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ALGEBRAIC STRUCTURES ASSOCIATED WITH SMOOTH MANIFOLDS

A Thesis

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ABSTRACT

The notions of abstract multi-derivatives, annihilators and imitators in rings are introduced.

Rings weakly associated, associated, or compact associated with T_1 spaces are defined. It is shown that a ring which is compact associated with a regular, locally compact T_1 space determines the space and that a ring which is associated with a regular compact T_1 space determines the space.

The ring of infinitely differentiable functions on a smooth or infinitely differentiable manifold is defined. This ring of functions is associated with the space of the manifold while the subring of infinitely differentiable functions with compact supports is compact associated with the space. In the last chapter the space of the manifold is constructed from the ring of all infinitely differentiable functions without requiring compactness.

Covariant and contravariant tensor fields on the manifold are defined in a purely algebraic way by using the ring of infinitely differentiable functions. Rings of covariant poly-tensors and contravariant poly-tensors are constructed by using the conventional notions of sum and outer product of tensors. If the manifold is compact, it is shown that these rings are associated with the space of the manifold and hence determine it. The rings of poly-tensors with compact supports also determine the space of the manifold even if it is not compact.

The Lie Ring of tangent vectors on an infinitely differentiable manifold and the Grassman rings are defined.

ALGEBRAIC STRUCTURES ASSOCIATED WITH SMOOTH MANIFOLDS

INTRODUCTION

In classical Riemannian geometry one generally studies the local properties or differential invariants of a differentiable manifold. Much of modern differential geometry, however, is concerned with relations between classical differential invariants and global properties of a manifold.

This thesis is an initial step in the study of those global properties of differentiable manifolds which may be obtained from the algebraic structure of the ring of differentiable functions on the manifold. Most of this work is concerned with infinitely differentiable manifolds; since, as is shown, operators can be defined on the ring of infinitely differentiable functions on these manifolds in a purely algebraic manner which determine contravariant and covariant tensor fields.

Rings are constructed from these operators, and it is shown that their algebraic structure determines the topology of the manifold. In showing that these rings of tensors determine the space, use is made of only a limited number of the relationships which exist between the ring and the space. In one case if these relationships exist between a ring and a topological space, then the ring is "associated" with the space. In another case the ring is "compact associated" with the space. If a ring is associated (compact associated) with a regular compact (locally compact) Hausdorf space, then its algebraic structure determines the topology of the space.

There are other cases of the existence of these relationships between rings and spaces. For example, the ring of all continuous functions on a normal Hausdorf space is associated with this same space; the ring of all infinitely differentiable functions on an infinitely differentiable manifold is associated with the space of the manifold and, hence, determines its topology if it is compact (as is known); the ring of all infinitely differentiable functions with compact supports on an infinitely differentiable manifold is compact associated with the manifold and determines its topology (as is also known).

By making use of some of the properties of associated rings the space of an infinitely differentiable manifold is constructed from the ring of all infinitely differentiable functions without requiring compactness.

This thesis is only the beginning of a possible study of differential geometry in the large. It is shown that certain rings constructed on a manifold determine its topology, but no results are obtained which relate particular algebraic invariants of these rings with corresponding spaces. The problem of constructing particular topological invariants, such as the homology groups, from the ring of infinitely differentiable functions or other rings associated or compact associated with an infinitely differentiable manifold will be the subject of later investigations.

PART I

RINGS, DERIVATIVES, AND ASSOCIATED SPACES

Chapter 1. Zero Commutative Rings and Abstract Multi-Derivatives

In this chapter we will discuss certain properties of rings and some of their operators which we will find useful in later chapters. In this paper a ring is assumed associative unless it is stated otherwise.

(1.1) Definition: An "abstract derivative" d on a ring R is an operator $d:R\to R$ such that:

(i) $d(r+s)=d(r)+d(s)$

(ii) $d(rs)=rd(s)+d(r)s$

for all r, s in R. An "abstract multi-derivative of degree n on R", $1\leq n$, is an operator $d:R^n\to R$ such that for any $i=1,\ldots, n$ and for any given subcollection $r_1,\ldots, \hat{r}_i,\ldots, r_n$ (i.e. not including r_i) the operator $d(r_1,\ldots, r_{i-1}, r, r_{i+1},\ldots, r_n):R\to R$ is an abstract derivative.

Hereafter in this chapter we will write $d(r_1,\ldots,r_{i-1},t,r_{i+1},\ldots,r_n)$, where t is an element or a subset of R, as $d(\ldots,t,\ldots)$ when we are discussing properties of d relating to an arbitrary fixed i.

(1.2) Lemma: If d is an abstract multi-derivative of degree n on a ring R, then for all i and all $r_1,\ldots,\hat{r}_i,\ldots,r_n$:

(i) $d(\ldots,0,\ldots)=0$.

(ii) $d(\ldots,-r,\ldots)=-d(\ldots, r,\ldots)$.

(iii) If R has an identity e and if m is any integer, then

$$d(\ldots,me,\ldots)=0.$$

The proofs follow directly from the definitions.

Since for every n, the zero operator $0:R^n \to 0$, 0 in R, is an abstract multi-derivative of degree n we define:

(1.3) <u>The zero operator is an "abstract multi-derivative of degree infinity"</u>.

(1.4) <u>Theorem</u>: If r is an element of the center of a ring R, i.e. r commutes with every element of R, and if d is an abstract multi-derivative of degree n on R, then for any i $d(\ldots,r,\ldots)$ is in the center of R for every $r_1,\ldots,\hat{r}_i,\ldots,r_n$.

<u>**Proof**</u>: Since the terms $r_1,\ldots,\hat{r}_i,\ldots,r_n$ serve only as parameters, it is sufficient to give the proof for $n=1$.

Let s be any element of R, then

$$\begin{aligned} d(rs) &= rd(s)+d(r)s = d(s)r+d(r)s \\ &= d(sr) \qquad\quad = d(s)r+sd(r). \end{aligned}$$

Hence

$$d(r)s = sd(r)$$

for any s in R.

q.e.d.

(1.5) <u>Definition</u>: A ring R is "differentiable" if there is an abstract derivative on R which is not the zero operator.

If R has a multi-derivative d of degree n which is not the zero operator, then by keeping n-1 of the arguments fixed d determines a non-zero abstract derivative.

There are rings which are not differentiable. It is easily shown that the ring of integers and the ring of rational numbers are not differentiable.

We can represent any continuous real valued function f as

$$f = h_a^2 - g_a^2 + a$$

where a is any constant in the domain of f and h_a and g_a are real valued continuous functions which vanish at all points p such that $f(p) = a$. Using this representation one can show that the ring of all real valued continuous functions on a Euclidean space is not a differentiable ring.

The set of all continuous functions on the real line with continuous derivatives of all orders is an example of a differentiable ring, since the ordinary derivative in this case is also an abstract derivative.

Whether or not the set of all real numbers is differentiable or not is an open question.

(1.6) Definition: A ring R is "a zero commutative ring" if for any r and s in R, $rs = 0$ if and only if $sr = 0$. A ring R is said to "have no square roots of zero" if for r in R, $rr = 0$ if and only if $r = 0$.

If R is a zero commutative ring and if r is an element of R, then by the "annihilator of r", written A(r), we mean the set of all s in R such that $rs = 0$, i.e. $sr = 0$. If S is a subset of R, we define

$$A(S) = \bigcap_{s \in S} A(s).$$

(1.7) Theorem: If r is an element of a zero commutative ring R, then A(r) is an ideal in R. If S is a subset of R, A(S) is an ideal in R.

Proof: Let s be any element of $A(r)$ and t any element of R, then

$$r(st)=(rs)t=0$$

and

$$0=t(sr)=(ts)r=r(ts).$$

Hence (st) and (ts) are in $A(r)$. If t and s are in $A(r)$, then $r(t-s)=0$ and $t-s$ is in $A(r)$. Hence $A(r)$ is an ideal.

Since $A(S)$ is an intersection of ideals, it is an ideal.

q.e.d.

(1.8) **Theorem:** If r is an element of a zero commutative ring R without square roots of zero, then $A(r)=R$ if and only if $r=0$.

Proof: If $r=0$, then obviously $A(r)=R$. If $A(r)=R$, then r is in $A(r)$ and $rr=0$. Since R has no square roots of zero, then r is zero.

q.e.d.

(1.9) **Theorem:** Let d be a multi-derivative of degree n on a zero commutative ring. If s is in $A(r_i)$ for any r_i, then ss is in $A(d(r_1,\ldots,r_n))$ for any $r_1,\ldots,\hat{r}_i,\ldots,r_n$.

Proof: If s is in $A(r_i)$, then

$$d(\ldots,r_i s,\ldots)=d(\ldots,0,\ldots)=0=r_i\, d(\ldots,s,\ldots)+d(r_1,\ldots,r_n)s.$$

Since $sr_i=0$, then after substituting in the above equation for zero we have:

$$0=s\cdot 0=(s)(d(r_1,\ldots,r_n)s)=(d(r_1,\ldots,r_n)s)(s)=d(r_1,\ldots,r_n)(ss),$$

and ss is in $A(d(r_1,\ldots,r_n))$.

q.e.d.

(1.10) <u>Theorem</u>: Let d be a multi-derivative of degree n on a zero commutative ring R without square roots of zero, then:

(i) For any r and s in R, $ssr=0$ only if $sr=0$ and $rss=0$ only if $rs=0$.

(ii) If s is in $A(r_i)$, then s is in $A(d(r_1,\ldots,r_n))$ for any $r_1,\ldots,\hat{r_i},\ldots,r_n$.

(iii) If S is a subset of R, then A(S) is contained in $A(d(\ldots,S,\ldots))$.

<u>Proof</u>: (i) If $ssr=0$, then

$$0=s(sr)=(sr)s$$

and

$$0=0\cdot r=(sr)(sr).$$

Since R has no square roots of zero, then $sr=0$. In the same way $rss=0$ implies $rs=0$.

(ii) If s is in $A(r_i)$, then by (1.9),

$$d(r_1,\ldots,r_n)ss=0.$$

By (i) above $d(r_1,\ldots,r_n)s=0$ and s is in $A(d(r_1,\ldots,r_n))$.

(iii) Proposition (iii) is a direct consequence of (ii).

q.e.d.

(1.11) <u>Definition</u>: If r is any element of a zero commutative ring R, then by I(r), "the imitator of r", we mean the set of all s in R whose annihilators contain the annihilator of r.

(1.12) Theorem: If r is any element of a zero commutative ring R, then I(r) is an ideal in R. If R has no square roots of zero, then I(r) is the zero ideal, (0), if and only if $r=0$.

Proof: (a) If t is in R and s is in I(r), then A(s) contains A(r) and $ws=sw=0$ for all w in A(r). Hence

$$0=(ws)t=w(st)=t(ws)=t(sw)=(ts)w=w(ts)$$

for all w in A(r). Therefore A(ts) and A(st) contain A(r), i.e. ts and st are in I(r).

If t and s are in I(r), then $wt=tw=0$ and $sw=ws=0$ for all w in A(r). Hence $w(t-s)=(t-s)w=0$ for all w in A(r). Therefore A(t-s) contains A(r) or (t-s) is in I(r). Therefore I(r) is an ideal.

(b) Let R be without square roots of zero. If $r=0$, then by (1.8) $A(r)=R$, but there is no non-zero s such that $A(s)=R$, hence $I(r)=(0)$. If $r\neq 0$, then $A(r)\supseteq A(r)$. Therefore r is in I(r) and I(r) is not a zero ideal.

q.e.d.

(1.13) Theorem: If d is a multi-derivative of degree n on a zero commutative ring R without square roots of zero and if r_i is an element of I(s), then $d(r_1,\ldots,r_n)$ is in I(s) for every $r_1,\ldots,\widehat{r_i},\ldots,r_n$.

Proof: If r_i is in I(s), then $A(r_i)$ contains A(s) and by (1.10, ii), $A(d(r_1,\ldots,r_n))$ contains A(s). Hence $d(r_1,\ldots,r_n)$ is in I(s).

q.e.d.

(1.14) Definition: If d is a multi-derivative of degree n and d' is a multi-derivative of degree n', we define the "product", dd', as an operator d" on $R^{n+n'}$ such that

$$d''(r_1,\ldots,r_{n+n'}) = d(r_1,\ldots,r_n)\cdot d'(r_{n+1},\ldots,r_{n+n'})$$

for all $r_1,\ldots,r_{n+n'}$ in R.

If $n = n'$, we define the "sum", d+d', as an operator d" on $R^n = R^{n'}$ and the "difference", $d - d' = d^*$ as an operator on R^n such that

$$d''(r_1,\ldots,r_n) = d(r_1,\ldots,r_n) + d'(r_1,\ldots,r_n)$$

$$d^*(r_1,\ldots,r_n) = d(r_1,\ldots,r_n) - d'(r_1,\ldots,r_n)$$

for all $r_1,\ldots,r_n$ in R.

We define the "product", rd, for r in R as an operator d" on R^n such that

$$d''(r_1,\ldots,r_n) = rd(r_1,\ldots,r_n)$$

for all $r_1,\ldots,r_n$ in R. We define $dr = rd$ (not to be confused with d(r)).

If d is a multi-derivative and 0 is the zero operator, we define:

$$d \cdot 0 = 0 \cdot d = 0,$$

$$d + 0 = 0 + d = d,$$

and for any r in R

$$r \cdot 0 = 0 \cdot r = 0.$$

(1.15) <u>Theorem</u>: Let d be a multi-derivative of degree n on a ring R, let d' be a multi-derivative of degree n' on R, and let r be an element of R.

(i) If $n = n'$, then $d + d'$ and $d - d'$ are multi-derivatives of degree $n = n'$ on R.

(ii) If R is commutative, then dd' (or $d'd$) is a multi-derivative of degree $n + n'$ on R.

(iii) If r is in the center of R (or if R is commutative), then $rd = dr$ is a multi-derivative of degree n on R.

The proofs follow immediately from the definition of multi-derivatives.

(1.16) <u>Definition</u>: Let R be a commutative differentiable ring. We define any non-zero element r of R as a "multi-derivative of degree zero". The zero element which we identify with the zero operator is a "multi-derivative of degree infinity".

A "poly-derivative" on R is a formal sum $\sum_{n=0}^{\infty} d_n$ where d_n is either a multi-derivative of degree n or the zero operator and almost all of the d_n's are zero operators.

If $\sum_n d_n$ and $\sum_n d'_n$ are two poly-derivatives on R, we define the "sum"

$$(\sum_n d_n) + (\sum_n d'_n) = \sum_n (d_n + d'_n)$$

where $d_n + d'_n$ is defined as in (1.14) and the "product"

$$(\sum_n d_n)(\sum_n d'_n) = (d_0 d'_0) + (d_0 d'_1 + d_1 d'_0) + (d_0 d'_2 + d_2 d'_0 + d_1 d'_1)$$
$$+ (d_0 d'_3 + d_3 d'_0 + d_1 d'_2 + d_2 d'_1) + \dots,$$

where the expressions in the parenthesis are given by (1.14).

(1.17) <u>Theorem</u>: If R is a commutative differentiable ring, then the sum or product of two poly-derivatives is a poly-derivative . The set of all poly-derivatives on R with the operations sum and product defined in (1.16) form a ring, the ring of poly-derivatives on R, which we denote by (R). The proof follows directly from the definitions.

(1.18) <u>Definition</u>: The "lower degree of a poly-derivative" $\sum_n d_n$ is the degree of the d_n having the lowest degree. The "upper degree" is the degree of the non-zero term having the highest degree. By (R, n) we mean the set of all poly-derivatives in R with lower degree greater than or equal to n. Under this definition (R) = (R,0).

(1.19) <u>Theorem</u>: If R is a commutative differentiable ring, then (R, n) is an ideal in (R) and is therefore a subring of (R) for any $n = 0, 1, \ldots$. Also (R, n) is an ideal in (R, m), if $m \leq n$.

(1.20) <u>Definition</u>: If B is any subset of a commutative differentiable ring R, then a multi-derivative d of degree n is said to "vanish on B" if for every $i = 1, \ldots, n$ and for every subcollection $r_1, \ldots, \hat{r}_i, \ldots, r_n$ the operator $d(\ldots, r, \ldots): R \to R$ sends B into the zero of R.

We define (R; B, m) as all poly-derivatives $\sum_n d_n$ from (R, m) such that d_n vanishes on B for $n > 0$.

(1.21) <u>Theorem</u>: If R is a commutative differentiable ring, then (R; B, m) is a subring in (R; k) if $0 \leq k \leq m$.

Chapter 2. Rings Associated with Topological Spaces

(2.1) Definition: A ring R is said to be "weakly associated with a T_1 topological space X" if for every r in R, there is a closed set Z(r) in X such that:

(i) $r=0$ if and only if $Z(r)=X$

(ii) $rs=0$ if and only if $Z(r)\cup Z(s)=X$.

(iii) If the point x is not in a closed set F, then there is an r in R such that F is contained in Z(r) but x is not in Z(r).

In the above definition, if we assume the ring R has no square roots of zero, then axiom (ii) implies axiom (i). Also, the space X is regular (i.e. a closed set F and a point x not in F have disjoint neighborhoods) if and only if there is an r in R such that F is contained in Int Z(r) but x is not in Z(r).

We will later add other axioms. In the following discussion we will study the relationship between annihilators of elements of R and the sets Int Z(r) and Cl Int Z(r).

(2.2) Theorem: If a ring R is weakly associated with a T_1 space X, then R is a zero commutative ring without square roots of zero.

Proof: From (2.1,ii), if $rs=0$, then $Z(r)\cup Z(s)=X$. Hence $Z(s)\cup Z(r)=X$, and $sr=0$. Likewise if $sr=0$, then $rs=0$, hence R is zero commutative.

If $rr=0$, then $X=Z(r)\cup Z(r)=Z(r)$ and $r=0$. Hence R has no square roots of zero.

q.e.d.

(2.3) Theorem: (i) If a ring R is weakly associated with a space consisting of one point, then the ring R has no divisors of zero.

(ii) Any ring without divisors of zero may be weakly associated with a space consisting of only one point.

Proof: (i) If r and s are both different from zero, then both Z(r) and Z(s) are empty. Hence $Z(r) \cup Z(s)$ is empty and rs is not zero.
(ii) If a ring R is without divisors of zero, we let $Z(0) = p$ and let Z(r) be the empty set if r is not zero. It is easily shown that R is weakly associated with the space of the point p.

q.e.d.

Since a ring R, weakly associated with a space X, is zero commutative without square roots of zero, we can make use of the annihilator and the imitator in studying these rings and their weakly associated spaces.

(2.4) Theorem: If r and s are any two elements in a ring R which is weakly associated with a T_1 space X, then:

(i) $\text{Int } Z(r) \subseteq \text{Int } Z(s)$ if and only if $A(r) \subseteq A(s)$

(ii) $\text{Int } Z(r) = \text{Int } Z(s)$ if and only if $A(r) = A(s)$

(iii) $\text{Int } Z(r) = \Lambda$ if and only if $A(r) = (0)$

where Λ is the empty set and (0) is the zero ideal.

Proof: (i) Assume Int Z(s) does not contain Int Z(r), then Z(s) does not contain Int Z(r) and there is a point x' in Int Z(r) which is not in $(X - \text{Int } Z(r)) \cup Z(s)$. By axiom (iii), there is an element t in R such that Z(t) contains $(X - \text{Int } Z(r)) \cup Z(s)$, but x' is not in Z(t). Since

$$Z(t) \cup Z(r) = X,$$

then $tr = rt = 0$ and t is in A(r).

Since x' is not in Z(t) or in Z(s), then

$$Z(t) \cup Z(s) \neq X$$

and $ts \neq 0$ and $st \neq 0$, i.e. t is not in A(s). Therefore A(s) does not contain A(r).

Assume A(s) does not contain A(r), then there is an element t in R such that $rt = tr = 0$ but $st \neq 0$ and $ts \neq 0$. Therefore

$$Z(t) \cup Z(r) = X,$$

but

$$Z(t) \cup Z(s) \neq X.$$

The non-empty open set $X-(Z(t) \cup Z(s))$ is in Int Z(r), but does not intersect Int Z(s). Hence Int Z(s) does not contain Int Z(r).

Therefore

$$\text{Int } Z(r) \subseteq \text{Int } Z(s) \quad \text{if and only if} \quad A(r) \subseteq A(s).$$

(ii) Proposition (ii) is a direct consequence of (i).
(iii) If Int Z(r) contains a point x', then there is an element t in R such that x' is not in Z(t), but Z(t) contains (X-Int Z(r)). Therefore t is not zero, but $Z(r) \cup Z(t) = X$, i.e. $rt = tr = 0$. Hence t is in A(r), and A(r) is not the zero ideal.

If A(r) is not the zero ideal, there is a non-zero s in R such that $rs = sr = 0$, i.e. $Z(s) \cup Z(r) = X$. The open set X-Z(s) is non-empty and is contained in Int Z(r). Therefore Int Z(r) is not empty.

q.e.d.

We observe that in the results of (2.4), the words Int may be replaced by Cl Int.

(2.5) Theorem: If $r_1,\dots,r_n$ is any finite collection of elements from a ring R which is weakly associated with a T_1 space X, then

$$\bigcap_{i=1}^{n} \operatorname{Int} Z(r_i)=\Lambda \quad \text{if and only if} \quad \bigcap_{i=1}^{n} A(r_i)=(0).$$

Proof: If $\bigcap_{i=1}^{n}\operatorname{Int} Z(r_i)$ is not empty, there is a point x' which is in every Int $Z(r_i)$. There is an s in R such that x' is not in Z(s), but

$$Z(s)\supseteq X-\bigcap_{i=1}^{n}\operatorname{Int} Z(r_i).$$

Then s is not zero, but $Z(s)\cup Z(r_i)=X$, i.e. $sr_i=r_is=0$ for all i. Hence s is in $A(r_i)$ for all i and $\bigcap_{i=1}^{n}A(r_i)$ is not the zero ideal.

If $\bigcap_{i=1}^{n}A(r_i)$ is not the zero ideal, then there is a non-zero element s in R such that $sr_i=r_is=0$ for all i. Then $Z(s)\cup Z(r_i)=X$ for all i, but Z(s) is not X. The open set X-Z(s) is not empty and is contained in every Int $Z(r_i)$. Hence $\bigcap_{i=1}^{n}\operatorname{Int} Z(r_i)$ is not empty.

q.e.d.

(2.6) Definition: If r is an element of a ring R which is weakly associated with a T_1 space X, then we define the closed set

$$S(r)=\operatorname{Cl}(X-Z(r))$$

as the "support" or "closed support of r". The set Int S(r) is the "open support of r".

(2.7) Lemma: If r is any element of a ring R which is weakly associated with a space X, then:

(i) $S(r)=X-\operatorname{Int} Z(r)$

(ii) $\operatorname{Int} S(r)=X-\operatorname{Cl}\operatorname{Int} Z(r)$

(iii) $S(r)=\operatorname{Cl}\operatorname{Int} S(r)$.

The proof readily follows from the fact that for any subset Y of a space X,

$$X - Cl(Y) = Int(X-Y).$$

(2.8) Lemma: If r and s are any two elements of a ring R weakly associated with a T_1 space X, then

(i) $Int\ S(r) \subseteq Int\ S(s)$ if and only if $Int\ Z(r) \supseteq Int\ Z(s)$

(ii) $I(r) \subseteq I(s)$ if and only if $A(r) \supseteq A(s)$.

The proof of this lemma is by a direct application of the definitions of the sets involved. As a direct consequence we have:

(2.9) Theorem: Under the hypothesis of (2.8)

(i) $Int\ S(r) \subseteq Int\ S(s)$ if and only if $I(r) \subseteq I(s)$

(ii) $Int\ S(r) = Int\ S(s)$ if and only if $I(r) = I(s)$.

(2.10) Theorem: If r is any element of a ring R weakly associated with a T_1 space X, then

$$Int\ S(r) = \Lambda \quad \text{if and only if} \quad r = 0.$$

Proof: If $Int\ S(r) = Cl(X - Z(r))$ is empty, then $Z(r) = X$ and $r = 0$. If $r = 0$, then $Z(r) = X = Cl\ Int\ Z(r)$ and $Int\ S(r) = \Lambda$.

q.e.d.

By theorem (1.12) $r = 0$ if and only if $I(r) = (0)$. Hence:

(2.11) Corollary: Under the hypothesis of (2.10)

$$Int\ S(r) = \Lambda \quad \text{if and only if} \quad I(r) = (0).$$

(2.12) Theorem: If $r_1, \ldots, r_n$ is any finite subset of a ring R, weakly associated with a T_1 space X, then

$$\bigcap_{i=1}^{n} Int\ S(r_i) = \Lambda \quad \text{if and only if} \quad \bigcap_{i=1}^{n} I(r_i) = (0).$$

Proof: If $\bigcap_{i=1}^{n} I(r_i)$ is not the zero ideal, then there is a non-zero s in R such that A(s) contains all $A(r_i)$. Hence Cl Int Z(s) contains all Cl Int $Z(r_i)$, but Z(s) is not X. Therefore

$$\bigcap_{i=1}^{n} \text{Int } S(r_i) = \bigcap_{i=1}^{n} (X\text{-Cl Int } Z(r_i)) \supseteq X\text{-Cl Int } Z(s) \neq \Lambda$$

and $\bigcap_{i=1}^{n}$ Int S(r) is not empty.

If the open set $\bigcap_{i=1}^{n}$ Int $S(r_i)$ is not empty, then the set $\bigcup_{i=1}^{n}$ Cl Int $Z(r_i) \neq X$. Let x' be in $\bigcap_{i=1}^{n}$ Int $S(r_i)$, then there is a non-zero s in R such that Z(s) contains Cl Int $Z(r_i)$ for every i but not x'. Hence Int Z(s) contains Int $Z(r_i)$ for every i, and A(s) contains every $A(r_i)$, i.e. s is a non-zero element of $\bigcap_{i=1}^{n} I(r_i)$ and $\bigcap_{i=1}^{n} I(r_i) \neq (0)$.

q.e.d.

(2.13) Definition: Let R be a zero commutative ring without square roots of zero. We say two elements r and r' in R are "disjunct" if for every pair of elements s and s' in R, there is an element t in R such that (t-s) is in A(r) and (t-s') is in A(r').

(2.14) Theorem: Let R be a zero commutative ring without square roots of zero. If two elements r and r' in R are disjunct, then

$$I(r) \cap I(r') = (0).$$

Proof: Let s be any element in $I(r) \cap I(r')$, then A(s) contains A(r) and A(r'). There is an element t in R such that (t-s) is in A(r) and $(t-0) = t$ is in A(r').

Since t is in A(r') and hence in A(s), then

$$st = ts = 0.$$

Since $(t-s)$ is in $A(r)$ and hence in $A(s)$, then

$$0 = s(t-s) = st - ss = -ss.$$

Since R has no square roots of zero, then $s = 0$ and the intersection of $I(r)$ and $I(r')$ is the zero ideal.

q.e.d.

(2.15) Corollary: If r and r' are two disjunct elements of a ring R weakly associated with a T_1 space X, then Int $S(r)$ and Int$S(r')$ are disjoint.

(2.16) Definition: Let R be a ring which is weakly associated with a T_1 space S. Consider the following axioms:

(iv) Two elements r and r' in R are disjunct if and only if the closed sets $S(r)$ and $S(r')$ are disjoint.

(v,a) If a point x of X is not in a closed set F of X, then there is an element r in R such that F is contained in $S(r)$ but x is not.

(v,b) The closed set $S(r)$ is compact for every r in R; and if x in X is not in a closed compact set F, then there is an element s in R such that F is contained in $S(s)$ but x is not.

If axioms (iv) and (v,a) are satisfied, then the ring R is said to be "associated with the space X". If axioms (iv) and (v,b) are satisfied, the ring R is said to be "compact associated with the space X".

We observed in (2.3) that any ring without divisors of zero may be weakly associated with a space consisting of only one point. Since the annihilator of any non-zero element of a ring without divisors of zero is the zero ideal, no two non-zero elements of such a ring are disjunct. Since $A(0) = R$, the zero and any non-zero element are disjunct.

Consequently we have:

(2.17) <u>Theorem:</u> (i) If a ring R is associated or compact associated with a space of one point, then the ring has no divisors of zero. (ii) Any ring without divisors of zero may be associated or compact associated with a space consisting of one point. (iii) No ring without divisors of zero can be weakly associated, compact associated, or associated with a Hausdorf space consisting of more than one point.

(2.18) <u>Theorem:</u> If a ring R is associated with a space X, then every closed set in X can be formed by the intersection of sets of the form S(r). If R is compact associated with X, then every compact closed set can be formed by the intersection of sets of the form S(r).

The proof follows directly from axiom (v,a) and (v, b).

We will now show that if the ring R is associated or compact associated with a space X, then under certain conditions the algebraic structure of R determines the topological structure of X.

(2.19) <u>Definition:</u> If R is a zero commutative ring without square roots of zero, then a subset B of R is said to have the "finite intersection property" if for every finite subcollection $r_1,\ldots,r_n$ of B, the intersection of the ideals $I(r_i)$ is not the zero ideal.

(2.20) <u>Lemma:</u> If a ring R is associated with a regular T_1 space or is compact associated with a locally compact, regular T_1 space, then:

(i) If the point x is not in the closed set F, then there is an r in R such that Int Z(r) contains F but x is not in Z(r).

(ii) If x and x' are distinct points of X, then there is an r' in R such that x' is in Int S(r'), but x is not in S(r').

(iii) If x and x' are distinct points of X, then there are elements r and r' of R such that x is in Int S(r), x' is in Int S(r'), but S(r) and S(r') do not meet, i.e. r and r' are disjunct.

(iv) Let R(x) be all elements r of R such that the point x of X is in Int S(r). If S(r') intersects S(r) for every r in R(x), then x is in S(r').

Proof: (i) If x is not in F and if X is regular, then there is an open set U(x) containing x, such that Cl(U(x)) does not intersect F. There is an element r in R such that Z(r) contains (X-U(x)) but not x. Since F is contained in Int(X-U(x)), then r is the desired element of R.
(ii) The point x is a closed set. The point x' has a neighborhood U whose closure does not contain x. If X is locally compact, U may be selected so that Cl(U) is compact. There is an r' in R such that S(r') contains Cl(U) but not x. Hence x' is in Int S(r'), and r' is the desired element.
(iii) We select r' as in (ii), then x' is in Int S(r'), but x is not in S(r'). By (i) there is an r in R such that x is not in Z(r) but Int Z(r) contains S(r'). Since S(r)=(X-Int Z(r)) and Int S(r)= X-Cl Int Z(r), then x is in Int S(r), but S(r') and S(r) do not intersect.
(iv) Suppose the point x of X is not in S(r'), then by (i) there is an element r in R such that Int Z(r) contains S(r'), but x is not in Z(r). Then S(r)=(X-Int Z(r) does not meet S(r'), but x is in Int S(r)= (X-Cl Int Z(r)) and r is in R(x). Hence S(r') does not meet S(r) for every r in R(x).

q.e.d.

(2.21) Definition: If R and R' are isomorphic zero commutative rings without square roots of zero, we will denote the elements of R by r,s,... and their isomorphic images by r', s',... . We will denote the annihilators in R' by A'(r') and the imitators by I'(r').

(2.22) Lemma: If R and R' are isomorphic zero commutative rings without square roots of zero, then the isomorphic image of I(r) is I'(r') (and conversely), and the isomorphic image of A(r) is A'(r') (and conversely). Also all set-theoretic relationships between these ideals are preserved by the isomorphism. If r and s are disjunct elements of R, then r' and s' are disjunct elements of R'.

(2.23) Lemma: A subset F of a locally compact Hausdorf space X is closed if and only if F intersects every compact subset of X in a compact subset.

The proof has been given by David Gale[2][1].

(2.24) Theorem: If the rings R and R' are compact associated with the regular, locally compact T_1 (hence Hausdorf) spaces X and X', respectively, and if R and R' are isomorphic under an isomorphism $f:R \rightarrow R'$, then the spaces X and X' are homeomorphic.

Proof: As indicated in definition (2.21) we will write $r' = f(r)$, $A'(r') = f(A(r))$, etc.

From (2.22) we observe that for any pair of elements r and s in R, that S(r) meets S(s) in X if and only if S'(r') meets S'(s') in X'.

For any point x in X, let R(x) be the set of all elements r of R

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1. Numbers in brackets refer to the bibliography.

such that x is in Int S(r). The collection of open sets $\{\text{Int } S(r), r \text{ in } R(x)\}$ has the finite intersection property, hence the subset R(x) of R has the finite intersection property. From lemma (2.22) the subset f(R(x)) of R' has the finite intersection property, and $\{\text{Int } S'(r'), r' \text{ in } f(R(x))\}$ has the finite intersection property. Since every S'(r') is compact, then the set,

$$F(x) \equiv \bigcap_{r' \varepsilon f(R(x))} S'(r'),$$

is not empty.

This set F(x) contains exactly one point. If F(x) contained at least two points x' and y', then there would be two elements r' and s' in R' such that Int S'(r') and Int S'(s') would contain x' and y', respectively, but S'(r') and S'(s') would be disjoint. But for every t' in f(R(x)), S'(t') meets S'(r') and S'(s'). Hence for every $t = f^{-1}(t')$ in R(x), S(t) meets S(r) and S(s). By (2.20, iv) the point x is in S(r) and S(s). But we have chosen r' and s', so that S(r) and S(s) do not meet which is a contradiction. Therefore F(x) contains exactly one point.

We will now show that the single valued transformation $F: X \to X'$ is a one-one transformation of X onto X'. We construct a single valued transformation $F': X' \to X$ in the same way that we constructed the transformation $F: X \to X'$.

We will show that $F'F(x) = x$ for all x in X. For any x in X suppose $F(x) = x'$ is in Int S'(r'), then S'(r') meets S'(t') for every t' in f(R(x)). Accordingly S(r) meets S(t) for every t in R(x) and by (2.20, iv) S(r) contains x. Therefore x is in $S(f^{-1}(r'))$ for every r' in R'(x'), i.e.

$$x = F'F(x),$$

for every x in X. Likewise

$$x' = FF'(x')$$

for every x' in X', and F maps X onto X'. Since F is single valued and has a single valued inverse F', then F and F' are one-one transformations.

We will now show that the transformation F sends every compact set onto a compact set. If F(x) is in S'(r') where r' is any element of R', then S'(r') meets S'(t') for every t' in f(R(x)) and S(r) meets S(t) for every t in R(x). By (2.20, iv) x is in S(r). Since F' is the inverse map of F, we have

$$F'(S'(r')) \subseteq S(r)$$

for every r'=f(r) in R'. Likewise

$$F(S(r)) \subseteq S'(r').$$

Since FF'(S'(r'))=S'(r') and since F and F' are one-one, then

$$F'(S'(r')) = S(r)$$

and

$$F(S(r)) = S'(r').$$

By (2.18) any compact set in X can be formed by an intersection of sets of the form S(r) which are compact. Applying the above results, the images of compact sets under the transformations F and F' are compact sets.

Since the spaces X and X' are regular T_1 (hence Hausdorf) spaces which are locally compact, then by lemma (2.23) it is easily shown that F and F' are closed mappings and hence are homeomorphisms.

q.e.d.

(2.25) <u>Corollary</u>: If the rings R and R' are associated with the regular, compact T_1 spaces X and X', respectively, and if R and R' are isomorphic, then X and X' are homeomorphic.

The corollary immediately follows from (2.24) for if R is associated with a compact space X, then R is compact associated with X which is also locally compact.

The converses to (2.25) and (2.24) do not hold for we have shown that <u>any</u> two rings without divisors of zero may be associated or compact associated with the same space.

A ring may be associated with a non-compact space and still determine its topology. For example, the ring of all continuous functions over a normal T_1 space is associated with this space; but Hewitt [3] has shown that if such a space satisfies the second axiom of countability, then the ring of all continuous functions determines the space.

PART II

RINGS ASSOCIATED WITH INFINITELY DIFFERENTIABLE MANIFOLDS

Chapter 3. Infinitely Differentiable Functions on Euclidean Space.[1]

In this chapter we will show that if $f(x^1,\dots, x^n)$ is a continuous function which is infinitely differentiable and is continuous in all derivatives of all orders everywhere on a Euclidean space R^n, then

$$f(x^1,\dots, x^n) = f(x^1{}_0,\dots, x^n{}_0) + \sum_{i=1}^{n} (\partial f(x^1{}_0,\dots, x^n{}_0)/\partial x^i)(x^i - x^i{}_0) \sum_{i,j=1}^{n} g_{ij}(x^1,\dots, x^n; x^1{}_0,\dots, x^n{}_0)\cdot(x^i - x^i{}_0)(x^j - x^j{}_0),$$

where the g_{ij}'s are continuous functions of the x^i's which are infinitely differentiable and are continuous in all derivatives of all orders everywhere on R and where the numbers $(x^1{}_0,\dots, x^n{}_0)$ are coordinates of an arbitrary fixed point. Without loss of generality we may assume the point $(x^1{}_0,\dots, x^n{}_0)$ to be the origin $(0,\dots, 0)$.

(3.1) <u>Definition</u>: By "$C_i{}^m(R^n)$" we mean the set of all continuous functions on R^n such that all derivatives not involving the coordinate x^i and all derivatives involving x^i at most m times exist and are continuous on R^n. By "$C_{i,0}{}^m(R^n)$" we mean all functions in $C_i{}^m(R^n)$ which vanish on the hyperplane $x^i = 0$ with all derivatives not involving x^i or involving x^i at most m times.

By "$D_i{}^m(R^n)$" we mean the set of all functions on R^n such that all derivatives not involving x^i and all derivatives involving x^i at most m times exist and are finite everywhere on R^n. By "$D_{i,0}{}^m(R^n)$" we mean all functions in $D_i{}^m(R^n)$ which vanish on the hyperplane $x^i = 0$ with all derivatives not involving x^i or involving x^i at most m times.

[1] The results of this chapter follow readily from classical analysis; but they were not found in the literature and were given here for completeness.

We have:

(3.2) (a) $D_i^1(R^n) \supset D_i^2(R^n) \supset \dots \supset D_i^m(R^n) \supset \dots$.

(b) $C_i^1(R^n) \supset C_i^2(R^n) \supset \dots \supset C_i^m(R^n) \supset \dots$.

(c) $D_{i,0}^1(R^n) \supset D_{i,0}^2(R^n) \supset \dots \supset D_{i,0}^m(R^n) \supset \dots$.

(d) $C_{i,0}^1(R^n) \supset C_{i,0}^2(R^n) \supset \dots \supset C_{i,0}^m(R^n) \supset \dots$.

(3.3) _If f is in $D_i^m(R^n)$, hence in $D_i^1(R^n)$, then f is continuous in x^i at every point of R^n._

In this chapter we will use the following notation: by $f(\dots,g^i,\dots)$, where g^i is any expression, we mean $f(x^1,\dots, x^{i-1},g^i,x^{i+1},\dots, x^n)$. By $f(\dots,0,\dots)$ we mean $f(x^1,\dots, x^{i-1},0, x^{i+1},\dots, x^n)$.

By the mean value theorem for functions of one variable:

(3.4) _If f is in $D_i^1(R^n)$, there is a function $\phi^i(x^1,\dots, x^n)$ such that_

$$f(x^1,\dots, x^n) = f(\dots,0,\dots) + (x^i)f_i^{(1)}(\dots,\phi^i(x^1,\dots, x^n),\dots)$$

for all $x^1,\dots, x^n$ where we define

$$"f_i^{(k)}" = \partial^k f/\partial (x^i)^k$$

and where

$$x^i < \phi^i(x^1,\dots, x^n) < 0 \qquad \text{for } x^i < 0$$
$$0 < \phi^i(x^1,\dots, x^n) < x^i \qquad \text{for } 0 < x^i$$
$$\phi^i(x^1,\dots, x^n) = 0 \qquad \text{for } x^i = 0.$$

(3.5) _Definition:_ We will denote $f_i^{(1)}(\dots,\phi^i,\dots)$ as $M_i(f)$, i.e.

$$M_i(f) = \begin{cases} (f(x^1,\dots, x^n)-f(\dots,0,\dots))/x^i & \text{for } x^i \neq 0 \\ f^{(1)}_i(\dots,0,\dots) & \text{for } x^i = 0. \end{cases}$$

(3.6) Theorem: If f is in $D_i^m(R^n)$, then for $x^i \neq 0, (M_i(f))^{(k)}_i$ exists and

$$(M_i(f))^{(k)}_i = a_{0k} f^{(k)}_i/(x^i)^1 + \dots + a_{k-1,k} f^{(1)}_i/(x^i)^k + a_{kk}(M_i f)/(x^i)^k$$

for $k \leq m$ where $a_{j,k} = (-1)^j \binom{k}{j}$.

Proof: From Leibnitz's rule for successive differentiation of a product we obtain

(3.7) $$(M_i(f))^{(k)}_i = a_{0k}\, f^{(k)}_i/(x^i)^1 + \dots + a_{k-1,k}\, f^{(1)}_i/(x^i)^k$$

$$a_{kk}\,(f-f(\dots,0,\dots))/(x^i)^{k+1}.$$

Substituting from definition (3.4) we obtain the desired result.

q.e.d.

(3.8) Theorem: If f is in $D_{i,0}{}^m(R^n)$, then there are functions $\phi^i{}_m(x^1,\dots, x^n)$ and $\theta^i{}_m(x^1,\dots, x^n)$ defined on R^n such that:

	$x^i < \phi^i{}_m(x^1,\dots, x^n) < 0$	for $x^i < 0$
(i)	$0 < \phi^i{}_m(x^1,\dots, x^n) < x^i$	for $0 < x^i$
	$\phi^i{}_m(x^1,\dots, x^n) = 0$	for $x^i = 0$
(ii)	$\lvert \theta^i{}_m(x^1,\dots, x^n)\rvert \leq 1$	for all $(x^1,\dots, x^n)$
(iii)	$f/(x^i)^m = \theta^i{}_m\, f^{(m)}_i(\dots,\phi^i{}_m,\dots)$	for $x^i \neq 0$.

Proof: If we take $\theta^i{}_m = 1$ and $\phi^i{}_1 = \phi^i$, the theorem has been established for $m = 1$.

Assume the proposition holds for $m = k$. Consider any f in $D_{i,0}{}^{k+1}(R\ )$, then f is in $D_{i,0}{}^k(R^n)$. For $x^i \neq 0$

$$f/(x^i)^{k+1} = f/(x^i)^k\,(x^i) = \theta^i{}_k\, f^{(k)}_i(\ldots,\phi^i{}_k,\ldots)/(x^i).$$

Since f is in $D_{i,0}{}^{k+1}(R^n)$, then $f^{(k)}_i$ has a finite derivative with respect to x^i everywhere and $f^{(k)}_i$ is continuous with respect to x^i at every point of R^n. By the mean value theorem for functions of one variable there is a function $\phi^*(x^1,\ldots, x^n)$ such that

$$x^i < \phi^*(x^1,\ldots, x^n) < 0 \qquad \text{for } x^i < 0$$
$$0 < \phi^*(x^1,\ldots, x^n) < x^i \qquad \text{for } 0 < x^i$$
$$\phi^*(x^1,\ldots, x^n) = 0 \qquad \text{for } x^i = 0$$

and

$$f^{(k)}_i(x^1,\ldots, x^n) = (x^i)\, f^{(k+1)}_i(\ldots,\phi^*,\ldots)$$

for all $(x^1,\ldots, x^n)$ in R^n. Hence

$$f^{(k)}_i(\ldots,\phi^i{}_k,\ldots) = \phi^i{}_k\, f^{(k+1)}_i(\ldots,\phi^*(\ldots,\phi^i{}_k,\ldots),\ldots).$$

By substitution

$$f/(x^i)^{k+1} = \theta^i{}_k \cdot (x^i)^{-1}\phi^i{}_k f^{(k+1)}_i(\ldots,\phi^*(\ldots,\phi^i{}_k,\ldots),\ldots).$$

Set

$$\theta^i{}_{k+1} = \begin{cases} \theta^i{}_k \cdot \phi^i{}_k/x^i & \text{for } x^i \neq 0 \\ 1 & \text{for } x^i = 0 \end{cases}$$

and

$$\phi^i{}_{k+1} = \phi^*(\ldots,\phi^i{}_k,\ldots).$$

It is easily verified that $\theta^i{}_{k+1}$ and $\phi^i{}_{k+1}$ satisfy (i), (ii), and (iii).

The proof follows by induction.

q.e.d.

(3.9) Corollary: If f is in $D_{i,0}^{k\ m}(R^n)$, then there are functions $\phi^i_{k,m}$ and $\theta^i_{k,m}$ which satisfy conditions (i) and (ii) of (3.8) respectively and such that

$$f^{(k)}_i/(x^i)^m = \theta^i_{k,m}\, f^{(k+m)}_i(\ldots,\phi^i_{k,m},\ldots).$$

(3.10) Lemma: Let $x=(x^1,\ldots, x^n)$ and $x_0=(x^1_0,\ldots, x^n_0)$ be points of R^n. If F(x) is a function on R such that

$$\lim_{x\to x_0} F(x)=0,$$

a(x) is any function on R^n which is bounded in some neighborhood of x_0, and b(x) is defined on R^n such that in some neighborhood of x_0 for some i:

$$\begin{aligned} x^i &< b(x^1,\ldots, x^n) < x^i_0 \qquad &\text{for } x^i < x^i_0 \\ x^i_0 &< b(x^i,\ldots, x^n) < x^i \qquad &\text{for } x^i_0 < x^i \\ & b(x^i,\ldots, x^n) = x^i_0 \qquad &\text{for } x^i = x^i_0, \end{aligned}$$

then

$$\lim_{x\to x_0} F(\ldots,b(x),\ldots) = 0$$

$$\lim_{x\to x_0} a(x)F(x) = 0$$

$$\lim_{x\to x_0} a(x)F(\ldots,b(x),\ldots) = 0$$

The proof of this lemma is an immediate consequence of the definition of a limit, for if x is in a given rectangular neighborhood of x_0, then the point $(\ldots,b(x),\ldots)$ is also in that neighborhood.

(3.11) Lemma: (i) If f is in $D_{i,0}^{m}(R^n)$ and if the coordinates $x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n$ are fixed, then

$$\lim_{x^i\to 0} f/(x^i)^m = 0$$

and

$$\lim_{x^i\to 0} f^{(k)}/(x^i)^{m-k} = 0 \qquad \text{for } 0<k<m.$$

(ii) If f is in $C_{i,0}^{m}(R^n)$, x is any point of R^n, and $x_0=(x^1{}_0,\ldots,x^n{}_0)$ is a fixed point of R^n such that $x^i{}_0=0$, and if we define

$$F = \begin{cases} f/(x^i)^m & \text{for } x^i\neq 0 \\ 0 & \text{for } x^i=0 \end{cases}$$

and for $0<k\leq m$

$$F_k = \begin{cases} f^{(k)}_i/(x^i)^{m-k} & \text{for } x^i\neq 0 \\ 0 & \text{for } x^i=0, \end{cases}$$

then

$$\lim_{x\to x_0} F = \lim_{x\to x_0} F_k = 0.$$

Proof: (i) Proposition (i) is a special case of a well-known theorem, but we will give a proof.

From (3.8)

$$f/(x^i)^m = f/(x^i)^{m-1}(x^i)$$
$$=(\theta^i{}_{m-1})(x^i)^{-1}(\phi^i{}_{m-1})f^{(m-1)}_i(\ldots,\phi^i{}_{m-1},\ldots)/\phi^i{}_{m-1}.$$

Since f is in $D_{i,0}^{m}(R^n)$, then for $x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n$ fixed

$$\lim_{x^i \to 0} f_i^{(m-1)}/x^i = 0,$$

and by (3.10)

$$\lim_{x^i \to 0} f/(x^i)^m = 0.$$

By (3.9)

$$f_i^{(k)}/(x^i)^{m-k}$$
$$=(\theta^i_{k,m-(k+1)})(x^i)^{-1}(\phi^i_{k,m-(k+1)})f_i^{(m-1)}(\ldots,\phi^i_{k,m-k+1},\ldots)/\phi^i_{k,m-(k+1)}.$$

(ii) If f is in $C_{i,0}{}^m(R^n)$, then

$$f/(x^i)^m = \theta^i_m \, f_i^{(m)}(\ldots,\phi^i_m,\ldots) \qquad \text{for } x^i \neq 0.$$

Since

$$\theta^i_m \, f_i^{(m)}(\ldots,\phi^i_m(\ldots,0,\ldots),\ldots) = 0 \qquad \text{for } x^i = 0,$$

then

$$F = \theta^i_m \, f_i^{(m)}(\ldots,\phi^i_m,\ldots) \qquad \text{for all } x.$$

By (3.10), since $f_i^{(m)}$ is continuous and vanishes for $x^i = 0$, then

$$\lim_{x \to x_0} F = 0.$$

Likewise

$$F_k = \theta^i_{k,m-k} \, f_i^{(m)}(\ldots,\phi^i_{k,m-k},\ldots),$$

and as before

$$\lim_{x \to x_0} F_k = 0.$$

q.e.d.

(3.12) <u>Lemma</u>: (i) If f is in $D_{i,0}^{m+1}(R^n)$, then the first m derivatives of $M_i(f)$ with respect to x^i exist and are finite everywhere and vanish for $x^i = 0$.

(ii) If f is in $C_{i,0}^{m+1}(R^n)$, then $M_i(f)$ and the first m derivatives of $M_i(f)$ with respect to x^i are continuous on R^n.

<u>Proof</u>: (i) Set $M_i(f) = F(x^1, \ldots, x^n)$. If $m = 1$, then for f in $D_{i,0}^{m+1} = D_{i,0}^2$ we have $F(\ldots, x^i = 0, \ldots) = f_i^{(1)}(\ldots, x^i = 0, \ldots) = 0$. Hence

$$(F(x^1, \ldots, x^n) - F(\ldots, x^i = 0, \ldots))/(x^i - 0) = F/x^i = f/(x^i)^2.$$

By (3.11) the right term goes to zero with x^i, hence $F_i^{(1)}$ exists and is zero for $x^i = 0$.

For $x^i \neq 0$, $F = f/x^i$ and is differentiable with respect to x^i for all $(x^1, \ldots, x^n)$ such that $x^i \neq 0$. Therefore if f is in $D_{i,0}^2$, then $F_i^{(1)}$ exists everywhere and vanishes for $x^i = 0$. The proposition is valid for $m = 1$.

Assume the proposition is valid for $m = k$. Consider f in $D_{i,0}^{(k+1)+1}$, then f is in $D_{i,0}^{(k+1)}$ and the first k partial derivatives of F with respect to x^i exist everywhere and vanish for $x^i = 0$. From (3.7)

$$(F_i^{(k)}(x^1, \ldots, x^n) - F_i^{(k)}(\ldots, x^i = 0, \ldots))/(x^i - 0)$$

$$= a_{0k}\, f_i^{(k)}/(x^i)^2 + \ldots + a_{k-1,\,k} f_i^{(1)}/(x^i)^{k+1} + a_{kk} \cdot f/(x^i)^{k+2}.$$

Since f is in $D_{i,0}^{k+2}(R^n)$, then by (3.11) the terms on the right go to zero with x^i and $F_i^{(k+1)}$ exists and is zero for $x^i = 0$. For $x^i \neq 0$, $F_i^{(k+1)}$ exists according to (3.6).

The proposition follows by induction.

(ii) If f is in $C_{i,0}^{m+1}$, then f is in $C_{i,0}^{1}$ and it is easily shown that $F = M_i(f)$ is continuous. From (i) F has the first m derivatives with respect to x^i and they vanish for $x^i = 0$. From (3.7) for $k \leq m$ and for $x^i \neq 0$,

$$F_i^{(k)} = a_{0k}\, f_i^{(k)}/(x^i)^1 + \dots + a_{k-1,k}\, f_i^{(1)}/(x^i)^k + a_{kk}\, f/(x^i)^{k+1},$$

and $F_i^{(k)}$ is continuous for $x^i \neq 0$. Applying (3.11) to the above equation we find that $F_i^{(k)}$ is continuous for $x^i = 0$.

q.e.d.

From definition (3.1) it immediately follows that:

(3.13) <u>If</u> f <u>is in</u> $C_{i,0}^{m}(R^n)$ <u>and</u> $j \neq i$, <u>then</u> $\partial f/\partial x^j$ <u>exists everywhere and is in</u> $C_{i,0}^{m}(R^n)$.

(3.14) <u>Lemma</u>: If f is in $C_{i,0}^{m}(R^n)$ and $j \neq i$, then $\partial(M_i(f))/\partial x^j$ exists and is continuous everywhere and

$$\partial(M_i(f))/\partial x^j = M_i(\partial f/\partial x^j).$$

<u>Proof</u>: Since for f in $C_{i,0}^{m}(R^n)$

$$M_i(f) = \begin{cases} f/x^i & \text{for } x^i \neq 0 \\ 0 & \text{for } x^i = 0, \end{cases}$$

then $\partial(M_i(f))/\partial x^j$ exists and is continuous everywhere and

$$\partial(M_i(f))/\partial x^j = \begin{cases} f_j^{(1)}/x^i & \text{for } x^i \neq 0 \\ 0 & \text{for } x^i = 0 \end{cases}$$
$$= M_i(\partial f/\partial x^j).$$

q.e.d.

(3.15) Lemma: (i) If all first derivatives of a function f exist everywhere and if $\partial^2 f/\partial x^i \partial x^j$ exists and is continuous everywhere, then $\partial^2 f/\partial x^j \partial x^i$ exists everywhere and

$$\partial^2 f/\partial x^i \partial x^j = \partial^2 f/\partial x^j \partial x^i.$$

(ii) If all k-th derivatives of f exist everywhere and if $\partial^{k+1} f/\partial x^{i_1} \dots \partial x^{i_k} \partial x^{i_{k+1}}$ exists and is continuous everywhere, then $\partial^{k+1} f/\partial x^{i_1} \dots \partial x^{i_{k-1}} \partial x^{i_{k+1}} \partial x^{i_k}$ exists everywhere and

$$\partial^{k+1} f/\partial x^{i_1}, \dots, \partial x^{i_{k-1}} \partial x^{i_{k+1}} \partial x^{i_k} = \partial^{k+1} f/\partial x^{i_1} \dots \partial x^{i_k} \partial x^{i_{k+1}}.$$

(iii) If all partial derivatives of f not involving x^i exist and are continuous everywhere, then we can permute the order of differentiation of these derivatives without changing their value.

The proof of this lemma is well known. See for example [4, pp262-268].

(3.16) Lemma: If f is in $C_{i,0}^{m+1}(R^n)$, then $M_i(f)$ is in $C_{i,0}^{m}(R^n)$.

Proof: For $n=1$, the theorem follows immediately from (3.12). For $n \geq 2$ set $F = M_i(f)$ and consider $\partial^p(\partial^k F/\partial x^{i_1} \dots \partial x^{i_k})/\partial (x^i)^p$ where $i_j \neq i$ and $0 \leq p \leq m$. Then by repeated application of (3.14)

$$(\partial^k F/\partial x^{i_1} \dots \partial x^{i_k}) = M_i(\partial^k f/\partial x^{i_1} \dots \partial x^{i_k}),$$

which exists and is continuous everywhere. By (3.13) $\partial^k f/\partial x^{i_1} \dots \partial x^{i_k}$ is in $C_{i,0}^{m+1}$, hence by (3.12) $\partial^p(\partial^k F/\partial x^{i_1} \dots \partial x^{i_k})/\partial (x^i)^p$ exists and is continuous everywhere for all $p \leq m$ and vanishes for $x^i = 0$. This result holds for all $i_1, \dots, i_k$, $k = 1, 2, \dots$ such that $i_j \neq i$ for all j and for all p such that $0 \leq p \leq m$.

We will now show that all derivatives involving x^i at most m times exist and are continuous everywhere. We consider derivatives of the form $F^{(1)\ldots(1)\,(p)}_{i_1\ldots i_k\,i} \equiv \partial^p(\partial^k F/\partial x^{i_1}\ldots\partial x^{i_k})/\partial(x^i)^p$ and show that we may permute the order of differentiation.

For $k=1$ and $p=1$, the proposition follows immediately from (3.15), hence for any $p=1,\ldots,m$

$$F^{(1)\,(p)}_{j\,\,i} = F^{(1)\,(1)\,(p-1)}_{i\,\,\,j\,\,\,\,i}.$$

Now by inductive argument on p one can show that

$$F^{(1)\,(p)}_{j\,\,i} = F^{(t)\,(1)\,(p-t)}_{i\,\,\,j\,\,\,\,i}$$

for all $t \leq p \leq m$.

Assume the proposition holds for $k=q$. Since $F^{(1)\ldots(1)}_{i_1\ldots i_{q+1}}$; $F^{(1)\ldots(1)\,(1)}_{i_1\ldots i_q\,i}$; and $F^{(1)\ldots(1)\,(1)}_{i_1\ldots i_{q+1}\,i}$ exist and are continuous everywhere, then by (3.15)

$$F^{(1)\ldots(1)\,(1)\,(1)}_{i_1\ldots i_q\,\,i\,\,\,i_{q+1}} = F^{(1)\ldots(1)\,(1)}_{i_1\ldots i_{q+1}\,i}.$$

But since the proposition holds for $k=q$ and since in $F^{(1)\ldots(1)}_{i_1\ldots i_{q+1}}$ the indices may be permuted, the indices $i_1,\ldots i_{q+1}, i$ may be permuted arbitrarily. As for $k=1$, we may now show the proposition for all p.

The validity of the proposition for all k and for $p=0,\ldots,m$ follows by induction.

Since all derivatives not involving x^i and all derivatives involving x^i at most m times exist and are continuous everywhere, then $F=M_i(f)$ is in $C_{i,0}{}^m(R^n)$.

q.e.d.

(3.17) <u>Theorem</u>: If f is in $C_i{}^{m+1}(R^n)$, then $M_i(f)$ is in $C_i{}^m(R^n)$.

<u>Proof</u>: We will give a proof for $n>1$ which must be modified slightly

for $n=1$. For f in $C_i^{m+1}(R^n)$ set

$$\underline{f}=f-f(\ldots,x^i=0,\ldots)-(x^i)^1\, f_i^{(1)}(\ldots,x^i=0,\ldots)/1!$$

$$-\ldots-(x^i)^{m+1}\, f_i^{(m+1)}(\ldots,x^i=0,\ldots)/(m+1)!$$

Every term on the right is in C_i^{m+1}. Also $\underline{f}$ and all partial derivatives of $\underline{f}$ not involving x^i vanish for $x^i=0$.

Consider a partial derivative involving x^i p times where $p\leq m+1$. Since $\underline{f}$ is in C_i^{m+1}, the differentiations with respect to x^i may be performed first as in the proof of (3.16). We obtain

$$\underline{f}_i^{(p)}=f_i^{(p)}-f_i^{(p)}(\ldots,x^i=0,\ldots)-(x^i)\, f_i^{(p+1)}(\ldots,x^i=0,\ldots)/1!$$

$$-\ldots-(x^i)^{m+1-p}\, f_i^{(m+1)}(\ldots,x^i=0,\ldots)/(m+1-p)!\ .$$

Hence $\underline{f}_i^{(p)}$ vanishes for $x^i=0$ and any partial derivative of $\underline{f}_i^{(p)}$ not involving x^i vanishes for $x^i=0$. Therefore $\underline{f}$ and all partial derivatives of $\underline{f}$ involving x^i at most m+1 times vanish at $x^i=0$, and $\underline{f}$ is in $C_{i,0}^{m+1}(R^n)$.

Set $\underline{F}=M_i(\underline{f})$, then $\underline{f}=(x^i)\,\underline{F}$. Set $F=M_i(f)$, then $(x^i)F=f-f(\ldots,x^i=0,\ldots)$ and

$$(x^i)F=(x^i)\underline{F}+(x^i)\, f_i^{(1)}(\ldots,x^i=0,\ldots)/1!\ +$$

$$\ldots+(x^i)^{m+1}\, f_i^{(m+1)}(\ldots,x^i=0,\ldots)/(m+1)!$$

For $x^i=0$, $F=f_i^{(1)}(\ldots,x^i=0,\ldots)$, hence for all x^i

$$F=\underline{F}+f_i^{(1)}(\ldots,x^i=0,\ldots)/1!+\ldots+(x^i)^m\, f_i^{(m+1)}(\ldots,x^i=0,\ldots)/(m+1)!\ .$$

Since $\underline{f}$ is in $C_{i,0}^{m+1}$, then $\underline{F}$ is in $C_{i,0}^{m}$ which is in C_i^m. Since $(x^i)^j\, f_i^{(j+1)}(\ldots,x^i=0,\ldots)/(j+1)!$ is in C_i^{m+1}, then F is in C_i^m.

q.e.d.

(3.18) <u>Definition:</u> In <u>this</u> <u>chapter</u> by $C^{\infty}(R^n)$ we mean the set of continuous functions on R^n all of whose derivatives exist everywhere and are continuous, i.e. $C^{\infty}(R^n)$ is the intersection of $C_i^{m}(R^n)$ for all $m=1,2,3,\ldots$.

The set $C^{\infty}(R^n)$ as defined above is independent of i.

(3.19) <u>Lemma:</u> If f is in $C^{\infty}(R^n)$, then $M_i(f)$ is in $C^{\infty}(R^n)$ for any i.

(3.20) <u>Lemma:</u> If f is in $C^{\infty}(R^n)$ and if $F_i = M_i(f)$, then

$$f(x^1,\ldots,x^n) = f(0,\ldots,0) + (x^1)F_1(x^1,\ldots,x^n)$$
$$+(x^2)F_2(0,x^2,\ldots,x^n) + \ldots + (x^i)F_i(0,\ldots,0,x^i,\ldots,x^n)$$
$$+(x^n)F_n(0,\ldots,0,x^n).$$

<u>Proof:</u> For $n=1$, the lemma is immediate. For $n>1$, since

$$(x^i)F_i(0,\ldots,0,x^i,\ldots,x^n) = f(0,\ldots,0,x^i,\ldots,x^n)$$
$$-f(0,\ldots,0,x^{i+1},\ldots,x^n),$$

the result is easily obtained by substitution.

q.e.d.

The functions $F_i(0,\ldots,0,x^i,\ldots,x^n)$, given above, are in $C^{\infty}(R^n)$. If we apply (3.19) to each $F_i(0,\ldots,0,x^i,\ldots,x^n)$, since $F_i(0,\ldots,0) = \partial f(0,\ldots,0)/\partial x^i$, we obtain:

(3.21) <u>Theorem:</u> If f is in $C^{\infty}(R^n)$, there are functions g_{ij} in $C^{\infty}(R^n)$ such that

$$f = f(0,\ldots 0) + \sum_{i=1}^{n} (x^i)\,\partial f(0,\ldots,0)/\partial x^i + \sum_{i,j=1}^{n} (x^i)(x^j)g_{ij},$$

for all $x^1,\dots,x^n$ in R^n.

As a generalization of (3.21) we have:

(3.22) Theorem: If f is in $C^\infty(R^n)$ and if $(x^1_0,\dots,x^n_0)$ is any fixed point of R^n, then there are functions g_{ij} in $C^\infty(R^n)$ such that

$$f = f(x^1_0,\dots,x^n_0) + \sum_{i=1}^{n} (x^i - x^i_0)\,\partial f(x^1_0,\dots,x^n_0)/\partial x^i + \sum_{i,j=1}^{n} (x^i - x^i_0)(x^j - x^j_0) g_{ij},$$

for all $x^1,\dots,x^n$ in R^n. (The g_{ij}'s in this theorem are not necessarily the same functions as in (3.21), since they depend on the point $(x^1_0,\dots,x^n_0)$.)

Chapter 4. Infinitely Differentiable Manifolds

In this chapter we will define infinitely differentiable manifolds and the rings of infinitely differentiable functions on these manifolds. We will show that this ring is associated with the underlying space of the manifold.

(4.1) Definition: An open covering $\{V_a\}$ of a topological space X is said to be "neighborhood finite" if every point x of X has a neighborhood U (x) which meets at most finitely many of the covering sets V_a.

If a covering is star finite, then it is neighborhood finite. If it is neighborhood finite, then it is point finite.

(4.2) Lemma: If $\{V_a\}$ is a neighborhood finite open covering of a space X, and if each V_a has a neighborhood finite open covering $\{U_{ab}\}$, then the set of all U_{ab} is a neighborhood finite open covering of X.

Proof: Let x be any point of X. The point x has a neighborhood W (x) which meets only the sets $V_1,\ldots,V_n$ of $\{V_a\}$. In each $V_i, i=1,\ldots,n$, which contains x, the point x has a neighborhood W_i (x) which meets only finitely many U_{ib}. Let G be th intersection of W and all W_i for which x is in V_i, then G is a neighborhood of x which meets only finitely many U_{ab}. Hence $\{U_{ab}\}$ is a neighborhood finite covering for X.

q.e.d.

In this chapter we will use the following special notation: X is a connected Hausdorf space, R is the set of real numbers, R^n is the n-dimensional Euclidean space, R(X) is the set of all real-valued continuous functions, and $C^{\infty}(I^n)$ is the set of all infinitely differentiable functions on open subintervals of R^n. In this chapter by an infinitely differentiable function we mean a continuous function whose derivatives of all orders are continuous. A function "vanishes on a set" if it vanishes at every point of the set.

(4.3) Definition: A subring $C^{\infty}(X)$ of R (X) is a "ring of infinitely differentiable functions on X", if:

(i) X has a countable, neighborhood finite open covering $\{U_a$, a in A$\}$. For each U_a there are n functions $x_a^1,\ldots,x_a^n$ in $C^{\infty}(X)$ such that for any f in $C^{\infty}(X)$, there is an f^a in $C^{\infty}(I^n)$ such that

$$f(q)= f^a(x_a^1(q),\ldots,x_a^n(q))$$

for all q in U_a and for all U_a.

(ii) The mapping $h_a:U_a \rightarrow R^n$ defined by $h_a(q)=(x_a^1(q),\ldots,x_a^n(q))$ is a homeomorphism onto an open interval $|r^i-r_a^i|<b_a^i$ of R^n, (r_a^i fixed).

(iii) If a function f in R (X) can be represented at every point p in X as in (i), then f is in $C^{\infty}(X)$.

(4.4) Definition: If a ring $C^{\infty}(X)$, as defined in (4.3), exists on a connected Hausdorf space X, then the space X with the ring $C^{\infty}(X)$ is an "infinitely differentiable" or "smooth manifold". The space X is the "underlying space of the manifold". The covering $\{U_a\}$ is a "coordinate covering for X", and the set of n-systems of functions $\{(x_a^1,\ldots,x_a^n)\}$ constitute a "coordinate system" or "system of coordinate functions for X".[1]

From the definition it immediately follows that an infinitely differentiable manifold is a topological manifold.

The ring $C^{\infty}(X)$ is not unique. Let X be the real line. Let the identity function X be the coordinate function, obtaining a ring $C^{\infty}_1(X)$. If we let $(x)^3$ be a coordinate function, obtaining a ring

[1] This definition is similar to that given by Chevalley [1] for analytic manifolds.

$C^{\infty}_3(X)$, then we see that the rings $C^{\infty}_1(X)$ and $C^{\infty}_3(X)$ are not the same.

The hypothesis that the covering $\{U_a\}$ is neighborhood finite is not actually necessary. Assume $\{U_a$, a in A$\}$ satisfies all other conditions of (4.3). We can cover X with a countable covering $\{V_{a'}$, a' in A'$\}$ such that every $V_{a'}$ is contained in some U_a where $a=\phi(a')$, and $h_{\phi(a')}(V_{a'})$ is a closed (i.e. compact) subinterval of $U_{\phi(a')}$. Wilder [5, pp. 130 and 169] has shown that the covering $\{V_{a'}\}$, has a countable, star finite refinement of open sets $\{G_b$, b in B$\}$. Since every G_b is contained in some $V_{\theta(b)}$ where $\theta(b)=a'$, then G_b can be covered by a countable, star finite collection of open sets $\{W_{b'b}\}$ such that $h_{\phi\theta(b)}(W_{b'b})$ is a subinterval of $h_{\phi\theta(b)}(U_{\phi\theta(b)})$. The set of all $\{W_{b'b}$, b in B$\}$ is a countable, neighborhood finite covering for X. If we take the functions $x_{\phi\theta(b)}{}^i$ as coordinates for each of the open sets $W_{b'b}$, then the covering $\{W_{b'b}\}$ is a coordinate covering for X, and the ring of infinitely differentiable functions generated will be the same as will be generated if $\{U_a\}$ is the coordinate covering.

(4.5) <u>Theorem</u>: Let X be an infinitely differentiable manifold. If $f^1,\ldots,f^m$ is a set of functions in $C^{\infty}(X)$ and U is an open set in X such that for every f in $C^{\infty}(X)$ there is an f^* in $C^{\infty}(I^m)$ such that

$$f(p)=f^*(f^1(p),\ldots,f^m(p))$$

for all p in U, then

(i) $m=n$

and

(ii) in every U_a which meets U, the Jacobian $D(f^{1,a},\ldots,f^{n,a})/D(x_a{}^1,\ldots,x_a{}^n)\neq 0$ everywhere in $U\cap U_a$.

Proof: Let a be fixed. For any p in $U \cap U_a$ we may write $f^i(p) = f^{i,a}(x_a^1(p), \ldots, x_a^n(p))$ where $i = 1, \ldots, m$. Also $x_a^j(p) = x_a^{j*}(f^1(p), \ldots, f^m(p))$ where $j = 1, \ldots, n$, i.e.

$$f^{i,a}(x_a^{1*}(u^1, \ldots, u^m), \ldots, x_a^{n*}(u^1, \ldots, u^m)) = u^i \qquad i = 1, \ldots, m;$$

$$x_a^{j*}(f^{1,a}(r^1, \ldots, r^n), \ldots, f^{n,a}(r^1, \ldots, r^n)) = r^j \qquad j = 1, \ldots, n;$$

for all $(u^1, \ldots, u^m)$ in R^m such that $u^i = f^i(p)$, p in $U \cap U_a$, and for all $(r^1, \ldots, r^n)$ in R^n such that $r^j = x_a^j(p)$, p in $U \cap U_a$.

For all such values of u^i and r^j we have

(i) $$\sum_{j=1}^{n} (\partial f^{i,a}/\partial r^j)(\partial x_a^{j*}/\partial u^k) = \delta^i_k \qquad i, k = 1, \ldots, m,$$

and

(ii) $$\sum_{i=1}^{m} (\partial x_a^{j*}/\partial u^i)(\partial f^{i,a}/\partial r^k) = \delta^j_k \qquad j, k = 1, \ldots, n.$$

Let A be the m by n matrix with elements $A_{ij} = (\partial f^{i,a}/\partial r^j)$, and let B be the n by m matrix with the elements $B_{ji} = (\partial x_a^{j*}/\partial u^i)$. From (i) we have $AB = E^m$ where E^m is the m by n identity matrix. From (ii) we have $BA = E^n$. Then,

$$n = m$$

for otherwise either $\det(AB) = 0$ which implies $AB \neq E^m$ or $\det(BA) = 0$ which implies $BA \neq E^n$.

Since $\det(AB) = 1 = \det(A)\det(B)$ everywhere in $U \cap U_a$, we have

$$\det(A) = \det(\partial f^{i,a}/\partial r^j) = D(f^{1,a}, \ldots, f^{n,a})/D(x_a^1, \ldots, x_a^n) \neq 0$$

for all p in $U \cap U_a$.

q.e.d.

(4.6) <u>Corollary</u>: If $a \neq a'$ and if U_a meets $U_{a'}$, then $D(x_a^1, \ldots, x_a^n)/D(x_{a'}^1, \ldots, x_{a'}^n) \neq 0$ on $U_a \cap U_{a'}$.

(4.7) <u>The underlying space X of an infinitely differentiable manifold is a second countable Tychonoff space, i.e. X is separable metric.</u> The proof immediately follows from the fact that X has a countable covering of open sets homeomorphic to Euclidean space.

(4.8) <u>Definition</u>: Let $\{U_a, \text{ a in } A\}$ be a coordinate covering for an infinitely differentiable manifold with an underlying space X. A covering $\{V_b, \text{ b in } B\}$ is an "admissable refinement of $\{U_a, \text{ a in } A\}$" if it is a countable, neighborhood finite open covering of X and if there is a single-valued map $\phi: B \to A$ such that :

(i) V_b is contained in $U_{\phi(b)}$

(ii) $h_{\phi(b)}(V_b)$ is an open interval in $h_{\phi(b)}(U_{\phi(b)})$.

(iii) The union of all V_b such that $\phi(b) = a$ covers U_a.

(We do not require that sets V_b with distinct subscripts be distinct.)

An admissable refinement is said to be "completely admissable" if $h_{\phi(b)}(\mathrm{Cl}(V_b))$ is a closed subinterval contained in $h_{\phi(b)}(U_{\phi(b)})$ for all b in B, i.e. $\mathrm{Cl}(V_b)$ is compact and is contained in $U_{\phi(b)}$.

(4.9) <u>Theorem</u>: If $\{U_a, \text{ a in } A\}$ is a coordinate covering for an infinitely differentiable manifold, then there exists a completely admissable refinement of $\{U_a, \text{ a in } A\}$.

<u>Proof</u>: Let $h_a(U_a) = I_a$ where I_a is an open Euclidean interval. Every I_a has a countable, star finite open covering by subintervals $\{I_{b',a}, \text{ b' in } B'(a)\}$ such that $\mathrm{Cl}(I_{b',a})$ is a closed (compact) subinterval in I_a for all b' in B'(a).

Set

$$B=\{(b',a)\}=\bigcup_{a\varepsilon A} B'(a)\times a,$$

then B is countable. Let ϕ be the "projection" of B onto A, i.e.

$$\phi(b',a)=a.$$

We denote the elements of B by b and set

$$V_b=\{p \text{ in } U_{\phi(b)} \mid h_{\phi(b)}(p) \text{ is in } I_b=I_{(b',a)}\},$$

then $\{V_b, \text{ b in B}\}$ is a completely admissable refinement of $\{U_a, \text{ a in A}\}$.

q.e.d.

(4.10) <u>Theorem</u>: If $\{U_a, \text{ a in A}\}$ is a coordinate covering for an infinitely differentiable manifold, then $\{U_a, \text{ a in A}\}$ has completely admissable refinements $\{V_b, \text{ b in B}\}$ and $\{W_b, \text{ b in B}\}$ such that $Cl(V_b)$ is contained in W_b for every b.

<u>Proof</u>: We can construct $\{V_b\}$ as in theorem (4.9), but so that given $I_{b',a}$ we can find an open subinterval $J_{b',a}$ of I_a such that $Cl(I_{b',a})$ is contained in $J_{b',a}$, the $J_{b',a}$'s form a star finite open covering of I_a, and $Cl(J_{b',a})$ is a closed subinterval, contained in I_a. The inverse images of the $J_{b',a}$'s form the W_b's.

q.e.d.

(4.11) <u>The index set B of a completely admissable refinement</u> $\{V, \text{ b in B}\}$ <u>is never finite</u>.

(4.12) <u>Theorem</u>: If $\{U_a, \text{ a in A}\}$ is a coordinate covering for an infinitely differentiable manifold, then $\{U_a, \text{ a in A}\}$ is an admissable (but not completely admissable) refinement of itself.

(4.13) <u>Theorem</u>: Let I be an open bounded interval in Euclidean space R^n and let I' be an open interval in R^n such that Cl(I') is compact and is contained in I, then there is an infinitely differentiable function $F(I, I')$ on R^n such that

(i) $F(I,I')=1$ on I'

(ii) $F(I,I')>0$ on I

(iii) $F(I,I')=0$ on $R-I$.

<u>Proof</u>: It is well known that the function which is 0 for $x=a$ and $\exp(-1/(x-a)^2)$ for $x\neq a$ is infinitely differentiable and vanishes with all of its derivatives at $x=a$. We will denote this function by $g(x,a)$. Also the function

$$F(x,a)=\begin{cases} g(x,a) & \text{for } x>a \\ 0 & \text{for } x\leq a \end{cases}$$

is infinitely differentiable everywhere, since at $x=a$ the right and left derivatives of all orders exist and are equal.

If $a<a'<b'<b$, then the function $F(-x,a')+F(x,a)F(-x,b)+F(x,b')$ is infinitely differentiable and is positive everywhere. Define

$$F(x;a,a',b',b)=F(x,a)F(-x,b)/(F(-x,a')+F(x,a)F(-x,b)+F(x,b')),$$

then the function $F(x;a,a',b',b)$ is one on $a'\leq x\leq b'$, zero on $x\leq a$ and $b\leq x$, and is positive on $a\leq x\leq a'$ and $b'\leq x\leq b$.

If the coordinates of points of R^n are $(r^1,\ldots,r^n)$, set

$$\left\{(r^1,\ldots,r^n)\mid a^i<r^i<b^i\right\} = I,$$

and

$$\left\{(r^1,\ldots,r^n)\mid a'^i<r^i<b'^i\right\} = I'.$$

The function

$$F(I,I')=\prod_{i=1}^{n} F(r^i;\ a^i,\ a'^i,\ b'^i,\ b^i)$$

is the desired function.

q.e.d.

(4.14) <u>Corollary</u>: If $\{U_a$, a in $A\}$ is a coordinate covering for an infinitely differentiable manifold and if $\{V_b$, b in $B\}$ and $\{W_b$, b in $B\}$ are completely admissable in $\{U_a$, a in $A\}$ such that $Cl(V_b)$ is contained in W_b for every b, then there are functions u_b in $C^\infty(X)$ such that

$$u_b \begin{cases} = 1 & \text{on } V_b \\ > 0 & \text{on } W_b \\ = 0 & \text{on } X-W_b. \end{cases}$$

(4.15) <u>Theorem</u>: If $\{U_a$, a in $A\}$ is a coordinate covering for an infinitely differentiable manifold, if $\{V_b$, b in $B\}$ is admissable in $\{U_a$, a in $A\}$, and if for every b there is a function g_b in $C^\infty(X)$ which vanishes on $X-V_b$, then the function $\sum_{b\in B} g_b$ is properly defined and is in $C^\infty(X)$.

<u>Proof</u>: Let p be any point of X. A neighborhood G(p) meets finitely many V_b: $V_1, \ldots, V_m$. In G we have

$$\sum_{b\in B} g_b = g_1 + \ldots + g_m.$$

The sum on the right gives a properly defined infinitely differentiable function. Since $\sum_{b\in B} g_b$ is a properly defined infinitely differentiable function in some neighborhood of every point, then it is in $C^\infty(X)$.

q.e.d.

(4.16) <u>Corollary</u>: Let u_b be defined as in (4.14), then $\sum_b u_b$, $1/\sum_b u_b$, and $u_{b'}/\sum_b u_b = v_{b'}$ are in $C^\infty(X)$. Also

$$v_{b'}\begin{cases} =0 & \text{on } X - W_{b'} \\ >0 & \text{on } W_{b'} \end{cases}$$

and $\sum_b v_b = 1$ on all X.

(4.17) <u>Theorem:</u> If F in $C^\infty(I^n)$ is an infinitely differentiable function on $h_{a'}(U_{a'})$, a' in A, and if a function g in $C^\infty(X)$ vanishes on X-W where Cl(W) is contained in $U_{a'}$, then the function

$$f = \begin{cases} F(x_{a'}^1, \ldots, x_{a'}^n)\cdot g & \text{on } U_{a'} \\ 0 & \text{on } X - U_{a'} \end{cases}$$

is in $C^\infty(X)$.

<u>Proof:</u> The function f is in $C^\infty(X)$ if for every a in A there is a function f^a, infinitely differentiable on $h_a(U_a)$, such that

$$f(p) = f^a(x_a^1(p), \ldots, x_a^n(p))$$

on U_a. We need only to show this for $a \neq a'$ and such that U_a meets $U_{a'}$.

We may set

$$f^a = \begin{cases} 0 & \text{on } h_a(U_a - Cl(W)) \\ F(x_{a'}^{1,a}, \ldots, x_{a'}^{n,a})\cdot(g^a) & \text{on } h_a(U_a \cap U_{a'}). \end{cases}$$

The function f^a is infinitely differentiable, since the zero function is infinitely differentiable on the open set $h_a(U_a - Cl(W))$ and $F(x_{a'}^{1,a}, \ldots, x_{a'}^{n,a})$ is infinitely differentiable on the open set $h_a(U_{a'} \cap U_a)$. The union of these two open sets is $h_a(U_a)$.

Hence

$$f^a(x_a^{\,1}(p),\ldots, x_a^{\,n}(p)) = \begin{cases} F(x_{a'}^{\,1}(p),\ldots, x_{a'}^{\,n}(p))\cdot g(p) & \text{on } U_a \cap U_{a'} \\ 0 & \text{on } U_a - U_{a'} \end{cases}$$

and $f = f^a(x_a^{\,1},\ldots, x_a^{\,n})$ on U_a. Therefore f is in $C^\infty(X)$.

q.e.d.

Some authors define infinitely differentiable manifolds in a different way than we have. They assume X is covered by a finite number of coordinate neighborhoods W_i, such that there exist homeomorphic maps h_i which map W_i onto an interval I_i. The maps $h_i h_j^{-1}$: $I_j \cap h_j(W_i \cap W_j) \rightarrow I_i \cap h_i(W_i \cap W_j)$ are assumed to be infinitely differentiable with non-zero Jacobians. A continuous function f on X will be infinitely differentiable on X if for every W_i there is an f^i which is infinitely differentiable on I_i such that $f = f^i(h_i)$ on W_i.

We can show that if a space X satisfies the above conditions, then it is an infinitely differentiable manifold in the sense of definition (4.4) and the infinitely differentiable functions in the above sense are the elements of $C^\infty(X)$. We first construct two coverings $\{U_{ia}\}$ and $\{V_{ia}\}$ for X which are completely admissable in $\{W_i\}$ and such that $Cl(U_{ia})$ is in V_{ia}. The construction is similar to that used in theorem (4.10). As in corollary (4.14) we construct functions u_{ia} which are one on U_{ia}, vanish on $X - V_{ia}$ and are infinitely differentiable in the above sense. Let $\phi_i^{\,j}$ be the projection of I_i onto the j-coordinate of I_i. We now define

$$x_{ia}^{\,j}(p) = \begin{cases} 0 & \text{on } X - W_i \\ u_{ia}(p)\cdot\phi_i^{\,j}(h_i(p)) & \text{on } W_i. \end{cases}$$

The covering $\{U_{ia}\}$ is a coordinate covering for X and the functions $x_{ia}{}^{j}(p)$ form a system of coordinate functions for X. A function f on X will be infinitely differentiable in the above sense if and only if it is in $C^{\infty}(X)$.

(4.18) <u>Lemma:</u> If F and F' are disjoint closed sets in the underlying space X of an infinitely differentiable manifold, then:

(i) There is a function f in $C^{\infty}(X)$ which is one on F and zero on F'.

(ii) The sets F and F' have disjoint neighborhoods, U(F) and U(F'), with disjoint closures, hence X is normal.

(iii) There is a function g in $C^{\infty}(X)$ which is one on a neighborhood of F and is zero on a neighborhood of F'.

<u>Proof</u>: (i) Let $\{V_b,\ b \text{ in } B\}$ be a completely admissable refinement of the coordinate covering of X. Since $h_{\phi(b)}Cl(V_b)$ is compact, the distance between the closed sets $h_{\phi(b)}(Cl(V_b)\cap F)$ and $h_{\phi(b)}(Cl(V_b)\cap F')$ (if both sets are non-empty) is positive. We will denote this distance by d(b) and the subset of B for which $Cl(V_b)$ meets both F and F' by B'. As in theorem (4.10) we can construct completely admissable refinements $\{W_c,\ c \text{ in } C\}$ and $\{W'_c,\ c \text{ in } C\}$ of $\{V_b,\ b \text{ in } B\}$ with a transformation $\psi: C \rightarrow B$ such that $h_{\phi\psi(c)}W'_c$ is a subinterval of $h_{\phi\psi(c)}V_{\psi(c)}$ and $Cl(W_c)$ is contained in W'_c. In addition these refinements can be so constructed that for $\psi(c)$ in B'

$$\text{diameter}(h_{\phi\psi(c)}W_c) < \text{diameter}(h_{\phi\psi(c)}W'_c) \leq \tfrac{1}{2}d(b).$$

Hence the closure of no element of the refinements $\{W_c\}$ and $\{W'_c\}$ intersects both F and F'.

Let C' be the set of all elements of C such that $Cl(W'_c)$ meets F. Hence $Cl(W'_c)$ does not meet F' if c is in C'.

From (4.14) for all c in C there are functions w_c in $C^\infty(X)$ which are one on W_c and vanish on $X-W'_c$. Since the covering $\{W'_c\}$ is neighborhood finite, the functions $\sum_{c\epsilon C'} w_c$ and $\sum_{c\epsilon C} w_c$ are in $C^\infty(X)$ and are equal on F, and $\sum_{c\epsilon C'} w_c$ vanishes on F'. Since $\sum_{c\epsilon C} w_c$ is different from zero everywhere, the function $f = \sum_{c\epsilon C'} w_c / \sum_{c\epsilon C} w_c$ is in $C^\infty(X)$, vanishes on F', and is one on F.

(ii) Using the function f constructed above we set

$$U(F) = \{p \text{ in } X \mid f(P) > 3/4\}$$

$$U(F') = \{p \text{ in } X \mid f(p) < 1/4\}.$$

These open sets are the desired neighborhoods.

(iii) Applying proposition (i) to Cl(U(F)) and Cl(U(F')) we obtain proposition (iii).

q.e.d.

(4.19) <u>Corollary</u>: If F is a closed set of the underlying space X of an infinitely differentiable manifold and p is any point of (X-F), then there is a function f in $C^\infty(X)$ which is zero on a neighborhood of F and is one on a neighborhood of p. The function (1-f), which is also in $C^\infty(X)$, is one on a neighborhood of F and is zero on a neighborhood of p.

(4.20) <u>Lemma</u>: If F and F' are disjoint closed sets in the underlying space X of an infinitely differentiable manifold and if F is compact, then:

(i) The closed sets F and F' have neighborhoods V(F) and V(F') with disjoint closures such that Cl(V(F)) is compact.

(ii) There is a function f in $C^\infty(X)$ which is one on F and zero on F' such that the subset of X on which f is different from zero has a compact closure which does not meet F'.

Proof: (i) Let $\{V_b, \text{ b in } B\}$ be a completely admissable refinement of the coordinate covering of X. Since F is compact, there is a finite subcollection of $\{V_b\}$ whose union, G, contains F. Since $Cl(V_b)$ is compact for all b, then Cl(G) is compact. Let U(F) and U(F') be defined as in (4.18,ii). Set

$$V(F) = U(F) \cap G$$

and

$$V(F') = U(F'),$$

then V(F) and V(F') are the desired neighborhoods.

(ii) From (4.18,i) there is a function f in $C^{\infty}(X)$ which one on F and vanishes on (X-V(F)). This function is the desired function.

q.e.d.

(4.21) Theorem: Let X be the underlying space of an infinitely differentiable manifold.

(a) If for any f in $C^{\infty}(X)$ we define

$$Z(f) = \{p \text{ in } X \mid f(p) = 0\},$$

then in the sense of definition (2.16) the ring $C^{\infty}(X)$ is associated with the space X.

(b) Also, the subring of $C^{\infty}(X)$ consisting of all functions in $C^{\infty}(X)$ with compact supports is compact associated with X.

Proof: (a) Axioms (i) and (ii) of (2.1) are obviously satisfied. From (4.19) axiom (iii) of (2.1) and axiom(v,a) of (2.16) follows immediately.

We now consider axiom (iv) of (2.16). For any f in $C^{\infty}(X)$, A(f) consists of all g in $C^{\infty}(X)$ which vanish on X-Z(f). But g vanishes on X-Z(f) if and only if it vanishes on $Cl(X-Z(f)) = X\text{-Int } Z(f) = S(f)$, i.e. $A(f)$ consists of all functions which vanish on $S(f)$. If the functions f and f' are disjunct, then for every g and g' in $C^{\infty}(X)$ there is a function h in $C^{\infty}(X)$ such that $(h-g)$ vanishes on $S(f)$ and $(h-g')$ vanishes on $S(f')$. If we consider g to be the zero function and g' to be one everywhere, we see that such a function h does not exist if the sets $S(f)$ and $S(f')$ meet. Hence if $S(f)$ and $S(f')$ are not disjoint, then f and f' are not disjunct.

Suppose $S(f)$ and $S(f')$ are disjoint, then there is a function w which vanishes on $S(f)$ but is one on $S(f')$. Consider any pair g and g' in $C^{\infty}(X)$. Set

$$h=(g)(1-w)+(g')(w),$$

then h is equal to g on $S(f)$ and is equal to g' on $S(f')$. Hence $(h-g)$ is in $A(f)$, while $(h-g')$ is in $A(f')$. Therefore if $S(f)$ and $S(f')$ are disjoint, then f and f' are disjunct.

(b) Axioms (i), (ii), and (iii) of (2.1) are shown as in (a) above. Axiom (v,b) of (2.16) follows from (4.20) since a point of X is a closed set.

We will now show that for any two functions f and f' in $C^{\infty}(X)$ with compact supports the sets $S(f)$ and $S(f')$ are disjoint if and only if f and f' are disjunct.

From (4.20) there is a function g with compact support which is one on $S(f)$. As in (a) if $S(f)$ and $S(f')$ meet, then there is no function h in $C^{\infty}(X)$ such that $(h-g)$ is in $A(f)$ and $(h-0)$ is in $A(f')$. Hence if $S(f)$ and $S(f')$ are not disjoint, then f and f' are not disjunct. If $S(f)$ and $S(f')$ are disjoint, then there are functions w and w' in $C^{\infty}(X)$ with compact supports such that w is one on $S(f)$ and vanishes on $S(f')$ while w' is one on $S(f')$ and vanishes on $S(f)$. Given any two g and g' in $C^\infty(X)$, set

$$h = (g)(w) + (g')(w'),$$

then h is in $C^\infty(X)$ and has a compact support. Also (h-g) vanishes on S(f) and is in A(f), while (h-g') vanishes on S(f') and is in A(f'). Hence if S(f) and S(f') are disjoint, then f and f' are disjunct. Therefore axiom (iv) of (2.16) is satisfied and the ring of all functions in $C^\infty(X)$ with compact supports is compact associated with the space X.

q.e.d.

(4.22) Corollary: If the underlying space X of an infinitely differentiable manifold is compact, then the ring $C^\infty(X)$ determines the topology of X. If X is the underlying space of any infinitely differentiable manifold, then the ring of all functions in $C^\infty(X)$ with compact supports determines the topology of X.

Chapter 5. Contravariant and Covariant Tensors on Infinitely Differentiable Manifolds

(5.1) **Definition:** A multi-derivative L of degree m on $C^\infty(X)$ is a "tangent operator of order m on $C^\infty(X)$" if for any constant function a and for any collection $f^1,\ldots,\widehat{f^i},\ldots,f^m$ of m-1 functions from $C^\infty(X)$,

$$L(f^1,\ldots,f^{i-1},a,f^{i+1},\ldots,f^m)=0$$

over all X for any $i=1,\ldots,m$. A tangent operator of order one is a "tangent vector".

As in Chapter 1 and Chapter 2 we will write $L(f^1,\ldots,f^{i-1},g^i,f^{i+1},\ldots,f^m)$, as $L(\ldots,g^i,\ldots)$, i.e. if we say a certain proposition holds for $L(\ldots,g^i,\ldots)$ for all $f^1,\ldots,\widehat{f^i},\ldots,f^m$, we mean this proposition holds for $L(f^i,\ldots,f^{i-1},g^i,f^{i+1},\ldots,f^m)$ for all $f^1,\ldots,\widehat{f^i},\ldots,f^m$.

We may easily show that the sum, difference, and product of any two tangent operators is a tangent operator, and the product of a tangent operator with a function is a tangent operator.

(5.2) **Theorem:** If L is a tangent operator of order m on $C^\infty(X)$, then:

(i) L is linear with respect to R in each of its variables, i.e. if a_i and b_i are constants, then

$$L(\ldots,a_i\, f^i+b_ig^i,\ldots)=a_iL(\ldots,f^i,\ldots)+b_iL(\ldots,g^i,\ldots).$$

(ii) If a function g vanishes on an open set G, then $L(\ldots,g,\ldots)$ vanishes on G for any collection $f^1,\ldots,\widehat{f^i},\ldots,f^m$ from $C^\infty(X)$.

(iii) For all $f^1,\ldots,f^m$ and $g^1,\ldots,g^m$ in $C^\infty(X)$

$$L(f^1,\ldots,f^m)-L(g^1,\ldots,g^m)=\sum_{i=1}^{n} L(g^1,\ldots,g^{i-1},f^i-g^i,f^{i+1},\ldots,f^m)$$

(iv) If $f^i = g^i$ on an open set G for all $i=1,\ldots,m$, then $L(f^1,\ldots,f^m)=L(g^1,\ldots,g^m)$ on G.

Proof: (i) The validity of (i) follows in the obvious manner from the definition of a multi-derivative and from definition (5.1).

(ii) In (1.10) we showed that

$$A(g)\subseteq A(L(\ldots,g,\ldots)).$$

Since $C^\infty(X)$ is associated with X, then

$$\operatorname{Int} Z(g)\subseteq \operatorname{Int} Z(L(\ldots,g,\ldots)).$$

(iii) Expanding the member on the right we obtain

$$\begin{aligned}
&\sum_{i=1}^{m} L(g^1,\ldots,g^{i-1},f^i-g^i,f^{i+1},\ldots,f^m)\\
&=\sum_{i=1}^{m} L(g^1,\ldots,g^{i-1},f^i,\ldots,f^m)-\sum_{i=1}^{m} L(g^1,\ldots,g^i,f^{i+1},\ldots,f^m)\\
&=L(f^1,\ldots,f^m)+\sum_{i=2}^{m} L(g^i,\ldots,g^{i-1},f^i,\ldots,f^m)\\
&-\sum_{j=2}^{m} L(g^1,\ldots,g^{j-1},f^j,\ldots,f^m)-L(g^1,\ldots,g^m)\\
&=L(f^1,\ldots,f^m)-L(g^1,\ldots,g^m).
\end{aligned}$$

(iv) From (ii) and (iii) we have (iv). q.e.d.

(5.3) Theorem: There exist tangent operators of every order on $C^\infty(X)$, which are not the zero operator.

Proof: Let $\{V_b,\ b \text{ in } B\}$ and $\{W_b,\ b \text{ in } B\}$ be completely admissable refinements of $\{U_a,\ a \text{ in } A\}$ such that $Cl(V_b)$ is contained in W_b and let u_b be defined as in (4.14). For a fixed b' in B, set $a'=\phi(b')$ and define

$$L(f^1,\ldots,f^m)=\begin{cases}(u_b)(\partial f^1/\partial x_{a'}{}^1)\cdots(\partial f^m/\partial x_{a'}{}^1) & \text{on } U_{a'}\\ 0 & \text{on } X-U_{a'},\end{cases}$$

then $L(f^1,\ldots,f^m)$ is in $C^\infty(X)$ for $f^1,\ldots,f^m$ in $C^\infty(X)$. By substitution of a constant a, f^i+g^i, and $f^i\cdot g^i$ for f^i it immediately follows that L is a non-trivial tangent operator of order m.

q.e.d.

One may define a contravariant tensor T of order m as an operator $T:(C^\infty(X))^m\to C^\infty(X)$ such that

$$T(f^1,\ldots,f^m)=\sum_{i_1\cdots i_m}(\partial f^1/\partial x_a^{i_1})\cdots(\partial f^m/\partial x_a^{i_m})T(x_a^{i_1},\ldots,x_a^{i_m})$$

on U_a.

It is easily shown that such an operator T is a tangent operator of order m. We will now show that any tangent operator of order m is a contravariant tensor of order m in the above sense.

(5.4) Theorem: If L is a tangent operator of order 1 on $C^\infty(X)$, i.e. is a tangent vector, then for any f in $C^\infty(X)$

$$L(f)=\sum_{i=1}^{n}(\partial f/\partial x^i{}_a)L(x^i{}_a)$$

on the coordinate neighborhood U_a.

Proof: Let $\{V_b,\ b \text{ in } B\}$ and $\{W_b,\ b \text{ in } B\}$ be completely admissable refinements of $\{U_a,\ a \text{ in } A\}$ such that $Cl(V_b)$ is in W_b, and let u_b be defined as in (4.14). Let p be any fixed point in U_a. There is a b' in B such that p is in $V_{b'}$ and $a=\phi(b')$. There is an f^a in $C^\infty(I^n)$ such that $f(q)=f^a(x_a{}^1(q),\ldots,x_a{}^n(q))$ for all q in U_a. From Chapter 3 we can write

$$f^a(x_a^1(q),\dots,x_a^n(q)) = f^a(x_a^1(p),\dots,x_a^n(p))$$
$$+\sum_i (\partial f^a(x_a^1(p),\dots,x_a^n(p))/\partial x_a^i)\cdot(x_a^i(q)-x_a^i(p))$$
$$+\sum_{ij} g_{ij}(x_a^1(q),\dots,x_a^n(q))\cdot(x_a^i(q)-x_a^i(p))\cdot(x_a^j(q)-x_a^j(p))$$

where g_{ij} is infinitely differentiable on $h_a(U_a)$. We define the following functions on X:

$$G_{ij}(q)=\begin{cases} g_{ij}(x_a^1(q),\dots,x_a^n(q))u_b & \text{on } U_a \\ 0 & \text{on } X-U_a, \end{cases}$$

then the G_{ij}'s are in $C^\infty(X)$ and

$$f\cdot u_b = f(p)\cdot u_b + \sum_i \partial f/\partial x_a^i\big|_p (x_a^i - x_a^i(p))\cdot u_b$$
$$+\sum_{ij} G_{ij}(x_a^i - x_a^i(p))\cdot(x_a^j - x_a^j(p))$$

on X. Since p is fixed and $f(p)$, $\partial f/\partial x_a^i\big|_p$ and $x_a^i(p)$ are constants, we have

$$L(f\cdot u_b) = u_b L(f) + fL(u_b) = f(p)L(u_b) + \sum_i \partial f/\partial x_a^i\big|_p \left(L(x_a^i)\cdot u_b + x_a^i L(u_b)\right)$$
$$+\sum_{ij} L(G_{ij})(x_a^i - x_a^i(p))(x_a^j - x_a^j(p))$$
$$+\sum_{ij} G_{ij}L(x_a^i)(x_a^j - x_a^j(p)) + \sum_{ij} G_{ij}(x_a^i - x_a^i(p))L(x_a^j).$$

On V_b the function u_b is one, hence $L(u_b)$ is zero on V_b and

$$L(f\cdot u_b)\big|_p = L(f)\big|_p = \sum_i (\partial f/\partial x_a^i)\big|_p \cdot L(x_a^i)\big|_p.$$

Since this equation is valid for any p in U_a, we have

$$L(f) = \sum_i (\partial f/\partial x_a^i)\, L(x_a^i)$$

on U_a.

q.e.d.

(5.5) **Corollary:** If L is a tangent operator of order m on $C^{\infty}(X)$, then for any collection $f^1,\ldots,f^m$ of m functions in $C^{\infty}(X)$ we have

$$L(f^1,\ldots,f^m)=\sum_{i_1\cdots i_m}(\partial f^1/\partial x_a^{i_1})\cdots(\partial f^m/\partial x_a^{i_m})L(x_a^{i_1},\ldots,x_a^{i_m})$$

on the coordinate neighborhood U_a.

Proof: If all but one variable of L is held fixed, then L is a tangent vector in that variable and we have

$$\begin{aligned}L(f^1,\ldots,f^m)&=\sum_{i_1=1}^{n}(\partial f^1/\partial x_a^{i_1})L(x_a^{i_1},f^2,\ldots,f^m)\\&=\sum_{i_1,i_2}(\partial f^1/\partial x_a^{i_1})(\partial f^2/\partial x_a^{i_2})L(x_a^{i_1},x_a^{i_2},f^3,\ldots,f^m)\\&=\ldots=\sum_{i_1\cdots i_m}(\partial f^1/\partial x_a^{i_1})\cdots(\partial f^m/\partial x_a^{i_m})L(x_a^{i_1},\ldots,x_a^{i_m}).\end{aligned}$$

q.e.d.

Since we have now shown that a contravariant tensor (or contravariant tensor field) defined as an operator is a tangent operator and vice versa, we will henceforth use the terms contravariant tensor and tangent operator interchangeably.

(5.6) Definition: We define an operator $N^b{}_{i_1\ldots i_m}$ by the equation

$$N^b{}_{i_1\ldots i_m}(f^1,\ldots,f^m)=\begin{cases}(\partial f^1/\partial x_a^{i_1})\cdots(\partial f^m/\partial x_a^{i_m})(u_b)^m & \text{on } U_a\\ 0 & \text{on } X-U_a\end{cases}$$

where $\{U_a,\ a \text{ in } A\}$ is a coordinate covering of X, $\{V_b,\ b \text{ in } B\}$ and $\{W_b,\ b \text{ in } B\}$ are completely admissable refinements of $\{U_a,\ a \text{ in } A\}$ such that W_b contains $Cl(V_b)$, $a=\phi(b)$, and u_b are the functions of $C^{\infty}(X)$ described in (4.14) which are one on V_b and vanish on $X-W_b$.

It is easily shown that $N^b{}_{i_1\ldots i_m}$ is a tangent operator of order m.

(5.7) Lemma: If $N^b{}_{i_1 \ldots i_m}$ and $N^b{}_{j_1 \ldots j_{m'}}$, are defined as in (5.6), then

$$N^b{}_{i_1 \ldots i_m j_1 \ldots j_{m'}} = (N^b{}_{i_1 \ldots i_m})(N^b{}_{j_1 \ldots j_{m'}}).$$

The proof follows directly from (5.6).

(5.8) Definition: If L is a contravariant tensor of order m, then the function $L(x_a{}^{i_1}, \ldots, x_a{}^{i_m})$ is the " $i_1 \ldots i_m$ coordinate of L on U_a in the coordinate system $x_a{}^1 \ldots x_a{}^m$".

(5.9) Definition: If X is the underlying space of an infinitely differentiable manifold and if Y is a subset of X, then the contravariant tensor L "vanishes on Y" if $L(f^1, \ldots, f^m)$ vanishes at every point of Y for every $f^1, \ldots, f^m$ in $C^\infty(X)$. Two contravariant tensors of the same order "are equal on Y" if their difference vanishes on Y.

(5.10) Lemma: If L is a contravariant tensor of order m on an infinitely differentiable manifold and if the tensors $N^b{}_{i_1 \ldots i_m}$ are defined as in (5.6), then the contravariant tensor, $\sum_{i_1 \cdots i_m} L(x_{\emptyset(b)}{}^{i_1}, \ldots, x_{\emptyset(b)}{}^{i_m}) \cdot N^b{}_{i_1 \ldots i_m}$, is equal to L on V_b.

The proof follows from (5.5) and the definition of $N^b{}_{i_1 \ldots i_m}$ since $(u_b)^m$ is one on V_b.

(5.11) Lemma: A contravariant tensor L vanishes at a point p:

(i) if and only if all coordinates of L in some coordinate system $x_a{}^1, \ldots, x_a{}^n$ such that U_a contains p vanish at p.

(ii) if and only if all coordinates of L in all coordinate systems $x_a{}^1, \ldots, x_a{}^m$ such that U_a contains p vanish at p.

(5.12) Lemma: If F is any compact subset of the underlying space X of an infinitely differentiable manifold and if k is any positive integer, then there is a contravariant tensor of order 2k which does not vanish at any point of F.

Proof: Let $N^b{}_1$ be the tangent vector described in (5.6) and let $V_1, \ldots, V_t$ be a finite covering of F_n by sets from $\{V_b,\ b \text{ in } B\}$. Consider the contravariant tensor $\sum_{i=1}^{n} (N^i{}_1)^{2k} = L$. Any p in F is in some $V_{b'}$ where b' is in $\{1, \ldots, t\}$, then for $x_a{}^1$ where $a = \phi(b')$, at p $(N^{i'}{}_1)^{2k}(x_a{}^1) \geq 0$ and $(N^{b'}{}_1)^{2k}(x_a{}^1) = 1$. Hence L does not vanish at any point of F.

q.e.d.

Since the covering $\{V_b,\ b \text{ in } B\}$ is neighborhood finite we can show that there is a contravariant tensor given by $\sum_{b \in B} (N^b{}_1)^{2k}$ which does not vanish at any point of X.

We will now define covariant tensors.

(5.13) Definition: A "covariant tensor $\underline{T}$ of order m" on an infinitely differentiable manifold X is a transformation of the set of all contravariant tensors of order m into $C^\infty(X)$ which is linear with respect to $C^\infty(X)$, i.e. $\underline{T}(L)$ is in $C^\infty(X)$ and

$$\underline{T}(fL + f'L') = f\underline{T}(L) + f'\underline{T}(L')$$

for all f and f' in $C^\infty(X)$ and for all contravariant tensors L and L' of order m on the manifold.

The zero operator, which satisfies the above conditions for every m, is a "covariant tensor of order infinity".

(5.14) Lemma: If $\underline{T}$ is a covariant tensor of order m on an infinitely differentiable manifold X and if L is a contravariant tensor of order m on X which vanishes at every point of an open set G, then the function $\underline{T}(L)$ vanishes at every point of G. If L and L' are contravariant tensors of order m which are equal on G, then the functions $\underline{T}(L)$ and $\underline{T}(L')$ are equal on G.

Proof: If p is any point in the open set G, then from (4.19) there is a function f in $C^{\infty}(X)$ which is one on X–G but vanishes at p. Since $L = fL$, then $\underline{T}(L) = f\underline{T}(L)$ vanishes at p. Hence $\underline{T}(L)$ vanishes on G.

If L and L' are equal on G, then their difference L–L' vanishes on G, and

$$\underline{T}(L) - \underline{T}(L') = \underline{T}(L-L')$$

vanishes on G, hence $\underline{T}(L)$ equals $\underline{T}(L')$ on G.

q.e.d.

(5.15) Definition: If $f^1,\ldots,f^m$ are any m functions in $C^{\infty}(X)$, then the gradient of $f^1,\ldots,f^m$ is the transformation G of the set of all contravariant tensors of order m into $C^{\infty}(X)$ given by

$$G(L) = L(f^1,\ldots,f^m)$$

for every contravariant tensor L or order m.

If ${x_a}^1,\ldots,{x_a}^n$ are coordinates of a coordinate neighborhood U_a, then we write the gradient of ${x_a}^{i_1},\ldots,{x_a}^{i_m}$ as ${G_a}^{i_1\cdots i_m}$.

(5.16) Lemma: If $f^1,\ldots,f^m$ are any m functions from $C^{\infty}(X)$, then the gradient of $f^1,\ldots,f^m$ is a covariant tensor of order m on $C^{\infty}(X)$.

The proof is straight forward and need not be given here.

(5.17) Definition and Lemma: If $\underline{T}$ and $\underline{T}'$ are covariant tensors of the same order m on an infinitely differentiable manifold with underlying space X and if f and f' are from $C^{\infty}(X)$, then we define the operators $f\underline{T} = \underline{T}f$, $\underline{T} + \underline{T}'$, and $\underline{T} - \underline{T}'$ by

$(f\underline{T})(L) = f\underline{T}(L)$

$(\underline{T}+\underline{T}')(L) = \underline{T}(L) + \underline{T}'(L)$

$(\underline{T} - \underline{T}')(L) = \underline{T}(L) - \underline{T}'(L)$

for all contravariant tensors L of order m. These operators are covariant tensors of order m.

The proof that these operators are covariant tensors of order m follows directly from the definitions.

(5.18) Definition: A covariant tensor $\underline{T}$ of order m on an infinitely differentiable manifold with underlying space X "vanishes on a subset Y of X "if $\underline{T}(L)$ vanishes at every point of Y for every contravariant tensor L or order m.

Two covariant tensors of the same order "are equal on Y " if their difference vanishes on Y.

(5.19) Theorem: If $\underline{T}$ is a covariant tensor of order m on an infinitely differentiable manifold with completely admissable coverings $\{V_b, \text{ b in B}\}$ and $\{W_b, \text{ b in B}\}$ such that W_b contains $Cl(V_b)$ and if the contravariant tensors $N^b{}_{i_1 \ldots i_m}$ are defined as in (5.6), then the covariant tensors $\underline{T}$ and $\underline{T}'$ given by

$$\underline{T}' = \sum_{i_1 \cdots i_m = 1}^{n} \underline{T}(N^b{}_{i_1 \ldots i_m}) G_{\emptyset(b)}{}^{i_1 \cdots i_m}$$

are equal on V_b.

Proof: Let L be any contravariant tensor of order m, then L is equal to $\sum_{i_1 \cdots i_m} L(x_a^{i_1}, \ldots, x_a^{i_m}) N^b{}_{i_1 \ldots i_m}$ on V_b where $a = \emptyset(b)$. Therefore

$$\underline{T} \sum L(x_a^{i_1}, \ldots, x_a^{i_m}) N^b{}_{i_1 \ldots i_m} = \sum L(x_a^{i_1}, \ldots, x_a^{i_m}) \; \underline{T}\, (N^b{}_{i_1 \ldots i_m})$$

$$= \sum \; \underline{T}(N^b{}_{i_1 \ldots i_m}) \; G_a{}^{i_1 \cdots i_m}(L) = \underline{T}'(L)$$

is equal to $\underline{T}(L)$ on V_b for every L. Therefore $\underline{T}$ is equal to $\underline{T}'$ on V_b.

q.e.d.

(5.20) Corollary: Under the hypothesis of (5.19) for any contravariant tensor L of order m the functions $\underline{T}(L)$ and $\sum_{i_1 \cdots i_m} \underline{T}(N^b{}_{i_1 \ldots i_m}) L(x_{\phi(b)}{}^{i_1}, \ldots, x_{\phi(b)}{}^{i_m})$ are equal on V_b.

(5.21) Definition: We call the functions $\underline{T}(N^b{}_{i_1 \ldots i_m})$ as given in theorem (5.19) the " $i_1 \ldots i_m$ coordinates of $\underline{T}$ in the coordinate system $x_{\phi(b)}{}^1, \ldots, x_{\phi(b)}{}^n$ on the neighborhood V_b ".

(5.22) Theorem: Under the hypothesis of (5.19)

$$\underline{T}(N^b{}_{i_1 \ldots i_m}) = \sum_{j_1 \cdots j_m} (\partial x_{a'}{}^{j_1} / \partial x_a{}^{i_1}) \cdots (\partial x_{a'}{}^{j_m} / \partial x_a{}^{i_m}) \underline{T}(N^{b'}{}_{j_1 \ldots j_m})$$

on the intersection of V_b and $V_{b'}$ where $a = \phi(b)$ and $a' = \phi(b')$.

Proof: Set

$$f^{j_1 \cdots j_m}_{i_1 \cdots i_m} = \begin{cases} (\partial x_{a'}{}^{j_1} / \partial x_a{}^{i_1}) \cdots (\partial x_{a'}{}^{j_m} / \partial x_a{}^{i_m})(u_b)^m & \text{on } U_{\phi(b)} \\ 0 & \text{on } X - W_b, \end{cases}$$

then $f^{j_1 \cdots j_m}_{i_1 \cdots i_m}$ is in $C^\infty(X)$. Set

$$L = \sum_{j_1 \cdots j_m} f^{j_1 \cdots j_m}_{i_1 \cdots i_m} N^{b'}{}_{j_1 \cdots j_m},$$

then on the intersection of V_b and $V_{b'}$

$$L(f^1, \ldots, f^m) = (\partial f^1 / \partial x_a{}^{i_1}) \cdots (\partial f^m / \partial x_a{}^{i_m}).$$

Hence L is equal to $N^b{}_{i_1 \cdots i_m}$ on $V_b \cap V_{b'}$, i.e. $\underline{T}(L) = \underline{T}(N^b{}_{i_1 \cdots i_m})$ on $V_b \cap V_{b'}$, but on $V_b \cap V_{b'}$

$$\underline{T}(L) = \sum_{j_1 \cdots j_m} (\partial x_{\phi(b')}{}^{j_1} / \partial x_{\phi(b)}{}^{i_1}) \cdots (\partial x_{\phi(b')}{}^{j_m} / \partial x_{\phi(b)}{}^{i_m}) \underline{T}(N^b{}_{j_1 \cdots j_m}).$$

q.e.d.

(5.23) **Theorem:** Let $\{V_b, b \text{ in } B\}$ and $\{W_b, b \text{ in } B\}$ be completely admissable coverings of an infinitely differentiable manifold with underlying space X. If to every coordinate system $x_{\emptyset(b)}^1,\ldots,x_{\emptyset(b)}^n$ we assign $(m)^2$ functions $f^b{}_{i_1\ldots i_m}$ from $C^\infty(X)$ such that

$$f^b{}_{i_1\ldots i_m} = \sum_{j_1\cdots j_m} f^{b'}{}_{j_1\ldots j_m}(\partial x_{\emptyset(b')}{}^{j_1}/\partial x_{\emptyset(b)}{}^{i_1})\cdots(\partial x_{\emptyset(b')}{}^{j_m}/\partial x_{\emptyset(b)}{}^{i_m})$$

on $V_b \cap V_{b'}$, then there is a covariant tensor $\underline{T}$ on X such that, for every b in B, $\underline{T}$ is equal to $\sum_{i_1\cdots i_m} f^b{}_{i_1\ldots i_m} G_{\emptyset(b)}{}^{i_1\cdots i_m}$ on V_b.

Proof: Let L be any contravariant tensor of order m. If p is any point of X and if V_b is any element of $\{V_b, b \text{ in } B\}$ which contains p, then we assign to p and b the value r(p,b) which is the value of the function $\sum_{i_1\cdots i_m} f^b{}_{i_1\ldots i_m} L(x_{\emptyset(b)}{}^{i_1},\ldots,x_{\emptyset(b)}{}^{i_m})$ evaluated at p. If b is fixed, then the function r(p,b) on V_b is equal on V_b to a function from $C^\infty(X)$. It is easily shown that r(p,b) has the same value for every b such that p is in V_b, i.e. r(p,b) determines a single valued function r(p) on X. Since r(p) is equal to some element of $C^\infty(X)$ on every V_b, then r(p) is in $C^\infty(X)$.

The transformation $L \to r(p)$ is a covariant tensor; for if $L' \to r'(p)$, then we see from above that $(fL + f'\,L') \to f\,r(p) + f'\,r'(p)$.

q.e.d.

(5.24) Lemma: A covariant tensor $\underline{T}$ on an infinitely differentiable manifold with an underlying space X vanishes at a point p of X:

(i) if and only if p is in some completely admissable neighborhood V_b such that all coordinates of $\underline{T}$ in the coordinate system $x_{\emptyset(b)}^1,\ldots,x_{\emptyset(b)}^n$ vanish at p.

(ii) if and only if all coordinates of $\underline{T}$ in all coordinate systems $x_{\emptyset(b)}^1,\ldots,x_{\emptyset(b)}^n$ such that V_b contains p vanish at p.

(5.25) Lemma: If F is a compact subset of the underlying space X of an infinitely differentiable manifold and if k is any positive integer, then there is a covariant tensor of order 2k which does not vanish at any point of F.

Proof: Let $\{V_b$, b in B$\}$ and $\{W_b$, b in B$\}$ be completely admissable refinements of the coordinate covering $\{U_a$, a in A$\}$ of X such that W_b contains $Cl(V_b)$. Let u_b be the functions defined in (4.14). Let $G_b{}^{1\ldots1}$ be the gradient of $x_{\phi(b)}{}^1$ taken 2k times. Let $V_1, \ldots, V_m$ be a finite covering of F, then $\underline{T} = \sum_{i=1}^{m} u_i G_i{}^{1\ldots1}$ is a covariant tensor of order 2k.

If p is any point of F in some $V_{i'}$, then at the point p

$$\underline{T}((N^{i'}{}_1)^{2k}) = u_{1'}(N^{i'}{}_1(x_{\phi(i')}{}^1))^{2k} + (\text{non-negative terms}).$$

Since the first term is one at p, then $\underline{T}$ does not vanish at p. Hence it follows that $\underline{T}$ does not vanish at any point of F.

q.e.d.

Since the covering $\{W_b$, b in B$\}$ is neighborhood finite, we can show that $\sum_{b \in B} u_b G_b{}^{1\ldots1}$ determines a covariant tensor of order 2k which does vanish at any point of X.

Chapter 6. Rings of Poly-tensors

In this chapter we will construct rings from the covariant and contravariant tensors over an infinitely differentiable manifold and show that these rings under certain conditions determine the underlying space of the manifold.

(6.1) Definition: "The ring of contravariant poly-tensors" over an infinitely differentiable manifold with an underlying space X is the ring $(C^{\infty}(X);\ R,0)$ as defined in (1.20) where R is the set of all real numbers. The elements of this ring are the forms $\sum_{m=0}^{\infty} L_m$ where L_m is either a contravariant tensor of order m or the zero operator, a tensor of order 0 is any non-zero element of $C^{\infty}(X)$, and almost all L_m are the zero operator. The ring $(C^{\infty}(X);\ R,k)$ is known as "the k-section of the ring of contravariant poly-tensors". We observe that these rings are not commutative.

A poly-tensor $\sum_m L_m$ is said "to vanish on the subset Y of X " if every L_m vanishes on Y.

In this chapter a contravariant tensor L of order m will be identified with the poly-tensor $\sum_k L_k$ for which L_k is the zero operator for all $k \neq m$ and for which $L_m = L$.

(6.2) Theorem: If the product of two contravariant poly-tensors in $(C^{\infty}(X);\ R,\ 0)$ vanishes at a point p of X, then at least one of the factors must vanish at the point p. If one of two poly-tensors vanishes at p, then their products vanish at p.

Proof: If L and L' are two contravariant tensors of order m and m' respectively neither of which vanish at the point p, then their product LL' does not vanish at p. For if L and L' do not vanish at p, then there are functions $f^1,\ldots,f^m$ and $f^{m+1},\ldots,f^{m+m'}$ such that $L(f^1,\ldots,f^m)$ and $L'(f^{m+1},\ldots,f^{m+m'})$ do not vanish at p. Hence $LL'(f^1,\ldots,f^{m+m'})$ does not vanish at p and LL' does not vanish at p. If one or both of the two tensors is of order zero, i.e. is a function from $C^\infty(X)$, then the product still does not vanish at the point p.

If one of two tensors of any order vanishes at p, then obviously their product vanishes at p.

Suppose $\sum_k L_k$ and $\sum_k L'_k$ are poly-tenosrs neither of which vanishes at p. Let L_m and $L'_{m'}$ be the tensors of lowest order in the respective poly-tensors which do not vanish at p, then the m+m' term (i.e. tensor of order m+m') of the poly-tensor $(\sum_k L_k)(\sum_k L'_k)$ consists of the product $L_m L'_m$ plus products which vanish at p. Therefore the m+m' term of the product does not vanish at p, and the product itself does not vanish at p.

If both $\sum_k L_k$ and $\sum_k L'_k$ vanish at p, then every L_k and L'_k vanishes at p and every term of the product of the two poly-tensors vanishes at p. Hence the product of these two poly-tensors vanishes at the point p. This statement still holds if we change the order of multiplication.

q.e.d.

Since for $k \geq 0$, the ring $(C^\infty(X); R,k)$ is a subring of $(C^\infty(X); R,0)$ we have:

(6.3) Corollary: The product of two contravariant poly-tensors in $(C^\infty(X); R,k)$ vanishes at a point p of X if and only if at least one of the factors vanishes at the point p.

(6.4) Theorem: Let X be the underlying space of an infinitely differentiable manifold, If for any poly-tensor $\sum_m L_m$ in $(C^\infty(X); R,k)$, we define $Z(\sum_m L_m)$ as the set of all points on which the poly-tensor vanishes and the support $S(\sum_m L_m)$ as the closure of the complement of $Z(\sum_m L_m)$, then the set of all poly-tensors in $(C^{\infty}(X); R,k)$ with compact supports will be a subring $(C^{\infty}(X); R,k)^*$ of $(C^{\infty}(X); R,k)$ and will be compact associated with the space X.

<u>Proof:</u> We first observe that $(C^{\infty}(X); R,k)^*$ is a subring of $(C^{\infty}(X); R,k)$. It is easily shown that if two poly-tensors vanish at a point, then their difference vanishes at that point. Therefore we can show that the support of the difference is contained in the union of the supports of the two given poly-tensors, i.e. the support of the difference of two poly-tensors with compact supports is compact.

From (6.3) we can easily show that the support of the product of two poly-tensors is in the intersection of the supports of the two given poly-tensors. Therefore the product of two poly-tensors with compact supports has a compact support. Therefore $(C^{\infty}(X); R,k)^*$ is a subring of $(C^{\infty}(X); R,k)$.

We will now show that $(C^{\infty}(X); R,k)^*$ is compact associated with X.

(2.1,i) The zero, 0, of $(C^{\infty}(X); R,k)^*$ (or of $(C^{\infty}(X); R,k)$) is the poly-tensor, $\sum_m L_m$, all of whose terms L_m are the zero tensor. If $\sum_m L_m = 0$, then every L_m vanishes on all X, hence $Z(0)=X$. Likewise if $Z(\sum_m L_m) = X$, then every L_m must vanish on all X and L_m is the zero tensor, i.e. $\sum_m L_m = 0$.

(2.1,ii) From (6.3) we have that the product of two poly-tensors vanishes at a point if and only if at least one of the factors vanishes at this point. Therefore for two poly-tensors $\sum_m L_m$ and $\sum_m L'_m$ we have

$$Z((\sum_m L_m)(\sum_m L'_m)) = Z(\sum_m L_m) \cup Z(\sum_m L'_m).$$

From (2.1,i) we have

$$(\sum_m L_m)(\sum_m L'_m) = 0 \quad \text{if and only if} \quad Z(\sum_m L_m) \cup Z(\sum_m L'_m) = X$$

(2.1,iii) From (5.12) for any positive integer m and any point p of X there is contravariant tensor of order 2m which does not vanish at p. If F is any closed set of X and if p is a point of X − F, then there is function f with a compact support which vanishes everywhere on F but not at p. If L_{2k} is a contravariant tensor of order 2k which does not vanish at p, then fL_{2k} is in $(C^{\infty}(X); R,k)^*$ and F is in $Z(fL_{2k})$ but p is not.

(2.16,iv) Let $r = \sum_m L_m$ and $r' = \sum_m L'_m$ be any two poly-tensors in $(C^{\infty}(X); R,k)^*$. The ideals A(r) and A(r') consist of all poly-tensors with compact supports which vanish on S(r) and S(r') respectively. Suppose the sets S(r) and S(r') are not disjoint. Since S(r), S(r'), and their union are compact, there is a poly-tensor in $(C^{\infty}(X); R,k)$ which does not vanish at any point of the union of S(r) and S(r'). Multiplying this poly-tensor with a function in $C^{\infty}(X)$ with a compact support which does not vanish at any point of S(r) or S(r'), we obtain a poly-tensor s in $(C^{\infty}(X); R,k)^*$ which does not vanish at any point of S(r) or S(r'). Since S(r) and S(r') intersect, there is no poly-tensor t such that (t−s) vanishes on S(r) and (t−0) = t vanishes on S(r'). Hence r and r' are not disjunct.

Suppose S(r) and S(r') are disjoint. There is a function f in $C^{\infty}(X)$ which is one on S(r) and vanishes on S(r'). If s and s' are any two poly-tensors in $(C^{\infty}(X); R,k)^*$, then the poly-tensor $t = fs + (1-f)s'$ is also in this ring. But (t−s) vanishes on S(r), i.e. (t−s) is in A(r), and (t−s') vanishes on S(r'), i.e. (t−s') is in A(r'). Therefore r and r' are disjunct.

(2.16;v,b) By definition the support of every element of $(C^{\infty}(X);\ R,k)^*$ is compact. If F is any compact subset of X and if p is a point in X-F, then there is a function f in $C^{\infty}(X)$ with a compact support S(f) such that S(f) contains F but not p. From (5.12) there is a contravariant tensor L_{2k} of order 2k which does not vanish at any point of F, i.e. F is in $S(L_{2k})$. The poly-tensor fL_{2k} is in $(C^{\infty}(X);\ R,k)^*$, and $S(fL_{2k})$ contains F but not p.

q.e.d.

(6.5) Corollary: If the underlying space X of an infinitely differentiable manifold is compact, then the ring $(C^{\infty}(X);\ R,k)$ is associated with the space X for any $k = 0, 1, \ldots$.

This corollary follows immediately; since if X is compact, then $(C^{\infty}(X);\ R,k) = (C^{\infty}(X);\ R,k)^*$ and compact association is association.

As an immediate consequence of (6.4), (6.5), (2.24), and (2.25) we have:

(6.6) Theorem: (a) If X and Y are the underlying spaces of two infinitely differentiable manifolds and if the rings $(C^{\infty}(X);\ R,k)^*$ and $(C^{\infty}(Y);\ R,k)^*$ of poly-tensors with compact supports are isomorphic for any $k = 0,1,2,\ldots$, then the spaces X and Y are homeomorphic.
(b) If X and Y are compact and if the rings $(C^{\infty}(X);\ R,k)$ and $(C^{\infty}(Y);\ R,k)$ are isomorphic for any $k = 0,1,2,\ldots$, then the spaces X and Y are homeomorphic.

It should be noted here, that if $k > 0$, then the rings $(C^{\infty}(X);\ R,k)$, and $(C^{\infty}(X);\ R,k)^*$ do not contain any elements of the ring $C^{\infty}(X)$.

We will now construct rings of covariant poly-tensors and show that for such rings that a theorem similar to (6.6) holds.

(6.7) **Definition and Lemma:** If $\underline{T}$ and $\underline{T}'$ are covariant tensors of order m and m', respectively, on an infinitely differentiable manifold with underlying space X, then there is exactly one covariant tensor $\underline{Q}$ of order $m+m'$ such that

$$\underline{Q}(LL') = (\underline{T}(L))(\underline{T}')L'))$$

for every contravariant tensor L of order m and L' of order m'. We define Q to be the "product $\underline{T}\ \underline{T}'$ of $\underline{T}$ and $\underline{T}'$ ".

Proof: Let $\{V_b$, b in B$\}$ and $\{W_b$, b in B$\}$ be completely admissable refinements of the coordinate covering of X such that W_b contains $Cl(V_b)$. Let the coordinates of $\underline{T}$ in V_b be given by the functions $f^b{}_{i_1 \ldots i_m}$ and the coordinates of $\underline{T}'$ be given by the functions $g^b{}_{i_{m+1} \ldots i_{m+m'}}$. To every b we assign the $(m+m')^2$ functions

$$h^b{}_{i_1 \ldots i_{m+m'}} = f^b{}_{i_1 \ldots i_m}\, g^b{}_{i_{m+1} \ldots i_{m+m'}} .$$

From (5.22) we can show that on $V_b \cap V_{b'}$

$$h^b{}_{i_1 \ldots i_{m+m'}} = \sum_{j_1 \cdots j_m} (\partial x_{\phi(b')}{}^{j_1} / \partial x_{\phi(b)}{}^{i_1}) \cdots \\ \cdots (\partial x_{\phi(b')}{}^{j_{m+m'}} / \partial x_{\phi(b)}{}^{i_{m+m'}})\, h^{b'}{}_{j_1 \ldots j_{m+m'}} .$$

From (5.23) there is a covariant tensor $\underline{Q}$ such that

$$\underline{Q} = h^b{}_{i_1 \ldots i_{m+m'}}\, G_{\phi(b)}{}^{i_1 \cdots i_{m+m'}}$$

on V_b for every b in B. By using (5.19) and the definition of the gradient tensor one can show that

$$\underline{Q}(LL') = \underline{T}(L) \cdot \underline{T}'(L')$$

for every contravariant tensor L of order m and L' of order m'.

We will now show that this tensor $\underline{Q}$ is unique.

Let the contravariant tensors $N^b{}_{i_1 \ldots i_k}$ be defined as in (5.6). From (5.7) and (5.10) any contravariant tensor L" of order m+m' is equal to the tensor $L''_b = \sum_{i_1 \cdots i_{m+m'}} L(x_a{}^{i_1}, \ldots, x_a{}^{i_{m+m'}}) \, N^b{}_{i_1 \ldots i_m} N^b{}_{i_{m+1}, \ldots i_{m+m'}}$ on V_b where $a = \phi(b)$. Suppose $\underline{Q}$ and $\underline{Q}'$ are covariant tensors of order m+m' such that

$$\underline{Q}(LL') = \underline{Q}'(LL')$$

for every L of order m and L' of order m', then it follows that

$$\underline{Q}L''_b = \underline{Q}' \, L''_b$$

for every b, i.e. $\underline{Q} = \underline{Q}'$ on every V_b, i.e. $\underline{Q} = \underline{Q}'$ on all X.

q.e.d.

We observe that the product of two covarianttensors is not always commutative.

(6.8) <u>Definition</u>: We define a covariant poly-tensor on an infinitely differentiable manifold with underlying space X to be a formal sum $\sum_{m=0}^{\infty} \underline{T}_m$ where $\underline{T}_m$ is either a covariant tensor of order m or the zero operator and almost all $\underline{T}_m$ are the zero operator. By a covariant tensor of order zero we mean any non-zero element of $C^{\infty}(X)$. We define the sum of two covariant poly-tensors $\sum_m \underline{T}_m$ and $\sum_m \underline{T}'_m$ to be the poly-tensor $\sum_m (\underline{T}_m + \underline{T}'_m)$. The product $(\sum_m \underline{T}_m)(\sum_m \underline{T}'_m)$ of these two poly-tensors is the poly-tensor $\sum_m \underline{T}''_m$ such that:

$$\underline{T}''_0 = \underline{T}_0 \, \underline{T}'_0$$

$$\underline{T}''_1 = \underline{T}_0 \, \underline{T}'_1 + \underline{T}_1 \, \underline{T}'_0$$

$$\underline{T}''_2 = \underline{T}_0 \, \underline{T}'_2 + \underline{T}_1 \, \underline{T}'_1 + \underline{T}_2 \, \underline{T}'_0$$

$\vdots$

etc.

By $\underline{T}\,f$ we mean $f\,\underline{T}$.

By the lower degree of a covariant poly-tensor $\sum_m \underline{T}_m$ we mean the order of the non-zero $\underline{T}_m$ of lowest order.

We denote the set of all covariant poly-tensors on X by $\underline{T}(X,0)$ and the set of all covariant poly-tensors whose lower degree is not less than k by $\underline{T}(X,k)$.

(6.9) Theorem: If X is the underlying space of an infinitely differentiable manifold, then the sets $\underline{T}(X,k)$ for $k=0,1,..$ are rings. If $k \leq k'$, then $\underline{T}(X,k')$ is an ideal in $\underline{T}(X,k)$.

The proof follows directly from the definitions.

(6.10) Definition: Let X be the underlying space of an infinitely differentiable manifold X. We say that a covariant poly-tensor $(\sum_m \underline{T}_m)$ "vanishes at a point p of X" if every $\underline{T}_m$ vanishes at p. By $Z(\sum_m \underline{T}_m)$ we mean the set of all p at which $(\sum_m \underline{T}_m)$ vanishes.

(6.11) Theorem: The product of two covariant poly-tensors $\sum_m \underline{T}_m$ and $\sum_m \underline{T'}_m$ of $\underline{T}(X,0)$ vanishes at a point p of X if and only if at least one of the two given poly-tensors vanishes at p. Hence

$$Z((\sum_m \underline{T}_m)(\sum_m \underline{T'}_m)) = Z(\sum_m \underline{T}_m) \cup Z(\sum_m \underline{T'}_m).$$

Proof: Suppose $\underline{T}$ and $\underline{T'}$ are covariant tensors of orders m and m' which do not vanish at p, then for some contravariant tensors L and L' the functions $\underline{T}(L)$ and $\underline{T'}(L')$ do not vanish at p. Hence $(\underline{T}\,\underline{T'})(LL')$ does not vanish at p and $\underline{T}\,\underline{T'}$ does not vanish at p.

If neither of two functions vanish at p, then their product does not. If neither a function nor a covarianttensor vanish at p, then their product does not. On the other hand if a covariant tensor or a function vanishes at p, then it is easily shown that its product with a function or a covariant tensor must vanish at p.

Suppose neither of two covariant poly-tensors $\sum_m \underline{T}_m$ and $\sum_m \underline{T}'_m$ vanish at p. Let $\underline{T}_k$ and $\underline{T}'_{k'}$ be the terms of lowest order which do not vanish at p, then the k+k' term of the product $(\sum_m \underline{T}_m)(\sum_m \underline{T}'_m)$ is $\underline{T}_k\, \underline{T}'_{k'}$ plus terms which vanish at p. Hence the k+k' term of the product does not vanish at p and the product of these two poly-tensors does not vanish at p.

On the other hand if one of the two poly-tensors vanishes at p. then the product will also.

q.e.d.

Since for every compact subset of the underlying space of a manifold there is a covariant tensor of order 2k which does not vanish on this set, then using (6.11) we can show in the same manner as we did for contravariant poly-tensors that:

(6.12) <u>Theorem:</u> If X is the underlying space of an infinitely differentiable manifold, then for every k the set $\underline{T}(X,k)^*$ of all elements of $\underline{T}(X,k)$ with compact supports is a ring and is compact associated with X. If X is compact, then $\underline{T}(X,k)$ is associated with X.

(6.13) <u>Theorem:</u> If X and Y are the underlying spaces of two infinitely differentiable manifolds and if for some k the rings $\underline{T}(X,k)^*$ and $\underline{T}(Y,k)^*$ are isomorphic, then the spaces X and Y are homeomorphic.

If X and Y are compact and if $\underline{T}(X,k)$ and $\underline{T}(Y,k)$ are isomorphic for some k, then the spaces X and Y are homeomorphic.

Again we observe that if k is greater than zero, then the rings $\underline{T}(X,k)$ and $\underline{T}(X,k)^*$ do not contain elements of $C^{\infty}(X)$.

In an m-differentiable manifold a contravariant vector (or vector field) may be defined as a transformation $L:C^m \to C^{m-1}$ such that

$$L(f)=\sum_i (\partial f/\partial x^i)L(x^i)$$

in every coordinate neighborhood where C^k, $(k \leq m)$, is the ring of k-fold continuously differentiable functions on the manifold. Higher order tensors may be defined in a similar way. One may show that every pair of disjoint closed sets has a characteristic function in C^m. One may also show that for every compact set and every positive integer k there is a contravariant tensor of order 2k which does not vanish on this set.

After defining the rings of contravariant poly-tensors, then one can show that for any k the ring of all poly-tensors with lower degree not less than k and with compact supports determines the topology of the manifold. If the manifold is compact, then its topology is determined by the ring of all poly-tensors of lower degree not less than k for any k.

A similar theorem for covariant tensors on an m-differentiable manifold can be given.

Chapter 7. The Lie Ring of Tangent Vectors and The Grassman Rings

If L and L' are tangent vectors (i.e. contravariant tensors of order one) on an infinitely differentiable manifold with underlying space X, then the transformation L*L': $C^{\infty}(X) \rightarrow C^{\infty}(X)$ defined by

$$L*L'(f) = L(L'(f))$$

is not a tangent vector, but the operator L*L'-L'*L which we denote by (L', L) is a tangent vector. We call (L', L) the "Lie product" of L and L'.

This product is not associative, but it is distributive with respect to addition and satisfies the following identities:

$$(L,L)=0,$$

$$((L,L'),L'')+((L',L''),L)+((L'',L),L')=0,$$

and

$$(L,L')=-(L',L).$$

These results are proved by Chevalley [1, pp. 83-84] for analytic manifolds. The proofs are the same for infinitely differentiable manifolds.

The set of all tangent vectors with the operations of addition and Lie multiplication form a non-associative ring, "the Lie ring of tangent vectors".

The question of whether or not this ring determines the topology of the underlying space is now being investigated.

Another algebraic object constructed on an infinitely differentiable manifold is the Grassman ring of contravariant tensors. In the remainder of the chapter by L_m, L'_m, etc. we mean contravariant tensors of order m. By P(n) we mean the ~~set~~ group of all permutations of the first n integers. By w we mean a representative of P(n). By w(1) we mean w acting on 1. The function e(w) is a function on P(n) which is 1 if w is an even permutation and -1 if w is an odd permutation, then $e(ww') = e(w)e(w')$. We also define the following operators:

(7.1) Definition: The operator A_m^w is a transformation of the set of all contravariant tensors of order m into itself such that if $L'_m = A_m^w L_m$, then

$$L'_m(f^1,\ldots,f^m) = L_m(f^{w(1)},\ldots,f^{w(m)}).$$

It is easily shown that L'_m is actually a contravariant tensor. The operator A_m which we call the "alternator" is a transformation of the set of all m-order contravariant tensors into itself such that

$$A_m = (1/m!)\sum_{w \in P(n)} e(w)\ A_m^w.$$

The operator A is a transformation of the ring $(C^\infty(X); R,1)$ of all poly-tensors of lower degree not less than one into itself such that

$$A(\Sigma_m L_m) = \Sigma_m A_m L_m.$$

We observe that if m is one, then A_m is the identity operator. One may also show that:

(7.2) Theorem: The set J of all poly-tensors $\Sigma_m L_m$ in $(C^\infty(X); R,1)$ such that $A(\Sigma_m L_m) = 0$ is an ideal in $(C^\infty(X); R,1)$.

For a proof see Chevalley [1, pp. 142-143]. The objects which Chevalley discusses are not contravariant poly-tensors in our sense, but the proof is valid since it depends onthe theory of permutation groups and not on the objects being permuted.

(7.3) <u>Definition:</u> The residue class ring $(C^{\infty}(X); R,1)/J$ is the "Grassman ring of contravariant tensors on X".

If $\underline{T}$ is a covariant tensor of order m, then the operator $A_m^W(\underline{T}) = \underline{T}'$ defined by

$$\underline{T}'(L_m) = \underline{T}(A_m^W(L_m))$$

is a covariant tensor of order m. In a manner similar to that used for contravariant tensors one obtains the "Grassman ring of covariant tensors" as the ring $\underline{T}(X,1)/J$ where J is the ideal consisting of all $\sum_m \underline{T}_m$ such that $A(\sum_m \underline{T}_m)$ vanishes.

Whether or not the Grassman rings on an infinitely differentiable manifold determine the underlying space has not been determined.

Chapter 8. Construction of the Underlying Space of a Manifold from the Ring of Infinitely Differentiable Functions on the Manifold

In chapter 4 we showed that the ring of infinitely differentiable functions on a manifold is associated with underlying space; consequently this ring determines the underlying space if it is compact. In this chapter we will show that the underlying space of a manifold can be constructed from the ring of infinitely differentiable functions even if the space is not compact.

In this chapter by X we will mean the underlying space of some infinitely differentiable manifold unless we specify otherwise.

(8.1) Definition: If J is any ideal in the ring $C^{\infty}(X)$, then by "Z(J)" we mean the intersections of all sets Z(f) for which f is in J. An ideal J is a "fixed ideal" if the set Z(J) is not empty. An ideal which is not fixed is a "free ideal".

If the space X is compact, then all proper ideals in $C^{\infty}(X)$ are fixed ideals, and the ideal J is maximal if and only if the set Z(J) consists of a single point. If the space X is not compact, then there are free proper ideals. However, we will be able to determine algebraically the fixed ideals in $C^{\infty}(X)$. From these fixed ideals we will construct the space.

(8.2) Lemma: If p is any point of the space X, there is a function f in $C^{\infty}(X)$ which vanishes at p and only at p.

Proof: In (4.13) we showed that if I is a bounded interval in Euclidean space and if I' is an interval whose closure is contained in I, then there is an infinitely differentiable function F with continuous derivatives of all orders which is one on I' but vanishes on the complement of I. The function which was constructed is also less than one on I-I'. The construction of this function is still valid if the interval I' is replaced by a single point p in I, then the function F is 1 at p, less than 1 at any other point, and vanishes on the complement of I.

With this function F, given any point p in X, we can construct a function g in $C^{\infty}(X)$ which is 1 at p but is less that 1 at all other points.

The function $f=(1-g)$ is the desired function.

q.e.d.

(8.3) Lemma: If a function f in $C^{\infty}(X)$ does not vanishat any point of a closed set F, then there is a function g in $C^{\infty}(X)$ such that $f\cdot g$ is equal to 1 on F.

Proof: Since the space X is normal (4.18), the disjoint sets F and Z(f) have disjoint neighborhoods U(F) and U(Z(f)) with disjoint closures. By (4.18), there is a function h which is 1 on Cl(U(F)) and vanishes on Cl(U(Z(f))). Define a function g on X by:

$$g=\begin{cases}(1/f)h & \text{on } X-Z(f)\\ 0 & \text{on } Cl(U(Z(f))).\end{cases}$$

It is easily shown that g is in $C^{\infty}(X)$, and $f\cdot g$ is one on F.

q.e.d.

(8.4) Lemma: If f and g are functions in $C^{\infty}(X)$, then f does not vanish at any point of S(g), i.e. Z(f) and S(g) are disjoint, if and only if the residue class f-A(g) has an inverse in the residue class ring $C^{\infty}(X)$-A(g) where A(g) is the annihilator of g.

Proof: The ideal $A(g)$ consists of all functions which vanish on $X-Z(g)$ hence on $S(g)$. If f does not vanish at any point of $S(g)$, then there is a function h in $C^{\infty}(X)$ such that $f\cdot h$ is 1 on $S(g)$, i.e. $(f\cdot h-1)$ is in $A(g)$. Therefore $f-A(g)$ has an inverse in $C^{\infty}(X)-A(g)$.

Conversely, if $f-A(g)$ has an inverse in $C^{\infty}(X)-A(g)$, then there is a function h in $C^{\infty}(X)$ such that $(f\cdot h-1)$ is in $A(g)$, i.e. $f\cdot h-1$ vanishes at every point of $S(g)$. Therefore f does not vanish at any point of $S(g)$.

q.e.d.

(8.5) Definition: If f is in $C^{\infty}(X)$, then by $H(f)$ we mean the set of all functions g in $C^{\infty}(X)$ such that $f-A(g)$ has an inverse in the ring $C^{\infty}(X)-A(g)$. We order the functions of $C^{\infty}(X)$ as follows:

$f \prec h$ if and only if $H(f) \supseteq H(h)$.

We say then that "f precedes h".

This ordering is reflexive but is not proper as we will see later.

If an element of $C^{\infty}(X)$ has an inverse, then in any residue class ring its residue class has an inverse. Hence:

(8.6) Lemma: If f is a unit (i.e. has an inverse) in $C^{\infty}(X)$, then $H(f)=C^{\infty}(X)$ and f precedes every g in $C^{\infty}(X)$. Since $C^{\infty}(X)$ has more than one unit, the ordering, $\prec$, is not a proper ordering.

(8.7) Definition: An ideal J in $C^{\infty}(X)$ is said to be "bounded" if there is a function f in $C^{\infty}(X)$ without an inverse in $C^{\infty}(X)$ such that f precedes every element h of J. By $K(X)$ we mean the set of all bounded ideals in $C^{\infty}(X)$. An ideal is a "bounded maximal ideal" if it is in $K(X)$ and is not properly contained by another ideal in $K(X)$.

By X' we mean the space (which might be empty) whose points are the bounded maximal ideals of K(X). A set F' in X' is closed if there is an ideal J in K(X) such that every ideal in F' contains J, but no bounded maximal ideal not in F' contains J.

The space X' which we defined above is determined solely by the algebraic structure of $C^{\infty}(X)$, hence:

(8.8) Lemma: If X and Y are the underlying spaces of two infinitely differentiable manifolds and if the rings $C^{\infty}(X)$ and $C^{\infty}(Y)$ are isomorphic, then the spaces X' and Y' are homeomorphic.

We will now show that the spaces X and X' are homeomorphic.

(8.9) Lemma: If f and g are two functions in $C^{\infty}(X)$, then

$$Z(f) \subseteq Z(g) \quad \text{if and only if} \quad f \prec g.$$

Proof: From lemma (8.4) and definition (8.5) we see that H(f) is the set of all functions h in $C^{\infty}(X)$ such that Z(f) and S(h) are disjoint.

If $Z(f) \subseteq Z(g)$, then $H(f) \supseteq H(g)$ and $f \prec g$.

Suppose there is a point p in Z(f) but not in Z(g), then there is a function h in $C^{\infty}(X)$ which is one at p but which vanishes on a neighborhood of Z(g). For this function h, Z(f) meets S(h) but Z(g) does not; hence h is in H(g) but not in H(f) and f does not precede g. Therefore if f precedes g, then Z(g) contains Z(f).

q.e.d.

A function f in $C^{\infty}(X)$ has no inverse in $C^{\infty}(X)$ if and only if Z(f) is not empty. Hence as a consequence of (8.9) we have:

(8.10) Lemma: The fixed ideals of the ring $C^{\infty}(X)$ are the bounded ideals. Since there are functions in $C^{\infty}(X)$ which vanish only at a given point, a bounded maximal ideal is the set of all functions which vanish at a point.

(8.11) <u>Theorem</u>: If X is the underlying space of an infinitely differentiable manifold, then the space X and the space X' as defined in (8.7) are homeomorphic.

<u>Proof</u>: The transformation $T: X \to X'$ which assigns to each point p of X the ideal consisting of all functions which vanish at p is one to one and onto. If F is a closed set of X and if I is the ideal consisting of all functions which vanish on F, then $Z(I) = F$ and T(F) is the set of all bounded maximal ideals which contain I. Hence T(F) is closed and the transformation T is closed.

Suppose F' is a closed set in X'. The set F' consists of all bounded maximal ideals which contain a bounded ideal I, then $Z(I) = T^{-1}(F')$ which is closed. Therefore T^{-1} is a closed transformation.

q.e.d.

As a consequence of (8.8) and (8.11),

(8.12) <u>Theorem</u>: If X and Y are the underlying spaces of two infinitely differentiable manifolds such that $C^{\infty}(X)$ and $C^{\infty}(Y)$ are isomorphic, then X and Y are homeomorphic.

Suppose X is the underlying space of an m-differentiable manifold and $C^m(X)$ is the set of all continuous functions on X which have all derivatives with respect to the admissable coordinates of X up to and including the m-th order and such that all of these derivatives are continuous. By methods essentially the same as we used for infinitely differentiable functions one can show that:

(i) Every pair of disjoint closed sets in X has a characteristic function in $C^m(X)$.

(ii) The ring $C^m(X)$ is associated with the space X.

(iii) Given any point p of X, there is a function in $C^m(X)$ which vanishes at p and only at p.

With these properties we can, as we did above for infinitely differentiable manifolds, show that:

(8.13) <u>Theorem</u>: If X is the underlying space of an m-differentiable manifold, then the ring $C^m(X)$ determines the space X.

We have observed that if X is a normal T_1 space, then the ring R(X) of all continuous functions on X is associated with X. The first countable spaces are the normal T_1 spaces which have the property that given any point p of X there is a continuous function on X which vanishes at p and only at p. By the same method of proof as we used for infinitely differentiable manifolds one can show:

(8.14) <u>Theorem</u>: If X is a first countable normal T_1 (hence Hausdorff) space, then the ring R(X) of continuous real functions on X determines the space.

Hewitt [3] has defined a hyper-real ideal as a maximal ideal I in R(X) such that the residue class ring R(X)-I is not isomorphic to the real numbers R but contains R as a proper subring. He defines a Q space as completely regular space X such that every maximal free ideal is hyper-real. He then shows that the ring R(X) determines the completely regular space X if and only if X is a Q space. He also shows that any second countable completely regular space is a Q space. From (8.14) we obtain:

(8.15) <u>Theorem</u>: Any first countable normal T_1 space is a Q space in the sense of Hewitt.

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