Then (Hill, H. Ravenel)

If $M$ is a stably framed manifold of Kervaire invariant 1,
then $\dim M$ is

2, 6, 14, 30, 62, 126.

Don't know what's going on!
Should be 6 !
(God made one every day)

The story of the problem

W- We start off by thinking about
h. classes of maps

$$S^{n+k} \to S^n$$

Pontryagin asked: what if we have more variables?
Pontryagin set up an amazing relationship between geometry and homotopy theory:

1930's \( \Omega \text{framed } S^n = \text{cobordism group of framed } k\text{-manifolds} \)

He used the classification of manifolds to understand in dim 0, 1, 2 homotopy groups of spheres!
\[ k = 0 \]

- point
- point

reproduces degree of a map,

\[ \text{so} \quad \mathbb{T}^{n} S^{n} = \mathbb{Z} \]

\[ k = 1 \]

Nice story: generalized notion of degree of a map

\[ M \rightarrow \text{Sphere} \]

Thinking about maps

\[ M^{n+1} \rightarrow S^{n} \]

was due to Steenrod. This one extra case.
led to the development of all htpy theory!

$k=2$ Pontyagin made a mistake!

\[ g=0 \]

bounds a disc.

\[
\text{change of framing} = \text{map into} \quad \text{general graph} \\
\uparrow \\
\Pi_2(\cdots) = 0
\]

so if genus = 0, must be trivial.

\[ g=1 \]

Brilliant idea

cut ?+ open here

\[
\text{sew in 2 discs to form sphere.}
\]
But obstruction!
\[ \phi : \mathbb{H}_1(\Sigma, \mathbb{Z}) \rightarrow \mathbb{Z}/2. \]

Pontryagin said dim is even, so...

\[ \nabla \nu \]

So always something in kernel, so you can lower genus. Therefore concluded

\[ \pi_{n+2} S^n = 0. \]

Mistake!

Mid 1930s.

1940s Whitehead calculated cyclic of order two.

Andrew Ranicki: Took 5 years.

The mistake is the map \( \phi \) is nonlinear in general!
Codd 60 surgery

weird loop.

Somehow makes $\phi$ not linear.

In fact, $\phi$ is quadratic!

$$\phi(xy) - \phi(x) - \phi(y) = \text{I}(x,y)$$

intersection pairing

In our example

$$\begin{array}{ccc}
\mathbb{Z}/2 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\mathbb{Z}/2 &
\end{array}$$

2 quadratic refinements.

Beautiful invariant of a quadratic function:

$$\text{Arf}(\phi) \in \mathbb{Z}/2$$

there are two framed Riemann surfaces
so $\mathbb{T}_{n+2} S^n = \mathbb{Z}/2$.

In modern terms, Pontryagin was trying to use surgery to convert mfd into sphere.

Q. In which dimension does every framed cobordism class contain a homotopy sphere? (in dim > 5, this is homeomorphic to sphere)

Answer: In every dim except those six!

Pontryagin only had a $1/\infty$ chance of making a mistake!

Invariants of quadratic forms/functions are deep and subtle!
1956: Milnor showed there were exotic 7-spheres.
A completely amazing thing!

1960: Kervaire studied Pontryagin's mistake.
Defined a quadratic refinement
\[ \phi : H_n(M^n, \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \]
when \( n \) is an odd number, and
\( M \) stably framed.

Kervaire: \( \Phi(M) = \text{Arf}(\phi) \).

Kervaire did two things:

- Showed
  \[ \Phi(M^{10}) = 0 \]

- Showed it could be defined when \( M \) not smooth
  constructed from PL manifold \( N \), and found
  \[ \Phi(N') \neq 0 \].
So an example of a PL manifold that couldn't be smoothed! Beautiful homotopy theory ideas.

At this point:

**Question:** In which dimensions can $\Phi(M)$ be nonzero?

Kervaire–Milnor (announced at international congress 1958) published 1963

(Nathaniel argues).

This is a beautiful paper. Introduced group

$$\Theta_n = \text{group of smooth structures on } S^n$$

under connected sum.

Related to hopy groups of spheres, Bernoulli numbers, ... great differential topology.

When $n$ even: signature $\rightarrow$ Bernoulli
don $n$ odd: $\rightarrow$ Kervaire invariant
Determined in terms of $\Theta_n S^0$ up to a factor of $2^n$, which depended on the Kervaire invariant!

So now we know: most of the time there are twice as many exotic spheres as we used to know!

There were 2 papers that approached this via stable theory.

1966 Brown-Peterson

\[ \Phi(M^{8k+2}) = 0 \quad k > 1 \]

used spin structures and techniques of K-theory.

Design a cohomology theory which produces an arithmetic sequence... then eliminate it! We used the same technique.

1969 Browder ... very deep

\[ \Phi(M^n) = 0 \text{ unless } n = 2^{j+1} - 2 \]
\[ \exists M^{2^{i+1}-2} \quad \text{with} \quad \Phi(M) \neq 0 \]

\[ \iff \exists \Theta_0 \in \Pi_2^{2^{i+1}-2} S^0 \]

representing an \( h_j^2 \) in the \underline{Adams spectral sequence}.

\[
\begin{align*}
h_j & \leftrightarrow \text{Hopf} \quad \text{m valued 1 class} \\
h_1 & \leftrightarrow S^3 \quad \Theta_1 = S^1 \times S^2 \\
h_2 & \leftrightarrow S^3 \quad \Theta_2 = S^2 \times S^2 \\
h_3 & \leftrightarrow S^1 \\
\end{align*}
\]

By relating it to Adams spectral sequence, this was a game-changer.

André: Is it conceivable you could use some techniques to eliminate 126?

M. Hopkins: We might be able to \underline{eliminate it},

but we couldn't verify its existence.
1968, 1984

\[ Θ_4 \text{ exists} \leftarrow \text{geometric construction (Jones)} \]

\[ Θ_5 \text{ exists}. \]

Very far from geometry, algebraic, you calculate the Adams spectral sequence and eliminate things.

The mindset was that those all exist.

It's a real game changer to say:

"we're looking for some construction which works in dim $2^i-2$ and it's beautiful"

No response.

But if you say,

"we're looking for 6 things,"

it's psychologically different! Lie groups, etc.
$E_6, E_7, E_8$.

Exciting question:

2, 6, 14, 30, 62, 126

\[ D_4, D_5, E_6, E_7, E_8 \]

Hans Duistermaat: said he'd seen those numbers pop up in Painlevé theory and they correspond to $D_4, D_5, E_6, E_7$ and $E_8$!

James (localize at 2)

\[ E \]

\[ \pi_k(S^n) \to \pi_{k+1}(S^{n+1}) \]

$\Phi \rightarrow \bigcirc$
Toda used this sequence to calculate first 14 homotopy groups of spheres. Showed Hopf invariant didn't exist.

Must come to grips with first place the $S^{2n+1}$ term is nonzero.

$\pi_{2n+1} S^{2n+1} \to \pi_{2n-1} S^n$

$1 \to [\ast i, i] \cdots$ Whitehead product.

Write $S^n \times S^n = S^n \vee S^n \cup e^{2n}$.

Contains the tangent bundle of the sphere.

Think as an exact couple. Two questions:

a) Is Whitehead product divisible by 2?

b) For which $j$ is $[i_n, i_n]$ in the image of $E^k$?

This breaks into three separate problems.
Question b) is equivalent to the vector fields on spheres problem. (15) (solved in 60's by Adams using K-theory).

a) When \( n \) is even \( \iff \) Hopf invariant one problem
\( n \) odd \( \iff \) Kervaire problem

Amazing that

So three fundamental problems in homotopy theory have

Consider following space:

\[ V_2(S^n) = \text{space of points } (a,b), a, b \in S^n \]

st: \( a \neq b, a \neq -b \)

\[ \text{homology } \cong \text{orthonormal 2-frames in } (S^{n-1}) \]?

Map from

\[ V_2(S^n) \]

\( (a,b) \mapsto (b,a) \).
Question: Is this map homotopic to the identity?

→ Like asking to divide Whitehead product by two.

eg dim 2:

2-sphere

rotate 180°

average them → midpoint

Can do this every time sphere has complex structure.

so at least $S^2, S^6 = \mathcal{O} G_2 / SU(3)$

has complex structure because

So an exceptional Lie group comes up. I don't know, may be something to this!

Ronicki: Pontryagin and Whitehead both got it independently in 1950. Defined quadratic formula geometrically. See webpage!
Mark's guess now is that 126 exists, possibly.

Something about $\Theta_5^2$.

In Turkey, has banknote with Arf invariant!