

3-Morphisms in the Degree 2 Degeneracy Maps of Street Nerves

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1 Preliminaries

First recall that the components of the left, middle, and right 2-unitors of C are invertible 3-morphisms of the form

$$\begin{array}{ccc}
 (\text{id}_C \otimes g) \otimes f & \xrightarrow{\lambda_g^C \otimes \text{id}_f} & g \otimes f \\
 \alpha_{\text{id}_C, g, f}^C \searrow & \Downarrow \lambda_{g, f}^C & \nearrow \lambda_{g \otimes f}^C \\
 & \text{id}_C \otimes (g \otimes f); &
 \end{array}
 \quad
 \begin{array}{ccc}
 (g \otimes \text{id}_B) \otimes f & \xrightarrow{\alpha_{g, \text{id}_B, f}^C} & g \otimes (\text{id}_B \otimes f) \\
 (\rho^C)_g^* \otimes \text{id}_f \uparrow & & \downarrow \lambda_{g, f}^C \\
 g \otimes f & \xrightarrow{\text{id}_{g \otimes f}} & g \otimes f; \\
 & & \text{id}_g \otimes \lambda_f^C
 \end{array}$$

$$\begin{array}{ccc}
 g \otimes f & \xrightarrow{\text{id}_g \otimes (\rho_f^C)^*} & g \otimes (f \otimes \text{id}_A) \\
 (\rho_{g \otimes f}^C)^* \searrow & \Downarrow \rho_{g, f}^C & \nearrow \alpha_{g, f, \text{id}_A}^C \\
 & (g \otimes f) \otimes \text{id}_A &
 \end{array}$$

2 Extra 2-Unitors

The left, middle, and right 2-unitors of C induce the following ‘extra 2-unitors’:

$$\begin{array}{ccc}
 (\text{id}_C \otimes g) \otimes f & \xrightarrow{\alpha_{\text{id}_C, g, f}^C} & \text{id}_C \otimes (g \otimes f) \\
 \downarrow \lambda_g^C \otimes \text{id}_f & \nearrow (\mathbb{X}_1^C)_{g,f} & \downarrow \text{id}_{\text{id}_C \otimes (g \otimes f)} \\
 g \otimes f & \xrightarrow{(\lambda_{g \otimes f}^C)^*} & \text{id}_C \otimes (g \otimes f)
 \end{array}$$

$$\begin{array}{ccc}
 g \otimes f & \xleftarrow{\text{id}_g \otimes \rho_f^C} & g \otimes (f \otimes \text{id}_A) & (g \otimes \text{id}_B) \otimes f & \xrightarrow{\alpha_{g, \text{id}_B, f}^C} & g \otimes (\text{id}_B \otimes f) \\
 \uparrow \rho_{g \otimes f}^C & \nearrow (\mathbb{P}_1^C)_{g,f} & \nearrow \alpha_{g,f, \text{id}_A}^C & \downarrow \rho_g^C \otimes \text{id}_f & \nearrow (\mathbb{P}_3^C)_{g,f} & \downarrow \text{id}_g \otimes \lambda_f^C \\
 (g \otimes f) \otimes \text{id}_A & & & g \otimes f & & g \otimes f.
 \end{array}$$

These are all pastings. Here are their constructions:

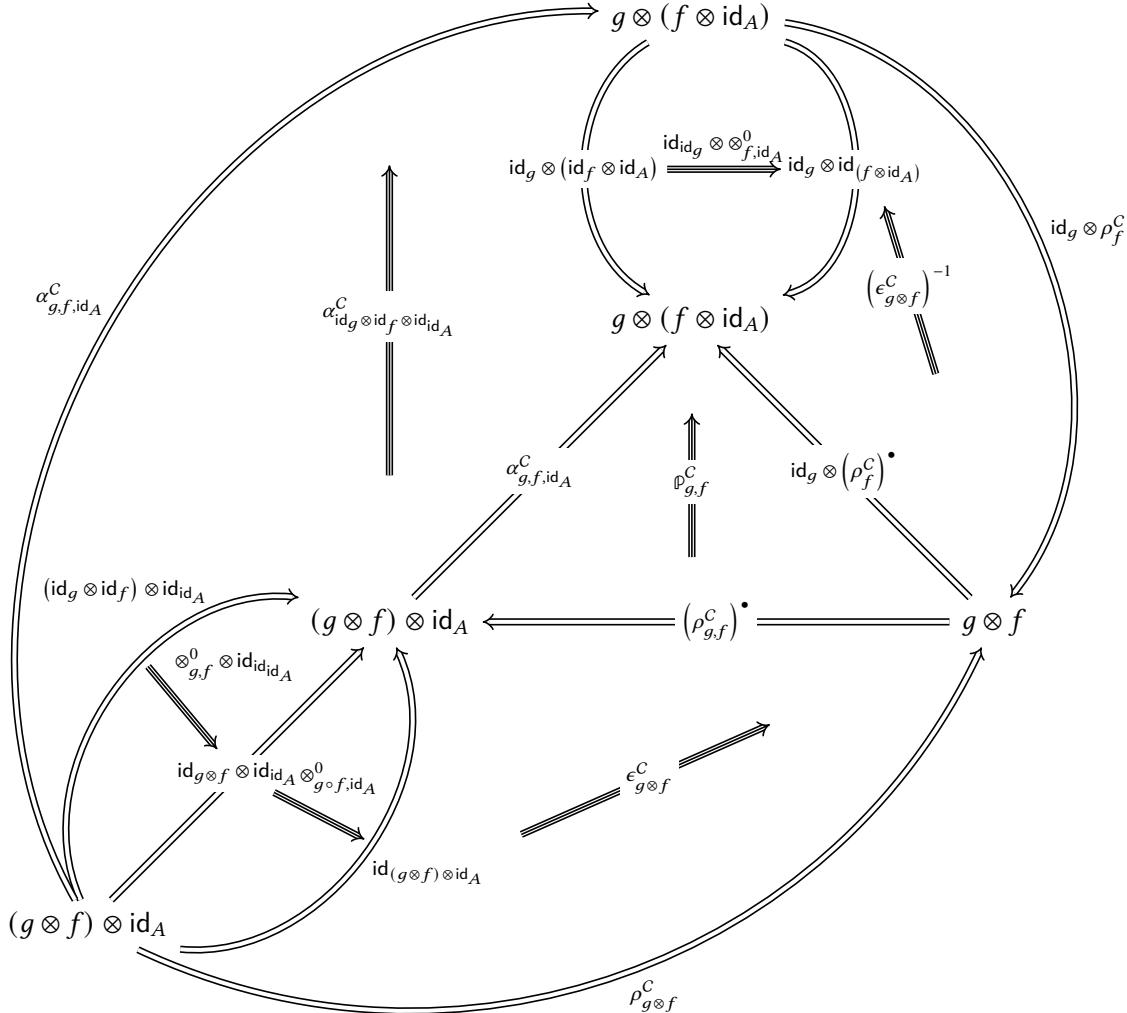
CONSTRUCTING \mathbb{X}_1^C

The component at f of \mathbb{X}_1^C is given by the pasting of the following diagram:

$$\begin{array}{ccc}
 (\text{id}_C \otimes g) \otimes f & \xrightarrow{\alpha_{\text{id}_C, g, f}^C} & \text{id}_C \otimes (g \otimes f) \\
 \downarrow \lambda_g^C \otimes \text{id}_f & \nearrow \lambda_{g,f}^C & \downarrow \text{id}_{\text{id}_C \otimes (g \otimes f)} \\
 g \otimes f & \xrightarrow{(\eta_{g \circ f}^C)^{-1}} & \text{id}_C \otimes (g \otimes f).
 \end{array}$$

CONSTRUCTING \mathbb{P}_1^C

The component at f of \wp_1^C is given by the pasting of the following diagram:

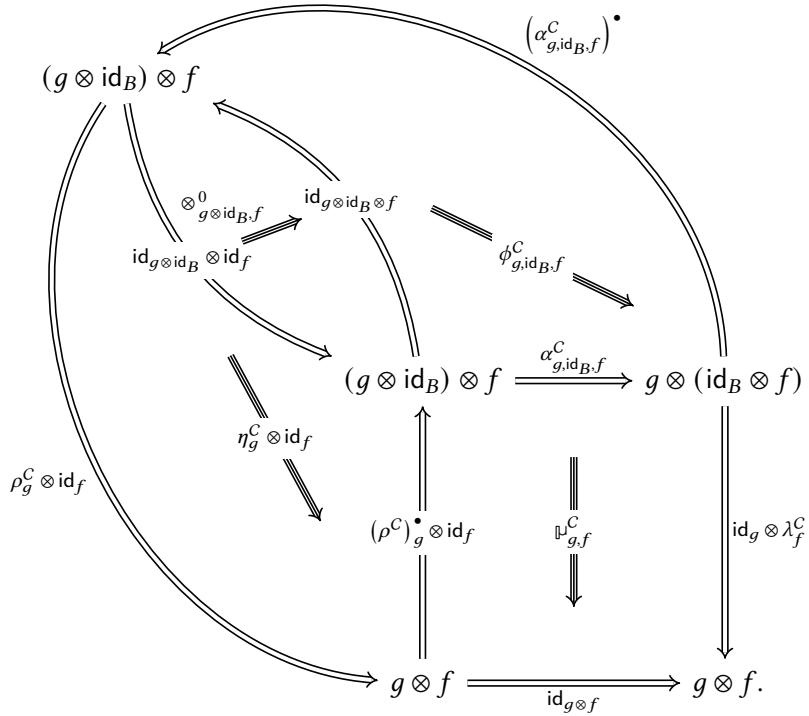


CONSTRUCTING \wp_3^C

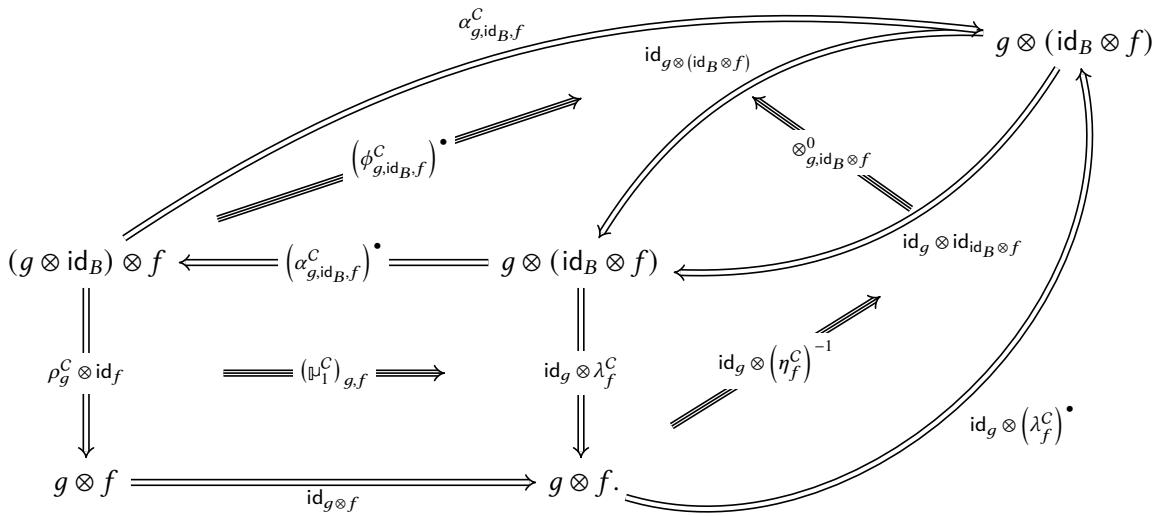
To construct \wp_3^C , we first define invertible 3-morphisms \wp_1^C and \wp_2^C as in the diagrams

$$\begin{array}{ccc}
 (g \otimes \text{id}_B) \otimes f & \xleftarrow{\quad (\alpha_{g,\text{id}_B,f}^C)^* \quad} & g \otimes (\text{id}_B \otimes f) & (g \otimes \text{id}_B) \otimes f & \xrightarrow{\quad \alpha_{g,\text{id}_B,f}^C \quad} & g \otimes (\text{id}_B \otimes f) \\
 \downarrow \rho_g^C \otimes \text{id}_f & \Downarrow (\wp_1^C)_{g,f} & \downarrow \text{id}_g \otimes \lambda_f^C & \downarrow \rho_g^C \otimes \text{id}_f & \uparrow (\wp_2^C)_{g,f} & \uparrow \text{id}_g \otimes (\lambda_f^C)^* \\
 g \otimes f & \xrightarrow{\quad \text{id}_{g \otimes f} \quad} & g \otimes f & g \otimes f & \xrightarrow{\quad \text{id}_{g \otimes f} \quad} & g \otimes f.
 \end{array}$$

1. μ_1^C : The component at f of μ_1^C is given by the following pasting diagram:



2. μ_2^C : The component at f of μ_2^C is given by the following pasting diagram:



3. μ_3^C : The component at f of μ_3^C is given by the pasting of the diagram

$$\begin{array}{ccc}
 (g \otimes \text{id}_B) \otimes f & \xrightarrow{\alpha_{g,\text{id}_B,f}^C} & g \otimes (\text{id}_B \otimes f) \\
 \rho_g^C \otimes \text{id}_f \Downarrow & \uparrow (\mu_2^C)_{g,f} & \uparrow \text{id}_g \otimes (\lambda_f^C)^\bullet \\
 g \otimes f & \xrightarrow{\text{id}_{g \otimes f}} & g \otimes f \\
 & \searrow \text{id}_{g \otimes f} & \swarrow \text{id}_{g \otimes f} \\
 & \text{id}_{g \otimes f} &
 \end{array}$$

in $\mathbf{Hom}_C[A, C]$, where $\lambda(g \circ f) : (\text{id}_{g \otimes f} \circ \text{id}_{g \otimes f}) \Rightarrow \text{id}_{g \otimes f}$ is the invertible 3-morphism defined as the composition

$$\begin{array}{c}
 \text{id}_{g \otimes f} \circ \text{id}_{g \otimes f} = \boxed{\quad} \\
 \downarrow \\
 (\otimes_{g,f}^0)^{-1} * (\otimes_{g,f}^0)^{-1} \\
 \downarrow \\
 (\text{id}_g \otimes \text{id}_f) \circ (\text{id}_g \otimes \text{id}_f) \\
 \downarrow \\
 \otimes_{(\text{id}_g \otimes \text{id}_f), (\text{id}_g \otimes \text{id}_f)}^2 \\
 \downarrow \\
 (\text{id}_g \circ \text{id}_g) \otimes (\text{id}_f \circ \text{id}_f) \\
 \downarrow \\
 \lambda^{\mathbf{Hom}_C(B,C)} \otimes \lambda^{\mathbf{Hom}_C(A,B)} \\
 \downarrow \\
 \text{id}_g \otimes \text{id}_f \\
 \downarrow \\
 \otimes_{g,f}^0 \\
 \downarrow \\
 \text{id}_{g \otimes f} \leftarrow \boxed{\quad}
 \end{array}$$

in $\mathbf{Hom}_C(B, C) \times \mathbf{Hom}_C(A, B)$.

3 The 3-Morphisms in the Degree 2 Degeneracy Maps of the Street Nerve

3.1 $s_0^2(\sigma)$

$$s_0^2(\sigma) = \begin{array}{ccc} & \text{Diagram showing two configurations of a commutative triangle with vertices } A, B, C. \\ & \text{Left configuration: } A \xrightarrow{\text{id}_A} A \xrightarrow{i} B, A \xrightarrow{k} C, A \xrightarrow{j} C. \\ & \text{Right configuration: } A \xrightarrow{\text{id}_A} A \xrightarrow{i} B, A \xrightarrow{j} C, A \xrightarrow{k} C. \\ & \text{A horizontal arrow between them is labeled: } ((\rho_1^C)_{i,j} \otimes \text{id}_\theta) \otimes (\rho_\theta^C)^{-1} \\ & \text{Below the diagram: where } ((\rho_1^C)_{i,j} \otimes \text{id}_\theta) \otimes (\rho_\theta^C)^{-1} \text{ is the pasting of the diagram} \end{array}$$

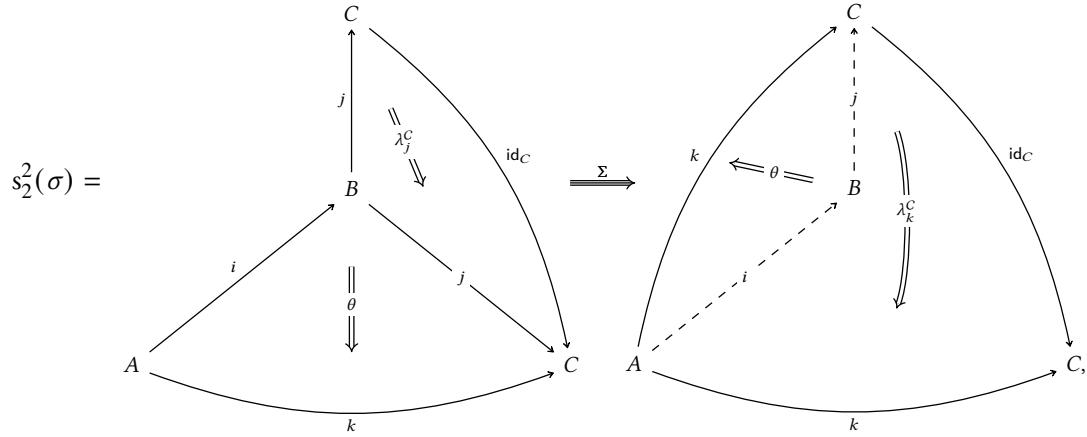
$$\begin{array}{ccc} (j \otimes i) \otimes \text{id}_A & \xrightarrow{\alpha_{j,i,\text{id}_A}^C} & j \otimes (i \otimes \text{id}_A) \\ \theta \otimes \text{id}_{\text{id}_A} \quad \quad \quad & \swarrow \quad \searrow & \quad \quad \quad (\rho_1^C)_{j,i} \\ k \otimes \text{id}_A & \xrightarrow{(\rho_\theta^C)^{-1}} & j \otimes i \\ \rho_k^C \quad \quad \quad & \quad \quad \quad \theta & \quad \quad \quad \end{array}$$

3.2 $s_1^2(\sigma)$

$$s_1^2(\sigma) = \begin{array}{c} \text{Diagram showing three nodes } A, B, C. \\ \text{Arrows: } A \xrightarrow{i} B, A \xrightarrow{k} C, B \xrightarrow{j} C. \\ \text{Isomorphisms: } \theta \text{ (horizontal double bar), } \rho_j^C \text{ (curved double bar).} \\ \text{Equations: } \text{id}_B \text{ (vertical arrow from } B \text{ to } B\text{), } \lambda_i^C \text{ (dashed curved arrow from } A \text{ to } C\text{).} \\ \text{Equation labels: } (\text{id}_\theta \otimes (\mu_3^C)_{j,i}) \otimes ((\rho_\theta^{\text{Hom}_C(A,C)})^{-1} \otimes \text{id}_{\rho_j^C \otimes \text{id}_i}), \theta \end{array}$$

where $(\text{id}_\theta \otimes (\mu_3^C)_{j,i}) \otimes ((\rho_\theta^{\text{Hom}_C(A,C)})^{-1} \otimes \text{id}_{\rho_j^C \otimes \text{id}_i})$ is the pasting of the diagram

$$\begin{array}{ccc} (j \otimes \text{id}_B) \otimes i & \xrightarrow{\alpha_{j,\text{id}_B,i}^C} & j \otimes (\text{id}_B \otimes i) \\ \rho_j^C \otimes \text{id}_i & \swarrow & \downarrow \text{id}_j \otimes \lambda_i^C \\ j \otimes i & \xrightarrow{\text{id}_{j \otimes i}} & j \otimes i \\ \theta & \searrow & \swarrow (\rho_\theta^{\text{Hom}_C(A,C)})^{-1} \\ & k & \end{array}$$

3.3 $s_2^2(\sigma)$ 

where Σ is the pasting of the diagram

