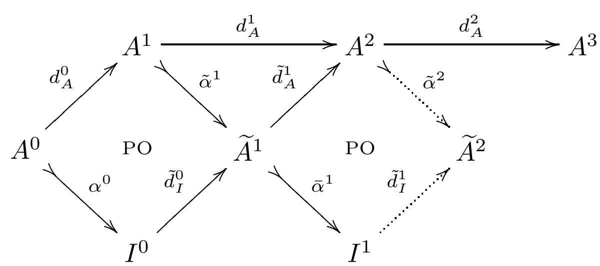
To see essential surjectivity we prove:

Lemma 9.7. *Let $\mathscr{A}$ be an abelian category with enough injectives. Every left bounded complex is quasi-isomorphic to a left bounded complex with injective components.*

Proof (Keller [Kel90, 4.1, Lemma, b)]). Let A be a complex with $A^n = 0$ for $n < 0$. Choose an embedding $\alpha^0 : A^0 \rightarrowtail I^0$ of A^0 into an injective and form the push-out diagram:

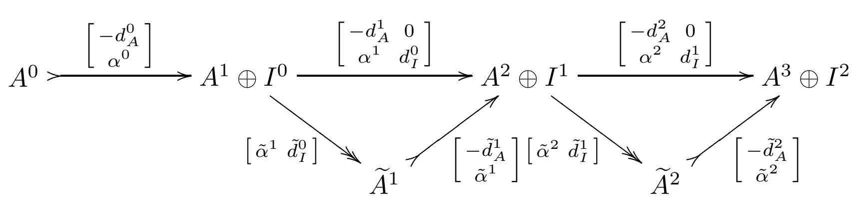


Since $d_A^1 d_A^0 = 0$ the universal property of push-out squares yields a unique morphism $\tilde{d}_A^1 : \widetilde{A}^1 \to A^2$ such that $\tilde{d}_A^1 \tilde{\alpha}^1 = d_A^1$ and $\tilde{d}_A^1 \tilde{d}_I^0 = 0$. Now choose an embedding $\bar{\alpha}^1 : \widetilde{A}^1 \rightarrowtail I^1$ of $\widetilde{A}^1$ into an injective and form the push-out under $\tilde{d}_A^1$ and $\bar{\alpha}^1$ to obtain



It follows from $d_A^2 d_A^1 = 0$ and the universal property of push-out squares that $d_A^2 \tilde{d}_A^1 = 0$. Again, by the universal property of push-out squares, we find a unique morphism $\tilde{d}_A^2 : \widetilde{A}^2 \to A^3$ such that $\tilde{d}_A^2 \tilde{\alpha}^2 = d_A^2$ and $\tilde{d}_A^2 \tilde{d}_I^1 = 0$. Now choose an embedding $\bar{\alpha}^2 : \widetilde{A}^2 \to I^2$ of $\widetilde{A}^2$ into an injective, form the push-out under $\tilde{d}_A^2$ and $\bar{\alpha}^2$, *etc.*

Put $I^n = 0$ for $n < 0$ and put $d_I^n = \bar{\alpha}^{n+1} \tilde{d}_I^n$ for $n \geq 0$, the reader will readily check that I is indeed a complex. Put $\alpha^n = \bar{\alpha}^n \tilde{\alpha}^n$ for $n > 0$ and $\alpha^n = 0$ for $n < 0$, this is obviously a chain map $\alpha : A \to I$. We claim that $\alpha : A \to I$ is a quasi-isomorphism. Indeed, its mapping cone is the complex



which is exact because the exactness of the sequences $\widetilde{A}^n \rightarrowtail A^{n+1} \oplus I^n \twoheadrightarrow \widetilde{A}^{n+1}$ follows from the universal property of push-out squares. □

LEMMA 9.8. *Let I be a left bounded complex of injectives and let $I \xrightarrow{f} A$ be a quasi-isomorphism. Then f has a left inverse in $\mathbf{K}(\mathscr{A})$.*

PROOF. Because f is a quasi-isomorphism, cone(f) is acyclic. By the comparison theorem, see [Wei94, 2.2.6, 2.2.7], the morphism

$$\operatorname{cone}(f) \xrightarrow{[1\ 0]} \Sigma I$$

is null homotopic by a map $[k\ s] : \operatorname{cone}(f) \to I$ of graded objects (*not* a chain map!). The second coordinate of the equation

$$\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}k & s\end{bmatrix}\begin{bmatrix}-d_I & 0\\ f & d_A\end{bmatrix} - d_I\begin{bmatrix}k & s\end{bmatrix} = \begin{bmatrix}-kd_I+sf-d_Ik & sd_A-d_Is\end{bmatrix}$$

shows that $s : A \to I$ is a chain map and the first coordinate of the equation proves that sf is homotopic to the identity of I via k. □

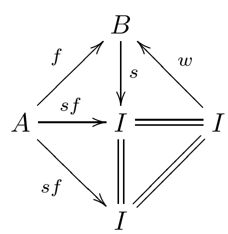
Now we are ready to prove full faithfulness of the functor $\mathbf{K}^+(\mathscr{I}) \to \mathbf{D}^+(\mathscr{A})$, we even prove:

LEMMA 9.9. *If I is a left bounded complex of injectives then*

$$\operatorname{Hom}_{\mathbf{D}(\mathscr{A})}(A, I) \cong \operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(A, I)$$

for each complex A.

PROOF. Let the fraction $A \xrightarrow{f} B \xleftarrow{w} I$ represent a morphism $A \to I$ in $\mathbf{D}$, so suppose w is a quasi-isomorphism. By the previous lemma, there is a left inverse s of w and the diagram



proves that $(w \setminus f) = (1 \setminus sf)$, in other words: every morphism $A \to I$ in $\mathbf{D}$ is represented by a morphism $A \to I$ in $\mathbf{K}$.

On the other hand, if two parallel morphisms $f, g : A \to I$ of $\mathbf{K}$ are identified in $\mathbf{D}$ then there exists a quasi-isomorphism $I \xrightarrow{t} B$ such that $tf = tg$. Again by the previous lemma t has a left inverse s, so $f = stf = stg = g$ in $\mathbf{K}$. □

EXERCISE 9.10. Suppose that every object of the abelian category $\mathscr{A}$ has an injective resolution of *finite length*. Prove that $\mathbf{K}^b(\mathscr{I}) \cong \mathbf{D}^b(\mathscr{A})$.

REMARK 9.11. So far, we have not worried about the smallness of Hom-sets in the derived category. The equivalence $\mathbf{D}^+(\mathscr{A}) \cong \mathbf{K}^+(\mathscr{I})$ shows that $\mathbf{D}^+$ has small Hom-sets provided that there are enough injectives. Since $\mathbf{D}^b \subset \mathbf{D}^+$ is a full subcategory, this holds for the bounded derived category as well. To prove that the unbounded derived category $\mathbf{D}(\mathscr{A})$ has small Hom-sets one needs further assumptions, such as existence and exactness of filtered colimits. If $\mathscr{A}$ is the category of modules over a ring or the category of sheaves on a topological space, this can be done quite painlessly, see e.g. [Wei94, Chapter 10.4] and [Spa88].

EXERCISE 9.12. Let $\mathscr{A}$ be an abelian category and let $\mathscr{E} \subset \mathscr{A}$ be a full additive subcategory which is closed under extensions in the sense that for every short exact sequence $B' \rightarrowtail A \twoheadrightarrow B''$ with $B', B'' \in \mathscr{E}$ also A lies in $\mathscr{E}$. In that case, $\mathscr{E}$ is called an *exact subcategory* of $\mathscr{A}$. Let $\mathscr{S}$ be the class of short exact sequences with all three objects in $\mathscr{E}$. Monics appearing in a sequence in $\mathscr{S}$ are called admissible monics, similarly for epics. Prove that $\mathscr{E}$ is an *exact category* in the sense of Quillen [Qui73]:

(i) Identity morphisms are both admissible monics and admissible epics.
(ii) The composition of two admissible monics (epics) is again an admissible monic (epic).
(iii) The push-out under an admissible monic and an arbitrary morphism exists in $\mathscr{E}$ and yields an admissible monic. The pull-back over an admissible epic and an arbitrary morphism exists and yields an admissible epic:

$$\begin{array}{ccc} E_0' & \rightarrowtail & E_0 \\ \downarrow & \text{PO} & \downarrow \\ E_1' & \rightarrowtail & E_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E_1 & \twoheadrightarrow & E_1'' \\ \downarrow & \text{PB} & \downarrow \\ E_0 & \twoheadrightarrow & E_0''. \end{array}$$

Remark 9.13. Exact categories can be characterized intrinsically, that is, without assuming that there is an ambient abelian category. See [Kel90, Appendix A] for a discussion of the axioms and see [Kel96] and [Nee90] for the construction of the derived category of an exact category.

Given an exact subcategory $\mathscr{E}$ of $\mathscr{A}$, we define its derived category $\mathbf{D}(\mathscr{E})$ to be the Verdier quotient $\mathbf{K}(\mathscr{E})/\mathbf{Ac}(\mathscr{E})$. Since the inclusion functor $i : \mathscr{E} \subset \mathscr{A}$ is exact, the universal property of $\mathbf{D}(\mathscr{E})$ yields a commutative diagram

$$\begin{array}{ccc} \mathbf{K}^*(\mathscr{E}) & \longrightarrow & \mathbf{K}^*(\mathscr{A}) \\ \downarrow & & \downarrow \\ \mathbf{D}^*(\mathscr{E}) & \xrightarrow{\iota^*} & \mathbf{D}^*(\mathscr{A}) \end{array}$$

of triangulated categories for each $* \in \{\ ,+,-,b\}$. We are interested in the properties of the functor ι^*.

Example 9.14. Let $\mathscr{A}$ be an abelian category. Then the full subcategory $\mathscr{I}$ of injective objects is an exact subcategory of $\mathscr{A}$. By the comparison theorem the category $\mathbf{Ac}^+(\mathscr{I})$ is equivalent to the zero category (every left bounded and acyclic complex of injectives is null-homotopic). In particular $\mathbf{K}^+(\mathscr{I}) = \mathbf{D}^+(\mathscr{I})$ and we already know that ι^+ is an equivalence of triangulated categories provided that $\mathscr{A}$ has enough injectives.

This example generalizes as follows:

Theorem 9.15 ([Kel96, Theorem 12.1]). *Let $\mathscr{E}$ be an exact subcategory of an abelian category $\mathscr{A}$ and consider the functor $\iota^+ : \mathbf{D}^+(\mathscr{E}) \to \mathbf{D}^+(\mathscr{A})$.*

(i) If for all $A' \in \mathscr{A}$ there exists a short exact sequence $A' \rightarrowtail E \twoheadrightarrow A''$ with $E \in \mathscr{E}$ then ι^+ is essentially surjective.
(ii) If for every short exact sequence $E' \rightarrowtail A \twoheadrightarrow A''$ with $E' \in \mathscr{E}$ there exists a commutative diagram

$$\begin{array}{ccccc} E' & \rightarrowtail & A & \twoheadrightarrow & A'' \\ \| & & \downarrow & & \downarrow \\ E' & \rightarrowtail & E & \twoheadrightarrow & E'' \end{array}$$

in which the lower row is exact and $E, E'' \in \mathscr{E}$ then ι^+ is fully faithful.

In particular, if the conditions in (i) and (ii) are both satisfied then ι^+ is an equivalence of triangulated categories.

EXERCISE 9.16. For a ring R consider the categories $R-\mathbf{Mod}$ of left R-modules and $\mathbf{Mod}-R$ of right R-modules. Recall that a left R-module F is *flat* if the functor $-\otimes_R F : \mathbf{Mod}-R \to \mathbf{Ab}$ is exact. If $\mathscr{F} \subset R-\mathbf{Mod}$ denotes the full subcategory of flat R-modules prove that $\mathbf{D}^-(\mathscr{F})$ is equivalent to $\mathbf{D}^-(R-\mathbf{Mod})$.

10. Derived Functors

Let us consider the simplest case in which we are given an additive functor $F : \mathscr{A} \to \mathscr{B}$ from an abelian category $\mathscr{A}$ with enough injectives to another abelian category $\mathscr{B}$. For every object $A \in \mathscr{A}$, choose an injective resolution $A \to I$, then the ith classical derived functor of F is given on objects by $R^iF(A) = H^i(F(I))$. Since we know that the natural functor $\varphi : \mathbf{K}^+(\mathscr{I}) \to \mathbf{D}^+(\mathscr{A})$ is an equivalence of triangulated categories, we can choose a quasi-inverse $\psi : \mathbf{D}^+(\mathscr{A}) \to \mathbf{K}^+(\mathscr{I})$. A simple way to give ψ is to choose for each left bounded complex A a quasi-isomorphism $r_A : A \to I_A$ to a left bounded complex of injectives. For convenience only, let us assume that $A = I_A$ if A is already a left bounded complex of injectives and that $r_{\Sigma A} = \Sigma r_A$ for all left bounded complexes A. Then ψ is obviously a triangle functor such that $\psi\varphi = \mathrm{id}_{\mathbf{K}^+(\mathscr{I})}$ and the natural isomorphism $\mathrm{id}_{\mathbf{D}^+(\mathscr{A})} \Rightarrow \varphi\psi$ is given by the collection of quasi-isomorphisms r_A.

Now define the (total) *right derived functor* $\mathbf{R}^+F : \mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{B})$ on objects by

$$\mathbf{R}^+F(A) = q_{\mathscr{B}}F\psi(A),$$

where $q_{\mathscr{B}} : \mathbf{K}^+(\mathscr{B}) \to \mathbf{D}^+(\mathscr{B})$ is the quotient functor and by abuse of notation, F denotes the restriction of $\mathbf{K}^+F$ to $\mathbf{K}^+(\mathscr{I})$. By construction it is clear that $\mathbf{R}^+F$ is a triangle functor. Moreover, we have:

PROPOSITION 10.1.

(i) There is a natural transformation $\alpha : q_{\mathscr{B}}F \Rightarrow \mathbf{R}^+Fq_{\mathscr{A}}$

$$\begin{array}{ccc} \mathbf{K}^+(\mathscr{A}) & \xrightarrow{F} & \mathbf{K}^+(\mathscr{B}) \\ \downarrow{\scriptstyle q_{\mathscr{A}}} & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle q_{\mathscr{B}}} \\ \mathbf{D}^+(\mathscr{A}) & \xrightarrow{\mathbf{R}^+F} & \mathbf{D}^+(\mathscr{B}) \end{array}$$

(ii) For every triangle functor $G : \mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{B})$ *and every natural transformation* $\gamma : q_{\mathscr{B}}F \Rightarrow Gq_{\mathscr{A}}$ *there is a unique* $\beta : \mathbf{R}^+F \Rightarrow G$ *such that* $\gamma = (\beta q_{\mathscr{A}}) \circ \alpha$.

PROOF. To construct the natural transformation α, simply recall that for each A in $\mathbf{K}^+(\mathscr{A})$ we have chosen a quasi-isomorphism $r_A : A \to I_A = \psi q_{\mathscr{A}}(A)$. Applying the functor $q_{\mathscr{B}}F$ yields the morphism $\alpha_A : q_{\mathscr{B}}F(A) \to \mathbf{R}^+Fq_{\mathscr{A}}(A)$. We leave it to the reader as an exercise to check that this is indeed a natural transformation in order to complete point *(i)*.

To prove point *(ii)*, we construct the diagram

$$\begin{array}{ccccc} A & \quad & q_{\mathscr{B}}F(A) & \xrightarrow{\gamma_A} & Gq_{\mathscr{A}}(A) \\ \downarrow{\scriptstyle r_A} & & \downarrow{\scriptstyle q_{\mathscr{B}}F(r_A)} & & \downarrow{\scriptstyle Gq_{\mathscr{A}}(r_A)} \\ I_A & & q_{\mathscr{B}}F(I_A) & \xrightarrow{\gamma_{I_A}} & Gq_{\mathscr{A}}(I_A). \end{array}$$

Observe first that $q_{\mathscr{B}}F(I_A) = \mathbf{R}^+F(A)$, second that $q_{\mathscr{B}}F(r_A) = \alpha_A$ and third that $q_{\mathscr{A}}(r_A)$ is an isomorphism, hence it makes sense to put $\beta_A = (Gq_{\mathscr{A}}(r_A))^{-1}\gamma_{I_A}$ and the reader will readily check that this is a natural transformation with the required property and that it is unique. □

The proposition gives the universal property of $\mathbf{R}^+F$ and thus justifies:

Definition 10.2. A *(total) right derived functor* of $F : \mathscr{A} \to \mathscr{B}$ is a pair $(\mathbf{R}^+F, \alpha)$ consisting of a triangle functor $\mathbf{R}^+F : \mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{B})$ and a natural transformation $\alpha : q_{\mathscr{B}}F \Rightarrow \mathbf{R}^+F$ having the universal property *(ii)* of the previous proposition.

Remark 10.3. By definition, the derived functor $(\mathbf{R}^+F, \alpha)$ is unique up to unique isomorphism.

Example 10.4 (Classical Derived Functors). Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between abelian categories and assume that $\mathbf{R}^+F$ exists. Considering an object $A \in \mathscr{A}$ as a complex considered in degree zero, we put $R^iF(A) = H^i(\mathbf{R}^+F(A))$ and obtain a family of functors $R^iF : \mathscr{A} \to \mathscr{B}$. By Example 8.3 each short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ sits in a distinguished triangle $A' \to A \to A'' \to \Sigma A$ in $\mathbf{D}^+(\mathscr{A})$. Now $\mathbf{R}^+F$ is a triangle functor, hence we obtain a distinguished triangle

$$\mathbf{R}^+F(A') \to \mathbf{R}^+F(A) \to \mathbf{R}^+F(A'') \to \Sigma\,\mathbf{R}^+F(A')$$

in $\mathbf{D}^+(\mathscr{B})$. Applying the homological functor H^i, we obtain the long exact sequence

$$R^iF(A') \to R^iF(A) \to R^iF(A'') \to R^{i+1}(A')$$

in $\mathscr{B}$. Obviously, this sequence is natural in the short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ so that we have constructed a δ-functor.

Example 10.5 (Ext). Consider an abelian category $\mathscr{A}$ with enough injectives. Let $A, B \in \mathscr{A}$ and choose an injective resolution $B \to I$ of B. We have seen in Lemma 9.9 that

$$\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(A, B) \cong \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(A, I)$$

and we can identify

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(A, I) \cong \mathbf{R}^+\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(A, -)(B),$$

where we consider $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(A, -)$ as a triangle functor $\mathbf{K}^+(\mathscr{A}) \to \mathbf{K}^+(\mathbf{Ab})$. It is therefore customary to write

$$\mathbf{Ext}^i(A, B) = \mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(A, \Sigma^i B)$$

and the classical Ext-bifunctor is given by

$$\mathrm{Ext}^i(A, B) = H^i(\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(A, B)) = H^0(\mathbf{Ext}^i(A, B)).$$

Compare also with Exercise 3.2.

Instead of exploiting the equivalence $\mathbf{K}^+(\mathscr{I}) \cong \mathbf{D}^+(\mathscr{A})$ for constructing $\mathbf{R}^+F$, we can use the generalization $\mathbf{D}^+(\mathscr{E}) \cong \mathbf{D}^+(\mathscr{A})$ for some exact subcategory $\mathscr{E}$ of $\mathscr{A}$.

Definition 10.6. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. An exact subcategory $\mathscr{E} \subset \mathscr{A}$ is (right) *adapted* to F if

(i) For every object $A' \in \mathscr{A}$ there exists an exact sequence $A' \rightarrowtail E \twoheadrightarrow A''$ with $E \in \mathscr{E}$.

(ii) For every short exact sequence $E' \rightarrowtail A \twoheadrightarrow A''$ with $E' \in \mathscr{E}$ there exists a commutative diagram

$$\begin{array}{ccccc} E' & \rightarrowtail & A & \twoheadrightarrow & A'' \\ \| & & \downarrow & & \downarrow \\ E' & \rightarrowtail & E & \twoheadrightarrow & E'' \end{array}$$

in which the lower row is exact and $E, E'' \in \mathscr{E}$.

(iii) The restriction of F to $\mathscr{E}$ is exact.

Let us explain the point of this definition: First, the inclusion $\mathscr{E} \subset \mathscr{A}$ induces a triangle functor $\varphi : \mathbf{D}^+(\mathscr{E}) \to \mathbf{D}^+(\mathscr{A})$ which is an equivalence of categories by Theorem 9.15. Second, the restriction $F : \mathbf{K}^+(\mathscr{E}) \to \mathbf{K}^+(\mathscr{B})$ factors uniquely over a triangle functor $\tilde{F} : \mathbf{D}^+(\mathscr{E}) \to \mathbf{D}^+(\mathscr{B})$ by the universal property of $\mathbf{D}^+(\mathscr{E})$. Choosing a quasi-inverse $\psi : \mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{E})$ of φ, we can construct the right derived functor by setting $\mathbf{R}^+F = \tilde{F}\psi$. By refining the proof of the previous proposition we obtain:

Theorem 10.7 (Generalized Existence Theorem). *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Suppose there is an exact subcategory $\mathscr{E} \subset \mathscr{A}$ which is adapted to F. Then the right derived functor $\mathbf{R}^+F$ exists and can be computed as above.*

Lemma 10.8. *Suppose that $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are additive functors between abelian categories and that the right derived functors $(\mathbf{R}^+F, \alpha)$, $(\mathbf{R}^+G, \beta)$ and $(\mathbf{R}^+(GF), \gamma)$ all exist. Then there exists a unique natural transformation*

$$\eta : \mathbf{R}^+(GF) \Rightarrow \mathbf{R}^+G \circ \mathbf{R}^+F$$

such that for each $A \in \mathbf{K}^+(\mathscr{A})$ the following diagram in $\mathbf{D}^+(\mathscr{C})$ is commutative:

$$\begin{array}{ccc} q_{\mathscr{C}}GF(A) & \xrightarrow{\beta_{F(A)}} & (\mathbf{R}^+G)q_{\mathscr{B}}F(A) \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \mathbf{R}^+G(\alpha_A)} \\ \mathbf{R}^+(GF)(q_{\mathscr{A}}A) & \xrightarrow{\eta_{q_{\mathscr{A}}A}} & \mathbf{R}^+G\,\mathbf{R}^+F(q_{\mathscr{A}}A). \end{array}$$

Proof. Apply the universal property of $(\mathbf{R}^+(GF), \gamma)$. □

Corollary 10.9 (Composition Theorem). *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be additive functors between abelian categories. Assume that there are exact subcategories $\mathscr{E} \subset \mathscr{A}$ and $\mathscr{F} \subset \mathscr{B}$ such that $\mathscr{E}$ is adapted to F and $\mathscr{F}$ is adapted to G. Moreover, assume that $F(\mathscr{E}) \subset \mathscr{F}$. Then:*

(i) The right derived functors $(\mathbf{R}^+F, \alpha)$, $(\mathbf{R}^+G, \beta)$ and $(\mathbf{R}^+(GF), \gamma)$ exist.
(ii) The unique natural transformation $\eta : \mathbf{R}^+(GF) \Rightarrow \mathbf{R}^+G \circ \mathbf{R}^+F$ given by Lemma 10.8 yields an isomorphism

$$\mathbf{R}^+(GF) \cong \mathbf{R}^+G \circ \mathbf{R}^+F$$

of functors.

Proof. Let us prove (i). By the generalized existence theorem, the derived functors $\mathbf{R}^+F$ and $\mathbf{R}^+G$ exist. Obviously, GF is exact on $\mathscr{E}$, so $\mathscr{E}$ is adapted to GF and hence $\mathbf{R}^+GF$ exists as well.

To prove (ii), just observe that for a complex $E \in \mathbf{K}^+(\mathscr{E})$ we have

$$\mathbf{R}^+(GF)(q_{\mathscr{A}}E) \cong q_{\mathscr{C}}GF(E) \cong \mathbf{R}^+G(q_{\mathscr{B}}F(E)) \cong \mathbf{R}^+G(\mathbf{R}^+F(q_{\mathscr{A}}(E)))$$

by our hypotheses. □

Remark 10.10 (Grothendieck Spectral Sequence). The conceptually simple isomorphism $\mathbf{R}^+(GF) \cong \mathbf{R}^+G \circ \mathbf{R}^+F$ underlies the Grothendieck spectral sequence, see [Wei94, 10.8.3]. More precisely, for every left bounded chain complex A the *hypercohomology spectral sequence* with E_2-page given by $E_2^{pq} = R^pG(H^q(\mathbf{R}^+F(A)))$ converges to $H^{p+q}(\mathbf{R}^+(GF)(A))$. If A is a complex concentrated in degree zero, the latter can be identified with $R^{p+q}(GF)(A)$ and the E_2-page is precisely the initial page of the Grothendieck spectral sequence.

Remark 10.11. Left derived functors can be treated by dualization of the entire section. A left derived functor of F is a pair $(\mathbf{L}^{-}F, \alpha)$ consisting of a triangle functor $\mathbf{L}^{-}F : \mathbf{D}^{-}(\mathscr{A}) \to \mathbf{D}^{-}(\mathscr{B})$ and a natural transformation $\alpha : \mathbf{L}^{-}Fq_{\mathscr{A}} \Rightarrow q_{\mathscr{B}}F$, *etc.*

11. A Brief Guide to the Literature

The idea of the derived category is an outcome of Grothendieck's re-foundation of algebraic geometry and homological algebra. The details were carried out by his student Verdier in his thesis, see [Ver96] and [Dea77, Appendice A]. A nice historical account of the development of the ideas is given in the first section of [Ill90].

A very good – if abstract – account of the main ideas is contained in Keller's survey [Kel96], largely without proofs but with precise references. We also recommend [Kel98].

The textbook-level introductions the author is aware of are: [Wei94, Chapter 10], [GM03, Chapters III, IV], [BGK+87, Chapitre I], [KS94, Chapter I], [KS06, Chapter 13], [Ive86].

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