Kevin Costello 3

<u>So far</u>: The derived critical locus of a function is a P_0-algebra, so it wants to quantize to E_0.

If we have a classical field theory, the derived space of solutions to EL yields a P_0 algebra in factorization algebras. So it wants to become a factorization algebra.

<u>Example</u>: $\phi \in C^\infty(M)$, $S(\phi) = \int \phi \Delta \phi$,

Derived space of solutions to EL is the complex

$$\underset{0}{C^\infty(M)} \xrightarrow{\Delta} \underset{1}{C^\infty(M)}$$

If $B \subseteq M$ is a ball, then

$$\mathcal{O}\left(EL^{\text{derived}}(B)\right) = \text{Symmetric algebra on dual}$$

$$= \prod_{n \geq 0} \mathrm{Hom}\left(\left(C^\infty(\mathrm{Int}\, B) \xrightarrow{\Delta} C^\infty(\mathrm{Int}\, B)\right), \mathbb{R} \right)^{S}$$

This is a commutative dga, and defines a commutative factorization algebra.

If $S(\phi) = \int \phi \Delta \phi + \phi^3$

we get the same algebra of functions, but the differential changes.

Yang-Mills: first consider the appropriate derived quotient of $\Omega^1(M) \otimes \mathfrak{g}$ by $\Omega^0(M) \otimes \mathfrak{g}$, and then take derived critical locus of YM action.

In physics literature, this is called the <u>BV formalism</u>.

What we get, when linearized, looks like

d star d.

$$\mathcal{E} = \underset{-1}{\Omega^0(M)_{\otimes \mathfrak{g}}} \xrightarrow{d} \underset{0}{\Omega^1(M)_{\otimes \mathfrak{g}}} \xrightarrow{d * d} \underset{1}{\Omega^3(M)_{\otimes \mathfrak{g}}} \xrightarrow{d} \underset{2}{\Omega^4(M)_{\otimes \mathfrak{g}}}$$

The algebra of functions is $\prod \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n}$ with differential including YM action.

<u>Theorem</u> If we take the derived space of sol^n's to the EL eq^n, look infinitesimally near a given solution (perturbing), then functions on this has a P_0-algebra structure in factorization algebras on M.

We'd like to quantize one of these.

This amounts to quantizing the action S into a solution of the "quantum master equation".

I've written a book about how to do this!

The quantum master eqn is not defined.

Requires machinery of counter-terms, Wilsonian effective action, to even talk about the QME.

<u>Thm</u> (joint with O. Gwilliam)

"Naive version". Consider the scalar field theory with an action

$$S(\phi) = \int \phi(\Delta\phi + m^2\phi) + \text{arbitrary local cubic} + \text{higher terms}$$

Let $\mathcal{F}_S$ be the classical fact. algebra associated to it. $\leftarrow$ P_0-algebra.

Let $Q^{(n)}(\mathcal{F}_S)$ be the set of quantizations defined mod $\hbar^{n+1}$.

(Lift of $\mathcal{F}_S$ to an algebra over $BD/\hbar^{n+1}$).

Then there is a sequence

$$T^{(n)} \to T^{(n-1)} \to \cdots \to T^{(1)} \to pt$$

where $T^{(n)}$ maps to $Q^{(n)}(\mathcal{F}_S)$ so obvious diagram commutes.

Where $T^{n} \to T^{n-1}$ is a torsor for the abelian group of local functions.

So: there is no canonical way to quantize, but if we quantize up to order n, we are free to add extra terms to Lagrangian.

$$T^{(\infty)} = \lim_{\leftarrow} T^{n},$$

then $T^{\infty} \cong \left\{ \sum_{k \geq 1} \hbar^k S^{(k)} \right\}$, $S^{(k)}$ is a local functional.

But, this is non-canonical.

More sophisticated version

Consider any reasonable classical theory, yielding classical fact. algebra $\mathcal{F}$.

Let $Q^{(n)}(\mathcal{F})$

= simplicial set of quantizations defined mod $\hbar^{n+1}$.

$\mathrm{Der}_{loc}(\mathcal{F})$ is the cochain complex of derivations of $\mathcal{F}$, preserving P_0 structure.

(in fact, local functionals on an "extended" space of fields).

<u>Theorem</u> There exists a sequence of simplicial sets

$$\cdots \to T^{(n)} \to T^{(n-1)} \to \cdots \to T^{(1)} \to pt$$

with maps $T^{(n)} \to Q^{(n)}(\mathcal{F}_s)$,

such that each $T^{(n)}$ fits into a homotopy fibre diagram

$$\begin{tikzcd} T^{(n)} \arrow[r] \arrow[d] & 0 \arrow[d] \\ T^{(n-1)} \arrow[r, "\mathrm{obs}"'] & \mathrm{Der}(\mathcal{F})[2] \end{tikzcd}$$

so the set of possible ways of quantizing at the next order is the number of ways you can kill the obstruction at this level.

I'm saying there are no analytic obstructions to quantization, but there may be homological obstructions (eg. anomalies in gauge field theory).

eg. famous β-function obstruction in ϕ^4 theory.

<u>Theorem</u> Let $\mathfrak{g}$ be a simple Lie algebra. Then there is a quantization of YM on $\mathbb{R}^4$, which is renormalizable in the Wilsonian sense, (behaves well under scaling). The set of such quantizations is isomorphic to

$$\hbar\, \mathbb{R}[\hbar].$$

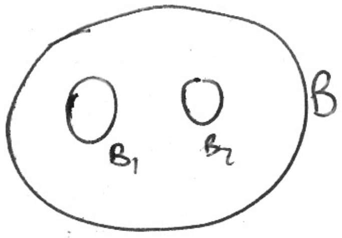
Where do correlation functions appear?

If $\mathcal{F}$ is a fact. algebra on M, corresponding to some QFT, then

$$\mathcal{F}(B) \underset{\substack{\uparrow \\ \text{think of}}}{=} \left\{ \begin{array}{c} \text{measurements we can} \\ \text{make on} \\ \text{the ball } B \end{array} \right\}$$

So the correlation functions:

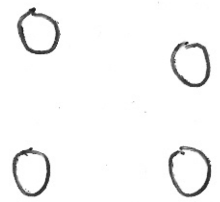
If $B_1, B_2 \subseteq B$ are disjoint,



the maps $\mathcal{F}(B_1) \otimes \mathcal{F}(B_2) \longrightarrow \mathcal{F}(B)$

corresponds to doing both observations on B_1 and B_2 inside B.

We'd like correlation functions to be cochain maps



$$\langle \quad \rangle : \mathcal{F}(B_1) \otimes \cdots \otimes \mathcal{F}(B_n) \longrightarrow \mathbb{R}$$

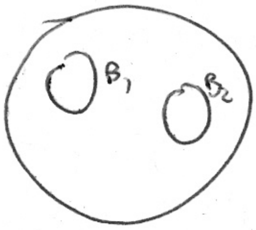
if $B_1, \dots, B_n$ are disjoint.

If $O_i \in \mathcal{F}(B_i)$, then

$\langle O_1, \dots O_n \rangle$ is a measurement of how observations O_i correlate

Want compatibility conditions:

If $B_1, B_2 \subseteq \tilde{B}$, then



the diagram should commute:

$$\begin{tikzcd}
\mathcal{F}(B_1) \otimes \cdots \otimes \mathcal{F}(B_n) \arrow[rr, "\langle\ \rangle"] \arrow[d] & & \mathbb{R} \\
\mathcal{F}(\tilde{B}) \otimes \mathcal{F}(B_3) \otimes \cdots \otimes \mathcal{F}(B_n) \arrow[urr, "\langle\ \rangle"'] & &
\end{tikzcd}$$

So far this is just a hope. But remarkably, cor. functions oftentimes uniquely det. by this property.

We can consider correlation functions with coefficients in any cochain complex; we require they must satisfy this eqn.

<u>Defn</u> (Beilonson-Drinfeld) the notoriously tough book "chiral algebras"

$$CH_*(M, \mathcal{F}) := \text{the universal recipient of correlation functions}$$

chiral homology

$$= \operatorname*{colim}_{\substack{B_1 \to \cdots \to B_n \subseteq M \\ \text{disjoint}}} \mathcal{F}(B_1) \otimes \cdots \otimes \mathcal{F}(B_n)$$