A SHORT NOTE ON ∞-GROUPOIDS AND THE PERIOD MAP FOR PROJECTIVE MANIFOLDS

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Abstract. We show how several classical results on the infinitesimal behaviour of the period map for smooth projective manifolds can be read in a natural and unified way within the framework of ∞-categories.

A common criticism of ∞-categories in algebraic geometry is that they are an extremely technical subject, so abstract to be useless in everyday mathematics. The aim of this note is to show in a classical example that quite the converse is true: even a naïve intuition of what an ∞-groupoid should be clarifies several aspects of the infinitesimal behaviour of the periods map of a projective manifold. In particular, the notion of Cartan homotopy turns out to be completely natural from this perspective, and so classical results such as Griffiths' expression for the differential of the periods map, the Kodaira principle on obstructions to deformations of projective manifolds, the Bogomolov-Tian-Todorov theorem, and Goldman-Millson quasi-abelianity theorem are easily recovered.

The use of the language of ∞-categories should not be looked at as providing new proofs for these results; namely, up to a change in language, our proofs verbatim reproduce arguments from the recent literature on the subject, particularly from the work of Marco Manetti and collaborators on dglas in deformation theory. Rather, by this change of language we change our point of view on the classical theorems above: in the perspective of ∞-sheaves from [Lu09b], all these theorems have a very simple *local* nature which can be naturally expressed in terms of ∞-groupoids (or, equivalently, of dglas); their classical *global* counterparts are then obtained by taking derived global sections. It is worth remarking that, if one prefers proofs which do not rely on the abstract machinery of ∞-categories, one can rework the arguments of this note in purely classical terms. Namely, once the abstract ∞-nonsense has suggested the "correct" local dglas, one can globalize them by means of an explicit model for the derived global sections, e.g., via resolutions by fine sheaves as in [FM09], or by the Thom-Sullivan-Whitney model as in [IM010].

Since most of the statements and constructions we recall in the paper are well known in the $(\infty, 1)$-categorical folklore, despite our efforts in giving credit, it is not unlikely we may have misattributed a few of the results; we sincerely apologize for this. We thank Ezra Getzler, Donatella Iacono, Marco Manetti, Jonathan Pridham, Carlos Simpson, Jim Stasheff, Bruno Vallette, Gabriele Vezzosi, and the nLab for suggestions and several inspiring conversations on the subject of this paper.

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Through the whole paper, $\mathbb{K}$ is a fixed characteristic zero field, all algebras are defined over $\mathbb{K}$ and local algebras have $\mathbb{K}$ as residue field. In order to keep our account readable, we will gloss over many details, particularly where the use of higher category theory is required.

1. From dglas to ∞-groupoids and back again

With any nilpotent dgla $\mathfrak{g}$ is naturally associated the simplicial set

$$\mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet),$$

where MC stands for the Maurer-Cartan functor mapping a dgla to the set of its Maurer-Cartan elements, and $\Omega_\bullet$ is the simplicial differential graded commutative associative algebra of polynomial differential forms on algebraic n-simplexes, for $n \geq 0$. The importance of this construction, which can be dated back to Sullivan's [Su77], relies on the fact that, as shown by Hinich and Getzler [Ge09, Hi97], the simplicial set $\mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet)$ is a Kan complex, or -to use a more evocative name- an ∞-groupoid. A convenient way to think of ∞-groupoids is as homotopy types of topological spaces; namely, it is well known[1] that any ∞-groupoid can be realized as the ∞-Poincaré groupoid, i.e., as the simplicial set of singular simplices, of a topological space, unique up to weak equivalence. Therefore, the reader who prefers to can substitute homotopy types of topological spaces for equivalence classes of ∞-groupoids. To stress this point of view, we'll denote the k-truncation of an ∞-groupoid $\mathbf{X}$ by the symbol $\pi_{\leq k}\mathbf{X}$. More explicitely, $\pi_{\leq k}\mathbf{X}$ is the k-groupoid whose j-morphisms are the j-morphisms of $\mathbf{X}$ for $j < k$, and are homotopy classes of j-morphisms of $\mathbf{X}$ for $j = k$. In particular, if $\mathbf{X}$ is the ∞-Poincaré groupoid of a topological space X, then $\pi_{\leq 0}\mathbf{X}$ is the set $\pi_0(X)$ of path-connected components of X, and $\pi_{\leq 1}\mathbf{X}$ is the usual Poincaré groupoid of X.

The next step is to consider an $(\infty, 1)$-category, i.e., an ∞-category whose hom-spaces are ∞-groupoids. This can be thought as a formalization of the naïve idea of having objects, morphisms, homotopies between morphisms, homotopies between homotopies, et cetera. In this sense, endowing a category with a model structure should be thought as a first step towards defining an $(\infty, 1)$-category structure on it.

Turning back to dglas, an easy way to produce nilpotent dglas is the following: pick an arbitrary dgla $\mathfrak{g}$; then, for any (differential graded) local Artin algebra A, take the tensor product $\mathfrak{g} \otimes \mathfrak{m}_A$, where $\mathfrak{m}_A$ is the maximal ideal of A. Since both constructions

$$\begin{aligned} \mathbf{DGLA} \times \mathbf{Art} &\to \textbf{nilpotent DGLA} \\ (\mathfrak{g}, A) &\mapsto \mathfrak{g} \otimes \mathfrak{m}_A \end{aligned}$$

and

$$\begin{aligned} \textbf{nilpotent DGLA} &\to \infty\textbf{-Grpd} \\ \mathfrak{g} &\mapsto \mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet) \end{aligned}$$

are functorial, their composition defines a functor

$$\mathrm{Def} : \mathbf{DGLA} \to \infty\textbf{-Grpd}^{\mathbf{Art}}.$$

The functor of Artin rings $\mathrm{Def}(\mathfrak{g})\colon \mathbf{Art} \to \infty\textbf{-Grpd}$ is called the formal ∞-groupoid associated with the dgla $\mathfrak{g}$. Note that $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}))$ is the usual set valued deformation functor associated with $\mathfrak{g}$, i.e., the functor

$$A \mapsto \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_A)/\text{gauge},$$

[1] At least in higher categories folklore

where the gauge equivalence of Maurer-Cartan elements is induced by the gauge action

$$e^{\alpha} * x = x + \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{\alpha})^n}{(n+1)!} \, ([\alpha, x] - d\alpha)$$

of $\exp(\mathfrak{g}^0 \otimes \mathfrak{m}_A)$ on the subset $\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_A)$ of $\mathfrak{g}^1 \otimes \mathfrak{m}_A$. However, due to the presence of nontrivial irrelevant stabilizers, the groupoid $\pi_{\leq 1}(\mathrm{Def}(\mathfrak{g}))$ is not equivalent to the action groupoid $\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}_A)//\exp(\mathfrak{g}^0 \otimes \mathfrak{m}_A)$, unless $\mathfrak{g}$ is concentrated in nonnegative degrees. We will come back to this later. Also note that the zero in $\mathfrak{g}^1 \otimes \mathfrak{m}_A$ gives a natural distinguished element in $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}))$: the isomorphism class of the trivial deformation. Since this marking is natural, we will use the same symbol $\pi_0(\mathrm{Def}(\mathfrak{g}))$ to denote both the set $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}))$ and the pointed set $\pi_0(\mathrm{Def}(\mathfrak{g}); 0)$.

A very good reason for working with ∞-groupoids valued deformation functors rather than with their apparently handier set-valued or groupoid-valued versions is the following folk statement, which allows one to move homotopy constructions back and forth between dglas and (homotopy types of) 'nice' topological spaces.

Theorem. *The functor* $\mathrm{Def} : \mathbf{DGLA} \to \infty\text{-}\mathbf{Grpd}^{\mathbf{Art}}$ *induces an equivalence of* $(\infty, 1)$*-categories.*

Here the $(\infty, 1)$-category structures involved are the most natural ones, and they are both induced by standard model category structures. Namely, on the category of dglas one takes surjective morphisms as fibrations and quasi-isomorphisms as weak equivalences, just as in the case of differential complexes, whereas the model category structure on the right hand side is induced by the standard model category structure on Kan complexes as a subcategory of simplicial sets. A sketchy proof of the above equivalence can be found in [Lu09a]; see also [Pr10].

2. Homotopy vs. gauge equivalent morphisms of dglas (with a detour into L_∞-morphisms)

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two (nilpotent) dglas. Then, from the $(\infty, 1)$-category structure on dglas, we have a natural notion of homotopy equivalence on the set of dgla morphisms $\mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$. Actually, in this form this is a too naïve statement. Indeed, in order to have a good notion of homotopy classes of morphisms one first has to perform a fibrant-cofibrant replacement of $\mathfrak{g}$ and $\mathfrak{h}$. In more colloquial terms, what one does is moving from the too narrow realm of strict dgla morphisms to the more flexible world of morphisms which preserve the dgla structure only up to homotopy; the formalization of this idea leads to the notion of L_∞-morphism, see, e.g., [LS93, Ko03]. Now, a notion of homotopy (and of higher homotopies) is well defined on the set of L_∞-morphisms between the dglas $\mathfrak{g}$ and $\mathfrak{h}$; this defines the ∞-groupoid $\mathrm{Hom}_\infty(\mathfrak{g}, \mathfrak{h})$. The definition of L_∞-morphism is best given in the language of differential graded cocommutative coalgebras. Namely, for a graded vector space V, let

$$C(V) = \bigoplus_{n \geq 1} (\otimes^n V)^{\Sigma_n} \subseteq \bigoplus_{n \geq 1} (\otimes^n V)$$

be the cofree graded cocommutative coalgebra without counit cogenerated by V, endowed with the standard coproduct

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{q_1+q_2=n} \sum_{\sigma \in \mathrm{Sh}(q_1,q_2)} \pm (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(q_1)}) \bigotimes (v_{\sigma(q_1+1)} \otimes \cdots \otimes v_{\sigma(n)}),$$

where σ ranges in the set of (q_1, q_2)-unshuffles and $\pm$ stands for the Koszul sign; an explicit determination for the signs in this and the following formulas can be found, e.g, in [LM95, Sc04]. If V is endowed with a dgla structure, then the differential of V can be seen as a linear morphism $Q_1^1 : V[1] \to V[2]$ and the Lie bracket of V as a linear morphism $Q_2^1 : (V[1] \otimes V[1])^{\Sigma_2} \to V[2]$, via the canonical identification $(V \wedge V)[2] \cong (V[1] \otimes V[1])^{\Sigma_2}$. Since $C(V[1])$ is cofreely generated by $V[1]$, the morphisms Q_1^1 and Q_2^1 uniquely extend to a degree 1 coderivation Q of $C(V[1])$, and the compatibility of the differental and the bracket of V translates into the condition $QQ = 0$, i.e., Q is a codifferential.

With the dglas $\mathfrak{g}$ and $\mathfrak{h}$ are therefore associated the differential graded cocommutative coalgebras $(C(\mathfrak{g}[1]), Q_\mathfrak{g})$ and $(C(\mathfrak{h}[1]), Q_\mathfrak{h})$, respectively. An L_∞-morphism between $\mathfrak{g}$ and $\mathfrak{h}$ is then defined as a coalgebra morphism $F : C(\mathfrak{g}[1]) \to C(\mathfrak{h}[1])$ compatible with the codifferentials, i.e., such that $FQ_\mathfrak{g} = Q_\mathfrak{h} F$. Since $C(\mathfrak{g}[1])$ is cofreely cogenerated by $\mathfrak{g}[1]$, a coalgebra morphism is completely determined by its Taylor coefficients, i.e. by the components $F_n^1 : (\otimes^n \mathfrak{g}[1])^{\Sigma_n} \to h[1]$. Similarly, the codifferentials $Q_\mathfrak{g}$ and $Q_\mathfrak{h}$ are completely determined by their Taylor coefficients which, as we have already remarked, are nothing but the differentials and the brackets of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Therefore, the equation $FQ_\mathfrak{g} = Q_\mathfrak{h} F$ is equivalent to the following set of equations involving only the morphisms F_n^1 and the dgla structures of $\mathfrak{g}$ and $\mathfrak{h}$:

$$d_\mathfrak{h} F_n^1(\gamma_1 \wedge \cdots \wedge \gamma_n) + \frac{1}{2} \sum_{\substack{q_1+q_2=n \\ \sigma \in \mathrm{Sh}(q_1,q_2)}} \pm [F_{q_1}^1(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(q_1)}), F_{q_2}^1(\gamma_{\sigma(q_1+1)} \wedge \cdots \wedge \gamma_{\sigma(q_1+q_2)})]_\mathfrak{h}$$
$$= \sum_i \pm F_n^1(\gamma_1 \wedge \cdots \wedge d_\mathfrak{g}\gamma_i \wedge \cdots \wedge \gamma_n)$$
$$+ \sum_{i<j} \pm F_{n-1}^1([\gamma_i, \gamma_j]_\mathfrak{g} \wedge \gamma_1 \wedge \cdots \wedge \widehat{\gamma_i} \wedge \cdots \wedge \widehat{\gamma_j} \wedge \cdots \wedge \gamma_{q+1}).$$

Note in particular that a dgla morphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ is, in a natural way, an L_∞-morphism between $\mathfrak{g}$ and $\mathfrak{h}$, of a very special kind: all but the first one of its Taylor coefficients vanish. One sometimes refers to this by saying that φ is a *strict* L_∞-morphisms between $\mathfrak{g}$ and $\mathfrak{h}$.

The equation defining L_∞-morphisms above manifestly looks like the Maurer-Cartan equation in a suitable dgla. This is not unexpected: by the equivalence between dglas and (formal) ∞-groupoids stated at the end of the previous section, there must be a dgla $\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h})$ such that $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h}) \otimes \Omega_\bullet)$ is equivalent to $\mathrm{Hom}_\infty(\mathfrak{g}, \mathfrak{h})$. What we see here is that the dgla $\underline{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{h})$ arises in a very natural way and admits a simple explicit description: it is the Chevalley-Eilenberg-type dgla given by the total dgla of the bigraded dgla

$$\underline{\mathrm{Hom}}^{p,q}(\mathfrak{g}, \mathfrak{h}) = \mathrm{Hom}_{\mathbb{Z}-\mathbf{Vect}}(\wedge^q \mathfrak{g}, \mathfrak{h}[p]) = \mathrm{Hom}^p(\wedge^q \mathfrak{g}, \mathfrak{h}),$$

endowed with the Lie bracket

$$[\, , \,]_{\underline{\mathrm{Hom}}} \colon \underline{\mathrm{Hom}}^{p_1,q_1}(\mathfrak{g}, \mathfrak{h}) \otimes \underline{\mathrm{Hom}}^{p_2,q_2}(\mathfrak{g}, \mathfrak{h}) \to \underline{\mathrm{Hom}}^{p_1+p_2,q_1+q_2}(\mathfrak{g}, \mathfrak{h})$$

defined by

$$[f, g]_{\underline{\mathrm{Hom}}}(\gamma_1 \wedge \cdots \wedge \gamma_{q_1+q_2}) = $$
$$= \sum_{\sigma \in \mathrm{Sh}(q_1,q_2)} \pm [f(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(q_1)}), g(\gamma_{\sigma(q_1+1)} \wedge \cdots \wedge \gamma_{\sigma(q_1+q_2)})]_\mathfrak{h},$$

with σ ranging in the set of (q_1, q_2)-unshuffles, and with the differentials

$$d_{1,0} \colon \underline{\mathrm{Hom}}^{p,q}(\mathfrak{g}, \mathfrak{h}) \to \underline{\mathrm{Hom}}^{p+1,q}(\mathfrak{g}, \mathfrak{h})$$

and

$$d_{0,1}\colon \underline{\mathrm{Hom}}^{p,q}(\mathfrak{g},\mathfrak{h}) \to \underline{\mathrm{Hom}}^{p,q+1}(\mathfrak{g},\mathfrak{h})$$

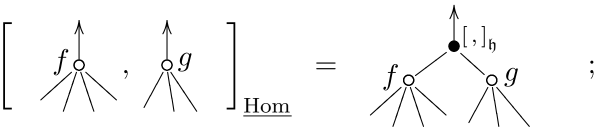
given by

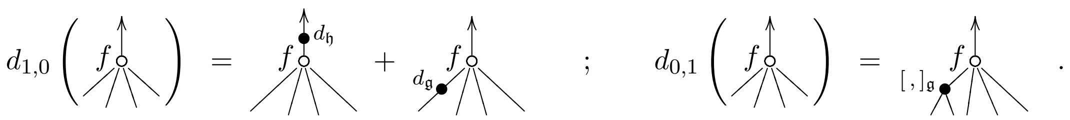
$$(d_{1,0}f)(\gamma_1\wedge\cdots\wedge\gamma_q) = d_{\mathfrak{h}}(f(\gamma_1\wedge\cdots\wedge\gamma_q)) + \sum_i \pm f(\gamma_1\wedge\cdots\wedge d_{\mathfrak{g}}\gamma_i\wedge\cdots\wedge\gamma_{q+1})$$

and

$$(d_{0,1}f)(\gamma_1\wedge\cdots\wedge\gamma_{q+1}) = \sum_{i<j} \pm f([\gamma_i,\gamma_j]_{\mathfrak{g}}\wedge\gamma_1\wedge\cdots\wedge\widehat{\gamma_i}\wedge\cdots\wedge\widehat{\gamma_j}\wedge\cdots\wedge\gamma_{q+1}).$$

These operations are best seen pictorially:





It should be remarked that the above construction is an instance of a more general phenomenon: if $\mathcal{O}$ is an operad, A is an $\mathcal{O}$-algebra, and B is a (differential graded) cocommutative coalgebra, then the space of linear mappings from B to A has a natural $\mathcal{O}$-algebra structure, see [Do07].

At the zeroth level, the equivalence $\mathrm{Hom}_\infty(\mathfrak{g},\mathfrak{h}) \simeq \mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g},\mathfrak{h})\otimes\Omega_\bullet)$ implies the following:

Proposition. *Let $f, g : \mathfrak{g} \to \mathfrak{h}$ be two L_∞-morphisms of dglas. Then f and g are gauge equivalent in* $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g},\mathfrak{h}))$ *if and only if f and g represent the same morphism in the homotopy category of dglas.*

Indeed, one immediately sees that $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g},\mathfrak{h}))$ is the set of L_∞-morphisms between $\mathfrak{g}$ and $\mathfrak{h}$ and, as we have already remarked, the set $\pi_{\leq 0}(\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g},\mathfrak{h})\otimes\Omega_\bullet))$ is somorphic to the quotient $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g},\mathfrak{h}))/\mathrm{gauge}$. On the other hand, $\pi_{\leq 0}(\mathrm{Hom}_\infty(\mathfrak{g},\mathfrak{h}))$ is the set of homotopy classes of L_∞-algebra morphisms between $\mathfrak{g}$ and $\mathfrak{h}$, i.e., the set of morphisms between $\mathfrak{g}$ and $\mathfrak{h}$ in the homotopy category of dglas.

We thank Jonathan Pridham for having shown us a proof of the equivalence between $\mathrm{Hom}_\infty(\mathfrak{g},\mathfrak{h})$ and $\mathrm{MC}(\underline{\mathrm{Hom}}(\mathfrak{g},\mathfrak{h})\otimes\Omega_\bullet)$, and Bruno Vallette for having addressed our attention to [Do07]. The same result holds, more in general, for the homotopy category of $\mathcal{O}$-algebras, where $\mathcal{O}$ is an operad, see [MV09, Pr09].

3. Cartan homotopies appear

Let now $\mathfrak{g}$ and $\mathfrak{h}$ be dglas and $\boldsymbol{i}\colon \mathfrak{g}\to\mathfrak{h}[-1]$ be a morphism of graded vector spaces. Then $\boldsymbol{i}$, and so also $-\boldsymbol{i}$, is an element of $\underline{\mathrm{Hom}}^{-1,1}(\mathfrak{g},\mathfrak{h})$, and so a degree zero element in the dgla $\underline{\mathrm{Hom}}(\mathfrak{g},\mathfrak{h})$. The gauge transformation $e^{-\boldsymbol{i}}$ will map the 0 dgla morphism to an L_∞-morphism $e^{-\boldsymbol{i}} * 0$ between $\mathfrak{g}$ and $\mathfrak{h}$. This L_∞-morphism will in general fail to be a dgla morphism (i.e., it will not be a strict L_∞-morphism) since its nonlinear components will be nontrivial. This is conveniently seen as follows: let $\boldsymbol{l} = d_{1,0}\boldsymbol{i}$; that is, $\boldsymbol{l}_a = d_{\mathfrak{h}}\boldsymbol{i}_a + \boldsymbol{i}_{d_{\mathfrak{g}}a}$ for any $a \in \mathfrak{g}$. Then the $(0,1)$-component of

$$e^{-\boldsymbol{i}} * 0 = \sum_{n=0}^{+\infty} \frac{(\mathrm{ad}_{-\boldsymbol{i}})^n}{(n+1)!}\,(d_{\underline{\mathrm{Hom}}}\boldsymbol{i}) = \sum_{n=0}^{+\infty} \frac{(\mathrm{ad}_{-\boldsymbol{i}})^n}{(n+1)!}\,(\boldsymbol{l} + \boldsymbol{i}_{[\,,\,]_{\mathfrak{g}}})$$

is just $\boldsymbol{l}$; the $(-1,2)$-component is

$$\boldsymbol{i}_{[\,,\,]_{\mathfrak{g}}} - \frac{1}{2}[\boldsymbol{i}, \boldsymbol{l}]_{\underline{\mathrm{Hom}}}$$

and, for $n \geq 3$ the $(1-n,n)$-component has two contributions, one of the form $[\boldsymbol{i}, [\boldsymbol{i}, \cdots, [\boldsymbol{i}, \boldsymbol{l}]_{\underline{\mathrm{Hom}}} \cdots]_{\underline{\mathrm{Hom}}}]_{\underline{\mathrm{Hom}}}$ and the other of the form $[\boldsymbol{i}, [\boldsymbol{i}, \cdots, [\boldsymbol{i}, \boldsymbol{i}_{[\,,\,]_{\mathfrak{g}}}]_{\underline{\mathrm{Hom}}} \cdots]_{\underline{\mathrm{Hom}}}]_{\underline{\mathrm{Hom}}}$. From this we see that all the nonlinear components of $e^{-\boldsymbol{i}} * 0$ vanish as soon as one imposes the two simple conditions

$$\boldsymbol{i}_{[a,b]_{\mathfrak{g}}} = \frac{1}{2}\big([\boldsymbol{i}_a, \boldsymbol{l}_b]_{\mathfrak{h}} \pm [\boldsymbol{i}_b, \boldsymbol{l}_a]_{\mathfrak{h}}\big) \qquad \text{and} \qquad [\boldsymbol{i}_a, [\boldsymbol{i}_b, \boldsymbol{l}_c]_{\mathfrak{h}}]_{\mathfrak{h}} = 0, \qquad \text{for all } a, b, c \in \mathfrak{g}.$$

A linear map $\boldsymbol{i}\colon \mathfrak{g} \to \mathfrak{h}[-1]$ satisfying the two conditions above will be called a Cartan homotopy. Up to our knowledge, this terminology has been introduced in [FM06, FM09], where the stronger conditions $\boldsymbol{i}_{[a,b]_{\mathfrak{g}}} = [\boldsymbol{i}_a, \boldsymbol{l}_b]_{\mathfrak{h}}$ and $[\boldsymbol{i}_a, \boldsymbol{i}_b]_{\mathfrak{h}} = 0$ were imposed. The name Cartan homotopy has an evident geometric origin: if $\mathcal{T}_X$ is the tangent sheaf of a smooth manifold X and Ω^*_X is the sheaf of complexes of differential forms, then the contraction of differential forms with vector fields is a Cartan homotopy

$$\boldsymbol{i}\colon \mathcal{T}_X \to \mathcal{E}nd^*(\Omega^*_X)[-1].$$

In this case, $\boldsymbol{l}_a$ is the Lie derivative along the vector field a, and the conditions $\boldsymbol{i}_{[a,b]} = [\boldsymbol{i}_a, \boldsymbol{l}_b]$ and $[\boldsymbol{i}_a, \boldsymbol{i}_b] = 0$, together with the defining equation $\boldsymbol{l}_a = [d_{\Omega^*_X}, \boldsymbol{i}_a]$ and with the equations $\boldsymbol{l}_{[a,b]} = [\boldsymbol{l}_a, \boldsymbol{l}_b]$ and $[d_{\Omega^*_X}, \boldsymbol{l}_a] = 0$ expressing the fact that $\boldsymbol{l}\colon \mathcal{T}_X \to \mathcal{E}nd^*(\Omega^*_X)$ is a dgla morphism, are nothing but the well-known Cartan identities involving contractions and Lie derivatives.

The above discussion can be summarized as follows.

Proposition. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two dglas. If $\boldsymbol{i}\colon \mathfrak{g} \to \mathfrak{h}[-1]$ is a Cartan homotopy, then $\boldsymbol{l} = d_{1,0}\boldsymbol{i}\colon \mathfrak{g} \to \mathfrak{h}$ is a dgla morphism gauge equivalent to the zero morphism via the gauge action of $e^{\boldsymbol{i}}$.*

4. Homotopy fibers (and the associated exact sequence)

Let now $\boldsymbol{i}\colon \mathfrak{g} \to \mathfrak{h}[-1]$ be a Cartan homotopy and $\boldsymbol{l}\colon \mathfrak{g} \to \mathfrak{h}$ be the associated dgla morphism. Then, the equation $e^{\boldsymbol{i}} * \boldsymbol{l} = 0$ implies that, for any subdgla $\mathfrak{n}$ of $\mathfrak{h}$ containing the image of $\boldsymbol{l}$, the morphism $\boldsymbol{l}\colon \mathfrak{g} \to \mathfrak{n}$ equalizes the diagram $\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h}$ up to a homotopy provided by the gauge action of $e^{\boldsymbol{i}}$. Hence we have a morphism to the homotopy limit:

$$\mathfrak{g} \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h} \right).$$

Taking Def's we obtain a natural transformation of ∞-groupoid valued functors:

$$\mathrm{Def}(\mathfrak{g}) \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathrm{Def}(\mathfrak{n}) \underset{\mathrm{Def}_0}{\overset{\mathrm{Def}_{\text{incl.}}}{\rightrightarrows}} \mathrm{Def}(\mathfrak{h}) \right).$$

The map $\mathrm{Def}_0\colon \mathrm{Def}(\mathfrak{n}) \to \mathrm{Def}(\mathfrak{h})$ is the constant map to the distinguished point 0 in $\mathrm{Def}(\mathfrak{h})$; therefore, the homotopy limit above is the homotopy fiber of $\mathrm{Def}_{\mathrm{incl.}}\colon \mathrm{Def}(\mathfrak{n}) \to \mathrm{Def}(\mathfrak{h})$ over the point 0, and we obtain a natural transformation

$$\mathrm{Def}(\mathfrak{g}) \xrightarrow{(\boldsymbol{l},e^{\boldsymbol{i}})} \mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0),$$

which at the zeroth level gives a natural transformation of **Set**-valued deformation functors

$$\mathcal{P}\colon \pi_{\leq 0}\,\mathrm{Def}(\mathfrak{g}) \to \pi_{\leq 0}\mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0).$$

The differential of $\mathcal{P}$ is easily computed: it is the linear map

$$H^1(\mathfrak{g}) \xrightarrow{H^1((\boldsymbol{l},e^{\boldsymbol{i}}))} H^1(\mathrm{holim}\left(\mathfrak{n} \overset{\mathrm{incl.}}{\underset{0}{\rightrightarrows}} \mathfrak{h} \right)).$$

Since the model category structure on dglas is the same as on differential complexes, we can compute the H^1 on the right hand side by taking the holimit in complexes. Then the natural quasi-isomorphism $\mathrm{holim}(\mathfrak{n} \overset{\mathrm{incl.}}{\underset{0}{\rightrightarrows}} \mathfrak{h}) \simeq (\mathfrak{h}/\mathfrak{n})[-1]$ tells us that the differential of $\mathcal{P}$ is just the map

$$H^1(\boldsymbol{i})\colon H^1(\mathfrak{g}) \to H^0(\mathfrak{h}/\mathfrak{n})$$

induced by the morphism of complexes $\boldsymbol{i}\colon \mathfrak{g} \to (\mathfrak{h}/\mathfrak{n})[-1]$. Also, the map

$$H^2(\boldsymbol{i})\colon H^2(\mathfrak{g}) \to H^1(\mathfrak{h}/\mathfrak{n})$$

maps the obstruction space of $\pi_{\leq 0}\,\mathrm{Def}(\mathfrak{g})$ (as a subspace of $H^2(\mathfrak{g})$) to the obstruction space of $\pi_{\leq 0}\mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0)$ (as a subspace of $H^1(\mathfrak{h}/\mathfrak{n})$). In particular, if $\pi_{\leq 0}\mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0)$ is smooth, and therefore unobstructed, the obstructions of the deformation functor $\pi_{\leq 0}\,\mathrm{Def}(\mathfrak{g})$ are contained in the kernel of the map $H^2(\boldsymbol{i})\colon H^2(\mathfrak{g}) \to H^1(\mathfrak{h}/\mathfrak{n})$.

To investigate the geometry of $\pi_{\leq 0}\mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0)$ note that, by looking at it as a pointed set, it nicely fits into the homotopy exact sequence

$$\pi_1(\mathrm{Def}(\mathfrak{n});0) \xrightarrow{\mathrm{Def}_{\mathrm{incl.}*}} \pi_1(\mathrm{Def}(\mathfrak{h});0) \to \pi_0(\mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0);0) \to \pi_0(\mathrm{Def}(\mathfrak{n});0),$$

so we get a canonical isomorphism

$$\pi_{\leq 0}\mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0) \simeq \frac{\pi_1(\mathrm{Def}(\mathfrak{h});0)}{\mathrm{Def}_{\mathrm{incl.}*}\pi_1(\mathrm{Def}(\mathfrak{n});0)}.$$

The group $\pi_1(\mathrm{Def}(\mathfrak{h});0)$ is the group of automorphisms of 0 in the groupoid $\pi_{\leq 1}(\mathrm{Def}(\mathfrak{h}))$. We have already remarked that this groupoid is not equivalent to the Deligne groupoid of $\mathfrak{h}$, i.e., the action groupoid for the gauge action of $\exp(\mathfrak{h}^0 \otimes \mathfrak{m}_A)$ on $\mathrm{MC}(\mathfrak{h} \otimes \mathfrak{m}_A)$, since the irrelevant stabilizer

$$\mathrm{Stab}(x) = \{dh + [x,h] \mid h \in \mathfrak{h}^{-1} \otimes \mathfrak{m}_A\} \subseteq \{a \in \mathfrak{h}^0 \otimes \mathfrak{m}_A \mid e^a * x = x\}$$

of a Maurer-Cartan element x may be nontrivial. However, the group $\pi_1(\mathrm{Def}(\mathfrak{h});0)$ only sees the connected component of 0, and on this connected component the irrelevant stabilizers are trivial as soon as the differential of the dgla $\mathfrak{h}$ vanishes on $\mathfrak{h}^{-1}$. This immediately follows from noticing that irrelevant stabilizers of gauge equivalent Maurer-Cartan elements are conjugate subgroups of $\exp(\mathfrak{h}^0 \otimes \mathfrak{m}_A)$, see, e.g., [Ma07]. In particular, if $\mathfrak{h}$ is a graded Lie algebra (which we can consider as a dgla with trivial differential), then $\pi_1(\mathrm{Def}(\mathfrak{h});0) \simeq \exp(\mathfrak{h}^0)$, where $\mathfrak{h}^0$ denotes the degree zero component of $\mathfrak{h}$. Similarly, since $\mathfrak{n}$ is a subdgla of $\mathfrak{h}$, one has $\pi_1(\mathrm{Def}(\mathfrak{n});0) \simeq \exp(\mathfrak{n}^0)$, and the group homomorphism

$\mathrm{Def}_{\mathrm{incl.}*}$ is just the inclusion. Therefore, when $\mathfrak{h}$ has trivial differential, the map induced at the zeroth level by $\mathrm{Def}(\mathfrak{g}) \to \mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0)$ is just the natural map

$$e^{\boldsymbol{i}} \colon \pi_{\leq 0} \mathrm{Def}(\mathfrak{g}) \to \exp(\mathfrak{h}^0)/\exp(\mathfrak{n}^0)$$

which sends a Maurer-Cartan element $\xi \in \mathfrak{g}^1 \otimes \mathfrak{m}_A$ to $e^{\boldsymbol{i}_\xi} \mod \exp(\mathfrak{n}^0)$. A particularly interesting case is when the pair $(\mathfrak{h}, \mathfrak{n})$ is formal,[2] i.e., if the inclusion of $\mathfrak{n}$ in $\mathfrak{h}$ induces an inclusion in cohomology and the two inclusions $H^*(\mathfrak{n}) \hookrightarrow H^*(\mathfrak{h})$ and $\mathfrak{n} \hookrightarrow \mathfrak{h}$ are homotopy equivalent. Indeed, in this case the pair $(\mathrm{Def}(\mathfrak{h}), \mathrm{Def}(\mathfrak{n}))$ will be equivalent to the pair $(\mathrm{Def}(H^*(\mathfrak{h})), \mathrm{Def}(H^*(\mathfrak{h})))$ and there will be an induced isomorphism between $\pi_1(\mathrm{Def}(\mathfrak{h}); 0)/\mathrm{Def}_{\mathrm{incl.}*}\pi_1(\mathrm{Def}(\mathfrak{n}); 0)$ and the smooth homogeneous space $\exp(H^0(\mathfrak{h}))/\exp(H^0(\mathfrak{n}))$. We can summarize the results described in this section as follows:

Proposition. *Let $\boldsymbol{i} \colon \mathfrak{g} \to \mathfrak{h}[-1]$ be a Cartan homotopy, let $\boldsymbol{l} \colon \mathfrak{g} \to \mathfrak{h}$ be the associated dgla morphism, and let $\mathfrak{n}$ be a subdgla of $\mathfrak{h}$ containing the image of $\boldsymbol{l}$. Then, if the pair $(\mathfrak{h}, \mathfrak{n})$ is formal, we have a natural transformation[3] of* **Set***-valued deformation functors*

$$\mathcal{P} \colon \pi_{\leq 0}(\mathrm{Def}(\mathfrak{g})) \to \exp(H^0(\mathfrak{h}))/\exp(H^0(\mathfrak{n}))$$

induced by the dgla map

$$\mathfrak{g} \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\mathrm{incl.}}{\rightrightarrows}} \mathfrak{h} \right).$$

In particular, since $\exp(H^0(\mathfrak{h}))/\exp(H^0(\mathfrak{n}))$ is smooth, the obstructions of the **Set***-valued deformation functor $\pi_{\leq 0}(\mathrm{Def}(\mathfrak{g}); 0)$ are contained in the kernel of the map $H^2(\boldsymbol{i}) \colon H^2(\mathfrak{g}) \to H^1(\mathfrak{h}/\mathfrak{n})$.*

This result can be nicely refined, by showing how the main result from [IM010] naturally fits into the discussion above. We have:

Proposition. *Let $(\mathfrak{h}, \mathfrak{n})$ be a formal pair of dglas. Then, the dgla $\mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\mathrm{incl.}}{\rightrightarrows}} \mathfrak{h} \right)$ is quasi-abelian. In particular there is a (non-canonical) quasi-isomorphism of dglas between $\mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\mathrm{incl.}}{\rightrightarrows}} \mathfrak{h} \right)$ and the abelian dgla obtained by endowing the complex $(\mathfrak{h}/\mathfrak{n})[-1]$ with the trivial bracket.*

To see this, notice that, since by hypothesis the inclusion $\mathfrak{n} \hookrightarrow \mathfrak{h}$ induces an inclusion $H^*(\mathfrak{n}) \hookrightarrow H^*(\mathfrak{h})$, the projection $\mathfrak{h}[-1] \to \mathfrak{h}/\mathfrak{n}[-1]$ admits a section $\boldsymbol{i}$ which is a morphism of complexes. Denote by $\mathfrak{g}$ the dgla obtained from the complex $\mathfrak{h}/\mathfrak{n}[-1]$ by endowing it with the trivial bracket. Then, the map of graded vector spaces $\boldsymbol{i} \colon \mathfrak{g} \to \mathfrak{h}[-1]$ is a Cartan homotopy whose associated dgla morphism is the zero map $0 \colon \mathfrak{g} \to \mathfrak{h}$. Therefore we have a dgla map

$$(\mathfrak{h}/\mathfrak{n})[-1] \xrightarrow{(0, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathfrak{n} \underset{0}{\overset{\mathrm{incl.}}{\rightrightarrows}} \mathfrak{h} \right).$$

[2]We are not sure whether this terminology is a standard one

[3]This natural transformation is not canonical: it depends on the choice of a quasi isomorphism $(\mathfrak{h}, \mathfrak{n}) \simeq (H^*(\mathfrak{h}), H^*(\mathfrak{n}))$. Also note that the tangent space at 0 on the right hand side is $H^0(\mathfrak{h})/H^0(\mathfrak{n})$; this is only apparently in contrast with the general result mentioned above that the tangent space at 0 to $\pi_{\leq 0}\mathrm{hoDef}^{-1}_{\mathrm{incl.}}(0)$ is $H^0(\mathfrak{h}/\mathfrak{n})$. Indeed, when $(\mathfrak{h}, \mathfrak{n})$ is a formal pair, the two vector spaces $H^0(\mathfrak{h})/H^0(\mathfrak{n})$ and $H^0(\mathfrak{h}/\mathfrak{n})$ are (non canonically) isomorphic.

Since $\boldsymbol{i}$ is a section to $\mathfrak{h}[-1] \to \mathfrak{h}/\mathfrak{n}[-1]$, the map in cohomology

$$H^*(\mathfrak{h}/\mathfrak{n})[-1] \xrightarrow{H^*(0,e^{\boldsymbol{i}})} H^*(\operatorname{holim}\left(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h} \right))$$

is identified with the identity of $H^*(\mathfrak{h}/\mathfrak{n})[-1]$ by the the natural quasi-isomorphism of complexes $\operatorname{holim}(\mathfrak{n} \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathfrak{h}) \xrightarrow{\sim} (\mathfrak{h}/\mathfrak{n})[-1]$.

5. From local to global, and classical (and generalized) periods

Assume now $\mathbb{K}$ is algebraically closed. Let X be a smooth projective manifold, and let $\mathcal{T}_X$ and Ω^*_X be the tangent sheaf and the sheaf of differential forms on X, respectively. The sheaf of complexes $(\Omega^*_X, d_{\Omega^*_X})$ is naturally filtered by setting $F^p\Omega^*_X = \oplus_{i\geq p}\Omega^i_X$. Finally, let $\mathcal{E}nd^*(\Omega^*_X)$ be the endomorphism sheaf of Ω^*_X and $\mathcal{E}nd^{\geq 0}(\Omega^*_X)$ be the subsheaf consisting of nonnegative degree elements. Note that $\mathcal{E}nd^{\geq 0}(\Omega^*_X)$ is a subdgla of $\mathcal{E}nd^*(\Omega^*_X)$, and can be seen as the subdgla of endomorphisms preserving the filtration on Ω^*_X.

Recall that the prototypical example of Cartan homotopy was the contraction of differential forms with vector fields $\boldsymbol{i} : \mathcal{T}_X \to \mathcal{E}nd^*(\Omega^*_X)[-1]$; the corresponding dgla morphism is $a \mapsto \boldsymbol{l}_a$, where $\boldsymbol{l}_a$ the Lie derivative along a. Explicitly, $\boldsymbol{l}_a = d_{\Omega^*_X}\circ\boldsymbol{i}_a+\boldsymbol{i}_a\circ d_{\Omega^*_X}$, and so $\boldsymbol{l}_a$ preserves the filtration. Therefore, we have a natural transformation[4]

$$\operatorname{Def}(\mathcal{T}_X) \xrightarrow{(\boldsymbol{l},e^{\boldsymbol{i}})} \operatorname{holim}\left(\operatorname{Def}(\mathcal{E}nd^{\geq 0}(\Omega^*_X)) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \operatorname{Def}(\mathcal{E}nd^*(\Omega^*_X)) \right).$$

The homotopy fiber on the right should be thought as a homotopy flag manifold. Let us briefly explain this. At least naïvely, the functor $\operatorname{Def}(\mathcal{E}nd^*(\Omega^*_X))$ describes the infinitesimal deformations of the differential complex Ω^*_X, whereas the functor $\operatorname{Def}(\mathcal{E}nd^{\geq 0}(\Omega^*_X))$ describes the deformations of the filtered complex $(\Omega^*_X, F^\bullet\Omega^*_X)$, i.e., of the pair consisting of the complex Ω^*_X *and* the filtration $F^\bullet\Omega^*_X$. Therefore, the holimit describes a deformation of the pair (complex, filtration) together with a trivialization of the deformation of the complex. Summing up, the contraction of differential forms with vector fields induces a map of deformation functors

$$\operatorname{Def}(\mathcal{T}_X) \to \operatorname{hoFlag}(\Omega^*_X; F^\bullet\Omega^*_X),$$

which we will call the *local periods map* of X.

To recover from this the classical periods map, we just need to take global sections. Clearly, since we are working in homotopy categories, these will be derived global sections. The morphism of sheaves $\boldsymbol{i} : \mathcal{T}_X \to \mathcal{E}nd^*(\Omega^*_X)[-1]$ induces a Cartan homotopy $\boldsymbol{i} : \mathbf{R}\Gamma\mathcal{T}_X \to \mathbf{R}\Gamma\mathcal{E}nd^*(\Omega^*_X)[-1]$; composing this with the dgla morphism $\mathbf{R}\Gamma\mathcal{E}nd^*(\Omega^*_X) \to \operatorname{End}^*(\mathbf{R}\Gamma\Omega^*_X)$ induced by the action of (derived) global sections of the endomorphism sheaf of Ω^*_X on (derived) global sections of Ω^*_X, we get a Cartan homotopy

$$\boldsymbol{i} : \mathbf{R}\Gamma\mathcal{T}_X \to \operatorname{End}^*(\mathbf{R}\Gamma\Omega^*_X)[-1].$$

[4]Of what? The correct answer would be of ∞-sheaves, see [Lu09b], but to keep this note as far as possible at an informal level we will content ourselves with noticing that, for any open subset U of X, there is a natural transformation of ∞-groupoids induced by the dgla map $\mathcal{T}_X(U) \xrightarrow{(\boldsymbol{l},e^{\boldsymbol{i}})}$ $\operatorname{holim}\left(\mathcal{E}nd^{\geq 0}(\Omega^*_X)(U) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathcal{E}nd^*(\Omega^*_X)\ (U)\right)$.

The image of the corresponding dgla morphism $\boldsymbol{l}$ (the derived globalization of Lie derivative) preserves the filtration $F^\bullet \mathbf{R}\Gamma\Omega^*_X$ induced by $F^\bullet\Omega^*_X$, so we have a natural map of ∞-groupoids

$$\mathrm{Def}(\mathbf{R}\Gamma\mathcal{T}_X) \to \mathrm{hoFlag}(\mathbf{R}\Gamma\Omega^*_X; F^\bullet\mathbf{R}\Gamma\Omega^*_X)$$

and, at the zeroth level, a map of **Set**-valued deformation functors

$$\mathcal{P}\colon \pi_{\leq 0}\,\mathrm{Def}(\mathbf{R}\Gamma\mathcal{T}_X) \to \pi_{\leq 0}\mathrm{hoFlag}(\mathbf{R}\Gamma\Omega^*_X; F^\bullet\mathbf{R}\Gamma\Omega^*_X)$$

The functor on the left hand side is the **Set**-valued functor of (classical) infinitesimal deformations of X; let us denote it by Def_X. If we denote by $\mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X; F^\bullet\mathbf{R}\Gamma\Omega^*_X)$ the subdgla of $\mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X)$ consisting of endomorhisms preserving the filtration, then the pair $(\mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X), \mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X; F^\bullet\mathbf{R}\Gamma\Omega^*_X))$ is formal.[5] Moreover, $H^0(\mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X)) = \mathrm{End}^0(H^*_{dR}(X;\mathbb{K}))$ and $H^0(\mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X; F^\bullet\mathbf{R}\Gamma\Omega^*_X)) = \mathrm{End}^0(H^*_{dR}(X;\mathbb{K}); F^\bullet H^*_{dR}(X;\mathbb{K}))$, where $F^\bullet H^*_{dR}(X;\mathbb{K})$ is the Hodge filtration on the algebraic de Rham cohomology of X. By results described in the previous section, this means that

$$\pi_{\leq 0}\mathrm{hoFlag}(\mathbf{R}\Gamma\Omega^*_X; F^\bullet\mathbf{R}\Gamma\Omega^*_X) \simeq \frac{\exp(\mathrm{End}^0(H^*_{dR}(X;\mathbb{K})))}{\exp(\mathrm{End}^0(H^*_{dR}(X;\mathbb{K}); F^\bullet H^*_{dR}(X;\mathbb{K})))}$$

and we recover the classical periods map of X

$$\mathcal{P}\colon\ \mathrm{Def}_X \to \mathrm{Flag}(H^*_{dR}(X;\mathbb{K}); F^\bullet H^*_{dR}(X;\mathbb{K})).$$

Also, the differential of $\mathcal{P}$ is the map induced in cohomology by the contraction of differential forms with vector fields,

$$H^1(\boldsymbol{i})\colon H^1(X,\mathcal{T}_X) \to \int_p \mathrm{Hom}^0\left(F^pH^*_{dR}(X;\mathbb{K}); \frac{H^*_{dR}(X;\mathbb{K})}{F^pH^*_{dR}(X;\mathbb{K})}\right),$$

a result originally proved by Griffiths [Gr68]. In the above formula, $\int_p$ denotes the end of the diagram

$$\mathrm{Hom}^0\left(F^pH^*_{dR}; \frac{H^*_{dR}}{F^pH^*_{dR}}\right) \to \mathrm{Hom}^0\left(F^pH^*_{dR}; \frac{H^*_{dR}}{F^{p+1}H^*_{dR}}\right) \leftarrow \mathrm{Hom}^0\left(F^{p+1}H^*_{dR}; \frac{H^*_{dR}}{F^{p+1}H^*_{dR}}\right)$$

Also, we have the following version of the so-called Kodaira principle (ambient cohomology annihilates obstructions): obstructions to classical infinitesimal deformations of X are contained in the kernel of

$$H^2(\boldsymbol{i})\colon H^2(X,\mathcal{T}_X) \to \int_p \mathrm{Hom}^1\left(F^pH^*_{dR}(X;\mathbb{K}); \frac{H^*(X;\mathbb{K})}{F^pH^*_{dR}(X;\mathbb{K})}\right).$$

In particular, if the canonical bundle of X is trivial, then the contraction pairing

$$H^2(X,\mathcal{T}_X) \otimes H^{n-2}(X,\Omega^1_X) \to H^n(X;\mathcal{O}_X) \simeq \mathbb{K}$$

is nondegenerate, and so classical deformations of X are unobstructed (Bogomolov-Tian-Todorov theorem, see [Bo78, Ti87, To89]). Following [IM010], one immediately obtains the following refinement, due in its original formulation to Goldman and Millson [GM90]: if the canonical bundle of X is trivial, then $\mathbf{R}\Gamma\mathcal{T}_X$ is a quasi-abelian dgla. To see this, just notice that the dgla map

$$\mathbf{R}\Gamma\mathcal{T}_X \xrightarrow{(\boldsymbol{l},e^{\boldsymbol{i}})} \mathrm{holim}\left(\ \mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X; F^\bullet\mathbf{R}\Gamma\Omega^*_X) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X)\ \right)$$

is injective in cohomology and the target is a quasi-abelian dgla. Indeed, if $f\colon \mathfrak{g}\to\mathfrak{h}$ is a dgla morphism, with $H^*(f)$ injective and $\mathfrak{h}$ quasi-abelian, then the diagram of dglas

[5] This is essentially a consequence of the E_1-degeneration of the Hodge-to-de Rham spectral sequence, see, e.g., [DI87, Fa88].

$$\begin{array}{ccccc} \mathfrak{g} & & & \mathfrak{k} & \\ & \searrow^{f} & \swarrow_{\sim} & & \searrow^{\sim} \\ & & \mathfrak{h} & & & V \end{array}$$

where V is a graded vector space considered as a dgla with trivial differential and bracket, can be completed to a homotopy commutative diagram

$$\begin{array}{ccccccc} & & \mathfrak{l} & & & & \\ & \swarrow^{\sim} & & \searrow & & & \\ \mathfrak{g} & & & & \mathfrak{k} & & \\ & \searrow^{f} & & \swarrow_{\sim} & & \searrow^{\sim} & \\ & & \mathfrak{h} & & & & V \\ & & & & & & & \searrow \\ & & & & & & & & W \end{array}$$

with W a graded vector space, and the composition $\mathfrak{l} \to W$ a quasi-isomorphism.

As a conclusion, we recast the description of a period map for generalized deformations from [FM09] in the language of this note. Let X be a smooth projective variety defined over the field $\mathbb{C}$ of the complex numbers, and denote by $\mathcal{P}oly_X^*$ the sheaf of dglas of *multivector fields* on X, given by $\mathcal{P}oly_X^j = \bigwedge^{1-j} \mathcal{T}_X$, endowed with the zero differential and with the Schouten-Nijenhuis bracket. Notice that $\mathcal{T}_X$ is a sub-Lie algebra of the dgla $\mathcal{P}oly_X^*$. The contraction of differential forms with multivector fields

$$\boldsymbol{i} : \mathcal{P}oly_X^* \to \mathcal{E}nd^*(\Omega_X^*)[-1]$$

is a Cartan homotopy, and the corresponding dgla morphism $\boldsymbol{l}$ is the Lie derivative along a multivector field, i.e., $\boldsymbol{l}_\xi = [d_{\Omega_X^*}, \boldsymbol{i}_\xi]$. It is immediate that the image of $\boldsymbol{l}$ is contained in the sub-sheaf of dglas:

$$\mathcal{E}nd_0^*(\Omega_X^*) = \{f \in \mathcal{E}nd^*(\Omega_X^*) \,|\, f(\ker d_{\Omega_X^*}) \subseteq \mathrm{Im}(d_{\Omega_X^*})\} \subset \mathcal{E}nd^*(\Omega_X^*),$$

and so we have a natural transformation:

$$\mathrm{Def}(\mathcal{P}oly_X^*) \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \mathrm{holim}\left(\mathrm{Def}(\mathcal{E}nd_0^*(\Omega_X^*)) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathrm{Def}(\mathcal{E}nd^*(\Omega_X^*)) \right),$$

which we can think of as a local period map for generalized deformations. As above, to go from local to global, we take the derived global sections; then, taking $\pi_{\leq 0}$ we obtain a natural morphism of **Set**-valued deformation functors:

$$\pi_{\leq 0}\,\mathrm{Def}(\mathbf{R}\Gamma\mathcal{P}oly_X^*) \xrightarrow{(\boldsymbol{l}, e^{\boldsymbol{i}})} \pi_{\leq 0}\,\mathrm{holim}\left(\mathrm{Def}(\mathrm{End}_0^*(\mathbf{R}\Gamma\Omega_X^*)) \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} \mathrm{Def}(\mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*)) \right).$$

On the left, $\pi_{\leq 0}\,\mathrm{Def}(\mathbf{R}\Gamma\mathcal{P}oly_X^*)$ is the functor $\widetilde{\mathrm{Def}}_X$ of generalized deformations of X. It is shown in [FM09], using the Dolbeault resolution as a model for $\mathbf{R}\Gamma\Omega_X^*$, and making use of the $\partial\overline{\partial}$-lemma, that the pair $(\mathrm{End}_0^*(\mathbf{R}\Gamma\Omega_X^*), \mathrm{End}^*(\mathbf{R}\Gamma\Omega_X^*))$ is quasi-isomorphic to the pair $(0, \mathrm{End}^*(H^*(X, \mathbb{C})))$. Hence, one obtains the period map for generalized deformations:

$$\widetilde{\mathcal{P}} : \widetilde{\mathrm{Def}}_X \to \exp(\mathrm{End}^0(H^*(X, \mathbb{C})).$$

The tangent map $d\widetilde{\mathcal{P}}$ is the contraction of differential forms with multivector fields, read at the cohomology level:

$$H^1(\boldsymbol{i}) : \big(\oplus_k H^k(X; \wedge^k \mathcal{T}_X)\big) \otimes \big(\oplus_{p,q} H^q(X, \Omega_X^p)\big) \to \oplus_{p,q,k} H^{q+k}(X, \Omega_X^{p-k}),$$

and obstructions to generalized deformations are contained in the kernel of the contraction

$$H^2(\boldsymbol{i}) : \big(\oplus_k H^{k+1}(X; \wedge^k \mathcal{T}_X)\big) \otimes \big(\oplus_{p,q} H^q(X, \Omega_X^p)\big) \to \oplus_{p,q,k} H^{q+k+1}(X, \Omega_X^{p-k}).$$

In particular, from this one recovers Barannikov-Kontsevich's result, that generalized deformations of a smooth projective Calabi-Yau manifold are unobstructed [BK98].

It is tempting to extend the construction of the period map for generalized deformations to the case of a smooth projective manifold defined on an arbitrary characteristic zero algebraically closed field $\mathbb{K}$,

$$\widetilde{\mathcal{P}} : \widetilde{\mathrm{Def}}_X \to \exp(\mathrm{End}^0(H^*_{dR}(X;\mathbb{K})).$$

To do this one only has to prove that $(\mathrm{End}^*_0(\mathbf{R}\Gamma\Omega^*_X), \mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X))$ is quasi-isomorphic to $(0, \mathrm{End}^*(H^*_{dR}(X,\mathbb{K}))$. Yet, to mimic the argument in [FM09] one needs an algebraic substitute of the $\partial\overline{\partial}$-lemma. A natural candidate for this is E_1-degeneracy of the Hodge-to-de Rham spectral sequence for a smooth projective manifold. It has however to be remarked that, while in the $\partial\overline{\partial}$-lemma the two differentials ∂ and $\overline{\partial}$ play perfectly interchangeable roles, see, e.g. [Hu05, Corollary 3.2.10], this is not true for the Čech and the de Rham differentials in the Čech-de Rham bicomplex $\check{C}^q(\mathcal{U}, \Omega^p_X)$ associated with an open cover $\mathcal{U}$ of X. In particular only one of the two spectral sequences associated with this bicomplex, namely the Hodge-to-de Rham spectral sequence, degenerates at E_1. This asymmetry seems to suggest that a purely algebraic proof of the quasi-isomorphism $(\mathrm{End}^*_0(\mathbf{R}\Gamma\Omega^*_X), \mathrm{End}^*(\mathbf{R}\Gamma\Omega^*_X)) \simeq (0, \mathrm{End}^*(H^*_{dR}(X,\mathbb{K}))$ could be a nontrivial result.

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