I Topological Field Theories, Quantum Groups, and Geometric Langlands By Jacob Lurie (MIT) May 26, 2009

Notation:

- G: reductive alg. group over $\mathbb{C}$, $G^\vee$ its Langlands dual (e.g. $G = GL_n = G^\vee$)
- X: alg. curve over $\mathbb{C}$.
- $\mathrm{Bun}_G(X)$ (G-bundles on X)
- $\mathrm{Loc}_G(X)$ (G-bundles on X with connections)

Conjecture (Geom. Langlands): $\mathrm{QCoh}(\mathrm{Loc}_G(X)) \cong \mathcal{D}\text{-mod}(\mathrm{Bun}_G(X))$.
Local systems

$\mathrm{Loc}_G(X) \simeq_{\text{anal.}} (BG)^X \simeq ((\pi_1(X) \to G)/\,\text{conj.})$

Remark I.1. On $\mathrm{Bun}_G(X)$ there is a canonical line bundle, called det. $\rightsquigarrow$ gives a twisted version $\mathcal{D} - \mathrm{mod}_c(\mathrm{Bun}_G)$ (twist by the c-th power of det). ■

Quantum Geometric Langlands: $\mathcal{D} - \mathrm{mod}_c(\mathrm{Bun}_G) \cong \mathcal{D} - \mathrm{mod}_{-\frac{1}{c}}(\mathrm{Bun}_{G^\vee})$
TFT

Recall: definition of Bord_n (symmetric monoidal (∞, n)-category)

Definition I.2. An (extended) TQFT is a $\otimes$-functor $\mathrm{Bord}_n \to \mathcal{C}$, where $\mathcal{C}$ is another $\otimes$ (∞, n)-category. ■

Definition I.3. If M is a m-manifold $m \leq n$, then an n framing of M is a trivialization $TM \oplus \underline{\mathbb{R}}^{n-m} \simeq \underline{\mathbb{R}}^n$ ■

Theorem I.4 (Framed Cobordism Hypothesis).

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \curvearrowleft O(n) \simeq \text{"}n\text{-dualizable objects in } \mathcal{C}\text{"} \curvearrowleft O(n) \quad Z \mapsto Z(\mathrm{pt})$$

Theorem I.5 (Non-framed Cob-Hyp.).

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \text{"}n\text{-dualizable objects in } \mathcal{C}\text{"} \quad Z \mapsto Z(\mathrm{pt})$$

Example of a 2-category: Alg: (objects: $\mathbb{C}$-algebras, morphisms: bimodules, 2-morphisms: maps of bimodules). Every object of Alg is 1-dualizable and it is 2-dualizable iff it is semi-simple $\rightsquigarrow$ boring!

Example of a $(\infty, 2)$-category: dgAlg (objects: dg-$\mathbb{C}$-algebras, morphisms dg-bimod. 2-morph.: maps of dg-bimod. 3-morph.: chain homotopies). Every objects in dgAlg is 1-dualizable and it is 2-dualizable iff it is a smooth and proper dg-algebra.

To increase the "category level": replace dg-vector spaces by dg-categories and an additional associative product.

Definition I.6. Braid: $(\infty, 4)$-category with

objects: braided monoidal dg-categories

morph.: monoidal dg-categories

2-morph.: dg-categories

3-morph.: functors

4-morph.: nat. transfromations

... and so on... ■

Relation to the representation category $\mathrm{Rep}_q(G)$ of a quantum group ...(missed something)... by dualizability, $\mathrm{Rep}_q(G)$ gives a 3d theory $Z : \mathrm{Bord}_3 \to \mathrm{Braid}$ with

$$Z(\Sigma) = \otimes_{x\in\Sigma} \mathrm{Rep}_q(G)$$

for Σ a closed surface.

Example I.7. $q = 1 \Rightarrow \mathrm{Rep}_q(G) = \mathrm{Rep}(G)$ and

$$\otimes_{x\in\Sigma} \mathrm{Rep}(G) \cong \mathrm{QCoh}(BG^{\Sigma})$$ ■

$Z(S^2) \simeq$ higher Drinfeld center of $\mathrm{Rep}_q(G)$.
Now S^2 has two incarnations in alg. geom:

- $S^2 \cong \mathbb{P}^1 \Rightarrow \mathcal{D}(\mathrm{Bun}_G(\mathbb{P}^1))$
- $S^2 \cong D^2 \coprod_{\dot{D}^2} D^2 \Rightarrow G[[t]]\backslash G((t))/G[[t]]$

$\mathrm{Gr}_G = G((t))/G[[t]]$ (affine Grassmannian) $\mathcal{D}(\mathrm{Bun}(D^2 \coprod_{\dot{D}^2} D^2)) \cong \mathcal{D}(\mathrm{GR}_G)^{G[[t]]}$ (Spherical Hecke Category or Satake Category)

There is a gemoetric Satake isomorphism (Mirkovic-Vilones) $\mathcal{D}_c(\mathrm{Gr}_G)^{G[[t]]} \simeq$ trivial if c is generic

$\mathcal{D}(\mathrm{Gr})$ is "like" a braided moniodal category. $\mathrm{Gr}_G(x,y) = \{G-\text{bundles trivialized on} X - \{x,y\}\} \simeq \{\mathrm{Gr}\times\mathrm{Gr} \text{ if } x \neq y\}$ or $\{\mathrm{Gr} \text{ if } x = y\}$

$\mathcal{D}_c(\mathrm{Gr}_G)$ can be simplified in two ways:

- Consider $G((t))$-equivariant $\mathcal{D}$-modules, which is by Kazhdan-Luztik equivalent to $\mathrm{Rep}_q(G)$.
- Take a full subcategory of "Whittake" objects, which also is equivalent to $\mathrm{Rep}_{q'}(G^\vee)$

$$\otimes_{x\in X} \mathrm{Whit} \leftarrow \mathcal{D}_c(\mathrm{Bun}_G(X)) \to \otimes_{x\in X} KL$$

where $\to$ is fully faithful

Table of analogies:

Topological (Betti)	**alg./geom. (de Rham)**	**physics (Kapuskin-Witten)**
$\mathrm{QCoh}(BG^X)$	$\mathrm{QCoh}(\mathrm{Loc}_G(X)) \cong \mathcal{D}-\mathrm{mod}(\mathrm{Bun}_G(X))$	The geom. Langlangs corresp. comes from an equivalence of 4d-TFTs ($n = 4$: super Yang-Mills in the GL-twist for G and $G^\vee$)
$q = e^{2\pi ic}$	twisting parameter $c \in \mathbb{C}$	c or $\mathbb{C}/(1,c)$
Comes from a TQFT?:		
Yes (for a boring $\mathcal{C}$)	No (but the same corresponding structures)	Maybe (with a very interesing $\mathcal{C}$)
$q \mapsto e^{\frac{4\pi}{\log(q)}}$	$c \mapsto \frac{1}{c}$	
$\mathrm{Rep}_q(G)$	$\mathcal{D}(\mathrm{Gr}_G)$	