Multispans

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1 Multispans

We define a multi-cospan as the image of a poset in a category $\mathcal{C}$ with finite colimits (and dually for multi-spans). We think of such a diagram in $\mathcal{C}$ as a hierarchical cell complex where morphisms describe inclusion of boundary pieces. A multispan is the dual concept.

Definition 1.1 (posets) *A poset is a small category D with $D(a,b)$ either empty or the singleton set, for all $a, b \in \mathrm{Obj}(D)$, i.e. a category enriched over $(\{\emptyset, \mathrm{pt}\}, \times)$. Let* Posets *be the full sub-category of* $\mathsf{Categories}$ *on posets and let* $\mathsf{Posets}_{\max}$ *the category whose objects are the posets that have a terminal object and whose morphisms are morphisms of posets that preserve the terminal object.*

A terminal object $\top$ is a poset D is an object such that $\forall a \in \mathrm{Obj}(D) : (a, \top) \simeq \mathrm{pt}$. A co-terminal object b is one for which $\forall a \neq b \in \mathrm{Obj}(D) : (a, b) \simeq \emptyset$. There is an obvious forgetful functor $U : \mathsf{Posets}_{\max} \to \mathsf{Posets}$ with a left adjoint $\overline{(\;)} : \mathsf{Posets} \to \mathsf{Posets}_{\max}$ which freely adjoins a terminal object to a given poset.

Definition 1.2 (multi-cospan) *A multi-cospan in a category $\mathcal{C}$ with finite colimits is a poset $D \in \mathsf{Posets}_{\max}$ and a functor $K : D \to \mathcal{C}$.*

Remark. We think of the object $K(\top) \in \mathcal{C}$, for $\top$ the terminal object of D, as a single top-dimensional cell in a hierarchical complex, in that we think of any morphism $K(a \hookrightarrow \top)$ in $\mathcal{C}$ for every $a \in D$ as embedding a boundary piece $K(a)$ labeled by a into the total space (but there is no requirement that $K(a \hookrightarrow \top)$ be a monomorphism) and think of each morphism $K(b \hookrightarrow a)$ for $b \hookrightarrow a$ in D as describing a boundary piece $K(b)$ of a boundary piece $K(a)$.

1.1 Composition

The idea is that multi-cospans are composed by first gluing them along a common sub-multi-cospan, then forming the colimit cocone over that, and finally picking a sub-mult-cospan in that, containing the tip of the colimit cocone.

Definition 1.3 (multi-cospan closure) *For $D \in$ Posets and $K : D \to \mathcal{C}$ a functor the multi-cospan closure of K is the unique multi-cospan*

$$\bar{K} : \bar{D} \to \mathcal{C}$$

such that

$$\begin{tikzcd} D \arrow[r, hook] \arrow[rr, bend right, "K"'] & \bar{D} \arrow[r, "\bar{K}"] & \mathcal{C} \end{tikzcd}$$

and

$$\begin{tikzcd} \{\top\} \arrow[r, hook] \arrow[rr, bend right, "\top \mapsto \mathrm{colim}_D K"'] & \bar{D} \arrow[r, "\bar{K}"] & \mathcal{C} \end{tikzcd}$$

Definition 1.4 (multi-cospan composition) *For $K_1 : D_1 \to \mathcal{C}$ and $K_2 : D_2 \to \mathcal{C}$ two multi-cospans in $\mathcal{C}$, and given a diagram $U(D_1) \hookleftarrow D_{\mathrm{glue}} \hookrightarrow U(D_2)$ in* Posets *of sub-poset inclusions respecting co-terminal objects, such that*

$$\begin{tikzcd} D_{\mathrm{glue}} \arrow[r] \arrow[d] & D_1 \arrow[d, "K_1"] \\ D_2 \arrow[r, "K_2"] & \mathcal{C} \end{tikzcd}$$

and given a morphism in $\mathsf{Posets}_{\max}$ $D_{\mathrm{comp}} \hookrightarrow \overline{D_1 \sqcup_{\mathrm{glue}} D_2}$ we say that composition of K_1 and K_2 along D_{glue} to D_{comp} is the multi-cospan

$$K_{\mathrm{comp}} : \; D_{\mathrm{comp}} \hookrightarrow \overline{D_1 \sqcup_{\mathrm{glue}} D_2} \xrightarrow{\overline{K_1 \sqcup_{\mathrm{glue}} K_2}} \mathcal{C}$$

Example. We obtain ordinary cospans and their composition by taking all multi-cospan domains to be

$$D = \left\{ \begin{tikzcd}[column sep=small] & \top & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \end{tikzcd} \right\}$$

and $D_{\mathrm{glue}} = \{\bullet\}$, so that

$$D \sqcup_{\mathrm{glue}} D = \left\{ \begin{tikzcd}[column sep=small] & t_1 & & t_2 & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$$

and

$$\overline{D \sqcup_{\mathrm{glue}} D} = \left\{ \begin{tikzcd}[column sep=small] & & \top & & \\ & t_1 \arrow[ur] & & t_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$$

and finally taking $D_{\mathrm{comp}} = D$ with $D_{\mathrm{comp}} \hookrightarrow \overline{D \sqcup_{\mathrm{glue}} D}$ given by $a_1 \mapsto a_1$ and $a_2 \mapsto a_3$.

Dually, if $\mathcal{C}$ is replaced by a category S^{op}, such multi-cospans in S^{op} are spans in S.

But already in this case we get a little more flexibility spans for handling cospans.

For definitenss, consider cospans $\mathcal{C} = \mathsf{Sets}^{\mathrm{op}}$.

Composition of the multispan $\bar{K}_1$ which is the closure of

$$K_1 := \left\{ \begin{tikzcd} \Psi_{\mathrm{in}} \arrow[d] & \Psi_{\mathrm{out}} \arrow[d] \\ X & Y \end{tikzcd} \right\}$$

with

$$K_2 := \left\{ \begin{tikzcd} & R \arrow[d] & \\ & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \right\}$$

along the preimages of X and Y *to* $\top$ is the tip of

$$\begin{tikzcd} & \langle \Psi_{\mathrm{in}} | R | \Psi_{\mathrm{out}} \rangle \arrow[d] \arrow[ddl] \arrow[ddr] & \\ & R \arrow[d] & \\ \Psi_{\mathrm{in}} \arrow[d] & X \times Y \arrow[dl] \arrow[dr] & \Psi_{\mathrm{out}} \arrow[d] \\ X & & Y \end{tikzcd}$$

and describes the contraction of the matrix K with the vectors Ψ_{in} and Ψ_{out}.

Next, consider two matrices R_1 and R_2 given by the multispan $\bar{K}_1$ which is the closure of

$$K_1 := \left\{ \begin{tikzcd} & R_1 \arrow[d] & R_2 \arrow[d] & \\ & X \times Y \arrow[dl] \arrow[d] & Y \times Z \arrow[d] \arrow[dr] & \\ X & Y & Y & Z \end{tikzcd} \right\}$$

and the consider the multispan

$$K_2 := \left\{ \begin{tikzcd} X \arrow[dd, no head, "="] & & X \times Z \arrow[ll] \arrow[rr] & & Z \arrow[dd, no head, "="'] \\ & & X \times Y \times Z \arrow[u] \arrow[ull] \arrow[urr] \arrow[dl] \arrow[dr] & & \\ & X \times Y \arrow[dl] \arrow[dr] & & Y \times Z \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd} \right\}.$$

Then composition *along* the lower zig-zag *to* the resulting top zig-zag yields the matrix product

$$K_{\mathrm{comp}} = \left\{ \begin{tikzcd} & R_1 \cdot R_2 \arrow[d] & \\ & X \times Z \arrow[dl] \arrow[dr] & \\ X & & Z \end{tikzcd} \right\}$$

These represent $\mathbb{N}$-valued matrices. Somewhat more interestingly, let k be some field and

$$K_1 := \left\{ \begin{tikzcd} & R_1 \arrow[d] & & R_2 \arrow[d] & \\ & (X \times Y) \times k \arrow[dl] \arrow[d] & & (Y \times Z) \times k \arrow[d] \arrow[dr] & \\ X & Y & & Y & Z \end{tikzcd} \right\}$$

two k-valued matrices to be composed with the multispan

$$K_2 := \left\{ \begin{tikzcd} & & (X \times Z) \times k \arrow[dll] \arrow[drr] & & \\ X \arrow[ddd, dash, "="] & & & & Z \arrow[ddd, dash, "="'] \\ & & (X \times Y \times Z) \times k \times k \arrow[ull] \arrow[urr] \arrow[uu, "(-)\cdot(-)"'] \arrow[dl] \arrow[dr] & & \\ & (X \times Y) \times k \arrow[dl] \arrow[dr] & & (Y \times Z) \times k \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd} \right\}.$$

Then the result is (after k-valued cardinality) the product of k-valued matrices.

1.2 Trace and co-trace

Definition 1.5 (co-trace) *If for $K : D \to \mathcal{C}$ a multi-cospan in which for a collection $\{a_i \in \mathrm{Obj}(D)\}_i$ of coterminal objects we have $K(a_i) \simeq X$ for all i and for X a given object of $\mathcal{C}$ the co-trace of K over $\{a_i\}$ is the composition of K with the co-span*

$$K_X : \left\{ \begin{tikzcd} & & \top & & \\ a_1 \arrow[urr] & a_2 \arrow[ur] & \cdots & a_i \arrow[ul] & \cdots \end{tikzcd} \right\} \xrightarrow{\mathrm{const}_X} \mathcal{C}$$

along the obvious identifications of the a_i to the diagram D with the a_i removed.

For spans the co-trace is called the trace.

Examples. Let $\mathcal{C} = \mathsf{Categories}$ and $\mathcal{I} := \{a \to b\}$ be the 1-globe (the (directed) interval) regarded as a co-span

$$\begin{tikzcd} & \mathcal{I} & \\ \mathrm{pt} \arrow[ur] & & \mathrm{pt} \arrow[ul] \end{tikzcd}$$

then the co-trace of $\mathcal{I}$ over pt is the the colimit over

$$\begin{tikzcd} & \mathcal{I} & \\ \mathrm{pt} \arrow[ur] \arrow[dr] & & \mathrm{pt} \arrow[ul] \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

which is $\mathbf{B}\mathbb{N}$ (the (directed) circle).

Dually, let $\mathcal{C} = \mathsf{Sets}^{\mathrm{op}}$ and consider spans of finite sets again, with

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

an $|X| \times |X|$-matrix, then the trace of this is the limit over

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & X \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

which is $\sqcup_{x \in X} Rx, x$, as expected.