Stacks, Gerbes and T-Duality

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Theorem (well-known)

For G a Lie group, principal G-bundles form a stack $\mathcal{B}un_G$ over the site of smooth manifolds.

Definition

A *stack* is a sheaf of categories, i.e. a presheaf of categories satisfying descent.

- Presheaf of categories: weak functor $\mathcal{F} : \mathcal{M}an^{op} \to \mathcal{C}at$.
- Grothendieck topology on $\mathcal{M}an$: surjective submersions
- Descent data for $\pi : Y \to M$: pair (X, g) with
  1. X is an object in $\mathcal{F}(Y)$
  2. $g : \mathrm{pr}_1^* X \to \mathrm{pr}_2^* X$ is an isomorphism in $\mathcal{F}(Y \times_M Y)$
  3. g satisfies the cocycle condition in $\mathcal{F}(Y \times_M Y \times_M Y)$
- Descent: for every surjective submersion $\pi : Y \to M$, the functor
$$\mathcal{F}(M) \to \mathcal{D}esc_\pi(\mathcal{F}) : X \mapsto (\pi^* X, \mathrm{id})$$
is an equivalence of categories.

Theorem (well-known)

For G a Lie group, principal G-bundles form a stack $\mathcal{B}un_G$ over the site of smooth manifolds.

Proof. $\mathcal{B}un_G$ is the stackification of the presheaf $\underline{\mathbb{B}G}$ represented by the Lie groupoid $\mathbb{B}G$,

$$\mathcal{B}un_G \cong \underline{\mathbb{B}G}^+.$$

□

Example

Lie groupoids:

- $\mathbb{B}G = \begin{pmatrix} G \\ \downarrow\downarrow \\ * \end{pmatrix}$, for G any Lie group.
- $M_{dis} = \begin{pmatrix} M \\ \downarrow\downarrow \\ M \end{pmatrix}$, for M any smooth manifold
- $M /\!\!/ H = \begin{pmatrix} H \times M \\ \downarrow\downarrow \\ M \end{pmatrix}$, for H acting smoothly on M.

Notice that $\mathbb{B}G = * /\!\!/ G$ and $M_{dis} = M /\!\!/ *$.

Some details of the proof:

- Every Lie groupoid Γ represents a presheaf $\underline{\Gamma}$ of categories:

$$\underline{\Gamma}(M) := \left(\begin{array}{c} C^\infty(M, \Gamma_1) \\ \downarrow\downarrow \\ C^\infty(M, \Gamma_0) \end{array} \right).$$

- Every presheaf $\mathcal{F}$ can be sheafified via the *Grothendieck plus construction*, to yield a stack $\mathcal{F}^+$.

- The objects of the category $\mathcal{F}^+(M)$ are triples (π, X, g) with
  – $\pi : Y \to M$ a surjective submersion
  – (X, g) a descent object for π

- The claim
$$\mathcal{B}un_G(M) \cong \underline{\mathbb{B}G}^+(M)$$
is now the usual correspondence between G-bundles and their transition functions.

Theorem (Stevenson [Ste00])

$\mathrm{U}(1)$*-bundle gerbes form a 2-stack* $\mathcal{B}dl\mathcal{G}rb_{\mathrm{U}(1)}$ *over the site of smooth manifolds.*

- New proof (Nikolaus-Schweigert [NS11]):

$$\mathcal{B}dl\mathcal{G}rb_{\mathrm{U}(1)} = (\mathbb{B}\mathcal{B}un_{\mathrm{U}(1)})^+ \cong \underline{\mathbb{B}\mathbb{B}\mathrm{U}(1)}^+.$$

- Hidden: monoidal structure of $\mathcal{B}un_{\mathrm{U}(1)}$ is used in order to form the presheaf $\mathbb{B}\mathcal{B}un_{\mathrm{U}(1)}$.
- Main problem with defining *non-abelian bundle gerbes*: $\mathcal{B}un_G$ is not monoidal unless G is abelian.

  In terms of the representing groups: there is no functor

$$\mathbb{B}G \times \mathbb{B}G \longrightarrow \mathbb{B}G$$

  making $\mathbb{B}G$ a monoid, unless G is abelian (Eckmann-Hilton).

Definition

A *Lie 2-group* is a Lie groupoid Γ together with smooth functors

$$m : \Gamma \times \Gamma \to \Gamma \quad \text{and} \quad i : \Gamma \to \Gamma$$

that satisfy the axioms of an associative unital binary operation and of a corresponding inversion, either strict or in a coherent way.

Example

- $\mathbb{B}G$ is a Lie 2-group if and only if G is abelian.
- M_{dis} is a Lie 2-group if and only if M is a Lie group.

Example

A *crossed module* of Lie groups is a Lie group homomorphism

$$t : H \to G$$

and a smooth action of G on H by Lie group automorphisms, such that

$$t(g \triangleright h) = gt(h)g^{-1} \quad \text{and} \quad t(h) \triangleright x = hxh^{-1}.$$

- We obtain a Lie 2-group

$$\begin{pmatrix} H \ltimes G \\ \downarrow\downarrow \\ G \end{pmatrix}$$

- The underlying Lie groupoid is $G /\!\!/ H$, where H acts on G by multiplication along t.
- Up to equivalence, every (strict) Lie 2-group is of this form (Brown-Spencer [BS76]).

Definition

Let Γ be a Lie groupoid. A *principal Γ-bundle* over a smooth manifold M consists of:

- a total space P with a surjective submersion $P \to M$
- a smooth anchor map $\phi : P \to \Gamma_0$
- $\gamma \in \Gamma$ acts on any $p \in P$ with $\phi(p) = s(\gamma)$, and $\phi(p\gamma) = t(\gamma)$.
- action preserves projection and is fibrewise free and transitive.

Lemma

Principal Γ-bundles over M form a stack $\mathcal{B}un_\Gamma$.

Proof. $\mathcal{B}un_\Gamma \cong \underline{\Gamma}^+$. □

Lemma

If Γ is a Lie 2-group, then $\mathcal{B}un_\Gamma$ is a monoidal stack.

Proof. $\Gamma \times \Gamma \to \Gamma$ induces $\mathcal{B}un_\Gamma \times \mathcal{B}un_\Gamma \to \mathcal{B}un_\Gamma$. □

Definition

Let Γ be a Lie 2-group. Then, the 2-stack of Γ-bundle gerbes is

$$\mathcal{B}dl\mathcal{G}rb_\Gamma := (\mathbb{B}\mathcal{B}un_\Gamma)^+ \cong \underline{\mathbb{B}\Gamma}^+ .$$

- $\Gamma = \mathbb{B}\mathrm{U}(1)$ gives $\mathcal{B}dl\mathcal{G}rb_{\mathrm{U}(1)}$ and $\Gamma = G_{dis}$ gives $\mathcal{B}un_G$.
- Concrete cocycle description: consider the objects of $\underline{\mathbb{B}\Gamma}^+(M)$ with respect to $Y := \coprod U_i$, for an open covering $U_i \subseteq M$:

  $$g_{ij} : U_i \cap U_j \to \Gamma_0 \quad , \quad \gamma_{ijk} : U_i \cap U_j \cap U_k \to \Gamma_1$$

  with

  $$\gamma_{ijk} : g_{jk} g_{ij} \to g_{ik} \quad \text{and} \quad \gamma_{ikl} \circ (\mathrm{id}_{g_{kl}} \cdot \gamma_{ijk}) = \gamma_{ijl} \circ (\gamma_{jkl} \cdot \mathrm{id}_{g_{ij}}).$$

- Classification by Giraud's non-abelian cohomology:

  $$\pi_0(\mathcal{B}dl\mathcal{G}rb_\Gamma(M)) \cong \pi_0(\underline{\mathbb{B}\Gamma}^+) \cong [M, B|\Gamma|] \cong \mathrm{H}^1(M, \Gamma).$$

Facts about Lie 2-groups:

- Every Lie 2-group Γ determines two groups:

  $\pi_0\Gamma :=$ the group of isomorphism classes of objects
  $\pi_1\Gamma := \mathrm{Aut}(1)$; this is always abelian

- There is an action of $\pi_0\Gamma$ on $\pi_1\Gamma$:

  $$\gamma \cdot [g] := [\mathrm{id}_g \cdot \gamma \cdot \mathrm{id}_{\gamma^{-1}}].$$

  If this action is trivial we say that Γ is *central*.

- For a crossed module $t : H \to G$, we have

  $$\pi_0\Gamma = G/t(H) \quad \text{and} \quad \pi_1\Gamma = \mathrm{Kern}(t) \subseteq H$$

  and the action is induced by the action of G on H.

Example

Let H be a Lie group. Consider the crossed module

$$i : H \to \mathrm{Aut}(H)$$

where $\mathrm{Aut}(H)$ acts on H by evaluation. The corresponding Lie 2-group is the automorphism 2-group of H, denoted $\mathbb{A}\mathrm{ut}(H)$.

- $\pi_0 \mathbb{A}\mathrm{ut}(H) = \mathrm{Out}(H)$ and $\pi_1 \mathbb{A}\mathrm{ut}(H) = Z(H)$.
- $\mathbb{A}\mathrm{ut}(H)$-bundles can be seen as H-bibundles, with the additional left action

$$h \cdot p := p \cdot \phi(p)(h)^{-1}.$$

The theory of non-abelian (bundle) gerbes started with the case of $\Gamma = \mathbb{A}\mathrm{ut}(H)$ in work of Breen-Messing [BM05], Aschieri-Cantini-Jurco [ACJ05], Laurent-Gengoux [CLGX09],...

Example

We consider the automorphism 2-group of $\mathrm{U}(1)$:

- $\pi_0 \mathbb{A}\mathrm{ut}(\mathrm{U}(1)) = \mathbb{Z}_2$ and $\pi_1 \mathbb{A}\mathrm{ut}(\mathrm{U}(1)) = \mathrm{U}(1)$
- π_0 acts on π_1 by inversion; thus, $\mathbb{A}\mathrm{ut}(\mathrm{U}(1))$ is *not* central.

- An $\mathbb{A}\mathrm{ut}(\mathrm{U}(1))$-bundle is a *graded* $\mathrm{U}(1)$*-bundle* $P \to M$, i.e. it is equipped with map $\epsilon : M \to \mathbb{Z}_2$.
- An $\mathbb{A}\mathrm{ut}(\mathrm{U}(1))$-bundle gerbe consists of graded $\mathrm{U}(1)$-bundles and a twisted bundle gerbe product,

$$\mathrm{pr}_{12}^* P \otimes \mathrm{pr}_{23}^* P^{\epsilon_{12}} \to \mathrm{pr}_{13}^* P.$$

Theorem (Mertsch [Mer20])

The K-theory twistings of Freed-Hopkins-Teleman are the 1-truncation of the stack $\mathcal{B}dl\mathcal{G}rb_{\mathbb{A}\mathrm{ut}(\mathrm{U}(1))}$.

Every Lie 2-group can be seen as an extension

$$\mathbb{B}\pi_1\Gamma \longrightarrow \Gamma \longrightarrow (\pi_0\Gamma)_{dis}$$

of Lie 2-groups, which is central if and only if Γ is central.

Theorem (Schommer-Pries [SP11])

Central extensions

$$\mathbb{B}U \longrightarrow \Gamma \longrightarrow K_{dis}$$

are classified by the smooth group cohomology $\mathrm{H}^3_{sm}(BK, U)$.

If $U = \mathrm{U}(1)$, this is cohomology group is just $\mathrm{H}^4(BK, \mathbb{Z})$.

Consider a central Lie 2-group Γ with central extension

$$\mathbb{B}\mathrm{U}(1) \to \Gamma \to (\pi_0\Gamma)_{dis}$$

- Via the Lie 2-group homomorphism $\Gamma \to (\pi_0\Gamma)_{dis}$, every Γ-bundle gerbe $\mathcal{G}$ determines an underlying $\pi_0\Gamma$-bundle $\pi_0\mathcal{G}$.
- Conversely, given a $\pi_0\Gamma$-bundle P, one can ask for liftings of P to a Γ-bundle gerbe $\mathcal{G}$, so that $P \cong \pi_0\mathcal{G}$.

There is a 2-group-theoretical analog to the lifting gerbe theory of Murray [Mur96]. The lifting 2-gerbe is well-known: it is the *Chern-Simons 2-gerbe* $\mathbb{CS}_\Gamma(P)$ associated to the "level" of Γ in $\mathrm{H}^4(B\pi_0\Gamma, \mathbb{Z})$ and the bundle P, as described by Carey-Johnson-Murray-Stevenson-Wang [CJM+05].

Theorem (Nikolaus-KW [NW13])

Let P be a principal $\pi_0\Gamma$-bundle. A lift of P to a Γ-bundle gerbe is the same as a trivialization of $\mathbb{CS}_\Gamma(P)$.

Example

- The *string group* is an (infinite-dimensional) Lie 2-group constructed by Baez-Crans-Schreiber-Stevenson [BCSS07]. It sits in a central extension

$$\mathbb{B}\mathrm{U}(1) \to \mathrm{String}(n) \to \mathrm{Spin}(n)_{dis}$$

  and is classified by $\frac{1}{2}p_1 \in \mathrm{H}^4(B\mathrm{Spin}(n), \mathbb{Z})$.

- A *string structure* on a spin manifold M is a lift of the spin-oriented frame bundle of M to a $\mathrm{String}(n)$-bundle gerbe.

- By the lifting theorem, this is the same as a trivialization of the Chern-Simons 2-gerbe, whose characteristic class is

$$\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z}).$$

  Hence, M admits string structures if and only if this class vanishes.

Example

- Let Λ be a free abelian group. We consider the vector space $\mathfrak{t} := \Lambda \times_{\mathbb{Z}} \mathbb{R}$ and the associated torus $T := \mathfrak{t}/\Lambda$, and a bilinear form
$$J : \Lambda \times \Lambda \to \mathbb{Z}.$$
- We define a crossed module by
$$\mathrm{U}(1) \times \Lambda \to \mathfrak{t} : (z, m) \mapsto m \otimes_{\mathbb{Z}} \mathbb{R},$$
with $\mathfrak{t}$ acting on $\mathrm{U}(1) \times \Lambda$ by $x \triangleright (z, m) := (z \cdot e^{2\pi \mathrm{i} J(x,m)}, m)$.
- The associated Lie 2-group $\mathbb{T}_J$ is called a *categorical torus*; it is a central extension
$$\mathbb{B}\mathrm{U}(1) \to \mathbb{T}_J \to T_{dis}.$$

Theorem ([Gan18])

$\mathbb{T}_J$ is classified by $J^{tr} + J \in S^2(\Lambda) \cong \mathrm{H}^4(BT, \mathbb{Z})$.

We consider the categorical torus associated to the bilinear form

$$J : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to \mathbb{Z}$$

with matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathbb{Z}^{2n \times 2n}.$$

The class of $J^{tr} + J$ in $\mathrm{H}^4(BT^{2n}, \mathbb{Z})$ is the *Poincaré class*,

$$P_n := \sum_{i=1}^{n} c_i \cup c_{i+n},$$

where c_i are the $2n$ generators (the Chern-classes).

Theorem (Nikolaus-KW [NW])

The categorical torus $\mathbb{TD} := \mathbb{T}_J$ *represents the 2-stack of T-duality correspondences:*

$$\mathrm{T\text{-}Corr} \cong \mathcal{B}dl\mathcal{G}rb_{\mathbb{TD}} \cong \underline{\mathbb{BTD}}^{+}.$$

A *correspondence* over M consists of:

- T-backgrounds $(E_1, \mathcal{G}_1)$ and $(E_2, \mathcal{G}_2)$, where $E_i \to M$ is a principal T^n-bundle, and $\mathcal{G}_i$ is a $\mathrm{U}(1)$-bundle gerbe over E_i.
- an isomorphism $\mathcal{D} : \mathrm{pr}_1^*\mathcal{G}_1 \to \mathrm{pr}_2^*\mathcal{G}_2$ over the correspondence space

$$\begin{array}{ccccc} & & E_1 \times_M E_2 & & \\ & \overset{\mathrm{pr}_1}{\swarrow} & & \overset{\mathrm{pr}_2}{\searrow} & \\ E_1 & & & & E_2 \\ & \searrow & & \swarrow & \\ & & M & & \end{array}$$

Lemma

Correspondences form a 2-stack Corr.

The *standard T-duality correspondence* $\mathcal{T}_X$ over X is between the trivial backgrounds $(M \times T^n, \mathcal{I})$, and the isomorphism over the correspondence space $M \times T^n \times T^n$ is given by the n-fold Poincaré bundle P, whose first Chern class is

$$J - J^{tr} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(2n, \mathbb{Z}) \cong \mathrm{H}^2(T^{2n}, \mathbb{Z}).$$

Definition

A correspondence $\mathcal{C}$ is called *T-duality correspondence*, if there exist isomorphisms $\mathcal{C}|_x \cong \mathcal{T}_x$ of correspondence diagrams, for each $x \in M$.

Lemma

T-duality correspondences form a sub-2-stack $\mathrm{T\text{-}Corr} \subseteq \mathrm{Corr}$.

Theorem (Nikolaus-KW [NW])

The categorical torus $\mathbb{TD} := \mathbb{T}_J$ represents the 2-stack of T-duality correspondences:

$$\mathrm{T\text{-}Corr} \cong \mathcal{B}dl\mathcal{G}rb_{\mathbb{TD}} \cong \underline{\mathbb{BTD}}^+ .$$

First proof – homotopy theory. Bunke-Rumpf-Schick [BRS06] computed a classifying space B for $\mathrm{T\text{-}Corr}$, namely the homotopy fibre of the map $\xi : BT^{2n} \to K(\mathbb{Z}, 4)$ that represents the Poincaré class $P_n \in \mathrm{H}^4(BT^{2n}, \mathbb{Z})$. On the other hand, the central extension

$$\mathbb{B}\mathrm{U}(1) \to \mathbb{TD} \to T^{2n}_{dis}$$

induces a fibre sequence

$$\ldots \longrightarrow K(\mathbb{Z}, 3) \longrightarrow B|\mathbb{TD}| \longrightarrow BT^{2n} \xrightarrow{\xi} K(\mathbb{Z}, 4) \longrightarrow \ldots$$

in which ξ appears, as P_n classifies $\mathbb{TD}$. Thus, $|\mathbb{BTD}| = B$. □

Theorem (Nikolaus-KW [NW])

The categorical torus $\mathbb{TD} := \mathbb{T}_J$ represents the 2-stack of T-duality correspondences:

$$\text{T-Corr} \cong \mathcal{B}dl\mathcal{G}rb_{\mathbb{TD}} \cong \underline{\mathbb{BTD}}^{+}.$$

Second proof – lifting theory. A $\mathbb{TD}$-bundle gerbe has an underlying $\pi_0\mathbb{TD} = T^{2n}$-bundle E, and it is a lift of E; hence, a trivialization of the Chern-Simons 2-gerbe $\mathbb{CS}_{\mathbb{TD}}(E)$. This, in turn, is a $\mathrm{U}(1)$-bundle gerbe $\mathcal{G}$ over E with extra structure: equivariant structures for the actions of $\{e\} \times T^n$ and $T^n \times \{e\}$. Consider

$$E_1 := E/_{\{e\}\times T^n} \quad \text{and} \quad E_2 := E/_{T^n\times\{e\}},$$

and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the descent bundle gerbes. We have $E = E_1 \times_M E_2$, and the isomorphism over E is

$$\mathrm{pr}_1^*\mathcal{G}_1 \cong \mathcal{G} \cong \mathrm{pr}_2^*\mathcal{G}_2.$$

This induces the equivalence $\mathcal{B}dl\mathcal{G}rb_{\mathbb{TD}}(M) \to \text{T-Corr}(M)$. □

Theorem (Nikolaus-KW [NW])

The categorical torus $\mathbb{TD} := \mathbb{T}_J$ represents the 2-stack of T-duality correspondences:

$$\mathrm{T\text{-}Corr} \cong \mathcal{B}dl\mathcal{G}rb_{\mathbb{TD}} \cong \underline{\mathbb{BTD}}^+.$$

Third proof – stack theory. We define a 2-functor

$$\underline{\mathbb{BTD}}(M) \to \mathrm{T\text{-}Corr}(M).$$

For objects, $* \mapsto \mathcal{T}_M$. A 1-morphism in $\underline{\mathbb{BTD}}(M)$ is a smooth map $M \to \mathbb{R}^{2n}$; each half $a_i : M \to \mathbb{R}^n$ induces an automorphism of the trivial T^n-bundle over M, together forming an automorphism of $\mathcal{T}_M$ with trivial bundle gerbe automorphism. This extends to a 2-functor. Since $\mathrm{T\text{-}Corr}$ is a 2-stack, the above 2-functor extends to a 2-stack morphism $\underline{\mathbb{BTD}}^+ \to \mathrm{T\text{-}Corr}$. By the T-duality condition, it is essentially surjective. □

We want to describe the left leg $(E_1, \mathcal{G}_1)$ and the right leg $(E_2, \mathcal{G}_2)$ of a T-duality correspondence in terms of Lie 2-groups.

From the third proof, we see that they satisfy the F_2 *condition*:

- they are locally trivial, i.e., locally isomorphic to $(M \times T^n, \mathcal{I})$
- the gluing is along identity bundle gerbe morphisms

Lemma

There is a strict, non-central Fréchet Lie 2-group $\mathbb{TB}^{\mathrm{F2}}$ *that classifies* F_2 *backgrounds; as a crossed module it is*

$$1 : C^\infty(T^n, \mathrm{U}(1)) \to T^n$$

where T^n *acts on* $C^\infty(T^n, \mathrm{U}(1))$ *by translation.*

Analogs of proofs 2 and 3 can be used to prove this lemma.

The projection to the left and right leg can be implemented as weak Lie 2-group homomorphisms

$$\mathbb{TB}^{\mathrm{F2}} \xleftarrow{\;lele\;} \mathbb{TD} \xrightarrow{\;rile\;} \mathbb{TB}^{\mathrm{F2}}$$

Theorem (Nikolaus-KW [NW])

These homomorphism induce bijections in non-abelian cohomology. In particular, every F_2 T-background has a T-dual.

This reproduces a result of Bunke-Rumpf-Schick [BRS06].

We want to consider a bigger class of backgrounds called F_1, which are only locally trivial. They do not have T-duals, these are "mysteriously missing".

Main idea: every F_1 background is locally F_2, and hence has locally defined T-duals and T-duality correspondences. We want to glue them to a global structure. Hence, we analyze the automorphisms of $\mathbb{TD}$.

Theorem (Nikolaus-KW, Ganter)

The automorphism 2-group of $\mathbb{TD}$ is a non-central extension

$$\mathbb{BZ}^{2n} \to \mathrm{Aut}(\mathbb{TD}) \to \mathrm{O}^{\pm}(n,n,\mathbb{Z}).$$

It splits canonically over the subgroup

$$\mathfrak{so}(n,\mathbb{Z}) \hookrightarrow \mathrm{O}(n,n,\mathbb{Z}) : B \mapsto \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

Thus, $\mathfrak{so}(n,\mathbb{Z})$ acts on $\mathbb{TD}$ by Lie 2-group homomorphisms.

Similarly, $\mathfrak{so}(n,\mathbb{Z})$ acts on $\mathbb{TB}^{\mathrm{F2}}$.

We form the semi-direct products

$$\mathbb{TB}^{\mathrm{F1}} := \mathbb{TB} \ltimes \mathfrak{so}(n,\mathbb{Z}) \quad \text{and} \quad \mathbb{TD}^{\frac{1}{2}geo} := \mathbb{TD} \ltimes \mathfrak{so}(n,\mathbb{Z}).$$

Lemma

The left leg $lele : \mathbb{TD} \longrightarrow \mathbb{TB}^{\mathrm{F2}}$ *is* $\mathfrak{so}(n,\mathbb{Z})$*-equivariant, and hence induces a homomorphism*

$$\mathbb{TD}^{\frac{1}{2}geo} \longrightarrow \mathbb{TB}^{\mathrm{F1}}.$$

The right leg does not extend to $\mathbb{TD}^{\frac{1}{2}geo}$, it is non-geometric. We call the objects represented by $\mathbb{TD}^{\frac{1}{2}geo}$ *half-geometric T-duality correspondences*.

Theorem (Nikolaus-KW [NW])

- $\mathbb{TB}^{\mathrm{F1}}$ *classifies* F_1 *backgrounds.*
- $\mathbb{TD}^{\frac{1}{2}geo} \longrightarrow \mathbb{TB}^{\mathrm{F1}}$ *induces a bijection in cohomology.*

In particular, every F_1-background is the left leg of a unique half-geometric T-duality correspondence!

Cocycle data of a half-geometric T-duality correspondence:

- data for two torus bundles: $a_{ij}, b_{ij} : U_i \cap U_j \to \mathbb{R}^n$
- matrices $B_{ij} \in \mathfrak{so}(n, \mathbb{Z})$ satisfying $B_{ik} = B_{ij} + B_{jk}$
- winding numbers for two tori: $n_{ijk}, m_{ijk} \in \mathbb{Z}^n$, such that

$$a_{ik} = n_{ijk} + a_{jk} + a_{ij}$$
$$b_{ik} = m_{ijk} + b_{jk} + b_{ij} + B_{jk} a_{ij}$$

- data for a gerbe: $t_{ijk} : U_i \cap U_j \cap U_k \to \mathrm{U}(1)$, subject to a complicated gluing condition depending on the matrices B_{ij}.

Example

Consider the F_1 background with $T^3 = T^2 \times S^1 \to S^1$ and

$$[\mathcal{G}] = \mathrm{pr}_1^*\gamma \cup \mathrm{pr}_2^*\gamma \cup \mathrm{pr}_3^*\gamma,$$

where $\gamma \in \mathrm{H}^1(S^1, \mathbb{Z})$ is a generator. Cocycle data is completely trivial, except for $B_{ij} \in \mathfrak{so}(2, \mathbb{Z})$, whose entries $B_{ij}^{12} = -B_{ij}^{21}$ are $\mathbb{Z}$-valued Čech cocycles in S^1, with class γ.

Remarks:

- Half-geometric T-duality correspondence allow to use higher-categorical geometry as an alternative to non-commutative geometry.
- A half-geometric T-duality correspondence has a well-defined twisted K-theory (the one of the well-defined left leg). It is related to the twisted K-theories of the locally defined geometric T-duals, via local Fourier-Mukai transforms.
- The Lie 2-group $\mathbb{TD} \ltimes \mathrm{Aut}(\mathbb{TD})$ is supposed to classify so-called *T-folds*. These have no geometric legs anymore. There is an exact sequence

  $$\mathbb{TD} \to \mathbb{TD}^{\frac{1}{2}geo} \to \mathfrak{so}(n, \mathbb{Z})_{dis}.$$

  It is expected that a T-fold still has a well-defined twisted K-theory.
- Our approach is able to treat F_0 backgrounds (not even locally trivial), by using Lie 2-groupoid-bundle gerbes.

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