Abelian Anyons on Flux-Quantized M5-Branes

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Abstract

While fractional quantum Hall systems provide the best experimental evidence yet of (abelian) anyons plausibly necessary for future fault-tolerant quantum computation, like all strongly-coupled quantum systems their physics is not deeply understood. However, generally a promising approach is to (holographically) realize such systems on branes in string/M-theory; and specifically an old argument by Hellerman & Susskind gives a sketch of fractional quantum Hall states arising via discrete light cone quantization of M5/M9-brane intersections.

Here we present a rigorous derivation of abelian anyon quantum states on M5$\perp$MO9-branes ("open M5-branes") on the discrete light cone, after globally completing the traditional local field content on the M5-worldvolume via a flux-quantization law compatible with the ambient 11d supergravity, specifically taken to be in unstable co-Homotopy cohomology ("Hypothesis H").

The main step in the proof uses a theorem of Okuyama to identify co-Homotopy moduli spaces with configuration spaces of strings with charged endpoints, and identifies their loop spaces with cobordism of framed links that, under topological light cone quantization, turn out to be identified with the regularized Wilson loops of abelian Chern-Simons theory.

Contents

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1 Introduction

Strongly-coupled quantum systems. The remarkable properties of quantum materials [KM17] on which rest the hopes notably of fault-tolerant quantum computation [Sau17][MySS24] are due to these quantum systems being *strongly coupled*. This requires their constituents to be *strongly correlated* [Fu12][FGSS20], hence that their quantum states not be small perturbations of the vacuum. Traditional and well-established perturbation- and mean-field-theory does not apply to such systems [BaSh10], while the exploration of the realm of *non-perturbative* quantum theory is still comparatively in its infancy [FS10][Str13]. However, a promising general approach is via "geometric engineering" [GK99] of strongly-coupled quantum systems on worldvolumes of branes in string/M-theory [Du96][Du99a]; or dually via the *holographic* (cf. [AG$^+$00][Nat15]) imprint that these branes leave in the ambient field of (super-)gravity (applied to quantum materials since [HKSS07], review in [Pi14][ZLSS15][Na17][HLS18]).

Anyonic topological order. Concretely, the holy grail of topological quantum matter [Sta20] is the understanding and realization of *anyonic topological order* [ZCZW19, §III][SS23c], where the adiabatic movement of configurations of defects (e.g. vortices in a quantum liquid) induces quantum state monodromies that are topologically invariant (cf. pointers in [MySS24, §3]) and as such shielded from the ubiquitous noise notoriously jeopardizing the coherence of quantum circuits [Sau17]. The best experimental evidence for anyons comes (as predicted [Ha84][ASW84][FH21]) from *fractional quantum Hall* (FQH) systems [Sto99][Gi04][Fu12, §14.2][Sta20, §6.2.1] where a planar electronic quantum liquid is placed in a strong transversal magnetic field at extremely low temperature. The observed quantum Hall anyons [NLGM20][Gl$^+$24] are abelian (non-abelian ones "seem to be realized in rather rare conditions" [Ste20]), hence do not by themselves implement a universal set of quantum gates, but still go at least a long way towards useful topological quantum computation [Pa06][Ll02][Wo10][WP11].

Models of the FQH effect. While good phenomenological models for the FQH effect have been developed (cf. [Ja14]), a microscopic understanding remains elusive (cf. [DS20, §1][Jac21, §1]). At the same time, it is remarkable that the FQH effect is universal (e.g., [Gi04, p. 133]) in that it is seen across diverse materials and independently of their impurities. This suggests that brane models are fundamentally worthwhile and should see the FQH effect fairly generically. In this respect the observation of [HS01] is noteworthy: These authors sketched an argument that the worldvolume physics of flat M5-branes carrying a constant H_3-flux density (as in [GSS24b, Ex. 3.14]) generally exhibits FQH at unit filling factor, and at fractional filling factor $1/(k+1)$ when placed near k M9-branes (namely when the corresponding D4-branes are placed near k D8-branes, cf. e.g. [Ha12]). While we will use rather different methods here, with precise definitions and rigorous proofs, our brane setup (§2) is similar to and the conclusions are compatible with [HS01]: We consider flat M5-branes intersecting MO9-planes (known as "open M5-branes") while carrying constant H_3-flux density, and we derive abelian anyon quantum states on their worldvolume. The novel ingredient that makes this work is a completion of the worldvolume field content by a *flux quantization law*.

The need for flux quantization. Our starting point here is the observation that previous discussions of brane physics have tended to ignore the key non-perturbative effect already in the classical theory, namely the "flux quantization" (see [SS24c][SS24a], building on e.g. [Al85][Fr00][Sa10]) of the (higher) worldvolume gauge field. In this context, the local field content traditionally considered on a single coordinate chart is globalized to include topological solitonic field configurations classified by some generalized cohomology theory (this in generalization of the familiar case of Dirac charge quantization of ordinary electromagnetism in ordinary integral cohomology (e.g. [Al85]), which famously implies the solitonic fields to be Dirac monopoles and Abrikosov vortices, reviewed in [SS24c, §2.1]). Especially for M5-branes, the flux quantization of their H_3-flux is more subtle than may have been commonly appreciated [SS24c, §4.3][GSS24b]. This is because it is twisted by the (G_4, G_7)-flux density of the ambient 11d supergravity background [GSS24b, (19)-(21)] that itself famously satisfies a non-linear Bianchi identity (review in [MiSc06, §3.1.3][GSS24a]) and as such has ([SS24c, §3.2][FSS23]) admissible flux-quantization only in non-abelian (i.e., unstable) generalized cohomology theories ([To02, Def. 6.0.6][Lu14, Def. 6][FSS23, §2]), which have not yet received wide attention yet. Hence, completing the theory by flux quantization may be expected to reveal previously unrecognized (explanations for) physical effects in the quantum physics on M5-branes and therefore, by extension, in quantum materials.

Cohomotopical charges on M5-branes. The simplest (in a sense) flux quantization law which is admissible for M5-branes in 11d supergravity turns out ([FSS20b, §3.7][FSS21a][GSS24b, (22)]) to be a twisted form of unstable co-Homotopy cohomology theory (classically considered by [Bo36][Pon38][Sp49], being the historical origin of the seminal Pontrjagin-Thom theorem, cf. [SS23a, §2.2, 3.2]). This means that the sectors of topological charges are identified with suitable homotopy classes of continuous maps from the spacetime/worldvolume domain to higher dimensional *spheres*. The hypothesis that this is the "correct" flux-quantization law to be used for 11d supergravity with M5-branes ("Hypothesis H", [FSS20b][GS21][SS23a], following [Sa13, §2.5]) is justified by theorems matching its implications to a list of common expectations about "M-theory" (reviewed in [SS24c, §4]; more in detail, here we use $\mathbb{Z}_2$-*equivariant* "Real" Cohomotopy [SS20b, p. 100][HSS19] appropriate for M-branes at MO9-orientifold planes [SS20a, §4]).

However, one may also turn this around and regard the cohomotopically flux-quantized M5-brane as the definition of a model for topological effects in strongly coupled quantum systems and investigate its predictions (as in [FSS21a][FSS21c]). This is what we do here regarding the emergence of abelian anyon quantum states, following [GSS24b].

The article is organized as follows:

– In §2 we introduce the configurations of M5-branes and their worldvolume solitons to be considered and briefly review the relevant aspects of their flux-quantization in co-Homotopy.

– In §3 we identify the resulting transverse position moduli of cohomotopically flux-quantized solitons with the configuration space of points in the plane and give a knot-theoretic description of its fundamental group in terms of cobordism classes of framed links, using a theorem by Okuyama [Ok05].
Our main Theorem 3.18 here is pure algebraic-topology/knot-theory and as such seems to be new and may be of interest in its own right (cf. Rem. 3.19).

– In §4 we put these pieces together and show that the topological soliton sector on holographic M5-branes under consideration is controlled by (abelian) Chern-Simons quantum field theory.

– In §5 we conclude and provide some outlook, such as in view of topological quantum computation with anyons.

Acknowledgements. We thank Sadok Kallel and Shingo Okuyama for comments on the material in §3, following the first preprint version of this article. In particular, Shingo Okuyama kindly informed us of the set of talk slides [Ok18] where diagrams similar to our Figure 2 as well as the claim of a Hopf generator representative equivalent to (26) already appear.

2 Brane configuration

Here we introduce the configurations of M5-branes and their worldvolume solitons to be considered (Figure 1), and along the way we briefly review the relevant aspects of their flux-quantization in co-Homotopy, mostly recalling from [GSS24b] and references given there.

Quantized charges on flat M5-Branes. In the special case of interest here, where the M5-brane worldvolume and its ambient spacetime are flat, and the supergravity C-field is trivial, the flux quantization on the worldvolume according to Hypothesis H is in plain (as opposed to twisted) 3-co-Homotopy [GSS24b, (141)]. This means [FSS23, Ex. 2.7] that the global charge is encoded by a continuous map $\chi : \Sigma^6 \to S^3$ from the worldvolume domain Σ^6, subject to the constraint that its "character" $\mathrm{ch}(\chi) : \Sigma^6 \xrightarrow{\chi} S^3 \xrightarrow{1} K(\mathbb{R},3)$ — the coHomotopical analog of the Chern character on K-theory, see [FSS23, Ex. 9.3], essentially being the pullback along χ of the volume form on S^3 [FSS20b, §3.7][FSS21a, §3] — coincides as an element of $H^3(\Sigma^6;\mathbb{R}) \simeq H^3_{\mathrm{dR}}(\Sigma^6)$ with the de Rham class of the B-field flux density H_3 on the worldvolume:

$$\underset{\textbf{Total flux}}{[H_3]} \;=\; \underset{\substack{\mathbb{R}\textbf{-rationalization}\\ \textbf{of total charge}}}{\mathrm{ch}(\chi)} \qquad \text{i.e. the following diagram commutes} \qquad \begin{tikzcd} & & \pi^3(\Sigma) \arrow[d, "\mathrm{ch}"] \\ * \arrow[urr, bend left, "\chi"] \arrow[r, "H_3"] & \Omega^3_{\mathrm{dR}}(\Sigma^6) \arrow[r, two heads, "{[-]}"] & H^3_{\mathrm{dR}}(\Sigma^6) \end{tikzcd} \tag{1}$$

The *complete* field content is given by a homotopy-theoretic enhancement of the diagram on the right, whose filling homotopy encodes how the flux density H_3 is related to the global charge χ by local *gauge potentials* B_2, see [GSS24b, §4.1] and see [SS24c, §3.3] for background.

Here, we are making use of the fact [FSS23, §2] that the Eilenberg-MacLane space $K(\mathbb{R},3)$ is a classifying space for ordinary real (de Rham) cohomology in analogy to how S^3 is the classifying space for 3-co-Homotopy (see [FSS23, §2]):

$$\textbf{Ordinary cohomology} \quad H^3_{\mathrm{dR}}(\Sigma^6) \;\simeq\; \pi_0 \mathrm{Maps}\big(\Sigma^6,\, K(\mathbb{R},3)\big)\,, \qquad \pi^3(\Sigma^6) \;:=\; \pi_0 \mathrm{Maps}\big(\Sigma^6,\, S^3\big) \quad \textbf{Co-Homotopy.}$$

Flux-quantized soliton moduli on flat M5-branes. Just like homotopies between maps to $K(\mathbb{Z},3)$ represent coboundaries in ordinary cohomology (cf. [FSS23, (2.3)]), homotopies between maps to S^3 represent the topological component of *gauge transformations* between flux-quantized B-field configurations. This means that (the path ∞-groupoid of) the mapping space $\boldsymbol{\pi}^3(\Sigma^6) \;:=\; \mathrm{Maps}^{*/}(\Sigma^6,\, S^3)$ plays the role of the topological sector of the global phase space BRST complex of the theory [SS24a]:

$$\begin{array}{cccc} \substack{\textbf{co-Homotopy}\\ \textbf{moduli space}} & \boldsymbol{\pi}^3(\Sigma) \;\equiv & \mathrm{Maps}^{*/}(\Sigma^6,\, S^3) & \substack{\textbf{Topological sector}\\ \textbf{of phase space}} \\ & \big\downarrow & & \\ \substack{\textbf{co-Homotopy}\\ \textbf{set}} & \pi^3(\Sigma^6) \;\equiv & \pi_0 \mathrm{Maps}^{*/}(\Sigma^6,\, S^3) & \textbf{Set of charges} \end{array} \tag{2}$$

The superscript indicates that we consider *pointed* maps, for any fixed basepoint in S^3 and for the basepoint in the worldvolume domain Σ^6 taken to be the "point at infinity" at which charges of localized solitons in the B-field are required to vanish ([SS23a, Rem. 2.3, Def. 3.16][SS23d, §A.2]).

For instance, consider the strong coupling regime the M5-brane is to be wrapped on the "decompactified" M-theory circle (e.g. [To96, p. 4][Na17, §26.5]), which we may model as the 1-point compactification [SS23a, (17)] $S^1_A := \mathbb{R}^1_{\cup\{\infty\}}$ of the real line by including its point at infinity. The moduli space becomes the based loop space of the moduli on the remaining 5-dimensional worldvolume Σ^5 of the corresponding D4-brane:

$$\mathrm{Maps}^{*/}\big(\underbrace{\Sigma^5 \wedge S^1_A}_{\Sigma^6},\, S^3\big) \;\simeq\; \Omega\, \mathrm{Maps}^{*/}(\Sigma^5,\, S^3)\,, \tag{3}$$

where "$\wedge$" denotes the smash product of pointed spaces [SS23d, (28)]. Moreover, to inspect potentially anyonic charges on the M5, we need to consider solitons of codimension=2 and hence take the D4 worldvolume domain to be further factored as $\Sigma^5 \;\equiv\; \mathbb{R}^{1,2}_{\sqcup\{\infty\}} \wedge \mathbb{R}^2_{\cup\{\infty\}}$ [SS24c, §2.2]. Here, the first factor is the soliton worldvolume (non-compactified, hence with a *disjoint* point-at-infinity), and the second factor is its transverse space (whose included point at infinity forces the charge to vanish at infinity and hence topologically stabilizes/localizes the solitons).

The equivariant moduli at the MO9-plane. Finally, for modeling the rigid longitudinal wrapping of the topological H_3-flux analogous to [HS01, (3.1)], we consider wrapping the brane on another circle factor $S^1_H := \mathbb{R}^1_{\cup\{\infty\}}$ equipped with $\mathbb{Z}_2$-reflection action whose fixed locus is to be the MO9-plane of heterotic M-theory [HW96]. This being an orienti-fold means that its charges are to be measured in the "Real" version of co-Homotopy [SS20a][HSS19][SS20b, Def. 5.28], hence that the moduli space (3) becomes that of pointed $\mathbb{Z}_2$-equivariant maps to S^3 equipped with reflection action on one of its coordinates:

$$\mathrm{Maps}^{*/}_{\mathbb{Z}_2}\big(\underbrace{\mathbb{R}^{1,1}_{\sqcup\{\infty\}} \wedge \mathbb{R}^2_{\cup\{\infty\}} \wedge S^1_H \wedge S^1_A}_{\Sigma^6},\, S^2 \wedge S^1_{\mathrm{sgn}}\big) \;\simeq\; \Omega\,\mathrm{Maps}^{*/}_{\mathbb{Z}_2}\big(\mathbb{R}^2_{\cup\{\infty\}} \wedge S^1_H,\, S^2 \wedge S^1_{\mathrm{sgn}}\big)\,. \tag{4}$$

(Since after passing to the naive quotient space $S^1_H/\mathbb{Z}_2 \simeq [0,1]$ this looks like M5-brane stretched along an interval, M5-branes wrapped on S^1_H have been called *open M5-branes* [BGT06, Fig. 3].)

Remarkably, the equivariant version [RS00, Thm. 2] of a classical theorem [Se73, Thm. 1] on configuration spaces (see [Ka24][SS22]) identifies the resulting moduli space of solitons on the D4-worldvolume with the *group completion* $\mathbb{G}$ of the *configuration space* $\mathrm{Conf}(\mathbb{R}^2 \times \mathbb{R}^1_{\mathrm{sgn}})^{\mathbb{Z}_2}$ of equivariant points inside (i.e., $\mathbb{Z}_2$-fixed finite subsets of) $\mathbb{R}^2 \times \mathbb{R}^1_{\mathrm{sgn}}$ [GSS24b, (154)]:

$$\mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}} \wedge S^1_{\mathrm{sgn}},\, S^2 \wedge \mathrm{S}^1_{\mathrm{sgn}}\big) \underset{\mathrm{whe}}{\simeq} \mathbb{G}\Big(\mathrm{Conf}\big(\mathbb{R}^2 \times \mathbb{R}^1_{\mathrm{sgn}}\big)^{\mathbb{Z}_2}\Big) \underset{\mathrm{whe}}{\simeq} \mathbb{G}\mathrm{Conf}(\mathbb{R}^3) \times \mathbb{G}\mathrm{Conf}(\mathbb{R}^2)\,. \tag{5}$$

Here, the "group completion" $\mathbb{G}$ roughly means that the points — which here we are to think of as the soliton worldvolumes localized in their transverse space inside the M5-worldvolume — are accompanied by corresponding "anti-points" (their homotopy-inverses under disjoint union) — hence negatively-charged solitons — with which they may undergo pair creation/annihilation. Interestingly, the precise statement, not widely appreciated before, is that this group completion is given by configurations of *strings* with charged endpoints; this is discussed in §3.

In any case, it is clear that for such a configuration of points to be $\mathbb{Z}_2$-invariant as a subset, its elements either are stuck at the O-plane in $\mathbb{R}^2 \times \{0\} \simeq \mathbb{R}^2$ else they sit in the bulk $\mathbb{R}^2 \times \big(\mathbb{R}^1 \setminus \{0\}\big)$ away from the O-plane and then appear in mirror pairs determined by the element in, say, $\mathbb{R}^2 \times \mathbb{R}^1_{>0} \simeq \mathbb{R}^3$ — and this explains the second equivalence in (5) [Xi06, (1.2), Thm. 4.1].

Note that the second factor on the right in (5) corresponds to the sector where the flux density H_3 is wrapped along the longitudinal direction S^1_H so that only a transverse 2-form component remains, in a flux-quantized refinement of the situation [HS01, (3.1)]. Since this remnant 2-form is the (electro-)magnetic flux density, we understand the second factor $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ as that of solitons in an FQH-like system, and our task (in §4) is that their quantum states are indeed anyonic.

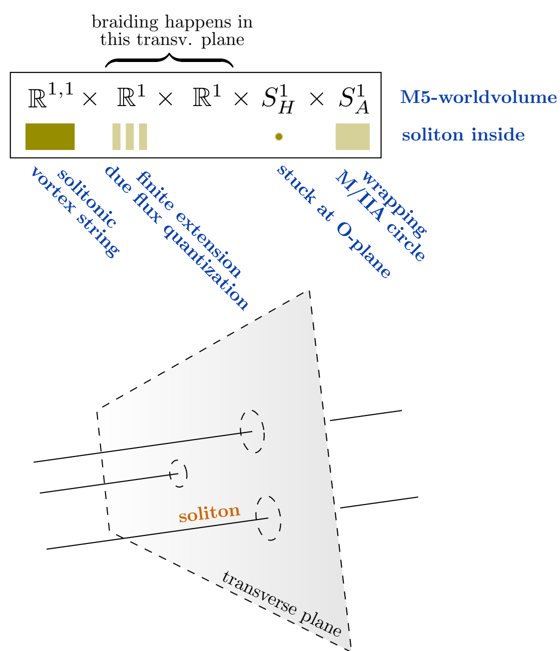


Figure 1. The brane-diagram of the solitons on M5-branes which carry anyonic quantum observables under Hypothesis H. Here, from right to left:

(i) Both the M5 and its worldvolume soliton are wrapped on the M/IIA circle S^1_A in order to admit topological lightcone quantization (cf. §4).

(ii) The M5-brane itself is moreover wrapped over the M/HET circle S^1_H, but their worldvolume solitons that we focus on are those that are stuck at an O-plane, i.e. at one of the fixed points in S^1_H (the others escape into the HW bulk and thus cannot be anyonic).

(iii) Due to a subtle effect of flux quantization in co-Homotopy discussed in §3, these solitons have *finite* extension along one of their would-be transverse directions inside the M5, as explained with Figure 2

(iv) Otherwise, after the compactification, the solitons look like strings that may move around each other in the transverse plane (not unlike Abrikosov vortex strings in a slab of type II superconducting material, cf. [SS24c, §2.1]).

With these brane/soliton configurations specified, we proceed to a careful analysis of their moduli space.

3 Configuration space

We recall that by $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) := \Omega B_{\sqcup}\mathrm{Conf}(\mathbb{R}^2)$ ([Se73, Def. 2.2, Ex.(b)]) we denote the group-completion of the configuration space of (un-labeled, un-ordered) points in the plane. Here we give a geometric description of its fundamental group in terms of cobordism classes of framed links.

This section is purely mathematical and, as such, may be read and may be of interest independently of the rest of the article. But readers who appreciate the physical "meaning" of $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ as the transverse position moduli (5) of solitons on M5 $\perp$ MO9 worldvolumes (according to §2) will readily recognize a vivid picture of interaction-processes of solitonic branes appearing as strings in their naïve transverse space (cf. Figure 2).

Configuration spaces of solitonic charges. One might expect $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ to be the configuration space of *signed* points in $\mathbb{R}^2$, where each point carries a charge in $\{\pm 1\}$, with the topology of the configuration space such that oppositely-charge points may undergo pair annihilation/creation. While this is the correct picture on the level of connected components, it turns out *not* to correctly capture the homotopy type of this space, as observed long ago in [McD75, p. 96].

However, it may not have come to be widely appreciated that something close *is* true: To get the correct moduli space, the points (hence the worldvolume solitons) need to be regarded as being of finite thickness [CW81] at least in one direction [Ok05], so that the points (which for us are the positions of worldvolume solitons in their transversal space, cf. Fig. 1.) are resolved to "strings" carrying charges at their ends.

We now discuss this in more detail (culminating in Thm. 3.18 below).

Beware that the group-completed plain configuration spaces considered here are different from the configuration spaces considered in [GS21]: The spaces there correspond to *intersections* of solitonic branes with codimension=1 branes (which induces an ordering of the points in the configuration, in contrast to the un-ordered configurations considered here).

Group-completed configuration space of points. For general background on configuration spaces of points, see [Wi20][Ka24][FH01].

Definition 3.1 (**Plain configuration space of points** [Se73, p. 215])**.** For $n \in \mathbb{N}$, we write $\mathrm{Conf}(\mathbb{R}^n)$ for the topological space of finite subsets of (i.e. configurations of plain points in) $\mathbb{R}^n$. This is a partial topological monoid under the partial operation

$$\mathrm{Conf}(\mathbb{R}^n) \times \mathrm{Conf}(\mathbb{R}^n) \xrightarrow{\sqcup} \mathrm{Conf}(\mathbb{R}^n) \tag{6}$$

which is defined when the pair of configurations is disjoint, in which case it is given by their union. We write

$$\mathbb{G}\mathrm{Conf}(\mathbb{R}^n) \;:=\; \Omega\big(B_{\sqcup}\mathrm{Conf}(\mathbb{R}^n)\big) \tag{7}$$

for the topological group completion of this partial monoid, namely the based loop space of the topological realization of its simplicial nerve.

(See [SS23d, §A.2] for the general topology of pointed spaces that we need here.)

Proposition 3.2 (**Group-completed configurations as iterated loops** [Se73, Thm. 1])**.** *The cohomotopy charge map ("scanning map") constitutes a weak homotopy equivalence between the group completion of the configuration space of plain points in $\mathbb{R}^n$ (Def. 3.1) and the n-fold based loop space of the n-sphere:*

$$\mathbb{G}\mathrm{Conf}(\mathbb{R}^n) \;\simeq\; \Omega^n S^n. \tag{8}$$

Definition 3.3 (**Configuration space of charged open strings** [Ok05, Def. 3.1-2])**.** For $n \in \mathbb{N}_{\geq 1}$, we write [1] $\mathrm{Conf}^I(\mathbb{R}^n)$ for the quotient by the equivalence relations indicated on the right of Figure 2 of the topological space of disjoint unions of (half-)open/closed line segments in $\mathbb{R}^n$ *parallel to the first coordinate axis*, where in Figure 2 a filled (black) circle indicates that the corresponding point is included in the interval, while an empty (white) circle indicates that it is not.

Proposition 3.4 (**Charged open strings as group-completion of plain points** [Ok05, Thm. 1])**.** *For $n \in \mathbb{N}_{\geq 1}$ there is a weak homotopy equivalence between the configuration space of charged open strings (Def. 3.3) and the group completion of the plain configuration space of plain points (Def. 3.1):*

$$\mathrm{Conf}^I(\mathbb{R}^n) \;\simeq\; \mathbb{G}\mathrm{Conf}(\mathbb{R}^n)\,. \tag{9}$$

[1] The space we denote $\mathrm{Conf}^I(\mathbb{R}^n)$ in Def. 3.3 would be denoted "$I_n(S^0)_{\mathbb{R}}$" in the notation of [Ok05].

Figure 2. Indicated in the left column is the equivalence relation ([McD75, p. 94]) controlling the configuration space of charged *points* in some $\mathbb{R}^n$, where configurations involving a positively and a negatively charged point are connected by a continuous path to the corresponding configuration where both of these points are absent (have mutually annihilated). This configuration space is close to but *not* (weak-homotopy) equivalent (by [McD75, p. 6]) to the group-completed configuration space $\mathbb{G}\mathrm{Conf}(\mathbb{R}^n)$.

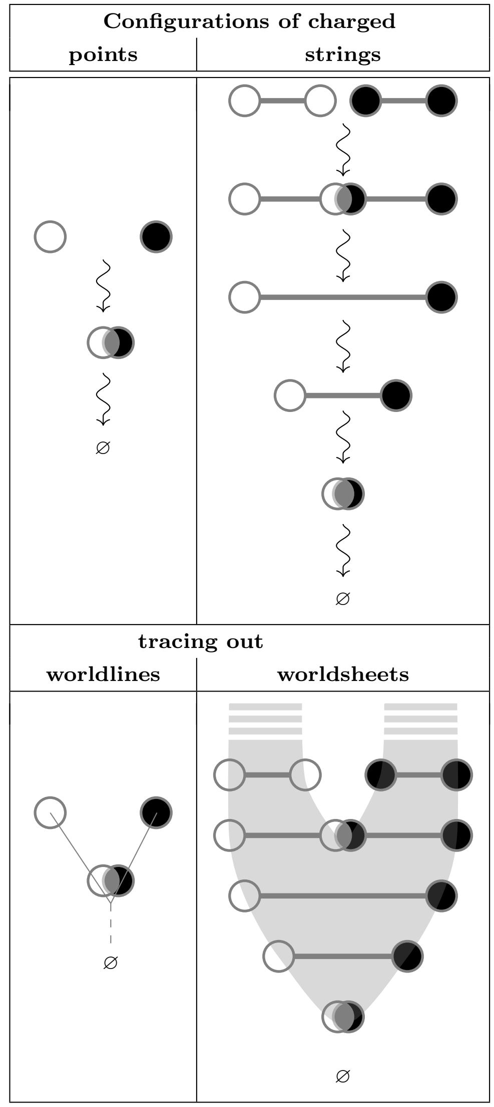
Indicated in the right column are the analogous relations (from [Ok05, Def. 3.1-2]) in the configuration space of charged "strings", where charged points are replaced by line segments of finite length, parallel to a fixed coordinate axis, whose endpoints are carrying charges. This configuration space *is* (weak-homotopy) equivalent to the group-completed configuration space $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ (by [Ok05, Thm. 1.1]).

(In both cases, the curvy lines indicate continuous paths in these configuration spaces, here realizing the pair-annihilation processes. Running along these paths in the opposite direction reflects the corresponding pair-creation processes.)

Notice that in both cases the physical processes are *grosso modo* the same — a pair of opposite charges mutually annihilate —, the difference being only that on the right the process is "smoothed out" in the familiar way in which string interactions resolve singularities in particle interactions — only that here we did not postulate this explicitly: it is derived by applying the result of [Ok05] to the consequence (5) of Hypothesis H.

This is curious because it means that what naively looks like (non-supersymmetric) solitonic 2-branes inside the M5-brane worldvolume is resolved via flux quantization in co-Homotopy to a kind of unstable open 3-branes (indicated by the broken block in the brane diagram Figure 1) – possibly to be interpreted as the decay products after supersymmetry breaking of the well-known supersymmetric 3-brane inside the M5 [HLW98], understood as the locus of 1/4BPS M5 $\perp$ M5-intersections [PT96].

(A discussion of stable such M5-defect 3-branes as anyons, we had previously given in [SS23b][SS23c], based on the *ordered* configuration spaces of points describing brane intersections according to [SS22]. This ordering allows for non-abelian anyons, as opposed to the abelian anyons found here, at the cost of more complicated brane configurations.)



Remark 3.5 (**Charged strings reflecting Cohomotopy moduli).** In summary, this identifies the n-Cohomotopy moduli vanishing at infinity on $\mathbb{R}^n$ with the Okuyama configuration space of charged open strings in $\mathbb{R}^n$:

$$\begin{aligned} \underset{\text{charged open strings}}{\text{Configuration space of}} \quad \mathrm{Conf}^I(\mathbb{R}^n) \;&\underset{(9)}{\simeq}\; \mathbb{G}\mathrm{Conf}(\mathbb{R}^n) \;\underset{(8)}{\simeq}\; \Omega^n S^n \\ &\simeq\; \mathrm{Maps}^{*/}\big(\mathbb{R}^n_{\cup\{\infty\}},\, S^n\big) \;\equiv\; \boldsymbol{\pi}^n\big(\mathbb{R}^n_{\cup\{\infty\}}\big) \quad \underset{\text{vanishing at infinity}}{\text{Cohomotopy moduli}}\,. \end{aligned} \tag{10}$$

This implies:

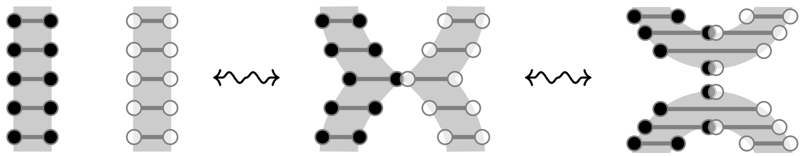
Proposition 3.6 (**Fundamental group of charged string configurations).** *The fundamental group of Okuyama's configuration space of charged open strings in the plane (Def. 3.3) is the group of integers:*

$$\pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big) \;\;\equiv\;\; \pi_0\big(\Omega_0\,\mathrm{Conf}^I(\mathbb{R}^2)\big) \;\underset{(10)}{\simeq}\; \pi_0\big(\Omega^3 S^2\big) \;\;\equiv\;\; \pi_3(S^2) \;\;\simeq\;\; \mathbb{Z}\,. \tag{11}$$

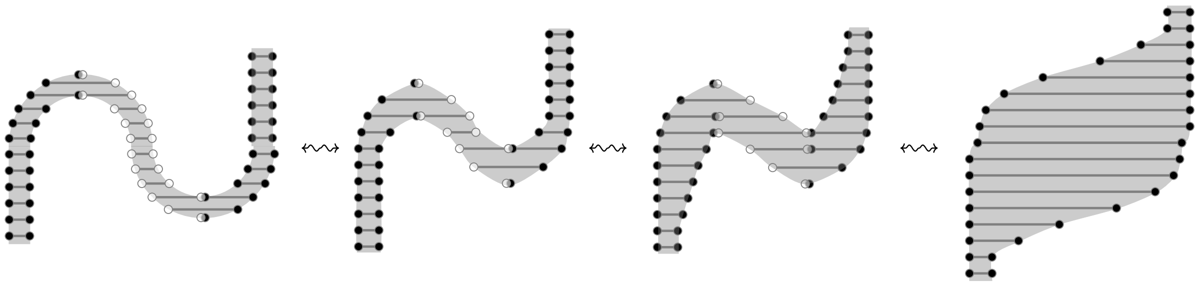
The generator on the right of (11) is well-known to be represented by the complex Hopf fibration $h_{\mathbb{C}} : S^3 \to S^2$. Our goal is to understand the corresponding generator on the left, i.e., the unit-charged open string loop whose composites and their reverses are deformation-equivalent to general charged open string loops.

A key observation for this identification is the following.

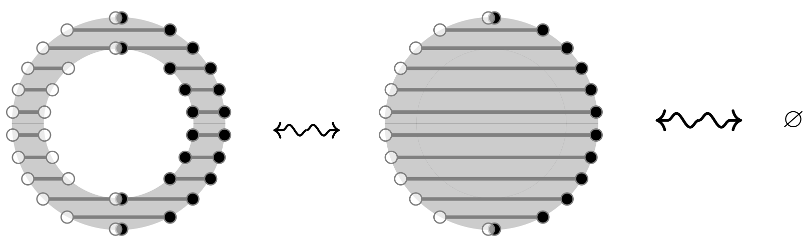
Example 3.7 (**Relations between charged open string worldsheets).** Continuous deformations of paths of charged open strings, i.e., continuous maps of the form $[0,1]^2 \longrightarrow \mathrm{Conf}^I(\mathbb{R}^n)$, subsume the following "moves" (and their images under the exchange of positive and negative charges):



(12)

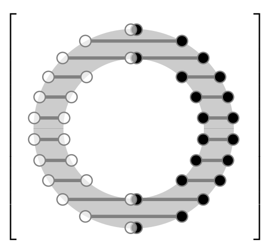


(13)



(14)

Here the third move (14), a path of based loops, implies that the class of the annulus worldsheet in the fundamental group of the configuration space vanishes:



$$= \; * \; \in \; \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^n)\big)\,.$$

Hence the annulus is not the generator of $\pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)$ that we are after, and we need to look further:

Loops in Okuyama's configuration space as framed oriented links. Our first observation now is that based loops in Okuyama's configuration space of charged open strings (Def. 3.3) may be identified with *framed oriented links* (cf. Figure F). For a general discussion of framed links, see for instance [Oh1, p. 15][EHI20].

Definition 3.8 (**Framed oriented links).**

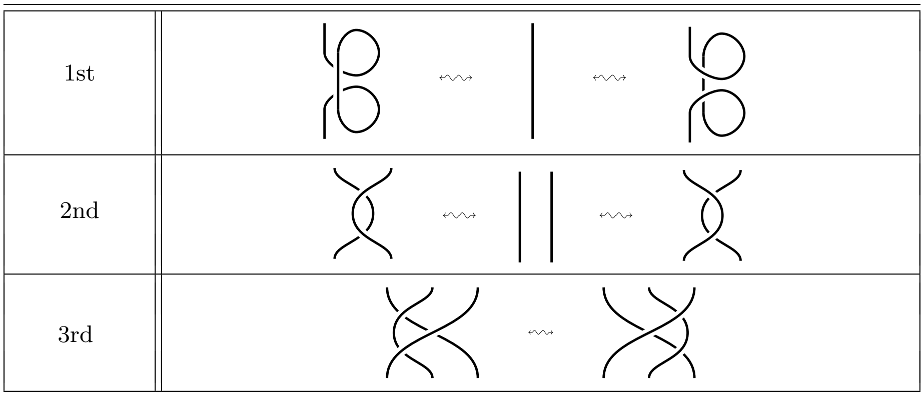
(i) A *framed oriented link diagram* is an immersion of k oriented circles $\big(S^1\big)^{\sqcup^k}$, for $k \in \mathbb{N}$, into the plane $\mathbb{R}^2$ with isolated crossings at Euclidean distance > 1 from each other, at each of which one of the two crossing segments is labeled as crossing *over*.

Here we demand in addition and without essential restriction of generality that no strictly horizontal segments appear, hence that the restriction of a link diagram to any $\mathbb{R}^1 \hookrightarrow \mathbb{R}^2$ parallel to $\mathbb{R} \times \{0\}$ consists of finitely many points – this is used in (17) below.

(ii) Two framed oriented link diagrams are regarded as equivalent if they may be transformed into each other by a sequence of isotopies (continuous paths in the space of framed link diagrams) and the three *Reidemeister moves* shown in Figure R.

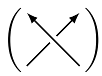
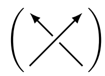
(iii) The *framed oriented links* are the corresponding equivalence classes of framed oriented link diagrams.

Figure R – Reidemeister moves for framed link diagrams (e.g. [Oh1, Thm. 1.8]).



Definition 3.9 (Crossing-, Linking- and Framing numbers).

(i) Any crossing in a framed oriented link diagram L (Def. 3.8) loclly is either of the following, up to local orientation-preserving diffeomorphism, which we assign the *crossing number* ± 1, respectively, as shown:

$\#$  $= +1\,,$ $\qquad \#$  $= -1\,.$ (15)

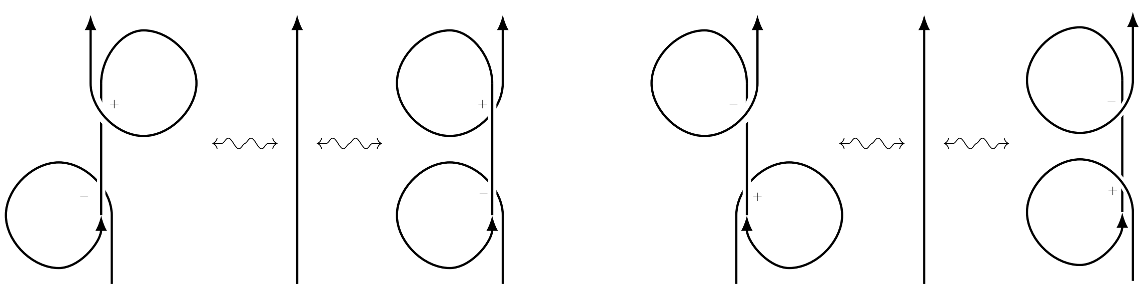
(ii) For $(L_i)_{i=1}^N$ the connected components of L, the *linking number* $\mathrm{lnk}(L_i, L_j)$ is *half* the sum of crossing numbers between L_i and L_j (cf. [Oh1, p. 7]).

(iii) The *framing number* $\mathrm{fr}(L_i)$ is the sum of crossing numbers of L_i with itself.

(iv) The sum $\#L$ of the crossing numbers of all crossings of L is hence the sum of all the framing and linking numbers:

$$\#(L) \;:=\; \sum_{\substack{c\,\in \\ \mathrm{crssngs}(L)}} \#(c) \;=\; \sum_i \mathrm{frm}(L_i) + \sum_{i,j} \mathrm{lnk}(L_i, L_j)\,. \tag{16}$$

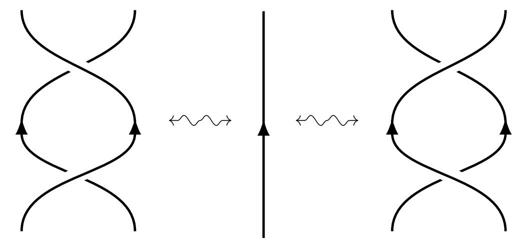
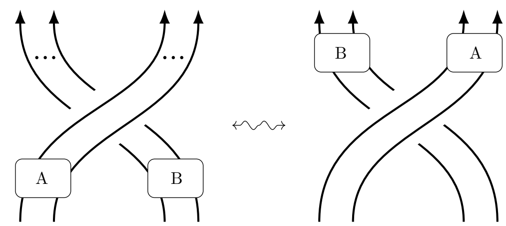
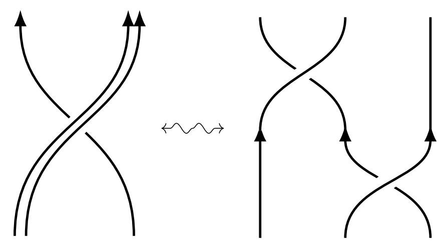
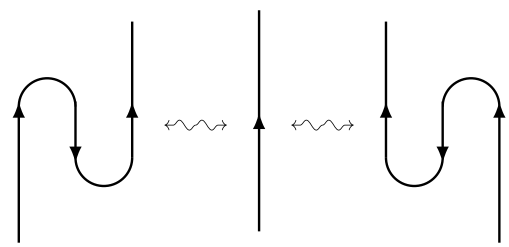
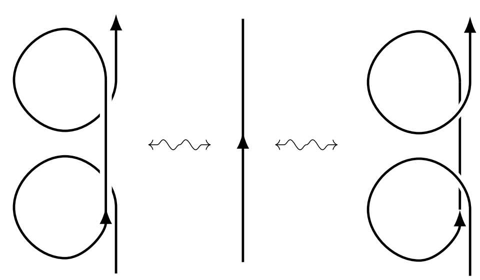
Example 3.10 (Invariance of framing number). The framing and linking numbers (Def. 3.9) are invariants depending only on the equivalence class of a framed oriented link diagram. The following moves show how successive self-crossings of opposite crossing number cancel out (by the Reidemeister moves):



In order to relate framed links in detail to string loops, we need a more combinatorial description of equivalence of link diagrams. This is provided by *functorial knot theory* [Ye01] via:

Proposition 3.11 (Shum's Theorem [Sh94]). *Framed oriented links* (Def. 3.8) *are equivalently the endomorphisms of* $\varnothing$ *in the category of framed oriented tangles, which is the ribbon category (aka tortile category) freely generated by a single object.*

What this means here (cf. [Ye01, §9.1]) is that for a function on link diagrams to descend to equivalence classes and hence to be a *link invariant*, it is sufficient that it respects (beyond wiggling of edges) the moves shown in Table D, which subsume the Reidemeister moves (Figure R) but also zig-zag yank moves to account for diagram isotopy combinatorially. For background on ribbon/tortile monoidal categories and the translation of their axioms to tangle diagrams as shown in Table D, see [Sel11] (going back to [JS93, Prop. 2.7] for the case of the 3rd Reidemeister move).

Ribbon category axiom	**Table D – Ribbon category presentation of framed oriented links** according to Shum's theorem (recalled as Prop. 3.11). Note that for the slide moves there is also the corresponding mirrored version, which we are not showing to save space.	Framed oriented link diagram move
braiding isomorphism		2nd Reidemeister move
braiding naturality		sliding moves ⇒ 3rd Reidemeister move
braiding hexagon law		
duality triangle law		zig-zag yank move
twist ribbon property		1st Reidemeister move

Definition 3.12 (Charged string loops as framed oriented links). From a framed oriented link diagram (Def. 3.8), we obtain a based loop in Okuyama's configuration space of charged strings in $\mathbb{R}^2$ (Def. 3.3) by thickening the underlying link to a string worldsheet (as illustrated in Figure F below):

$$\mathrm{FrmdOrntdLnkDgrm} \longrightarrow \Omega_0\,\mathrm{Conf}^I(\mathbb{R}^2)\,. \tag{17}$$

Note that this is well-defined due to our condition in Def. 3.8 that link diagrams have well-separated crossings and no straight horizontal segments: These conditions imply that the intersection of the link diagram with any horizontal line $\mathbb{R}^1 \hookrightarrow \mathbb{R}^2$ is a finite set of points, and that as we move the horizontal line vertically, these points (i) move, (ii) cross, (iii) merge, or (iv) emerge over well-separated intervals, which translate to the corresponding string worldsheets, where the orientation of the link determines the charges on the endpoints of these strings.

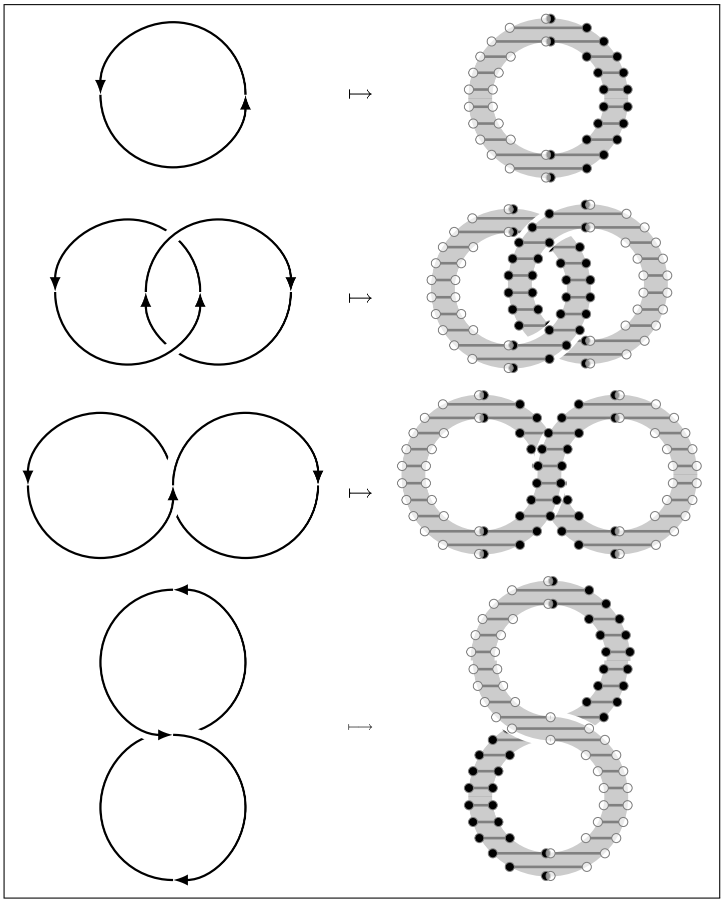
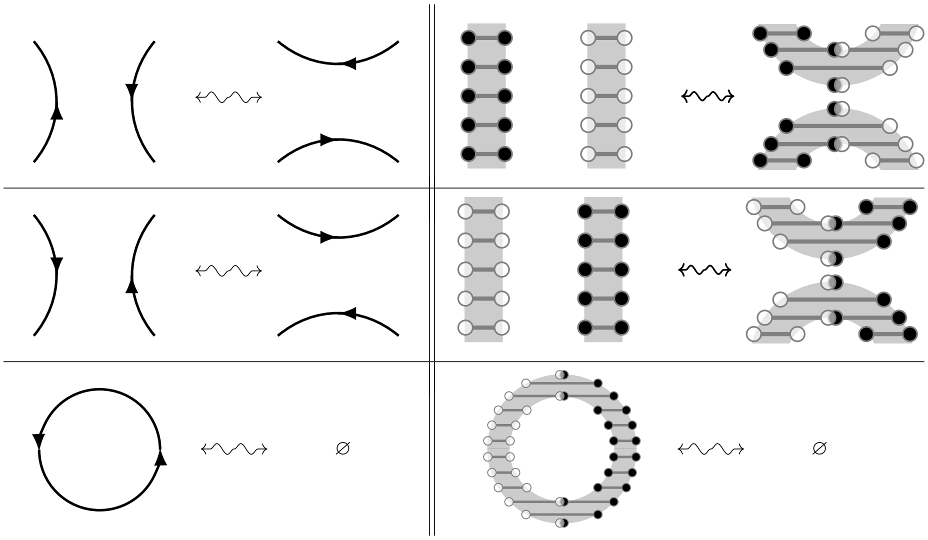


Figure F – Charged string loops as framed oriented links. Notice how it is the stringy nature of the loops of configurations on the right (via Def. 3.3) that reflects the "blackboard framing" of the link diagrams on the left. This framing would be absent for configurations of charged *points* as on the left of Figure 2.

Example 3.13 (Link cobordism). The first and third move of charged open string worldsheets from Ex. 3.7 relate diagrams whose pre-images under (17) are framed oriented link diagrams as shown in the following moves shown on the left:



(18)

These relations (which go beyond those from Table D defining basic link diagram equivalence) are known, respectively, as the *birth/death* move and the *fusion moves* ([Kh00, §6.3][Ja04, Fig. 15], cf. [Lo24, Fig. 12]) or *oriented saddle point* moves (e.g. [Kau15, Fig. 16]) generating (on top of usual link diagram equivalence) the relation of *link cobordism*[2].

[2]Beware that early authors (e.g. [Ho68][CS80]) say "link cobordism" for what is now called "link concordance", namely for cylindrical cobordisms only. In this case, the corresponding equivalence classes of links are non-trivial. The modern use of "link cobordism" for actual cobordisms considered here seems to originate with [Kh00, §6.3], cf. [Lo24, Fig 12]. With this notion, all (framed) links are equivalent to (framed) unknots (Lem. 3.16 below), and hence the broader interest in general link cobordism is instead in characterizing the cobordisms themselves, notably through their associated homomorphism between Khovanov homologies [Ja04].

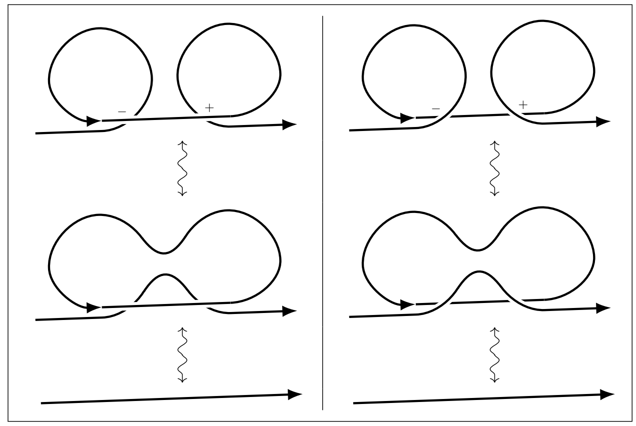
Proposition 3.14 (String loop classes as link invariants). *The map (17) descends to equivalence classes, here sending framed oriented links (instead of their representing diagrams) to elements in the fundamental group of the stringy configuration space:*

$$\mathrm{FrmdOrntdLnk} \longrightarrow\!\!\!\!\!\rightarrow \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)\,. \tag{19}$$

Proof. It is clear the sliding rules and hence the 1st and 2nd Reidemeister moves in Table D are respected. The zig-zag move is respected by (13).

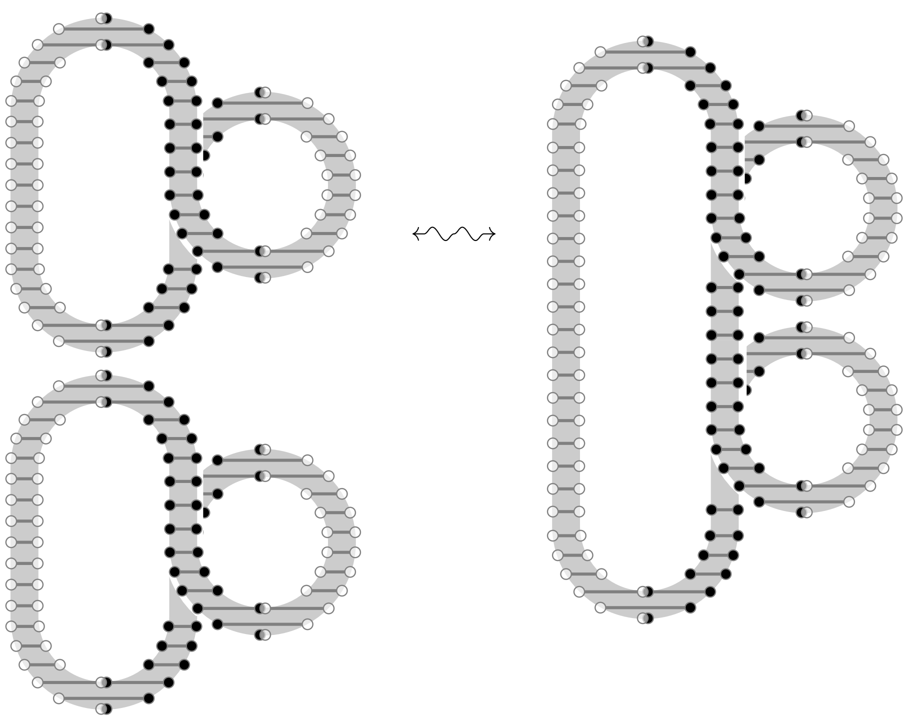
What remains to be shown is that also the 1st Reidemeister move is respected.
For this, it is sufficient to show that the extra moves (18) imply the 1st Reidemeister move.
That this is the case is shown on the right.

That the map (19) thus established is surjective is implied by the following analysis, culminating in Thm. 3.18 below.



□

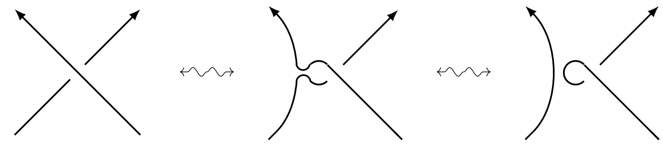
Example 3.15 (Group of stringy images of framed unknots). The images of the framed unknots under (19) constitute an integer subgroup $\mathbb{Z} \subset \mathbb{Z} \simeq \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)$ (cf. Prop. 3.6) whose group operation corresponds to the addition of framing number (Def. 3.9). For instance, the following is the move corresponding to the equation $1 + 1 = 2$ in this subgroup:



In fact, this subgroup inclusion is surjective (26), hence exhausts the full fundamental group, by the following further analysis.

Lemma 3.16 (**Framed links are cobordant to framed unknots).** *Every framed oriented link is related by the stringy moves* (18) *to a framed oriented unknot.*

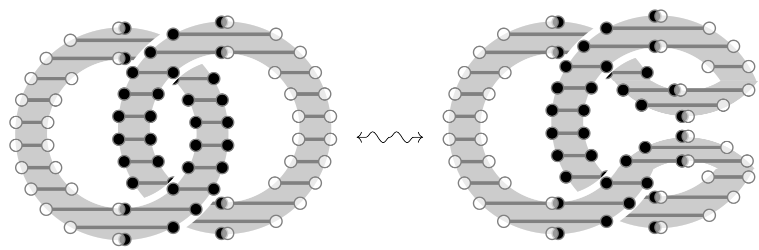
Proof. Using the zig-zag move (13) and the saddle move (18), every crossing may be turned into an avoided crossing of a straight edge with a twisted edge, like this:



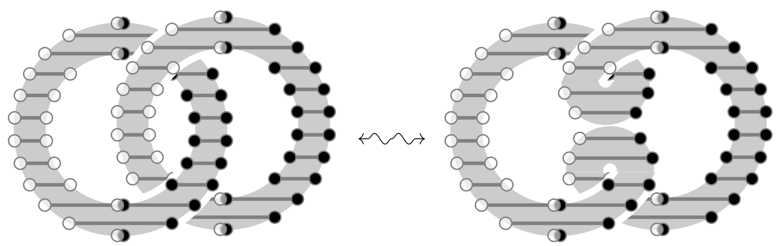
(20)

Applying such a move to all crossings of a given link diagram yields a framed unlink. Then forming the connected sum of its connected components (as in Ex. 3.15) yields a framed unknot. □

Example 3.17 (**Framed links turned into framed unknots).** The Hopf link becomes the unknot with framing ± 2 by applying the saddle move either on the right or in the middle, depending on the given orientations:

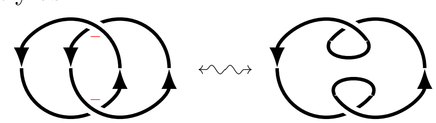


(21)



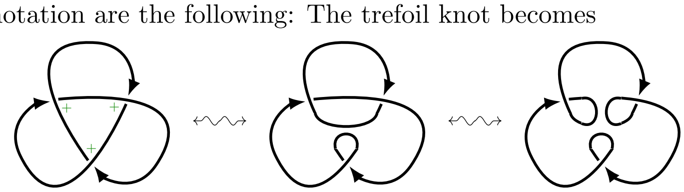
(22)

If we understand the stringy moves applied already to the corresponding framed link diagrams, then we may draw the above example more succinctly as



(23)

Further examples in this notation are the following: The trefoil knot becomes



(24)

and the figure-eight knot becomes

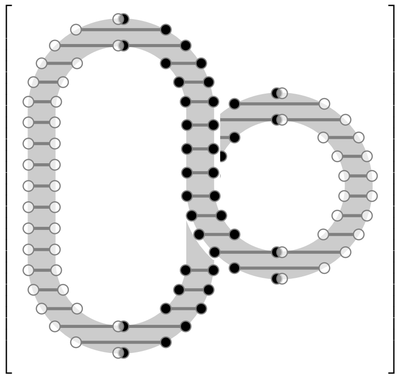
(25)

Using all this, finally, we have:

Theorem 3.18 (**Charged open string loops classified by crossing number).** *The map* (19) *from framed oriented links to the fundamental group of Okuyama's configuration space of charged open strings in the plane is, under the latter's identification with the integers* (11), *given by sending a link L to its total crossing number $\#(L)$* (16)*:*

$$\begin{array}{ccc} \mathrm{FrmdOrntdLnk} & \longrightarrow & \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big) \simeq \mathbb{Z}\,. \\ L & \longmapsto & \#(L) \end{array}$$

Proof. By Lem. 3.16, the image of L is equivalently a framed unknot via the saddle moves (18). Since all framed unknots are multiples of the unit-framed unknot, by Ex. 3.15, this exhibits the unit framed unknot as the generator



$$= \;\; 1 \quad \in \; \mathbb{Z} \; \simeq \; \pi_1\big(\mathrm{Conf}^I(\mathbb{R}^2)\big)\,. \tag{26}$$

(which hence corresponds to the Hopf fibration under the identification of Prop. 3.6).

Moreover, since the saddle move (20) used in Lem. 3.16 manifestly preserves total crossing numbers # (16), the total crossing number of the resulting unknot (being its framing number) is that of L (cf. Ex. 3.17), and hence it represents the $\#(L)$-fold multiple of the generator (26). □

Remark 3.19 (Comparison to Pontrjagin theorem).
(i) Under the equivalences of Prop. 3.6, Thm. 3.18 is similar to the statement of the Pontrjagin theorem (review in [SS23a, §3.2][SS20a, §2.1]) specialized to codimension=2 submanifolds in $\mathbb{R}^3$, which says that Cohomotopy cocycles $\mathbb{R}^3_{\cup\{\infty\}} \to S^2$ essentially correspond to closed 1-dimensional submanifolds in $\mathbb{R}^3$ (hence: links) equipped with *normal framing* and that coboundaries (homotopies) between such cocycles correspond to cobordism between such normally framed links.
(ii) By carefully translating between the different notions of framings – the framing in the sense of framed links as above in Def. 3.8 is not the same as a normal framing, but closely related (and both are of course different from tangential framing of the links) – this statement matches the above, and Thm. 3.18 may be viewed as a re-proof of Pontrjagin's theorem in these dimensions (cf. [Br93, p. 126]) from Okuyama's theorem [Ok05].

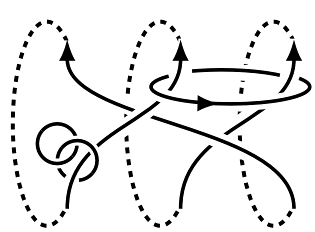
(iii) Besides the transparent diagrammatic analysis shown above, for our purposes this re-proof makes manifest the relation both to solitonic 3-branes insides M5-branes (as per Figure 1) and to anyon/anti-anyon braids of vanishing total charge (as per Figure C).

Remark 3.20 (Loops based in the n-component).
(i) Since the group completed configuration space $\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$ is, by construction, a topological group, it follows abstractly that all its connected components are, in particular, weakly homotopy equivalent, hence so are those of the weakly equivalent stringy configuration space $\mathrm{Conf}^I(\mathbb{R}^2)$, by Prop. 3.4, and hence so are the loop spaces based on any of these connected components:

$$\underset{n,n' \in \mathbb{Z}}{\forall} \qquad \Omega_n \,\mathrm{Conf}^I(\mathbb{R}^2) \;\; \simeq \;\; \Omega_{n'} \,\mathrm{Conf}^I(\mathbb{R}^2)\,.$$

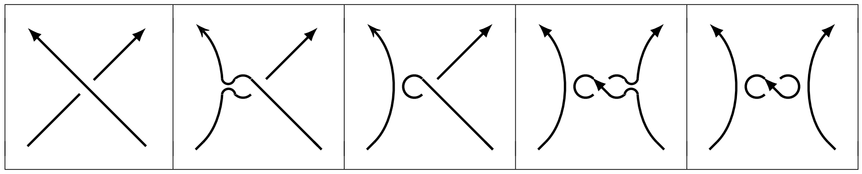
(ii) More concretely, we may now exhibit this equivalence in terms of the interpretation of loops in $\mathrm{Conf}^I(\mathbb{R}^2)$ as framed links that we have established. Or rather, this interpretation applies to the loops in the 0-charge sector, while loops in the charge=n sector may be understood more generally as *braids* on n strands interlinked with any number of links.

Figure L. An example of a loop in $\mathrm{Conf}^I(\mathbb{R}^2)$ based in the component of total charge $n = 3$.



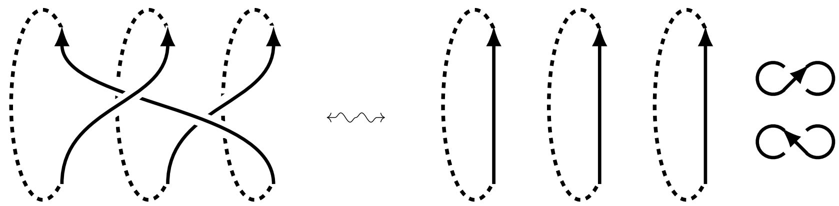
$$\in \quad \Omega_3 \,\mathrm{Conf}^I(\mathbb{R}^2).$$

(iii) To see the homotopy equivalence of the form $\Omega_n \,\mathrm{Conf}^I(\mathbb{R}^2) \simeq \Omega_0 \,\mathrm{Conf}^I(\mathbb{R}^2)$ (and hence also all the others) in terms of such "framed link-braids" (Figure L) being equivalently framed links with un-braids, and hence equivalently just framed links, observe that the saddle move from Lem. 3.16 in the following symmetrized form



(27)

allows to un-braid any braid-link at the cost of picking up a corresponding collection of further framed link components, e.g.:



(28)

This concludes our general analysis of $\Omega\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)$. In the next section we come back to understanding this as the moduli space of cohomotopically flux-quantized solitons on open M5-branes and use the above Thm. 3.18 to determine these pure quantum states of these solitons.

4 Quantum observables

We now put all the pieces together and show that the quantized topological soliton sector on holographic M5-branes under consideration is controlled by (abelian) Chern-Simons theory.

Topological Quantum Observables. First, we briefly recall the general notion of (discrete light cone) quantum observables on topological charge sectors of flux-quantized higher gauge fields according to [SS23d] (previously applied to Hanany-Witten brane configurations in [SS22][CSS23][Col23]). Given a flux quantization law $\mathcal{A}$ for higher gauge fields on a spacetime domain X equipped with the structure of an S^1-fibration $X \to Y$ (as in M/IIA duality), the corresponding algebra of light-cone quantum observables on the topological charge sectors may be understood to be the homology of a based loop space of the $\mathcal{A}$-cocycle space on Y:

$$\mathrm{QObs}_\bullet \;\equiv\; H_\bullet\big(\Omega\,\mathrm{Maps}^{*/}(Y,\,\mathcal{A});\,\mathbb{C}\big)\,, \tag{29}$$

equipped with the *Pontrjagin product*

$$\begin{array}{ccccc}
\mathrm{QObs}_\bullet \otimes \mathrm{QObs}_\bullet & & \xrightarrow{\quad(-)\cdot(-)\quad} & & \mathrm{QObs}_\bullet \\
\| & & & & \| \\
H_\bullet\big(\Omega\,\mathrm{Maps}^{*/}(Y,\mathcal{A})\big)\otimes H_\bullet\big(\Omega\,\mathrm{Maps}^{*/}(Y,\mathcal{A})\big) & \underset{\text{Künneth isomorphism}}{\simeq} & H_\bullet\Big(\Omega\,\mathrm{Maps}^{*/}(Y,\mathcal{A})\times\Omega\,\mathrm{Maps}^{*/}(Y,\mathcal{A})\Big) & \xrightarrow[\text{loop concatenation}]{\text{push along}} & H_\bullet\big(\Omega\,\mathrm{Maps}^{*/}(Y,\mathcal{A})\big)
\end{array} \tag{30}$$

and with the anti-involution

$$(-)^* \;:\; \mathrm{Obs}_\bullet \longrightarrow \mathrm{Obs}_\bullet \tag{31}$$

given by push-forward of homology along reversal of loops followed by complex conjugation (reflecting discrete light cone time reversal).

Remark 4.1 (**Nature of the topological observables).**
(i) In (29) the classifying space $\mathcal{A}$ is (just) the "topological realization" (the "shape", see [SS21, §3.3]) of the full higher moduli stack of on-shell flux quantized higher gauge fields, the latter being a homotopy fiber product of $\mathcal{A}$ with the classifying sheaf $\Omega^1_{\mathrm{dR}}(-;\mathfrak{a})_{\mathrm{clsd}}$ of on-shell flux densities (as explained in [SS24c, §3.3][SS24a][GSS24a] with technical details in [FSS23, §9]).
(ii) This means that the observables in (29) do not resolve the actual higher gauge field configurations but only their topological soliton sectors, whence they are "topological observables" only, which is what we are interested in here.
(iii) Moreover, where in the traditional path-integral picture the non-commutative product operation on quantum observables reflects their successive temporal ordering, the Pontrjagin product (30) orders by windings of observed configurations along the S^1-circle fiber, which hence plays the role of the boosted circle fiber in discrete light cone quantization, cf. [SS23d, p. 8].

Therefore the sector of the quantum states that suffice to take expectation values of these topological quantum observables are, for short, the *topological quantum states*:

Topological quantum states. Given a star-algebra of quantum observables, the corresponding quantum states are embodied by the expectation values that they induce, which are linear forms ρ on observables subject to (1.) reality, (2.) semi-positivity, and (3.) normalization (e.g. [Mey95, §I.1.1][Wa10, §7][La17, Def. 2.4], exposition in [Gl11, p. 11]):

$$\mathrm{QStates}_\bullet \;:=\; \left\{ \rho \,:\, \mathrm{Obs}_\bullet \xrightarrow[\text{linear}]{} \mathbb{C} \;\middle|\; \underset{\mathcal{O}\in\mathrm{Obs}_\bullet}{\forall}\Big(\underbrace{\rho\big(\mathcal{O}^*\big) = \rho(\mathcal{O})^*}_{\text{reality}},\; \underbrace{\rho\big(\mathcal{O}^*\cdot\mathcal{O}\big) \geq 0 \in \mathbb{R}\hookrightarrow\mathbb{C}}_{\text{(semi-)positivity}}\Big),\; \underbrace{\rho(1) = 1}_{\text{normalization}} \right\}. \tag{32}$$

This subsumes all *mixed* states ("density matrices"). Among them, the *pure* states (those which form a Hilbert space of states) are characterized as not being convex combinations of other states.

Note that (the expectation value of) a state ρ in (32) is *not required* to preserve the algebra product, those that do are called *multiplicative states*:

$$\rho : \mathrm{Obs}_\bullet \to \mathbb{C} \text{ is multiplicative} \qquad :\Leftrightarrow \qquad \underset{\mathcal{O},\mathcal{O}'\in\mathrm{Obs}_\bullet}{\forall} \;\rho\big(\mathcal{O}\cdot\mathcal{O}'\big) \;=\; \rho\big(\mathcal{O}\big)\,\rho\big(\mathcal{O}'\big)\,. \tag{33}$$

For these, we will need the following general fact:

Lemma 4.2 (**Multiplicative states are pure** (e.g. [Zhu93, Ex. 13.3-4][Wa10, Lem. 7.20-21]))**.** *Every multiplicative state* (33) *is pure; and on central observables the multiplicative states coincide with the pure states.*

Quantum states of solitons on holographic open M5-branes. Specializing this to the present case of solitons stuck at the O-planes of holographic open M5-branes wrapped on S^1_A (according to Figure 1) with their B-field flux quantized in equivariant 3-Cohomotopy (5), Prop. (3.4) gives that the topological quantum observables (29) here are:

$$\mathrm{QObs}_\bullet \;\equiv\; H_\bullet\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2);\,\mathbb{C}\big)\,.$$

From the base case of the Hurewicz theorem, this means that in degree$=0$ these topological quantum observables form the space

$$\mathrm{Obs}_0 \;=\; \mathbb{C}\big[\pi_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big)\big]$$

and, as such, are represented by compactly supported functions

$$\mathcal{O} \;:\; \pi_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big) \longrightarrow \mathbb{C}\,. \tag{34}$$

Now, by Thm. 3.18, these quantum observables detect the total crossing number $\#L$ of the links L which the solitons (of vanishing total charge at the O-plane in the M5-brane) form in their transverse space as discrete light-cone evolution moves them along the along S^1_A. A choice of $\mathbb{C}$-linear basis of the topological quantum observables is hence given by:

$$\mathrm{Obs}_0 \;\simeq\; \Big\langle \mathcal{O}_n \;:\; [L] \;\mapsto\; \delta\big(\#(L), n\big)\Big\rangle_{n\in\mathbb{Z}}\,, \tag{35}$$

in which the Pontrjagin product (30) and star-operation (31) is readily found to be

$$\mathcal{O}_n\cdot\mathcal{O}_{n'} \;=\; \mathcal{O}_{n+n'}\,, \qquad \big(\mathcal{O}_n\big)^* \;=\; \mathcal{O}_{-n}\,. \tag{36}$$

Proposition 4.3 (**The pure topological quantum states in degree=0).** *The (expectation values of) pure quantum states* (32) *on* QObs_0 (35) *are precisely the linear maps of the form*

$$\begin{array}{ccc} \mathrm{QObs}_0 & \xrightarrow{\;\rho_k\;} & \mathbb{C} \\ \mathcal{O}_n & \longmapsto & \exp\big(\tfrac{2\pi\mathrm{i}}{k}\,n\big) \end{array} \tag{37}$$

for any

$$k \in \mathbb{R}\setminus\{0\}\,. \tag{38}$$

Proof. With (36) and by Lem. 4.2, a pure state ρ on the commutative observables Obs_0 restricts to and is fixed by a group homomorphism

$$\rho\big(\mathcal{O}_{n+n'}\big) \;=\; \rho\big(\mathcal{O}_n\cdot\mathcal{O}_{n'}\big) \;=\; \rho\big(\mathcal{O}_n\big)\,\rho\big(\mathcal{O}_{n'}\big)$$

from the additive group of integers to the multiplicative group of non-vanishing (due to the normalization condition) complex numbers, hence:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{C}^\times \\ n & \longmapsto & \rho\big(\mathcal{O}_n\big) \;=\; \rho(\mathcal{O}_1)^n\,. \end{array} \tag{39}$$

Moreover, using also the reality condition (32) gives that $\rho(\mathcal{O}_1)$ is unitary

$$\rho\big(\mathcal{O}_1\big)^* \underset{(32)}{=} \rho\big(\mathcal{O}_1^*\big) \underset{(36)}{=} \rho\big(\mathcal{O}_{-1}\big) \underset{(39)}{=} \rho\big(\mathcal{O}_1\big)^{-1}$$

and hence of the claimed form (37).

It just remains to observe that every map of the form (37) really is (the expectation value of) a quantum state (32), which follows readily. □

Remark 4.4 (**Pure topological quantum states as wave-functions).** Being linear forms on 0-homology $\mathrm{Obs}_0 \equiv H_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big)$, the pure topological quantum states (37) are naturally identified with 0-cocycles in $H^0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big)$ and as such are functions on our soliton configuration space of the form

$$\begin{array}{ccc} \Omega\,\mathrm{Conf}^I(\mathbb{R}^2) \twoheadrightarrow \pi_0\big(\Omega\,\mathrm{Conf}^I(\mathbb{R}^2)\big) & \longrightarrow & \mathbb{C} \\ L & \longmapsto & \exp\big(\tfrac{2\pi\mathrm{i}}{k}\,\#(L)\big) \end{array} \tag{40}$$

in that their evaluation on a 0-chain representing the homology class $\mathcal{O}_n$ — namely on any (framed, oriented) link L with total crossing number n (16) — is $\exp\big(\tfrac{2\pi\mathrm{i}}{k}\,\#(L)\big) \;=\; \exp\big(\tfrac{2\pi\mathrm{i}}{k}\,n\big)$.

This is remarkable because it coincides with the known form of quantum states/observables of abelian Chern-Simons theory:

Remark 4.5 (**Identification with quantum observables of** U(1)**-CS theory).**
(i) For Chern-Simons theory with abelian gauge group U(1) it is widely understood by appeal to path-integral arguments ([Wi89, p. 363][FK89, p. 169] following [Pol88]) that

- the quantum states of the gauge field are labeled by a *level* [3] $k \in \mathbb{R} \setminus \{0\}$,
- the quantum observables are labeled by framed links L,
  often considered as equipped with labels (charges) q_i on their ith connected component L_i

and the expectation value of these observables in these states is the charge-weighted exponentiated framing- and linking numbers (Def. 3.9) as follows ([Wi89, p. 363], cf. review e.g. in [MPW19, (5.1)]):

$$\mathrm{W}_k(L) \;\;=\;\; \exp\left(\tfrac{2\pi \mathrm{i}}{k}\Big(\textstyle\sum_i q_i^2\,\mathrm{frm}(L_i) + \sum_{i,j} q_i q_j\,\mathrm{lnk}(L_i, L_j)\Big)\right). \tag{41}$$

(ii) However, with the charges q_i being integers, we may equivalently replace a q_i-charged component L_i with q_i unit-charged parallel copies of L_i, and hence assume without loss of generality that $\forall_i\;\; q_i = 1$. With this, we may observe that the Chern-Simons expectation values (41) coincide exactly with our pure topological quantum states (40):

$$W_k(L) \;=\; \exp\left(\tfrac{2\pi \mathrm{i}}{k}\Big(\sum_i \mathrm{frm}(L_i) + \sum_{i,j}\mathrm{lnk}(L_i, L_j)\Big)\right) \underset{(16)}{=} \exp\left(\tfrac{2\pi \mathrm{i}}{k}\#(L)\right).$$

In conclusion, we have established the following:

Fact. *With flux quantization on flat M5-branes taken to be in 3-Cohomotopy* (§2)*, the pure topological quantum states* (Prop. 4.3, Rem. 4.4) *of B-field solitons stuck on O-planes in open holographic M5-branes wrapped on* S^1_A (Figure 1) *are in any total charge sector (Rem. 3.20) exactly those of abelian Chern-Simons theory* (Rem. 4.5).

Remark 4.6 (**Comparison to the literature).**
(i) Specifically, the emergence of U(1)-Chern-Simons theory on M5-branes has previously been argued in [MPW19] by inspection of Wilson loops in $D = 5$ super Yang-Mills theory. The realization of non-abelian Chern-Simons knot invariants on suitably wrapped M5-branes has previously been argued in [Wi12][GS12], see also [NO16, §1.1].

(ii) It may be noteworthy that in these previous references, going back to [Pol88][Wi89], the all-important framing of links is imposed in an *ad hoc* manner in order to work around an ill-defined expression appearing from the path-integral arguments (going back to [Pol88, p. 326]), whereas above we use only well-defined constructions and the framing instead emerges automatically (under Hypothesis H) by careful analysis of the moduli of solitons on M5-branes, via Okuyama's theorem.

[3] Note that the level quantization, which for non-abelian compact gauge groups forces the level k to be an integer, does not apply in the abelian case considered here (cf. e.g. [FK89, p. 169]) so that the level may indeed be any non-zero real number, just as in (38).

5 Conclusion

Motivated (in §1) by suggestions that desired but subtle anyonic quantum states as expected and apparently observed in fractional quantum Hall materials could be elucidated by their geometric engineering on M5-branes, we have (in §2) briefly recalled relevant aspects of the recent global completion [GSS24b] of the field content on (here: flat and "open") M5-worldvolumes by flux quantization laws, specifically by the candidate law of co-Homotopy theory [SS24c].

The main result (in §3) is an analysis of the topological sector of the phase space BRST complex of the corresponding solitonic excitations on the flat M5 $\perp$ MO9-configurations (Figure 1), identified with loops in the group-completed configuration space of points in the plane (representing the solitons in their transverse space). By unraveling a possibly underappreciated result by Okuyama [Ok05], we could identify (Prop. 3.14) these loops with stringy loops (Figire 2) forming framed links (Prop. 3.14) and show (Thm. 3.18) that their (gauge) equivalence classes are labeled by the framing number plus twice their linking number. (A variant of the classical Pontrjagin theorem, Rem. 3.19, which may be of interest in its own right.)

Finally observing (in §4) that just this invariant gives the quantum observables of abelian Chern-Simons theory (Rem. 4.5) we found from algebraic quantum theory (Prop. 4.3) that the pure topological quantum states of our solitons (according to the topological light cone quantization of [SS23d]) are exactly those of abelian Chern-Simons theory, exhibiting our quantized solitons as abelian anyons.

Generally, this result amplifies that crucial topological effects in strongly-coupled/correlated quantum systems engineered on branes in string/M-theory depend on the global completion of (higher) gauge fields by flux quantization laws in non-abelian differential cohomology [SS24c]. This is only beginning to be understood, requiring machinery from equivariant geometric homotopy theory that is still relatively novel [SS20b][FSS23] (quick exposition in [Sc24]), certainly in its application to quantum physics.

Concretely, it is interesting that the link diagrams (of soliton configurations) – that emerged in §3 and were identified in §4 as worldlines of anyonic quantum defects – are just the kind of processes envisioned in many texts on quantum-computational processes based on anyon braiding:

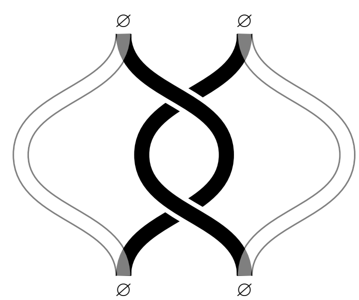
Figure C. In the traditional picture of anyon braiding processes implementing topological quantum computations (e.g. [Kau02, Fig. 17][FKLW03, Fig. 2][Ro16, Fig. 2][DMNW17, Fig. 2][RW18, Fig. 3][Ro22, Fig. 1]), the computation is:

(i) initialized by creating anyon/anti-anyon pairs out of the vacuum $\varnothing$,

(ii) executed by adiabatically braiding their worldlines,

(iii) read-out by annihilating the anyons again into the vacuum $\varnothing$.

This means that the computation is encoded by a *link diagram* and that its result is the corresponding Wilson loop observable, just as here we naturally found realized on M5-branes (albeit only for the abelian case).



Outlook. While abelian anyons are not universal for topological quantum gates by themselves, they become so already when combined with quantum measurement gates (see [Pa06][Ll02][Wo10][WP11]). Therefore, experimental realization in the form of manipulatable *solitons* (as found here on holographic M5-branes) would be a major step towards fault-tolerant quantum computation, and a microscopic holographic understanding of their nature should eventually be conducive to overcoming the present impasse in laboratory realizations of anyons.

In order to obtain a more complete such microscopic holographic understanding of (abelian) anyons, one will need to pass beyond the sector of topological quantum observables considered here (cf. Rem. 4.1) in order to resolve the quantum dynamics also of the local B-field gauge potentials (cf. [GSS24b, §4.1]) and of the fluctuations of the M5-brane worldvolume. We have been preparing the ground for this in [GSS24c], and we hope to take further steps in this direction in the future.

Declarations:

Data availability – There is no data associated with this manuscript.

Competing interests – The authors declare that they have no conflict of interest.

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