A SHORT NOTE ON ∞-GROUPOIDS AND THE PERIOD MAP FOR PROJECTIVE MANIFOLDS

DOMENICO FIORENZA AND ELENA MARTINENGO

ABSTRACT. We show how several classical results on the infinitesimal behaviour of the period map for smooth projective manifolds can be read in a natural and unified way within the framework of ∞-categories.

A common criticism of ∞-categories in algebraic geometry is that they are an extremely technical subject, so abstract to be useless in everyday mathematics. The aim of this note is to show in a classical example that quite the converse is true: even a naïve intuition of what an ∞-groupoid should be clarifies several aspects of the infinitesimal behaviour of the periods map of a projective manifold. In particular, the notion of Cartan homotopy turns out to be completely natural from this perspective, and so classical results such as Griffiths' expression for the differential of the periods map, the Kodaira principle on obstructions to deformations of projective manifolds, the Bogomolov-Tian-Todorov theorem, and Goldman-Millson quasi-abelianity theorem are easily recovered.

The use of the language of ∞-categories should not be looked at as providing new proofs for these results; namely, up to a change in language, our proofs verbatim reproduce arguments from the recent literature on the subject, particularly from the work of Marco Manetti and collaborators on dglas in deformation theory. Rather, by this change of language we change our point of view on the classical theorems above: in the perspective of ∞-sheaves from [Lu09a], all these theorems have a very simple *local* nature which can be naturally expressed in terms of ∞-groupoids (or, equivalently, of dglas); their classical *global* counterparts are then obtained by taking derived global sections. It is worth remarking that, if one prefers proofs which do not rely on the abstract machinery of ∞-categories, one can rework the arguments of this note in purely classical terms. Namely, once the abstract ∞-nonsense has suggested the "correct" local dglas, one can globalize them by means of an explicit model for the derived global sections, e.g., via resolutions by fine sheaves as in [FM09], or by the Thom-Sullivan-Whitney model as in [IM010].

Since most of the statements and constructions we recall in the paper are well known in the $(\infty, 1)$-categorical folklore, despite our efforts in giving credit, it is not unlikely we may have misattributed a few of the results; we sincerely apologize for this. We thank the referee for accurate remarks which helped us a lot in improving the present paper, and Ezra Getzler, Donatella Iacono, Marco Manetti, Jonathan Pridham, Carlos Simpson, Jim Stasheff, Bruno Vallette, Gabriele Vezzosi, and the nLab for suggestions and several inspiring conversations on the subject of this paper.

1

Through the whole paper, $\mathbb{K}$ is a fixed characteristic zero field, all algebras are defined over $\mathbb{K}$ and local algebras have $\mathbb{K}$ as residue field. In order to keep our account readable, we will gloss over many details, particularly where the use of higher category theory is required.

1. FROM DGLAS TO ∞-GROUPOIDS AND BACK AGAIN

With any nilpotent dgla $\mathfrak{g}$ is naturally associated the simplicial set

$$\mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet),$$

where MC stands for the Maurer-Cartan functor mapping a dgla to the set of its Maurer-Cartan elements, and $\Omega_\bullet$ is the simplicial differential graded commutative associative algebra of polynomial differential forms on algebraic n-simplexes, for $n \geq 0$. The importance of this construction, which can be dated back to Sullivan's [Su77], relies on the fact that, as shown by Hinich and Getzler [Ge09, Hi97], the simplicial set $\mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet)$ is a Kan complex, or -to use a more evocative name- an ∞-groupoid. A convenient way to think of ∞-groupoids is as homotopy types of topological spaces; namely, it is well known[1] that any ∞-groupoid can be realized as the ∞-Poincaré groupoid, i.e., as the simplicial set of singular simplices, of a topological space, unique up to weak equivalence. Therefore, the reader who prefers to can substitute homotopy types of topological spaces for equivalence classes of ∞-groupoids. To stress this point of view, we'll denote the k-truncation of an ∞-groupoid $\mathbf{X}$ by the symbol $\pi_{\leq k}\mathbf{X}$. More explicitely, $\pi_{\leq k}\mathbf{X}$ is the k-groupoid whose j-morphisms are the j-morphisms of $\mathbf{X}$ for $j < k$, and are homotopy classes of j-morphisms of $\mathbf{X}$ for $j = k$. In particular, if $\mathbf{X}$ is the ∞-Poincaré groupoid of a topological space X, then $\pi_{\leq 0}\mathbf{X}$ is the set $\pi_0(X)$ of path-connected components of X, and $\pi_{\leq 1}\mathbf{X}$ is the usual Poincaré groupoid of X.

The next step is to consider an $(\infty, 1)$-category, i.e., an ∞-category whose hom-spaces are ∞-groupoids. This can be thought as a formalization of the naïve idea of having objects, morphisms, homotopies between morphisms, homotopies between homotopies, et cetera. In this sense, endowing a category with a model structure should be thought as a first step towards defining an $(\infty, 1)$-category structure on it.

Turning back to dglas, an easy way to produce nilpotent dglas is the following: pick an arbitrary dgla $\mathfrak{g}$; then, for any differential graded local Artin algebra A, take the tensor product $\mathfrak{g} \otimes \mathfrak{m}_A$, where $\mathfrak{m}_A$ is the maximal ideal of A. Since both constructions

$$\mathbf{DGLA} \times \mathbf{dgArt} \to \mathbf{nilpotent\ DGLA}$$

$$(\mathfrak{g}, A) \mapsto \mathfrak{g} \otimes \mathfrak{m}_A$$

and

$$\mathbf{nilpotent\ DGLA} \to \infty\mathbf{-Grpd}$$

$$\mathfrak{g} \mapsto \mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet)$$

are functorial, their composition defines a functor

$$\mathrm{Def} : \mathbf{DGLA} \to \mathbf{Formal}\ \infty\mathbf{-Grpd},$$

where, by definition, a formal ∞-groupoid is a functor $\mathbf{dgArt} \to \infty\mathbf{-Grpd}$. Note that $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}))$ is the usual set valued deformation functor associated with $\mathfrak{g}$, i.e., the functor

$$A \mapsto \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_A)/\mathrm{gauge},$$

---

[1]At least in higher categories folklore

where the gauge equivalence of Maurer-Cartan elements is induced by the gauge action

$$e^\alpha * x = x + \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_\alpha)^n}{(n+1)!} \left([\alpha, x] - d\alpha\right)$$

of $\exp(\mathfrak{g}^0 \otimes \mathfrak{m}_A)$ on the subset $\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_A)$ of $\mathfrak{g}^1 \otimes \mathfrak{m}_A$. However, due to the presence of nontrivial irrelevant stabilizers, the groupoid $\pi_{\leq 1}(\mathrm{Def}(\mathfrak{g}))$ is not equivalent to the action groupoid $\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_A)/\!\!/\exp(\mathfrak{g}^0 \otimes \mathfrak{m}_A)$, unless $\mathfrak{g}$ is concentrated in nonnegative degrees. We will come back to this later. Also note that the zero in $\mathfrak{g}^1 \otimes \mathfrak{m}_A$ gives a natural distinguished element in $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}))$: the isomorphism class of the trivial deformation. Since this marking is natural, we will use the same symbol $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}))$ to denote both the set $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}))$ and the pointed set $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}); 0)$.

It is important to remark that the functors of the form $\mathrm{Def}(\mathfrak{g})$ are very special ones among all formal ∞-groupoids. To begin with, $\mathrm{Def}(\mathfrak{g})(\mathbb{K}) = \{0\}$ and so, in particular, $\mathrm{Def}(\mathfrak{g})(\mathbb{K})$ is a homotopically trivial ∞-groupoid. Another characterzing property of the functors of the form $\mathrm{Def}(\mathfrak{g})$ among formal ∞-groupoids is that, under suitable assumptions, they commute with homotopy pullbacks; see [Lu11, Pr10] for a precise statement. In other words, if we call "formal moduli problems" those formal ∞-groupoids which satisfy the two conditions we have just observed for $\mathrm{Def}(\mathfrak{g})$, what we are saying is that Def is actually a functor

$$\mathrm{Def} : \textbf{DGLA} \to \textbf{Formal moduli problems}.$$

And a very good reason for working with ∞-groupoids valued deformation functors rather than with their apparently handier set-valued or groupoid-valued versions is the following remarkable result, which allows one to move homotopy constructions back and forth between dglas and formal moduli problems.

Theorem (Pridham-Lurie). *The functor* $\mathrm{Def} : \textbf{DGLA} \to \textbf{Formal moduli problems}$ *is an equivalence of* $(\infty, 1)$-*categories.*

Here the $(\infty, 1)$-category structures involved are the most natural ones, and they are both induced by standard model category structures. Namely, on the category of dglas one takes surjective morphisms as fibrations and quasi-isomorphisms as weak equivalences, just as in the case of differential complexes, whereas the model category structure on the right hand side is induced by the standard model category structure on Kan complexes as a subcategory of simplicial sets. A proof of the above equivalence can be found in [Lu11, Pr10].

We will often identify a dgla $\mathfrak{g}$ with the functor $\textbf{dgArt} \to \textbf{nilpotent DGLA}$ it defines by the rule $A \mapsto \mathfrak{g} \otimes \mathfrak{m}_A$. With this in mind, we will occasionally apply constructions that generally only make sense for nilpotent dglas (such as exp) to arbitrary dglas. What we mean in these cases is that the construction is applied not to a single dgla, but to the functor from $\textbf{dgArt}$ to nilpotent dglas it defines. The same kind of consideration applies to our somehow colloquial use of the expression "∞-groupoid" in the following sections; namely, by that we will occasionally mean "formal ∞-groupoid", or even "formal stack in ∞-groupoids". The precise meaning to be given to "∞-groupoid" will always be clear from the context.

2. Tangent spaces and obstructions

If $\mathbf{X}$ is a formal moduli problem, then the simplicial set $\mathbf{X}(\mathbb{K}[\epsilon]/(\epsilon^2))$ has a natural structure of simplicial vector space, and so, via the Dold-Kan correspondence, it is equivalent to the datum of a chain complex: the tangent complex $T\mathbf{X}$ of $\mathbf{X}$. Passing from $\mathbf{X}$ to the associated classical moduli problem $\pi_{\leq 0}\mathbf{X}$, the only datum we read of the tangent complex is its homotopy class, i.e., since we are working on a field, its cohomology. In particular, we have a natural isomorphism

$$T\pi_{\leq 0}\operatorname{Def} \xrightarrow{\sim} H^1$$

of functors $\mathbf{DGLA} \to \mathbf{Vector\ spaces}$ between the tangent space to the classical moduli problem associated to a dgla and the first cohomology group of the dgla seen as a cochain complex. Let us rephrase this in a more explicit form. As we noticed in the previous section, $\pi_{\leq 0}\operatorname{Def}(\mathfrak{g})$ is the functor of Artin rings $A \mapsto \operatorname{MC}(\mathfrak{g} \otimes \mathfrak{m}_A)/\text{gauge}$, hence

$$T\pi_{\leq 0}\operatorname{Def}(\mathfrak{g}) = \operatorname{MC}\left(\mathfrak{g} \otimes (\epsilon)/(\epsilon^2)\right)/\text{gauge} \cong H^1(\mathfrak{g}).$$

This isomorphism is natural. Namely, given a morphism $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ of dglas, let us write Φ for the induced morphism of classical moduli problems,

$$\Phi = \pi_{\leq 0}\operatorname{Def}(\varphi) \colon \pi_{\leq 0}\operatorname{Def}(\mathfrak{g}) \to \pi_{\leq 0}\operatorname{Def}(\mathfrak{h}).$$

Then the differential of Φ,

$$d\Phi \colon T\pi_{\leq 0}\operatorname{Def}(\mathfrak{g}) \to T\pi_{\leq 0}\operatorname{Def}(\mathfrak{h})$$

is naturally identified with

$$H^1(\varphi) \colon H^1(\mathfrak{g}) \to H^1(\mathfrak{h}).$$

The second cohomology group H^2 defines a natural obstruction theory for $\pi_{\leq 0}\operatorname{Def}$, i.e., obstructions for the classical moduli problem $\pi_{\leq 0}\operatorname{Def}(\mathfrak{g})$ are naturally identified with elements in $H^2(\mathfrak{g})$, see [Ma02]. Note that this does not mean that each element in $H^2(\mathfrak{g})$ represents an obstruction: one can have dglas with nontrivial H^2 governing unobstructed deformation problems. The naturally of the obstruction theory given by the second cohomology groups means that, if $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of of dglas, the induced morphism in cohomology,

$$H^2(\varphi) \colon H^2(\mathfrak{g}) \to H^2(\mathfrak{h}),$$

maps obstructions for the classical moduli problem $\pi_{\leq 0}\operatorname{Def}(\mathfrak{g})$ to obstructions for the classical moduli problem $\pi_{\leq 0}\operatorname{Def}(\mathfrak{h})$. In particular, if the moduli problem $\pi_{\leq 0}\operatorname{Def}(\mathfrak{h})$ is unobstructed (e.g., if the functor $\pi_{\leq 0}\operatorname{Def}(\mathfrak{h})$ is smooth), then

$$\operatorname{Obstructions}\left(\pi_{\leq 0}\operatorname{Def}(\mathfrak{g})\right) \subseteq \ker\left(H^2(\varphi) \colon H^2(\mathfrak{g}) \to H^2(\mathfrak{h})\right).$$

3. Homotopy vs. gauge equivalent morphisms of dglas (with a detour into L_∞-morphisms)

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two dglas. The hom-space $\operatorname{Hom}_\infty(\mathfrak{g}, \mathfrak{h})$ of morphisms between $\mathfrak{g}$ and $\mathfrak{h}$ in the $(\infty, 1)$-category of dglas is conveniently modelled as the simplicial set $\operatorname{MC}(\underline{\operatorname{Hom}}(\mathfrak{g}, \mathfrak{h}) \otimes \Omega_\bullet)$, where $\underline{\operatorname{Hom}}(\mathfrak{g}, \mathfrak{h})$ is the Chevalley-Eilenberg-type dgla associated with the pair $(\mathfrak{g}, \mathfrak{h})$. It is given as the total dgla of the bigraded dgla

$$\underline{\operatorname{Hom}}^{p,q}(\mathfrak{g}, \mathfrak{h}) = \operatorname{Hom}_{\mathbb{Z}-\mathbf{Vect}}(\wedge^q \mathfrak{g}, \mathfrak{h}[p]) = \operatorname{Hom}^p(\wedge^q \mathfrak{g}, \mathfrak{h}),$$

endowed with the Lie bracket

$$[\,,\,]_{\underline{\operatorname{Hom}}} \colon \underline{\operatorname{Hom}}^{p_1,q_1}(\mathfrak{g}, \mathfrak{h}) \otimes \underline{\operatorname{Hom}}^{p_2,q_2}(\mathfrak{g}, \mathfrak{h}) \to \underline{\operatorname{Hom}}^{p_1+p_2,q_1+q_2}(\mathfrak{g}, \mathfrak{h})$$

defined by

$$[f, g]_{\underline{\mathrm{Hom}}}(\gamma_1 \wedge \cdots \wedge \gamma_{q_1+q_2}) =$$
$$= \sum_{\sigma \in \mathrm{Sh}(q_1, q_2)} \pm [f(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(q_1)}), g(\gamma_{\sigma(q_1+1)} \wedge \cdots \wedge \gamma_{\sigma(q_1+q_2)})]_{\mathfrak{h}},$$

with σ ranging in the set of (q_1, q_2)-unshuffles and and $\pm$ standing for the Koszul sign, and with the differentials

$$d_{1,0} \colon \underline{\mathrm{Hom}}^{p,q}(\mathfrak{g}, \mathfrak{h}) \to \underline{\mathrm{Hom}}^{p+1,q}(\mathfrak{g}, \mathfrak{h})$$

and

$$d_{0,1} \colon \underline{\mathrm{Hom}}^{p,q}(\mathfrak{g}, \mathfrak{h}) \to \underline{\mathrm{Hom}}^{p,q+1}(\mathfrak{g}, \mathfrak{h})$$

given by

$$(d_{1,0}f)(\gamma_1 \wedge \cdots \wedge \gamma_q) = d_{\mathfrak{h}}(f(\gamma_1 \wedge \cdots \wedge \gamma_q)) + \sum_i \pm f(\gamma_1 \wedge \cdots \wedge d_{\mathfrak{g}}\gamma_i \wedge \cdots \wedge \gamma_{q+1})$$

and

$$(d_{0,1}f)(\gamma_1 \wedge \cdots \wedge \gamma_{q+1}) = \sum_{i<j} \pm f([\gamma_i, \gamma_j]_{\mathfrak{g}} \wedge \gamma_1 \wedge \cdots \wedge \widehat{\gamma_i} \wedge \cdots \wedge \widehat{\gamma_j} \wedge \cdots \wedge \gamma_{q+1}).$$

An explicit determination for the signs in the above formulas can be found, e.g, in [LM95, Sc04]. These operations are best seen pictorially:

Note that 0-simplices in $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h}) \otimes \Omega_\bullet)$ are those degree 1 elements f in $\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h})$ such that

$$d_{\mathfrak{h}}f_n(\gamma_1 \wedge \cdots \wedge \gamma_n) + \frac{1}{2} \sum_{\substack{q_1+q_2=n \\ \sigma \in \mathrm{Sh}(q_1, q_2)}} \pm [f_{q_1}(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(q_1)}), f_{q_2}(\gamma_{\sigma(q_1+1)} \wedge \cdots \wedge \gamma_{\sigma(q_1+q_2)})]_{\mathfrak{h}}$$

$$= \sum_i \pm f_n(\gamma_1 \wedge \cdots \wedge d_{\mathfrak{g}}\gamma_i \wedge \cdots \wedge \gamma_n)$$

$$+ \sum_{i<j} \pm f_{n-1}([\gamma_i, \gamma_j]_{\mathfrak{g}} \wedge \gamma_1 \wedge \cdots \wedge \widehat{\gamma_i} \wedge \cdots \wedge \widehat{\gamma_j} \wedge \cdots \wedge \gamma_{q+1}),$$

where we have written $f = \sum_{n \geq 1} f_n$ with f_n in $\underline{\mathrm{Hom}}^{1-n,n}(\mathfrak{g}, \mathfrak{h})$. Here one recognizes the explicit equation characterizing L_∞-morphisms between $\mathfrak{g}$ and $\mathfrak{h}$ one finds in, e.g., [LS93, Ko03].

Next, as we remarked in the previous section, the set $\pi_{\leq 0}(\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h}) \otimes \Omega_\bullet))$ is isomorphic to the quotient $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h}))/\mathrm{gauge}$. Thus we obtain the following proposition.

Proposition. *Let $f, g : \mathfrak{g} \to \mathfrak{h}$ be two L_∞-morphisms of dglas. Then f and g are gauge equivalent in $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h}))$ if and only if f and g represent the same morphism in the homotopy category of dglas.*

It should be remarked that the above construction is an instance of a more general phenomenon: if $\mathcal{O}$ is a differential graded operad, A is an $\mathcal{O}$-algebra, and B is a (differential graded) cocommutative coalgebra, then the space of linear mappings from B to A has a natural $\mathcal{O}$-algebra structure, see [Do07]. Also, the analogue of the above proposition holds, more in general, for the homotopy category of $\mathcal{O}$-algebras, see [MV09, Pr09]. We thank Jonathan Pridham and Bruno Vallette for this remark.

4. Cartan homotopies appear

Let now $\mathfrak{g}$ and $\mathfrak{h}$ be dglas and $\boldsymbol{i} \colon \mathfrak{g} \to \mathfrak{h}[-1]$ be a morphism of graded vector spaces. Let $\boldsymbol{l} = d_{1,0}\boldsymbol{i}$; that is, $\boldsymbol{l}_a = d_\mathfrak{h}\boldsymbol{i}_a + \boldsymbol{i}_{d_\mathfrak{g} a}$ for any $a \in \mathfrak{g}$. The morphism $\boldsymbol{i}$ is called a Cartan homotopy[2] if it satisfies the two conditions

$$\boldsymbol{i}_{[a,b]_\mathfrak{g}} = [\boldsymbol{i}_a, \boldsymbol{l}_b]_\mathfrak{h} \qquad \text{and} \qquad [\boldsymbol{i}_a, \boldsymbol{i}_b]_\mathfrak{h} = 0, \qquad \text{for all } a, b \in \mathfrak{g}.$$

This name has an evident geometric origin: if $\mathcal{T}_X$ is the tangent sheaf of a smooth manifold X and Ω_X^* is the sheaf of complexes of differential forms, then the contraction of differential forms with vector fields is a Cartan homotopy

$$\boldsymbol{i} \colon \mathcal{T}_X \to \mathcal{E}nd^*(\Omega_X^*)[-1].$$

In this case, $\boldsymbol{l}_a$ is the Lie derivative along the vector field a, and the conditions $\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, \boldsymbol{l}_b]$ and $[\boldsymbol{i}_a, \boldsymbol{i}_b] = 0$, together with the defining equation $\boldsymbol{l}_a = [d_{\Omega_X^*}, \boldsymbol{i}_a]$ and with the equations $\boldsymbol{l}_{[a,b]} = [\boldsymbol{l}_a, \boldsymbol{l}_b]$ and $[d_{\Omega_X^*}, \boldsymbol{l}_a] = 0$ expressing the fact that $\boldsymbol{l} \colon \mathcal{T}_X \to \mathcal{E}nd^*(\Omega_X^*)$ is a dgla morphism, are nothing but the well-known Cartan identities involving contractions and Lie derivatives.

It is a straightforward computation to see that, if $\boldsymbol{i}$ is a Cartan homotopy, then the degree zero morphism of graded vector spaces $\boldsymbol{l} \colon \mathfrak{g} \to \mathfrak{h}$ is actually a dgla morphism. Actually much more is true: the dgla morphism $\boldsymbol{l}$ is gauge equivalent to zero, and $e^{\boldsymbol{i}}$ is a homotopy between $\boldsymbol{l}$ and 0. To see this, just compute

$$e^{-\boldsymbol{i}} * 0 = \sum_{n=0}^{+\infty} \frac{(\mathrm{ad}_{-\boldsymbol{i}})^n}{(n+1)!} \, (d_{\underline{\mathrm{Hom}}}\boldsymbol{i}) = \sum_{n=0}^{+\infty} \frac{(\mathrm{ad}_{-\boldsymbol{i}})^n}{(n+1)!} \, (\boldsymbol{l} + \boldsymbol{i}_{[,]_\mathfrak{g}}).$$

Then one sees that the $(0,1)$-component of $e^{-\boldsymbol{i}} * 0$ is just $\boldsymbol{l}$; the $(-1,2)$-component is

$$\boldsymbol{i}_{[,]_\mathfrak{g}} - \frac{1}{2}[\boldsymbol{i}, \boldsymbol{l}]_{\underline{\mathrm{Hom}}}$$

and, for $n \geq 3$ the $(1-n, n)$-component has two contributions, one of the form $[\boldsymbol{i}, [\boldsymbol{i}, \cdots, [\boldsymbol{i}, \boldsymbol{l}]_{\underline{\mathrm{Hom}}} \cdots]_{\underline{\mathrm{Hom}}}]_{\underline{\mathrm{Hom}}}$ and the other of the form $[\boldsymbol{i}, [\boldsymbol{i}, \cdots, [\boldsymbol{i}, \boldsymbol{i}_{[,]_\mathfrak{g}}]_{\underline{\mathrm{Hom}}} \cdots]_{\underline{\mathrm{Hom}}}]_{\underline{\mathrm{Hom}}}$. Threfore, if $\boldsymbol{i}$ is a Cartan homotopy, then all the nonlinear components of $e^{-\boldsymbol{i}} * 0$ vanish and one finds $e^{-\boldsymbol{i}} * 0 = \boldsymbol{l}$, i.e.,

$$e^{\boldsymbol{i}} * \boldsymbol{l} = 0.$$

The above discussion can be summarized as follows.

Proposition. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two dglas. If $\boldsymbol{i} \colon \mathfrak{g} \to \mathfrak{h}[-1]$ is a Cartan homotopy, then $\boldsymbol{l} = d_{1,0}\boldsymbol{i} \colon \mathfrak{g} \to \mathfrak{h}$ is a dgla morphism gauge equivalent to the zero morphism via the gauge action of $e^{\boldsymbol{i}}$.*

---

[2]Up to our knowledge, this terminology has been introduced in [FM06, FM09].

5. HOMOTOPY FIBERS (AND THE ASSOCIATED EXACT SEQUENCE)

Let now $i\colon \mathfrak{g} \to \mathfrak{h}[-1]$ be a Cartan homotopy and $l\colon \mathfrak{g} \to \mathfrak{h}$ be the associated dgla morphism. Then, the equation $e^i * l = 0$ implies that, for any subdgla $\mathfrak{n}$ of $\mathfrak{h}$ containing the image of l, the morphism $l\colon \mathfrak{g} \to \mathfrak{n}$ equalizes the diagram $\mathfrak{n} \overset{\text{incl.}}{\underset{0}{\rightrightarrows}} \mathfrak{h}$ up to a homotopy provided by the gauge action of e^i. Hence we have a morphism to the homotopy limit:

$$\mathfrak{g} \xrightarrow{(l,e^i)} \operatorname{holim}\left(\mathfrak{n} \overset{\text{incl.}}{\underset{0}{\rightrightarrows}} \mathfrak{h} \right).$$

Taking Def's we obtain a natural transformation of ∞-groupoid valued functors:

$$\operatorname{Def}(\mathfrak{g}) \xrightarrow{(l,e^i)} \operatorname{holim}\left(\operatorname{Def}(\mathfrak{n}) \overset{\operatorname{Def}_{\text{incl.}}}{\underset{\operatorname{Def}_0}{\rightrightarrows}} \operatorname{Def}(\mathfrak{h}) \right).$$

The map $\operatorname{Def}_0\colon \operatorname{Def}(\mathfrak{n}) \to \operatorname{Def}(\mathfrak{h})$ is the constant map to the distinguished point 0 in $\operatorname{Def}(\mathfrak{h})$; therefore, the homotopy limit above is the homotopy fiber of $\operatorname{Def}_{\text{incl.}}\colon \operatorname{Def}(\mathfrak{n}) \to \operatorname{Def}(\mathfrak{h})$ over the point 0, and we obtain a natural transformation

$$\operatorname{Def}(\mathfrak{g}) \xrightarrow{(l,e^i)} \operatorname{hoDef}_{\text{incl.}}^{-1}(0),$$

which at the zeroth level gives a natural transformation of **Set**-valued deformation functors

$$\mathcal{P}\colon \pi_{\leq 0}\operatorname{Def}(\mathfrak{g}) \to \pi_{\leq 0}\operatorname{hoDef}_{\text{incl.}}^{-1}(0).$$

The differential of $\mathcal{P}$ is easily computed: it is the linear map

$$H^1(\mathfrak{g}) \xrightarrow{H^1((l,e^i))} H^1(\operatorname{holim}\left(\mathfrak{n} \overset{\text{incl.}}{\underset{0}{\rightrightarrows}} \mathfrak{h} \right)).$$

Since the model category structure on dglas is the same as on differential complexes, we can compute the H^1 on the right hand side by taking the holimit in complexes. Then the natural quasi-isomorphism $\operatorname{holim}(\ \mathfrak{n} \overset{\text{incl.}}{\underset{0}{\rightrightarrows}} \mathfrak{h}\ ) \simeq (\mathfrak{h}/\mathfrak{n})[-1]$ tells us that the differential of $\mathcal{P}$ is just the map

$$H^1(i)\colon H^1(\mathfrak{g}) \to H^0(\mathfrak{h}/\mathfrak{n})$$

induced by the morphism of complexes $i\colon \mathfrak{g} \to (\mathfrak{h}/\mathfrak{n})[-1]$. Also, the map

$$H^2(i)\colon H^2(\mathfrak{g}) \to H^1(\mathfrak{h}/\mathfrak{n})$$

maps the obstruction space of $\pi_{\leq 0}\operatorname{Def}(\mathfrak{g})$ (as a subspace of $H^2(\mathfrak{g})$) to the obstruction space of $\pi_{\leq 0}\operatorname{hoDef}_{\text{incl.}}^{-1}(0)$ (as a subspace of $H^1(\mathfrak{h}/\mathfrak{n})$). In particular, if $\pi_{\leq 0}\operatorname{hoDef}_{\text{incl.}}^{-1}(0)$ is smooth, and therefore unobstructed, the obstructions of the deformation functor $\pi_{\leq 0}\operatorname{Def}(\mathfrak{g})$ are contained in the kernel of the map $H^2(i)\colon H^2(\mathfrak{g}) \to H^1(\mathfrak{h}/\mathfrak{n})$.

To investigate the geometrical aspects of the map $\mathcal{P}$, note that, by looking at $\pi_{\leq 0}\operatorname{hoDef}_{\text{incl.}}^{-1}(0)$ as a pointed set, it nicely fits into the homotopy exact sequence

$$\pi_1(\operatorname{Def}(\mathfrak{n});0) \xrightarrow{\operatorname{Def}_{\text{incl.}*}} \pi_1(\operatorname{Def}(\mathfrak{h});0) \to \pi_0(\operatorname{hoDef}_{\text{incl.}}^{-1}(0);0) \to \pi_0(\operatorname{Def}(\mathfrak{n});0),$$

so we get a canonical isomorphism between the preimage of the distinguished point 0 under the map $\pi_0(\operatorname{hoDef}_{\text{incl.}}^{-1}(0);0) \to \pi_0(\operatorname{Def}(\mathfrak{n});0)$ and the quotient set

$$\frac{\pi_1(\operatorname{Def}(\mathfrak{h});0)}{\operatorname{Def}_{\text{incl.}*}\pi_1(\operatorname{Def}(\mathfrak{n});0)}.$$

The group $\pi_1(\mathrm{Def}(\mathfrak{h}); 0)$ is the group of automorphisms of 0 in the groupoid $\pi_{\leq 1}(\mathrm{Def}(\mathfrak{h}))$. We have already remarked that this groupoid is not equivalent to the Deligne groupoid of $\mathfrak{h}$, i.e., the action groupoid for the gauge action of $\exp(\mathfrak{h}^0 \otimes \mathfrak{m}_A)$ on $\mathrm{MC}(\mathfrak{h} \otimes \mathfrak{m}_A)$, since the irrelevant stabilizer

$$\mathrm{Stab}(x) = \{dh + [x, h] \mid h \in \mathfrak{h}^{-1} \otimes \mathfrak{m}_A\} \subseteq \{a \in \mathfrak{h}^0 \otimes \mathfrak{m}_A \mid e^a * x = x\}$$

of a Maurer-Cartan element x may be nontrivial. However, the group $\pi_1(\mathrm{Def}(\mathfrak{h}); 0)$ only sees the connected component of 0, and on this connected component the irrelevant stabilizers are trivial as soon as the differential of the dgla $\mathfrak{h}$ vanishes on $\mathfrak{h}^{-1}$. This immediately follows from noticing that irrelevant stabilizers of gauge equivalent Maurer-Cartan elements are conjugate subgroups of $\exp(\mathfrak{h}^0 \otimes \mathfrak{m}_A)$, see, e.g., [Ma07]. In particular, if $\mathfrak{h}$ is a graded Lie algebra (which we can consider as a dgla with trivial differential), then $\pi_1(\mathrm{Def}(\mathfrak{h}); 0) \simeq \exp(\mathfrak{h}^0)$, where $\mathfrak{h}^0$ denotes the degree zero component of $\mathfrak{h}$. Similarly, since $\mathfrak{n}$ is a subdgla of $\mathfrak{h}$, one has $\pi_1(\mathrm{Def}(\mathfrak{n}); 0) \simeq \exp(\mathfrak{n}^0)$, and the group homomorphism $\mathrm{Def}_{\mathrm{incl}.*}$ is just the inclusion. Moreover, when $\mathfrak{h}$ has trivial differential, $e^\alpha * 0 = 0$ for any α in $\mathfrak{h}^0 \otimes \mathfrak{m}_A$. Therefore, the composition $\pi_{\leq 0}\mathrm{Def}(\mathfrak{g}) \to \pi_{\leq 0}\mathrm{hoDef}_{\mathrm{incl}.}^{-1}(0) \to \pi_{\leq 0}\mathrm{Def}(\mathfrak{n})$ maps the whole of $\pi_{\leq 0}\mathrm{Def}(\mathfrak{g})$ onto the distinguished point 0 of $\pi_{\leq 0}\mathrm{Def}(\mathfrak{n})$ and so it induces a natural map $\pi_{\leq 0}\mathrm{Def}(\mathfrak{g}) \to \exp(\mathfrak{h}^0)/\exp(\mathfrak{n}^0)$. This is nothing but the natural map

$$e^{\boldsymbol{i}} \colon \pi_{\leq 0}\mathrm{Def}(\mathfrak{g}) \to \exp(\mathfrak{h}^0)/\exp(\mathfrak{n}^0)$$

which sends a Maurer-Cartan element $\xi \in \mathfrak{g}^1 \otimes \mathfrak{m}_A$ to $e^{\boldsymbol{i}\xi} \mod \exp(\mathfrak{n}^0 \otimes \mathfrak{m}_A)$. A particularly interesting case is when the pair $(\mathfrak{h}, \mathfrak{n})$ is formal,[3] i.e., if the inclusion of $\mathfrak{n}$ in $\mathfrak{h}$ induces an inclusion in cohomology and the two inclusions $H^*(\mathfrak{n}) \hookrightarrow H^*(\mathfrak{h})$ and $\mathfrak{n} \hookrightarrow \mathfrak{h}$ are homotopy equivalent. Indeed, in this case the pair $(\mathrm{Def}(\mathfrak{h}), \mathrm{Def}(\mathfrak{n}))$ will be equivalent to the pair $(\mathrm{Def}(H^*(\mathfrak{h})), \mathrm{Def}(H^*(\mathfrak{h})))$ and there will be an induced isomorphism between $\pi_1(\mathrm{Def}(\mathfrak{h}); 0)/\mathrm{Def}_{\mathrm{incl}.*}\pi_1(\mathrm{Def}(\mathfrak{n}); 0)$ and the smooth homogeneous space $\exp(H^0(\mathfrak{h}))/\exp(H^0(\mathfrak{n}))$. We can summarize the results described in this section as follows:

Proposition. *Let $\boldsymbol{i} \colon \mathfrak{g} \to \mathfrak{h}[-1]$ be a Cartan homotopy, let $\boldsymbol{l} \colon \mathfrak{g} \to \mathfrak{h}$ be the associated dgla morphism, and let $\mathfrak{n}$ be a subdgla of $\mathfrak{h}$ containing the image of $\boldsymbol{l}$. Then, if the pair $(\mathfrak{h}, \mathfrak{n})$ is formal, we have a natural transformation[4] of* **Set***-valued deformation functors*

$$\mathcal{P} \colon \pi_{\leq 0}(\mathrm{Def}(\mathfrak{g})) \to \exp(H^0(\mathfrak{h}))/\exp(H^0(\mathfrak{n}))$$

induced by the dgla map

$$\mathfrak{g} \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\mathrm{incl}.}{\rightrightarrows}} \mathfrak{h} \right).$$

In particular, since $\exp(H^0(\mathfrak{h}))/\exp(H^0(\mathfrak{n}))$ is smooth, the obstructions of the **Set***-valued deformation functor $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}); 0)$ are contained in the kernel of the map $H^2(\boldsymbol{i}) \colon H^2(\mathfrak{g}) \to H^1(\mathfrak{h}/\mathfrak{n})$.*

This result can be nicely refined, by showing how the main result from [IM010] naturally fits into the discussion above. We have:

---

[3]We are not sure whether this terminology is a standard one

[4]This natural transformation is not canonical: it depends on the choice of a quasi isomorphism $(\mathfrak{h}, \mathfrak{n}) \simeq (H^*(\mathfrak{h}), H^*(\mathfrak{n}))$. Also note that the tangent space at 0 on the right hand side is $H^0(\mathfrak{h})/H^0(\mathfrak{n})$; this is only apparently in contrast with the general result mentioned above that the tangent space at 0 to $\pi_{\leq 0}\mathrm{hoDef}_{\mathrm{incl}.}^{-1}(0)$ is $H^0(\mathfrak{h}/\mathfrak{n})$. Indeed, when $(\mathfrak{h}, \mathfrak{n})$ is a formal pair, the two vector spaces $H^0(\mathfrak{h})/H^0(\mathfrak{n})$ and $H^0(\mathfrak{h}/\mathfrak{n})$ are (non canonically) isomorphic.

Proposition. *Let $(\mathfrak{h}, \mathfrak{n})$ be a formal pair of dglas. Then, the dgla* $\mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h} \right)$ *is quasi-abelian. In particular there is a (non-canonical) quasi-isomorphism of dglas between* $\mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h} \right)$ *and the abelian dgla obtained by endowing the complex* $(\mathfrak{h}/\mathfrak{n})[-1]$ *with the trivial bracket.*

To see this, notice that, since by hypothesis the inclusion $\mathfrak{n} \hookrightarrow \mathfrak{h}$ induces an inclusion $H^*(\mathfrak{n}) \hookrightarrow H^*(\mathfrak{h})$, the projection $\mathfrak{h}[-1] \to \mathfrak{h}/\mathfrak{n}[-1]$ admits a section i which is a morphism of complexes. Denote by $\mathfrak{g}$ the dgla obtained from the complex $\mathfrak{h}/\mathfrak{n}[-1]$ by endowing it with the trivial bracket. Then, the map of graded vector spaces $i\colon \mathfrak{g} \to \mathfrak{h}[-1]$ is a Cartan homotopy whose associated dgla morphism is the zero map $0\colon \mathfrak{g} \to \mathfrak{h}$. Therefore we have a dgla map

$$(\mathfrak{h}/\mathfrak{n})[-1] \xrightarrow{(0,e^i)} \mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h} \right).$$

Since i is a section to $\mathfrak{h}[-1] \to \mathfrak{h}/\mathfrak{n}[-1]$, the map in cohomology

$$H^*(\mathfrak{h}/\mathfrak{n})[-1] \xrightarrow{H^*(0,e^i)} H^*(\mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h} \right))$$

is identified with the identity of $H^*(\mathfrak{h}/\mathfrak{n})[-1]$ by the the natural quasi-isomorphism of complexes $\mathrm{holim}(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h}) \xrightarrow{\sim} (\mathfrak{h}/\mathfrak{n})[-1]$.

6. FROM LOCAL TO GLOBAL, AND CLASSICAL (AND GENERALIZED) PERIODS

Assume now $\mathbb{K}$ is algebraically closed. Let X be a smooth projective manifold, and let $\mathcal{T}_X$ and Ω_X^* be the tangent sheaf and the sheaf of differential forms on X, respectively. The sheaf of complexes $(\Omega_X^*, d_{\Omega_X^*})$ is naturally filtered by setting $F^p\Omega_X^* = \oplus_{i \geq p}\Omega_X^i$. Finally, let $\mathcal{E}nd^*(\Omega_X^*)$ be the endomorphism sheaf of Ω_X^* and $\mathcal{E}nd^{\geq 0}(\Omega_X^*)$ be the sub-sheaf consisting of nonnegative degree elements. Note that $\mathcal{E}nd^{\geq 0}(\Omega_X^*)$ is a subdgla of $\mathcal{E}nd^*(\Omega_X^*)$, and can be seen as the subdgla of endomorphisms preserving the filtration on Ω_X^*.

Recall that the prototypical example of Cartan homotopy was the contraction of differential forms with vector fields $i : \mathcal{T}_X \to \mathcal{E}nd^*(\Omega_X^*)[-1]$; the corresponding dgla morphism is $a \mapsto l_a$, where l_a the Lie derivative along a. Explicitly, $l_a = d_{\Omega_X^*} \circ i_a + i_a \circ d_{\Omega_X^*}$, and so l_a preserves the filtration. Therefore, we have a natural transformation[5]

$$\mathrm{Def}(\mathcal{T}_X) \xrightarrow{(l,e^i)} \mathrm{holim}\left(\mathrm{Def}(\mathcal{E}nd^{\geq 0}(\Omega_X^*)) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathrm{Def}(\mathcal{E}nd^*(\Omega_X^*)) \right).$$

The homotopy fiber on the right should be thought as a homotopy flag manifold. Let us briefly explain this. At least naïvely, the functor $\mathrm{Def}(\mathcal{E}nd^*(\Omega_X^*))$ describes the infinitesimal deformations of the differential complex Ω_X^*, whereas the functor $\mathrm{Def}(\mathcal{E}nd^{\geq 0}(\Omega_X^*))$ describes the deformations of the filtered complex $(\Omega_X^*, F^\bullet\Omega_X^*)$, i.e., of the pair consisting of the complex Ω_X^* *and* the filtration $F^\bullet\Omega_X^*$. Therefore, the holimit describes a

---

[5]Of what? The correct answer would be of ∞-sheaves, see [Lu09a], but to keep this note as far as possible at an informal level we will content ourselves with noticing that, for any open subset U of X, there is a natural transformation of ∞-groupoids induced by the dgla map $\mathcal{T}_X(U) \xrightarrow{(l,e^i)} \mathrm{holim}\left(\mathcal{E}nd^{\geq 0}(\Omega_X^*)(U) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathcal{E}nd^*(\Omega_X^*)\,(U) \right).$

deformation of the pair (complex, filtration) together with a trivialization of the deformation of the complex. Summing up, the contraction of differential forms with vector fields induces a map of deformation functors

$$\mathrm{Def}(\mathcal{T}_X) \to \mathrm{hoFlag}(\Omega_X^*; F^\bullet \Omega_X^*),$$

which we will call the *local periods map* of X.

To recover from this the classical periods map, we just need to take global sections. Clearly, since we are working in homotopy categories, these will be derived global sections. The morphism of sheaves $\boldsymbol{i} \colon \mathcal{T}_X \to \mathcal{E}nd^*(\Omega_X^*)[-1]$ induces a Cartan homotopy $\boldsymbol{i} \colon \mathbf{R}\Gamma\mathcal{T}_X \to \mathbf{R}\Gamma\mathcal{E}nd^*(\Omega_X^*)[-1]$; composing this with the dgla morphism $\mathbf{R}\Gamma\mathcal{E}nd^*(\Omega_X^*) \to \mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*)$ induced by the action of (derived) global sections of the endomorphism sheaf of Ω_X^* on (derived) global sections of Ω_X^*, we get a Cartan homotopy

$$\boldsymbol{i} \colon \mathbf{R}\Gamma\mathcal{T}_X \to \mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*)[-1].$$

The image of the corresponding dgla morphism $\boldsymbol{l}$ (the derived globalization of Lie derivative) preserves the filtration $F^\bullet \mathbf{R}\Gamma\Omega_X^*$ induced by $F^\bullet \Omega_X^*$, so we have a natural map of ∞-groupoids

$$\mathrm{Def}(\mathbf{R}\Gamma\mathcal{T}_X) \to \mathrm{hoFlag}(\mathbf{R}\Gamma\Omega_X^*; F^\bullet \mathbf{R}\Gamma\Omega_X^*)$$

and, at the zeroth level, a map of **Set**-valued deformation functors

$$\mathcal{P} \colon \pi_{\leq 0} \mathrm{Def}(\mathbf{R}\Gamma\mathcal{T}_X) \to \pi_{\leq 0}\mathrm{hoFlag}(\mathbf{R}\Gamma\Omega_X^*; F^\bullet \mathbf{R}\Gamma\Omega_X^*)$$

The functor on the left hand side is the **Set**-valued functor of (classical) infinitesimal deformations of X; let us denote it by Def_X. If we denote by $\mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*; F^\bullet \mathbf{R}\Gamma\Omega_X^*)$ the subdgla of $\mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*)$ consisting of endomorhisms preserving the filtration, then the pair $(\mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*), \mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*; F^\bullet \mathbf{R}\Gamma\Omega_X^*))$ is formal.[6] Moreover, $H^0(\mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*)) = \mathrm{End}^0(H_{dR}^*(X; \mathbb{K}))$ and $H^0(\mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*; F^\bullet \mathbf{R}\Gamma\Omega_X^*)) = \mathrm{End}^0(H_{dR}^*(X; \mathbb{K}); F^\bullet H_{dR}^*(X; \mathbb{K}))$, where $F^\bullet H_{dR}^*(X; \mathbb{K})$ is the Hodge filtration on the algebraic de Rham cohomology of X. By results described in the previous section, this means that the preimage of the distinguished point 0 under the map

$$\pi_{\leq 0}\mathrm{hoFlag}(\mathbf{R}\Gamma\Omega_X^*; F^\bullet \mathbf{R}\Gamma\Omega_X^*) \to \pi_{\leq 0} \mathrm{Def}(\mathcal{E}nd^{\geq 0}(\Omega_X^*))$$

is the quotient set

$$\frac{\exp(\mathrm{End}^0(H_{dR}^*(X; \mathbb{K})))}{\exp(\mathrm{End}^0(H_{dR}^*(X; \mathbb{K}); F^\bullet H_{dR}^*(X; \mathbb{K})))}$$

and we recover the classical periods map of X

$$\mathcal{P} \colon \mathrm{Def}_X \to \mathrm{Flag}(H_{dR}^*(X; \mathbb{K}); F^\bullet H_{dR}^*(X; \mathbb{K})).$$

Also, the differential of $\mathcal{P}$ is the map induced in cohomology by the contraction of differential forms with vector fields,

$$H^1(\boldsymbol{i}) \colon H^1(X, \mathcal{T}_X) \to \int_p \mathrm{Hom}^0\left(F^p H_{dR}^*(X; \mathbb{K}); \frac{H_{dR}^*(X; \mathbb{K})}{F^p H_{dR}^*(X; \mathbb{K})}\right),$$

a result originally proved by Griffiths [Gr68]. In the above formula, $\int_p$ denotes the end of the diagram

$$\mathrm{Hom}^0\left(F^p H_{dR}^*; \frac{H_{dR}^*}{F^p H_{dR}^*}\right) \to \mathrm{Hom}^0\left(F^p H_{dR}^*; \frac{H_{dR}^*}{F^{p+1} H_{dR}^*}\right) \leftarrow \mathrm{Hom}^0\left(F^{p+1} H_{dR}^*; \frac{H_{dR}^*}{F^{p+1} H_{dR}^*}\right)$$

---

[6]This is essentially a consequence of the E_1-degeneration of the Hodge-to-de Rham spectral sequence, see, e.g., [DI87, Fa88].

Also, we have the following version of the so-called Kodaira principle (ambient cohomology annihilates obstructions): obstructions to classical infinitesimal deformations of X are contained in the kernel of

$$H^2(\boldsymbol{i})\colon H^2(X, \mathcal{T}_X) \to \int_p \mathrm{Hom}^1\left(F^p H^*_{dR}(X; \mathbb{K}); \frac{H^*(X; \mathbb{K})}{F^p H^*_{dR}(X; \mathbb{K})}\right).$$

In particular, if the canonical bundle of X is trivial, then the contraction pairing

$$H^2(X, \mathcal{T}_X) \otimes H^{n-2}(X, \Omega^1_X) \to H^n(X; \mathcal{O}_X) \simeq \mathbb{K}$$
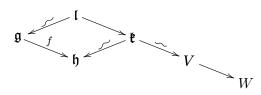
is nondegenerate, and so classical deformations of X are unobstructed (Bogomolov-Tian-Todorov theorem, see [Bo78, Ti87, To89]). Following [IM010], one immediately obtains the following refinement, due in its original formulation to Goldman and Millson [GM90]: if the canonical bundle of X is trivial, then $\mathbf{R}\Gamma\mathcal{T}_X$ is a quasi-abelian dgla. To see this, just notice that the dgla map

$$\mathbf{R}\Gamma\mathcal{T}_X \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X; F^\bullet \mathbf{R}\Gamma\Omega^*_X) \underset{0}{\overset{\mathrm{incl.}}{\rightrightarrows}} \mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X)\right)$$

is injective in cohomology and the target is a quasi-abelian dgla. Indeed, if $f\colon \mathfrak{g} \to \mathfrak{h}$ is a dgla morphism, with $H^*(f)$ injective and $\mathfrak{h}$ quasi-abelian, then the diagram of dglas

where V is a graded vector space considered as a dgla with trivial differential and bracket, can be completed to a homotopy commutative diagram

with W a graded vector space, and the composition $\mathfrak{l} \to W$ a quasi-isomorphism.

As a conclusion, we recast the description of a period map for generalized deformations from [FM09] in the language of this note. Let X be a smooth projective variety defined over the field $\mathbb{C}$ of the complex numbers, and denote by $\mathcal{P}oly^*_X$ the sheaf of dglas of *multivector fields* on X, given by $\mathcal{P}oly^j_X = \bigwedge^{1-j} \mathcal{T}_X$, endowed with the zero differential and with the Schouten-Nijenhuis bracket. Notice that $\mathcal{T}_X$ is a sub-Lie algebra of the dgla $\mathcal{P}oly^*_X$. The contraction of differential forms with multivector fields

$$\boldsymbol{i}\colon \mathcal{P}oly^*_X \to \mathcal{E}nd^*(\Omega^*_X)[-1]$$

is a Cartan homotopy, and the corresponding dgla morphism $\boldsymbol{l}$ is the Lie derivative along a multivector field, i.e., $\boldsymbol{l}_\xi = [d_{\Omega^*_X}, \boldsymbol{i}_\xi]$. It is immediate that the image of $\boldsymbol{l}$ is contained in the sub-sheaf of dglas:

$$\mathcal{E}nd^*_0(\Omega^*_X) = \{f \in \mathcal{E}nd^*(\Omega^*_X) \mid f(\ker d_{\Omega^*_X}) \subseteq \mathrm{Im}(d_{\Omega^*_X})\} \subset \mathcal{E}nd^*(\Omega^*_X),$$

and so we have a natural transformation:

$$\mathrm{Def}(\mathcal{P}oly^*_X) \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathrm{Def}(\mathcal{E}nd^*_0(\Omega^*_X)) \underset{0}{\overset{\mathrm{incl.}}{\rightrightarrows}} \mathrm{Def}(\mathcal{E}nd^*(\Omega^*_X))\right),$$

which we can think of as a local period map for generalized deformations. As above, to go from local to global, we take the derived global sections; then, taking $\pi_{\leq 0}$ we obtain a natural morphism of **Set**-valued deformation functors:

$$\pi_{\leq 0} \operatorname{Def}(\mathbf{R}\Gamma\mathcal{P}oly_X^*) \xrightarrow{(l,e^i)} \pi_{\leq 0} \operatorname{holim}\left(\operatorname{Def}(\operatorname{End}_0^*(\mathbf{R}\Gamma\Omega_X^*)) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \operatorname{Def}(\operatorname{End}^*(\mathbf{R}\Gamma\Omega_X^*)) \right).$$

On the left, $\pi_{\leq 0} \operatorname{Def}(\mathbf{R}\Gamma\mathcal{P}oly_X^*)$ is the functor $\widetilde{\operatorname{Def}}_X$ of generalized deformations of X. It is shown in [FM09], using the Dolbeault resolution as a model for $\mathbf{R}\Gamma\Omega_X^*$, and making use of the $\partial\bar{\partial}$-lemma, that the pair $(\operatorname{End}_0^*(\mathbf{R}\Gamma\Omega_X^*), \operatorname{End}^*(\mathbf{R}\Gamma\Omega_X^*))$ is quasi-isomorphic to the pair $(0, \operatorname{End}^*(H^*(X, \mathbb{C})))$. Hence, one obtains the period map for generalized deformations:

$$\widetilde{\mathcal{P}} : \widetilde{\operatorname{Def}}_X \to \exp(\operatorname{End}^0(H^*(X, \mathbb{C})).$$

The tangent map $d\widetilde{\mathcal{P}}$ is the contraction of differential forms with multivector fields, read at the cohomology level:

$$H^1(\boldsymbol{i}) : \left(\oplus_k H^k(X; \wedge^k \mathcal{T}_X)\right) \otimes \left(\oplus_{p,q} H^q(X, \Omega_X^p)\right) \to \oplus_{p,q,k} H^{q+k}(X, \Omega_X^{p-k}),$$

and obstructions to generalized deformations are contained in the kernel of the contraction

$$H^2(\boldsymbol{i}) : \left(\oplus_k H^{k+1}(X; \wedge^k \mathcal{T}_X)\right) \otimes \left(\oplus_{p,q} H^q(X, \Omega_X^p)\right) \to \oplus_{p,q,k} H^{q+k+1}(X, \Omega_X^{p-k}).$$

In particular, from this one recovers Barannikov-Kontsevich's result, that generalized deformations of a smooth projective Calabi-Yau manifold are unobstructed [BK98].

It is tempting to extend the construction of the period map for generalized deformations to the case of a smooth projective manifold defined on an arbitrary characteristic zero algebraically closed field $\mathbb{K}$,

$$\widetilde{\mathcal{P}} : \widetilde{\operatorname{Def}}_X \to \exp(\operatorname{End}^0(H_{dR}^*(X; \mathbb{K})).$$

To do this one only has to prove that $(\operatorname{End}_0^*(\mathbf{R}\Gamma\Omega_X^*), \operatorname{End}^*(\mathbf{R}\Gamma\Omega_X^*))$ is quasi-isomorphic to $(0, \operatorname{End}^*(H_{dR}^*(X, \mathbb{K}))$. Yet, to mimic the argument in [FM09] one needs an algebraic substitute of the $\partial\bar{\partial}$-lemma. A natural candidate for this is E_1-degeneracy of the Hodge-to-de Rham spectral sequence for a smooth projective manifold. It has however to be remarked that, while in the $\partial\bar{\partial}$-lemma the two differentials ∂ and $\bar{\partial}$ play perfectly interchangeable roles, see, e.g. [Hu05, Corollary 3.2.10], this is not true for the Čech and the de Rham differentials in the Čech-de Rham bicomplex $\check{C}^q(\mathcal{U}, \Omega_X^p)$ associated with an open cover $\mathcal{U}$ of X. In particular only one of the two spectral sequences associated with this bicomplex, namely the Hodge-to-de Rham spectral sequence, degenerates at E_1. This asymmetry seems to suggest that a purely algebraic proof of the quasi-isomorphism $(\operatorname{End}_0^*(\mathbf{R}\Gamma\Omega_X^*), \operatorname{End}^*(\mathbf{R}\Gamma\Omega_X^*)) \simeq (0, \operatorname{End}^*(H_{dR}^*(X, \mathbb{K}))$ could be a nontrivial result.

References

[BK98] S. Barannikov, M. Kontsevich. *Frobenius manifolds and formality of Lie algebras of polyvector fields.* Internat. Math. Res. Notices **1998**, no. 4, 201–215

[Bo78] F. Bogomolov. *Hamiltonian Kählerian manifolds.* Dokl. Akad. Nauk SSSR **243** (1978) 1101-1104. Soviet Math. Dokl. **19** (1979) 1462-1465.

[DI87] P. Deligne, L. Illusie. *Relévements modulo p^2 et décomposition du complexe de de Rham.* Invent. Math. **89** (1987) 247-270.

[Do07] V. A. Dolgushev. *Erratum to: "A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold";* `arXiv:math/0703113`.

[Fa88] G. Faltings. *p-adic Hodge theory.* J. Amer. Math. Soc. **1** (1988) 255-299.

[FM06] D. Fiorenza, M. Manetti. *L_∞-algebras, Cartan homotopies and period maps;* `math/0605297`

[FM09] D. Fiorenza, M. Manetti. *A period map for generalized deformations.* Journal of Noncommutative Geometry, Vol. 3, No. 4 (2009), 579-597; `arXiv:0808.0140`.

[Ge09] E. Getzler. *Lie theory for nilpotent L_∞-algebras.* Ann. of Math., **170**, (1), (2009), 271-301; `arXiv:math/0404003v4`.

[GM90] W. M. Goldman, J. J. Millson. *The homotopy invariance of the Kuranishi space.* Illinois J. Math. **34** (1990) 337-367.

[Gr68] P. A. Griffiths. *Periods of integrals on algebraic manifolds. II. Local study of the period mapping.* Amer. J. Math. **90** (1968) 805-865.

[Hi97] V. Hinich. *Descent of Deligne groupoids.* Internat. Math. Res. Notices, (1997), **5**, 223-239.

[Hu05] D. Huybrechts. *Complex geometry. An introduction.* Universitext. Springer-Verlag, Berlin, 2005. xii+309 pp.

[IM010] D. Iacono, M. Manetti.*An algebraic proof of Bogomolov-Tian-Todorov theorem* Deformation Spaces. Vol. 39 (2010), p. 113-133; `arXiv:0902.0732`

[Ko03] M. Kontsevich. *Deformation quantization of Poisson manifolds.* Lett. Math. Phys. 66 (2003), no. 3, 157–216; `arXiv:q-alg/9709040`

[LM95] T. Lada, M. Markl. *Strongly homotopy Lie algebras.* Comm. Algebra 23 (1995), no. 6, 2147–2161; `arXiv:hep-th/9406095`

[LS93] T. Lada, J. Stasheff. *Introduction to SH Lie algebras for physicists.* Internat. J. Theoret. Phys. 32 (1993), no. 7, 1087-1103; `arXiv:hep-th/9209099`

[Lu09a] J. Lurie. *Higher Topos Theory.* Princeton University Press (2009); `arXiv:math/0608040`.

[Lu09b] J. Lurie. *A survey of elliptic cohomology.* Algebraic topology, 219–277, Abel Symp., 4, Springer, Berlin, 2009; `http://www.math.harvard.edu/~lurie/papers/survey.pdf`

[Lu11] J. Lurie. *Derived Algebraic Geometry X: Formal Moduli Problems* `http://www.math.harvard.edu/ lurie/papers/DAG-X.pdf`

[Ma02] M. Manetti. *Extended deformation functors.* Int. Math. Res. Not. 2002, no. 14, 719-756; `arXiv:math/9910071`

[Ma07] M. Manetti. *Lie description of higher obstructions to deforming submanifolds.* Ann. Sc. Norm. Super. Pisa Cl. Sci. **6** (2007) 631-659; `arXiv:math.AG/0507287`

[MV09] S. Merkulov, B. Vallette. *Deformation theory of representation of prop(erad)s I and II.* J. Reine Angew. Math. **634** (2009), 51106 and J. Reine Angew. Math. **636** (2009), 123–174; `arXiv:0707.0889`

[Pr10] J. P. Pridham. *Unifying derived deformation theories.* Adv. Math. **224** (2010), no.3, 772-826; `arXiv:0705.0344`

[Pr09] J. P. Pridham. *The homotopy theory of strong homotopy algebras and bialgebras*; `arXiv:0908.0116`

[Sc04] F. Schuhmacher. *Deformation of L_∞-Algebras*; `arXiv:math/0405485`

[Su77] D. Sullivan. *Infinitesimal computations in topology.* Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269-331.

[Ti87] G. Tian. *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric.* Mathematical Aspects of String Theory (San Diego, 1986), Adv. Ser. Math. Phys. 1, World Sci. Publishing, Singapore, (1987), 629-646.

[To89] A. N. Todorov. *The Weil-Petersson geometry of the moduli space of $SU(3)$ (Calabi-Yau) Manifolds I.* Commun. Math. Phys., **126**, (l989), 325-346.

Dipartimento di Matematica "Guido Castelnuovo", Università di Roma "La Sapienza", P.le Aldo Moro 5, I-00185 Roma Italy.
E-mail address: `fiorenza@mat.uniroma1.it`

Institut für Mathematik und Informatik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany
E-mail address: `elenamartinengo@gmail.com`