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1 Open question

The open question of M-theory is to figure out the "missing microscopic degrees of freedom". Hints as to where these should hide include the following:

- at orbifold fixed points locally of the form $\mathbb{R}^{2,1} \times (\mathbb{H} \times \mathbb{H})//G_{\mathrm{ADE}}$ the solitonic M2-brane with worldvolume locally of the form $\mathbb{R}^{2,1} \times \{0\}$ picks up non-abelian gauge enhancement;

  (for G_{ADE} in the A-series, i.e. $G_{\mathrm{ADE}} = \mathbb{Z}/n$, this is shown by the ABJM model [2], for general G this is shown by the classification of possible worldvolume superpotentials [8] and of possible BPS spacetime geometries [9])

- at orbifold fixed points locally of the form $\mathbb{R}^{6,1} \times \mathbb{H}//G_{\mathrm{ADE}}$ the KK-monopole/D6-brane with worldvolume locally of the form $\mathbb{R}^{6,1} \times \{0\}$ appears and exhibits gauge enhancement as shown by type IIA string theory

  (for G_{ADE} in the A-series and D-series this is due to [10], for G_{ADE} exceptional, this is [1, section 6.4])

- at orbifold fixed points locally of the form $\mathbb{R}^{9,1} \times S^1/\mathbb{Z}_2$ the heterotic string appears and exhibits E_8-gauge enhancement.

  (This is argued via spacetime anomaly cancellation in [5] and via anomaly cancellation on the open nonabelian BLG M2-brane in [6]

Notice that in the case of the M2-brane and also in the case of higher charged KK-monopoles / multiple M2-branes, then the orbifold singularity is also a spacetime singularity and hence excluded from the solution to 11d supergravity (like any black hole singularity). But the auxiliary arguments (ABJM model in the first case, type IIA limit in the second) show that *something* needs to be present in M-theory *at* that singularity, having the "microscopic degrees of freedom" that carry the gauge enhancement.

2 Interior solutions

Of course the canonical way to resolve a black hole singularity in 4d GR is to replace the idealized point-localized mass/charge with a realistic finite charge distribution, spherically symmetric of finite radius. In the exterior of the support of the mass/charge distribution then the solution is the ordinary Reissner-Nordstrom solution, while the *interior solution* now provides a smooth resolution of the singularity (e.g. [4]).

Here in the interior, the dual Faraday tensor $\tilde{F}_2 = \star F_2$ is no longer closed, $d\tilde{F}_2 \neq 0$. For a *homogeneous* charge dsitribution it must be proportional to

$$\tilde{F}_2 \propto r^3 \mathrm{dvol}_{S^2}$$

so that

1. it is SO(3) invariant;
2. the total charge inside the sphere S_r^3 of radius r is proportional to $Q_r \propto \int_{S^2} \tilde{F}$ (Gauss's law).

Of course towards the boundary, where the interior solution goes smoothly into the exterior solution the r-dependence of $\tilde{F}$ will have to decay to zero, but very close to the center the above will be a good approximation.

3 M2-brane

Let's consider such an "interior solution" for the "black" M2-brane, by "resolving" the 2+1-dimensional singularity analogously. In 11d sugra we are not free to add a matter energy-momentum tensor at will, and so the result will not be a solution to 11d supergravity. However, the whole point of the consideration is that

1. we consider only a tiny (e.g. Plack scale) neighbourhood of the previous singularity;
2. we *want* to find deviation from plain supergravity (that's the whole point, to identity what these could be)

Hence consider $\mathbb{R}^{2,1} \times \mathbb{R}^8$ as a tiny open neighbourhood of a would-be BPS M2-brane locus. By direct analogy to the above situation of interior solutions for Reissner-Nordstrom black hole, we expect a dual Faraday tensor G_7 with component on $\mathbb{R}^8$ proportional to

$$G_7 \propto r^8 \mathrm{dvol}_{S^7} \,.$$

Now, while we will not impose the supergravity equation of motion $G_7 = \star G_4$, we do impose the kinematic constraint

$$dG_7 = G_4 \wedge G_4$$

because this follows purely from the charge structure of the M-branes [3].

Hence we need to find a 4-form on $\mathbb{R}^8$ such that

$$G_4 \wedge G_4 \propto r^7 \mathrm{dvol}_{S^7} \,.$$

Moreover, since we are not really on $\mathbb{R}^8$ itself, but on a G_{ADE}-orbifold of it, we need this form to be G_{ADE}-invariant.

Let's assume we are considering a $> 1/2$-BPS M2-brane solution (outside the "interior solution"). By [9, section 3] this means that $G_{\mathrm{ADE}} \subset \mathrm{SU}(2)$ acts via the diagonal of the canonical action of SU(2) on

$$\mathbb{R}^8 \simeq \mathbb{H} \times \mathbb{H} \simeq \mathbb{C}^2 \times \mathbb{C}^2 \,.$$

So we need G_4 to be invariant under that action. By [7, p.18 (ii)] this diagonal SU(2)-action factors through the canonical $SO(8)$-action via the inclusion of Spin(7)

$$\mathrm{SU}(2) \hookrightarrow \mathrm{Spin}(7) \hookrightarrow \mathrm{SO}(8) \hookrightarrow \mathrm{GL}(8) \,.$$

Hence it is sufficient to find a Spin(7)-invariant 4-form ω_4. Now of course the inclusion Spin(7) $\hookrightarrow$ GL(8) is *defined* as the stabilizer of a 4-form $\Phi_0 \in \Omega^4(\mathbb{R}^8)$.

One way to define it: let $\phi_0 \in \Omega^3(\mathbb{R}^7)$ be the "associative 3-form" with components given by the structure constants of the imaginary octonions. Then define $\Phi_0 \in \Omega^4(\mathbb{R}^7 \times \mathbb{R})$ to be

$$\Phi_0 := dx_8 \wedge \phi_0 + \star_{\mathbb{R}^8} \phi_0 \,.$$

This Φ_0 is the Spin(7)-invariant, hence in particular SU(2)-invariant, hence in particular G_{ADE}-invariant.

Is its square also a source term for the 7-form G_7 that we are after?

Yes, by [7, def. 2.4] there is a unique decomposition

$$\Phi_0 = r^3 dr \wedge \phi + r^4 \star_{S^7} \phi$$

for

$$\phi \in \Omega^3(S^7)\,,$$

such that ϕ defines a (nearly parallel) G_2-structure on S^7. This means in particular that

$$\phi \wedge \star\phi \propto \mathrm{dvol}_{S^7}\,.$$

Hence it follows that

$$\Phi_0 \wedge \Phi_0 \propto r^7 \mathrm{dvol}_{S^7}\,.$$

That's precisely the condition we need.

In conclusion, we see that due to ADE-equivariance the possible "interior solutions" for black M2-branes close to the center are very constrained, but that a consistent solution for the 4-flux is given in the vicinity $\mathbb{R}^8$ of the brane locus by the Spin(7)-invariant 4-form

$$G_4 = \Phi_0\,.$$

I don't know if there are other 4-forms G_4 that satisfy the two constraints

1. $G_4 \wedge G_4 \propto r^7 \mathrm{dvol}_{S^7}$;
2. G_4 is invariant under SU(2) $\subset$ Spin(7)

but since the solution by Φ_4 is so nice, let's consider fixing this.

Then we would say that rationally the M-brane charge data on a spacetime with black M2-brane singularity *included* must be a diagram of the form

$$\begin{tikzcd}
 & \mathbb{R}^{2,1} \times D^8 \arrow[dl, hook'] \arrow[dr, "{(\Phi_0,\, c r^8 \mathrm{dvol}_{S^4})}"] & \\
\mathbb{R}^{2,1} \times \mathbb{R}^8 \arrow[rr, "{(G_4, G_7)}"'] & & \mathbf{\Omega}(-, \mathfrak{l}(S^4))
\end{tikzcd},$$

where now D^8 denotes a small open ball around the origin in $\mathbb{R}^8$ (and c is some constant prefactor).

This expresses rational 4-sphere valued forms subject to the constraint that in a small vicinity of the M2-brane locus they coincide with the "interior solution" which we just found.

4 The KK-monopole/D6-brane

Similarly, the KK-monopole with worldvolume $\mathbb{R}^{6,1} \times \{0\}$ is supposed to sit at the orbifold fixed point of $\mathbb{R}^{6,1} \times \mathbb{H}/\!/G_{\mathrm{ADE}}$.

After compactification on the $\mathbb{R}^6$-factor this expresses a black hole solution in 5d-supergravity on $\mathbb{R}^{0,1} \times \mathbb{H}/\!/G_{\mathrm{ADE}}$. Supergravity in 5d has a 2-form G_2 and 3-form G_3 subect to the constraint

$$dG_3 = G_2 \wedge G_2 \,.$$

So now the analogous reasoning as before applies just with dimensions reduced.

By the analogous reasoning as before, we consider a tiny thickening of the singular locus $\mathbb{R}^{0,1} \times \mathbb{D}^4$. Here we ask for a G_{ADE}-invariant 2-form G_2 on $\mathbb{H} \simeq \mathbb{C}^2$ such that $G_2 \wedge G_2 \propto r^3 \mathrm{dvol}_{S^3} \propto \mathrm{dvol}_{\mathbb{H}}$.

Such certainly exists: the holomorphic volume form $\Omega_2 = dz^1 \wedge dz^2$ is SU(2)-invariant, and so

$$G_2 \propto \Omega_2 + \overline{\Omega}_2$$

satisfies our conditions.

So we would impose

$$\begin{tikzcd} & \mathbb{R}^{0,1} \times D^4 \arrow[dl, hook'] \arrow[dr, "{(\Omega_2 + \overline{\Omega}_2,\, c r^3 \mathrm{dvol}_{S^2})}"] & \\ \mathbb{R}^{0,1} \times \mathbb{R}^4 \arrow[rr, "{(G_2, G_3)}"'] & & \mathbf{\Omega}(-, \mathfrak{l}(S^2)) \end{tikzcd} ,$$

5 Bredon equivariant cohomology?

Above we argued that the situation for the solitonic M2-brane with interior included is expressed by a diagram of the form

$$\begin{tikzcd} \mathbb{R}^{2,1} \times D^8 \arrow[d, hook] \arrow[dr, "{(\Phi_0,\, c r^8 \mathrm{dvol}_{S^4})}"] & \\ \mathbb{R}^{2,1} \times \mathbb{R}^8 \arrow[r, "{(G_4, G_7)}"'] & \mathbf{\Omega}(-, \mathfrak{l}(S^4)) \end{tikzcd}$$

and the situation for the KK-monopole by

$$\begin{tikzcd} \mathbb{R}^{0,1} \times D^4 \arrow[d, hook] \arrow[dr, "{(\Omega_2 + \overline{\Omega}_2,\, c r^3 \mathrm{dvol}_{S^2})}"] & \\ \mathbb{R}^{0,1} \times \mathbb{R}^4 \arrow[r, "{(G_2, G_3)}"'] & \mathbf{\Omega}(-, \mathfrak{l}(S^2)) \end{tikzcd} ,$$

Now since $\mathbb{R}^{2,1} \times \{0\}$ and $\mathbb{R}^{0,1} \times \{0\}$ appearing here are also the fixed-point loci of the G_{ADE}-action, the above situation reminds one of a cocycle in "differential Bredon cohomology", hence in the ∞-topos

$$\mathrm{PSh}(\mathrm{Orb}_{G_{\mathrm{ADE}}}, \mathrm{Sh}_\infty(\mathrm{SmoothMfd}))$$

of presheaves on the *orbit category* of G_{ADE}, with values in smooth stacks.

The orbit category Orb_G is the category of G-spaces of the form G/H for closed subgroups $H \hookrightarrow G$, and morphisms being G-equivariant maps between these. Hence every topological space

X with G-action becomes a presheaf on Orb_G by assigning to G/H the H-fixed point space of X. Here in the differential geometric context we might want to assign formal neighbourhoods of fixed point spaces. So our spacetime would become a presheaf of the form

$$\begin{tikzcd} G/G & \mathbb{R}^{2,1} \times D^8 \arrow[d] \\ G/1 \arrow[u] & \mathbb{R}^{2,1} \times \mathbb{R}^8 \end{tikzcd}$$

(where we are for the moment displaying only the trivial and the full subgroup).

For E some other presheaf on the orbit category, then a Bredon equivariant cocycle would be a diagram of the form

$$\begin{tikzcd} G/G & \mathbb{R}^{2,1} \times D^8 \arrow[r] \arrow[d] & E_G \arrow[d] \arrow[dl, Rightarrow] \\ G/1 \arrow[u] & \mathbb{R}^{2,1} \times \mathbb{R}^8 \arrow[r] & E_1 \end{tikzcd} .$$

for $E_\bullet : \mathrm{Orb}_G \to \mathrm{Sh}_\infty(\mathrm{SmthMfd})$ some coefficients.

This is a kind of relative cocycle, relative to fixed point sets. It looks a little bit alike the diagram which we found above, but is also a little different. (?)

For instance we would like to take $E_1 = \mathbf{\Omega}(-, \mathfrak{l}(S^4))$. But then there is not really a choice for E_G that would reproduce the above. (?)

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