1. Bulk algebra

2. Module category

3. Outlook - logarithmic conformal field theories (non-simply simple)

Recap:

\[ \text{bicat: } V \text{ a v.o.a, } C = \text{Rep} V, \text{ a MTC} \]

\[ C \text{ a MTC, } \Rightarrow \text{ bicategory } \quad \text{ where objects are special symm } \text{ Frob algebras, } A,B, \]

\[ \text{ morphisms are } A\text{bbimodule}, \text{ defect line} \]

\[ A \in \text{Rep} V \]

\[ \text{Frob objects occurring elsewhere :} \]

\[ A \]

\[ (U, \phi \in \text{Hom}_A(A \otimes U, A)) \]

\[ \Rightarrow \text{Hom}(U, A) \]

i.e. there's a space of boundary fields / open states

\[ H_{A,A} = A. \text{ i.e. } A \text{ represents } U \mapsto \text{Hom}(U,A) \]
1. Bulk algebra.

2. Module category.

3. Outlook - logarithmic conformal field theories (non-semisimple).

Recaf:

Data: \( V \) a v.o.a, \( C = \text{Rep} V \), a MTC

\( C \) a MTC, bicategory - whose objects are special symm Frob algebras, \( A, B \), morphisms are Afdimodules, defect lines.

\( A \in \text{Rep} V \), Frobenius algebras occurring elsewhere.

\[
\begin{array}{c}
\sum A
\
(u, \phi \in \text{Hom}_A(A \otimes U, A))
\
\Rightarrow \text{Hom}(U, A)
\
i.e. \text{there's a space of boundary fields/open states,}
\
H_{A,A} = A. \text{ i.e. } A \text{ represents } U \mapsto \text{Hom}(U, A)
\end{array}
\]
\[ \psi \in \text{Hom}_A(M \otimes U, N) \]

\[ H_{MN} = M^o \otimes_A N. \]

What happens with:

\[ A \]

\[ \text{becomes} \]

\[ \text{two marked points} \]

\[ \xrightarrow{\text{inverse braiding}} \]

\[ \text{inverse twist} \]

\[ A \otimes C \rightarrow C \]

\[ \varepsilon_{R \otimes V} \leftarrow \varepsilon_{R \otimes V \boxtimes R \otimes V} \]

expect:

1) \[ C \in \mathcal{C} \otimes \mathcal{C} \]

2) \[ C \text{ a commutative Frobenius algebra} \]
Functor:

\[ R : C \rightarrow C_+ \otimes C_- \]

think of it as the adjoint of

\[ C_+ \otimes C_- \rightarrow C \]

\[ U \times V \rightarrow U \otimes V. \]

Properties:

if \( A \) is a ssFA in \( C \), then \( R(A) \) is a ssFA special sym Froh. algebra in \( C_+ \otimes C_- \).

We need to take its center to get a commutative ssFA.

\[ \text{Defn. The } \wedge \text{centre of an alg. } A \text{ in } C \text{ is the maximal subobject } C_L \text{ of } \]

\[ C \text{ of } A \text{ such that} \]

\[ \text{we made a choice of braiding!} \]

Also have right centre \( C_R \) of \( A \). Need not have \( C_L \cong C_R \), or even Morita-equivalent.
Def. The full centre $\mathbb{Z}(A)$ of a ssFA $A$ in $C$ is $C_e(R(A)) \oplus C_+$ \text{ with } C_+ \otimes C_-.

Need to take left centre for later. This defn more symmetric.

**Properties of $\mathbb{Z}(A)$**

- $\mathbb{Z}(A)$ is commutative ssFA in $C_+ \otimes C_-$.
- $C_e(A) \times 11$ and $11 \times C_R(A)$ are subalgebras of $\mathbb{Z}(A)$.
- $\mathbb{Z}(A) = \bigoplus_{ij \in I} (U_i^* \times U_j^*) \oplus \mathbb{Z}_{ij}(A)$ the m"uller invariant matrix.

**Thm.** The number of iso-clases of simple $A$-left modules = $\text{tr}[Z_{ij}]$.

\[ Z_{ij} = \dim \text{Hom}_{A/A} (U_i \otimes^+ A \otimes^+ U_j, A) \]

this links to Christopher talk.
$Z(A)$ defines a closed CFT

**correlation function**: multilinear map

$$Z(A) \otimes \cdots \otimes Z(A) \rightarrow \mathbb{C}.$$

What have we done

**boundary** $(A)$, constructed closed CFT $Z(A)$

from unlabelled $M$, $M^V \otimes_A M$.

Now $M$ and $A$ are Manta-equivalent.

Would like to verify $Z(A)$ independent of Manta.

**Thm (Kong, Runkel 07)** $\mathcal{C}$ a mod. tens. cat.,

$A, B$ simple ssFaa. Then

$$A \sim_{\text{meq.}} B \Rightarrow Z(A) \cong Z(B) \text{ as algebras}$$

(not necessarily as Frob algebras)
eg, eg for ssFA in Vec, we have

\[ Z(A) \cong \text{End}(\text{id}_{A-\text{mod}}) \]

\[ \therefore \text{ Morita equivariant.} \]

The converse holds too: \( Z(A) \cong Z(B) \implies A \cong B \).

Thm (Kong, RO8) \( C \) a mod. tors. cat., \( C \) a commutative simple ssFA in \( C_+ \otimes C_- \) st. \( \dim C = \text{Dim } \mathfrak{C} \) (mod. inv.),

then:

i) exists ssFA \( A \in C \) st. \( C \cong Z(A) \).

ii) \( T(C) = \bigoplus A \mathfrak{c} \),

\[ \uparrow \]

\( A \) a simple ssFA,

\( \# U \mapsto U \mathfrak{c} \)

all Morita equivalent.

Any one can be used, i.e.

\[ C \cong Z(\text{all summands in } T(C)) \]

Every modular invariant CFT with left/right chiral sym given by \( \mathfrak{V} \) is part of an open/closed CFT.
2. Module categories

A ring, a right module is $M \times R \rightarrow M$

category.

**Defn**  C, M be abelian, C-linear categories. C a tensor cat.

M is a right module category over C if:

- bilinear $\otimes : M \times C \rightarrow M$
- st. associative, $1 \in C$ acts as unit.
- up to coherent iso. (mixed pentagon, triangle)

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**Examples:**

* $M = C$, $\otimes = \otimes$.

* A an algebra in C, A-mod is a right-module cat over C.

$$A \otimes U = A \otimes_U$$
in C.

**Thm (Osmik 01)** C a mtc (don't need braiding), M semisimple, indecomposable ($\neq M, \oplus M_2$). Then $M \sim A$-mod for A an algebra in C.
Get $A$ via

\[ \text{Internal homs:} \]

$M$ a module cat over $C$,

$M, M, N \in M$,

$\text{Hom}_M(M, N)$ is object in $C$ representing the functor

\[ U \longrightarrow \text{Hom}_M(M \otimes U, N). \]

* associative composition

\[ \text{Hom}_M(M, N) \otimes \text{Hom}_M(K, M) \longrightarrow \text{Hom}_M(K, N) \]

in part $\text{End}_M(M)$ is an algebra in $C$.

This is how you get $A$.

In fact,

\[ \text{Fun}(M, N) \supseteq \text{Nat} \]

\[ \uparrow \]

demand $F(M) \otimes U \Rightarrow F(M \otimes U)$

\[ \text{Fun}(A\text{-mod}, B\text{-mod}) \Rightarrow B\text{-}A\text{ mod as } \otimes\text{-cat.} \]
objects are

- "good" module cat over $\mathcal{C}$, i.e. equiv to $A\text{-mod}$ for some $ssFa A$ in $\mathcal{C}$.

- morphisms are $\text{Fun}(M, N)$.

- space of boundary fields $M, N \in M = \text{collection of all boundary conditions of the given CFT compatible with } \mathcal{V}.$

$\text{Urs} \text{ want to think of, ala Ben-Zvi,}$

$$C\text{-Mod} \to \text{Mod}_C$$

$\uparrow$

objects are $C$-enriched categories, profunctors between them, ...

"C a mtc, $M$ a "good" m"
\[ \alpha^\pm : C \to \text{End}(M) \]

\[ \alpha^\pm(U) : (M \to M \otimes U) \]

The \( \pm \) refers to equipping it as an endofunctor of \( M \),

(need braiding).

\[ \text{End}(M) \times C \otimes C \to \text{End}(M) \]

\[ F \times (U \otimes V) \to \alpha^+(U) \cdot F \cdot \alpha^-(V) \]

**Statement:** \( Z_M = \text{End}(\text{id}_M) \)

**Theorem:** \( Z_{A\text{-mod}} \cong Z(A) \).

So you can recover