Superconnections and index theory

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1. Superconnections
2. Index theorem
3. Sketch some proofs

Definition (Quillen) A superconnection $\nabla$ on a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $\nabla_{\mathcal{V}}$ is an odd derivation on $\Omega^{*}(M, \mathcal{V}).$

(An ordinary connection is a degree 1 derivation on $\Omega^{*}(M, \mathcal{V})$)

Superconnections form an affine space modeled on $\Omega(M, \text{End}\mathcal{V})^{\text{odd}}.$

$\text{End}(\mathcal{V})$ has a grading

\[ \begin{array}{c|c} \text{even} & \text{odd} \\ \hline x & x \\ \hline x & x \end{array} \]

So a superconnection can be written as

\[ \nabla = \omega_0 + \nabla + \omega_2 + \omega_3 \]

where $\omega_i$ in different degrees (?)
Why superconnections? For K-theory.

In K-theory we support,

\[ V \xrightarrow{f} W \]

where \( f \) is nonsingular.

For superconnection, have

\[ \nabla' = (\frac{1}{f} f^*) + \nabla + \cdots \]

so

\[ V. \xrightarrow{f} V' \]

so nicely represents class.

\[ \text{ch}(\nabla') = \text{Tr} \ e^{\nabla'^2} \]

Chern character form.
When $\nabla^2$ has support somewhere, then $\text{Tr} \nabla^2 \equiv 3$

decays exponentially fast off support ... good.

2. Index theory

**Defn.** Let $M$ be smooth, Riemannian, spin. The Dirac operator associated to the data of a complex unitary spin bundle with superconnection

$$\begin{pmatrix} V, \nabla \\ M \end{pmatrix}$$

is defined by:

$$\mathcal{D}(V) : \Gamma(S \otimes V) \overset{\nabla + i\eta}{\longrightarrow} \Omega(M, S \otimes V)$$

by Levi-Civita

Clifford multiplication $\rightarrow \alpha(\cdot) : \mathcal{D}(V) \rightarrow \Gamma(S \otimes V)$

• If $M$ is closed, $\mathcal{D}$ is elliptic, and formally self-adjoint.

So looks like $\mathcal{D}(V) = \left( \begin{pmatrix} \alpha \end{pmatrix} \right)$
Expect a thm

\[ \text{index} \left( \mathcal{D}(\nabla) \right) = \int_M \hat{\text{A}}(S^m) \text{ch}(\nabla) \]

In fact, this is a corollary of the Atiyah-Singer index thm.

Because a superconnection can be homotoped to zero, so

\[ \text{index} \left( \mathcal{D}(\nabla) \right) = \int_M \hat{\text{A}}(S^m) \text{ch}(\nabla) \]

So they don't tell us about topology, but about geometry.

**Local index theorems**

McKean-Singer:

\[ \text{Tr} e^{-t \mathcal{D}(\nabla)^2} = \text{index} \left( \mathcal{D}(\nabla) \right) \]

Heat kernel

\[ e^{-t \mathcal{D}(\nabla)^2} \psi(x) = \int_M P_t(x,y) \psi(y) \, dy \]
$$\text{Tr} \ e^{-t D(x)^2} = \int_M \text{Tr} \ p_t(x,x) \ dv_{x, t}.$$ (5)

**Thm** (Bott, Patodi, Singer, …)

$$\lim_{t \to 0} \text{Tr} p_t(x,x) \ dv_{x, t} = (2\pi t)^{-n/2} \left[ \hat{A} (\Omega^*) \text{ch}(\Omega) \right]$$

Might imagine with a bold noda, it's true but no... a superconnection diverges.

Need to introduce scaling (each form of different degree scales differently)

$$\nabla^s = |s|^{-1/2} \omega_0 + \nabla^+ |s|^{-1/2} \omega_2 + |s| \omega_3 + \cdots$$

Then it does hold (the local index thm) … due to Berezin, using stochastic techniques.

He tried hard for one without stochastic techniques.
Families. A Riemannian map is a triple

\((\pi, g, P)\)

where \(\pi: M \to B\) is a proper submersion of smooth manifolds

\(g\) metric on \(T(M/B)\)

\(P\) is roughly a connection,

\[ P: T(M) \to T(M/B) \]

We ask that fibres be closed + spin.

Now we can make a family of Dirac operators.

Recall top. index then gives you class in K-theory of base.

Bismut did this at level of forms.
$T^*_x(V)$: the fibre of $y \in B$ is

smooth sections $\rightarrow \Gamma_y(\Sigma^n \otimes V)$

Bismut: If we have a connection on the top, you get a superconnection on the bottom.

$\text{tr}, \nabla = \text{tr}, D + \text{tr}, \omega$.

Bismut told us what to do with $D$.

I can tell you what to do with $\omega$!

So this is a main motivation for superconnections

$T^*_x\omega(\xi_1, \ldots, \xi_i) = \mathbb{C}^{n_B}((\xi_1), (\xi_2), \ldots, (\xi_i) \omega_i)$

This is superconnection on base.

Can scale the fibres $T \mapsto T^e$ (multiply metric).

Theorem (Alex)

$$\lim_{t \to 0} T^*_t(V) \chi \frac{\text{tr}}{\dim}$$

$$= (2\pi i)^{-\dim n_B} T^*_x \left[ \hat{A}(\Sigma^n) \chi(V) \right]$$
I was interested in determinant line bundles. You end up with \( \eta \)-invariants.

With superconnections, spectral things not so easy!

(don't commute).

2 choices: Spectral defn \rightarrow no nice geometric theorems

or what I do:

"Geometric defn"

Want to have curvature

= 2-form part, etc.

Look at \( D, \Omega \) inside superconnection, have \( \left( \frac{f^*}{f} \right) \).
Associate to this

\[(\det V, \det W, D)\]

\[\downarrow\]

\[\mathcal{M}\]

the bundle with connection \(\nabla\), whose curvature is

\[\text{curv}(\nabla) = \text{ch} \nabla.\]