Alternative to yesterday's axioms:

Replace $B(M)$ by embeddings $(\overline{D}^n, M)$

$$\sim \quad \text{Embeddings} \left( \overline{D}^n, M \right) \times \text{Emb} \left( \overline{D}^n \cup \cdots \cup \overline{D}^n, \overline{D}^n \right)_{k \text{ times}}$$

Basic idea

Factorization algebras form a symmetric monoidal category.

If $F, F'$ are factorization algebras, then

$$(F \otimes F')(B) = F(B) \otimes F'(B^*)$$

**Def.** A classical factorization algebra is a commutative algebra in the category of factorization algebras.

[Recall, an $E_{\infty}$ object in $E_n$ algebras is an $E_{\infty}$-algebra.]

Idea: We want to associate a classical fact. alg to a classical field theory as follows:
Suppose we have a classical field theory, e.g. space of fields is sections of vector bundle $E \rightarrow M$

$$S : \Pi(M, E) \rightarrow \mathbb{R}$$

is the classical action.

$S$ is local: obtained by $\int$ of a Lagrangian.

If $B \subset M$ is a ball, let

$$\text{EL}(B) = \left\{ \phi \in \Pi(\text{Interior } B, E) \right\}$$

which satisfy the Euler-Lagrange Equations

Don Freed: You're working in Euclidean signature?

Costello: Yes. We hope we can Wick rotate later.

Rough idea

The classical $\mathfrak{f}$ algebra $F_S$ associated to $S$ assigns to $B$ the algebra of functions on the set of solutions to $\text{EL}$.

$\Theta(\text{EL}(B))$. 
We want maps

\[ F_s(B_1) \otimes \cdots \otimes F_s(B_n) \rightarrow F_s(B_{n+1}) \]

if \( B_1 \perp \cdots \perp B_n \subseteq B_{n+1} \).

We have a map

\[ \text{EL}(B_{n+1}) \rightarrow \text{EL}(B_1) \times \cdots \times \text{EL}(B_n) \]

This yields a map

\[ \Theta(\text{EL}(B_1)) \otimes \cdots \otimes \Theta(\text{EL}(B_n)) \rightarrow \Theta(\text{EL}(B_{n+1})) \]

as desired.

**Simple example:**

Fields are \( C^\infty \)-functions on \( M \).

\[ S(\phi) = \int_M \phi \Delta \phi \]

Euler-Lagrange eq. \( \Delta \phi = 0 \)

\( \text{EL}(B) = \) Harmonic functions on \( \text{Int} B \)

\[ \Theta(\text{EL}(B)) := \bigoplus_{n>0} \text{Hom}(\text{EL}(B) \otimes \mathbb{R}^n, \mathbb{R}) S_n \]

i.e. formal power series.
where Hom means continuous linear maps, $\otimes$ is composition.

Later, we'll see we really need to take the derived space of EL solns.

Why does this classical factorization algebra want to become just a fact. algebra?

Fact. algebras form a symmetric monoidal category.

The $\mathbb{E}_0$ operad is defined by $\mathbb{E}_0(n) = \emptyset$ if $n > 1,$ and $\mathbb{E}_0(1) = \text{pt}.$

An $\mathbb{E}_0$-algebra in vector spaces is just a vector space with an element.

Forgot to mention that fact. algebras need to have a unit, a section of $F$ on $B(M),$ which is a unit for the product.

So: an $\mathbb{E}_0$-algebra in Fact. alg. is just a Fact. algebra!
Beilinson + Drinfeld define an operad over the ring $\mathbb{R}[[[t]]]$ as follows:

- generated by $\cdot$, a comm. product
- $\{\{\}_3\}$, a Poisson bracket of degree +1
- with differential $d(\cdot) = t \cdot \{\{\}_3\}$.

Call this the BD operad.

$$\frac{BD}{hBD} = \text{operad of comm. algebras } \{\{\}_3\} \text{ deg. } +1.$$ 

$$H_* \left( \frac{BD(n) \otimes \mathbb{R}[[h]]}{\mathbb{R}[h]} \right) = 0.$$
So, $BD \otimes \mathbb{R}(\hbar) \cong E_0$.

[Aside: BV operad if framed $E_2$.
But this has nothing to do with the Batalin-Vilkovisky formalism.
The BV operad is really the BD operad!]

Def\textsuperscript{\textnormal{\textdagger}}. The $P_0$ (or $Poisson_0$) operad is the operad of \textup{\textquotesingle\textquotesingle} Poisson algebras with $\mathcal{E}$ of degree $-1$.

so, $P_0 = BD/\hbar$.

General fact

Let $M$ be a manifold, $f: M \to \mathbb{R}$.

Then $\Theta(\text{Derived critical locus of } f)$ is a $P_0$-algebra.

The critical locus = $Z(df)$.

So $\Theta(\text{critical locus}) = \Theta(M) / \text{image } (\Gamma(M, TM)^\vee \to C^\infty(M))$. 
The derived critical locus has functions the dga

\[ \cdots \rightarrow \Gamma(M, \wedge^k TM) \rightarrow \Gamma(M, TM) \rightarrow \Theta(M) \]

If \( f \) is Morse, this is equiv. to usual setup.

In general captures more info.

This is the same as polyvector fields \( \Gamma(M, \wedge TM) \)

\[ \wedge^k TM \text{ is in deg } -k \]

with \( \text{diff} \ \nu df \).

Now,

\[ \Gamma(M, \wedge TM) \]

has Schouten bracket, which is of deg +1.

This "wants" to become \( E_0 \).

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\[ T^*M \quad \text{graph of } df \]

\[ \text{Crit}(f) = \Gamma(df) \wedge M \]

Derived critical locus = derived intersection.
Observation:

If $M$ has a measure, then $\Theta (\text{Crit } \circ (f))$

has a canonical quantization to an $E_0$-algebra. algebra over BD

The quantization is

\[
\left( \Gamma (M, \Lambda TM), \nu \text{df } + \hbar \Delta \right)
\]

the BV operator arises whenever $M$

has a measure.

\[
\Delta X = \text{Div} X \quad \text{if } X \text{ is a vector field}
\]
This is also done by Kevin Walker (blob homology) or Jacob Lurie (topological chiral homology).

**Lemma** For a massive scalar field,

\[ CH_*(M, \mathcal{F}) \cong \mathbb{R}[[h]] \]

≠ very exciting!

In general,

\[ CH_*(M, \mathcal{F}) \]

looks like measures on the space of critical points of the classical action.

If we perturb around isolated critical point,

\[ CH_*(M, \mathcal{F}) \cong \mathbb{R}[[h]] \]

In this situation, correlation functions exist and are unique.

**General program**: Correlation functions define a measure on space of classical solutions which we perturb around.

It's strange: we don't really perform the path integral, we "quantize", and this does it "automatically".
Q: Where's the propagator?
A: Some QME's written down,
    something about renormalization.
So far: The derived critical locus of a function is a $P_0$-algebra, so it wants to quantize to $E_0$.

If we have a classical field theory, the derived space of solutions to EL yields a $P_0$ algebra in factorization algebras. So it wants to become a factorization algebra.

Example: $\phi \in C^\infty(M)$, $S(\phi) = \int \phi \Delta \phi$,

Derived space of solutions to EL is the complex

\[ C^\infty(M) \xrightarrow{\Delta} C^\infty(M) \]

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If $B \subseteq M$ is a ball, then

$\Theta(\text{EL}_{\text{derived}}(B)) = \text{symmetric algebra on dual}$

\[
\prod_{n \geq 0} \text{Hom}(C^\infty(M), C^\infty(\text{int } B))
\]
This is a commutative algebra, and defines a commutative factorization algebra.

If \( S(\phi) = \int \phi \Delta \phi + \phi^3 \)

we get the same algebra of functions, but the differential changes.

Yon- Mills: first consider the appropriate derived quotient of \( \Omega^*(M) \otimes g \) by \( \Omega^0(M) \otimes g \) , and then take derived critical locus of YM action.

In physics literature, this is called the BV formalism.

What we get, when linearized, looks like

\[
E = \Omega^0(M) \otimes g \xrightarrow{d} \Omega^1(M) \otimes g \xrightarrow{\frac{d}{d\theta}} \Omega^2(M) \otimes g \xrightarrow{d} \Omega^4(M) \otimes g
\]

\(-1 \quad 0 \quad 1 \quad 2\)

The algebra of functions is \( \bigoplus_1 \text{Hom}(E^{\otimes n}, IR)^{S_n} \)

with differential including YM action.