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Superconnections and index theory.

Plan: (1) Superconnections. (2) Index theory. (3) Sketch some proofs.

Definition: A superconnection $\nabla$ on $\mathbf{Z}/2$-graded vector bundle $V \to M$ is an odd derivation on $\Omega^*(M, V)$.

Superconnections form an affine space over $\Omega(M, \mathrm{End}(V))^{\mathrm{odd}}$. A reminder: $\mathrm{End}(V)$ have both even and odd part. Hence a superconnection can be written in the form $\omega_0 + \nabla + \omega_2 + \cdots$. In particular if the superconnection is unitary, then $\omega_0 = \begin{pmatrix} 0 & f^* \\ f & 0 \end{pmatrix}$ for some $f\colon V^0 \to V^1$. There is a Chern character form for a superconnection: $\mathrm{ch}(V) = \mathrm{tr}(\exp(\nabla^2))$.

Index theory.

Definition: Let $M$ be a smooth Riemannian and spin manifold. The Dirac operator associated to $(V, \nabla) \to M$ is defined by $D(\nabla)\colon \Gamma(\mathbf{S} \otimes V) \to \Omega(M, \mathbf{S} \otimes V) \to \Gamma(\mathbf{S} \otimes V)$. The first map is induced by the superconnection. It is equal to $\nabla \otimes 1 \oplus 1 \otimes \nabla$. The second map is given by multiplication in the Clifford algebra (map forms into the corresponding Clifford algebra).

The Dirac operator is an elliptic formally self-adjoint operator: $D(\nabla) = \begin{pmatrix} 0 & \mathbf{D}^*(\nabla) \\ \mathbf{D}(\nabla) & 0 \end{pmatrix}$.

Theorem (Atiyah-Singer): $\mathrm{index}(\mathbf{D}(\nabla)) = \int_M \hat{A}(\Omega^M)\mathrm{ch}(\nabla) = \mathrm{index}(\mathbf{D}(\nabla)) = \int_M \hat{A}(\Omega^M)\mathrm{ch}(\nabla)$

Atiyah-Singer theorem says that $\mathrm{tr}(\exp(-tD(\nabla)^2)) = \mathrm{index}(\mathbf{D}(\nabla))$. We have

$$\exp(-tD(\nabla)^2)\psi(x) = \int_M p_t(x, y)\psi(y)dy.$$

Hence $\mathrm{tr}(\exp(-tD(\nabla)^2)) = \int_M \mathrm{tr}\, p_t(x, x)d\omega$. Now

$$\lim_{t\to 0} \mathrm{tr}\, p_t(x, x)d\omega = (2\pi i)^{-n/2}(\hat{A}(\Omega^M)\mathrm{ch}(\nabla)).$$

This is not true for superconnection. Getzler: $\mathrm{tr}(\exp(-tD(\nabla^s)^2)) = \int_M \mathrm{tr}\, p_{t,s}(x, x)d\omega$ and

$$\lim_{t\to 0} \mathrm{tr}\, p_{t,t^{-1}}(x, x)d\omega = (2\pi i)^{-n/2}(\hat{A}(\Omega^M)\mathrm{ch}(\nabla)).$$

Definition: A Riemannian mapis a triple $(\pi, g^{M/B}, P)$, where $\pi\colon M \to B$ is a map (submersion?), $g^{M/B}$ is a metric on $T(M/B)$ and $P\colon T(M) \to T(M/B)$.

Assume that the fibers are closed and spin. Bismut's construction for superconnections follows. We have $\lim_{t\to 0} \mathrm{ch}(\pi^t_! \nabla) = (2\pi i)^{-\dim M/B}\pi_*(\hat{A}(\Omega^{M/B}\mathrm{ch}(\nabla))$.