1. Cycles, relations
2. Integration
3. ch, c1, γ Adams (? Bruce)
4. Riemann Roch

The model: cycles

Want to construct

\[(K, R, I, a, S)\]

as in Thomas' talk.

**Cycle:** \((E, p)\)

\(E: \text{a geometric family} = \left(\begin{array}{c}
E \\
\downarrow \\
B
\end{array}\right)
\)

**Example:** From \(W = \left(\begin{array}{c}
W \\
\downarrow \\
B
\end{array}\right)\) _C-vector bundle_, \(h^W, \nabla^W\)

Can make a geometric family.
\[
W = \left( B, \begin{array}{c}
0 \\
\uparrow \\
B \\
\end{array}, 0, TB, W \right)
\]

"0-dim fibers!"

What is \( p \)?

\[ p \in \mathcal{M}(B, K) = C^\infty(B, \Lambda^* T^* B \otimes K^*) \]

If you have a geometric family, have local index form

\[ \mathcal{M}(E) = \int \hat{A} \text{ch} \]

In the \( W \) example,

\[ \mathcal{M}(W) = \text{ch}(\nabla W) \]

usual thing from local index theory.
b) Relations:

\[ H(\varepsilon) \quad \text{family of Hilbert spaces} \]

\[ H(\varepsilon)_b = L^a(E_b, W_{E_b}) \]

\[ D(\varepsilon) \quad \text{family of Dirac operators} \]

\[ Q(\varepsilon) \quad \text{family of smoothing operators} \]

s.t. \quad D(\varepsilon) + Q(\varepsilon) \quad \text{invertible}

\[ \Rightarrow \text{can define } \eta \text{-form} \]

\[ \eta(\varepsilon_t) \in \Omega(B, K) \]

with property

\[ d\eta = \omega(\varepsilon), \]

\[ [\omega(\varepsilon)] = \text{ch}(\text{index } D(\varepsilon)) \]
We say 
\[(\varepsilon, \rho) \sim 0\]

if there exists \(\varepsilon_t\) such that
\[\rho = \eta(\varepsilon_t)\]

A. Henriques: I'm lost.

U. Bunke: Take g. family \((\varepsilon, 0) + (\varepsilon^{\text{op}}, 0) \sim 0\)

The claim is that this
\[= (\varepsilon \cup \varepsilon^{\text{op}}, 0)\]

Take \(\ker(D(\varepsilon)) \oplus \ker(D(\varepsilon^{\text{op}}))\)

\[\text{Lemma: } \eta((\varepsilon \cup \varepsilon^{\text{op}})_t) = 0.\]

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\[R(\varepsilon, \rho) = \Omega(\varepsilon) - d\rho.\]
\[I(\varepsilon, \rho) = \text{index}(D(\varepsilon)).\]
\[\alpha(\omega) = (\phi, -\omega).\]
What is η-form?

\[ η(ξ_t) = \int Tr \frac{∂}{∂t} A_ξ(ξ_t) e^{-A_ξ^2(ξ_t)} \]

If we have

\[ W = (W, h^W, ∇W) \]

\[ → [W] = (W, 0). \]

Interpretation (sorry... Integration)

Take vertical bundle. If it has Spin \( ^C \) structure, it induces ...

\[ p: A → B \quad \text{proper submersion} \]

\[ T^v_p \text{ spin}^C → K-orientation. \]

Need smooth refinement of orientation!
Wish list

\[ \mathcal{O} \{ \text{smooth orientation} \} \rightarrow \text{tensors for } \Omega^*(A, k)/_{\text{ind}} \]

\[ \mathcal{O} \{ \text{top. orientations} \} \]

Want also composition

\[
\begin{array}{ccc}
A & \rightarrow & \text{compose + pull-back.} \\
\downarrow p & & \\
B & & \\
\downarrow q & & \\
C & & \\
\end{array}
\]

André: it is a refinement of the coh. theory $K(\text{smooth orientation})$?

Ulrich: No, I couldn't get that.
\[ R(\hat{\sigma}) = \hat{A}^c(\hat{\bigwedge}) - d\alpha \in \Omega^0(A, k) \]
\[
\hat{\rho}^0 : K^*(A) \longrightarrow K^{*-m}(B)
\]

\[
m = \dim T^v p
\]

\[
\hat{\rho}^0(\mathcal{E}, p) = \left( \rho^! \mathcal{E}, \int A^c(\mathcal{V})^\wedge p + \int A^\wedge \mathcal{R}(\mathcal{E}, p) \right)
\]

actually only true in the adiabatic limit

where you scale down fibers.

**Theorem (Burke, Schick)**

This is:

- well-defined
- functorial
- compatible with products

This is the natural home of invariants.

E.g., Assume \( T^v p \) is stably framed.

Prop \( p \) has a canonical smooth orientation.

"odd counterterm"
\( \hat{\Theta} = \left[ A, \tilde{A}^\circ (\tilde{\Delta}, \tilde{\Delta}^{fr}) \right] \)

Can look at
\[
e(\pi) = \hat{\rho}_! \hat{\Theta} (1) \in K_{\text{flat}}^- (B) \]
\[\cong K \mathbb{L} / \mathbb{Z}^{m+1} \]

If \( B \) is a point, modd,
\[
e(\pi) = \text{Adams } e \text{-invariant } \in K \mathbb{L} / \mathbb{Z}^{2l}
\]
(uses APS interpretation)

Because of nice bordism formula,
easy to calculate in many cases.
Thm. Have natural lifts

\[ \hat{\mathcal{C}}_i : K \to \hat{\mathbb{Z}}^{2i} \]

\[ \hat{\mathcal{C}}_i \text{ odd} : K^1 \to \hat{\mathbb{Z}}^{2i-1} \]

\[ 1 + \hat{\mathcal{C}}_i \mathbb{S} = : \mathcal{C} \]

preserves group structure.

\[ \int \hat{\mathcal{C}}_i = \hat{\mathcal{C}}_i \text{ odd} \int \]

Also, in older Bunke–Schick paper, have

\[ \hat{\text{ch}} : K \to \hat{\mathbb{H}Q} \]

More recently, have Adams operations

\[ \hat{\psi}^k : \hat{K} \left[ \frac{1}{\pi} \right] \to \hat{K} \left[ \frac{1}{\pi} \right] \]

only on compact

\[ \hat{\psi}^k \circ \hat{\psi}^l = \hat{\psi}^{kl} \]
Andre: Okay, it's

Ulrich: It's Istanbul! (?)

Here, this is $\mathbb{Z}$-graded.

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Riemann-Roch

\[
\begin{array}{c}
A \\ \hat{\theta}
\end{array}
\xrightarrow{p} B
\]

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Classical

\[
\begin{array}{c}
K(A) \\ \rho!
\end{array}
\xrightarrow{\text{ch}}
\begin{array}{c}
H \mathbb{Q} (A) \\ \int_{A^c (\tau_v)} \nu...
\end{array}
\]

\[
\begin{array}{c}
K(B) \\ \text{ch}
\end{array}
\xrightarrow{\text{ch}}
\begin{array}{c}
H \mathbb{Q} (B)
\end{array}
\]

Now we have extensions. Put hats over everything.

Exists $\hat{H} \mathbb{Q} (A)$, need $\hat{A}$.

Thm (Riemann-Roch) \cite{B-S}. This diagram commutes.
One proves such a thing by showing that the difference couldn't exist!

How about Adams operation?

\[ K[A][\frac{1}{k}] \xrightarrow{\psi^k} K(A)[\frac{1}{k}] \xleftarrow{Adams\ correction} K(B)[\frac{1}{k}] \]

\[ \Pi \xrightarrow{^0} K(B)[\frac{1}{k}] \xrightarrow{\psi^k} K(B)[\frac{1}{k}] \]

Some equation concerning Freed-Melrose,?

One consequence:

Prop: If \( \hat{\mathcal{D}} \) comes from a stolde framing, then

\[ \hat{\rho}(\hat{\mathcal{D}}) = 1 \]

Cor: \[ \psi^k e(p) = e(p) \]

in special case: \( k^r \left( \frac{k^r - 1}{m=2r-1} \right) \)