

Multispans

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1 Multispans

We define a multi-cospan as the image of a poset in a category \mathcal{C} with finite colimits (and dually for multi-spans). We think of such a diagram in \mathcal{C} as a hierarchical cell complex where morphisms describe inclusion of boundary pieces. A multispan is the dual concept.

Definition 1.1 (posets) A poset is a small category D with $D(a, b)$ either empty or the singleton set, for all $a, b \in \text{Obj}(D)$, i.e. a category enriched over $(\{\emptyset, \text{pt}\}, \times)$. Let Posets be the full sub-category of Categories on posets and let Posets_{\max} the category whose objects are the posets that have a terminal object and whose morphisms are morphisms of posets that preserve the terminal object.

A terminal object \top in a poset D is an object such that $\forall a \in \text{Obj}(D) : (a, \top) \simeq \text{pt}$. A co-terminal object b is one for which $\forall a \neq b \in \text{Obj}(D) : (a, b) \simeq \emptyset$. There is an obvious forgetful functor $U : \text{Posets}_{\max} \rightarrow \text{Posets}$ with a left adjoint $(\) : \text{Posets} \rightarrow \text{Posets}_{\max}$ which freely adjoins a terminal object to a given poset.

Definition 1.2 (multi-cospan) A multi-cospan in a category \mathcal{C} with finite colimits is a poset $D \in \text{Posets}_{\max}$ and a functor $K : D \rightarrow \mathcal{C}$.

Remark. We think of the object $K(\top) \in \mathcal{C}$, for \top the terminal object of D , as a single top-dimensional cell in a hierarchical complex, in that we think of any morphism $K(a \hookrightarrow \top)$ in \mathcal{C} for every $a \in D$ as embedding a boundary piece $K(a)$ labeled by a into the total space (but there is no requirement that $K(a \hookrightarrow \top)$ be a monomorphism) and think of each morphism $K(b \hookrightarrow a)$ for $b \hookrightarrow a$ in D as describing a boundary piece $K(b)$ of a boundary piece $K(a)$.

1.1 Composition

The idea is that multi-cospans are composed by first gluing them along a common sub-multi-cospan, then forming the colimit cocone over that, and finally picking a sub-multi-cospan in that, containing the tip of the colimit cocone.

Definition 1.3 (multi-cospan closure) For $D \in \text{Posets}$ and $K : D \rightarrow \mathcal{C}$ a functor the multi-cospan closure of K is the unique multi-cospan

$$\bar{K} : \bar{D} \rightarrow \mathcal{C}$$

such that

$$D \xrightarrow{\quad} \bar{D} \xrightarrow{\quad \bar{K} \quad} \mathcal{C}$$

$$\quad \quad \quad \searrow \quad \quad \nearrow$$

$$\quad \quad \quad K \quad \quad \quad$$

and

$$\{\top\} \xrightarrow{\quad} \bar{D} \xrightarrow{\quad \bar{K} \quad} \mathcal{C}$$

$$\quad \quad \quad \searrow \quad \quad \nearrow$$

$$\quad \quad \quad \top \mapsto \text{colim}_D K \quad \quad \quad$$

Definition 1.4 (multi-cospan composition) For $K_1 : D_1 \rightarrow \mathcal{C}$ and $K_2 : D_2 \rightarrow \mathcal{C}$ two multi-cospans in \mathcal{C} , and given a diagram $U(D_1) \longleftarrow D_{\text{glue}} \longrightarrow U(D_2)$ in Posets of sub-poset inclusions respecting co-terminal objects, such that

$$\begin{array}{ccc} D_{\text{glue}} & \longrightarrow & D_1 \\ \downarrow & & \downarrow K_1 \\ D_2 & \xrightarrow{K_2} & \mathcal{C} \end{array}$$

and given a morphism in Posets_{\max} $D_{\text{comp}} \hookrightarrow \overline{D_1 \sqcup_{\text{glue}} D_2}$ we say that composition of K_1 and K_2 along D_{glue} to D_{comp} is the multi-cospan

$$K_{\text{comp}} : D_{\text{comp}} \hookrightarrow \overline{D_1 \sqcup_{\text{glue}} D_2} \xrightarrow{\overline{K_1 \sqcup_{\text{glue}} K_2}} \mathcal{C}$$

Example. We obtain ordinary cospans and their composition by taking all multi-cospan domains to be $D = \left\{ \begin{array}{c} \top \\ \nearrow a_1 \quad \nwarrow a_2 \end{array} \right\}$ and $D_{\text{glue}} = \{\bullet\}$, so that $D \sqcup_{\text{glue}} D = \left\{ \begin{array}{c} t_1 \quad t_2 \\ \nearrow a_1 \quad \nwarrow a_2 \quad \nearrow a_3 \end{array} \right\}$ and $\overline{D \sqcup_{\text{glue}} D} = \left\{ \begin{array}{c} \top \\ \nearrow t_1 \quad \nwarrow t_2 \\ \nearrow a_1 \quad \nwarrow a_2 \quad \nearrow a_3 \end{array} \right\}$ and finally taking $D_{\text{comp}} = D$ with $D_{\text{comp}} \hookrightarrow \overline{D \sqcup_{\text{glue}} D}$ given by $a_1 \mapsto a_1$ and $a_2 \mapsto a_3$.

Dually, if \mathcal{C} is replaced by a category S^{op} , such multi-cospans in S^{op} are spans in S .

But already in this case we get a little more flexibility spans for handling cospans.

For definiteness, consider cospans $\mathcal{C} = \text{Sets}^{\text{op}}$.

Composition of the multispan \bar{K}_1 which is the closure of

$$K_1 := \left\{ \begin{array}{c} \Psi_{\text{in}} \quad \Psi_{\text{out}} \\ \downarrow \quad \downarrow \\ X \quad Y \end{array} \right\}$$

with

$$K_2 := \left\{ \begin{array}{c} R \\ \downarrow \\ X \times Y \\ \swarrow \quad \searrow \\ X \quad Y \end{array} \right\}$$

along the preimages of X and Y to \top is the tip of

$$\begin{array}{c} \langle \Psi_{\text{in}} | R | \Psi_{\text{out}} \rangle \\ \swarrow \quad \downarrow \quad \searrow \\ \Psi_{\text{in}} \quad X \times Y \quad \Psi_{\text{out}} \\ \downarrow \quad \swarrow \quad \searrow \quad \downarrow \\ X \quad Y \end{array}$$

and describes the contraction of the matrix K with the vectors Ψ_{in} and Ψ_{out} .

Next, consider two matrices R_1 and R_2 given by the multispan \bar{K}_1 which is the closure of

$$K_1 := \left\{ \begin{array}{c} R_1 \quad R_2 \\ \downarrow \quad \downarrow \\ X \times Y \quad Y \times Z \\ \swarrow \quad \searrow \quad \downarrow \quad \searrow \\ X \quad Y \quad Y \quad Z \end{array} \right\}$$

and then consider the multispan

$$K_2 := \left\{ \begin{array}{c} X \longleftarrow X \times Z \longrightarrow Z \\ \swarrow \quad \uparrow \quad \searrow \\ X \times Y \times Z \\ \swarrow \quad \searrow \\ X \times Y \quad Y \times Z \\ \swarrow \quad \searrow \\ X \quad Y \quad Z \end{array} \right\}.$$

Then composition *along* the lower zig-zag *to* the resulting top zig-zag yields the matrix product

$$K_{\text{comp}} = \left\{ \begin{array}{c} R_1 \cdot R_2 \\ \downarrow \\ X \times Z \\ \swarrow \quad \searrow \\ X \quad Z \end{array} \right\}$$

These represent \mathbb{N} -valued matrices. Somewhat more interestingly, let k be some field and

$$K_1 := \left\{ \begin{array}{ccc} R_1 & & R_2 \\ \downarrow & & \downarrow \\ (X \times Y) \times k & & (Y \times Z) \times k \\ \swarrow \quad \downarrow & & \downarrow \quad \searrow \\ X & Y & Y & Z \end{array} \right\}$$

two k -valued matrices to be composed with the multispans

$$K_2 := \left\{ \begin{array}{ccc} & (X \times Z) \times k & \\ & \swarrow \quad \searrow & \\ X & & Z \\ & \uparrow (-) \cdot (-) & \\ & (X \times Y \times Z) \times k \times k & \\ & \swarrow \quad \searrow & \\ (X \times Y) \times k & & (Y \times Z) \times k \\ \swarrow \quad \searrow & & \swarrow \quad \searrow \\ X & Y & Z \end{array} \right\}.$$

Then the result is (after k -valued cardinality) the product of k -valued matrices.

1.2 Trace and co-trace

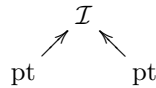
Definition 1.5 (co-trace) *If for $K : D \rightarrow \mathcal{C}$ a multi-cospan in which for a collection $\{a_i \in \text{Obj}(D)\}_i$ of coterminial objects we have $K(a_i) \simeq X$ for all i and for X a given object of \mathcal{C} the co-trace of K over $\{a_i\}$ is the composition of K with the co-span*

$$K_X : \left\{ \begin{array}{c} \top \\ \nearrow \quad \nwarrow \\ a_1 \quad a_2 \quad \dots \quad a_i \quad \dots \end{array} \right\} \xrightarrow{\text{const}_X} \mathcal{C}$$

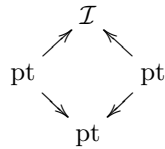
along the obvious identifications of the a_i to the diagram D with the a_i removed.

For spans the co-trace is called the trace.

Examples. Let $\mathcal{C} = \text{Categories}$ and $\mathcal{I} := \{a \rightarrow b\}$ be the 1-globe (the (directed) interval) regarded as a co-span

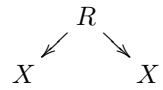


then the co-trace of \mathcal{I} over pt is the the colimit over

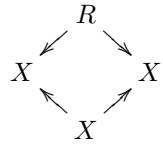


which is \mathbf{BN} (the (directed) circle).

Dually, let $\mathcal{C} = \mathbf{Sets}^{\text{op}}$ and consider spans of finite sets again, with



an $|X| \times |X|$ -matrix, then the trace of this is the limit over



which is $\sqcup_{x \in X} Rx, x$, as expected.