Orthomodularity in dagger biproduct categories

John Harding

Abstract

Abramsky and Coecke [2] have recently introduced an approach to finite dimensional quantum mechanics based on strongly compact closed categories with biproducts. In this note it is shown that the projections of any object A in such a category forms an orthoalgebra *Proj* A. Sufficient conditions are given to ensure this orthoalgebra is an orthomodular poset. A notion of a preparation for such an object is given in [2], and it is shown that each preparation induces a finitely additive map from $ProjA$ to the unit interval of the semiring of scalars for this category. The tensor product for the category is shown to induce an orthoalgebra bimorphism $ProjA \times ProjB \to Proj(A \otimes B)$ that shares some of the properties required of a tensor product of orthoalgebras.

These results are established in a setting more general than that of strongly compact closed categories. Many are valid in dagger biproduct categories, others require also a symmetric monoidal tensor compatible with the dagger and biproduct structure. Examples are considered for several familiar strongly compact closed categories.

1 Introduction

Abramsky and Coecke [2] introduced an axiomatic approach to finite dimensional quantum mechanics based on strongly compact closed categories with biproducts. In this setting they are able to develop many features familiar to quantum mechanics, including scalars, measurements, probabilities, as well as tensor products for treating compound systems. They also find interesting links to linear logic, and develop a graphical calculus that has appealing application to matters such as quantum information protocols. For a good introduction to this area see [1, 3, 21, 22].

Our aim is to connect the approach of Abramsky and Coecke to the quantum logic approach to the axiomatics of quantum mechanics initiated by Birkhoff and von Neumann [4, 14, 17, 20, 19]. It turns out that many features of the quantum logic approach fit very nicely within the framework of Abramsky and Coecke's method, as we briefly describe below.

For an object A in a strongly compact closed category, the projections *Proj* A form a type of orthomodular structure known as an orthoalgebra (abbreviated: OA). This OA is shown to belong to an enveloping orthomodular poset, and under a natural assumption about idempotents splitting they coincide. Scalars are maps from the tensor unit to itself. These scalars form a quasiordered semiring with unit interval $[0,1]_C$. Preparations of A are certain maps from the tensor unit to A and each preparation induces a finitely additive measure from *Proj* A to $[0,1]_C$. Finally, the

OA *Proj* $(A \otimes B)$ has many of the properties one would ask of a tensor product of the OAs *Proj* A and *Proj* B.

Our results do not require the full strength of strongly compact closed categories with biproducts. To obtain basic properties of *Proj* A we require only a dagger biproduct category [21]. This is a category C with biproduct $\oplus$ (an operation that acts both as a distinguished product and coproduct) and a period two contravariant functor $\dagger : C \to C$, called the adjoint, that is compatible with the biproduct. Further results require also a symmetric monoidal tensor $\otimes$ that is compatible with the adjoint and biproduct. In particular, tensor distributes over biproducts. We do not use the strong compact closed property of Abramsky and Coecke, and it is not apparent what impact this condition has at the level of *Proj* A.

The categorical and quantum logic approaches are somewhat complementary; the categorical approach is built to deal with the compound systems and processes the quantum logic approach struggles with, while the quantum logic approach is designed to deal with properties of isolated systems which are not primitive in the categorical approach. It seems advantageous for both approaches to be be combined in a common setting.

From a practical standpoint, having quantum logic built into the categorical approach allows access to a large body of work. This may point to refinements of the conditions one places on the categories such as the splitting of idempotents mentioned above. Further, the categorical approach will have to be modified to accommodate infinite dimensional quantum mechanics. This realization of quantum logic within the categorical approach is based on the simple notion of direct product decompositions, and may be sufficiently resilient to persist through, and help guide, such modifications.

This paper is organized in the following fashion. In the second section we provide background on dagger biproduct categories. In the third we introduce the weak projections of an object A in a dagger biproduct category. These are certain self-adjoint idempotents of A. We show that these weak projections of A naturally form an orthomodular poset. Here the idea is similar to the familiar idea from quantum logic that the idempotents of a ring form an orthomodular poset [11, 15]. In this case we are taking certain idempotents of the semiring of endomorphisms of A.

In the fourth section we introduce the projections of A. These are certain weak projections that arise from biproduct decompositions of A. We show the projections of A form an OA *Proj* A. The structure placed on these projections comes from the notion of one decomposition refining another as in [11]. Projections and weak projections are related in the fifth section, and it is shown that the two notions coincide if self-adjoint idempotents strongly split.

In the sixth section we give basics of dagger symmetric monoidal categories with biproducts (abbreviated: DSMB-categories). These are dagger biproduct categories with a tensor $\otimes$ that is compatible with both the dagger structure and the biproduct structure. These are more general than the strongly compact closed categories of Abramsky and Coecke. In the seventh section we review the fact that the endomorphisms of the tensor unit in such a category form a commutative semiring, called the semiring of scalars, and define a quasiordering on this semiring. The unit interval $[0,1]_C$ of the category is the unit interval in this semiring. It is then shown that the preparations of an object A, as defined by Abramsky and Coecke [2], give rise to finitely additive measures $Proj\ A \to [0,1]_C$ which we call states.

In the eighth section we consider tensor products of the OAs $Proj\ A$ and $Proj\ B$. We show the OA $Proj\ (A \otimes B)$ has a number of the more physically motivated conditions one would ask of a tensor product. In particular, there is a bimorphism into this OA, and certain states on $Proj\ A$ and $Proj\ B$ lift to a state on $Proj\ (A \otimes B)$, at least when states are considered as mappings into the unit interval $[0,1]_C$ of the category, rather than into the usual real unit interval.

In the ninth section the notions we have discussed are considered in the categories *Rel* of sets and relations, the category *FDHilb* of finite dimensional Hilbert spaces, and in the category Mat_K of matrices over a field K. In *Rel* things behave somewhat classically with $Proj\ A$ being the Boolean algebra of subsets of A, and in *FDHilb* we have $Proj\ \mathcal{H}$ is the usual orthomodular lattice of closed subspaces of $\mathcal{H}$. In Mat_K there is interesting behavior with an example of $Proj\ m$ being an orthomodular lattice that is not modular. The final section contains concluding remarks.

It is hoped that this paper is of interest to people working on the categorical foundations of quantum mechanics, and to ones working in quantum logic. We have tried to present the results in a manner that is accessible to both groups. For experts on one side or the other, please have patience with the pedestrian approach. Good references for the categories we consider are [2, 10, 18, 21, 22] and for aspects of quantum logic considered here see [7, 14, 20].

2 Dagger Biproduct Categories

We provide basic definitions and results. For a complete account of these categories and their properties see [1, 2, 3, 21, 22], and for general references on those aspects of category theory most pertinent here, see [10, 18].

Definition 2.1 *A dagger category C is a category with an involutive contravariant functor $\dagger : C \to C$, called adjoint, that is the identity on objects. Specifically*

1. $A^\dagger = A$ *for any object* A
2. *If* $f : A \to B$ *then* $f^\dagger : B \to A$
3. $id_A^\dagger = id_A$
4. $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$
5. $f^{\dagger\dagger} = f$.

Definition 2.2 *An object* 0 *in a category is a zero object if it is both initial and terminal. For such a zero object, for each pair of objects* A, B *there is a unique morphism* $A \to 0 \to B$ *that we denote* $0_{A,B}$.

Definition 2.3 *A category with biproducts is a category with zero* 0 *such that for any* $A_1, \ldots, A_n$ *there is an object* $A_1 \oplus \cdots \oplus A_n$ *that is both a product with projections* $\pi_i : A_1 \oplus \cdots \oplus A_n \to A_i$ *and a coproduct with injections* $\mu_i : A_i \to A_1 \oplus \cdots \oplus A_n$ *where*

$$\pi_i \circ \mu_j = \begin{cases} 1_{A_i} & \text{if } i = j \\ 0_{A_j,A_i} & \text{if } i \neq j \end{cases}$$

In a category, a family of objects may have many different biproducts. We use $A_1 \oplus \cdots \oplus A_n$ and μ_i, π_i for one chosen biproduct. As in [10] we use $< f_i >$ for a morphism into the product completing a cone, $[g_i]$ for a morphism from the coproduct completing the cone, and as product morphisms and coproduct morphisms agree in a category with biproducts, we use $\bigoplus h_i$ for these [10]. This is stated precisely below.

Definition 2.4 *If* $A \xrightarrow{f_i} A_i$, $B_i \xrightarrow{g_i} B$ *and* $A_i \xrightarrow{h_i} B_i$ *for* $i = 1, 2$ *we define* $< f_1, f_2 >: A \to A_1 \oplus A_2$, $[g_1, g_2] : B_1 \oplus B_2 \to B$, *and* $h_1 \oplus h_2 : A_1 \oplus A_2 \to B_1 \oplus B_2$ *to be the unique morphisms with*

1. $\pi_i \circ < f_1, f_2 >= f_i$
2. $[g_1, g_2] \circ \mu_i = g_i$
3. $\pi_i \circ (h_1 \oplus h_2) = h_i \circ \pi_i$ *and* $(h_1 \oplus h_2) \circ \mu_i = \mu_i \circ h_i$.

For objects A, B we use $C(A, B)$ for the homset of all morphisms from A to B. In any category with biproducts each homset carries the structure of a commutative monoid in an essentially unique way [10, pg. 310]. We outline this below.

Definition 2.5 *For* $f, g : A \to B$, *define* $f + g = [1_B, 1_B] \circ (f \oplus g) \circ < 1_A, 1_A >$.

$$A \to A \oplus A \xrightarrow{f \oplus g} B \oplus B \to B$$

Proposition 2.6 *If* C *has biproducts, the operation* $+$ *on* $C(A, B)$ *satisfies*

1. $+$ *is commutative and associative.*
2. $0_{A,B}$ *is an identity element for* $+$.
3. $(f + g) \circ h = f \circ h + g \circ h$ *and* $f \circ (g + h) = f \circ g + f \circ h$.

We next describe the matrix calculus for categories with biproducts. This also can be found in [10, pg. 317]. This matrix calculus will be our primary tool for calculations in such categories.

Proposition 2.7 *In a category with biproducts, any* $f : A_1 \oplus \cdots \oplus A_m \to B_1 \oplus \cdots \oplus B_n$ *is determined by the matrix* $F = (f_{ij})$ *where* $f_{ij} : A_j \to B_i$ *is* $f_{ij} = \pi_i \circ f \circ \mu_j$.

$$F = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nm} \end{pmatrix}$$

To illustrate, consider the identity map on $A \oplus B$. This map has a representation a 2 by 2 matrix. Computing the ij^{th} entry of this matrix as $\pi_i \circ 1 \circ \mu_j$, the conditions in the Definition 2.3 give this matrix as $\left(\begin{smallmatrix} 1_A & 0_{B,A} \\ 0_{A,B} & 1_B \end{smallmatrix}\right)$. Often we omit the subscripts on the identity and zero maps and simply write the matrix for the identity map on $A \oplus B$ as the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Similarly, the matrix for the morphism $\mu_1 \circ \pi_1$ from $A \oplus B$ to itself is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, the matrix for $\mu_2 \circ \pi_2$ is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and the matrix for the zero map from $A \oplus B$ to itself is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Obvious extensions to $A_1 \oplus \cdots \oplus A_n$ hold.

Proposition 2.8 *Suppose* $A_1 \oplus \cdots \oplus A_k \xrightarrow{e,f} B_1 \oplus \cdots \oplus B_m \xrightarrow{g} C_1 \oplus \cdots \oplus C_n$ *have matrices* E, F, G. *Then*

1. $e + f$ *has matrix* $E + F$
2. $g \circ f$ *has matrix* GF.

Here matrix addition and multiplication are defined in the natural way using + and composition for the addition and multiplication of the entries.

For example, suppose $f, g : A \to B$. Then $[1, 1] \circ (f \oplus g) \circ < 1, 1 >$ in matrix form becomes $\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ which simplifies to $f + g$ as expected. As another illustration, the identity below follows by computing the matrices of each morphism.

Proposition 2.9 $\mu_1 \circ \pi_1 + \cdots + \mu_n \circ \pi_n$ *is the identity map on* $A_1 \oplus \cdots \oplus A_n$.

Definition 2.10 *A dagger biproduct category is a category* C *with biproducts and a dagger* † *where*

1. $\pi_i^\dagger = \mu_i$
2. $\mu_i^\dagger = \pi_i$
3. $0_{A,B}^\dagger = 0_{B,A}$

The matrix calculus extends in a natural way to this setting.

Proposition 2.11 *If $f : A_1 \oplus \cdots \oplus A_m \to B_1 \oplus \cdots \oplus B_n$ has matrix $F = (f_{ij})$, then the adjoint $f^\dagger$ has matrix $F^\dagger = (f^\dagger_{ji})$. So $F^\dagger$ is the transpose with the adjoint taken of each entry.*

$$F^\dagger = \begin{pmatrix} f^\dagger_{11} & f^\dagger_{21} & \cdots & f^\dagger_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ f^\dagger_{1m} & f^\dagger_{2m} & \cdots & f^\dagger_{mn} \end{pmatrix}$$

Proof. $(f^\dagger)_{ij} = \pi_i \circ f^\dagger \circ \mu_j = (\mu_i^\dagger \circ f^\dagger \circ \pi_j^\dagger) = (\pi_j \circ f \circ \mu_i)^\dagger = (f_{ji})^\dagger$. ∎

Proposition 2.12 *If C is a dagger biproduct category, then*

1. $< f, g >^\dagger = [f^\dagger, g^\dagger]$
2. $[f, g]^\dagger = < f^\dagger, g^\dagger >$
3. $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$
4. $(f + g)^\dagger = f^\dagger + g^\dagger$

3 Weak Projections

Throughout this section we assume C is a dagger biproduct category and that A is an object in C. We will show that the collection of all weak projections of A forms an orthomodular poset. The motivating example is the well known fact that the projection operators of a Hilbert space form an orthomodular lattice.

Definition 3.1 *$p : A \to A$ is a weak projection if there is $p' : A \to A$ such that*

1. *Both p, p' are idempotent and self-adjoint.*
2. $pp' = 0 = p'p$.
3. $p + p' = 1$.

We set $Proj_w A$ to be the set of all weak projections on A.

Proposition 3.2 *If p is a weak projection, p' is unique, so $' : Proj_w A \to Proj_w A$.*

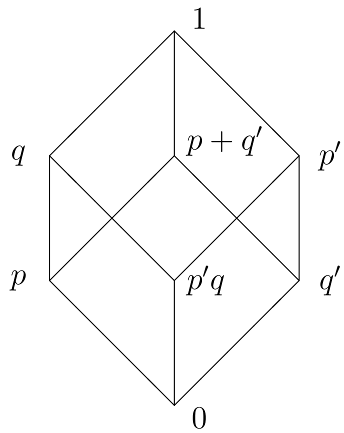
Proof. Suppose that p', p'' are two such morphisms. We then have $p' = p' \circ 1_A = p'(p + p'') = p'p + p'p'' = p'p''$, and $p'' = 1_A \circ p'' = (p + p')p'' = p'p''$. ∎

Definition 3.3 *Define $\leq_w$ on $Proj_w A$ by $p \leq_w p$ iff $pq = p = qp$.*

Lemma 3.4 *If $p \leq_w q$ then*

1. $pq = p = qp$
2. $p'q' = q' = q'p'$
3. $pq' = 0 = q'p$
4. $p'q = qp'$
5. $p + p'q = q$ *and* $q' + p'q = p'$.

This is shown in the figure below where all nodes are weak projections.



Proof. 1. This is from the definition. 2. $q' = q'(p + p') = q'(qp + p') = q'p'$ and $q' = (p + p')q' = (pq + p')q' = p'q'$. 3. $pq' = pp'q' = 0 = q'p'p = q'p$. 4. Note $1 = (p + p')(q + q') = p + p'q + q'$ and $1 = (q + q')(p + p') = p + qp' + q'$. So $p'q = p'q(p + qp' + q') = p'qp'$ and $qp' = (p + p'q + q')qp' = p'qp$. 5. By 4, $(p + p'q)^\dagger = p^\dagger + q^\dagger p'^\dagger = p + p'q$ and $(p + p'q)(p + p'q) = p + p'q$. So $p + p'q$ is self-adjoint and idempotent. But $(p + p'q)q' = 0 = q'(p + p'q)$ and $p + p'q + q' = (p + p')(q + q') = 1$. So by the uniqueness in Proposition 3.2 $p + p'q = q$. That $q' + p'q = p'$ is similar. ∎

Definition 3.5 $(P, \leq, 0, 1, \perp)$ *An orthomodular poset (abbreviated:* OMP*) if*

1. $(P, \leq, 0, 1)$ *is a bounded poset.*
2. $\perp : P \to P$ *is order inverting, period two, and $x^\perp$ is a complement of x.*
3. $x \leq y \Rightarrow x, y^\perp$ *have a least upper bound* $x \vee y^\perp$.
4. $x \leq y \Rightarrow x \vee (x \vee y^\perp)^\perp = y$.

Orthomodular posets [20] are the building blocks of the quantum logic approach to the foundations of quantum mechanics. They serve as models of the *Yes-No* propositions of a quantum mechanical system. The partial ordering $\leq$ reflects that one proposition implies another, orthocomplementation $\perp$ gives the negation of a proposition, and for orthogonal propositions ($x \leq y^\perp$) their join $x \vee y$ gives their disjunction. Mackey [17] provided an argument why the propositions of a quantum system should form an OMP. It is difficult to argue that arbitrary propositions should have a disjunction, this is why OMPs are used rather than their lattice counterparts orthomodular lattices.

Theorem 3.6 $(Proj_w\, A, \leq_w, {}', 0, 1)$ *is an orthomodular poset (*OMP*). Further, when elements* p, q *are orthogonal, their join is given by* $p \vee q = p + q$.

Proof. First, 0 and 1 are self-adjoint idempotents with $0 \circ 1 = 0 = 1 \circ 0$ and $0 + 1 = 1$. So $0, 1$ are weak projections with $0' = 1$. Also, for any weak projection p we have $0 \circ p = 0 = p \circ 0$ and $p \circ 1 = p = 1 \circ p$, so $0 \leq_w p$ and $p \leq_w 1$. We next show $\leq_w$ is a partial order. Suppose p, q, r are weak projections. As p is idempotent we have $p \leq_w p$. Suppose $p \leq_w q$ and $q \leq_w p$. Then $pq = p = qp$ and $qp = q = pq$, so $p = q$, giving antisymmetry. Suppose $p \leq_w q$ and $q \leq_w r$. Then $pr = (pq)r = p(qr) = pq = p = qp = (rq)p = r(qp) = rp$, so $p \leq_w r$.

Consider the map $'$. If $p \leq_w q$, the above lemma gives $q'p' = q' = p'q'$, so $q' \leq_w p'$. Thus $'$ is order inverting, and it is period two by definition. Suppose $p, p' \leq_w q$. Then $q = (p+p')q = pq + p'q = p + p' = 1$. This shows $p \vee p' = 1$, and as $'$ is order inverting and period two, $p \wedge p' = 0$.

Suppose $p \leq_w q$. We claim $p + q'$ is the least upper bound of p, q'. First, by Lemma 3.4, we know $p + q'$ is a weak projection with companion $p'q$. As $p(p + q') = p = (p + q')p$ and $q'(p + q') = q' = (p + q')p$ we have $p + q'$ is an upper bound of p, q'. If r is another such upper bound, then $(p+q')r = pr + q'r = p + q' = rp + rq' = r(p + q')$. So $p + q' \leq r$ showing $p + q'$ is the least upper bound. Finally, if $p \leq_w q$, by Lemma 3.4 $p \vee (p \vee q')' = p + (p + q')' = p + p'q = q$. ∎

Remark 3.7 *It is well-known that the idempotents of a commutative ring with unit form a Boolean algebra. This construction can be extended [11, 14, 15] to show that the idempotents of a ring with unit form an* OMP*. The above result may be viewed as an extension to the setting of a semiring, i.e. a commutative semigroup equipped with a multiplication that distributes over addition. One takes the idempotents* e *that have a companion* e' *that behave like* $1 - e$*, namely, that satisfies* $ee' = 0$ *and* $e + e' = 1$. *The existence of a dagger is not a vital part of this construction, rather it something tolerated by the construction.*

4 Projections

In this section we specialize the weak projections of the previous section to involve the dagger structure in an essential way and link more closely with the work of Abramsky and Coecke [2]. Again we assume A is an object in a dagger biproduct category C. We freely employ the matrix calculus for such categories, often using lower case letters such as p for a morphism and the corresponding upper case letter P for its matrix.

Definition 4.1 *$u : A \to B$ is unitary if it is an isomorphism and $u^\dagger = u^{-1}$.*

Proposition 4.2 *The composition of unitaries is unitary.*

Definition 4.3 *A standard projection matrix on $A_1 \oplus \cdots \oplus A_n$ is a matrix where off-diagonal entries are 0 and each diagonal entry is either 0 or 1. A permutation matrix is a matrix for the obvious morphism $p : A_1 \oplus \cdots \oplus A_n \to A_{\sigma(1)} \oplus \cdots \oplus A_{\sigma(n)}$ for some permutation σ of $1, \ldots, n$. Such a permutation matrix is one whose entries are all either 0 or 1, and each row and column has exactly one 1.*

Proposition 4.4 *Each permutation matrix is unitary.*

Definition 4.5 *$p : A \to A$ is a projection if there is a unitary $u : A \to A_1 \oplus A_2$ satisfying the following equivalent conditions.*

1. $p = u^\dagger \mu_1 \pi_1 u$
2. $P = U^\dagger \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U$

Let Proj A be the set of all projections on A.

Proposition 4.6 *$Proj A \subseteq Proj_w A$.*

Proof. If p is a projection, p has matrix $U^\dagger \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U$ for some unitary $u : A \to A_1 \oplus A_2$. Let $p' : A \to A$ have matrix $U^\dagger \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) U$. Simple matrix calculations show that p, p' are self-adjoint idempotents with $pp' = 0 = p'p$ and $p + p' = 1$. ∎

The definition of a projection says that in matrix form it can be represented $U^\dagger \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U$. One might ask whether a matrix representation such as $U^\dagger \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) U$ also yields a projection. This is the case. Indeed, as the following result shows, projections are obtained from any $U^\dagger S U$ where S is a standard projection matrix, meaning that S is all 0's except for some of its diagonal entries which are 1's.

Proposition 4.7 *If $u : A \to A_1 \oplus \cdots \oplus A_n$ is unitary then for any distinct $i_1, \ldots, i_k$, the morphism $u^\dagger(\mu_{i_1}\pi_{i_1} + \cdots + \mu_{i_k}\pi_{i_k})u$ is a projection.*

Proof. We treat the typical case $p = u^\dagger(\mu_1\pi_1 + \mu_3\pi_3)u$ where $u : A \to A_1 \oplus A_2 \oplus A_3$ and leave the reader to formulate the general argument. The matrix for p is given by $U^\dagger\left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right)U$. Consider the morphism $v : A_1 \oplus A_2 \oplus A_3 \to (A_1 \oplus A_3) \oplus A_2$ whose matrix is given by $V = \left(\begin{smallmatrix}\mu_1&0&\mu_2\\0&1&0\end{smallmatrix}\right)$. We recall $\mu_1 : A_1 \to A_1 \oplus A_3$ and $\mu_2 : A_3 \to A_1 \oplus A_3$ are the biproduct injections. By Definition 2.10 and Proposition 2.11, $V^\dagger = \left(\begin{smallmatrix}\pi_1&0\\0&1\\\pi_2&0\end{smallmatrix}\right)$. So

$$VV^\dagger = \begin{pmatrix}\mu_1\pi_1 + \mu_2\pi_2 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad V^\dagger V = \begin{pmatrix}\pi_1\mu_1 & 0 & \pi_1\mu_2\\ 0 & 1 & 0\\ \pi_2\mu_1 & 0 & \pi_2\mu_2\end{pmatrix}$$

Using Proposition 2.9 and Definition 2.3, both of these are identity matrices, and it follows that V is unitary. Then VU is unitary, giving that $U^\dagger V^\dagger\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)VU$ is a projection. But $U^\dagger V^\dagger\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)VU = U^\dagger\left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right)U$. ∎

Definition 4.8 *Projections p, q on A are orthogonal, written $p \perp q$, if there is a unitary $u : A \to A_1 \oplus A_2 \oplus A_3$ with $p = u^\dagger\mu_1\pi_1 u$ and $q = u^\dagger\mu_3\pi_3 u$.*

A proof very similar to that of Proposition 4.7 above provides the following.

Proposition 4.9 *If $p = u^\dagger\mu_i\pi_i u$ and $q = u^\dagger\mu_j\pi_j u$ for some $i \neq j$ and some unitary $u : A \to A_1 \oplus \cdots \oplus A_n$ and $i \neq j$, then p, q are orthogonal.*

While the sum $p + q$ of arbitrary projections need not be a projection, it follows from Proposition 4.7 that the sum of orthogonal projections is a projection. Therefore the restriction of $+$ to orthogonal pairs of projections yields a partial operation on $Proj\ A$. The following is a standard notion in quantum logic [7].

Definition 4.10 *An orthoalgebra (abbreviated:* OA*) is a set X with constants $0, 1$, a binary relation $\perp$ called orthogonality, and a partial binary operation $\oplus$ defined for orthogonal pairs and called orthogonal sum, satisfying*

1. *If $f \perp g$ then $g \perp f$ and $f \oplus g = g \oplus f$.*
2. *For each $f \in X$ there is a unique $f' \in X$ with $f \perp f'$ and $f \oplus f' = 1$.*
3. *If $f \perp f$ then $f = 0$.*
4. *If $e \perp f$ and $(e \oplus f) \perp g$, then $f \perp g$, $e \perp (f \oplus g)$ and $(e \oplus f) \oplus g = e \oplus (f \oplus g)$.*

There is a close relationship between OAs and OMPs that we discuss in detail in the following section. Here our objective is to show that $Proj A$ forms an OA.

Lemma 4.11 $0, 1$ *are projections.*

Proof. Let $u : A \to 0 \oplus A$ have matrix $U = \left(\begin{smallmatrix} 0_{A,0} \\ 1_A \end{smallmatrix}\right)$. Then $U^\dagger = \left(\begin{smallmatrix} 0_{0,A} & 1_A \end{smallmatrix}\right)$. As 0 is initial, there is exactly one morphism from 0 to itself, so $0_{A,0}0_{0,A} = 1_0$. This yields that $UU^\dagger = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $U^\dagger U = \left(\begin{smallmatrix} 1 \end{smallmatrix}\right)$, showing U is unitary. Note $U^\dagger \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U = \left(\begin{smallmatrix} 0 \end{smallmatrix}\right)$ and $U^\dagger \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) U = \left(\begin{smallmatrix} 1 \end{smallmatrix}\right)$, giving that $0, 1$ are projections. ∎

Lemma 4.12 *If $p, q \in ProjA$ and $p \perp q$, then $q \perp p$ and $p + q = q + p$.*

Proof. Definition 4.8 and Proposition 4.9 show that if $p \perp q$ then $q \perp p$, and we know $+$ is commutative. ∎

Lemma 4.13 *If $p \in ProjA$ there is a unique $p' \in ProjA$ with $p \perp p'$ and $p + p' = 1$.*

Proof. Suppose p is a projection given by $U^\dagger \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U$. Then for p' having matrix $U^\dagger \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) U$, Proposition 4.7 gives p' is a projection and Proposition 4.9 gives $p \perp p'$. Clearly $p + p' = 1$, and this gives existence. For uniqueness, Proposition 4.6 shows such p, p' are a weak projection and its complement, so uniqueness follows by Proposition 3.2. ∎

Lemma 4.14 *If $p \in ProjA$ and $p \perp p$, then $p = 0$.*

Proof. Suppose $p \perp p$. By definition, there is a unitary $u : A \to A_1 \oplus A_2 \oplus A_3$ with $p = u^\dagger \mu_1 \pi_1 u$ and $p = u^\dagger \mu_3 \pi_3 u$. So $p = pp = 0$. ∎

Proposition 4.15 *If $w : A \to B$ is unitary, there is a map $\varphi : ProjA \to ProjB$ defined by $\varphi p = wpw^\dagger$. This map is a bijection, satisfies $p \perp q$ iff $\varphi p \perp \varphi q$, as well as $\varphi(p + q) = \varphi p + \varphi q$ whenever $p \perp q$.*

Proof. This is a simple consequence of the definitions and the fact that the composite of unitaries is unitary. ∎

Lemma 4.16 *Suppose $p, q, r \in ProjA$ and $p \perp q$, $p + q \perp r$. Then $q \perp r$, $p \perp q + r$ and $(p + q) + r = p + (q + r)$.*

Proof. If $w : A \to B$ is unitary and $\varphi : Proj\ A \to Proj\ B$ is the map given by Proposition 4.15, then for $p, q, r \in Proj\ A$ we have $p \perp q$ and $p + q \perp r$ iff $\varphi p \perp \varphi q$ and $\varphi p + \varphi q \perp \varphi r$, and we have $q \perp r$ and $p \perp q + r$ iff $\varphi q \perp \varphi r$ and $\varphi p \perp \varphi q + \varphi r$. So to verify our result for p, q, r, it is sufficient to choose some unitary $w : A \to B$ and prove it for $\varphi p, \varphi q, \varphi r$. In particular, as we consider $p \perp q$, there is a unitary $w : A \to A_1 \oplus A_2 \oplus A_3$ with $p = w^\dagger \mu_1 \pi_1 w$ and $q = w^\dagger \mu_3 \pi_3 w$. Then using this unitary w, we have $\varphi p = \mu_1 \pi_1$ and $\varphi q = \mu_3 \pi_3$. In sum, we may assume without loss of generality that $A = A_1 \oplus A_2 \oplus A_3$, and that p, q, r are projections on A with

$p = \mu_1\pi_1$, $q = \mu_3\pi_3$, and $p + q \perp r$. We must show $q \perp r$ and $p \perp q + r$. That $(p+q)+r = p+(q+r)$ is obvious as $+$ is always associative.

Establishing our result requires a series of calculations; we first make a few guiding remarks. The projections $p = \mu_1\pi_1$ and $q = \mu_3\pi_3$ on $A = A_1 \oplus A_2 \oplus A_3$ come from the natural projections onto A_1 and A_3. If $A_2 = X \oplus Y$, then the projection r of A coming from a projection onto one of the factors X or Y satisfies $p + q \perp r$. In our proof, we show that all such morphisms r with $p + q \perp r$ essentially arise this way up to some unitary isomorphisms.

We begin the calculations. As $p + q \perp r$ there is a unitary $u : A_1 \oplus A_2 \oplus A_3 \to B_1 \oplus B_2 \oplus B_3$ with $p+q = u^\dagger \tilde{\mu}_1 \tilde{\pi}_1 u$ and $r = u^\dagger \tilde{\mu}_3 \tilde{\pi}_3 u$. Here $\tilde{\mu}_i, \tilde{\pi}_i$ are the injections and projections associated with $B_1 \oplus B_2 \oplus B_3$. In matrix form these conditions become

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \quad \text{and} \quad R = U^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} U \tag{4.1}$$

Writing U in component form and multiplying the first of these matrix equations on the left by U, we obtain the following after simple matrix multiplications.

$$\begin{pmatrix} u_{11} & 0 & u_{13} \\ u_{21} & 0 & u_{23} \\ u_{31} & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

Therefore we know that U and its adjoint $U^\dagger$ look as follows.

$$U = \begin{pmatrix} u_{11} & 0 & u_{13} \\ 0 & u_{22} & 0 \\ 0 & u_{32} & 0 \end{pmatrix} \quad \text{and} \quad U^\dagger = \begin{pmatrix} u_{11}^\dagger & 0 & 0 \\ 0 & u_{22}^\dagger & u_{32}^\dagger \\ u_{13}^\dagger & 0 & 0 \end{pmatrix} \tag{4.3}$$

Computing, we obtain

$$UU^\dagger = \begin{pmatrix} u_{11}u_{11}^\dagger + u_{13}u_{13}^\dagger & 0 & 0 \\ 0 & u_{22}u_{22}^\dagger & u_{22}u_{32}^\dagger \\ 0 & u_{32}u_{22}^\dagger & u_{32}u_{32}^\dagger \end{pmatrix} \tag{4.4}$$

as well as

$$U^\dagger U = \begin{pmatrix} u_{11}^\dagger u_{11} & 0 & u_{11}^\dagger u_{13} \\ 0 & u_{22}^\dagger u_{22} + u_{32}^\dagger u_{32} & 0 \\ u_{13}^\dagger u_{11} & 0 & u_{13}^\dagger u_{13} \end{pmatrix} \tag{4.5}$$

As U is unitary, both $UU^\dagger$ are identity matrices. This provides the following: (a) $u_{11}u_{11}^\dagger + u_{13}u_{13}^\dagger = 1_{B_1}$, (b) $u_{22}u_{22}^\dagger = 1_{B_2}$, (c) $u_{32}u_{32}^\dagger = 1_{B_3}$, (d) $u_{22}u_{32}^\dagger = 0$,

(e) $u_{32}u_{22}^\dagger = 0$, (f) $u_{11}^\dagger u_{11} = 1_{A_1}$, (g) $u_{22}^\dagger u_{22} + u_{32}^\dagger u_{32} = 1_{A_2}$, (h) $u_{13}^\dagger u_{13} = 1_{A_3}$, (i) $u_{11}^\dagger u_{13} = 0$, and (j) $u_{13}^\dagger u_{11} = 0$.

To digest these conditions, note they say the morphism $A_1 \oplus A_3 \to B_1$ with matrix $\begin{pmatrix} u_{11} & u_{13} \end{pmatrix}$ is unitary and the morphism $A_2 \to B_2 \oplus B_3$ with matrix $\begin{pmatrix} u_{22} \\ u_{32} \end{pmatrix}$ is unitary. Therefore $A_1 \oplus A_2 \oplus A_3$ is unitarily isomorphic to $A_1 \oplus (B_2 \oplus B_3) \oplus A_3$, and u behaves like the obvious morphism $A_1 \oplus A_2 \oplus A_3 \to (A_1 \oplus A_3) \oplus B_2 \oplus B_3$ that uses the isomorphism $\begin{pmatrix} u_{22} \\ u_{32} \end{pmatrix}$ to split A_2 into $B_2 \oplus B_3$.

We next define $v : A_1 \oplus A_2 \oplus A_3 \to A_3 \oplus (A_1 \oplus B_2) \oplus B_3$ to be the unique morphism whose matrix V and its adjoint $V^\dagger$ are given by

$$V = \begin{pmatrix} 0 & 0 & 1_{A_3} \\ \mu_1 & \mu_2 u_{22} & 0 \\ 0 & u_{32} & 0 \end{pmatrix} \quad \text{and} \quad V^\dagger = \begin{pmatrix} 0 & \pi_1 & 0 \\ 0 & u_{22}^\dagger \pi_2 & u_{32}^\dagger \\ 1_{A_3} & 0 & 0 \end{pmatrix} \tag{4.6}$$

Here we use μ_i and π_i for the canonical injections and projections associated with $A_1 \oplus B_2$. In particular, $A_1 \xrightarrow{\mu_1} A_1 \oplus B_2$, $A_2 \xrightarrow{u_{22}} B_2 \xrightarrow{\mu_2} A_1 \oplus B_2$ and $A_2 \xrightarrow{u_{32}} B_3$.

To see that V is unitary we compute

$$VV^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_1\pi_1 + \mu_2 u_{22} u_{22}^\dagger \pi_2 & \mu_2 u_{22} u_{32}^\dagger \\ 0 & u_{32} u_{22}^\dagger \pi_2 & u_{32} u_{32}^\dagger \end{pmatrix} \tag{4.7}$$

Note $\mu_1\pi_1 + \mu_2 u_{22} u_{22}^\dagger \pi_2 = \mu_1\pi_1 + \mu_2\pi_2$ by condition (b) above, and by Proposition 2.9 this is the identity map $1_{A_1 \oplus B_2}$. Also, $u_{32}u_{32}^\dagger = 1_{B_3}$ by (c), $u_{32}u_{22}^\dagger \pi_2 = 0$ by (e), and $\mu_2 u_{22} u_{32}^\dagger = 0$ by (d). Note also

$$V^\dagger V = \begin{pmatrix} \pi_1\mu_1 & \pi_1\mu_2 u_{22} & 0 \\ u_{22}^\dagger \pi_2 \mu_1 & u_{22}^\dagger \pi_2 \mu_2 u_{22} + u_{32}^\dagger u_{32} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.8}$$

As $\pi_i\mu_j = \delta_{ij}$ we have $\pi_1\mu_1 = 1_{A_1}$, $\pi_1\mu_2 u_{22} = 0$, and $u_{22}^\dagger \pi_2 \mu_1 = 0$. We also have $u_{22}^\dagger \pi_2 \mu_2 u_{22} + u_{32}^\dagger u_{32} = u_{22}^\dagger u_{22} + u_{32}^\dagger u_{32} = 1_{A_2}$ by (g). This shows that V is unitary.

Making computations with this unitary V, we find

$$V^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V = \begin{pmatrix} 0 & \pi_1 & 0 \\ 0 & u_{22}^\dagger \pi_2 & u_{32}^\dagger \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.9}$$

and

$$V^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V = \begin{pmatrix} 0 & \pi_1 & 0 \\ 0 & u_{22}^\dagger \pi_2 & u_{32}^\dagger \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u_{32}^\dagger u_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.10}$$

The matrix in equation (4.9) is that of $q = \mu_3\pi_3$. Equations (4.1) and (4.3), give

$$R = \begin{pmatrix} u_{11}^\dagger & 0 & 0 \\ 0 & u_{22}^\dagger & u_{32}^\dagger \\ u_{13}^\dagger & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{11} & 0 & u_{13} \\ 0 & u_{22} & 0 \\ 0 & u_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u_{32}^\dagger u_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.11}$$

and it follows that the matrix in equation (4.10) is R. So q and r are orthogonal via the unitary v, that is, $q \perp r$.

It remains to show $p \perp q + r$. To do so, we must construct another unitary. Let $w : A_1 \oplus A_2 \oplus A_3 \to A_1 \oplus B_2 \oplus (A_3 \oplus B_3)$ be the unique morphism whose matrix W and its adjoint $W^\dagger$ are given by

$$W = \begin{pmatrix} 1_{A_1} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & \mu_2 u_{32} & \mu_1 \end{pmatrix} \quad \text{and} \quad W^\dagger = \begin{pmatrix} 1_{A_1} & 0 & 0 \\ 0 & u_{22}^\dagger & u_{32}^\dagger \pi_2 \\ 0 & 0 & \pi_1 \end{pmatrix} \tag{4.12}$$

Here we use μ_i and π_i for the canonical injections and projections associated with $A_3 \oplus B_3$. In particular, $A_2 \xrightarrow{u_{32}} B_3 \xrightarrow{\mu_2} A_3 \oplus B_3$ and $A_3 \xrightarrow{\mu_1} A_3 \oplus B_3$. Computing, we have

$$WW^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{22}u_{22}^\dagger & u_{22}u_{32}^\dagger \pi_2 \\ 0 & \mu_2 u_{32} u_{22}^\dagger & \mu_2 u_{32} u_{32}^\dagger \pi_2 + \mu_1 \pi_1 \end{pmatrix} \tag{4.13}$$

and

$$W^\dagger W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{22}^\dagger u_{22} + u_{32}^\dagger \pi_2 \mu_2 u_{32} & u_{32}^\dagger \pi_2 \mu_1 \\ 0 & \pi_1 \mu_2 u_{32} & \pi_1 \mu_1 \end{pmatrix} \tag{4.14}$$

Then using the properties (a) through (i) given after equation (4.5) we see that $WW^\dagger$ and $W^\dagger W$ both evaluate to identity matrices, so W is unitary. Computing,

$$W^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{22}^\dagger & u_{32}^\dagger \pi_2 \\ 0 & 0 & \pi_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.15}$$

and

$$W^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{22}^\dagger & u_{32}^\dagger \pi_2 \\ 0 & 0 & \pi_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu_2 u_{32} & \mu_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u_{32}^\dagger u_{32} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.16)$$

The matrix in equation (4.15) is that of $p = \mu_1\pi_1$ and the matrix in equation (4.16) is the that of $q + r$. Thus, $p \perp q + r$, concluding the proof of the lemma. ∎

Using Lemmas 4.11, 4.12, 4.13, 4.14, 4.16, we have the following.

Theorem 4.17 *$(Proj\ A, 0, 1, \perp, +)$ is an orthoalgebra.*

5 Relating Projections and Weak Projections

It is well known [7] that any orthoalgebra X carries a partial ordering given by $x \leq y$ if there is z with $x \perp z$ and $x \oplus z = y$. We have seen in Proposition 4.6 that each projection is a weak projection, hence the partial ordering $\leq_w$ on the weak projections given by Definition 3.3 restricts to a partial ordering on the projections as well. In this section, we investigate the connection between these two partial orderings, and the connection between the orthoalgebra $Proj\ A$ and the orthomodular poset $Proj_w\ A$. Our strongest results will come under the additional assumption that self-adjoint idempotents strongly split in the category C. In this natural setting, we show that the OA of projections and the OMP of weak projections coincide.

Definition 5.1 *Define $\leq$ on $Proj\ A$ by $p \leq q$ iff there exists a projection r with $p \perp r$ and $p+r = q$, and let $\leq_w$ be the partial ordering on $Proj_w\ A$ defined by $p \leq_w q$ iff $pq = p = qp$.*

Proposition 5.2 *If $p, q \in Proj\ A$, then $p \leq q \Rightarrow p \leq_w q$.*

Proof. If $p \leq q$ there is r with $p \perp r$ and $p + r = q$. As $p \perp r$ there is a unitary $u : A \to A_1 \oplus A_2 \oplus A_3$ with $p = u^\dagger \mu_1 \pi_1 u$, $r = u^\dagger \mu_3 \pi_3 u$, and so $q = u^\dagger(\mu_1\pi_1 + \mu_3\pi_3)u$. It is then routine to verify $pq = p = qp$, hence $p \leq_w q$. ∎

For an OA X, the partial ordering $\leq$ on X described above makes X into an orthocomplemented poset where the orthocomplement x' of x is the unique element with $x \perp x'$ and $x \oplus x' = 1$. In this orthocomplemented poset, if $x \perp y$ are orthogonal elements, then $x \oplus y$ is a minimal, but not necessarily least, upper bound of x, y. If $x \oplus y$ is the least upper bound of x, y whenever $x \perp y$, then the orthocomplemented poset given by X is an OMP. In this case, the OA structure of X can be recovered from the orthocomplemented poset given by X as there is only one minimal upper bound for each orthogonal x, y. On the other hand, every OMP gives rise to an OA where $x \perp y$ iff $x \leq y'$ and $x \oplus y$ is the join of x, y when x, y are orthogonal. So OMPs naturally correspond to the class of OAs where $x \oplus y$ is the least upper bound of x, y for every orthogonal x, y. All these facts are found in [7]. In the following, we naturally consider an OA as an orthocomplemented poset, and an OMP as an OA.

Definition 5.3 *For* OA*s P, Q, a map $f : P \to Q$ is called an* OA *morphism if $f(0) = 0$, and $a \perp b \Rightarrow f(a) \perp f(b)$ and $f(a \oplus b) = f(a) \oplus f(b)$.*

Proposition 5.4 *The inclusion map $i : Proj\ A \to Proj_w\ A$ is an* OA *morphism.*

Proof. Suppose p, q are projections with $p \perp q$. By the definition of orthogonality of projections, there is a unitary $u : A \to A_1 \oplus A_2 \oplus A_3$ with $p = u^\dagger \mu_1 \pi_1 u$ and $q = u^\dagger \mu_3 \pi_3 u$. The orthocomplement q' of q in the OMP $Proj_w\ A$ is the unique weak projection with $qq' = 0 = q'q$, and it follows that $q' = u^\dagger(\mu_1\pi_1 + \mu_2\pi_2)u$. Then a simple calculation gives $pq' = p = q'p$, so $p \leq_w q'$, and therefore p, q are orthogonal in the OMP $Proj_w\ A$. In $Proj\ A$ the orthosum of the orthogonal elements p, q is given by $p + q$. In the OMP $Proj_w\ A$, the orthosum of the orthogonal elements p, q is their join, which by Theorem 3.6 is given by $p + q$. ∎

Remark 5.5 *The inclusion map $i : ProjA \to Proj_w A$ is a one-one* OA *morphism and Proposition 5.2 shows i is order preserving (in fact every* OA *morphism is order preserving). But we do not have that $p \leq_w q \Rightarrow p \leq q$, so it might not be an order embedding. This explains why $Proj\ A$ may be an* OA *but not an* OMP*. Suppose p, q are orthogonal in $Proj\ A$. So their orthosum $p + q$ is a minimal upper bound of p, q in $Proj\ A$. As the inclusion is order preserving, p, q are orthogonal also in the* OMP *$Proj_w A$, and $p + q$ is their join in this* OMP*. If we take a projection r that is an upper bound of p, q in $Proj\ A$, then r is an upper bound of p, q in $Proj_w\ A$, hence $p + q \leq_w r$. However, we may fail to have $p + q \leq r$ in $Proj\ A$ as there may fail to be a unitary isomorphism to witness this. So $p + q$ may be a minimal upper bound of p, q in $Proj\ A$ rather than a minimum upper bound.*

We next consider matters under an additional assumption regarding the self-adjoint idempotents in C. It is common practice to consider conditions related to splitting of idempotents in a category [10, 18], and that the condition we consider naturally arises in Selinger's work as well [22]. As a final comment, we note that each weak projection is by definition a self-adjoint idempotent. linger's work as well [22]. As a final comment, we note that each weak projection is by definition a self-adjoint idempotent.

Definition 5.6 *Self-adjoint idempotents strongly split in C if for each self-adjoint idempotent $e : A \to A$, there is an $f : A \to B$ with $e = f^\dagger f$ and $1_B = ff^\dagger$.*

Theorem 5.7 *If self-adjoint idempotents strongly split in C, then for each object A, every weak projection of A is a projection and $p \leq q$ iff $p \leq_W q$. Therefore the* OA *Proj A coincides with the* OMP *$Proj_w$ A.*

Proof. Suppose p is a weak projection for A with orthocomplement p'. Then as p, p' are self-adjoint idempotents, there are objects B, C and morphisms $f : A \to B$ and $g : A \to C$ with $f^\dagger f = p$, $ff^\dagger = 1_B$, $g^\dagger g = p'$ and $gg^\dagger = 1_C$. Consider the morphism $u : A \to B \oplus C$ with matrix U where $U = \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ and $U^\dagger = \left(\begin{smallmatrix} f^\dagger & g^\dagger \end{smallmatrix}\right)$. Then

$$UU^\dagger = \begin{pmatrix} ff^\dagger & fg^\dagger \\ gf^\dagger & gg^\dagger \end{pmatrix} \quad \text{and} \quad U^\dagger U = \begin{pmatrix} f^\dagger f + g^\dagger g \end{pmatrix} \tag{5.17}$$

Note $ff^\dagger = 1_B$ and $gg^\dagger = 1_C$. Also $fg^\dagger = (ff^\dagger)fg^\dagger(gg^\dagger) = f(f^\dagger f)(g^\dagger g)g^\dagger = fpp'g^\dagger = 0$, and similarly $gf^\dagger = 0$. This shows the first of the above matrices is an identity matrix. As $f^\dagger f + g^\dagger g = p + p' = 1_A$ the second is also an identity matrix. Thus u is unitary. Clearly $U^\dagger \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U = \left(\begin{smallmatrix} f^\dagger f \end{smallmatrix}\right) = \left(\begin{smallmatrix} p \end{smallmatrix}\right)$ and $U^\dagger \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) U = \left(\begin{smallmatrix} g^\dagger g \end{smallmatrix}\right) = \left(\begin{smallmatrix} p' \end{smallmatrix}\right)$. It follows that $p,' p$ are projections given by the unitary u.

We have still to show that if p, q are projections with $p \leq_w q$, then $p \leq q$. To do so, we must use the algebraic conditions given by Lemma 3.4 to build a unitary v realizing $p \leq q$. In particular, we use the fact that $p'q$ is a weak projection, that the product of any two of $p, p'q, q'$ is 0, and that the sum $p + p'q + q' = 1$, all provided by Lemma 3.4.

As $p, p'q, q'$ are weak projections, they are self-adjoint idempotents. So there are objects B, C, D and morphisms $f : A \to B$, $g : A \to C$, and $h : A \to D$ with $f^\dagger f = p$, $ff^\dagger = 1_B$, $g^\dagger g = p'q$, $gg^\dagger = 1_C$, $h^\dagger h = q'$ and $hh^\dagger = 1_D$. We consider then the morphism $v : A \to B \oplus C \oplus D$ with matrix $V = \left(\begin{smallmatrix} f \\ g \\ h \end{smallmatrix}\right)$ and $V^\dagger = \left(\begin{smallmatrix} f^\dagger & g^\dagger & h^\dagger \end{smallmatrix}\right)$. Then

$$VV^\dagger = \begin{pmatrix} ff^\dagger & fg^\dagger & fh^\dagger \\ gf^\dagger & gg^\dagger & gh^\dagger \\ hf^\dagger & hg^\dagger & hh^\dagger \end{pmatrix} \quad \text{and} \quad V^\dagger V = \begin{pmatrix} f^\dagger f + g^\dagger g + h^\dagger h \end{pmatrix} \tag{5.18}$$

Each of $ff^\dagger, gg^\dagger, hh^\dagger$ is an identity map, and as the product of any two of $p, p'q, q'$ is 0, calculations similar to the ones above show the off-diagonal entries of the first matrix, such as $fg^\dagger$ are all 0. So the first of these matrices is an identity matrix. But $f^\dagger f + g^\dagger g + h^\dagger h = p + p'q + q' = 1$, so the second is also an identity matrix. So V is unitary. One sees that $V^\dagger \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) V = \left(\begin{smallmatrix} f^\dagger f \end{smallmatrix}\right) = \left(\begin{smallmatrix} p \end{smallmatrix}\right)$ and $V^\dagger \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) V = \left(\begin{smallmatrix} h^\dagger h \end{smallmatrix}\right) = \left(\begin{smallmatrix} q' \end{smallmatrix}\right)$. This shows p, q' are orthogonal in the OA *Proj* A, and this implies $p \leq q$. ∎

6 Dagger Symmetric Monoidal Structure

In this section we give background on the categories we consider in the remainder of the paper, dagger symmetric monoidal categories with biproducts. These are the dagger biproduct categories considered earlier equipped with a tensor $\otimes$ that is compatible with the dagger and biproduct structure as described below. They are weaker than the the strongly compact closed categories with biproducts of Abramsky and Coecke [2]. None of our results require the symmetry of the tensor, but it seems so natural we have included it anyway.

Definition 6.1 *For a category C, a bifunctor $\otimes : C \times C \to C$ is a functor from the product category $C \times C$ to C. Specifically, this means*

1. *For objects A, B of C there is an object $A \otimes B$ of C.*
2. *For morphisms $f : A \to A'$ and $g : B \to B'$ there is $f \otimes g : A \otimes B \to A' \otimes B'$.*
3. *$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g')$ when the composites are defined.*
4. *$1_A \otimes 1_B = 1_{A \otimes B}$*

Definition 6.2 *A symmetric monoidal category is a category C with a bifunctor $\otimes$ called tensor product, an object I called the tensor unit, and natural isomorphisms*

$$\begin{aligned} \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ \sigma_{A,B} &: A \otimes B \to B \otimes A \\ \lambda_A &: A \to I \otimes A \\ \rho_A &: A \to A \otimes I \end{aligned}$$

where these natural isomorphisms satisfy standard coherence conditions [18, pg. 158]. Among these conditions is the requirement $\lambda_I = \rho_I$.

We next consider categories with some combination of a dagger $\dagger$, a biproduct $\oplus$, and a tensor $\otimes$ that are in some sense compatible. The first instance of this was Definition 2.10 where dagger biproduct categories were defined.

Definition 6.3 *A dagger symmetric monoidal category C is a symmetric monoidal category with with a dagger $\dagger : C \to C$ that satisfies*

1. *$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$*
2. *$\alpha^\dagger_{A,B,C} = \alpha^{-1}_{A,B,C}$*
3. *$\sigma^\dagger_{A,B} = \sigma^{-1}_{A,B}$*
4. *$\lambda^\dagger_A = \lambda^{-1}_A$ and $\rho^\dagger_A = \rho^{-1}_A$*

These dagger symmetric monoidal categories are considered in [21]. We next consider categories that combine a symmetric monoidal structure $\otimes$ and a biproduct structure $\oplus$. We connect the two through the additive structure $+$ that the biproduct induces on each homset $C(A, B)$. The close connection between the additive structure and the biproduct structure is detailed in [10, pg. 310].

Definition 6.4 *A symmetric monoidal category with biproducts is a category with symmetric monoidal structure given by $\otimes$ and biproduct structure given by $\oplus$ so that for any $f, f' : A \to B$ and $g, g' : C \to D$ we have*

1. $f \otimes (g + g') = (f \otimes g) + (f \otimes g')$ *and* $(f + f') \otimes g = (f \otimes g) + (f' \otimes g)$
2. $f \otimes 0 = 0$ *and* $0 \otimes g = 0$

In a category with biproducts, a functor $F : C \to C$ is additive [10, pg. 312] if the induced map $C(A, B) \to C(FA, FB)$ is a monoid homomorphism for each A, B.

Lemma 6.5 *If a category C has a symmetric monoidal structure given by $\otimes$ and a biproduct structure given by $\oplus$, then C is a symmetric monoidal category with biproducts iff for each object A, the functors $A \otimes -$ and $- \otimes A$ are additive.*

Proof. As $\otimes$ is a bifunctor and composition distributes over $+$, the first condition is equivalent to $1_A \otimes (g + g') = (1 \otimes g) + (1 \otimes g')$ and $(f + f') \otimes 1 = (f \otimes 1) + (f' \otimes 1)$. Thus these conditions are equivalent to having $A \otimes -$ and $- \otimes A$ additive. ∎

Definition 6.6 *A category C is a dagger symmetric monoidal biproduct category (abbreviated:* DSMB*-category) if it has a dagger †, tensor $\otimes$, and biproduct $\oplus$ and is simultaneously a dagger category with biproducts, a dagger symmetric monoidal category, and a dagger category with biproducts.*

In a DSMB-category we have use of all the properties in Section 2 as well as those in the definitions above. We next compare these categories with the strongly compact closed categories with biproducts of Abramsky and Coecke [2] which are also called biproduct dagger compact closed categories by Selinger [21].

Proposition 6.7 DSMB-*categories are more general than the strongly compact closed categories with biproducts of Abramsky and Coecke.*

Proof. Each strongly compact closed category with biproducts has a dagger †, tensor $\otimes$ and biproduct $\oplus$. That it is a dagger symmetric monoidal category, and a dagger biproduct category is outlined in [21] and follows directly from [2]. It remains to show $\otimes$ and $\oplus$ satisfy the conditions of Definition 6.4, or by Lemma 6.5, that the functors $A \otimes -$ and $- \otimes A$ are additive. Any strongly compact closed category is compact closed, hence a symmetric monoidal closed category, and this implies that these functors $A \otimes -$ and $- \otimes A$ have a right and left adjoint respectively. It follows by [10, pg. 318] that they are both additive. ∎

7 Scalars and States

In this section we review the known results that the scalars in a DSMB-category form a commutative semiring, and we consider the notion of positivity of morphisms to define a quasiorder on this semiring. We then consider the unit interval $[0,1]_C$ in this quasiordered semiring, and use this to define a notion of finitely additive measures, or states, on the orthostructures *Proj A* we constructed earlier.

Definition 7.1 *A scalar in a symmetric monoidal category is a morphism $s : I \to I$.*

The set of scalars is the homset $C(I, I)$, and this naturally forms a monoid under composition. It is well-known that in any monoidal category this monoid is commutative [16]. We give the proof below as we need a detail for later results.

Lemma 7.2 *If s, t are scalars in a symmetric monoidal category, then*

$$s \circ t = I \xrightarrow{\lambda_I} I \otimes I \xrightarrow{s \otimes t} I \otimes I \xrightarrow{\lambda_I^{-1}} I = t \circ s \tag{7.19}$$

Proof. Consider the following diagram.

$$\begin{array}{ccccccc}
I & \xrightarrow{\rho_I} & I \otimes I & = & I \otimes I & \xrightarrow{\lambda_I^{-1}} & I \\
{\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s \otimes 1} & & {\scriptstyle 1 \otimes t}\downarrow & & \downarrow{\scriptstyle t} \\
I & \xrightarrow[\lambda_I]{\rho_I} & I \otimes I & \xrightarrow{s \otimes t} & I \otimes I & \xrightarrow[\rho_I^{-1}]{\lambda_I^{-1}} & I \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle 1 \otimes t} & & {\scriptstyle s \otimes 1}\uparrow & & \uparrow{\scriptstyle s} \\
I & \xrightarrow{\lambda_I} & I \otimes I & = & I \otimes I & \xrightarrow{\rho_I^{-1}} & I
\end{array}$$

Here we are using the coherence condition that $\lambda_I = \rho_I$ of Definition 6.2. As λ and ρ are natural isomorphisms, the two squares on the left of the diagram commute, as λ^{-1} and ρ^{-1} are natural isomorphisms, the two squares on the right of the diagram commute, and as $\otimes$ is a bifunctor, the two squares in the middle commute. It follows that the top path agrees with the middle and bottom path, giving the result. ∎

Recall that composition distributes over sum in any category with biproducts. This gives the following.

Corollary 7.3 *In an* DSMB*-category the scalars $C(I, I)$ are a commutative semiring under $\circ, +, 0, 1$ with involution $\dagger$ satisfying $(s \circ t)^\dagger = t^\dagger \circ s^\dagger$ and $(s + t)^\dagger = s^\dagger + t^\dagger$.*

Definition 7.4 *A scalar s is positive if there is a morphism $\alpha : I \to A$ with $s = \alpha^\dagger\alpha$.*

Proposition 7.5 *In a* DSMB*-category*

1. $0, 1$ *are positive scalars*
2. *If s is a positive scalar, then $s^\dagger = s$*
3. *If s, t are positive scalars, so are $s + t$ and $s \circ t$.*

Thus the set $C^+(I, I)$ of positive scalars is a sub-involutive semiring of $C(I, I)$.

Proof. 1. $0 = 0^\dagger 0$ and $1 = 1^\dagger 1$. 2. If s is positive, then $s = \alpha^\dagger\alpha$, for some α, so $s^\dagger = \alpha^\dagger\alpha^{\dagger\dagger} = \alpha^\dagger\alpha$. 3. Suppose s, t are positive with $s = \alpha^\dagger\alpha$ and $t = \beta^\dagger\beta$ for some $\alpha : I \to A$ and $\beta : I \to B$. Consider $f, g : I \to A \oplus B$ with matrices $\left(\begin{smallmatrix}\alpha\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\\beta\end{smallmatrix}\right)$, so the matrix for $f + g$ is $\left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right)$. Then $(f + g)^\dagger(f + g) = \left(\begin{smallmatrix}\alpha^\dagger & \beta^\dagger\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right) = \alpha^\dagger\alpha + \beta^\dagger\beta = s + t$. This shows $s + t$ is positive. Finally, we show $s \circ t$ is positive. Using (7.19) we have $s \circ t = \lambda_I^{-1}(s \otimes t)\lambda_I = \lambda_I^{-1}(\alpha^\dagger \otimes \beta^\dagger)(\alpha \otimes \beta)\lambda_I$. Then using the condition $\lambda_I^{-1} = \lambda_I^\dagger$ of Definition 6.3, this equals $[(\alpha \otimes \beta)\lambda_I]^\dagger[(\alpha \otimes \beta)\lambda_I]$. So $s \circ t$ is positive. ∎

Definition 7.6 *For scalars s, t define $s \leq t$ iff $s + p = t$ for some positive scalar p.*

Proposition 7.7 *The relation $\leq$ is a quasiordering on $C(I, I)$ that satisfies (i) if $s_1 \leq t_1$ and $s_2 \leq t_2$ then $s_1 + s_2 \leq t_1 + t_2$, and (ii) if $s \leq t$ and $p \geq 0$ then $ps \leq pt$.*[1]

Proof. As 0 is positive, $\leq$ is reflexive, and as the positives are closed under $+$, we have $\leq$ is transitive, hence a quasiorder. Statement (i) follows as the sum of positives is positive, and statement (ii) follows as $p \geq 0$ means p is positive, and the fact that the product of positives is positive. ∎

Definition 7.8 *In a* DSMB*-category C, we define the unit interval to be*

$$[0, 1]_C = \{p : p \text{ is a scalar and } 0 \leq p \leq 1\}.$$

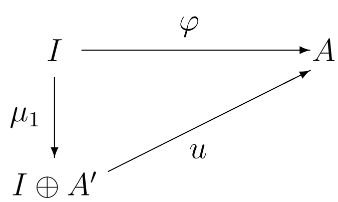
We next turn our attention to states. Recall that in quantum logic, it is common to use the term state in different ways. A unit vector in a Hilbert space $\mathcal{H}$ is often called a pure state of $\mathcal{H}$, and an additive mapping from the lattice of projection operators of $\mathcal{H}$ to the real unit interval is called a state on the OML of projections. Gleason's theorem [6, 20] provides the tie between these notions. Here we replace pure states of A with normal morphisms and preparations defined below, and states on *Proj* A with finitely additive measures into the unit interval $[0, 1]_C$ of the category.

Definition 7.9 *A normal morphism of A is a $\varphi : I \to A$ with $\varphi^\dagger\varphi = 1$.*

[1] One can further show that the equivalence relation induced by the quasiorder is a congruence on the semiring of positive elements, and that the quotient is a partially ordered semiring under the induced partial order.

In the category of finite dimensional Hilbert spaces and linear maps, a normal morphism φ on $\mathcal{H}$ is a map $\varphi : \mathbb{C} \to \mathcal{H}$ with $\varphi^\dagger\varphi = 1$, and these correspond to unit vectors in $\mathcal{H}$. Each unit vector induces a special biproduct decomposition of $\mathcal{H}$, and this is the idea behind Abramsky and Coecke's definition of a preparation [2].

Definition 7.10 *A preparation of A is a morphism $\varphi : I \to A$ for which there is an object A' and unitary $u : I \oplus A' \to A$ making the following diagram commute.*



Proposition 7.11 *Each preparation on A is a normal morphism.*

Proof. For a preparation φ we have an object A' and unitary $u : I \oplus A' \to A$ with $\varphi = u \circ \mu_1$. Then $\varphi^\dagger\varphi = \mu_1^\dagger u^\dagger u \mu_1 = \pi_1\mu_1 = 1$. ∎

A finitely additive measure, or state, on an OA P is a map $\sigma : P \to [0,1]$ satisfying (i) $\sigma(0) = 0$, (ii) $\sigma(1) = 1$, and (iii) if $x \perp y$, then $\sigma(x \oplus y) = \sigma(x) + \sigma(y)$. We generalize these definitions by replacing the real unit interval with $[0,1]_C$.

Definition 7.12 *A state on the OA $Proj\ A$ is a function $s : Proj\ A \to [0,1]_C$ into the unit interval of scalars in C that satisfies (i) $\sigma(0) = 0$, (ii) $\sigma(1) = 1$, and (iii) if $p \perp q$ then $\sigma(p+q) = \sigma(p) + \sigma(q)$. States on the OMP $Proj_w\ A$ are defined in an identical manner.*

Proposition 7.13 *Each state on $Proj_w\ A$ restricts to a state on $Proj\ A$.*

Proof. This follows as $p \perp q$ in $Proj\ A$ implies $p \perp_w q$ in $Proj_w\ A$. ∎

Proposition 7.14 *Each normal morphism φ of A, hence each preparation of A, yields a state σ_φ of $Proj_w\ A$ where*

$$\sigma_\varphi : Proj_w\ A \to [0,1]_C \quad \textit{is given by} \quad \sigma_\varphi(p) = \varphi^\dagger p \varphi.$$

Further, this state σ_φ restricts to a state on $Proj\ A$.

Proof. As p is a weak projection, it is a self-adjoint idempotent. So $\varphi^\dagger p\varphi = \varphi^\dagger p^\dagger p\varphi = (p\varphi)^\dagger(p\varphi)$, showing that $\varphi^\dagger p\varphi$ is a positive scalar. So $\sigma_\varphi(p) \geq 0$. For p' the orthocomplement of p, we have $\varphi^\dagger p'\varphi$ is a positive scalar. Note $\varphi^\dagger p\varphi + \varphi^\dagger p'\varphi = \varphi^\dagger(p+p')\varphi = \varphi^\dagger\varphi$ which equals 1 as φ is a normal morphism. It follows that $\varphi^\dagger p\varphi \leq 1$. So σ_φ is a map into the unit interval $[0,1]_C$. Clearly $\sigma_\varphi(0) = 0$ and $\sigma_\varphi(1) = 1$. If $p \perp_w q$, then $\sigma_\varphi(p+q) = \varphi^\dagger(p+q)\varphi = \sigma_\varphi(p) + \sigma_\varphi(q)$. ∎

Remark 7.15 *It is relatively common practice in quantum logic to consider states mapping orthostructures into partially ordered abelian groups. Indeed, this is a central ingredient in Foulis's work on* OA*'s and unigroups [8]. It would be of interest to see if there are natural conditions on our categories moving us closer to this situation. In particular, it would be desirable to know when the quasiorder $\leq$ is a partial order, and when the additive monoid structure on the positive scalars is cancellative.*

8 Tensor Products

In this section, we consider objects A, B in a DSMB-category C, and show that the orthoalgebra *Proj* $(A \otimes B)$ has many of the properties one would ask of a tensor product of the orthoalgebras *Proj* A and *Proj* B.

Definition 8.1 *For* OA*s A, B, C a map $f : A \times B \to C$ is called a bilinear mapping if for all $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$ we have*

1. $a_1 \perp a_2 \Rightarrow f(a_1, b) \perp f(a_2, b)$ *and* $f(a_1 \oplus a_2, b) = f(a_1, b) \oplus f(a_2, b)$
2. $b_1 \perp b_2 \Rightarrow f(a, b_1) \perp f(a, b_2)$ *and* $f(a, b_1 \oplus b_2) = f(a, b_1) \oplus f(a, b_2)$
3. $f(1, 1) = 1$

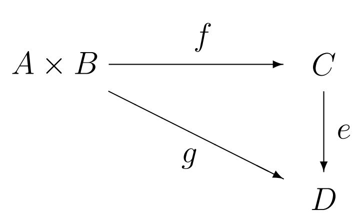
The common definition of a tensor product of OAs [5, 9] is via a universal property involving bilinear maps, much as one defines tensor products of modules. Specifically, the tensor product of OAs A and B is a map $f : A \times B \to C$ satisfying conditions T1 and T3 below. Several other properties [5, 9] arise in discussions of tensor products, and the physical motivation of these conditions is more apparent than that of the universal property. If A and B represent OAs of propositions of two physical systems and C represents the propositions of the compound system, direct physical reasoning asks for a map $f : A \times B \to C$ satisfying at least conditions T1 and T4 below.

Definition 8.2 *For* OA*s A, B, C and $f : A \times B \to C$ consider the conditions:*

T1 *f is bilinear*

T2 *C is generated as an* OA *by the image of f*

T3 *For any bilinear map $g : A \times B \to D$ there is an* OA*-morphism $e : C \to D$ making the following diagram commute.*



T4 *If σ, τ are states on A, B, then there is a state ω on C with $\omega(a,b) = \sigma(a)\tau(b)$ for all $a \in A, b \in B$.*

T5 *States on C are determined by their value on the image of f.*

We remark that the states referred to in conditions T4 and T5 are maps σ into the real unit interval $[0,1]$ that satisfy $x \perp y \Rightarrow \sigma(x \oplus y) = \sigma(x) + \sigma(y)$, and the multiplication $\sigma(a)\tau(b)$ in condition T4 is ordinary multiplication of real numbers. When interpreting these conditions on orthoalgebras $Proj\ A, Proj\ B$ in a DSMB-category C, we replace these states with states into the unit interval of this category $[0,1]_C$ as in Definition 7.12.

Proposition 8.3 *In a* DSMB*-category, if p, q are weak projections of A, B, then $p \otimes q$ is a weak projection of $A \otimes B$.*

Proof. First, suppose p, q are weak projections with p', q' their orthocomplements. Then $p = pp = p^\dagger$, $pp' = 0$ and $p + p' = 1$, with similar conditions for q. As $\otimes$ is a bifunctor, $(a \otimes b)^\dagger = a^\dagger \otimes b^\dagger$, and composition distributes over $+$, we obtain that $p \otimes q$ and $r = p \otimes q' + p' \otimes q + p' \otimes q'$ are self adjoint idempotents with $pr = 0$ and $p + r = 1$. Thus $p \otimes q$ is a weak projection. ∎

Proposition 8.4 *In a* DSMB*-category, if p, q are projections of A, B, then $p \otimes q$ is a projection of $A \otimes B$.*

Proof. Suppose $u : A \to A_1 \oplus A_2$ and $v : B \to B_1 \oplus B_2$ are unitary isomorphisms with $p = u^\dagger \mu_1 \pi_1 u$ and $q = v^\dagger \mu_1 \pi_1 v$. Note, μ_1, π_1 are used in different roles in these expressions, they come from the biproduct $A_1 \oplus A_2$ in the first expression, and $B_1 \oplus B_2$ in the second. Throughout the proof, the reader must determine injections and projections from context. Note also, as $\otimes$ is a biproduct and $(a \otimes b)^\dagger = a^\dagger \otimes b^\dagger$, it follows that $u \otimes v$ is unitary as well.

Define $w : (A_1 \oplus A_2) \otimes (B_1 \otimes B_2) \to (A_1 \otimes B_1) \oplus (A_2 \otimes B_1) \oplus (A_1 \otimes B_2) \oplus (A_2 \otimes B_2)$ to be the morphism whose matrix is given by

$$W = \begin{pmatrix} \pi_1 \otimes \pi_1 \\ \pi_2 \otimes \pi_1 \\ \pi_1 \otimes \pi_2 \\ \pi_2 \otimes \pi_2 \end{pmatrix} \quad \text{and} \quad W^\dagger = \begin{pmatrix} \mu_1 \otimes \mu_1 & \mu_2 \otimes \mu_1 & \mu_1 \otimes \mu_2 & \mu_2 \otimes \mu_2 \end{pmatrix}$$

The morphism $\pi_1 \otimes \pi_1$ in the top row of the matrix for W is the morphism from $(A_1 \oplus A_2) \otimes (B_1 \otimes B_2)$ to $A_1 \otimes B_1$ given by tensoring the projections $\pi_1 : A_1 \oplus A_2 \to A_1$ and $\pi_1 : B_1 \oplus B_2 \to B_1$. Its adjoint is $\pi_1^\dagger \otimes \pi_1^\dagger = \mu_1 \otimes \mu_1$ and so forth.

In computing $WW^\dagger$ each entry is of the form $(\pi_i \otimes \pi_j)(\mu_k \otimes \mu_l)$ which equals $\pi_i \mu_k \otimes \pi_j \mu_l$. If $i = k$ and $j = l$ this equals $1 \otimes 1 = 1$, otherwise at least one of the morphisms in the tensor product is zero, so by Definition 6.4 the result is 0. Thus $WW^\dagger$ is a 4×4 identity matrix. The matrix $W^\dagger W$ has one entry that can be written $[(\mu_1\pi_1 \otimes \mu_1\pi_1) + (\mu_2\pi_2 \otimes \mu_1\pi_1)] + [(\mu_1\pi_1 \otimes \mu_2\pi_2) + (\mu_2\pi_2 \otimes \mu_2\pi_2)]$. Applying Definition 6.4 and the fact that $\mu_1\pi_1 + \mu_2\pi_2 = 1$ this becomes $(1 \otimes \mu_1\pi_1) + (1 \otimes \mu_2\pi_2)$, and by the same argument this equals $1 \otimes 1 = 1$. Thus $W^\dagger W$ is a 1×1 identity matrix, and this shows w is unitary.

As $u \otimes v$ and w are unitary, we have that $w \circ (u \otimes v)$ is unitary. Note

$$W^\dagger \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} W = (\mu_1\pi_1 \otimes \mu_1\pi_1)$$

Therefore $(u \otimes v)^\dagger w^\dagger \mu_1\pi_1 w(u \otimes v) = (u^\dagger \otimes v^\dagger)(\mu_1\pi_1 \otimes \mu_1\pi_1)(u \otimes v)$ and this is equal to $u^\dagger\mu_1\pi_1 u \otimes v^\dagger\mu_1\pi_1 v$, and hence to $p \otimes q$. So $p \otimes q$ is a projection. ∎

Definition 8.5 *For objects A, B in a* DSMB*-category, define*

1. $\Gamma_w : Proj_w A \times Proj_w B \to Proj_w(A \otimes B)$ *by* $\Gamma_w(p, q) = p \otimes q$
2. $\Gamma : Proj\ A \times Proj\ B \to Proj(A \otimes B)$ *by* $\Gamma(p, q) = p \otimes q$

Note that Propositions 8.3 and 8.4 show these are well-defined.

Proposition 8.6 $\Gamma_w : Proj_w A \times Proj_w B \to Proj_w(A \otimes B)$ *is bilinear.*

Proof. Suppose p_1, p_2 are weak projections of A with $p_1 \perp_w p_2$ and q is a weak projection of B. Note $p_1 \perp_w p_2$ means $p_1 \leq_w p_2'$ where p_2' is the orthocomplement of p_2, and by Definition 3.3 this means $p_1 p_2' = p_1 = p_2' p_1$. In the proof of Proposition 8.3 we showed $r = p_2 \otimes q' + p_2' \otimes q + p_2' \otimes q'$ is the orthocomplement of $p_2 \otimes q$ in $Proj_w A \otimes B$. A simple calculation gives $(p_1 \otimes q)r = p_1 p_2' \otimes qq = p_1 \otimes q$ and similarly $r(p_1 \otimes q) = p_1 \otimes q$. So $p_1 \otimes q \leq_w r$, giving $p_1 \otimes q \perp_w p_2 \otimes q$. Clearly $(p_1 + p_2) \otimes q = p_1 \otimes q + p_2 \otimes q$, and this provides the first condition of Definition 8.1. The second condition follows by symmetry, and the third is $1 \otimes 1 = 1$, which is valid as $\otimes$ is a bifunctor. ∎

Proposition 8.7 $\Gamma : Proj\ A \times Proj\ B \to Proj\ (A \otimes B)$ *is bilinear.*

Proof. Suppose p_1, p_2 are projections of A with $p_1 \perp p_2$, and q is a projection of B. By Definitions 4.8 and 4.5 there are unitaries $u : A \to A_1 \oplus A_2 \oplus A_3$ and $v : B \to B_1 \oplus B_2$ with $p_1 = u^\dagger \mu_1\pi_1 u$, $p_2 = u^\dagger \mu_3\pi_3 u$ and $q = v^\dagger \mu_1\pi_1 v$. Again, the reader keeps track of the various injections and projections μ_i, π_i by context.

Consider $w : (A_1 \oplus A_2 \oplus A_3) \otimes (B_1 \oplus B_2) \to (A_1 \otimes B_1) \oplus \cdots \oplus (A_3 \otimes B_2)$ where

$$W = \begin{pmatrix} \pi_1 \otimes \pi_1 \\ \pi_2 \otimes \pi_1 \\ \vdots \\ \pi_3 \otimes \pi_2 \end{pmatrix} \quad \text{and} \quad W^\dagger = \begin{pmatrix} \mu_1 \otimes \mu_1 & \mu_2 \otimes \mu_1 & \cdots & \mu_3 \otimes \mu_2 \end{pmatrix}$$

Using arguments similar to those in Proposition 8.4 we find $WW^\dagger$ is a 6×6 identity matrix and $W^\dagger W$ is a 1×1 identity matrix, so w is unitary. Then $w(u \otimes v)$ is also unitary. We note that

$$W^\dagger \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} W = (\mu_1\pi_1 \otimes \mu_1\pi_1)$$

To avoid confusion we let $\tilde{\mu_i}, \tilde{\pi_i}$ $i = 1, \ldots 6$ be the injections and projections for the biproduct $(A_1 \otimes B_1) \oplus \cdots \oplus (A_3 \otimes B_2)$. In morphism form, the above equation says $w^\dagger \tilde{\mu_1}\tilde{\pi_1} w = \mu_1\pi_1 \otimes \mu_1\pi_1$, giving $(u \otimes v)^\dagger w^\dagger \tilde{\mu_1}\tilde{\pi_1} w(u \otimes v) = u^\dagger \mu_1\pi_1 u \otimes v^\dagger \mu_1\pi_1 v$ and therefore is equal to $p_1 \otimes q$. Similarly, as $w^\dagger \tilde{\mu_3}\tilde{\pi_3} w = \mu_3\pi_3 \otimes \mu_1\pi_1$ we have $(u \otimes v)^\dagger w^\dagger \tilde{\mu_3}\tilde{\pi_3} w(u \otimes v) = p_2 \otimes q$. Proposition 4.9 then gives $p_1 \otimes q \perp p_2 \otimes q$. Clearly $(p_1 + p_2) \otimes q = p_1 \otimes q + p_2 \otimes q$ giving the first condition of Definition 8.1. The second condition follows by symmetry, and the third is $1 \otimes 1 = 1$, which is valid as $\otimes$ is a bifunctor. ∎

Remark 8.8 *We have shown that the map $Proj\ A \times Proj\ B \to Proj(A \otimes B)$ satisfies the condition T1 one requires of a tensor product of* OA*s, with the corresponding result holding also for weak projections. A later example shows T2 need not hold, and it does not seem likely that T3 will be satisfied, at least without further conditions on the category. These conditions are more algebraically inspired, and less physically motivated than conditions T4 and T5 involving states. We next see that rudimentary versions of T4 hold, namely ones where we restrict consideration to states taking values in the unit interval $[0,1]_C$ of the category and arising from normal morphisms or preparations. It doesn't seem that stronger versions of T4, or T5, need hold without further conditions on the category.*

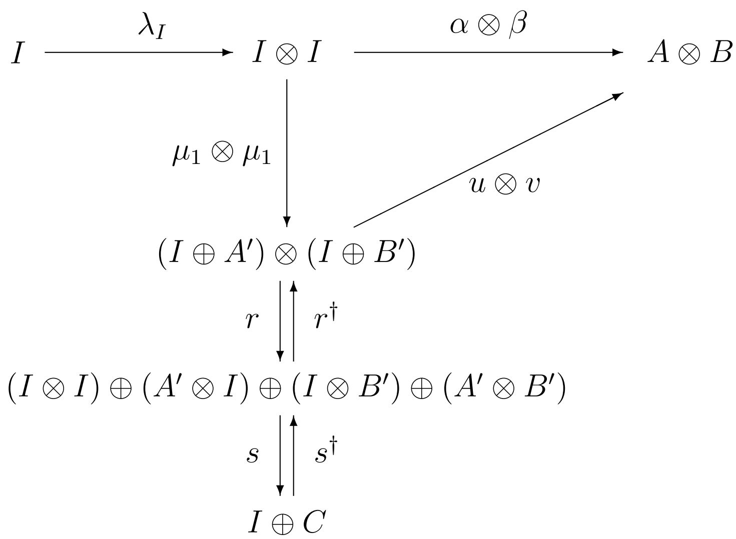
The reader might want to review Definitions 7.9 and 7.10 of normal morphisms and preparations $I \xrightarrow{\alpha} A$ and Proposition 7.14 showing each such normal morphism and preparation α induces a state σ_α on $Proj_w A$ and on $Proj\ A$.

Proposition 8.9 *For normal morphisms* $I \xrightarrow{\alpha} A$ *and* $I \xrightarrow{\beta} B$

1. $\gamma = (\alpha \otimes \beta) \circ \lambda_I$ *is a normal morphism of* $A \otimes B$.
2. *If* α, β *are preparations, so also is* γ.
3. σ_γ *is a state on* $Proj_w(A \otimes B)$ *with* $\sigma_\gamma(p \otimes q) = \sigma_\alpha(p)\sigma_\beta(q)$.
4. σ_γ *restricts to a state on* $Proj(A \otimes B)$.

Proof. For the first statement, Definition 6.3 gives $\lambda_I^{-1} = \lambda_I^\dagger$, and a calculation shows $[(\alpha \otimes \beta)\lambda_I]^\dagger[(\alpha \otimes \beta)\lambda_I] = \lambda_I^{-1}(\alpha^\dagger \otimes \beta^\dagger)(\alpha \otimes \beta)\lambda_I = \lambda_I^{-1}(\alpha^\dagger\alpha \otimes \beta^\dagger\beta)\lambda_I$. Then as α, β are normal morphisms, this equals $\lambda_I^{-1}(1 \otimes 1)\lambda_I = 1$. So γ is normal.

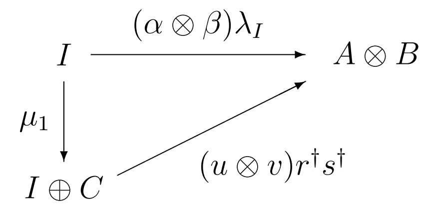
For the second statement, as α, β are preparations there are A', B' and unitaries $u : I \oplus A' \to A$ and $v : I \oplus B' \to B$ with $\alpha = u \circ \mu_1$ and $\beta = v \circ \mu_1$. So the triangle in the diagram below commutes.



In this diagram $C = (A' \otimes I) \oplus (I \otimes B') \oplus (A' \otimes B')$ and r, s have matrices

$$R = \begin{pmatrix} \pi_1 \otimes \pi_1 \\ \pi_2 \otimes \pi_1 \\ \pi_1 \otimes \pi_2 \\ \pi_2 \otimes \pi_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \lambda_I^{-1} & 0 & 0 & 0 \\ 0 & \mu_1 & \mu_2 & \mu_3 \end{pmatrix}$$

Simple calculations show r, s are unitary and the matrix for $sr(\mu_1 \otimes \mu_1)\lambda_I$ is $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Thus $sr(\mu_1 \otimes \mu_1)\lambda_I$ is the injection $I \xrightarrow{\mu_1} I \oplus C$. Further, $(u \otimes v)r^\dagger s^\dagger$ is unitary and the diagram below commutes. So $\gamma = (\alpha \otimes \beta)\lambda_I$ is a preparation of $A \otimes B$.



For the third statement, Proposition 7.14 shows σ_γ is a state. We need only show $\sigma_\gamma(p \otimes q) = \sigma_\alpha(p) \circ \sigma_\alpha(q)$. By definition, $\sigma_\gamma(p \otimes q) = \gamma^\dagger(p \otimes q)\gamma$. As $\lambda_I^\dagger = \lambda_I^{-1}$, this becomes $\lambda_I^{-1}(\alpha^\dagger \otimes \beta^\dagger)(p \otimes q)(\alpha \otimes \beta)\lambda_I$, which equals $\lambda_I^{-1}(\alpha^\dagger p\alpha \otimes \beta^\dagger q\beta)\lambda_I$. By definition of $\sigma_\alpha, \sigma_\beta$ this becomes $\lambda_I^{-1}(\sigma_\alpha(p) \otimes \sigma_\beta(q))\lambda_I$. Equation (7.19) provides this expression equals $\sigma_\alpha(p) \circ \sigma_\beta(q)$, as required. The fourth statement follows directly as each state of $Proj_w(A \otimes B)$ restricts to a state on $Proj(A \otimes B)$. ∎

Remark 8.10 *Normal morphisms are sufficient to build finitely additive states, but the examples below show that preparation may be closer to what one would want.*

9 Examples

In this section we consider examples of orthostructures of projections and their states in several categories. We look at the category Rel of sets and relations, $FDHilb$ of finite-dimensional Hilbert spaces, and the category Mat_K whose objects are natural numbers and whose morphisms are matrices over the field K. Each example is not only a DSMB-category, but even a strongly compact closed category with biproducts. The first two behave in a regular fashion, the third exhibits some pathology.

9.1 The category *Rel*

In this category, objects are sets, and the morphisms from a set A to a set B are the binary relations $R \subseteq A \times B$ from A to B. Composition of morphisms is usual composition of relations, and the identity morphisms are identity functions considered as relations in the usual way. This category has a unique zero object, the emptyset, and the zero map from A to B is the empty relation. The following is trivial to verify from Definition 2.1.

Proposition 9.1 *Rel is a dagger category where $R^\dagger$ is the relational converse.*

For sets A_1, A_2, their disjoint union $A_1 \uplus A_2$ is $A_1 \times \{1\} \cup A_2 \times \{2\}$. We let μ_i, π_j be the relations $A_i \xrightarrow{\mu_i} A_1 \uplus A_2 \xrightarrow{\pi_j} A_j$ defined by $\mu_i = \{(a,(a,i)) : a \in A_i\}$ and $\pi_j = \{((a,j),a) : a \in A_j\}$ and note that μ_i and π_i are converses of each other.

Proposition 9.2 *Rel is a dagger biproduct category with dagger being converse and biproducts being disjoint unions.*

Proof. For morphisms $A_i \xrightarrow{R_i} B$ one checks $[R_1, R_2] = \{((a,i),b)|(a,b) \in R_i\}$ is the unique morphism from $A_1 \uplus A_2$ to B with $[R_1, R_2] \circ \mu_i = R_i$, and for $B \xrightarrow{S_i} A_i$ one checks $< S_1, S_2 >= \{(b,(a,i)) : (b,a) \in S_i\}$ is the unique morphism from B to $A_1 \uplus A_2$ with $\pi_i \circ < S_1, S_2 >= S_i$. A simple calculation gives $\pi_i \circ \mu_i$ is the identity relation if $i = j$ and is empty, hence the zero morphism, if $i \neq j$. So this provides a biproduct structure. As μ_i and π_i are converses of one another and $0^\dagger_{A,B} = 0_{B,A}$, this yields a dagger biproduct category. ∎

We consider the additive structure on homsets. For $R_1, R_2 : A \to B$, recall $R_1 \vee R_2$ is the relation from A to B defined by $a(R_1 \vee R_2)b$ iff aR_1b or aR_2b.

Proposition 9.3 $R_1 + R_2 = R_1 \vee R_2$.

Proof. By definition 2.5 $R_1 + R_2 = [1_B, 1_B] \circ (R_1 \oplus R_2) \circ < 1_A, 1_A >$. From above, $< 1_A, 1_A >= \{(a,(a,i)) : i = 1,2\}$, $[1_B, 1_B] = \{((b,i),b) : i = 1,2\}$ and $R_1 \oplus R_2$ is the unique morphism with $\pi_i \circ (R_1 \oplus R_2) = R_i \circ \pi_1$ and $(R_1 \oplus R_2) \circ \mu_i = \mu_i \circ R_i$. So $R_1 \oplus R_2 = \{((a,i),(b,i)) : (a,b) \in R_i\}$. So $a(R_1 + R_2)b$ iff aR_1b or aR_2b. ∎

For objects A_1, A_2 let the tensor product $A_1 \otimes A_2$ be the usual Cartesian product, and for relations $A_i \xrightarrow{R_i} B_i$ let $R_1 \otimes R_2 = \{((a_1,b_1),(a_2,b_2)) : a_1R_1b_1 \text{ and } a_2R_2b_2\}$. It is a simple matter to see $\otimes$ is a bifunctor. Let the unit $I = \{*\}$ be a particular one-element set, and let $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\sigma_{A,B} : A \otimes B \to B \otimes A$, $\lambda_A : A \to I \otimes A$ and $\rho_A : A \otimes I$ be the obvious bijections considered as relations.

Proposition 9.4 *Rel is a* DSMB*-category with tensor being Cartesian product.*

Proof. That $\otimes$ yields a symmetric monoidal structure is similar to the situation for sets. As † is converse, it is clear that $(R \otimes S)^\dagger = R^\dagger \otimes S^\dagger$, and as the $\alpha, \sigma, \lambda, \rho$ are bijections, their converses are their inverses, so they are unitary. So the dagger is compatible with $\otimes$. To see that $\otimes$ is compatible with the additive structure note $R \otimes (S_1 + S_2) = R \otimes S_1 + R \otimes S_2$ as $+$ is given by $\vee$, and as the zero morphism is the empty relation, $R \otimes 0 = 0$. ∎

Proposition 9.5 *Rel is a strongly closed category with biproducts.*

Proof. This is established in [2]. ∎

As $I = \{*\}$ is a singleton, there are two morphisms from I to itself, $0, 1$, with both positive. The above description of $+$ gives $1 + 1 = 1$, establishing the following.

Proposition 9.6 *The unit interval* $[0,1]$ *in the category Rel is the set* $\{0,1\}$ *with the obvious partial ordering, addition being* max*, and multiplication being the ordinary multiplication.*

We next consider projections. For a set A, a relation $A \xrightarrow{R} A$ is a self-adjoint idempotent if $R \circ R = R$ and $R = R^\dagger$, which means that R is symmetric and transitive. For such R, let its support be $Supp(R) = \{a \in A : aRa\}$, and note that $a \in Supp(R)$ iff aRa' for some $a \in A$. Suppose R, R' are self-adjoint idempotents on A with $RR' = 0 = R'R$ and $R + R' = 1$. As $R + R' = R \vee R' = 1$, we have $aRa' \Rightarrow a = a'$, so R and R' are completely determined by their supports. The condition $RR' = 0$ implies these supports are disjoint, and the condition $R + R' = 1$ implies that their union is all of A. So each weak projection and its partner are determined by a subset of A and its complement, and one easily sees that each subset and its complement arise this way. Further, for weak projections R and S, we have $RS = S$ iff $Supp(S) \subseteq Supp(R)$. We have shown the following.

Proposition 9.7 *In the category Rel, the* OMP $Proj_w A$ *is isomorphic to the power set of* A*, hence is Boolean.*

Recall that a self-adjoint idempotent $A \xrightarrow{e} A$ strongly splits if there is $A \xrightarrow{f} B$ with $e = f^\dagger f$ and $1_B = ff^\dagger$.

Proposition 9.8 *Self-adjoint idempotents strongly split in the category Rel.*

Proof. Suppose $A \xrightarrow{R} A$ is a self adjoint idempotent. Let $A' = Supp(R)$ and let R' be the restriction of R to A'. Then R' is an equivalence relation on A' and we may consider $B = A'/R'$. Define a relation $A \xrightarrow{S} B$ by setting $aS(a'/R')$ iff aRa'. One checks that S is well defined, that $S^\dagger \circ S = R$ and $S \circ S^\dagger = 1_B$. ∎

Corollary 9.9 *In Rel, we have* $Proj\ A = Proj_w\ A$.

We next consider normal morphisms and preparations of an object A in *Rel*. Recall that a normal morphism is an $\{*\} \xrightarrow{\varphi} A$ with $\varphi^\dagger \circ \varphi = 1$. Any relation from $\{*\}$ to A is determined by the set of elements related to $*$, which we denote $Im(\varphi)$, and the relation φ will be a normal morphism iff this set is non-empty. The condition for φ to be a preparation is more stringent, there must be a set A' and a unitary $u : \{*\} \uplus A' \to A$ with $\varphi = u \circ \mu_1$. As unitaries in *Rel* are precisely bijections, φ is a preparation iff there is a single element of A to which $*$ is related. We have shown the following.

Proposition 9.10 *Normal morphisms of* A *correspond to non-empty subsets of* A*, and preparations of* A *correspond to singleton subsets of* A*.*

Normal morphisms and preparations φ of A induce states $\sigma_\varphi : Proj\ A \to [0,1]_C$ where $\sigma_\varphi(R) = \varphi^\dagger R \varphi$. If φ corresponds to the non-empty subset $T \subseteq A$, so $*\varphi a$ iff $a \in T$, and R corresponds to the subset S of A, so aRb iff $a = b$ and $a \in S$, then we compute $\sigma_\varphi(R) = 1$ iff $S \cap T \neq \emptyset$. This gives the following.

Proposition 9.11 *Identifying Proj A with its power set $\mathcal{P}(A)$, the states on Proj A given by normal morphisms are ones mapping all elements of a proper principal ideal of $\mathcal{P}(A)$ to 0 and all other elements to 1. The states arising from preparations are the two-valued homomorphisms mapping all elements of a principal prime ideal given by a coatom to 0 and all other elements to 1.*

Consider now the tensor product $\Gamma : Proj\ A \times Proj\ B \to Proj\ (A \otimes B)$ mapping (R, S) to $R \otimes S$ and recall $R \otimes S = \{((a, b), (a', b')) : aRa' \text{ and } bSb'\}$. Then if R is the projection corresponding to the subset $A' \subseteq A$ and S is the projection corresponding to the subset $B' \subseteq B$, we have $R \otimes S$ is the projection of $A \times B$ corresponding to the subset $A' \times B' = \{(a, b) : a \in A' \text{ and } b \in B'\}$. This shows the following.

Proposition 9.12 *Identifying Proj A, Proj B and Proj$(A \otimes B)$ with the power sets of A, B and $A \times B$, the tensor product of these orthostructures in this category is the embedding $\mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \times B)$ sending $(A', B') \rightsquigarrow A' \times B'$.*

Remark 9.13 *Roughly, the behavior is classical in the category Rel. For finite sets, the orthostructures one obtains are finite Boolean algebras and the states obtained from preparations are homomorphisms into the two-element Boolean algebra. Further, the tensor product satisfies conditions T1-T5. For infinite sets the Boolean algebras are the power set Boolean algebras, states from preparations are exactly the complete homomorphisms into the two-element Boolean algebra, and the tensor product behaves well if we consider complete generation and complete maps.*

9.2 The category FDHilb

This is the prime example. Objects are finite dimensional complex (or real) Hilbert spaces, and morphisms are linear transformations. The dagger structure is given by the usual adjoint of a map, the biproduct structure and tensor product are the usual ones. The additive structure on a homset is given by the usual addition of linear maps. The tensor unit is the the field $\mathbb{C}$. The scalars are naturally identified with $\mathbb{C}$, with the positive scalars being the positive real numbers, and the unit interval $[0, 1]_C$ being the usual real unit interval.

Proposition 9.14 *Self-adjoint idempotents strongly split, so weak projections and projections agree, and Proj $\mathcal{H}$ is the* OML *of projection operators of $\mathcal{H}$. Further, as the spaces involved are finite dimensional, this* OML *is even a modular ortholattice.*

Proof. If $e : \mathcal{H} \to \mathcal{H}$ is a self-adjoint idempotent of $\mathcal{H}$ then its image $\mathcal{H}'$ is a Hilbert space, and the obvious map $f : \mathcal{H} \to \mathcal{H}'$ satisfies $f^\dagger f = e$ and $ff^\dagger = 1_{\mathcal{H}'}$. So weak projections and projections agree. That weak projections are projection operators on $\mathcal{H}$ is by definition, and the ordering and orthocomplementation in *Proj* $\mathcal{H}$ are defined as is standard when considering the OML of projection operators. ∎

Normal morphisms are linear maps $\varphi : \mathbb{C} \to \mathcal{H}$ with $\varphi^\dagger\varphi$ being the identity map. Such φ is determined by $\varphi(1) = v$ and $\varphi^\dagger\varphi$ being the identity implies v is a unit vector. All unit vectors arise this way. For such φ there is a unitary isomorphism $u : S \oplus S^\perp \to \mathcal{H}$ where S is the subspace spanned by v. This shows each normal morphism is a preparation. The connection between unit vectors and states of $\mathcal{H}$ is well-known through Gleason's theorem [6], giving the following.

Proposition 9.15 *Normal morphisms and preparations of $\mathcal{H}$ coincide. The resulting states on $Proj\ \mathcal{H}$ are exactly the ones that cannot be expressed as a non-trivial convex combination of states.*

Finally, the tensor product $Proj(\mathcal{H}_1 \otimes \mathcal{H}_2)$ has the properties T1-T5. This is the motivating example for these conditions.

Remark 9.16 *A real problem is the restriction to finite dimensional Hilbert spaces, as quantum mechanics involves infinite dimensional Hilbert spaces in an essential way. This is a problem that is not easily remedied. The existence of adjoints is closely tied to completeness of the inner product space and boundedness of the maps. One might consider the category of Hilbert spaces and bounded linear maps, but this leaves out the position operator (which is not bounded).*

9.3 The category Mat_K

Here the objects are natural numbers, the morphisms from m to n are the $m \times n$ matrices (m columns and n rows) with entries from the field K, and composition of morphisms is usual matrix multiplication. We note that an $m \times 0$ or $0 \times n$ matrix has no entries, so there is exactly one such matrix. This shows that 0 is a zero object.

Proposition 9.17 *Mat_K is a dagger biproduct category where $\dagger$ is transpose and $m \oplus n$ is given by addition $m + n$ with the canonical injections and projections being the matrices having block form $\mu_1 = \left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right)$, $\mu_2 = \left(\begin{smallmatrix} 0 \\ I_n \end{smallmatrix}\right)$, $\pi_1 = \left(\begin{smallmatrix} I_m & 0 \end{smallmatrix}\right)$ and $\pi_2 = \left(\begin{smallmatrix} 0 & I_n \end{smallmatrix}\right)$.*

Proof. That transpose gives a dagger category is obvious. If $m \xrightarrow{M} k$ and $n \xrightarrow{N} k$, then the unique morphism $[M, N]$ from the coproduct completing the cone has block form $\left(\begin{smallmatrix} M & N \end{smallmatrix}\right)$, and if $k \xrightarrow{M} m$ and $k \xrightarrow{N} n$, then the unique morphism $< M, N >$ into the coproduct completing the cone has block form $\left(\begin{smallmatrix} M \\ N \end{smallmatrix}\right)$. That $\pi_i\mu_j = \delta_{ij}$ and $\mu_i^\dagger = \pi_i$ are easily seen. ∎

For matrices P, Q, one can check that $P \oplus Q$ is the matrix with block form $\left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right)$. From this it follows that the addition on a homset $Mat_K(m, n)$ is given by usual matrix addition.

Proposition 9.18 *Mat_K is a* DSMB*-category with tensor product $m \otimes n$ given by multiplication on objects, and $R \otimes S$ being usual Kronecker product of matrices.*

Proof. That $\otimes$ is a bifunctor amounts to well known properties of Kronecker products. For the natural isomorphisms α, λ, ρ, given m, n, p let $\alpha_{m,n,p} : mnp \to mnp$, $\lambda_m : m \to m\cdot 1$ and $\rho_m : m \to 1\cdot m$ be identity maps. That these are natural amounts to the associativity $(R\otimes S)\otimes T = R\otimes(S\otimes T)$ of Kronecker product, and the obvious conditions $R \otimes (1) = R$ and $(1) \otimes S = S$. So Mat_K is a strict monoidal category.

The natural isomorphism σ for symmetry is more delicate. Given $R : m \to m'$ and $S : n \to n'$, there is a permutation matrix $P_{m,n} : mn \to nm$ depending only on m, n and a permutation matrix $P_{m',n'} : m'n' \to n'm'$ depending only on m', n' with $P_{m',n'} \circ (R \otimes S) = (S \otimes R) \circ P_{m,n}$. The idea behind the permutation matrix $P_{m,n}$ is to permute $a_1b_1, \ldots, a_1b_n, \ldots, a_mb_1, \ldots a_mb_n$ into $a_1b_1, \ldots, a_mb_1, \ldots, a_1b_n, \ldots, a_mb_n$. Set $\sigma_{m,n} = P_{m,n}$, and note that the above gives the naturality of σ. Showing the compatibility condition involving σ [18, pg. 180] is a chore.

This shows Mat_K is a symmetric monoidal category, and we have seen above it is a dagger biproduct category. As Kronecker product distributes over matrix addition on both sides, $R \otimes 0 = 0$ and $0 \otimes S = 0$, we have Mat_K is a DSMB-category. ∎

Proposition 9.19 *Mat_K is a strongly compact closed category with biproducts.*

Proof. We follow Selinger [21] where strongly compact closed categories with biproducts are called biproduct dagger compact closed categories. We first show Mat_K is compact closed. As Mat_K is a symmetric monoidal category with the natural isomorphisms α, λ, ρ given by identity maps, this means we must define for each object n an object n^* and morphisms $\eta_n : 1 \to n^* \otimes n$ and $\epsilon_n : n \otimes n^* \to 1$ so that (i) $(\epsilon_n \otimes 1_n) \circ (1_n \otimes \eta_n) = 1_n$ and (ii) $(1_{n*} \otimes \epsilon_n) \circ (\eta_n \otimes 1_{n^*}) = 1_{n^*}$.

Let $n^* = n$. We define $\epsilon_n : n\cdot n \to 1$ to be the matrix with one row and n^2 entries formed from the $n \times n$ identity matrix I_n by placing its rows one after another.

$$\epsilon_n = (\underbrace{1\,0\,\cdots\,0}_{n}\;\;\underbrace{0\,1\,\cdots\,0}_{n}\;\;\cdots\;\;\underbrace{0\,\cdots\,0\,1}_{n})$$

More precisely, $\epsilon_n = (a_{11} \ldots a_{1n} \;\ldots\; a_{n1} \ldots a_{nn})$ where $a_{ij} = \delta_{ij}$. Set η_n to be the transpose of ϵ_n. Then in block form $(\epsilon_n \otimes I_n) \circ (I_n \otimes \eta_n)$ becomes

$$\begin{pmatrix} \epsilon_n & 0 & \cdots & 0 \\ 0 & \epsilon_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_n \end{pmatrix} \begin{pmatrix} I_n \\ \vdots \\ 0 \\ \hline \vdots \\ \hline 0 \\ \vdots \\ I_n \end{pmatrix} \tag{9.20}$$

Note ϵ_n times the first block of the above matrix (the portion above the line) equals the first row of the identity matrix I_n, that ϵ_n times the second block equals the second row the identity matrix, and so forth. Thus equation (9.20) evaluates to the identity matrix I_n, showing that (i) holds. A similar argument shows $(I_n \otimes \epsilon_n) \circ (\eta_n \otimes I_n) = I_n$, hence (ii) holds as well. Therefore Mat_K is compact closed.

We know Mat_K is a dagger symmetric monoidal category that is compact closed. To show it is a dagger compact closed category [21] we must show that for each n (iii) $\sigma_{n,n^*} \circ \epsilon_n^\dagger = \eta_n$. As $\epsilon_n^\dagger = \eta_n$ and σ_{n,n^*} is the permuation matrix $P_{n,n}$, we must show $P_{n,n} \circ \eta_n = \eta_n$. This amounts to showing the column vector η_n is fixed by $P_{n,n}$. Recall $P_{n,n}$ is the permutation matrix taking $a_1b_1, \ldots, a_1b_n, \ldots, a_nb_1, \ldots, a_nb_n$ to $a_1b_1, \ldots, a_nb_1, \ldots, a_1b_n, \ldots, a_nb_n$. But this leaves the a_ib_i fixed, so $P_{n,n}$ leaves the non-zero entries of η_n fixed, and permutes the zeros. So Mat_K is dagger compact closed. The further properties needed to be a biproduct dagger compact closed category were already established when we showed it was a dagger biproduct category. ∎

Proposition 9.20 *The scalars are the morphisms from I to itself, hence the 1×1 matrices, and therefore the semiring of scalars is isomorphic to the field K. The positive scalars are exactly the ones that are sums of squares of elements of K.*

Proof. We have only to show the statement about positivity. But this follows as a scalar s is positive iff it is of the form $\alpha^\dagger\alpha$ for some $1 \xrightarrow{\alpha} n$. But such α is a column matrix with entries $x_1, \ldots, x_n$ so $\alpha^\dagger\alpha = (x_1^2 + \cdots + x_n^2)$. ∎

Remark 9.21 *If K has finite characteristic, then as 1 is a square we have $0 \leq 1$ and $1 \leq 0$, so the unit interval in this case has a quasiorder that relates all elements to one another. Clearly this is not such a useful notion of an ordering.*

We next consider various notions of projections in the category Mat_K. First, the morphisms from m to n are exactly the $m \times n$ matrices over K, hence are exactly the linear transformations from K^m to K^n expressed as matrices using the standard bases. Thus the idempotent endomorphisms $Idem\ m$ of m are the idempotents of the endomorphism ring of K^m. It is well known that this forms an OMP [11, 15] with partial ordering $M \leq N$ iff $MN = M = NM$ and orthocomplement $M' = I - M$. We then have the following.

Proposition 9.22 *The idempotent endomorphisms $Idem\ m$ of m form an* OMP. *The weak projections $Proj_w m$ are a sub-*OMP *of this, and the projections $Proj\ m$ are a sub-*OA *of this.*

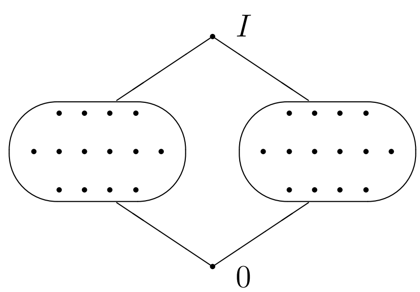
We consider the specific case where the field is $\mathbb{Z}_2$ and $m = 4$, and describe the orthostructure $Proj\ m$. Note $4 = 1 \oplus 1 \oplus 1 \oplus 1$ and that all projections of 4 are obtained as $U^\dagger SU$ for some unitary $u : 4 \to 1 \oplus 1 \oplus 1 \oplus 1$ and some standard projection matrix S. Recall a standard projection matrix is one of all 0's and 1's with off-diagonal entries all 0. For a fixed unitary u, the projections $U^\dagger S U$, where S ranges over all standard projection matrices, form a Boolean subalgebra of $\mathit{Proj}\ 4$. The atoms of this Boolean subalgebra are

$$U^\dagger \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} U \quad U^\dagger \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} U \quad U^\dagger \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix} U \quad U^\dagger \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix} U.$$

As any permutation matrix P is unitary, we have PU is unitary for any unitary U, and the Boolean algebras for PU and U agree. We say two unitary isomorphisms are equivalent if one is obtained from the other by a permutation matrix in this way. One can check that there are two non-equivalent 4×4 unitary matrices shown below.

$$U = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0 \end{pmatrix}$$

The Boolean algebras for U and V have only the zero matrix and the identity matrix in common (it is not difficult to verify this using symmetry). So $\mathit{Proj}\ 4$ consists of two 16-element Boolean algebras pasted together along $0, I$ as shown below. This means $\mathit{Proj}\ 4$ is the horizontal sum of two 16-element Boolean algebras, and therefore is an orthomodular lattice that is not modular.



We next consider the matter of preparations of 4. These are morphisms $1 \xrightarrow{\varphi} 4$ so that there is a unitary $u : 1 \oplus 3 \to 4$ with $\varphi = u \circ \mu_1$. From the above description of the matrix for μ_1, in this case a column vector with just the first spot 1 and the rest 0, the preparations are exactly the column vectors that arise as the first column of some 4×4 unitary matrix. The state arising from a preparation $\sigma_\varphi : \mathit{Proj}\ 4 \to \mathbb{Z}_2$ satisfies $\sigma_\varphi(P) = \varphi^\dagger P \varphi$. In the case that φ is the first column of the identity matrix, the state $\sigma_\varphi(P)$ simply takes the entry in the top left corner of P. In total, there are eight such preparations yielding eight states.

Finally, we remark that the tensor product behaves in an unusual fashion. Up to permutation, the identity is the only unitary 2×2 matrix, so $\mathit{Proj}\ 2$ is a four-element Boolean algebra whose elements are $\left(\begin{smallmatrix} 1&0\\0&1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1&0\\0&0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0&0\\0&1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0&0\\0&0 \end{smallmatrix}\right)$. But the tensor product $2 \otimes 2 = 4$, and the map $\Gamma : \mathit{Proj}\ 2 \times \mathit{Proj}\ 2 \to \mathit{Proj}\ 4$ takes (P, Q) to

$P \otimes Q$. As each of P, Q is a standard projection matrix and $P \otimes Q$ is their Kronecker product, each $P \otimes Q$ is also a standard projection matrix. So Γ maps entirely into the one of the two 16-element Boolean subalgebras of $Proj\ 4$.

In effect, the tensor product of these two four-element Boolean algebras $Proj\ 2$ is a sixteen-element Boolean algebra just as in the classical case, but with a phantom sixteen-element Boolean algebra pasted on to form $Proj\ 4$. This tensor product does not satisfy the condition T2 one might seek in a tensor product of OAs.

As a final comment, note that the self-adjoint idempotents do not split in this category. Indeed, $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is a self-adjoint idempotent, and therefore is a weak projection with partner $I - P$. But P is not a projection as I is the only 3×3 unitary and P is not a standard projection matrix.

Remark 9.23 *The situation for $Proj\ m$ in the setting of Mat_K is not settled. It is not determined whether $Proj\ m$ is always an* OML *or* OMP*, or whether it can be a proper* OA*. It is also not determined whether the preparations provide a full set of states. These questions may be of interest in quantum logic as the $Proj\ m$ provide an interesting source of* OA*s. There is also a close connection between the unitary group $\mathcal{O}_m$ (also called the orthogonal group) of $m \times m$ matrices over $\mathbb{Z}_2$ and self-dual codes [13]. Perhaps the connection between the* OA*s $Proj\ m$ and the groups $\mathcal{O}_m$ could be of interest in the study of these groups as well.*

10 Conclusions

The work of Abramsky and Coecke [2] points the way to developing a foundation for quantum mechanics based in category theory. It would be most desirable to extend their work from the finite-dimensional setting to the general one by adapting the types of categories one considers.

There is a basic and very portable method to link aspects of quantum logic to such a categorical approach. One views the direct product decompositions of an object in the category as propositions of the system represented by that object. The key idea being that refinement of decompositions yields a partial ordering and a resulting orthostructure. While this approach does not work in an arbitrary category, it does seem to hold under fairly mild assumptions — it is the idea underlying the occurrence of orthomodularity in dagger biproduct categories, and holds in many other natural settings as well [12].

In developing a categorical foundation for general quantum mechanics, it may be wise to consider this link to quantum logic, and view conditions on the category in this context as well. For instance, the condition of self-adjoint idempotents strongly splitting implies the projections form an OMP rather than an OA. Another area of interest is having natural categorical conditions that ensure a good supply of states on these orthostructures of decompositions.

There may be something to be learned from the experience with quantum logic. Quantum logic began with the seminal paper of Birkhoff and von Neumann [4] who proposed using an abstract modular ortholattice (MOLs) to serve as the propositions of a quantum mechanical system. To von Neumann, the emphasis on modularity was key as it provided a link to projective geometry. But the assumption of modularity was appropriate for the propositions of a quantum system only in the finite-dimensional setting. If one restricts attention to this area, quantum logic does very well indeed as there is a tight link between finite dimensional modular ortholattices and projective geometries.

To cope with the general case, focus in quantum logic shifted to more general orthostructures such as OMLs and OMPs. While there are connections between MOLs and OMLs, experience has taught us that these are truly different creatures. Perhaps this reflects basic differences between phenomenon in finite-dimensional quantum mechanics and the those in the general case.

One might expect the job of extending the categorical foundation to general quantum mechanics to be a substantial one. But there are reasons for optimism. In particular, it is encouraging that this approach allows different aspects such as isolated systems, compound systems, and processes, to be treated at the same time.

Acknowledgments

Thanks are due to Samson Abramsky and Bob Coecke. In particular, discussions with Bob Coecke were helpful with some of the main results regarding projections and with examples dealing with Mat_K, and Samson Abramsky pointed to the usefulness of splitting idempotents.

References

[1] S. Abramsky, *Abstract scalars, loops, and free traced and strongly compact closed categories*,

[2] S. Abramsky and B. Coecke, *A categorical semantics of quantum protocols*, Proceedings of the 19th annual IEEE Symposium on Logic in Computer Science (LiCS'04), IEEE Computer Science Press, 2004, 415-425. (extended version at arXiv: quant-ph/0402130)

[3] S. Abramsky and B. Coecke, Abstract physical traces, Theory and Applications of Categories, 12 (2005), 111-124.

[4] G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Annals of Mathematics 37 (1936), 823-843.

[5] A. Dvurečenskij, *Tensor products of difference posets*, Trans. Amer. Math. Soc. 347 (3) (1995), 1043-1057.

[6] A. Dvurečenskij, *Gleason's Theorem and Its Applications*, Kluwer, 1993.

[7] D. J. Foulis, R. J. Greechie, and G. T. Rüttimann, *Filters and supports in orthoalgebras*, Inter. J. of Theor. Phys. 31 (5), (1992), 789-807.

[8] D. J. Foulis, *Algebraic measure theory*, Atti Sem. Mat. Fis. Univ. Modena 48 (2), (2000), 435-461.

[9] D. J. Foulis and M. K. Bennett, *Tensor product of orthoalgebras*, Order 10 (1993), 271-282.

[10] H. Herrlich and G. Strecker, Category Theory: an Introduction, Allyn and Bacon, 1973.

[11] J. Harding, *Decompositions in quantum logic*, Trans. Amer. Math. Soc. 348 (5) (1996), 1839-1862.

[12] J. Harding, *Orthomodularity of decompositions in a categorical setting*, Inter. J. of Theor. Phys. 45 (6), (2006), 1117-1128.

[13] G. Janusz, *Parameterization of self-dual codes by orthogonal matrices*, Finite Fields and their Applications 13, (2007), 450-491.

[14] G. Kalmbach, *Orthomodular Lattices*, Academic Press, London, New York, 1983.

[15] F. Katrnoška, *Logics of idempotents of rings*, Proc. Second Winter School on Measure Theory Ján Liptovský, (1990) Jednota Slovensk. Mat. Fyz., Bratislava, 1990, 100-104.

[16] G. M. Kelly and M. L. Laplaza, *Coherence for compact closed categories,* J. of Pure and Applied Algebra 19, (1980), 193-213.

[17] G. Mackey, *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York, 1963.

[18] S. MacLean, *Categories for the Working Mathematician*, Springer Graduate Texts in Mathematics 5, 1971.

[19] C. Piron, *Foundations of Quantum Physics*, Benjamin, Reading, Mass., 1976.

[20] P. Pták and S. Pulmanová, *Orthomodular Structures as Quantum Logics*, Fundamental Theories of Physics 44, Kluwer, 1991.

[21] P. Selinger, *Dagger compact closed categories and completely positive maps (extended abstract)*, Electronic Notes in Theoretical Computer Science 170 (2007), 139-163.

[22] P. Selinger, *Idempotents in dagger categories (extended abstract)*, to be published in Electronic Notes in Theoretical Computer Science.

[23] A. Wilce, *Quantum logic and quantum probability*, Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/qt-quantlog/, Feb. 2002; revised February, 2006.

John Harding
Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003, USA
jharding@nmsu.edu
www.math.nmsu.edu/∼jharding