Engineering of Anyons on M5-Probes via Flux-Quantization

Hisham Sati [a,b] Urs Schreiber [a]
hsati@nyu.edu us13@nyu.edu

January 6, 2025

Abstract

Comprehensive flux-quantization of the M5-brane's tensor field – consistent with its non-linear self-duality and with its twisting by the bulk C-field – exists only in little-studied non-abelian generalized cohomology theories, notably in a twisted equivariant (and "twistorial") form of unstable Cohomotopy ("Hypothesis H") — but it is only through such flux-quantization that the field is globally completed, with well-defined solitonic configurations and torsion charges.

In these lecture notes we review the construction for an audience familiar with the general notion of flux-quantization (for which see [44]). The key result is a rigorous first-principles derivation of anyonic topological order on (single) magnetized M5-branes probing Seifert orbi-singularities ("geometric engineering" of anyons), which we motivate from open theoretical problems in the field of quantum computing.

Lecture notes (2nd part) for the lecture series
Introduction to Hypothesis H
held at
45th Winter School on Geometry and Physics
Srni, Czechia (18-25 Jan 2024)

Contents

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[a] Mathematics, Division of Science; and
Center for Quantum and Topological Systems,
NYUAD Research Institute,
New York University Abu Dhabi, UAE.

[b] The Courant Institute for Mathematical Sciences, NYU, NY.

The authors acknowledge the support by *Tamkeen* under the *NYU Abu Dhabi Research Institute grant* `CG008`.

1 Motivation: Better Anyon Theory

While the hopes associated with the idea of *quantum computing* [81][49] are hard to over-state [45][8][91], there are good arguments that commercial-value quantum computing will ultimately require quantum hardware exhibiting *anyonic topological order* [119][43]. But microscopic theoretical derivations, from first principles, of such anyonic quantum states in strongly-coupled quantum systems had remained sketchy, which may explain the dearth of experimental realizations to date.

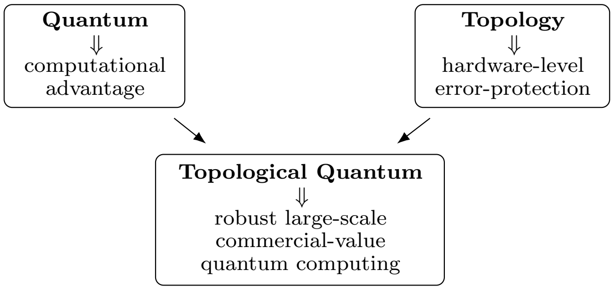
What we review here (from [99][102][47]) is a rigorous theoretical account via "geometric engineering on M-branes" subject to a previously neglected step of "flux-quantization" (the latter surveyed in [44]).

First, we expand on the motivation a little further:

Ultimate need for Topological Quantum Protection. Despite the fascinating reality of presently available Noisy Intermediate-Scale Quantum computers (NISQ [89]) and despite the mid-term prospect of their stabilization at the software-level via Quantum Error Correction (QEC [70][90], at heavy cost of available system scale), serious arguments [59][22][68][23][24][54][37] and experience [14] suggest that large-scale quantum computation is hardly attainable by incremental optimization of NISQ architectures, but [15] [1] that more fundamental quantum principles will need to be exploited – notably *topological* error *protection* already at the hardware-level [63][34][104][103] in order to suppress quantum errors occurring in the first place.

While topological quantum protection is thus possibly indispensable for achieving commercial-value quantum computing, its ambitious development, in theory and practice, is in fact far from mature, is in need of new ideas and of further analysis, and leaves much room for development.

Since this is not always made clear, to amplify this point:



(i) Theoretical challenges: While quantum theorists now routinely deal with the algebraic structure (namely: braided fusion categories) commonly *expected* [64] to describe interaction of anyon species *in toto*, the *microscopic* first-principles understanding of the formation of anyonic topological order as solitonic states in the many-body (electron) dynamics of quantum materials has remained at most sketchy, even in the best-understood case of the fractional quantum Hall effect [106], cf. [56]. [2]

In fact, this is an instance of the general open problem of analytically establishing gapped bound states in any strongly coupled/correlated quantum system: The problem of formulating non-perturbative quantum field theory [5][26]. The analogous issue in particle physics (there called the *Yang-Mills mass gap* problem [82]) has been recognized as being profound enough to be declared one of seven "Millennium Problems" [11].

(ii) Practical challenges: But without a robust theoretical prediction of anyonic solitons in actual quantum materials, it remains unclear where and how to look for them. As an unfortunate result, experimentalists have turned attention to mere stand-ins, such as "Majorana zero modes" at the ends of super/semi-conducting nonowires ([62][74] which, even if the doubts about their detection were to be removed [16], are by construction immobile and hence do not serve as hardware-protected quantum braid gates) and quantum-simulation of anyons on NISQ architectures ([55][33, Fig. 5], which might serve as software-level QEC but again offers no hardware-level protection.

In short: **Foundation and implementation** of topological quantum computing as a plausible long-term pathway to actual quantum value **deserves and admits thorough re-investigation**.

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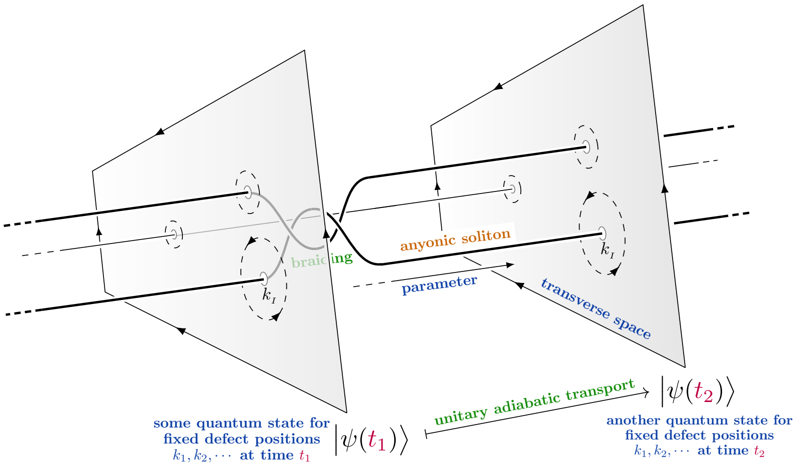
[1] [15]: "The qubit systems we have today are a tremendous scientific achievement, but they take us no closer to having a quantum computer that can solve a problem that anybody cares about. [...] What is missing is the breakthrough [...] bypassing quantum error correction by using far-more-stable qubits, in an approach called topological quantum computing."

[2] [56, p. 3]: "Though the Laughlin function very well approximates the true ground state at $\nu = 1/q$, the physical mechanism of related correlations and of the whole hierarchy of the FQHE remained, however, still obscure. [...] The so-called HH (Halperin–Haldane) model of consecutive generations of Laughlin states of anyonic quasiparticle excitations from the preceding Laughlin state has been abandoned early because of the rapid growth of the daughter quasiparticle size, which quickly exceeded the sample size. [...] the Halperin multicomponent theory and of the CF model advanced the understanding of correlations in FQHE, however, on phenomenological level only. CFs were assumed to be hypothetical quasi-particles consisting of electrons and flux quanta of an auxiliary fictitious magnetic field pinned to them. The origin of this field and the manner of attachment of its flux quanta to electrons have been neither explained nor discussed."

Concretely, the intrinsic tension haunting the traditional quantum computing paradigm is that:

(i) quantum gates are *implemented via interaction* of subsystems,
(ii) while quantum coherence requires *avoiding all interaction*.

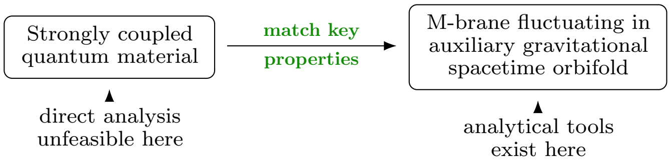
The idea of topological protection is to cut this Gordian knot by *quantum gates operating without interaction.* The physical principle that allows this to work [3][4][34, p 6][87, p 50] is the *quantum adiabatic theorem* [92]: Gapped quantum systems frozen at absolute zero in one of several ground states, but dependent on external parameters, will defy interaction with noise quanta below the energy gap and yet have their ground state transformed by sufficiently gentle tuning of the parameters: a *holonomic quantum gate.* This is *topological* if it is invariant under local deformations of parameter paths, and thus protected also against classical noise. For an *anyonic braid gate* the parameters in question are the positions of solitons in a 2-dimensional transverse space within a quantum material.



The remaining problem is to understand how such anyonic solitons may actually arise in quantum materials.

Improved Anyon Models via Geometric Engineering on M-branes. A remarkable solution to the otherwise elusive microscopic analysis of strongly-coupled/correlated quantum systems emerges in the guise of "geometric engineering" [61][10] of quantum fields on "M-branes" probing orbifold singularities, whereby the given dynamics is (partially) mapped onto the fluctuations of Membranes (whence *M-theory* [20]), and of higher-dimensional "M5-branes" [47], propagating within an auxiliary higher-dimensional gravitating spacetime orbifold [96].

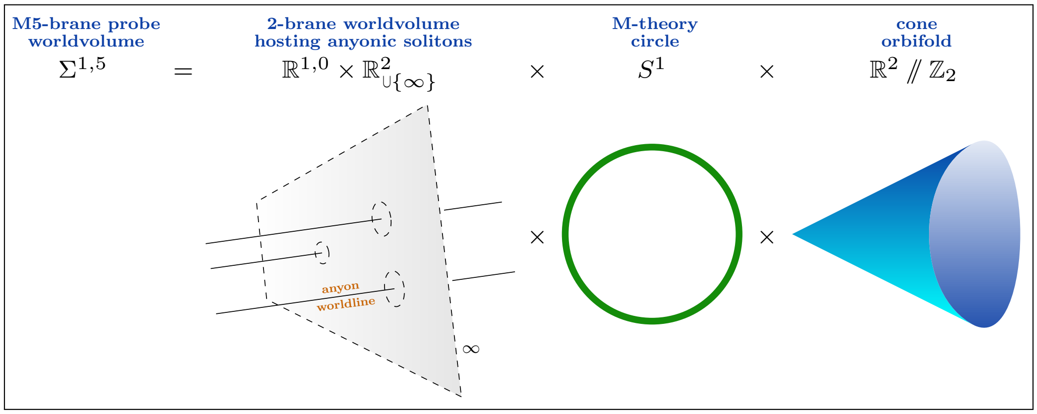
Geometric engineering of quantum systems on M-branes provides tools for analyzing otherwise elusive strongly coupled/correlated quantum phenomena.



This procedure is most famous in the (unrealistic) limit of large rank and hence of large numbers $N \to \infty$ of coincident such branes, where it extracts quantum correlators and quantum phase transitions entirely from classical gravitational asymptotics ("holographic duality" [1]). The application to quantum materials [118][52] is now well-studied, notably in the case of quantum critical superconductors engineered in M-theory [53][35][36][50][18][19][2].

But we have established [47][99][100][102] that after implementing a previously neglected step of "flux quantization" [44] on the M5-brane wordlvolume, there provably appear general solitonic and specifically anyonic quantum states already in the more realistic situation of single ($N = 1$) coincident branes. (Similar results for $N = 2$ had previously only been conjectured by appeal to an expected but notoriously undefined effective quantum field theory on coincident M5-branes.)

Brane diagram for geometric engineering of anyons on single M5-branes wrapping an orbi-singularity [102]:
It is a subtle mechanism of *flux-quantization* [44] of the self-dual tensor-field on the M5 [47] that stabilizes [99] its anyonic soliton configurations.



Here, we will review and explain how this works for an audience assumed to be familiar with the general mechanism of *flux quantization* as surveyed in [44]

A broad lesson following immediately from this successful geometric engineering of topological qbits is the plausible existence of more exotic anyonic states than traditionally envisioned: Namely the "duality symmetry" [88][20, §6] of M-theory predicts that any geometrically engineered quantum system has "dual" incarnations with isomorphic quantum observables but entirely different geometric realization, where ordinary space is replaced by more abstract parameter spaces. Notably "T-duality" [110][28][48] applied to topological quantum materials has been argued [77][78][51] to exchange the roles of ordinary space with that of reciprocal "momentum space".

(2) Novel experimental pathways towards anyons. Indeed, while anyonic solitons are traditionally envisioned as being localized in "position space" (meaning that the anyon cores are points in the plane of the crystal lattice) the physical principle behind topological quantum gates — namely [3][4][34, p 6][87, p 50] the *quantum adiabatic theorem* [92] — is unspecific to position space and only requires the material's Hamiltonian to *depend on any continuous parameters* (such as external voltage or strain) varying in any abstract parameter space.

The general physical conditions for topological quantum gates given by the *quantum adiabatic theorem*, listed (a) - (e) on the right, are much more general than traditionally considered for anyon braid gates — the latter are only the special case where the parameters are configurations of points in the plane of the 2D crystal lattice.

(a) **Ground state degeneracy** (when frozen at absolute zero, the system still has more than one state to be in, even up to phase).

(b) **Spectral gap** (quanta of energy smaller than a given gap $\epsilon > 0$ cannot excite these ground states).

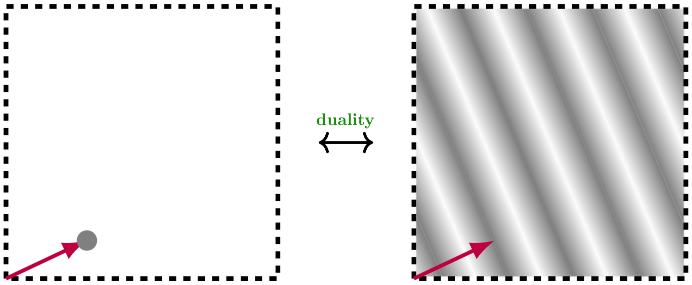
(c) **Control parameters** (the above properties hold for a range of continuously tunable external parameters).

(d) **Parameter topology** (there exist closed parameter paths that cannot be continuously contracted).

(e) **Local invariance** (continuously deformed parameter paths induce the same transformation on ground states).

This means that, in principle, the possibilities in which anyonic quantum states could arise in the laboratory are far more general than what has been explored to date.

Concretely, a key example of alternative parameters for ground states of a quantum material are points in their reciprocal *momentum space*: This is the space of (quasi-)momenta, hence of wave-vectors for plane quasiparticle waves going through the crystalline material.



Position space:
a point is a position
on the crystal lattice

Momentum space:
a point is a plane wave
on the crystal lattice

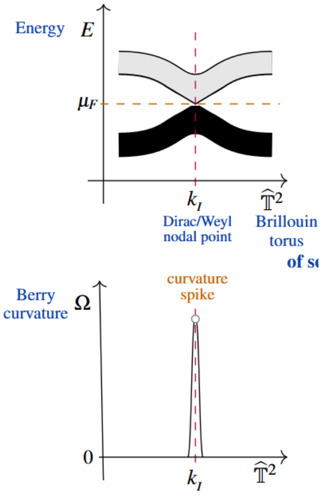
We have observed before that candidate anyon-like solitons localized (not in position space but) in momentum space are plausible both theoretically [43] as well as experimentally [117][108][58] and may have been hiding in plain sight: as band nodes of (interacting) topological semimetals.

Indeed, momentum space naturally features key properties that are typically assumed for anyon braid gates but remain elusive in position space:

(i) toroidal base topology is routinely assumed [111][112][71] in order to achieve the required ground-state degeneracy, but is quite unrealistic in position space, even more so when meant to be punctured by defect anyons — while the momentum space of a crystal is automatically a torus (the *Brillouin torus*).

(ii) stable defect points need special engineering in position space but arise automatically in momentum space in the guise of *band nodes* of topological semi-metals [43, Fig. 6]

(iii) defect point movement in a *controlled* way is necessary for braid gates but remains elusive in position space, while band nodes in momentum space have already shown to be movable in a varierty of systems, by tuning of external parameters (e.g. strain).



The geometric engineering of anyons discussed here goes towards providing also fundamental theoretical underpinning of the possibility of more "exotic" anyon realizations than have traditionally been envisioned.

2 Flux-Quantization on M5-Probes

The first task now is to understand the flux-quantization on M5-brane probes, according to [30][31][101].

We will not (need to) explain in full detail the (super-)geometry of probe branes nor of their (super-)gravity backgrounds (full discussion is in [46][47]), but do offer the following broad dictionary, for orientation: [3]

M5-Brane probes (namely *sigma-model* branes, in contrast to *black branes*) are 5-dimensional objects propagating in a gravitational target space X (the "bulk"), along trajectories that are modeled by (super-)immersions of their 6D (and $\mathcal{N} = (2,0)$) worldvolume (super-)manifolds Σ

$$\begin{tikzcd}
\substack{\textbf{probe M5-brane} \\ \textbf{(super-)worldvolume}} & \Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \arrow[r, "\phi_s", "\substack{\textbf{trajectory} \\ \textbf{(super-)immersion}}"'] & X^{1,10\,|\,\mathbf{32}} & \substack{\textbf{target/background} \\ \textbf{(super-)spacetime}}
\end{tikzcd} \tag{1}$$

Here the admissible ("on-shell", meaning: satisfying the appropriate equations of motion) immersions ϕ_s are controlled by the (super-)geometry of X – namely the brane's trajectory is subject to the gravitational- and Lorentz-forces exerted by the field content of X – but X itself remains unaffected by the choice of ϕ_s – meaning that the (gravitational) *back-reaction* of the brane on its ambient spacetime is neglected; this is what makes the brane but a *probe* of the *background* X.

Thereby the probe brane (Σ, ϕ_s) plays a double role:

(i) on the one hand it is like a (higher-dimensional) fundamental particle, an "observer" of the bulk X in the sense of mathematical relativity,

(ii) on the other hand it is itself a (super-)spacetime with its own (quantum) field content:

Remarkably, the magic of super-geometry makes such purely super-geometric immersions ϕ_s (1) embody not just the naïve (temporal-)spatial worldvolume trajectory, but also a 3-flux density H_3^s *on* Σ [47, §3.3]. This is (on-shell) the notorious "self-dual" flux density whose accurate quantization (traditionally neglected) is our main concern here.

This second aspect is what we are concerned with for the purpose of modeling strongly-coupled quantum systems:

The (1+3)D worldvolume $M^{1,3}$ of a quantum material – or, for the intent of modeling anyons, the effectively $(1+2)D$-worldvolume $M^{1,2}$ of a sheet-like material (e.g. an atomic mono-layer akin to graphene) – is to be identified with a sub-quotient of the brane worldvolume, typically with a fixed locus (orbifold singularity) inside the base of a fibration (Kaluza-Klein reduction).

$$\begin{tikzcd}
 & \substack{\textbf{M5-brane} \\ \textbf{worldvolume}} & \substack{\textbf{ambient bulk} \\ \textbf{spacetime}} \\
 & \Sigma^{1,5} \arrow[r, "\phi"] \arrow[d, two heads] & Y^{1,10} \arrow[d, two heads, "\substack{\textbf{M/IIA-} \\ \textbf{fibration}}"] \\
M^{1,2} \arrow[r, hook, "\substack{\textbf{orbi-} \\ \textbf{singularity}}"] & \Sigma^{1,4} \arrow[r] & X^{1,9} \\
\substack{\textbf{quantum material} \\ \textbf{worldvolume}} & \substack{\textbf{D4-brane} \\ \textbf{worldvolume}} &
\end{tikzcd}$$

Their flux-quantization (to recall from [44]) is then encoded in a choice of a fibration $\mathcal{A} \xrightarrow{p} \mathcal{B}$ of classifying spaces, subject to the constraint that the Bianchi identities for the (duality-symmetric) flux densities on bulk and brane are the closure/flatness condition on $\mathfrak{l}p$-valued differential forms, where $\mathfrak{l}(-)$ forms *Whitehead L_∞-algebras* of these classifying fibrations (dual to their minimal relative Sullivan model).

Given such a choice, the topological sector of the higher gauge fields on bulk and brane are given by maps from the brane-immersion into the classifying fibration:

$$\boxed{\begin{tikzcd}
\textbf{brane} \;\; \Sigma \arrow[r, dashed, "\substack{\textbf{densities of} \\ \textbf{brane fluxes}}"] \arrow[d, "\phi"', "\textbf{immersion}"] & \Omega^1_{\mathrm{dR}}(-; \mathfrak{l}_{\mathcal{B}}\mathcal{A})_{\mathrm{clsd}} \arrow[d, "(\mathfrak{l}p)_*"] & \Sigma \arrow[r, dashed, "\substack{\textbf{charges of} \\ \textbf{brane fields}}"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"', "\substack{\textbf{fibration of} \\ \textbf{classifying spaces}}"] \arrow[dl, phantom, "\Leftarrow"] \\
\textbf{bulk} \;\; X \arrow[r, dashed, "\substack{\textbf{densities of} \\ \textbf{bulk fluxes}}"'] & \Omega^1_{\mathrm{dR}}(-; \mathfrak{l}\mathcal{B})_{\mathrm{clsd}} & X \arrow[r, dashed, "\substack{\textbf{charges of} \\ \textbf{bulk fields}}"'] & \mathcal{B}
\end{tikzcd}}$$

With these comments on perspective out of the way, **the plan of this section** are the following topics:

(1.) Bianchi identities on magnetized M5-probes

(2.) Flux quantization in Twistorial Cohomotopy

(3.) Aside: Projective Spaces and their Fibrations

(4.) Orbi-worldvolumes and Equivariant charges

The first step of flux-quantization is to identify the Bianchi identities satisfied by the flux densities:

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[3] All brane concepts we consider are well-defined and all conclusions have proofs – at no point do we rely on informal string theory folklore beyond motivation.

Bianchi identities on M5-Probes of 11D SuGra via super-geometry. Consider the 11D super-tangent space

$$\underset{\text{super-Minkowski}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \hookrightarrow \underset{\text{super-Poincaré}}{\mathfrak{isom}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)} \twoheadrightarrow \underset{\text{Lorentz}}{\mathfrak{so}(1,10)}$$

with its super-invariant 1-forms (cf. [46, §2.1])

$$\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \;\simeq\; \underset{\text{super-transl. invar. forms}}{\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{R}^{1,10|\mathbf{32}}\big)^{\mathrm{li}}} \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10}\end{array}\right] \Big/ \left(\begin{array}{lcl}\mathrm{d}\,\Psi^\alpha &=& 0 \\ \mathrm{d}\,E^a &=& \big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\end{array}\right).$$

Remarkably, the quartic Fierz identities entail that [17][79][46, Prop. 2.73]:

$$\left.\begin{array}{lcl} G_4^0 &:=& \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)\,E^{a_1}E^{a_2} \\ G_7^0 &:=& \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)\,E^{a_1}\cdots E^{a_5}\end{array}\right\} \in \underset{\text{fully super-invariant forms}}{\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)^{\mathrm{Spin}(1,10)}} \qquad \text{satisfy :} \qquad \begin{array}{lcl}\mathrm{d}\,G_4^0 &=& 0 \\ \mathrm{d}\,G_7^0 &=& \tfrac{1}{2}G_4^0\,G_4^0\end{array}$$

To globalize this situation, say that an **11D super-spacetime** X is a super-manifold equipped with a super-Cartan connection, locally on an open cover $\widetilde{X} \twoheadrightarrow X$ given by

$$\left.\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10} \\ \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{10}\end{array}\right\} \in \Omega^1_{\mathrm{dR}}\big(\widetilde{X}\big) \qquad \begin{array}{l}\text{such that the} \\ \text{super-torsion} \\ \text{vanishes}\end{array} \qquad \mathrm{d}\,E^a - \Omega^a{}_b\,E^b \;=\; \big(\overline{\Psi}\,\Gamma^a\,\Psi\big),$$

and say that **C-field super-flux** on such a super-spacetime are super-forms with these co-frame components:

$$\boxed{\begin{array}{lclcl} G_4^s &:=& G_4 + G_4^0 &:=& \tfrac{1}{4!}(G_4)_{a_1\cdots a_4}E^{a_1}\cdots E^{a_4} + \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)E^{a_1}\,E^{a_2} \\ G_7^s &:=& G_7 + G_7^0 &:=& \tfrac{1}{7!}(G_4)_{a_1\cdots a_7}E^{a_1}\cdots E^{a_7} + \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)E^{a_1}\cdots E^{a_5}\end{array}}$$

Theorem [46, Thm. 3.1]: On an 11D super-spacetime X with C-field super-flux (G_4^s, G_7^s):

The duality-symmetric super-Bianchi identity $\left\{\begin{array}{lcl}\mathrm{d}\,G_4^s &=& 0 \\ \mathrm{d}\,G_7^s &=& \tfrac{1}{2}\,G_4^s\,G_4^s\end{array}\right\}$ *is equivalent to the full 11D SuGra equations of motion!*

Next, on the super-subspace $\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+} \xhookrightarrow{\phi_0} \mathbb{R}^{1,10\,|\,\mathbf{32}}$ fixed by the involution $\Gamma_{012345} \in \mathrm{Pin}^+(1,10)$ we have:

$$H_3^0 \;:=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,H_3^0 \;=\; \phi_0^*\,G_4^0$$

To globalize this situation, say that a super-immersion $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow{\phi_s} X^{1,10\,|\,\mathbf{32}}$ is **1/2BPS M5** if it is "locally like" ϕ_0, and say that **B-field super-flux** on such an M5-probe is a super-form with these co-frame components:

$$\boxed{H_3^s \;:=\; H_3 + H_3^0 \;:=\; \tfrac{1}{3!}(H_3)_{a_1a_2a_3}e^{a_1}\,e^{a_2}\,e^{a_3} + 0} \qquad \big(e^{a<6} \;:=\; \phi_s^*E^a\big)$$

Theorem [47, §3.3]: On a super-immersion ϕ_s with B-field super-flux H_3^s:

The super-Bianchi identity $\big\{\mathrm{d}\,H_3^s \;=\; \phi_s^*G_4^s\big\}$ *is equivalent to the M5's B-field equations of motion.*

In particular, the (non-linear self-)duality conditions on the ordinary fluxes are *implied*: $G_4 \leftrightarrow G_7$ and $H_3 \leftrightarrow H_3$.

Seeing from this that also trivial tangent super-cochains may have non-trivial globalization, observe next that:

$$F_2^0 \;:=\; \big(\overline{\psi}\,\psi\big) \;=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,F_2^0 \;=\; 0$$

Globalizing this to $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+}$ via

$$\boxed{F_2^s \;:=\; F_2 + F_2^s \;:=\; \tfrac{1}{2}(F_2)_{a_1a_2}\,e^{a_1}e^{a_2} + 0}$$

we have on top of the above:

Theorem [102, p 7]: *The super-Bianchi identity* $\big\{\mathrm{d}\,F_2^s \;=\; 0\big\}$ *is equivalent to the Chern-Simons E.O.M.:* $F_2 = 0$.

Flux quantization in Twistorial Cohomotopy. In summary, a remarkable kind of higher super-Cartan geometry locally modeled on the 11D super-Minkowski spacetime $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ entails that on-shell 11D supergravity probed by magnetized 1/2BPS M5-branes implies and is entirely governed by these Bianchi identities on super-flux densities:

$$\begin{array}{lll}
\textbf{A-field} & \mathrm{d}\,F_2^s \;=\; 0 \qquad\qquad \mathrm{d}\,G_4^s \;=\; 0 & \textbf{C-field} \\
\textbf{self-dual B-field} & \mathrm{d}\,H_3^s \;=\; \phi_s^* G_4^s + \theta\, F_2^s\, F_2^s \qquad \mathrm{d}\,G_7^s \;=\; \tfrac{1}{2} G_4^s\, G_4^s & \textbf{dual C-field} \\
\textbf{M5 probe} & \Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow[\text{1/2BPS immersion}]{\phi_s} X^{1,11\,|\,\mathbf{32}} & \textbf{SuGra bulk}
\end{array} \tag{2}$$

Here we have observed that the Green-Schwarz term $F_2^s F_2^s$ may equivalently be included for any theta-angle $\theta \in \mathbb{R}$ without affecting the equations of motion (since, recall, the CS e.o.m. $F_2^s = 0$ is already implied by $\mathrm{d}\,F_2^s \,=\, 0$).

But non-vanishing theta-angle does affect the admissible flux-quantization laws and hence the global solitonic and torsion charges of the fields. The choice of flux quantization according to *Hypothesis H* [30][31] is the following:

Admissible fibrations of classifying spaces for cohomology theories with the above character images (2). The homotopy quotient of S^7 is (i) for $\theta = 0$ by the trivial action and (ii) for $\theta \neq 0$ by the principal action of the complex Hopf fibration.

$$\begin{tikzcd}
\boxed{\theta = 0} & S^7 /\!\!/_0 \mathrm{U}(1) \simeq S^7 \times \mathbb{C}P^\infty \arrow[r, two heads] \arrow[d, two heads] & S^7 \arrow[r, two heads, "h_{\mathbb{H}}", "\text{$\mathbb{H}$-Hopf fibration}"'] \arrow[d, two heads, "\text{$\mathbb{C}$-Hopf fibration}"] & \mathbb{H}P^1 \arrow[d, equal] \\
\boxed{\theta \neq 0} & S^7 /\!\!/ \mathrm{U}(1) \arrow[r, "\sim"] & \mathbb{C}P^3 \arrow[r, two heads, "t_{\mathbb{H}}", "\text{Twistor fibration}"'] & \mathbb{H}P^1
\end{tikzcd}$$

Proof. This may be seen as follows [31, Lem. 2.13]:

Since the real cohomology of projective space is a truncated polynomial algebra,

$$\begin{array}{ll}
H^\bullet\big(\mathbb{C}P^n;\mathbb{R}\big) \simeq \mathbb{R}\big[\overbrace{c_1}^{\text{deg}=2}\big]/\big(c_1^{n+1}\big) & H^\bullet\big(\overbrace{\mathbb{C}P^\infty}^{\simeq B\mathrm{U}(1)};\mathbb{R}\big) \simeq \mathbb{R}[c_1] \\
H^\bullet\big(\mathbb{H}P^n;\mathbb{R}\big) \simeq \mathbb{R}\big[\underbrace{\tfrac{1}{2}p_1}_{\text{deg}=4}\big]/\big(p_1^{n+1}\big) & H^\bullet\big(\underbrace{\mathbb{C}P^\infty}_{\simeq B\mathrm{Sp}(1) \simeq B\mathrm{SU}(2)};\mathbb{R}\big) \simeq \mathbb{R}[\tfrac{1}{2}p_1]
\end{array}$$

---

the minimal dgc-algebra model for $\mathbb{C}P^n$ needs a closed generator f_2 to span the cohomology and a generator h_{2n+1} in order to truncate it; analogously for $\mathbb{H}P^n$.

$$\mathrm{CE}\big(\mathfrak{l}\,\mathbb{C}P^n\big) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_{2n+1} \end{bmatrix} \Big/ \left(\begin{array}{ll} \mathrm{d}\,f_2 & = 0 \\ \mathrm{d}\,h_{2n+1} & = (f_2)^{n+1} \end{array}\right)$$

$$\mathrm{CE}\big(\mathfrak{l}\,\mathbb{H}P^n\big) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} g_4 \\ g_{4n+3} \end{bmatrix} \Big/ \left(\begin{array}{ll} \mathrm{d}\,g_4 & = 0 \\ \mathrm{d}\,g_{4n+3} & = (g_4)^{n+1} \end{array}\right)$$

Furthermore, since the second Chern class of an $S\big(\mathrm{U}(1)^2\big)$-bundle is minus the cup square of the first Chern class (by the Whitney sum rule)

$$\begin{array}{ccc}
B\mathrm{U}(1) & \xrightarrow{B\left(c \mapsto \mathrm{diag}(c, c^*)\right)} & B\mathrm{SU}(2) \\
-(c_1)^2 & \longleftarrow\!\shortmid & \tfrac{1}{2}p_1 = c_2
\end{array}$$

---

the minimal model of $\mathbb{C}P^3$ *relative* to that of $\mathbb{H}P_1$ needs to adjoin to the latter not only f_2 but also a generator h_3 imposing this relation in cohomology.

$$\mathrm{CE}\big(\mathfrak{l}_{{}_{\mathbb{H}P^1}}\mathbb{C}P^3\big) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_3 \\ g_4 \\ g_7 \end{bmatrix} \Big/ \left(\begin{array}{ll} \mathrm{d}\,f_2 & = 0 \\ \mathrm{d}\,h_3 & = g_4 + f_2 f_2 \\ \mathrm{d}\,g_4 & = 0 \\ \mathrm{d}\,g_7 & = \tfrac{1}{2} g_4\, g_4 \end{array}\right)$$

The resulting fibration of L_∞-algebras is manifestly just that classifying the desired Bianchi identities (2) (we are showing the case $\theta \neq 0$, which by isomorphic rescaling may be taken to be $\theta = 1$):

$$\begin{tikzcd}
\Sigma^6 \arrow[r, dashed] \arrow[d, "\phi"'] & \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}_{{}_{\mathbb{H}P^1}}\mathbb{C}P^3\big)_{\mathrm{clsd}} \arrow[d, "(\mathfrak{l}\,t_{\mathbb{H}})_*"] & \Leftrightarrow & \Omega^\bullet_{\mathrm{dR}}\big(\Sigma^6\big) & \mathrm{CE}\big(\mathfrak{l}_{{}_{\mathbb{H}P^1}}\mathbb{C}P^3\big) \arrow[l, dashed] \arrow[d, "(\mathfrak{l}\,t_{\mathbb{H}})^*"] & \Leftrightarrow & \begin{array}{l} F_2 \\ H_3 \end{array} \in \Omega^\bullet_{\mathrm{dR}}(\Sigma^6) \left| \begin{array}{ll} \mathrm{d}\,F_2 & = 0 \\ \mathrm{d}\,H_3 & = G_4 + F_2\,F_2 \end{array} \right. \\
\Sigma^{11} \arrow[r, dashed] & \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}\,\mathbb{H}P^1\big)_{\mathrm{clsd}} & & \Omega^\bullet_{\mathrm{dR}}(X^{11}) \arrow[u, "\phi^*"] & \mathrm{CE}\big(\mathfrak{l}\,\mathbb{H}P^1\big) \arrow[l, dashed] & & \begin{array}{l} G_4 \\ G_7 \end{array} \in \Omega^\bullet_{\mathrm{dR}}\big(X^{11}\big) \left| \begin{array}{ll} \mathrm{d}\,G_4 & = 0 \\ \mathrm{d}\,G_7 & = \tfrac{1}{2} G_4\, G_4 \end{array} \right.
\end{tikzcd}$$

Aside: Projective Spaces and their Fibrations – some classical facts.
Consider:

division algebras $\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$ generically denoted $\mathbb{K} \in \big\{\mathbb{R}, \mathbb{C}, \mathbb{H}\big\}$

groups of units $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ understood with the multiplicative group structure

projective spaces $\mathbb{K}P^n := \big(\mathbb{K}^{n+1} \setminus \{0\}\big)/\mathbb{K}^\times$

higher spheres $S^n \simeq \big(\mathbb{R}^{n+1} \setminus \{0\}\big)/\mathbb{R}_{>0}$

$\mathbb{K}$-Hopf fibrations are the quotient co-projections induced by $\iota : \mathbb{R}_{>0} \hookrightarrow \mathbb{K}$

The classical Hopf fibrations $h_{\mathbb{K}}$ are:

$$\begin{tikzcd}
S^0 \simeq \mathbb{R}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\mathrm{ker}"] & \\
S^1 \arrow[d, two heads, "h_{\mathbb{R}}"] & \simeq \big(\mathbb{R}^2\setminus\{0\}\big)/\mathbb{R}_{>0} \arrow[d, two heads, "\iota_*"] \\
S^1 & \simeq \underbrace{\big(\mathbb{R}^2\setminus\{0\}\big)/\mathbb{R}^\times}_{\mathbb{R}P^1}
\end{tikzcd}$$

$$\begin{tikzcd}
S^1 \simeq \mathbb{C}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\mathrm{ker}"] & \\
S^3 \arrow[d, two heads, "h_{\mathbb{C}}"] & \simeq \big(\mathbb{C}^2\setminus\{0\}\big)/\mathbb{R}_{>0} \arrow[d, two heads, "\iota_*"] \\
S^2 & \simeq \underbrace{\big(\mathbb{C}^2\setminus\{0\}\big)/\mathbb{C}^\times}_{\mathbb{C}P^1}
\end{tikzcd}$$

$$\begin{tikzcd}
S^3 \simeq \mathbb{H}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\mathrm{ker}"] & \\
S^7 \arrow[d, two heads, "h_{\mathbb{H}}"] & \simeq \big(\mathbb{H}^2\setminus\{0\}\big)/\mathbb{R}_{>0} \arrow[d, two heads, "\iota_*"] \\
S^4 & \simeq \underbrace{\big(\mathbb{H}^2\setminus\{0\}\big)/\mathbb{H}^\times}_{\mathbb{H}P^1}
\end{tikzcd}$$

The Hopf fibrations in higher dimensions are the attaching maps exhibiting the topological cell-complex structure of projective spaces [83], from which the (cellular) cohomology follows readily.

$$\begin{tikzcd}
S\big(\mathbb{K}^{n+1}\big) \arrow[r] \arrow[d, "h_{\mathbb{K}}"'] & \ast \arrow[d] \arrow[dl, Rightarrow, "\text{(po)}"] \\
\mathbb{K}P^n \arrow[r, hook] & \mathbb{K}P^{n+1}
\end{tikzcd}$$

Further factor-fibrations arise by factoring the Hopf fibrations via the stage-wise quotienting along

$$\mathbb{R}_{>0} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}.$$

Notably, the classical quaternionic Hopf fibration $h_{\mathbb{H}}$ factors through a higher-dimensional complex Hopf fibration followed by the
Calabi-Penrose twistor fibration $t_{\mathbb{H}}$ [31, §2].

Equivariantization: Since the quotienting is by right actions, these fibrations are equivariant under the left action of

$$\mathrm{Spin}(5) \simeq \mathrm{Sp}(2) := \big\{ g \in \mathrm{GL}_2(\mathbb{H}) \,\big\vert\, g^\dagger \cdot g = \mathrm{e} \big\}\,.$$

$$\begin{tikzcd}
S^1 \simeq \mathbb{C}^\times/\mathbb{R}_{>0} \arrow[dr] & & \\
& S^7 \simeq \big(\mathbb{H}^2\setminus\{0\}\big)/\mathbb{R}_{>0} \arrow[dd, "h_{\mathbb{C}}"] \arrow[dddd, bend right=40, "h_{\mathbb{H}}"'] & \\
S^2 \simeq \mathbb{H}^\times/\mathbb{C}^\times \arrow[dr] & & \textbf{complex Hopf fibration} \\
& \mathbb{C}P^3 \simeq \big(\mathbb{H}^2\setminus\{0\}\big)/\mathbb{C}^\times \arrow[dd, "t_{\mathbb{H}}"] & \\
\textbf{quaternionic Hopf fibration} & & \textbf{Calabi-Penrose twistor fibration} \\
& \mathbb{H}P^1 \simeq \big(\mathbb{H}^2\setminus\{0\}\big)/\mathbb{H}^\times &
\end{tikzcd}$$

For example, the involution $\sigma := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Sp}(2)$ swaps the two copies of $\mathbb{H}$:

$$\begin{tikzcd}
\mathbb{C}P^3 \arrow[rrr, "t_{\mathbb{H}}"] \arrow[ddd, "\sigma"] \arrow[dr, equal] & & & \mathbb{H}P^1 \arrow[ddd, "\sigma"] \arrow[dl, equal] \\
& \big(\mathbb{H}\times\mathbb{H}\setminus\{0\}\big)/\mathbb{C}^\times \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \big(\mathbb{H}\times\mathbb{H}\setminus\{0\}\big)/\mathbb{H}^\times \arrow[d, shift left] \arrow[d, shift right] & \\
& \big(\mathbb{H}\oplus\mathbb{H}\setminus\{0\}\big)/\mathbb{C}^\times \arrow[r] & \big(\mathbb{H}\oplus\mathbb{H}\setminus\{0\}\big)/\mathbb{H}^\times & \\
\mathbb{C}P^3 \arrow[rrr, "t_{\mathbb{H}}"'] \arrow[ur, equal] & & & \mathbb{H}P^1 \arrow[ul, equal]
\end{tikzcd}$$

The resulting $\mathbb{Z}_2$-fixed locus is the 2-sphere:

$$\begin{tikzcd}
\big(\mathbb{C}P^3\big)^{\mathbb{Z}_2} \arrow[d, "(t_{\mathbb{H}})^{\mathbb{Z}_2}"] & \simeq & \big(\mathbb{H}\setminus\{0\}\big)/\,\mathbb{C}^\times \arrow[d] & \simeq & S^2 \arrow[d] \\
\big(\mathbb{H}P^1\big)^{\mathbb{Z}_2} & \simeq & \big(\mathbb{H}\setminus\{0\}\big)/\,\mathbb{H}^\times & \simeq & \ast
\end{tikzcd}$$

This is the 2-sphere coefficient that will end up being responsible for stabilizing anyons on orbi-worldvolumes!
We next discuss how this comes about.

Orbi-worldvolumes and Equivariant charges. Flux-quantization generalizes to *orbifolds* [4] by generalizing the cohomology of the charges to *equivariant cohomology* [96].

In terms of classifying spaces this simply means that all spaces are now equipped with the action of a finite group G and all maps are required to be G-equivariant.
We take $G := \mathbb{Z}_2$ and the classifying fibration to be the **twistor fibration** $p := t_{\mathbb{H}}$ equivariant under swapping the $\mathbb{H}$-summands,

$$\begin{tikzcd}
\text{orbi-worldvolume} & \overset{G\circlearrowright}{\Sigma} \arrow[r, dashed, "\text{orbi-brane charges}"] \arrow[d, "\phi"'] & \overset{G\circlearrowright}{\mathcal{A}} : \arrow[r, equal] \arrow[d, "p"'] \arrow[dl, Rightarrow, dashed] & \overset{\mathbb{Z}_2\circlearrowright}{\mathbb{C}P^3} \arrow[d, "t_{\mathbb{H}}"'] \\
\text{orbi-spacetime} & \underset{G\circlearrowleft}{X} \arrow[r, dashed, "\text{orbi-bulk charges}"'] & \underset{G\circlearrowleft}{\mathcal{B}} : \arrow[r, equal] & \underset{\mathbb{Z}_2\circlearrowleft}{S^4}
\end{tikzcd}$$

equivariant classifying fibration...

...for equivariant twistorial Cohomotopy

and the brane/bulk orbifold we take to be as on p. 3:

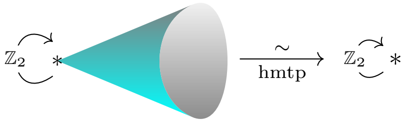
The orbi-brane diagram for a flat M5-brane wrapped on a trivial Seifert-fibered orbi-singularity. Shaded is the $\mathbb{Z}_2$-fixed locus/orbi-singularity.

We are adjoining the *point at infinity* to the space $\mathbb{R}^2_{\cup\{\infty\}} \underset{\text{homeo}}{\simeq} S^2$ which is thereby designated as transverse to any worldvolume solitons to be measured in reduced cohomology.

$$\begin{tikzcd}
\overset{\mathbb{Z}_2\circlearrowright}{\Sigma} \arrow[d, hook, "\phi"'] & := & \mathbb{R}^{1,0} \times \mathbb{R}^2_{\cup\{\infty\}} \times S^1 & \times & \overset{\mathbb{Z}_2\circlearrowright}{\mathbb{R}^2_{\mathrm{sgn}}} & & \\
\underset{\mathbb{Z}_2\circlearrowleft}{X} & := & \mathbb{R}^{1,0} \times \mathbb{R}^2_{\cup\{\infty\}} \times S^1 & \times & \overset{\mathbb{Z}_2\circlearrowright}{\mathbb{R}^2_{\mathrm{sgn}}} & \times & \mathbb{R}^5
\end{tikzcd}$$

time — trnsvrs space to solitons — M/IIA-circle — orbi-cone — trnsvrs space to M5-brane

But since the cone $\mathbb{Z}_2 \circlearrowright \mathbb{R}^2_{\mathrm{sgn}}$ is equivariantly contractible,



the inclusion of the $\mathbb{Z}_2$-fixed loci is actually a homotopy equivalence

$$\begin{tikzcd}
\Sigma^{\mathbb{Z}_2} \arrow[r, "\sim", "\text{hmtp}"'] \arrow[d, "\phi^{\mathbb{Z}_2}"'] & \overset{\mathbb{Z}_2\circlearrowright}{\Sigma} \arrow[d, "\phi"] \\
X^{\mathbb{Z}_2} \arrow[r, "\sim", "\text{hmtp}"'] & \underset{G\circlearrowleft}{X}
\end{tikzcd}$$

Therefore our equivariant classifying maps are determined up to equivariant homotopy by their restriction to the fixed-locus and hence the charges are *localized on the orbi-singularity* where they take values in 2-Cohomotopy:

$$\left\{
\begin{tikzcd}
\overset{\mathbb{Z}_2\circlearrowright}{\Sigma} \arrow[r, dashed] \arrow[d, "\phi"'] & \overset{\mathbb{Z}_2\circlearrowright}{\mathbb{C}P^3} \arrow[d, "t_{\mathbb{H}}"] \arrow[dl, Rightarrow, dashed] \\
\underset{\mathbb{Z}_2\circlearrowleft}{X} \arrow[r, dashed, "\text{charges on orbifold}"'] & \underset{\mathbb{Z}_2\circlearrowleft}{S^4}
\end{tikzcd}
\right\}
\simeq
\left\{
\begin{tikzcd}
\Sigma^{\mathbb{Z}_2} \arrow[r, dashed] \arrow[d, "\phi^{\mathbb{Z}_2}"'] & (\mathbb{C}P^3)^{\mathbb{Z}_2} \arrow[r, equal] \arrow[d, "t_{\mathbb{H}}^{\mathbb{Z}_2}"] \arrow[dl, Rightarrow, dashed] & S^2 \arrow[d] \\
X^{\mathbb{Z}_2} \arrow[r, dashed] & (S^4)^{\mathbb{Z}_2} \arrow[r, equal] & *
\end{tikzcd}
\right\}
\simeq
\left\{
\begin{tikzcd}
\mathbb{R}^2_{\cup\{\infty\}} \times S^1 \arrow[r, dashed] & S^2
\end{tikzcd}
\right\}$$

charges localized on orbi-singularity

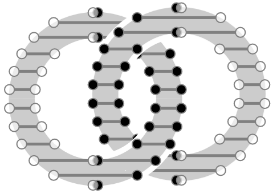
charges in 2-Cohomotopy of B-field solitons on M5 orbi-singularity

Moduli space of worldvolume solitons. To be precise, the solitonic charges are to be measured in the *reduced* 2-Cohomotopy classified by *pointed maps*, enforcing the condition that solitonic fields *vanish at infinity* [44, §2.2].

In the strongly-coupled situation, where the M/IIA circle de-compactifies to $\mathbb{R}^1$, the vanishing-at-infinity must also be applied here, whence the moduli space of topological solitons is the loop space of the reduced 2-Cohomotopy moduli of the transverse space:

$$\text{moduli space of solitons on M5 orbi-singularity}\quad \mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}} \wedge S^1,\, S^2\big) \;\simeq\; \Omega\,\mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big)\quad \text{loop space of moduli space of solitons on D4 orbi-singularity}$$

Outlook. Strinkingly, as we explain next, this is equivalently a space of *worldsheets of strings* in $\mathbb{R}^3$ with unit *charged endpoints* forming oriented *framed links*! [99]



Such link diagrams are just the envisioned topological quantum circuit protocols, and their framing regularizes the anyonic phase observables ("Wilson loop observables").

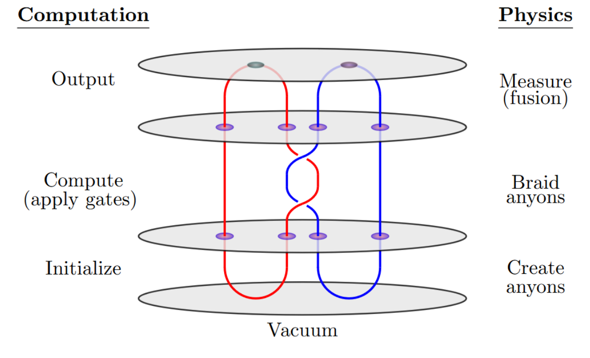


FIGURE 1. A single iteration of a topological quantum computation via braiding.

Figure from Rowell ([94], following [93, Fig. 2]).

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[4] For brevity we consider here only "very good" orbifolds, namely global quotients of manifolds by the action of a finite group G. This is sufficient for the present purpose and anyways the case understood by default in the string theory literature.

3 Cohomotopy charge of Solitons

Remarkably, there is an equivalence between *Cohomotopy* of spacetime/worldvolumes and *Cobordism* classes of submanifolds behaving like solitonic branes carrying the corresponding Cohomotopy charge [97, §2.2] [95, §2.1]:

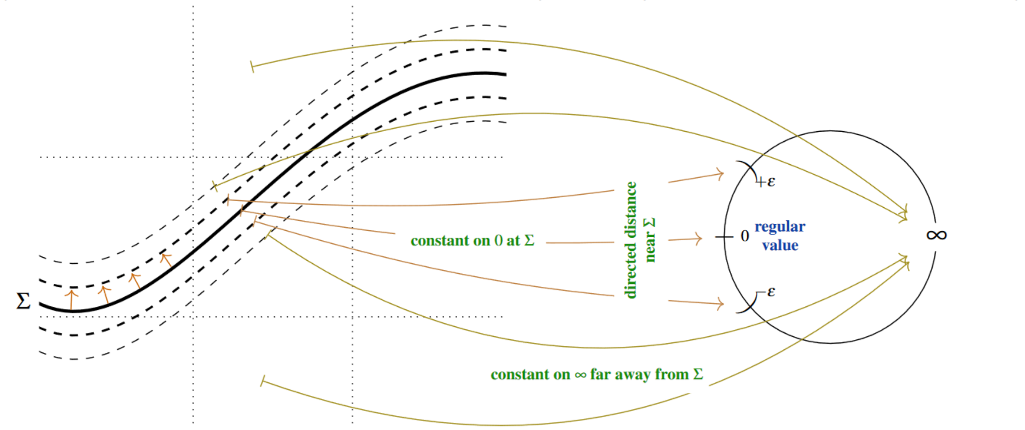
The **Pontrjagin theorem** [66][65, §IX] identifies the unstable n-Cohomotopy of a closed manifold with the cobordism classes of its normally framed submanifolds of co-dimension n.

$$\begin{tikzcd}
\text{framed unstable } n\text{-Cobordism of } M^d & \text{unstable } n\text{-Cohomotopy of } M^d \\
\mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big) := \mathrm{NFramedSubmflds}_{d-n}\big(M^d\big)_{/\mathrm{brdsm}} \arrow[r, shift left=2, "\text{directed asymptotic distance}"] \arrow[r, phantom, "\simeq"] \arrow[r, bend left=30, "\text{assign Cohomotopy charge}"] & \mathrm{Maps}\big(M^d, S^n\big)_{/\mathrm{hmtpy}} =: \pi^n\big(M^d\big). \arrow[l, shift left=2, "\text{pre-image of regular value 0}"] \arrow[l, bend left=30, "\text{reconstruct submanifold from its charge}"] \\
 & \text{closed manifold}
\end{tikzcd}$$

The **Cohomotopy charge** of a normally framed submanifold (aka *scanning map* or *Pontrjagin-Thom collapse*) is represented by mapping points of the ambient space to their directed distance if inside a tubular neighbourhood, else to ∞.

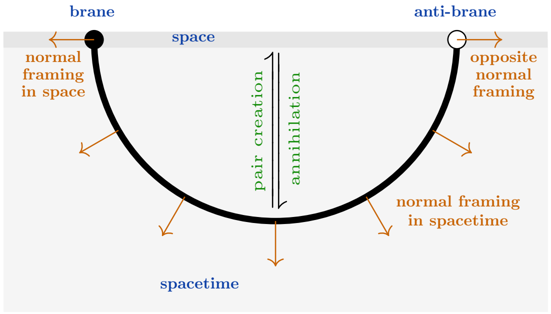
Conversely, every Cohomotopy class is represented by a smooth map with 0 a regular value, whose pre-image is a normally framed submanifold with that Cohomotopy charge.

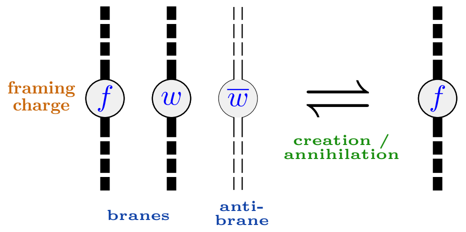
$$\begin{tikzcd}
\Sigma^{d-n} \arrow[r, hook] & M^d \arrow[r, "c", "\substack{\text{directed asymptotic distance from } \Sigma \\ \simeq \text{ cocycle representing Cohomotopy charge of } \Sigma}"'] & \mathbb{R}^n_{\mathrm{cpt}} = S^n \\
\substack{\text{closed submanifold,} \\ \text{normally framed}} & \text{manifold} & \substack{\text{Cohomotopy classifying space} \\ (n\text{-sphere})}
\end{tikzcd}$$



Under this relation, homotopy of charge maps corresponds to nrml. framed **cobordism** of submnflds.

The cobordism relation exhibits a form of pair creation/annihilation of submanifolds carrying opposite Cohomotopy charges.





When making more ambient dimensions available, the cobordism classes eventually (quickly) exhibit **stabilization** on abelian cobordism cohomology groups. (This might relate *Hypothesis H* to Vafa's *cobordism conjecture* cf. [97, §4]).

make more ambient spatial dimensions available to bordisms ⟶

$$\begin{tikzcd}
 & \Sigma^{d-n} \subset M^d \arrow[r, hook, "(\mathrm{id},0)"] & M^d \times \mathbb{R}^1 \arrow[r, hook, "(\mathrm{id},0)"] & M^d \times \mathbb{R}^2 \arrow[r, hook, dotted] & M^d \times \mathbb{R}^\infty & \\
\substack{\text{unstable} \\ \text{framed} \\ \text{Cobordism}} & \mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big) \arrow[r, "\sigma"] \arrow[d, dashed, "\text{Cohomotopy charge map}"] & \mathrm{Cob}^{n+1}_{\mathrm{Fr}}\big(M^d \times \mathbb{R}^1\big) \arrow[r, "\sigma"] \arrow[d, "\simeq"] & \mathrm{Cob}^{n+2}_{\mathrm{Fr}}\big(M^d \times \mathbb{R}^2\big) \arrow[r, dotted] \arrow[d, "\simeq"] & \widetilde{M\mathrm{Fr}}^n\big(M^d_{\mathrm{cpt}}\big) \arrow[d, equal, "\text{stable Pontrjagin-Thom isom.}"] & \substack{\text{stable} \\ \text{framed} \\ \text{Cobordism}} \\
\substack{\text{unstable} \\ \text{Cohomotopy}} & \widetilde{\pi}^n\big(M^d_{\mathrm{cpt}}\big) \arrow[r, "(-)\wedge S^1"] & \widetilde{\pi}^{n+1}\big(\big(M^d \times \mathbb{R}^1\big)_{\mathrm{cpt}}\big) \arrow[r, "(-)\wedge S^1"] & \widetilde{\pi}^{n+1}\big(\big(M^d \times \mathbb{R}^2\big)_{\mathrm{cpt}}\big) \arrow[r, dotted] & \widetilde{\mathbb{S}}^n\big(M^d_{\mathrm{cpt}}\big) & \substack{\text{stable} \\ \text{Cohomotopy}}
\end{tikzcd}$$

suspension homomorphism ⟶

This "linearized" Cohomotopy/Cobordism is a **form of K-theory**: algebraic K-theory over the "absolute base field $\mathbb{F}_1$" (cf. [12, Thm. 5.9][7, Cor. 2.25]).

$$\begin{tikzcd}
\substack{\text{non-abelian} \\ \text{Cohomotopy}} & \substack{\text{stable} \\ \text{Cohomotopy}} & & \substack{\text{stable framed} \\ \text{Cobordism}} \\
\pi^\bullet \arrow[r, "\text{linearize}", "\text{(i.e.: stabilize)}"'] & \mathbb{S}^\bullet \arrow[rr, equal, "\text{Pontrjagin \& Thom}"] \arrow[dr, equal, "\text{Barratt-Priddy \& Quillen}"'] & & \mathrm{MFr}^\bullet \arrow[dl, equal] \\
 & & K\mathbb{F}_1^\bullet & \\
 & & \substack{\text{algebraic K-theory of} \\ \text{“field with one element”}} &
\end{tikzcd}$$

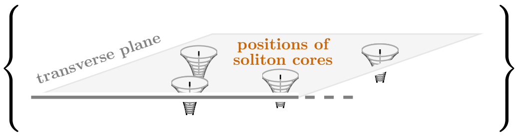
Thus flux quantization in Cohomotopy lifts to M-theory the same arguments that motivated topological K-theory in type II string theory: its character map reproduces the Bianchi identities & its equivalence relation models (anti-)brane pair-creation/annihilation.

Moduli space of soliton configurations. But the Pontrjagin theorem concerns only the total cohomotopical charge, identifying it with the *net* (anti-)brane content. Beyond that we have the whole *moduli space* of charges (considered now specialized to our 2D transverse space) and **Segal's theorem** [105] says that the cohomotopy charge map (scanning map) identifies this with a moduli space of brane positions, namely with the *group-completed configuration space of points* [13][113][40]:

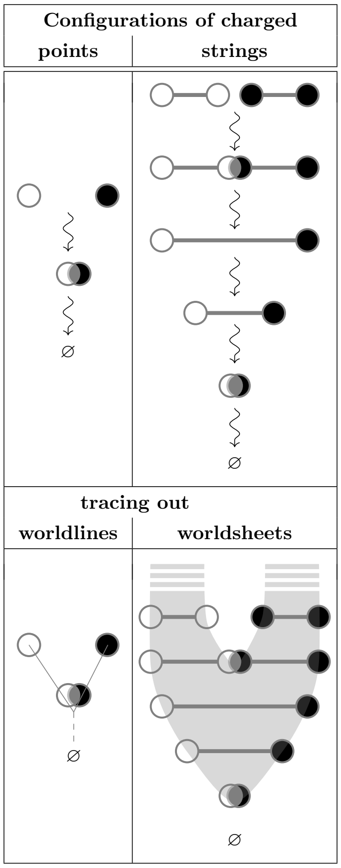
$$\begin{tikzcd}
\text{moduli space of solitonic brane charges} & \mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big) \arrow[r, "{[-]}", "\text{net charge}"'] & \pi^2\big(\mathbb{R}^2_{\cup\{\infty\}}\big) \arrow[dr, "\sim"', "\text{Hopf degree theorem}"] & \text{2-Cohomotopy} \\
 & & & \mathbb{Z} \\
\text{moduli space of solitonic brane positions} & \mathbb{G}\,\mathrm{Conf}(\mathbb{R}^2) \arrow[uu, "\text{Segal theorem}", "\wr"'] \arrow[r, "{[-]}", "\text{net brane content}"'] & \mathrm{Cob}^2_{\mathrm{Fr}}(\mathbb{R}^2) \arrow[uu, "\text{Pontrjagin theorem}", "\wr"'] \arrow[ur, "\sim"] & \\
 & \text{group-completed config space} & \text{2-Cobordism (unstable)} &
\end{tikzcd}$$

where the *configuration space of points* is the space of finite subsets of $\mathbb{R}^2$ – here understood as the space of positions of cores of solitons of unit charge $+1$,

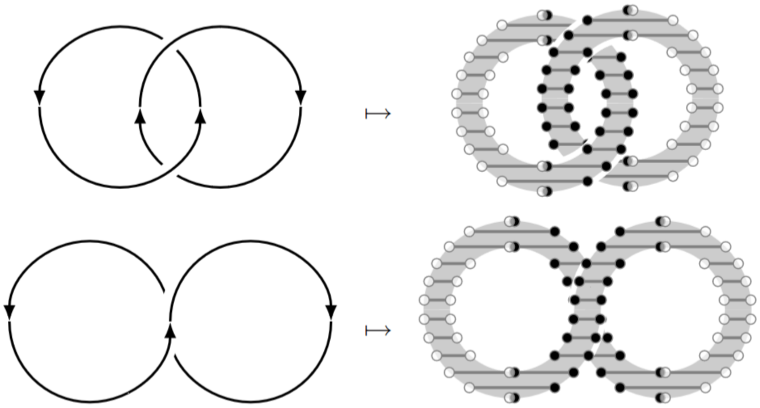
$$\mathrm{Conf}(\mathbb{R}^2) \;=\;$$



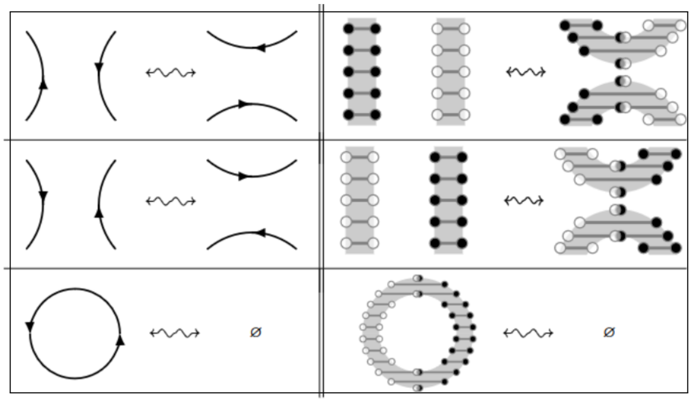
and its *group completion* $\mathbb{G}(-)$ is the topological completion of the topological partial monoid structure given by disjoint union of soliton configurations.
Naïvely this is given by including also **anti-solitons** in the form of configurations of $\pm$-*charged points*, topologized such as to allow for their pair annihilation/creation as shown in the left column on the right.
Remarkably, closer analysis reveals [84] that the group completion $\mathbb{G}(-)$ produces configurations of **strings** (extending parallel to one axis in $\mathbb{R}^3$) **with charged endpoints** whose pair annihilation/creation is smeared-out to string worldsheets as shown in the right column.

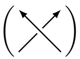
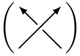


This means [99] that the **vacuum-to-vacuum soliton scattering processes**, forming the loop space $\Omega\,\mathbb{G}\,\mathrm{Conf}(\mathbb{R}^2)$, are identified with *framed links* ([85, pp 15]), for instance



subject to *link cobordism* (cf. [72]):



It follows [99, Thm 3.17] that the charge of a soliton scattering process L is the sum over crossings of the *crossing number* $\#$ $= +1$, $\#$ $= -1$, which equals the linking+framing number:

$$\begin{tikzcd}
\Omega\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "\sim"] & \Omega\mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}} S^2\big) \arrow[r, "{[-]}"] & \pi_3(S^2) \simeq \mathbb{Z} \\
L \arrow[rr, mapsto, "\text{total crossing number} =", "\text{linking + framing number}"'] & & \#L
\end{tikzcd}$$

But this is precisely the **Wilson loop observable** of L **in** (abelian) **Chern-Simons theory**! [99, §4] As we explain next.

4 The topological Quantum States

To summarize so far, we have seen that the topological sector of the flux-quantized phase space of solitons on magnetized M5-probes Σ wrapping Seifert orbi-singularities is

$$\underbrace{\mathrm{Maps}\begin{pmatrix} \Sigma & \mathbb{C}P^3 \\ \downarrow, & \downarrow \\ X & S^4 \end{pmatrix}^{\mathbb{Z}_2}}_{\substack{\text{topological sector}\\ \text{of flux-quantized}\\ \text{phase space}}} \simeq \underbrace{\mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}} \wedge S^1,\, S^2\big)}_{\substack{\text{2-Cohomotopy}\\ \text{cocycle space}}} \simeq \underbrace{\Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)}_{\substack{\text{loop space of}\\ \text{group-completed}\\ \text{configuration space}}} \xrightarrow{[-]} \pi_0\,\Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \simeq \underbrace{\mathbb{Z}}_{\substack{\text{net}\\ \text{charge}}}$$

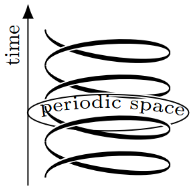
$$L \longmapsto \#L$$

The topological quantum states of this system now follow [98][99, §4] by general algebraic quantum theory:

The gauge-invariant topological **observables** form the (higher) homology of this space $\mathrm{Obs}_\bullet := H_\bullet(\Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\,\mathbb{C})$ making a (star-)algebra under concatenation (reversion) of loops — the *Pontrjagin algebra*.

$$\begin{tikzcd}
\Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "\text{loop reversal}", "\mathrm{rev}"'] & \Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \\
H_\bullet(\Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\,\mathbb{C}) \arrow[r, "\text{Pontr. antipode}", "\mathrm{rev}_*"'] \arrow[dr, "\substack{\text{Hermitian conjugation}\\ \text{of quantum observables}}"'] & H_\bullet(\Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\,\mathbb{C}) \arrow[d, "\overline{(\text{-})}"', "\substack{\text{cmplx}\\ \text{cnjgtn}}"] \\
 & H_\bullet(\Omega\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\,\mathbb{C})
\end{tikzcd}$$

This means that time-reversal goes along with reversal of looping around the M/IIA-circle, whence we are dealing with a version of *discrete light-cone quantization* in their topological sectors.



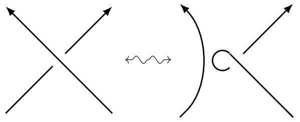
The basic ordinary (degree=0) observables detect the deformation class of a framed link L.

$$\begin{array}{lcccc}
\mathrm{Obs}_0 & \xrightarrow{\sim} & \mathbb{C}\Big[\pi_0\big(\Omega\mathbb{G}\mathrm{Conf}(\mathbb{R}^2)\big)\Big] & \xrightarrow{\sim} & \mathbb{C}[\mathbb{Z}] \\
\mathcal{O}_L & := & \delta_{[L]} & = & \delta_{\#L} \\
\mathcal{O}_L \cdot \mathcal{O}_{L'} & = & \delta_{L \sqcup L'} & = & \delta_{\#L + \#L'}
\end{array}$$

Since these observables commute among each other, their *pure* topological quantum states are their (real & positive) algebra homomorphisms:

$$\begin{tikzcd}
\mathrm{PureQStates}_0 \arrow[d, hook] \arrow[r, phantom, "\underset{\substack{\text{on commuting}\\ \text{observables}}}{\simeq}"] & \Big\{\rho \,:\, \mathrm{Obs}_0 \xrightarrow{\text{homo}} \mathbb{C} \,\Big|\, \rho \in \mathrm{MixedQStates}_0\Big\} \\
\mathrm{MixedQStates}_0 \arrow[r, phantom, ":="] & \Big\{\rho \,:\, \mathrm{Obs}_0 \xrightarrow{\text{linear}} \mathbb{C} \,\Big|\, \underset{\mathcal{O} \in \mathrm{Obs}_\bullet}{\forall} \Big(\underbrace{\rho(\mathcal{O}^*) = \rho(\mathcal{O})^*}_{\text{reality}},\ \underbrace{\rho(\mathcal{O}^* \cdot \mathcal{O}) \geq 0 \in \mathbb{R} \hookrightarrow \mathbb{C}}_{\text{(semi-)positivity}}\Big),\ \underbrace{\rho(1) = 1}_{\text{normalization}}\Big\}.
\end{tikzcd}$$

Therefore pure topological states $|m\rangle$ are determined by an **anyonic phase** $\exp(\pi \mathrm{i}/m)$ assigned to any crossing,



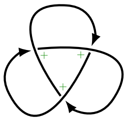
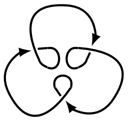
accumulating to the exponentiated crossing number

$$\begin{array}{ccc}
\mathrm{Obs}_0 & \xrightarrow{\langle m|-|m\rangle} & \mathbb{C} \\
\mathcal{O}_L & \longmapsto & e^{\frac{\pi \mathrm{i}}{m}\#L}
\end{array}$$

The resulting expectation values

$$\langle m|\mathcal{O}_L|m\rangle = \exp\big(\tfrac{\pi \mathrm{i}}{m}\#L\big) = \exp\Bigg(\frac{\pi \mathrm{i}}{m}\Big(\sum_{i \neq j \in \pi_0(L)} \underbrace{\mathrm{lnk}(L_i, L_j)}_{\substack{\text{linking}\\ \text{numbers}}} + \sum_{i \in \pi_0(L)} \underbrace{\mathrm{frm}(L_i)}_{\substack{\text{framing}\\ \text{numbers}}}\Big)\Bigg)$$

are [99, §4] just those of *Wilson loop observables* in "spin" *Chern-Simons theory*, as **expected for abelian anyons**!

For example: $\Bigg\langle m \Bigg|$  $\Bigg| m \Bigg\rangle = \Bigg\langle m \Bigg|$  $\Bigg| m \Bigg\rangle = \exp\big(\pi \mathrm{i}\,\tfrac{3}{m}\big)$

Applying the GNS-construction to such state produces a 1-dimensional Hilbert space $\overbrace{\mathbb{C}[\theta, \theta^{-1}]}^{\mathbb{C}[\mathbb{Z}]}/\big(e^{\pi \mathrm{i}/m} - \theta\big) \simeq \mathbb{C}$, which is as expected for the quantum states of abelian Chern-Simons theory on $\mathbb{R}^2_{\cup\{\infty\}}$. (More on this on p 13.)

Remark. These *solitonic* anyons are *not* yet the controllable/parameterized defect anyons that could be used for topological braid quantum gates operating by adiabatic movement of anyonic *defects* or (quasi-)*holes*. But the latter arise as defect points among the former, we come to this on p. 14.

Anyonic topological order on Flux-quantized M5-probes. We now identify the promised topological order on M5-probes flux-quantized in equivariant twistorial Cohomotopy, by considering M5s wrapping closed surfaces:

Anyonic quantum observables on closed surfaces. Consider now a *closed orientable surface* Σ^2_g of genus $g \in \mathbb{N}$ to replace the previous factor $\mathbb{R}^2_{\cup\{\infty\}}$ in the brane diagram:

$$\overset{\mathbb{Z}_2}{\circlearrowright}\Sigma^{1,6} := \mathbb{R}^{1,0} \times \Sigma^2_g \times S^1 \times \overset{\mathbb{Z}_2}{\circlearrowright}\mathbb{R}^2_{\mathrm{sgn}}$$

Directly analogous analysis as before gives that the topological quantum observables on the flux-quantized self-dual tensor field form the group algebra of the fundamental group of the 2-cohomotopy moduli space in the kth connected component

$$\mathrm{Obs}_0\big(\Sigma^2_g\big) \;:=\; H_0\Big(\Omega_k \,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big);\, \mathbb{C}\Big) \;\simeq\; \mathbb{C}\Big[\pi_0\Omega_k\,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big)\Big]\,, \tag{3}$$

where $k \in \mathbb{N}$ is the degree of the classifying maps, corresponding under the Pontrjagin theorem to a net number of k (anti-)solitons on Σ^2_g.

Theorem (using [39, Thm 1][67, Thm 1][60, Cor 7.6]). This group of 2-cohomotopy charge sectors is identified as *twice* the integer Heisenberg group extension (cf. [69]) of $\mathbb{Z}^{2g}$ by $\mathbb{Z}_{2|k|}$ [5]:

$$\pi_0\Omega_k\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big) \;\simeq\; \left\{ \big(\vec{a},\, \vec{b},\, [n]\big) \,\in\, \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z}_{2|k|}\,,\; \begin{array}{l} \big(\vec{a},\, \vec{b},\, [n]\big)\cdot\big(\vec{a}',\, \vec{b}',\, [n']\big) \;= \\ \big(\vec{a}+\vec{a}',\, \vec{b}+\vec{b}',\, [n+n'+\vec{a}\cdot\vec{b}'-\vec{a}'\cdot\vec{b}]\big) \end{array} \right\} \;=:\; \widehat{\mathbb{Z}^{2g}}$$

Ground state degeneracy. Hence the observable group-algebra Obs_0 for $g=1$, $\Sigma^2_1 = T^2$, has generators

$$\left\{ \begin{array}{lcl} W_a &:=& (1,0,[0]) \\ W_b &:=& (0,1,[0]) \\ \zeta &:=& (0,0,[1]) \end{array} \right\} \quad \text{subject to the relations} \quad \left\{ \begin{array}{l} W_a \cdot W_b \;=\; \zeta^2\, W_b \cdot W_a \\ \zeta^{2k} = 1 \\ [\zeta, -] = 0 \end{array} \right\}.$$

This algebra is just the observable algebra expected [109, (5.28)] for anyonic topological order on the torus as described by abelian Chern-Simons theory at *level* k. The unique non-trivial irrep has dimension k, this being the expected *ground state degeneracy* on the torus:

Hilbert space of quantum states on the torus

$$\mathcal{H}_{T^2} := \mathrm{Span}\Big(\big|[n]\big\rangle,\, [n] \in \mathbb{Z}_{|k|}\Big) \;\in\; \mathrm{Obs}_0(T^2)\mathrm{Modules}\,,\; \dim\big(\mathcal{H}_{T^2}\big) \;=\; k\,,$$

$$\begin{aligned} W_a\big|[n]\big\rangle &:= e^{2\pi\mathrm{i}n/k}\big|[n]\big\rangle \\ W_b\big|[n]\big\rangle &:= \big|[n+1]\big\rangle \\ \zeta\big|[n]\big\rangle &:= e^{\pi\mathrm{i}/k}\big|[n]\big\rangle \end{aligned}$$

Generally, writing $(\vec{e}_i \in \mathbb{Z}^g)_{1=1}^g$ for the canonical basis vectors, the observable group-algebra Obs_0 for general g has generators

$$\left\{ \begin{array}{lcl} W^i_a &:=& (\vec{e}_i, 0, [0]) \\ W^i_b &:=& (0, \vec{e}_j, [0])\,, 1 \leq i \leq g \\ \zeta &:=& (0,0,[1]) \end{array} \right\} \quad \text{subject to the relations} \quad \left\{ \begin{array}{l} W^i_a \cdot W^j_b \;=\; \delta^{ij}\zeta^2\, W^j_b \cdot W^i_a \\ \zeta^{2k} = 1 \\ \text{all other commutators vanish} \end{array} \right\}$$

The non-trivial irrep $\mathcal{H}_g$ of this algebra has dimension $|k|^g$ as expected [75, p 40] for abelian anyonic topological order on Σ^2_g.

Hilbert space of quantum states on genus=g surface

$$\mathcal{H}_{\Sigma^2_g} \;\in\; \mathrm{Obs}_0(\Sigma^2_g)\mathrm{Modules}\,, \qquad \dim\big(\mathcal{H}_{\Sigma^2_g}\big) \;=\; |k|^g\,,$$

Modular equivariance. Strikingly, in this construction modular symmetry **is manifest**, since the looped mapping space is canonically acted on by the mapping class group MCG of Σ^2_g (cf. [25, §2.1]), simply by precomposition of maps! Inspection of the above theorem (cf. [39, bottom of p 153]) shows that this MCG-action action identifies indeed as the canonical action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\widehat{\mathbb{Z}^{2g}}$.

$$\begin{tikzcd} \overbrace{\pi_0\mathrm{Homeos}\big(\Sigma^2_g\big)}^{\mathrm{MCG}(\Sigma^2_g)} \arrow[d, "{[25,\ \S 6.3]}"] \arrow[r, phantom, "\circlearrowleft"] & \pi_0\Omega_k\,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big) \arrow[d, phantom, "\wr|"] \\ \mathrm{Sp}_{2g}(\mathbb{Z}) \arrow[r, phantom, "\circlearrowleft"] & \widehat{\mathbb{Z}^{2g}} \end{tikzcd}$$

Moreover, this modular action on quantum observables induces a modular action on quantum states by compensating intertwiners. For $g=1$ one readily checks that these intertwiners are just the modular transformations known [75, pp 65] from abelian Chern-Simons theory:

modular action on observables and on states

$$m(W)\cdot m\big(\big|[n]\big\rangle\big) \;=\; m\Big(W\big|[n]\big\rangle\Big)\,, \qquad \forall \left\{ \begin{array}{lcl} m &\in& \mathrm{Sp}_{2g}(\mathbb{Z}) \\ W &\in& \widehat{\mathbb{Z}^{2g}} \\ \big|[n]\big\rangle &\in& \mathcal{H}_g \end{array} \right.$$

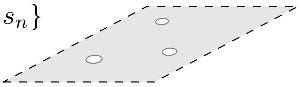
$$S\Big(\big|[n]\big\rangle\Big) \;=\; \frac{1}{\sqrt{|k|}}\sum_{[\widehat{n}]} e^{2\pi\mathrm{i}\frac{n\,\widehat{n}}{k}}\big|[\widehat{n}]\big\rangle\,, \qquad T\Big(\big|[n]\big\rangle\Big) \;=\; e^{\mathrm{i}\pi\frac{n^2}{k}}\big|[n]\big\rangle\,.$$

---

[5] Here $\mathbb{Z}_n := \mathbb{Z}/(n)$ (with $\mathbb{Z}_0 = \mathbb{Z}$) are the (in-)finite cyclic groups.

Quasi-hole defects via punctured worldvolumes. It is now immediate to bring *adiabatically movable anyon defects* into the picture, namely "*quasi-holes*" in FQH jargon (cf. [109, pp 85]): points where solitons are *absent*. To reflect this, we simply further generalize the surfaces Σ^2_g to their n-punctured versions obtained by deleting the positions of a subset of points – thus literally creating holes!

$$\Sigma^2_{g,n} \;\simeq\; \Sigma^2_{g,n} \setminus \{s_1, \cdots s_n\} \quad \text{for } \{s_1, \cdots s_n\} \subset \Sigma^2_g$$



That these holes are indeed void of our dynamic solitons is elegantly enforced by identifying all their positions with the point-at-infinity.

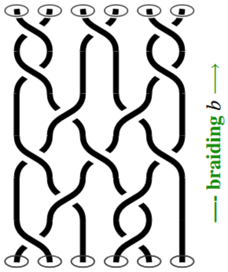
$$\big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}} \qquad \text{e.g.: } \mathbb{R}^2_{\cup\{\infty\}} \;\simeq\; \big(\Sigma^2_{0,1}\big)_{\cup\{\infty\}}$$

In this generality, our previous brane diagram now is:

$$\overset{\mathbb{Z}_2 \curvearrowright}{\Sigma^{1,6}} := \;\mathbb{R}^{1,0} \;\times\; \big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}} \;\times\; S^1 \;\times\; \overset{\mathbb{Z}_2 \curvearrowright}{\mathbb{R}^2_{\mathrm{sgn}}}\,,$$

and, by the same argument as before, the algebra of topological quantum observables on cohomotopically flux-quantized fields becomes:

$$\mathrm{Obs}_0\big(\Sigma^2_{g,n}\big) \;:=\; H_0\Big(\Omega_k\, \mathrm{Maps}^{*/}\big((\Sigma^2_{g,n})_{\cup\{\infty\}},\, S^2\big);\, \mathbb{C}\Big)\,.$$



A more explicit description of this algebra of observables may not be available at the moment. But we can immediately see that these are quantum observables on non-abelian anyons:

Braid group action. This algebra of observables is faithfully acted on by the mapping class group of the punctured surface – again simply by precomposition of maps.
But, with punctures, that group is now an extension (cf. [76, Thm. 3.13]) of the plain mapping class group by the surface *braid group* moving the defects/holes around each other!

$$\underset{\text{surface braid group}}{1 \to \mathrm{Br}_n(\Sigma^2_g)} \hookrightarrow \underset{\substack{\text{mapping class group}\\ \text{of punctured surface}}}{\pi_0 \mathrm{Homeos}^{*/}_{\mathrm{or}}\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big)} \twoheadrightarrow \underset{\substack{\text{mapping class group}\\ \text{of plain surface}}}{\mathrm{MCG}(\Sigma^2_g)} \to 1$$

$\left(\begin{array}{l}\text{In deducing this, we observed that}\\ \mathrm{Homeos}^{*/}\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big) \;\simeq\; \mathrm{Homeos}\big(\Sigma^2_{g,n}\big)\\ \text{since } (-)_{\cup\{\infty\}} \text{ is functorial on homeos.}\end{array}\right)$

Anyon-braiding on M5s as a quantum-gravitational effect. Noting that the mapping class group is equivalently the group of large *diffeomorphisms* of the punctured surface (cf. [25, p 45]),

$$\pi_0 \mathrm{Homeos}^{*/}_{\mathrm{or}}\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big) \;\simeq\; \pi_0\, \mathrm{Diffeos}_{\mathrm{or}}(\Sigma^2_{g,n})$$

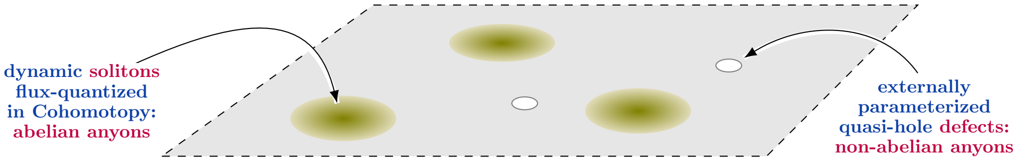
we see that braiding of anyonic defects is reflected in equipping the moduli spaces of cohomotopical charges on the brane worldvolume with the action by diffeomorphisms, hence by passing to the action *groupoid* of moduli quotiented by diffeos.

$$\mathrm{GnrlCovariantModuli}(\Sigma) \;\simeq\; \mathrm{Moduli}(\Sigma) \sslash \mathrm{Diffeos}(\Sigma)$$

This is the hallmark of *generally covariant* systems (cf. [21]), such as our probe branes.

Solitonic vs. Defect anyons. By the previous discussion we are to think of $\mathrm{Obs}_0(\Sigma^2_{g,n})$ as the quantum observables on abelian *solitonic anyons* propagating on the punctured surface $\Sigma^2_{g,n}$. But the ("adiabatic") dependence of these observables on the external parameters of n qasi-hole positions make these collectively represent non-abelian braiding of punctures.

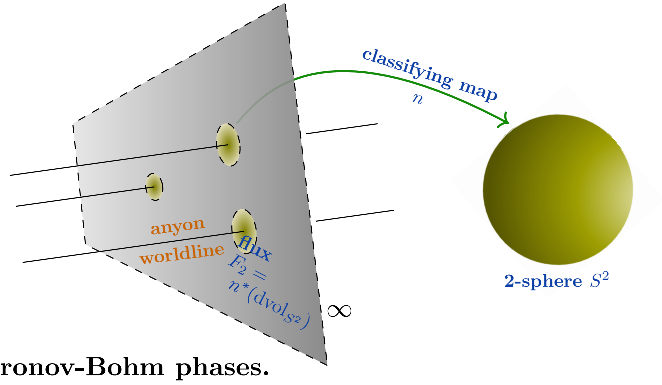
anyons as seen in Cohomotopy	nature	number	braiding
solitonic anyons	concentrations of flux density	net charge, CS-level: k	by (LC-)time evolution
defect anyons	punctures in worldvolume	n in $\Sigma^2_{g,n}$	by worldvolume diffeomorphisms



Conclusion – New theory of anyonic topological order, engineered on flux-quantized M5s. In summary, we have seen that global completion by flux-quantization of 11D supergravity with M5-probes (here: in equivariant twistorial cohomotopy – "Hypothesis H"), makes the quantized topological sector of the self-dual tensor field on M5-probes (wrapping Seifert orbi-singularities) reproduce key phenomena of abelian Chern-Simons theory thought of as an effective field theory for abelian anyons in fractional quantum Hall (FQH) systems:

(i) Flux tubes bound to anyons. The central assumption in the traditional heuristic understanding of the FQHE is that the anyonic solitons have flux quanta "attached" to them [106, pp 883]. It is crucially this assumption which motivates and justifies abelian Chern-Simons theory as an effective field theory for FQH anyons, since variation of the sum of the abelian Chern-Simons term with the standard source term predicts that the gauge field flux is localized at the source particles (cf. [109, (5.25)][114, (3.6)]).

In contrast, in the approach discussed here **this effect is a consequence of cohomotopical flux-quantization**, via the Pontrjagin theorem: The classifying map of the 2-Cohomotopy charge identifies an open neighbourhood of each anyon with the 2-sphere minus its point at infinity, and the flux density F_2 is the pullback of the sphere's volume form along this map (cf. p 20), hence supported on just these open neighbourhoods.



(ii) Anyons subject to each other's Aharonov-Bohm phases. Traditional discussion furthermore assumes from these attached flux tubes that the anyons must pick up Aharonov-Bohm quantum phases when circling around each other. While this is plausible, rigorous quantum field-theoretic derivation of this statement may not have found much attention.

In contrast, in the approach discussed here, this effect is again a direct consequence of cohomotopical flux-quantization, now via algebro-topological theorems of Segal and others, which serve to identify the cohomotopy charge moduli space with configuration spaces of soliton cores, whose fundamental group reflects the anyon braid phases (and thereby also the ground state degeneracy / topological order).

$$\begin{tikzcd}
\pi_0 \mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}},\, \boxed{S^2}\big) \arrow[r, "\sim"] \arrow[d, phantom, sloped, "\simeq"] & \pi_0 \mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}},\, B^2\mathbb{Z}\big) \arrow[d, phantom, sloped, "\simeq"] \\
\mathbb{Z} \arrow[r, phantom, "\text{same net charges...}"] & \mathbb{Z}
\end{tikzcd}$$

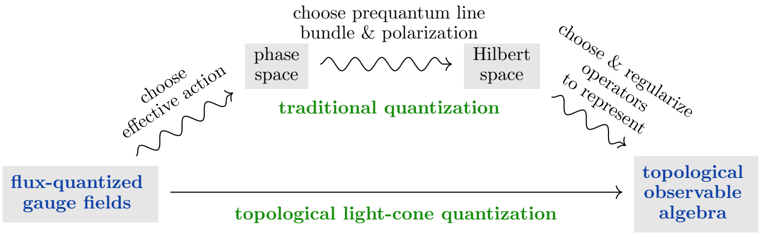
$$\begin{tikzcd}
\pi_1 \mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}},\, \boxed{S^2}\big) \arrow[r] \arrow[d, phantom, sloped, "\simeq"] & \pi_1 \mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}},\, B^2\mathbb{Z}\big) \arrow[d, phantom, sloped, "\simeq"] \\
\underset{\text{config space}}{\pi_1 \mathbb{G}\mathrm{Conf}(\mathbb{R}^2)} \arrow[r, phantom, "\text{...but different moduli}"] & \underset{\text{no structure}}{1}
\end{tikzcd}$$

Note how both these effects come about by changing the traditional flux-quantization of the Chern-Simons field from the classifying space for complex line bundles to just its first "cell". This preserves the quantization of charges but makes their moduli exhibit anyonic effects.

$$\begin{tikzcd}
\underset{\substack{\text{classifying space}\\ \text{for 2-Cohomotopy}}}{\boxed{S^2}} \simeq \mathbb{C}P^1 \arrow[r, hook, "\text{1st cell inclusion}"'] & \underset{\substack{\text{classifying space for}\\ \text{ordinary 2-cohomology}}}{\mathbb{C}P^\infty \simeq B^2\mathbb{Z}}
\end{tikzcd}$$

(iii) Topological order. The traditional rigorous way of establishing topological order is by applying geometric quantization to Wilson line observables, with respect to some effective action, which is a somewhat convoluted process involving ad-hoc choices and regularizations.

In contrast, in the approach discussed here the quantum observables obtain immediately, without further choices, from the topological light-cone quantization of the flux-quanized moduli space (as its Pontrjagin homology algebra).



Here the looping Ω_k that drives this quantum dynamics reflects dependence of moduli on the M/IIA circle.(!)

(iv) Non-abelian defect anyons. The traditional hypothesis about defect anyons in FQH systems, embodied by Laughlin- and Read & Moore wavefunction models, is that these are *quasi-holes* where the dynamical abelian anyons are absent – but a derivation of this expectation from microscopic electron dynamics is missing (cf. [109, §3, 4]). In contrast, in the approach presented here, the non-abelian anyonic nature of quasi-hole defects follows as a (modest) quantum-gravitational effect on the M5-worldvolumes, where it is the worldvolume diffeomorphism symmetry that on punctured worldvolumes translates into the braid group action on the quantum state space.

A Background on Homotopy Theory

Some notions used in the main text, to establish notation and give basic pointers to the literature.

Homotopy theory (cf. [107]). For $f_0, f_1 : X \to Y$ a pair of continuous maps between (topological) spaces a *homotopy* $\eta : f_0 \Rightarrow f_1$ is a continous deformation between them, namely a continuous map $\eta : [0,1] \times X \to Y$ such that

$$\begin{array}{l} \eta(0,x) = f_0(x), \\ \eta(1,x) = f_1(x), \end{array} \quad \text{denoted} \quad \begin{tikzcd} X \arrow[r, bend left, "f_0"{name=U}] \arrow[r, bend right, "f_1"'{name=D}] & Y \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}.$$

For example, a square "homotopy-commutative diagram"

$$\begin{tikzcd} \Sigma \arrow[r, dashed, "b"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"] \arrow[dl, Rightarrow, dashed, "\eta"] \\ X \arrow[r, dashed, "c"'] & \mathcal{B} \end{tikzcd} \quad \text{means that} \quad \begin{array}{l} \eta : [0,1] \times \Sigma \to \mathcal{B} \\ \eta(0,s) = p\big(b(s)\big), \\ \eta(1,s) = c\big(\phi(s)\big). \end{array}$$

If one declares – and we do – to work in a "convenient" full sub-category of all topological spaces (such as that of *compactly generated* or of *Delta-generated* topological spaces, cf. [41, p 21, 131]) then the topological space $\mathrm{Maps}(X,Y)$ of all continuous maps $X \to Y$ satisfies the adjointness relation $\big\{P \to \mathrm{Maps}(X,Y)\big\} \;\simeq\; \big\{P \times X \to Y\big\}$. For $P \equiv [0,1]$ this says that homotopies are equivalently paths in mapping spaces, and that homotopy-classes of maps are the mapping spaces' path-connected components: $\pi_0 \, \mathrm{Maps}(X,Y) \;\simeq\; \mathrm{Maps}(X,Y)_{/\mathrm{hmtp}}$.

Since homotopies are maps themselves, there are homotopies-between-homotopies and ever higher-homotopies.

Thereby topological spaces constitute a model for **higher categorical symmetry** namely for higher groupoids. As such, they represent both cohomology as well as higher gauge fields in the topological sector. [6]

cohomology	cocycle	coboundary	higher coboundary	...
homotopy	$X \xrightarrow{f} \mathcal{B}$	$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$	$\begin{tikzcd} X \arrow[r, bend left=60, "f"{name=U}] \arrow[r, bend right=60, "f'"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, Rightarrow, bend right, "\eta"'{name=L}] \arrow[from=U, to=D, Rightarrow, bend left, "\eta'"{name=R}] \arrow[from=L, to=R, Rightarrow] \end{tikzcd}$	...
physics	field	gauge transf.	higher gauge transf.	...

In this vein, spaces are homotopy-*equivalent* $\mathcal{B} \simeq \mathcal{B}'$ if they are gauge-equivalent namely if we have maps $\begin{tikzcd} \mathcal{B} \arrow[r, shift left, "f"] & \mathcal{B}' \arrow[l, shift left, "g"] \end{tikzcd}$ with $\begin{array}{l} g \circ f \Rightarrow \mathrm{id}_{\mathcal{B}} \\ f \circ g \Rightarrow \mathrm{id}_{\mathcal{B}'} \end{array}$ For example $\mathbb{R}^n \simeq *$ in homotopy theory, reflecting the fact that there is no non-trivial topological sector for fields on $\mathbb{R}^n$.

For actually computing homotopy classes of maps — hence cohomology, hence gauge-equivalence classes of fields in the topological sector — tools from *model category theory* are indispensable, which largely say how to "absorb homotopies into spaces" (cf. [32, §1]).

E.g., if $p : \mathcal{A} \to \mathcal{B}$ is a *Serre fibration*, such as a fiber bundle, and Σ is a *cell complex*, such as a manifold, then sections-up-to-homotopy of p pulled back to Σ are homotopy equivalent to plain sections:

$$\left\{ \begin{tikzcd} \Sigma \arrow[r, dashed, "b"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"] \arrow[dl, Rightarrow, dashed, "\eta"] \\ X \arrow[r, "c"'] & \mathcal{B} \end{tikzcd} \right\}_{\big/\mathrm{hmtp}} \overset{\substack{\Sigma \in \mathrm{Cof} \\ p \in \mathrm{Fib}}}{\simeq} \left\{ \begin{tikzcd} \Sigma \arrow[r, dashed, "b"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"] \arrow[dl, phantom, "="] \\ X \arrow[r, "c"'] & \mathcal{B} \end{tikzcd} \right\}_{\big/\mathrm{hmtp}}$$

This shows for instance that the twistor fibration is the classifying fibration for a twisted form of 2-Cohomotopy over a brane worldvolume: Its homotopy-sections are equivalently plain sections, and hence locally maps to the 2-sphere fiber.

Pointed homotopy theory (cf. [57, §3]). To reflect the condition that *solitonic fields are localized* in that they *vanish at infinity* we

- equip domain spaces X with a *point at infinity*, $\infty_X \in X$,
- equip classifying spaces $\mathcal{B}$ with a *point representing zero*, $0_{\mathcal{B}} \in \mathcal{B}$,
- require maps $f : (X, \infty_X) \to (\mathcal{B}, 0_{\mathcal{B}})$ to respect these base points

so that maps literally vanish at infinity

$$\begin{tikzcd} X \arrow[r, "c"] & \mathcal{B} \\ \{\infty_X\} \arrow[u, hook] \arrow[r] & \{0_{\mathcal{B}}\} \arrow[u, hook] \end{tikzcd}.$$

For instance, to make fields on $\mathbb{R}^n$ vanish at infinity, we adjoin its would-be "point at infinity" to it (jargon: "one-point compactification") to obtain $\mathbb{R}^n_{\cup\{\infty\}} \simeq S^n$. On the other hand, if we want fields on some X without a vanishing condition, we may adjoin a *disjoint* point-at-infinity, then pointed maps $X_{\sqcup\{\infty\}} \to \mathcal{B}$ are ordinary $X \to \mathcal{B}$. E.g.:

$$\overset{\textbf{based loop space}}{\mathrm{Maps}^{*/}(\mathbb{R}^1_{\cup\{\infty\}}, \mathcal{B}) \;=\; \Omega\mathcal{B}}, \qquad \overset{\textbf{free loop space}}{\mathrm{Maps}^{*/}(S^1_{\sqcup\{\infty\}}, X) =: \mathcal{L}\mathcal{B}}, \qquad \overset{\textbf{maps out of contractible}}{\mathrm{Maps}^{*/}(\mathbb{R}^1_{\sqcup\{\infty\}}, \mathcal{B}) \;=\; \mathcal{B}}$$

Given a pair of pointed spaces (X, ∞_X), (Y, ∞_Y), in their product space $X \times Y$ any point should be regarded as being at infinity which is so with respect to either factor space; this yields the *smash product*:

$X \wedge Y := \frac{X \times Y}{\{\infty_X\} \times Y \,\cup\, X \times \{\infty_Y\}}$ to which the sub-space $\mathrm{Maps}^{*/}(-,-)$ of pointed maps is again adjoint: $\big\{P \xrightarrow{\mathrm{pntd}} \mathrm{Maps}^{*/}(X,Y)\big\} \;\simeq\; \big\{P \wedge X \xrightarrow{\mathrm{pntd}} Y\big\}$.

For example, $S^n \wedge S^m \;\simeq\; \mathbb{R}^n_{\cup\{\infty\}} \wedge \mathbb{R}^m_{\cup\{\infty\}} \;\simeq\; (\mathbb{R}^n \times \mathbb{R}^m)_{\cup\{\infty\}} \simeq S^{n+m}$, so that for instance:

$$\mathrm{Maps}^{*/}\big(X \wedge S^1, \mathcal{B}\big) \;\simeq\; \mathrm{Maps}^{*/}\Big(S^1, \mathrm{Maps}^{*/}\big(X, \mathcal{B}\big)\Big) \;=:\; \Omega\,\mathrm{Maps}^{*/}\big(X, \mathcal{B}\big).$$

---

[6] Beyond the topological sector, full higher gauge fields are still represented by maps $X \to \mathcal{B}$ etc., only that now $\mathcal{B}$ is no longer just a topological space but a "smooth ∞-stack", cf. [27].

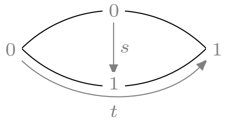
B Background on TED Cohomotopy

Gauge potentials in twistorial Cohomotopy — and the Green-Schwarz mechanism.

Consider the Whitehead L_∞-algebra of the twistor fibration $\mathbb{C}P^3 \xrightarrow{t_{\mathbb{H}}} \mathbb{H}P^1 \simeq S^4$,

$$\mathrm{CE}\big(\mathfrak{l}_{S^4}\mathbb{C}P^3\big) = \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} f_2 \\ h_3 \\ g_4 \\ g_7 \end{array}\right] \Big/ \left(\begin{array}{lcl} \mathrm{d}\, f_2 &=& 0 \\ \mathrm{d}\, h_3 &=& g_4 + f_2\, f_2 \\ \mathrm{d}\, g_4 &=& 0 \\ \mathrm{d}\, g_7 &=& \tfrac{1}{2} g_4\, g_4 \end{array}\right),$$

and bigons parameterized like this:



Theorem ([46, pp 23][47, §4.1]). Given a manifold U_i (generically: a coordinate chart):
(0.) Closed $\mathfrak{l}_{S^4}\mathbb{C}P^3$-valued differential forms are in natural bijection with **flux densities** of this form:

$$\left\{ \begin{tikzcd} U_i \arrow[d, "{(F_2,H_3,G_4,G_7)}"] \\ \Omega^1_{\mathrm{dR}}\big(-;\mathfrak{l}_{S^4}\mathbb{C}P^3\big)_{\mathrm{clsd}} \end{tikzcd} \right\} \begin{array}{c} i_0 \circ p_0 \,=\, \mathrm{id} \\ \xrightarrow{\;p_0\;} \\ \xleftarrow[\;i_0\;]{} \\ p_0 \circ i_0 \,=\, \mathrm{id} \end{array} \left\{ \begin{array}{l|l} F_2 \,\in\, \Omega^2_{\mathrm{dR}}(U_i) & \mathrm{d}\,F_2 \,=\, 0 \\ H_3 \,\in\, \Omega^3_{\mathrm{dR}}(U_i) & \mathrm{d}\,H_3 \,=\, G_4 \,+\, F_2\,F_2 \\ G_4 \,\in\, \Omega^4_{\mathrm{dR}}(U_i) & \mathrm{d}\,G_4 \,=\, 0 \\ G_7 \,\in\, \Omega^7_{\mathrm{dR}}(U_i) & \mathrm{d}\,G_7 \,=\, \tfrac{1}{2}\,G_4\,G_4 \end{array} \right\}$$

(1.) Given one of these, its set of coboundaries (null-concordances) naturally retracts onto the set of **gauge potentials** of this form:

$$\left\{ \begin{tikzcd} U_i \arrow[rr] \arrow[d, "{(F_2,H_3,G_4,G_7)}"'] & & \ast \arrow[d, "0"] \arrow[dll, Rightarrow, dashed, "{(\widehat{F}_2,\widehat{H}_3,\widehat{G}_4,\widehat{G}_7)}"] \\ \Omega^1_{\mathrm{dR}}\big(-;\mathfrak{l}_{S^4}\mathbb{C}P^3\big)_{\mathrm{clsd}} \arrow[rr, "\eta^{\int}"] & & \int\Omega^1_{\mathrm{dR}}\big(-;\mathfrak{l}_{S^4}\mathbb{C}P^3\big)_{\mathrm{clsd}} \end{tikzcd} \right\} \begin{array}{c} \xrightarrow{\;p_1\;} \\ \xleftarrow[\;i_1\;]{} \\ p_1 \circ i_1 \,=\, \mathrm{id} \end{array} \left\{ \begin{array}{l|l} A_1 \,\in\, \Omega^1_{\mathrm{dR}}(U_i) & \mathrm{d}\,A_1 \,=\, F_2 \\ B_2 \,\in\, \Omega^2_{\mathrm{dR}}(U_i) & \mathrm{d}\,B_2 \,=\, H_3 - C_3 - A_1\,F_2 \\ C_3 \,\in\, \Omega^3_{\mathrm{dR}}(U_i) & \mathrm{d}\,C_3 \,=\, G_4 \\ C_6 \,\in\, \Omega^6_{\mathrm{dR}}(U_i) & \mathrm{d}\,G_6 \,=\, G_7 - \tfrac{1}{2} C_3\,G_4 \end{array} \right\}$$

$$\left(\begin{array}{lcl} \widehat{F}_2 &:=& t\,F_2 \,+\, \mathrm{d}t\,A_1 \\ \widehat{H}_3 &:=& t\,H_3 \,+\, \mathrm{d}t\,B_2 \,+\, (t^2 - t)A_1F_2 \\ \widehat{G}_4 &:=& t\,G_4 \,+\, \mathrm{d}t\,C_3 \\ \widehat{G}_7 &:=& t^2\,G_7 \,+\, 2t\mathrm{d}t\,C_6 \end{array} \right) \begin{array}{c} \longmapsto \\ \longleftarrow\!\shortmid \end{array} \left(\begin{array}{lcl} A_1 &:=& \int_{[0,1]} \widehat{F}_2 \\ B_2 &:=& \int_{[0,1]} \Big(\widehat{H}_3 - \big(\int_{[0,-]} \widehat{F}_2\big)\widehat{F}_2\Big) \\ C_3 &:=& \int_{[0,1]} \widehat{G}_4 \\ C_6 &:=& \int_{[0,1]} \Big(\widehat{G}_7 - \tfrac{1}{2}\big(\int_{[0,-]} \widehat{G}_4\big)\widehat{G}_4\Big) \end{array} \right)$$

(2.) Given a pair of these, the set of higher coboundaries (2nd-order concordances) between them naturally retracts onto the set of **gauge transformations** of this form:

$$\left\{ \begin{tikzcd}[column sep=huge] 0 \arrow[r, Rightarrow, bend left=50, "{(\widehat{F}_2,\widehat{H}_3,\widehat{G}_4,\widehat{G}_7)}"] \arrow[r, Rightarrow, bend right=50, "{(\widehat{F}'_2,\widehat{H}'_3,\widehat{G}'_4,\widehat{G}'_7)}"'] \arrow[r, phantom, "{\left(\begin{smallmatrix}\widehat{\widehat{F}}_2,\,\widehat{\widehat{H}}_3\\ \widehat{\widehat{G}}_4,\,\widehat{\widehat{G}}_7\end{smallmatrix}\right)\Downarrow}"] & (F_2, H_3, G_4, G_7) \end{tikzcd} \right\} \begin{array}{c} \xrightarrow{\;p_2\;} \\ \xleftarrow[\;i_2\;]{} \\ p_2 \circ i_2 \,=\, \mathrm{id} \end{array} \left\{ \begin{array}{l|l} \alpha_0 \,\in\, \Omega^0_{\mathrm{dR}}(U_i) & \mathrm{d}\,\alpha_0 \,=\, A'_1 - A_1 \\ \beta_1 \,\in\, \Omega^1_{\mathrm{dR}}(U_i) & \mathrm{d}\,\beta_1 \,=\, B'_2 - B_2 + \gamma_2 + \alpha_0\,F_2 \\ \gamma_2 \,\in\, \Omega^2_{\mathrm{dR}}(U_i) & \mathrm{d}\,\gamma_2 \,=\, C'_3 - C_3 \\ \gamma_5 \,\in\, \Omega^5_{\mathrm{dR}}(U_i) & \mathrm{d}\,\gamma_5 \,=\, C'_6 - C_6 - \tfrac{1}{2}C'_3\,C_3 \end{array} \right\}$$

$$\left(\begin{array}{l} \widehat{\widehat{F}}_2 \,:=\, t\,F_2 \,+\, \mathrm{d}t\,A_1 \,+\, s\,\mathrm{d}t\big(A'_1 - A_1\big) \,-\, \mathrm{d}s\,\mathrm{d}t\,\alpha_0 \\ \widehat{\widehat{H}}_3 \,:=\, t\,H_3 \,+\, \mathrm{d}t\,B_2 \,+\, s\,\mathrm{d}t\,(B'_2 - B_2) \,-\, \mathrm{d}s\,\mathrm{d}t\,\beta_1 \\ \qquad + (t^2 - t)A_1F_2 + (t^2 - t)s(A'_1 - A_1)F_2 \\ \qquad\; +(t^2 - t)\mathrm{d}s\,\alpha_0F_2 \\ \widehat{\widehat{G}}_4 \,:=\, t\,G_4 \,+\, \mathrm{d}t\,C_3 \,+\, s\,\mathrm{d}t\big(C'_3 - C_3\big) \,-\, \mathrm{d}s\,\mathrm{d}t\,\gamma_2 \\ \widehat{\widehat{G}}_7 \,:=\, t^2\,G_7 \,+\, 2t\mathrm{d}t\,C_6 \,+\, 2st\mathrm{d}t(C'_6 - C_6) \\ \qquad - 2\mathrm{d}s\,t\mathrm{d}t\big(\gamma_5 \,+\, \tfrac{1}{2}\gamma_2\,C_3\big) \end{array} \right) \begin{array}{c} \longmapsto \\ \longleftarrow\!\shortmid \end{array} \left(\begin{array}{l} \alpha_0 := \int_{s\in[0,1]} \int_{t\in[0,1]} \widehat{\widehat{F}}_2 \\ \beta_1 := \int_{s\in[0,1]} \int_{t\in[0,1]} \Big(\widehat{\widehat{H}}_3 \,-\, \big(\int_{t'\in[0,-]} \widehat{\widehat{F}}_2\big)\widehat{\widehat{F}}_2\Big) \\ \gamma_2 := \int_{s\in[0,1]} \int_{t\in[0,1]} \widehat{\widehat{G}}_4 \\ \gamma_5 := \int_{s\in[0,1]} \int_{t\in[0,1]} \Big(\widehat{\widehat{G}}_7 - \tfrac{1}{2}\big(\int_{t'\in[0,-]} \widehat{\widehat{G}}_4\big)\widehat{\widehat{G}}_4\Big) \\ \qquad -\tfrac{1}{2}\gamma_2\,C_3 \end{array} \right)$$

Notice the expression for flux density subject to an (abelian) Green-Schwarz mechanism: $H_3 \,=\, \mathrm{d}\,B_2 \,+\, A_1F_2 \,+\, C_3$.

Proof. With the blue terms discarded, this is the statement of [46, pp 23][47, §4.1]. We compile the full argument:

To see that p_1 is well-defined:

- for C_3, C_6 this is [46, (70)],
- for A_1 it works just as for C_3,
- for B_2 we compute, in generalization of [47, below (138)], like this:

$$\begin{aligned} \mathrm{d}\, B_2 \;\; &\equiv \;\; \mathrm{d} \int_{[0,1]} \Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]} \widehat{F}_2 \big) \widehat{F}_2 \Big) \\ &= \;\; \underbrace{\iota_1^* \Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]} \widehat{F}_2 \big) \widehat{F}_2 \Big)}_{H_3 - A_1\, F_2} - \underbrace{\iota_0^* \Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]} \widehat{F}_2 \big) \widehat{F}_2 \Big)}_{=0} - \int_{[0,1]} \underbrace{\mathrm{d} \Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]} \widehat{F}_2 \big) \widehat{F}_2 \Big)}_{\widehat{G}_4} \\ &= \;\; H_3 \; - \; A_1\, F_2 \; - \; C_3 \,. \end{aligned}$$

To see that i_1 is well-defined:

- for $\widehat{G}_4, \widehat{G}_7$ this is [46, (72)],
- for $\widehat{F}_2$ it works just as for $\widehat{G}_4$,
- for $\widehat{H}_3$ we compute, in generalization of [47, further below (138)], as follows:

$$\left. \begin{aligned} &\mathrm{d}\big(tH_3 + \mathrm{d}t\, B_2 + (t^2 - t) A_1 F_2 \big) \\ &= \; \mathrm{d}t\, H_3 + tG_4 + tF_2 F_2 \\ &\quad\; - \mathrm{d}t\, H_3 + \mathrm{d}t\, C_3 + \mathrm{d}t\, A_1 F_2 \\ &\quad\; + \mathrm{d}\big((t^2 - t) A_1 F_2 \big) \end{aligned} \right\} \text{hence indeed: } \mathrm{d}\, \widehat{H}_3 \;=\; \underbrace{t\, G_4 + \mathrm{d}t\, C_3}_{\widehat{G}_4} + \overbrace{\underbrace{(tF_2 + \mathrm{d}t\, A_1)}_{\widehat{F}_2} \underbrace{(tF_2 + \mathrm{d}t\, A_1)}_{\widehat{F}_2}}^{= \mathrm{d}(t^2 A_1 F_2)}$$

Moreover, it is immediate from inspection that $\iota_1^* \widehat{H}_3 \;=\; H_3$ and $\iota_0^* \widehat{H}_3 = 0$.

To see that $p_1 \circ i_1 = \mathrm{id}$:

- for C_3, C_6 this is [46, below (72)],
- for A_1 this works just as for C_3,
- for B_2 we immediately compute:

$$\int_{[0,1]} \Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]} \widehat{F}_2 \big) F_2 \Big) \;\; = \;\; \underbrace{\int_{[0,1]} \mathrm{d}t\, B_2}_{B_2} - \int_{[0,1]} \underbrace{t A_1\, \mathrm{d}t\, A_1}_{=0} \;\; = \;\; B_2 \,.$$

To see that p_2 is well-defined:

- for $\widehat{\widehat{G}}_4, \widehat{\widehat{G}}_7$ this is [46, (74-5)],
- for $\widehat{\widehat{F}}_2$ this works just as for $\widehat{\widehat{F}}_2$,
- for $\widehat{\widehat{H}}_3$ we compute, in generalization of [47, below (140)], as follows:

$$\begin{aligned} \mathrm{d}\beta_1 \;\; &\equiv \;\; \mathrm{d} \int_{s \in [0,1]} \int_{t \in [0,1]} \Big(\widehat{\widehat{H}}_3 - \big(\textstyle\int_{t' \in [0,-]} \widehat{\widehat{F}}_2 \big) \widehat{\widehat{F}}_2 \Big) \\ &= \;\; \iota_{s=1}^* \int_{t \in [0,1]} \big(\widehat{\widehat{H}}_3 - \cdots \big) - \iota_{s=0}^* \int_{t \in [0,1]} \big(\widehat{\widehat{H}}_3 - \cdots \big) - \int_{s \in [0,1]} \mathrm{d} \int_{t \in [0,1]} \big(\widehat{\widehat{H}}_3 - \cdots \big) \\ &= \;\; \int_{t \in [0,1]} \iota_{s=1}^* \big(\widehat{\widehat{H}}_3 - \cdots \big) - \int_{t \in [0,1]} \iota_{s=0}^* \big(\widehat{\widehat{H}}_3 - \cdots \big) - \int_{s \in [0,1]} \iota_{t=1}^* \big(\widehat{\widehat{H}}_3 - \cdots \big) + \int_{s \in [0,1]} \int_{t \in [0,1]} \mathrm{d} \big(\widehat{\widehat{H}}_3 - \cdots \big) \\ &= \;\; \int_{t \in [0,1]} \big(\widehat{H}'_3 - \cdots \big) - \int_{t \in [0,1]} \big(\widehat{H}_3 - \cdots \big) \; + \; \Big(\int_{s \in [0,1]} \int_{t \in [0,1]} \widehat{\widehat{F}}_2 \Big) F_2 + \int_{s \in [0,1]} \int_{t \in [0,1]} \widehat{\widehat{G}}_4 \\ &= \;\; B'_2 - B_2 + \alpha_0\, F_2 + \gamma_2 \,. \end{aligned}$$

To see that i_2 is well-defined:

- for γ_2, γ_5 this is [46, (76)],
- for α_0 this works just as for γ_2,

- for β_1 we compute as follows:

$$
\begin{array}{rcl}
\mathrm{d}\left(t\,H_3 \,+\, \mathrm{d}t\,B_2 \,+\, s\,\mathrm{d}t\,(B_2' - B_2) \,-\, \mathrm{d}s\,\mathrm{d}t\,\beta_1\right) & = & \overbrace{t\,G_4 \,+\, \mathrm{d}t\,C_3 \,+\, s\,\mathrm{d}t(C_3' - C_3) \,-\, \mathrm{d}s\,\mathrm{d}t\,\gamma_2}^{\widehat{\widehat{G}}_4} \\
 & & +\,t\,F_2F_2 \,+\, \mathrm{d}t\,A_1\,F_2 \,+\, s\,\mathrm{d}t(A_1' - A_1)F_2 \,-\, \mathrm{d}s\,\mathrm{d}t\,\alpha_0\,F_2 \\[1em]
\mathrm{d}\left(\begin{array}{l}(t^2 - t)A_1\,F_2 \,+\, (t^2 - t)s(A_1' - A_1)F_2 \\ \quad +\,(t^2 - t)\,\mathrm{d}s\,\alpha_0\,F_2\end{array}\right) & = & \overbrace{t^2\,F_2\,F_2 \,+\, 2t\mathrm{d}t\,A_1F_2 \,+\, 2t\mathrm{d}t\,s(A_1' - A_1) \,+\, 2t\mathrm{d}t\,\mathrm{d}s\,\alpha_0\,F_2}^{\widehat{\widehat{F}}_2\,\widehat{\widehat{F}}_2} \\
 & & -\,t\,F_2\,F_2 \,-\, \mathrm{d}t\,A_1\,F_2 \,-\, \mathrm{d}t\,s(A_1' - A_1)F_2 \,-\, \mathrm{d}t\,\mathrm{d}s\,\alpha_0\,F_2 \\
\hline
\mathrm{d}\,\widehat{\widehat{H}}_3 & = & \widehat{\widehat{G}}_4 \,+\, \widehat{\widehat{F}}_2\,\widehat{\widehat{F}}_2\,.
\end{array}
$$

Moreover, it is immediate from inspection that $\iota_{s=0}^*\widehat{\widehat{H}}_3 \,=\, \widehat{H}_3$, $\iota_{s=1}^*\widehat{\widehat{H}}_3 \,=\, \widehat{H}_3'$ and $\iota_{t=0}^* \,=\, 0$, $\iota_{t=1}^* \,=\, H_3$.

To see that $p_2 \circ i_2 \,=\, \mathrm{id}$, we directly compute,
first

$$
\begin{aligned}
\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{G}}_4 \;&=\; \textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}(-\mathrm{d}s\,\mathrm{d}t\,\gamma_2) \;=\; \gamma_2 \\
\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{F}}_2 \;&=\; \textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}(-\mathrm{d}s\,\mathrm{d}t\,\alpha_0) \;=\; \alpha_0
\end{aligned}
$$

then

$$
\begin{aligned}
&\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\Big(\widehat{\widehat{G}}_7 - \tfrac{1}{2}\big(\int_{t'\in[0,t]}\widehat{\widehat{G}}_4\big)\widehat{\widehat{G}}_4\Big) \,-\, \tfrac{1}{2}\,\gamma_2\,C_3 \\
&=\; \textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{G}}_7 \,-\, \tfrac{1}{2}\int_{s\in[0,1]}\int_{t\in[0,1]}\big(t\,C_3 + st(C_3' - C_2) + t\mathrm{d}s\,\gamma_2\big)\big(t\,G_4 \,+\, \mathrm{d}t\,C_3 \,+\, s\,\mathrm{d}t(C_3' - C_3) - \mathrm{d}s\,\mathrm{d}t\,\gamma_2\big) \\
&\quad -\tfrac{1}{2}\,\gamma_2\,C_3 \\
&=\; \big(\gamma_5 \,+\, \tfrac{1}{2}\gamma_2C_3\big)\underbrace{-\tfrac{1}{2}C_3\gamma_2 - \tfrac{1}{4}(C_3' - C_3)\gamma_2 + \tfrac{1}{2}\,\gamma_2\,C_3 + \tfrac{1}{4}\,\gamma_2\,(C_3' - C_3)}_{0} - \tfrac{1}{2}\,\gamma_2\,C_3 \\
&=\; \gamma_5
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&\textstyle\int_{s\in[0,1]}\,\int_{t\in[0,1]}\,\big(\int_{t'\in[0,-]}\,\widehat{\widehat{F}}_2\big)\widehat{\widehat{F}}_2 \\
&=\; \textstyle\int_{s\in[0,1]}\,\int_{t\in[0,1]}\,\big(tA_1 + st(A_1' - A_1) \,+\, t\mathrm{d}s\,\alpha_0\big)\big(t\,F_2 \,+\, \mathrm{d}t\,A_1 \,+\, s\,\mathrm{d}t(A_1' - A_1) \,-\, \mathrm{d}s\,\mathrm{d}t\,\alpha_0\big) \\
&=\; \tfrac{1}{2}A_1\alpha_0 + \tfrac{1}{4}(A_1' - A_1)\alpha_0 - \tfrac{1}{2}\alpha_0A_1 - \tfrac{1}{4}\alpha_0(A_1' - A_1) \\
&=\; 0
\end{aligned}
$$

so that also

$$
\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\Big(\widehat{\widehat{H}}_3 - \big(\int_{t'\in[0,-]}\widehat{\widehat{F}}_2\big)\widehat{\widehat{F}}_2\Big) \;=\; \int_{s\in[0,1]}\int_{t\in[0,1]}(-\mathrm{d}s\,\mathrm{d}t\;\beta_1) \;=\; \beta_1\,. \qquad \square
$$

Cocycles in differential 2-Cohomotopy and the abelian Chern-Simons invariant on the 3-Sphere. Notice that the Bianchi identities encoded by 2-Cohomotopy are the characteristic property of the abelian Chern-Simons term:

$$\mathrm{CE}\big(\mathfrak{l}S^2\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_3 \end{bmatrix} \Big/ \left(\begin{aligned} \mathrm{d}\, f_2 \;&=\; 0 \\ \mathrm{d}\, h_3 \;&=\; f_2 f_2 \end{aligned} \right) \qquad \Rightarrow \qquad \Omega^1_{\mathrm{dR}}\big(X;\, \mathfrak{l}S^2\big)_{\mathrm{clsd}} \;\simeq\; \left\{ \begin{array}{l} F_2 \,\in\, \Omega^2_{\mathrm{dR}}(X) \\ H_3 \,\in\, \Omega^3_{\mathrm{dR}}(X) \end{array} \;\middle\vert\; \begin{array}{l} \mathrm{d}\, F_2 \;=\; 0 \\ \mathrm{d}\, H_3 \;=\; F_2\, F_2 \end{array} \right\}$$

We may bring this out more concretely:

Gauge-field configurations on $\mathbb{R}^3$ flux-quantized in 2-Cohomotopy and vanishing in a neighbourhood of infinity are cocycles in differential 2-Cohomotopy on $\mathbb{R}^3_{\cup\{\infty\}}$, hence dashed homotopies as shown on the right [44, §3.3].

$$\begin{tikzcd}
 & \mathbb{R}^3_{\cup\{\infty\}} \arrow[dl, dashed, "\text{flux density valued in } \mathfrak{l}S^2"', "(F_2{,}H_3)"] \arrow[dr, dashed, "\text{charge in 2-Cohomotopy}", "n"'] & \\
\Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^2\big)_{\mathrm{clsd}} \arrow[dr, "\eta^{\int}"'] & & \int S^2 \arrow[dl, "\mathbf{ch}"] \arrow[ll, Rightarrow, dashed, shorten=25mm, "(\mathbf{A}_1{,}\mathbf{B}_2)"', "\text{gauge potentials in diff. 2-Cohomotopy}"] \\
 & \int \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^2\big)_{\mathrm{clsd}} &
\end{tikzcd}$$

Theorem. For each $[n] \in \pi^2\big(\mathbb{R}^3_{\cup\{\infty\}}\big) \simeq \mathbb{Z}$ this exists with $H_3 = 0$ and $[n] = \int_{\mathbb{R}^3} A_1\, F_2$ the Chern-Simons invariant.

Lemma. *On a smooth manifold Σ, every cocycle α in rational 3-Cohomotopy is represented by a globally defined differential form H_3,*

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, dashed, "H_3"'] & & \int \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^3\big)_{\mathrm{clsd}} \arrow[dl, phantom, ""{name=T}] \\
 & \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^3\big)_{\mathrm{clsd}} \arrow[ur, "\eta^{\int}"'] &
\end{tikzcd}$$

Proof of the Lemma. Since $\mathfrak{l}S^3 \;\simeq\; \mathfrak{l}B^3\mathbb{Q}$ this is just the degree=3 case of the statement that cocycles in de Rham hyper-cohomology have global representatives on smooth manifolds (using partitions of unity). □

Proof of the Theorem. Stereographic projection provides a homeomorphism $\mathbb{R}^3_{\cup\{\infty\}} \xrightarrow{\sim} S^3$ which is smooth away from the point at infinity, which we may slightly deform to a smooth degree=1 map that is constant on a neighbourhood of infinity. Since $\pi^2(S^3) \simeq \pi_2(S^3) \simeq \mathbb{Z}$ we may find a smooth map $n : S^3 \to S^2$, with compact support away from the base point, so that $\mathbb{R}^3_{\cup\{\infty\}} \to S^3 \xrightarrow{n} S^2$ represents the charge $[n]$.

Now the 2-cohomotopical character map for charges on S^3, shown in black, factors as shown in blue (by naturality of rationalization), which furthermore factors as shown in orange (by the above Lemma).

$$\begin{tikzcd}
\mathbb{R}^3_{\cup\{\infty\}} \arrow[r] & S^3 \arrow[r, "\eta^{\int}"] \arrow[d, "n\cdot\mathrm{dvol}_{S^3}"'] & \int S^3 \arrow[r, "n"] \arrow[d, "\mathbf{ch}_{S^3}"] \arrow[dl, Rightarrow, shorten=5mm] & \int S^2 \arrow[d, "\mathbf{ch}_{S^2}"] \arrow[dl, Rightarrow, shorten=5mm] \\
 & \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}S^3)_{\mathrm{clsd}} \arrow[r, "\eta^{\int}"] & \int\Omega^1_{\mathrm{dR}}(-;\mathfrak{l}S^3)_{\mathrm{clsd}} \arrow[r, "(\mathfrak{l}n)_*"] & \int\Omega^1_{\mathrm{dR}}(-;\mathfrak{l}S^2)_{\mathrm{clsd}}
\end{tikzcd} \tag{4}$$

Hence to get a differential cocycle as desired it is sufficient to exhibit gauge potentials (A_1, B_2) encoding a concordance filling the diagram on the right

$$\begin{tikzcd}
\mathbb{R}^3_{\cup\{\infty\}} \arrow[r] & S^3 \arrow[r, "n"] \arrow[dd, "(F_2{,}H_3{=}0)"'] \arrow[dr, "n\cdot\mathrm{dvol}_{S^3}" description] & S^2 \arrow[r, "\eta^{\int}"] & \int S^2 \arrow[dd, "\mathbf{ch}_2"] \arrow[dl, Rightarrow, shorten=5mm, "(4)"] \\
 & & \Omega^1_{\mathrm{dR}}(-;\, \mathfrak{l}S^3)_{\mathrm{clsd}} \arrow[dr, "(\mathfrak{l}n)_*\circ\,\eta^{\int}" description] \arrow[dl, Rightarrow, shorten=10mm, "(A_1{,}B_2)"'] & \\
 & \Omega^1_{\mathrm{dR}}(-;\, \mathfrak{l}S^2)_{\mathrm{clsd}} \arrow[rr, "\eta^{\int}"'] & & \int\Omega^1_{\mathrm{dR}}(-;\, \mathfrak{l}S^2)_{\mathrm{clsd}}
\end{tikzcd}$$

But, since $H^2_{\mathrm{dR}}(S^3) \;=\; 0$, and by the *Whitehead integral formula* (cf. [38, p 134][9, p 228][29, pp 19]) there exists:

$$\left\{ \begin{array}{lcl} A_1 & \in & \Omega^1_{\mathrm{dR}}(S^3) \\ B_2 & \in & \Omega^2_{\mathrm{dR}}(S^3) \end{array} \right. \quad \text{s.t.} \quad \begin{array}{lcl} \mathrm{d}\, A_1 & = & F_2 \;:=\; n^*\mathrm{dvol}_{S^2} \\ \mathrm{d}\, B_2 & = & n\cdot\mathrm{dvol}_{S^3} - A_1 F_2 \end{array}$$

From this we get the the desired concordance:

$$(0, n\cdot\mathrm{dvol}_{S^3}) \Rightarrow (F_2, 0) \quad : \quad \left\{ \begin{array}{lcl} \widehat{F}_2 & := & t\, F_2 \,+\, \mathrm{d}t\, A_1 \\ \widehat{H}_3 & := & (t-1)n\,\mathrm{dvol}_{S^3} \,+\, \mathrm{d}t\, B_2 \,+\, (t^2-t)A_1 F_2\,. \end{array} \right. \qquad \begin{array}{l} \big(\widehat{F}_2, \widehat{H}_3\big)|_{t=0} \;=\; (0, n\cdot\mathrm{dvol}_{S^3}) \\ \big(\widehat{F}_2, \widehat{H}_3\big)|_{t=1} \;=\; (F_2, 0) \\ \mathrm{d}\,\widehat{F}_2 \;=\; 0\,,\; \mathrm{d}\,\widehat{H}_3 \;=\; \widehat{F}_2\,\widehat{F}_2\,. \end{array}$$ □

Cartesian M5-Probes charged in Cohomotopy. The equations of motion for a(n orbifolded) cartesian M5-probe demand that the flux $H_3 = \mathrm{const}$ [47, Ex. 3.14], and thus its solitonic vanishing-at-infinity implies $H_3 = 0$. The above theorem says that such solutions still support non-vanishing cohomotopical charge, in fact that the vanishing of H_3 forces the charge to be carried by the Chern-Simons invariant of the auxiliary gauge field $\mathbf{A}_1$ that is brought in by the cohomotopical flux quantization.

References

[1] Aharony, O., Gubser, S., Maldacena, J., Ooguri, H., Oz, Y., *Large N Field Theories, String Theory and Gravity*, Phys. Rept. **323** (2000) 183-386 [doi:10.1016/S0370-1573(99)00083-6, arXiv:hep-th/9905111]

[2] An, Y.-S., Li L., Yang, F.-G., Yang, R.-Q., *Interior Structure and Complexity Growth Rate of Holographic Superconductor from M-Theory*, J. High Energ. Phys. **2022** 133 (2022) [`arXiv:2205.02442`], [`doi:10.1007/JHEP08(2022)133`].

[3] Arovas, D. P., Schrieffer, R., Wilczek, F., *Fractional Statistics and the Quantum Hall Effect*, Phys. Rev. Lett. **53** (1984) 722 [`doi:10.1103/PhysRevLett.53.722`].

[4] Arovas, D. P., Schrieffer, R., Wilczek, F., Zee, A., *Statistical mechanics of anyons*, Nuclear Physics B **251** (1985) 117-126 [`doi:10.1016/0550-3213(85)90252-4`].

[5] Bakulev, A. P., Shirkov, D., *Inevitability and Importance of Non-Perturbative Elements in Quantum Field Theory*, Proceedings of the 6th Mathematical Physics Meeting, Belgrade (2010) 27–54 [`arXiv:1102.2380` [`ISBN:978-86-82441-30-4`]

[6] Balachandran, A. P., Srivastava, A. M., *Chern-Simons Dynamics and the Quantum Hall Effect* [`arXiv:hep-th/9111006`].

[7] Beardsley, J., Nakamura, S., *Projective Geometries and Simple Pointed Matroids as $\mathbb{F}_1$-modules* [`arXiv:2404.04730`].

[8] Beterov, I. I., *Progress and Prospects in the Field of Quantum Computing*, Optoelectron. Instrument. Proc. **60** (2024) 74–83 [`doi:10.3103/S8756699024700043`].

[9] Bott, R., Tu, L., *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics **82**, Springer (1982) [`doi:10.1007/978-1-4757-3951-0`].

[10] Bourjaily, J. L., Espahbodi S., *Geometrically Engineerable Chiral Matter in M-Theory* [`arXiv:0804.1132`].

[11] Clay Math Institute, *The Millennium Prize Problems* [`www.claymath.org/millennium-problems`].

[12] Chu, C., Lorscheid, O., Santhanam,R., *Sheaves and K-theory for $\mathbb{F}_1$-schemes*, Advances in Mathematics **229** 4 (2012) 2239-2286 [`arXiv:1010.2896`] [`doi:10.1016/j.aim.2011.12.023`].

[13] Cohen, F. R., *Introduction to configuration spaces and their applications*, in: *Braids*, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore **19** (2009) 183-261 [`doi:10.1142/9789814291415_0003`].

[14] DARPA, *Quantum Benchmarjing Initiative* (2024) [`www.darpa.mil/work-with-us/quantum-benchmarking-initiative`]

[15] Das Sarma, S., *Quantum computing has a hype problem*, MIT Tech Review (March 2022) [`www.technologyreview.com/2022/03/28/1048355/quantum-computing-has-a-hype-problem/`]

[16] Das Sarma, S., *In search of Majorana*, Nature Physics **19** (2023) 165-170 [`arXiv:2210.17365`], [`doi:10.1038/s41567-022-01900-9`].

[17] D'Auria, R., Fré, P., *Geometric Supergravity in $D = 11$ and its hidden supergroup*, Nucl. Phys. B **201** (1982), 101-140, [`doi:10.1016/0550-3213(82)90376-5`].

[18] Donos, A., Gauntlett, J. P., Sonner, J., Withers, B., *Competing orders in M-theory: superfluids, stripes and metamagnetism*, J. High Energ. Phys. **2013** 108 (2013) [`arXiv:1212.0871`], [`doi:10.1007/JHEP03(2013)108`].

[19] Donos, A., Gauntlett, J. P., Pantelidou, C., *Semi-local quantum criticality in string/M-theory*, J. High Energ. Phys. **2013** 103 (2013) [`arXiv:1212.1462`], [`doi:10.1007/JHEP03(2013)103`].

[20] M. Duff, *The World in Eleven Dimensions: Supergravity, Supermembranes and M-theory*, IoP (1999) [`ISBN:9780750306720`].

[21] F. Dul: *General Covariance from the Viewpoint of Stacks*, Lett Math Phys **113** (2023) 30 [`arXiv:2112.15473`], [`doi:10.1007/s11005-023-01653-3`].

[22] Dyakonov, M. I., *Prospects for quantum computing: extremely doubtful*, Int. J. of Modern Physics: Conf. Series **33** (2014) 1460357 [`arXiv:1401.3629`], [`doi:10.1142/S2010194514603573`].

[23] Ezratty O., *Where are we heading with NISQ?* [`arXiv:2305.09518`]

[24] Ezratty O., *Where are we heading with NISQ?*, blog post (2023) [`www.oezratty.net/wordpress/2023/where-are-we-heading-with-nisq`].

[25] Farb, B., Margalit, D., *A primer on mapping class groups*, Princeton University Press (2012) [`ISBN:9780691147949`], [`doi:j.ctt7rkjw`].

[26] Ferraz, A., Gupta, K. S., Semenoff, G. W., Sodano, P. (eds), *Strongly Coupled Field Theories for Condensed Matter and Quantum Information Theory*, Springer Proceedings in Physics **239**, Springer (2020) [`doi:10.1007/978-3-030-35473-2`].

[27] Fiorenza, D., Sati, H. , Schreiber, U., *A higher stacky perspective on Chern-Simons theory*, in *Mathematical Aspects of Quantum Field Theories* Springer (2014) 153-211 [`doi:10.1007/978-3-319-09949-1`].

[28] Fiorenza, D., Sati, H., Schreiber, U., *T-Duality from super Lie n-algebra cocycles for super p-branes*, Adv. Theor. Math. Phys **22** 5 (2018) [`arXiv:1611.06536`], [`doi:10.4310/ATMP.2018.v22.n5.a3`].

[29] Fiorenza, D., Sati, H., Schreiber, U., *Twisted Cohomotopy implies M5 WZ term level quantization*, Commun. Math. Phys. **384** (2021), 403-432, [`doi:10.1007/s00220-021-03951-0`], [`arXiv:1906.07417`].

[30] Fiorenza, D., Sati, H., Schreiber, U., *Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds*, Commun. Math. Phys. **377** (2020), 1961-2025, [`doi:10.1007/s00220-020-03707-2`], [`arXiv:1904.10207`].

[31] Fiorenza, D., Sati, H. , Schreiber, U., *Twistorial Cohomotopy Implies Green-Schwarz anomaly cancellation*, Rev. Math. Phys. **34** 05 (2022) 2250013, [`doi:10.1142/S0129055X22500131`], [`arXiv:2008.08544`].

[32] Fiorenza, D., Sati, H., Schreiber, U., *The Character map in Nonabelian Cohomology — Twisted, Differential and Generalized*, World Scientific, Singapore (2023), [`doi:10.1142/13422`], [`arXiv:2009.11909`].

[33] Foss-Feig, M., Pagano, G., Potter, A. C., Yao, N. Y.: *Progress in Trapped-Ion Quantum Simulation*, Annual Reviews of Condensed Matter Physics (2024) [`arXiv:2409.02990`], [`doi:10.1146/annurev-conmatphys-032822-045619`].

[34] Freedman, M., Kitaev, A., Larsen, M., Wang, Z., *Topological quantum computation*, Bull. Amer. Math. Soc. **40** (2003) 31-38 [`arXiv:quant-ph/0101025`], [`doi:10.1090/S0273-0979-02-00964-3`].

[35] Gauntlett, J. P., Sonner, J., Wiseman, T., *Holographic superconductivity in M-Theory*, Phys. Rev. Lett. **103** (2009) 151601 [`arXiv:0907.3796`], [`doi:10.1103/PhysRevLett.103.151601`].

[36] Gauntlett, J., Sonner, J., Wiseman, T., *Quantum Criticality and Holographic Superconductors in M-theory*, J. High Energ. Phys. **2010** 60 (2010) [`arXiv:0912.0512`], [`doi:10.1016/j.physletb.2009.12.017`].

[37] Gent, E, *Quantum Computing's Hard, Cold Reality Check*, IEEE Spectrum (Dec. 2023) [`spectrum.ieee.org/quantum-computing-skeptics`]

[38] Griffiths, P., Morgan, J., *Rational Homotopy Theory and Differential Forms*, Progress in Mathematics **16**, Birkhäuser (1981, 2013) [`doi:10.1007/978-1-4614-8468-4`].

[39] Hansen, V. L.: *On the Space of Maps of a Closed Surface into the 2-Sphere*, Mathematica Scandinavica **35** (1974) 149-158 [`doi:10.7146/math.scand.a-11542`], [`jstor:24490694`].

[40] Kallel, S., *Configuration spaces of points: A user's guide*, Encyclopedia of Mathematical Physics 2nd ed. **4** (2025) [`arXiv:2407.11092`], [`doi:10.1016/B978-0-323-95703-8.00211-1`].

[41] Sati, S. Schreiber, S., *Equivariant principal ∞-bundles* [`arXiv:2112.13654`].

[42] Sati, S. Schreiber, S., *Anyonic Defect Branes and Conformal Blocks in Twisted Equivariant Differential K-Theory*, Rev. Math. Phys. **35** 06 (2023) 2350009, [`doi:10.1142/S0129055X23500095`], [`arXiv:2203.11838`].

[43] Sati, H., Schreiber, U., *Anyonic topological order in TED K-theory*, Reviews in Mathematical Physics (2023) **35** 03 (2023) 2350001 [`arXiv:2206.13563`], [`doi:10.1142/S0129055X23500010`].

[44] Sati, H., Schreiber, U., *Flux quantization*, Encyclopedia of Mathematical Physics (2nd ed.) **4** (2025), 281-324, [`doi:10.1016/B978-0-323-95703-8.00078-1`], [`arXiv:2402.18473`].

[45] Gill, S. G., et al., *Quantum Computing: Vision and Challenges* [`arXiv:2403.02240`].

[46] G. Giotopoulos, H. Sati, U. Schreiber, *Flux Quantization on 11d Superspace*, J. High Energy Phys. **2024** (2024) 82, [`doi:10.1007/JHEP07(2024)082`], [`arXiv:2403.16456`].

[47] Giotopoulos, G., Sati, H. Schreiber, U., *Flux Quantization on M5-Branes* Journal of High Energy Physics **2024** 140 (2024) [`doi:10.1007/JHEP10(2024)140`], [`arXiv:2406.11304`].

[48] Giotopoulos, G., Sati, H., Schreiber, U. *Super L_∞ T-Duality and M-theory* [`arXiv:2411.10260`].

[49] Grumblin, E., Horowitz, M. (eds.), *Quantum Computing: Progress and Prospects*, The National Academies Press (2019) [`doi:10.17226/25196`], [`ISBN:9780309479691`].

[50] Gubser, S. S., Pufu, S. S., Rocha, F. D., *Quantum critical superconductors in string theory and M-theory*, Phys. Lett. B **683** (2010) 201-204 [`arXiv:0908.0011`], [`doi:10.1016/j.physletb.2009.12.017`].

[51] Hannabuss, K. C., *T-duality and the bulk-boundary correspondence*, Journal of Geometry and Physics **124** (2018) 421-435 [`arXiv:1704.00278`], [`doi:10.1016/j.geomphys.2017.11.016`].

[52] Hartnoll, S., Lucas, A., Sachdev, S., *Holographic quantum matter*, MIT Press (2018) [`arXiv:1612.07324`], [`ISBN:9780262348010`].

[53] Herzog, C. P., Kovtun, P., Sachdev, S., Thanh, D. Son, *Quantum critical transport, duality, and M-theory*, Phys. Rev. D **75** (2007) 085020 [`arXiv:hep-th/0701036`], [`doi:10.1103/PhysRevD.75.085020`].

[54] Hoefler, T., Haener, T., Troyer, M., *Disentangling Hype from Practicality: On Realistically Achieving Quantum Advantage*, Communications of the ACM **66** 5 (2023) 82-87 [`doi:10.1145/3571725`], [`arXiv:2307.00523`].

[55] Iqbal, M., Tantivasadakarn, N., Verresen, R. et al., *Non-Abelian topological order and anyons on a trapped-ion processor*, Nature **626** (2024) 505–511 [`doi:10.1038/s41586-023-06934-4`].

[56] Jacak, J. E.: *Topological approach to electron correlations at fractional quantum Hall effect*, Annals of Physics **430** (2021) 168493 [`doi:10.1016/j.aop.2021.168493`]

[57] James, I. M., *General Topology and Homotopy Theory*, Springer (1984) [`doi:10.1007/978-1-4613-8283-6`].

[58] Jiang, B., Bouhon, A., Lin, Z.-K., Zhou, X., Hou, B., Li, F., Slager, R.-J., Jiang, J.-H., *Experimental observation of non-Abelian topological acoustic semimetals and their phase transitions*, Nature Physics **17** (2021) 1239-1246 [`doi;10.1038/s41567-021-01340-x`], [`arXiv:2104.13397`].

[59] Kak, S., *Prospects for Quantum Computing*, talk at CIFAR Nanotechnology program meeting, Halifax (November 2008) [`arXiv:0902.4884`].

[60] Kallel, S., *Configuration Spaces and the Topology of Curves in Projective Space*, in: *Topology, Geometry, and Algebra: Interactions and new directions*, Contemporary Mathematics **279**, AMS (2001) 151–175 [`doi:10.1090/conm/279`].

[61] Katz, S., Klemm, A., Vafa, C. *Geometric Engineering of Quantum Field Theories*, Nucl. Phys. B **497** (1997) 173-195 [`doi:10.1016/S0550-3213(97)00282-4`], [`arXiv:hep-th/9609239`].

[62] Kitaev, A., *Unpaired Majorana fermions in quantum wires*, Physics-Uspekhi **44** 10S (2001) 131-136 [`arXiv:cond-mat/0010440`], [`doi:10.1070/1063-7869/44/10S/S29`].

[63] Kitaev, A., *Fault-tolerant quantum computation by anyons*, Annals Phys. **303** (2003) 2-30 [`doi:10.1016/S0003-4916(02)00018-0`], [`arXiv:quant-ph/9707021`].

[64] Kitaev, A., *Anyons in an exactly solved model and beyond*, Annals of Physics **321** 1 (2006) 2-111 [`doi:10.1016/j.aop.2005.10.005`].

[65] Kosinski, A., *Differential manifolds*, Academic Press (1993) [`ISBN:978-0-12-421850-5`]

[66] Pontrjagin, L., *Classification of continuous maps of a complex into a sphere*, Communication I, Doklady Akademii Nauk SSSR **19** 3 (1938) 147-149

[67] Larmore, L. L., Thomas, E.: *On the Fundamental Group of a Space of Sections*, Mathematica Scandinavica **47** 2 (1980) 232-246 [`jstor:24491393`].

[68] Lau J. W. Z., Lim, K. H., Shrotriya H., Kwek, L. C., *NISQ computing: where are we and where do we go?*, AAPPS Bull. **32** 27 (2022) [`doi:10.1007/s43673-022-00058-z`]

[69] Lee, S. T., Packer, J. A., *The Cohomology of the Integer Heisenberg Groups*, J. of Algebra **184** 1 (1996) 230-250 [`doi:10.1006/jabr.1996.0258`].

[70] Lidar, D. A., Brun, T. A. (eds.), *Quantum Error Correction*, Cambridge University Press (2013) [`ISBN:9780521897877`], [`doi:10.1017/CBO9781139034807`].

[71] Liu, S., *Anyon quantum dimensions from an arbitrary ground state wave function*, Nature Communications **15** (2024) 5134 [`doi:10.1038/s41467-024-47856-7`], [`arXiv:2304.13235`].

[72] Lobb, A., *A feeling for Khovanov homology*, Notices of the AMS **71** 5 (2024) [`doi:10.1090/noti2928`].

[73] Lloyd, S., *Quantum computation with abelian anyons*, Quantum Information Processing **1** 1/2 (2002), 13-18, [`doi:10.1023/A:1019649101654`], [`arXiv:quant-ph/0004010`].

[74] Lutchyn, R. M. Sau, J. D., Das Sarma, S., *Majorana Fermions and a Topological Phase Transition in Semiconductor-Superconductor Heterostructures*, Phys. Rev. Lett. **105** (2010) 077001 [`doi:10.1103/PhysRevLett.105.077001`].

[75] Manoliu, M., *Abelian Chern-Simons theory*, J. Math. Phys. **39** (1998) 170-206 [`arXiv:dg-ga/9610001`], [`doi:10.1063/1.532333`].

[76] G. Massuyeau: *Lectures on Mapping Class Groups, Braid Groups and Formality* (2021) [`massuyea.perso.math.cnrs.fr/notes/formality.pdf`].

[77] Mathai, V., Thiang, G. C., *T-Duality of Topological Insulators*, J. Phys. A: Math. Theor. **48** (2015) 42FT02 [`arXiv:1503.01206`], [`doi:10.1088/1751-8113/48/42/42FT02`].

[78] Mathai, V., Thiang, G. C., *T-Duality Simplifies Bulk-Boundary Correspondence*, Commun. Math. Phys. **345** (2016) 675–701
[`arXiv:1505.05250`], [`doi:10.1007/s00220-016-2619-6`].

[79] Naito, S., Osada, K., Fukui, T., *Fierz Identities and Invariance of Eleven-dimensional Supergravity Action*, Phys. Rev. D **34** (1986), 536-552, [`doi:10.1103/PhysRevD.34.536`]; Erratum Ibid. **35** (1986), 1536-1536, [`doi:10.1103/PhysRevD.35.1536`].

[80] Nakamura, J., et al., *Direct observation of anyonic braiding statistics*, Nat. Phys. **16** (2020) 931–936 [`doi:10.1038/s41567-020-1019-1`].

[81] Nielsen, M. A., Chuang, I. L., *Quantum computation and quantum information*, Cambridge University Press (2000) [`doi:10.1017/CBO9780511976667`].

[82] nLab, *Yang-Mills mass gap* [`ncatlab.org/nlab/show/Yang-Mills+mass+gap`]

[83] nLab, *cell structure of projective spaces* [`ncatlab.org/nlab/show/cell+structure+of+projective+spaces`].

[84] Okuyama, S., *The space of intervals in a Euclidean space*, Algebr. Geom. Topol. **5** (2005) 1555-1572 [`arXiv:math/0511645`], [`doi:10.2140/agt.2005.5.1555`].

[85] Ohtsuki, T., *Quantum Invariants – A Study of Knots, 3-Manifolds, and Their Sets*, World Scientific (2001) [`doi:10.2140/agt.2005.5.1555`].

[86] Pachos, J. K., *Quantum computation with abelian anyons on the honeycomb lattice*, Int. J. Quantum Information **4** 6 (2006), 947-954, [`doi:10.1142/S0219749906002328`], [`arXiv:quant-ph/0511273`].

[87] Pachos, J. K., *Introduction to Topological Quantum Computation*, Cambridge University Press (2012) [`doi:10.1017/CBO9780511792908`].

[88] Polchinski, J., *Dualities of Fields and Strings*, Studies in History and Philosophy of Modern Physics **59** (C) (2017) 6-20 [`arXiv:1412.5704`],

[89] J. Preskill, *Quantum Computing in the NISQ era and beyond*, Quantum **2** 79 (2018) [`arXiv:1801.00862`], [`doi:10.22331/q-2018-08-06-79`].

[90] J. Preskill, *Crossing the Quantum Chasm: From NISQ to Fault Tolerance*, talk at Q2B 2023, Silicon Valley (2023) [`ncatlab.org/nlab/files/Preskill-Crossing.pdf`].

[91] J. Preskill, *Beyond NISQ: The Megaquop Machine*, talk at *Q2B 2024 Silicon Valley* (Dec. 2024) [`www.preskill.caltech.edu/talks/Preskill-Q2B-2024.pdf`].

[92] Rigolin, G., Ortiz, G., *The Adiabatic Theorem for Quantum Systems with Spectral Degeneracy*, Phys. Rev. A **85** 062111 (2012) [`arXiv:1111.5333`], [`doi:10.1103/PhysRevA.85.062111`].

[93] Rowell, E. C., *An Invitation to the Mathematics of Topological Quantum Computation*, J. Phys.: Conf. Ser. **698** (2016) 012012 [`doi:10.1088/1742-6596/698/1/012012`], [`arXiv:1601.05288`].

[94] Rowell, E. C., *Braids, Motions and Topological Quantum Computing* [`arXiv:2208.11762`].

[95] Sati, S., Schreiber, U., *Equivariant Cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys. **156** (2020), 103775, [`doi:10.1016/j.geomphys.2020.103775`], [`arXiv:1909.12277`].

[96] Sati, H., Schreiber, U., *Proper Orbifold Cohomology* [`arXiv:2008.01101`].

[97] Sati, H., Schreiber, U., *M/F-Theory as Mf-Theory*, Rev. Math. Phys. **35** 10 (2023) [`arXiv:2103.01877`] [`doi:10.1142/S0129055X23500289`].

[98] Sati, H., Schreiber, U., *Quantum Observables of Quantized Fluxes* Annales Henri Poincaré (2024) [`arXiv:2312.13037`], [`doi:10.1007/s00023-024-01517-z`].

[99] Sati, H., Schreiber, U., *Abelian Anyons on Flux-Quantized M5-Branes* [`arXiv:2408.11896`].

[100] Sati, H., Schreiber, U., *Topological QBits in Flux-Quantized Supergravity*, in: *Quantum Gravity and Computation* Routledge (2025) [`arXiv:2411.00628`].

[101] H. Sati, U. Schreiber, *The character map in Equivariant Twistorial Cohomotopy*, in: *Applied Algebraic Topology*, Beijing J. Pure Appl. Math. (2025) [`arXiv:2011.06533`].

[102] Sati, H., Schreiber, U., *Anyons on M5-Probes of Seifert 3-Orbifolds via Flux-Quantization* [`arXiv:2411.16852`].

[103] Sati, H., Valera, S., *Topological Quantum Computing*, Encyclopedia of Mathematical Physics 2nd ed **4** (2025) 325-345 [`doi:10.1016/B978-0-323-95703-8.00262-7`].

[104] Sau S., *A Roadmap for a Scalable Topological Quantum Computer*, Physics **10** 68 (2017) [physics.aps.org/articles/v10/68]

[105] Segal, G., *Configuration-spaces and iterated loop-spaces*, Invent. Math. **21** (1973) 213-221 [`doi:10.1007/BF01390197`].

[106] Störmer, H. L., *Nobel Lecture: The fractional quantum Hall effect*, Rev. Mod. Phys. **71** (1999) 875 [`doi:10.1103/RevModPhys.71.875`].

[107] Strom, J., *Modern classical homotopy theory*, Graduate Studies in Mathematics **127**, American Mathematical Society (2011) [`ams:gsm/127`]

[108] Tiwari, A., Bzdušek, T., *Non-Abelian topology of nodal-line rings in PT-symmetric systems*, Phys. Rev. B **101** (2020) 195130 [`doi:10.1103/PhysRevB.101.195130`].

[109] Tong, D., *The Quantum Hall Effect*, lecture notes (2016) [`arXiv:1606.06687`].

[110] Waldorf, K., *Geometric T-duality: Buscher rules in general topology*, Ann. Henri Poincaré **25** (2024) 1285–1358 [`arXiv:2207.11799`], [`doi:10.1007/s00023-023-01295-0`].

[111] Wen, X.-G.: *Vacuum Degeneracy of Chiral Spin State in Compactified Spaces*, Phys. Rev. B **40** 7387 (1989) [`doi:10.1103/PhysRevB.40.7387`].

[112] Wen, X.-G., Dagotto, E., Fradkin, E., *Anyons on a torus*, Phys. Rev. B **42** (1990) 6110 [`doi:10.1103/PhysRevB.42.6110`].

[113] Williams, L., *Configuration Spaces for the Working Undergraduate*, Rose-Hulman Undergraduate Mathematics Journal **21** 1 (2020) Article 8 [`arXiv:1911.11186`], [`rhumj:vol21/iss1/8`].

[114] Witten, E., *Three Lectures On Topological Phases Of Matter*, La Rivista del Nuovo Cimento **39** (2016) 313-370 [`arXiv:1510.07698`], [`doi:10.1393/ncr/i2016-10125-3`].

[115] Wootton, J. R., *Dissecting Topological Quantum Computation*, PhD thesis, Leeds (2010), [`etheses:1163`].

[116] Wootton, J. R., Pachos, J. K., *Universal Quantum Computation with Abelian Anyon Models*, Electron. Notes Theor. Comput. Sci. **270** 2 (2011), 209-218, [`doi:10.1016/j.entcs.2011.01.032`], [`arXiv:0904.4373`].

[117] Wu, Q., Soluyanov, A. A., Bzdušek, T., *Non-Abelian band topology in noninteracting metals*, Science **365** (2019) 1273-1277 [`doi:10.1126/science.aau8740`], [`arXiv:1808.07469`].

[118] Zaanen, J., Liu, Y., Sun, Y.-W., Schalm, K., *Holographic Duality in Condensed Matter Physics*, Cambridge University Press (2015) [`doi:10.1017/CBO9781139942492`].

[119] Zeng, B., Chen, X., Zhou, D.-L., Wen, X.-G., *Quantum Information Meets Quantum Matter – From Quantum Entanglement to Topological Phases of Many-Body Systems*, Quantum Science and Technology (QST), Springer (2019) [`arXiv:1508.02595`], [`doi:10.1007/978-1-4939-9084-9`].