Engineering of Anyons on M5-Probes via Flux-Quantization

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Abstract

These extended lecture notes survey a novel derivation of anyonic topological order (as seen in fractional quantum Hall systems) on single magnetized M5-branes probing Seifert orbi-singularities ("geometric engineering" of anyons), which we motivate from fundamental open problems in the field of quantum computing.

The rigorous construction is non-Lagrangian and non-perturbative, based on previously neglected global completion of the M5-brane's tensor field by flux-quantization consistent with its non-linear self-duality and its twisting by the bulk C-field: This exists only in little-studied non-abelian generalized cohomology theories, notably in a twisted equivariant (and "twistorial") form of unstable Cohomotopy ("Hypothesis H").

As a result, topological quantum observables form Pontrjagin homology algebras of mapping spaces from the orbi-fixed worldvolume into a classifying 2-sphere. Remarkably, theorems of algebraic topology imply from this the quantum observables and modular functor of abelian Chern-Simons theory, as well as braid group actions on defect anyons of the kind envisioned as hardware for topologically protected quantum gates.

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Contents

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1 Motivation: Better Anyon Theory

While the hopes associated with the idea of *quantum computing* [92][55] are hard to over-state [48][9][105], there are good arguments that commercial-value quantum computing will ultimately require quantum hardware exhibiting *anyonic topological order* [144][116]. But microscopic theoretical derivations, from first principles, of such anyonic quantum states in strongly-coupled quantum systems had remained sketchy, which may explain the dearth of experimental realizations to date.

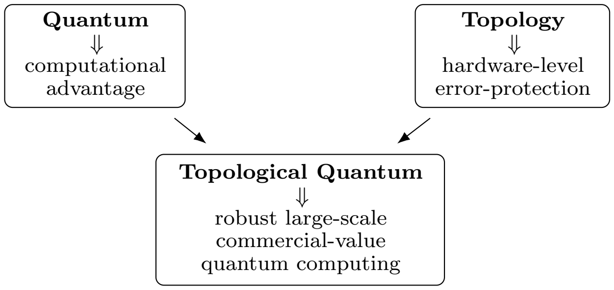
What we review here (based on [119][122][50]) is a rigorous theoretical account via "geometric engineering on M-branes" subject to a previously neglected step of "flux-quantization" (the latter surveyed in [117]).

First, we expand on the motivation a little further:

Ultimate need for Topological Quantum Protection. Despite the fascinating reality of presently available Noisy Intermediate-Scale Quantum computers (NISQ [103]) and despite the mid-term prospect of their stabilization at the software-level via Quantum Error Correction (QEC [79][104], at heavy cost of available system scale), serious arguments [68][28][77][29][30][63][47][132] and experience [20] suggest that large-scale quantum computation is hardly attainable by incremental optimization of NISQ architectures, but [21] [1] that more fundamental quantum principles will need to be exploited – notably *topological* error *protection* already at the hardware-level [73][43][124][123] in order to suppress quantum errors occurring in the first place.

While topological quantum protection is thus possibly indispensable for achieving commercial-value quantum computing, its ambitious development, in theory and practice, is in fact far from mature, is in need of new ideas and of further analysis, and leaves much room for development.

Since this is not always made clear, to amplify this point:



(i) Theoretical challenges: While quantum theorists now routinely deal with the algebraic structure (namely: braided fusion categories) commonly *expected* [74] to describe interaction of anyon species *in toto*, the *microscopic* first-principles understanding of the formation of anyonic topological order as solitonic states in the many-body (electron) dynamics of quantum materials has remained at most sketchy, even in the best-understood case of the fractional quantum Hall effect [126], cf. [65]. [2]

In fact, this is an instance of the general open problem of analytically establishing gapped bound states in any strongly coupled/correlated quantum system: The problem of formulating non-perturbative quantum field theory [6][32]. The analogous issue in particle physics (there called the *Yang-Mills mass gap* problem [93]) has been recognized as being profound enough to be declared one of seven "Millennium Problems" [18].

(ii) Practical challenges: But without a robust theoretical prediction of anyonic solitons in actual quantum materials, it remains unclear where and how to look for them. As an unfortunate result, experimentalists have turned attention to mere stand-ins, such as "Majorana zero modes" at the ends of super/semi-conducting nonowires ([72][84] which, even if the doubts about their detection were to be removed [22], are by construction immobile and hence do not serve as hardware-protected quantum braid gates) and quantum-simulation of anyons on NISQ architectures ([64][41, Fig. 5], which might serve as software-level QEC but again offers no hardware-level protection.

In short: **Foundation and implementation** of topological quantum computing as a plausible long-term pathway to actual quantum value **deserves and admits thorough re-investigation**.

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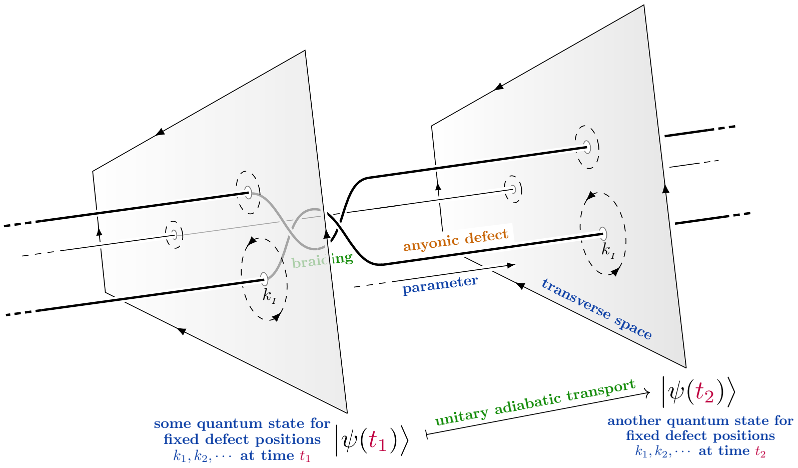
[1] [21]: "The qubit systems we have today are a tremendous scientific achievement, but they take us no closer to having a quantum computer that can solve a problem that anybody cares about. [...] What is missing is the breakthrough [...] bypassing quantum error correction by using far-more-stable qubits, in an approach called topological quantum computing."

[2] [65, p. 3]: "Though the Laughlin function very well approximates the true ground state at $\nu = 1/q$, the physical mechanism of related correlations and of the whole hierarchy of the FQHE remained, however, still obscure. [...] The so-called HH (Halperin–Haldane) model of consecutive generations of Laughlin states of anyonic quasiparticle excitations from the preceding Laughlin state has been abandoned early because of the rapid growth of the daughter quasiparticle size, which quickly exceeded the sample size. [...] the Halperin multicomponent theory and of the CF model advanced the understanding of correlations in FQHE, however, on phenomenological level only. CFs were assumed to be hypothetical quasi-particles consisting of electrons and flux quanta of an auxiliary fictitious magnetic field pinned to them. The origin of this field and the manner of attachment of its flux quanta to electrons have been neither explained nor discussed."

Concretely, the intrinsic tension haunting the traditional quantum computing paradigm is (cf. [15, p 272][132, p 3]) that:

> (i) quantum gates are *implemented via interaction* of subsystems,
> (ii) while quantum coherence requires *avoiding all interaction.*

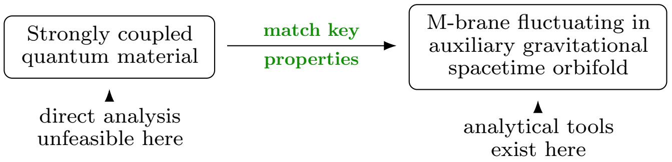
The idea of topological protection is to cut this Gordian knot by *quantum gates operating without interaction.* The physical principle that allows this to work [3][4][43, p 6][98, p 50] is the *quantum adiabatic theorem* [107]: Gapped quantum systems frozen at absolute zero in one of several ground states, but dependent on external parameters, will defy interaction with noise quanta below the energy gap and yet have their ground state transformed by sufficiently gentle tuning of the parameters: a *holonomic quantum gate.* This is *topological* if it is invariant under local deformations of parameter paths, and thus protected also against classical noise. For an *anyonic braid gate* the parameters in question are the positions of defects in a 2-dimensional transverse space within a quantum material.



The remaining problem is to develop a precise mathematical theory describing these *anyons.*

Improved Anyon Models via Geometric Engineering on M-branes. A remarkable approach to the otherwise elusive microscopic analysis of such strongly-coupled/correlated quantum systems emerges in the guise of "geometric engineering" [71][12] of quantum fields on "M-branes" probing orbifold singularities, whereby the given dynamics is (partially) mapped onto the fluctuations of Membranes (whence *M-theory* [26]), and of higher-dimensional "M5-branes" [50], propagating in an auxiliary higher-dimensional gravitating spacetime orbifold [111].

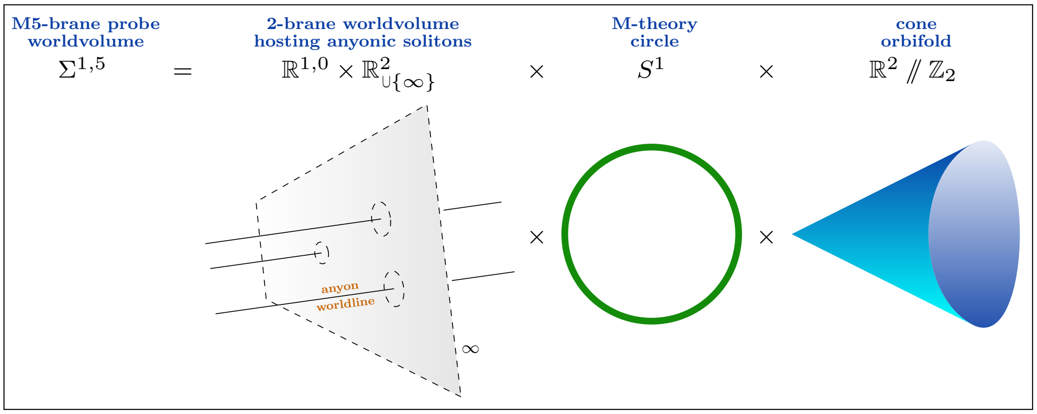
Geometric engineering of quantum systems on M-branes provides tools for analyzing otherwise elusive strongly coupled/correlated quantum phenomena.



This procedure is most famous in the (unrealistic) limit of large rank and hence of large numbers $N \to \infty$ of coincident such branes, where it extracts quantum correlators and quantum phase transitions entirely from classical gravitational asymptotics ("holographic duality" [1]). The application to quantum materials [143][59] is now well-studied, notably in the case of quantum critical superconductors engineered in M-theory [62][45][46][56][24][25][2].

But we have established [50][119][120][122] that after implementing a previously neglected step of "flux quantization" [117] on the M5-brane worldvolume, there provably appear general solitonic and specifically anyonic quantum states already in the more realistic situation of single ($N = 1$) coincident branes. (Similar results for $N = 2$ had previously only been conjectured [16] by appeal to the expected but notoriously undefined effective quantum field theory on coincident M5-branes.)

Brane diagram for geometric engineering of anyons on single M5-branes wrapping an orbi-singularity [122]:
It is a subtle mechanism of *flux-quantization* [117] of the self-dual tensor-field on the M5 [50] that stabilizes [119] its anyonic soliton configurations.



Here, we review and explain how this works, for an audience assumed to be familiar with the general mechanism of *flux quantization* as surveyed in [117]. First to recall the traditional theory of fractional quantum Hall anyons:

Quantum Hall effect ([102][13][126][99]). In a very thin and hence effectively 2-dimensional sheet Σ^2 of (semi-)conducting material, carrying magnetic flux density B:

the energy of electron states is quantized by *Landau levels* $i \in \mathbb{N}$ as $E \;=\; \hbar\omega_B(i + \tfrac{1}{2})\,,$

where each level comprises of one state per magnetic flux quantum: $n_{\mathrm{deg}} \;=\; B/\Phi_0\,;$ [131, (4-12)]

The Lorentz force on a longitudinal electron current J_x at filling fraction ν is compensated in equilibrium by an electric *Hall field*: $E_y \;=\; \tfrac{1}{\nu} J_x\,.$

Integer quantum Hall effect. Therefore, Fermi theory of idealized *free* electrons predicts the system to be a conductor away from the energy gaps between a completely filled and the next empty Landau level, hence away from the number of electrons being integer multiples of the number of flux quanta, where longitudinal conductivity should vanish.

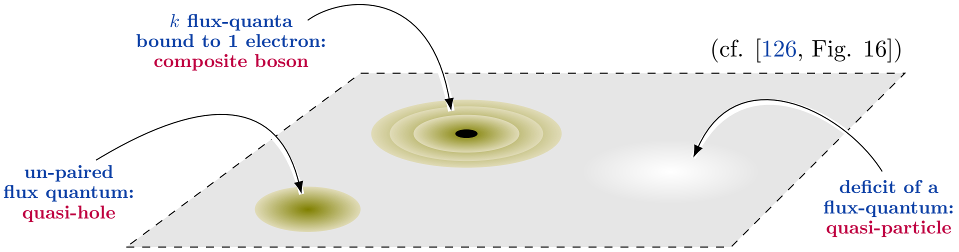
$$n_{\mathrm{el}} = \nu B/\Phi_0\,, \quad \nu \in \mathbb{N}.$$

This is indeed observed — in fact the vanishing conductivity is observed in sizeable neighbourhoods of the critical filling fractions ("Hall plateaux", attributed to subtle disorder effects).

Fractional quantum Hall effect (FQHE). But in reality the electrons are far from free. While there is little theory for strongly interacting quantum systems, experiment shows that the Fermi idealization breaks down at low enough temperature, where longitudinal conductivity decreases also in neighbourhoods of certain *fractional* filling factors ν.

$\nu \in \mathbb{Q}$, prominently for $\nu = 1/k\,, k \in 2\mathbb{N}+1$.

The traditional heuristic idea is that at these filling fractions the interacting electrons each form a kind of bound state with k flux quanta, making "composite bosons" (cf. [145]) that as such condense to produce an insulating mass gap, even inside the Landau level.



(cf. [126, Fig. 16])

Anyonic quasi-particles. This heuristic model suggests that in the Hall plateau neighbourhood *around* such filling fraction there are *unpaired* flux quanta effectively "bound to" $1/k$th of a (missing) electron: called "quasi-particles" ("quasi-holes"). These quasi-particles/holes evidently have fractional charge $\pm e/k$ and are expected to be anyonic with fractional pair exchange phase $e^{\mathrm{i}\pi/k}$. This phase has been experimentall observed [91].

Effective abelian Chern-Simons theory. The traditional proposal for an effective field theory description of k-fractional quantum Hall systems postulates that the effective field is a 1-form potential a for the electric current density 2-form J, itself minimally coupled to the *quasi-hole current* j, and with effective dynamics encoded by the level=k Chern-Simons (CS) Lagrangian [145][136].

$$\begin{array}{lcll}
\text{electron current density 2-form} & J & = \overset{\text{current 3-vector}}{\vec{J}} \lrcorner \overset{\text{volume form}}{\mathrm{dvol}} \;=:\; \mathrm{d}\,a & \text{effective gauge field} \\
\text{quasi-particle current density 2-form} & j & = \vec{j} \lrcorner \mathrm{dvol} & \\
\text{background flux density 2-form} & F & = \mathrm{d}\,A & \text{external gauge field} \\
\text{effective Lagrangian density 3-form} & L & := k \underbrace{\tfrac{1}{2} a\,\mathrm{d}a}_{\mathrm{CS}(a)} - \underbrace{A\,\mathrm{d}a}_{A\,J} + a\,j & [136, (2.11)]
\end{array}$$

Its Euler-Lagrange equations of motion

$$\frac{\delta L}{\delta a} = 0 \quad \Leftrightarrow \quad \boxed{J \;=\; \tfrac{1}{k}\big(F - j\big)}$$

at longitudinal electron current and static quasi-particles

$$\begin{aligned}
J &\equiv J_0\,\mathrm{d}x\,\mathrm{d}y - J_x\,\mathrm{d}t\,\mathrm{d}y \\
j &\equiv j_0\,\mathrm{d}x\,\mathrm{d}y \\
F &\equiv B\,\mathrm{d}x\,\mathrm{d}y - E_y\,\mathrm{d}t\,\mathrm{d}y
\end{aligned}$$

express just the hallmark properties of the FQHE at $\nu = 1/k$.

$$\Leftrightarrow \left\{ \begin{array}{lcll}
J_x & = & \tfrac{1}{k} E_y & \Leftrightarrow \;\text{Hall conductivity law at } 1/k \text{ filling} \\
J_0 & = & \tfrac{1}{k} B & \Leftrightarrow \;\text{each electron binds to } k \text{ flux quanta, but} \\
 & & -\tfrac{1}{k} j_0 & \; 1/k\text{th electron missing for each quasi-hole}
\end{array} \right. \qquad (!)$$

Conceptual problems. But this can only be a *local* description, on a single chart (as is common for Langrangian field theories): Neither J nor F may admit global coboundaries a and A, respectively. Instead, both must be subjected to some kind of flux-quantization. For F this must be classical Dirac charge quantization, which however is incompatible with integrality of J when $k \neq 1$ (cf. [139, p. 35][130, p 159]). But without this, the implications break concerning topological order from abelian CS theory (ground state degeneracy, modular functoriality, ...).

Question: *Is there a non-Lagrangian theory for quasi-particles of properly flux-quantized FQH systems?* Yes!...

The main result to be discussed here is that the key features of the anyonic topological order as seen in fractional quantum Hall systems are consistently, rigorously and naturally reflected by the topological light-cone quantization of the self-dual tensor field on M5-brane probes of certain orbi-singularities in 11D supergravity — once the subtle (non-abelian) flux-quantization of this field is properly taken care of, which is the key step that has not previously received attention.

Further aspects. In fact, fractional quantum Hall systems exhibit further remarkable properties which have not previously been reflected in their effective (Chern-Simons) descriptions, but which are naturally reflected in the M5-brane model:

(1.) hidden supersymmetry
(2.) hidden T-duality

We close this introduction by briefly indicating these two phenomena:

N-Electron ground states of quantum Hall systems. While a microscopic derivation of fractional quantum Hall ground states Ψ remains missing, phenomenologically successful Ansätze exist:[3]

at odd filling fraction $\nu = 1/q$, $q \in 2\mathbb{N}+1$, the **Laughlin wavefunction**

$$\Psi_{\mathrm{La}}\big(z^1, \cdots, z^N\big) \;:=\; \textstyle\prod_{i<j}\big(z^i - z^j\big)^q \,\exp\big(-\tfrac{1}{\ell_B^2}\sum_i \big|z^i\big|^2\big)$$

at even filling fraction $\nu = 1/q$, $q \in 2\mathbb{N}$, the **Read-Moore wavefunction**

$$\Psi_{\mathrm{RM}}\big(z^1, \cdots, z^N\big) \;:=\; \mathrm{Pf}\Big(\tfrac{1}{z^{\bullet_1} - z^{\bullet_2}}\Big)\,\Psi_{\mathrm{La}}\big(z^1, \cdots z^N\big)\,,$$

Here the *Pfaffian* Pf of a skew-symmetric $N \times N$ matrix A is the Bererzinian integral over anti-commuting variables $(\theta^i)_{i=1}^N$:

$$\mathrm{Pf}(A) \;:=\; \underset{\substack{\text{pick coefficient of}\\ \text{top } \theta\text{-power}}}{\textstyle\int\big(\prod_i \mathrm{d}\theta^i\big)} \,\exp\big(\tfrac{1}{2}A_{ij}\,\theta^i\theta^j\big)\,.$$

Hidden super-geometry of quantum Hall systems.
This suggests to promote the plane $\mathbb{C}^1$ to the super-space $\mathbb{C}^{1|1}$ with its *super-translation group* structure

$$(z, \theta) + (z', \theta') \;=\; \big(z + z' + \theta\theta',\, \theta + \theta'\big)$$

Here the **super-Laughlin state**

$$\Psi_{\mathrm{sLa}}\big((z^1, \theta^1), \cdots, (z^N, \theta^N)\big) \;:=\; \textstyle\prod_{i<j}\big(z^i - z^j - \theta^i\theta^j\big)^q \,\exp\big(-\tfrac{1}{\ell_B^2}\sum_i \big|z^i\big|^2\big)$$

exhibits the Read-Moore state as a super-partner to the Laughlin state (up to normalization) [60][54, (13)]:

$$\begin{tikzcd}[column sep=huge]
\underset{\text{fermionic for odd } q}{\textbf{Laughlin state}}\;\Psi_{\mathrm{La}} & \underset{\substack{\textbf{super-}\\ \textbf{Laughlin state}}}{\Psi_{\mathrm{sLa}}} \arrow[l, mapsto, "\text{lowest component}"', "0 \mapsfrom \theta" description] \arrow[r, mapsto, "\text{top component}", "\int \prod_i \mathrm{d}\theta^i" description] & \Psi_{\mathrm{RM}}\;\underset{\text{fermionic for even } q}{\textbf{Moore-Read state}}
\end{tikzcd}$$

Collective excitations. The Moore-Read state is known to have two *density-wave excitations* for wave-vectors $k \in \mathbb{C}$:

the **magneto-roton state**

$$\Psi_{\mathrm{MR},k}\big(z^1, \cdots, z^N\big) \;:=\; \textstyle\sum_i \exp\big(-\mathrm{i}\overline{k}\partial_{z^i}\big)\exp\big(-\tfrac{\mathrm{i}}{2}\overline{k}z^i\big)\,\Psi_{\mathrm{MR}}\big(z^1, \cdots, z^N\big)$$

the **neutral fermion** state

$\Psi_{\mathrm{NF},k}$ which originally did not have a closed expression

But lifting the magneto-roton state to super-space, for super-wavevector $(k, \kappa) \in \mathbb{C}^{1,1}$

$$\Psi_{\mathrm{MR},(k,\kappa)}\big(z^1, \cdots, z^N\big) \;:=\; \textstyle\int\big(\prod_i \mathrm{d}\theta^i\big)\sum_i \exp\big(-\mathrm{i}\overline{k}\partial_{z^i}\big)\exp\big(-\tfrac{\mathrm{i}}{2}\overline{k}z^i\big)\exp\big(-\tfrac{\mathrm{i}}{2}\overline{\kappa}\theta^i\big)\,\Psi_{\mathrm{sLa}}\big((z^1, \theta^1), \cdots, (z^N, \theta^N)\big)$$

it reproduces the magneto-roton state for even N, and the neutral fermion mode when an $(N+1)$st electron is added [54]:

$$\begin{tikzcd}[column sep=huge]
\underset{\textbf{state}}{\textbf{magneto-roton}}\;\Psi_{\mathrm{MR},k} & \underset{\substack{\textbf{super-}\\ \textbf{density excitation}}}{\Psi_{\mathrm{MR},(k,\kappa)}} \arrow[l, mapsto, "\text{no electron added}"', "\text{even } N" description] \arrow[r, mapsto, "\text{add an electron}", "\text{odd } N" description] & \kappa\Psi_{\mathrm{NF},k}\;\underset{\textbf{mode}}{\textbf{neutral-fermion}}
\end{tikzcd}$$

Hidden super-symmetry in fractional quantum Hall systems. This super-unification predicts hidden supersymmetry in fractional quantum Hall systems — which is indeed (numerically) observed [106][81] (also [5, §5]).

This all suggests that an accurate model for fractional quantum Hall systems should in fact itself *originate on superspace*, and this is what we start with now.

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[3] For N electrons in an effectively 2D material, and assumed to be completely spin-polarized by the transverse magnetic field, their wavefunction Ψ is a skew-symmetric (by Pauli exclusion) $\mathbb{C}$-valued function of N complex numbers $(z^i \in \mathbb{C})_{i=1}^N$. We omit normaliztion. For the Read-Moore state N must (for $\mathrm{Pf}(-)$ to be defined) be even (which is harmless since N is a macroscopic number of electrons).

2 Flux-Quantization on M5-Probes

The first task now is to understand the flux-quantization on M5-brane probes, according to [37][39][121].

We will not (need to) explain in full detail the (super-)geometry of probe branes nor of their (super-)gravity backgrounds (full discussion is in [49][50]), but do offer the following broad dictionary, for orientation: [4]

M5-Brane probes (namely *sigma-model* branes, in contrast to *black branes*) are 5-dimensional objects propagating in a gravitational target space X (the "bulk"), along trajectories that are modeled by (super-)immersions of their 6D (and $\mathcal{N} = (2,0)$) worldvolume (super-)manifolds Σ

$$\begin{tikzcd}
\underset{\textbf{(super-)worldvolume}}{\textbf{probe M5-brane}} & \Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \arrow[r, "\phi_s", "\substack{\textbf{trajectory}\\ \textbf{(super-)immersion}}"'] & X^{1,10\,|\,\mathbf{32}} & \underset{\textbf{(super-)spacetime}}{\textbf{target/background}}
\end{tikzcd} \tag{1}$$

Here the admissible ("on-shell", meaning: satisfying the appropriate equations of motion) immersions ϕ_s are controlled by the (super-)geometry of X – namely the brane's trajectory is subject to the gravitational- and Lorentz-forces exerted by the field content of X – but X itself remains unaffected by the choice of ϕ_s – meaning that the (gravitational) *back-reaction* of the brane on its ambient spacetime is neglected; this is what makes the brane but a *probe* of the *background* X.

Thereby the probe brane (Σ, ϕ_s) plays a double role:

(i) on the one hand it is like a (higher-dimensional) fundamental particle, an "observer" of the bulk X in the sense of mathematical relativity,

(ii) on the other hand it is itself a (super-)spacetime with its own (quantum) field content:

Remarkably, the magic of super-geometry makes such purely super-geometric immersions ϕ_s (1) embody not just the naïve (temporal-)spatial worldvolume trajectory, but also a 3-flux density H_3^s *on* Σ [50, §3.3]. This is (on-shell) the notorious "self-dual" flux density whose accurate quantization (traditionally neglected) is our main concern here.

This second aspect is what we are concerned with for the purpose of modeling strongly-coupled quantum systems:

The (1+3)D worldvolume $M^{1,3}$ of a quantum material – or, for the intent of modeling anyons, the effectively $(1+2)D$-worldvolume $M^{1,2}$ of a sheet-like material (e.g. an atomic mono-layer akin to graphene) – is to be identified with a sub-quotient of the brane worldvolume, typically with a fixed locus (orbifold singularity) inside the base of a fibration (Kaluza-Klein reduction).

$$\begin{tikzcd}
& \underset{\textbf{worldvolume}}{\textbf{M5-brane}} & \underset{\textbf{spacetime}}{\textbf{ambient bulk}} \\
& \Sigma^{1,5} \arrow[r, "\phi"] \arrow[d, ->>] & Y^{1,10} \arrow[d, ->>, "\substack{\textbf{M/IIA-}\\ \textbf{fibration}}"] \\
M^{1,2} \arrow[r, hook, "\substack{\textbf{orbi-}\\ \textbf{singularity}}"] & \Sigma^{1,4} \arrow[r] & X^{1,9} \\
\underset{\textbf{worldvolume}}{\textbf{quantum material}} & \underset{\textbf{worldvolume}}{\textbf{D4-brane}} &
\end{tikzcd}$$

Their flux-quantization (to recall from [117]) is then encoded in a choice of a fibration $\mathcal{A} \xrightarrow{p} \mathcal{B}$ of classifying spaces, subject to the constraint that the Bianchi identities for the (duality-symmetric) flux densities on bulk and brane are the closure/flatness condition on $\mathfrak{l}p$-valued differential forms, where $\mathfrak{l}(-)$ forms *Whitehead L_∞-algebras* of these classifying fibrations (dual to their minimal relative Sullivan model).

Given such a choice, the topological sector of the higher gauge fields on bulk and brane are given by maps from the brane-immersion into the classifying fibration:

With these comments on perspective out of the way, **the plan of this section** are the following topics:

(1.) Bianchi identities on magnetized M5-probes

(2.) Flux quantization in Twistorial Cohomotopy

(3.) Aside: Projective Spaces and their Fibrations

(4.) Orbi-worldvolumes and Equivariant charges

$$\begin{tikzcd}
\textbf{brane} & \Sigma \arrow[r, dashed, "\substack{\textbf{densities of}\\ \textbf{brane fluxes}}"] \arrow[d, "\phi"', "\text{immersion}"] & \Omega^1_{\mathrm{dR}}(-; \mathfrak{l}_{\mathcal{B}}\mathcal{A})_{\mathrm{clsd}} \arrow[d, "(\mathfrak{l}p)_*"] & \Sigma \arrow[r, dashed, "\substack{\textbf{charges of}\\ \textbf{brane fields}}"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"', "\substack{\text{fibration of}\\ \text{classifying spaces}}"] \arrow[dl, Rightarrow, dashed, shorten <=8mm, shorten >=8mm] \\
\textbf{bulk} & X \arrow[r, dashed, "\substack{\textbf{densities of}\\ \textbf{bulk fluxes}}"'] & \Omega^1_{\mathrm{dR}}(-; \mathfrak{l}\mathcal{B})_{\mathrm{clsd}} & X \arrow[r, dashed, "\substack{\textbf{charges of}\\ \textbf{bulk fields}}"'] & \mathcal{B}
\end{tikzcd}$$

The first step of flux-quantization is to identify the Bianchi identities satisfied by the flux densities:

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[4] All brane concepts we consider are well-defined and all conclusions have proofs – at no point do we rely on informal string theory folklore beyond motivation.

Bianchi identities on M5-Probes of 11D SuGra via super-geometry. Consider the 11D super-tangent space

$$\underset{\text{super-Minkowski}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \hookrightarrow \underset{\text{super-Poincaré}}{\mathfrak{isom}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)} \twoheadrightarrow \underset{\text{Lorentz}}{\mathfrak{so}(1,10)}$$

with its super-invariant 1-forms (cf. [49, §2.1]):

$$\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \;\simeq\; \underset{\text{super-transl. invar. forms}}{\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{R}^{1,10|\mathbf{32}}\big)^{\mathrm{li}}} \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32}\\ (E^a)_{a=0}^{10}\end{array}\right]\Big/\left(\begin{array}{lcl}\mathrm{d}\,\Psi^\alpha &=& 0\\ \mathrm{d}\,E^a &=& \big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\end{array}\right).$$

Remarkably, the quartic Fierz identities entail that [23][90][49, Prop. 2.73]:

$$\left.\begin{array}{lcl} G_4^0 &:=& \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)\,E^{a_1}E^{a_2} \\ G_7^0 &:=& \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)\,E^{a_1}\cdots E^{a_5} \end{array}\right\} \in \underset{\text{fully super-invariant forms}}{\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)^{\mathrm{Spin}(1,10)}} \qquad \text{satisfy :} \qquad \begin{array}{lcl}\mathrm{d}\,G_4^0 &=& 0\\ \mathrm{d}\,G_7^0 &=& \tfrac{1}{2}G_4^0\,G_4^0\end{array}$$

To globalize this situation, say that an **11D super-spacetime** X is a super-manifold equipped with a super-Cartan connection, locally on an open cover $\widetilde{X} \twoheadrightarrow X$ given by

$$\left.\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32}\\ (E^a)_{a=0}^{10}\\ \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{10}\end{array}\right\} \in \Omega^1_{\mathrm{dR}}\big(\widetilde{X}\big) \qquad \begin{array}{l}\text{such that the}\\ \text{super-torsion}\\ \text{vanishes}\end{array} \qquad \mathrm{d}\,E^a - \Omega^a{}_b\,E^b \;=\; \big(\overline{\Psi}\,\Gamma^a\,\Psi\big),$$

and say that **C-field super-flux** on such a super-spacetime are super-forms with these co-frame components:

$$\boxed{\begin{array}{lclcl} G_4^s &:=& G_4 + G_4^0 &:=& \tfrac{1}{4!}(G_4)_{a_1\cdots a_4}E^{a_1}\cdots E^{a_4} + \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)E^{a_1}\,E^{a_2} \\ G_7^s &:=& G_7 + G_7^0 &:=& \tfrac{1}{7!}(G_4)_{a_1\cdots a_7}E^{a_1}\cdots E^{a_7} + \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)E^{a_1}\cdots E^{a_5} \end{array}}$$

Theorem [49, Thm. 3.1]: On an 11D super-spacetime X with C-field super-flux (G_4^s, G_7^s):

The duality-symmetric super-Bianchi identity $\left\{\begin{array}{lcl}\mathrm{d}\,G_4^s &=& 0\\ \mathrm{d}\,G_7^s &=& \tfrac{1}{2}\,G_4^s\,G_4^s\end{array}\right\}$ *is equivalent to* *the full 11D SuGra equations of motion!*

Next, on the super-subspace $\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+} \xhookrightarrow{\phi_0} \mathbb{R}^{1,10\,|\,\mathbf{32}}$ fixed by the involution $\Gamma_{012345} \in \mathrm{Pin}^+(1,10)$ we have:

$$H_3^0 \;:=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,H_3^0 \;=\; \phi_0^*\,G_4^0$$

To globalize this situation, say that a super-immersion $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow{\phi_s} X^{1,10\,|\,\mathbf{32}}$ is ½**BPS M5** if it is "locally like" ϕ_0, and say that **B-field super-flux** on such an M5-probe is a super-form with these co-frame components:

$$\boxed{H_3^s \;:=\; H_3 + H_3^0 \;:=\; \tfrac{1}{3!}(H_3)_{a_1a_2a_3}e^{a_1}\,e^{a_2}\,e^{a_3} + 0} \qquad \big(e^{a<6} := \phi_s^* E^a\big)$$

Theorem [50, §3.3]: On a super-immersion ϕ_s with B-field super-flux H_3^s:

The super-Bianchi identity $\big\{\mathrm{d}\,H_3^s \;=\; \phi_s^* G_4^s\big\}$ *is equivalent to* *the M5's B-field equations of motion.*

In particular, the (non-linear self-)duality conditions on the ordinary fluxes are *implied*: $G_4 \leftrightarrow G_7$ and $H_3 \leftrightarrow H_3$.

Seeing from this that also trivial tangent super-cochains may have non-trivial globalization, observe next that:

$$F_2^0 \;:=\; \big(\overline{\psi}\,\psi\big) \;=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,F_2^0 \;=\; 0$$

Globalizing this to $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+}$ via

$$\boxed{F_2^s \;:=\; F_2 + F_2^s \;:=\; \tfrac{1}{2}(F_2)_{a_1a_2}\,e^{a_1}e^{a2} + 0}$$

we have on top of the above:

Theorem [122, p 7]:

The super-Bianchi identity $\big\{\mathrm{d}\,F_2^s \;=\; 0\big\}$ *is equivalent to* *the Chern-Simons E.O.M.:* $F_2 \;=\; 0.$

Flux quantization in Twistorial Cohomotopy. In summary, a remarkable kind of higher super-Cartan geometry locally modeled on the 11D super-Minkowski spacetime $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ entails that on-shell 11D supergravity probed by magnetized 1/2BPS M5-branes implies and is entirely governed by these Bianchi identities on super-flux densities:

$$\begin{array}{cllc}
\textbf{A-field} & \mathrm{d}\,F_2^s \;=\; 0 & \mathrm{d}\,G_4^s \;=\; 0 & \textbf{C-field} \\
\begin{array}{c}\textbf{self-dual}\\ \textbf{B-field}\end{array} & \mathrm{d}\,H_3^s \;=\; \phi_s^* G_4^s \,+\, \theta\, F_2^s\, F_2^s & \mathrm{d}\,G_7^s \;=\; \tfrac{1}{2} G_4^s\, G_4^s & \begin{array}{c}\textbf{dual}\\ \textbf{C-field}\end{array} \\
\textbf{M5 probe} & \Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow[\textbf{1/2BPS immersion}]{\phi_s} X^{1,10\,|\,\mathbf{32}} & & \textbf{SuGra bulk}
\end{array} \tag{2}$$

Here we have observed that the Green-Schwarz term $F_2^s F_2^s$ may equivalently be included for any theta-angle $\theta \in \mathbb{R}$ without affecting the equations of motion (since, recall, the CS e.o.m. $F_2^s = 0$ is already implied by $\mathrm{d}\,F_2^s \;=\; 0$).

But non-vanishing theta-angle does affect the admissible flux-quantization laws and hence the global solitonic and torsion charges of the fields. The choice of flux quantization according to *Hypothesis H* [37][39] is the following:

Admissible fibrations of classifying spaces for cohomology theories with the above character images (2). The homotopy quotient of S^7 is (i) for $\theta = 0$ by the trivial action and (ii) for $\theta \neq 0$ by the principal action of the complex Hopf fibration.

$$\begin{tikzcd}
\boxed{\theta = 0} & S^7 /\!\!/_0\, \mathrm{U}(1) \simeq S^7 \times \mathbb{C}P^\infty \arrow[r, two heads] \arrow[d, two heads] & S^7 \arrow[r, two heads, "h_{\mathbb{H}}", "\textbf{$\mathbb{H}$-Hopf fibration}"'] \arrow[d, two heads, "\textbf{$\mathbb{C}$-Hopf fibration}" description] & \mathbb{H}P^1 \arrow[d, equal] \\
\boxed{\theta \neq 0} & S^7 /\!\!/ \mathrm{U}(1) \arrow[r, "\sim"] & \mathbb{C}P^3 \arrow[r, two heads, "t_{\mathbb{H}}", "\textbf{Twistor fibration}"'] & \mathbb{H}P^1
\end{tikzcd}$$

Proof. This may be seen as follows [39, Lem. 2.13]:

Since the real cohomology of projective space is a truncated polynomial algebra,

$$\begin{array}{ll}
H^\bullet\big(\mathbb{C}P^n;\, \mathbb{R}\big) \;\simeq\; \mathbb{R}\big[\overbrace{\,c_1\,}^{\mathrm{deg}=2}\big]/\big(c_1^{n+1}\big) & H^\bullet\big(\overbrace{\mathbb{C}P^\infty}^{\simeq B\mathrm{U}(1)};\, \mathbb{R}\big) \;\simeq\; \mathbb{R}[c_1] \\
H^\bullet\big(\mathbb{H}P^n;\, \mathbb{R}\big) \;\simeq\; \mathbb{R}\big[\underbrace{\tfrac{1}{2}p_1}_{\mathrm{deg}=4}\big]/\big(p_1^{n+1}\big) & H^\bullet\big(\underbrace{\mathbb{H}P^\infty}_{\substack{\simeq B\mathrm{Sp}(1) \simeq B\mathrm{SU}(2) \\ \simeq B\mathrm{Spin}(3)}};\, \mathbb{R}\big) \;\simeq\; \mathbb{R}[\tfrac{1}{2}p_1]
\end{array}$$

the minimal dgc-algebra model for $\mathbb{C}P^n$ needs a closed generator f_2 to span the cohomology and a generator h_{2n+1} in order to truncate it; analogously for $\mathbb{H}P^n$.

$$\mathrm{CE}\big(\mathfrak{l}\,\mathbb{C}P^n\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} f_2 \\ h_{2n+1} \end{array}\right] \Big/ \left(\begin{array}{lcl} \mathrm{d}\,f_2 & = & 0 \\ \mathrm{d}\,h_{2n+1} & = & (f_2)^{n+1} \end{array}\right)$$

$$\mathrm{CE}\big(\mathfrak{l}\,\mathbb{H}P^n\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} g_4 \\ g_{4n+3} \end{array}\right] \Big/ \left(\begin{array}{lcl} \mathrm{d}\,g_4 & = & 0 \\ \mathrm{d}\,g_{4n+3} & = & (g_4)^{n+1} \end{array}\right)$$

Furthermore, since the second Chern class of an $S\big(\mathrm{U}(1)^2\big)$-bundle is minus the cup square of the first Chern class (by the Whitney sum rule)

$$\begin{array}{ccc}
B\mathrm{U}(1) & \xrightarrow{B\left(c \mapsto \mathrm{diag}(c, c^*)\right)} & B\mathrm{SU}(2) \\
-(c_1)^2 & \longleftarrow\!\shortmid & \tfrac{1}{2}p_1 = c_2
\end{array}$$

the minimal model of $\mathbb{C}P^3$ *relative* to that of $\mathbb{H}P^1$ needs to adjoin to the latter not only f_2 but also a generator h_3 imposing this relation in cohomology.

$$\mathrm{CE}\big(\mathfrak{l}_{{}_{\mathbb{H}P^1}}\mathbb{C}P^3\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} f_2 \\ h_3 \\ g_4 \\ g_7 \end{array}\right] \Big/ \left(\begin{array}{lcl} \mathrm{d}\,f_2 & = & 0 \\ \mathrm{d}\,h_3 & = & g_4 + f_2\, f_2 \\ \mathrm{d}\,g_4 & = & 0 \\ \mathrm{d}\,g_7 & = & \tfrac{1}{2} g_4\, g_4 \end{array}\right)$$

The resulting fibration of L_∞-algebras is manifestly just that classifying the desired Bianchi identities (2) (we are showing the case $\theta \neq 0$, which by isomorphic rescaling may be taken to be $\theta = 1$):

$$\begin{tikzcd}
\Sigma^6 \arrow[r, dashed] \arrow[d, "\phi"'] & \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}_{{}_{\mathbb{H}P^1}}\mathbb{C}P^3\big)_{\mathrm{clsd}} \arrow[d, "(\mathfrak{l}\, t_{\mathbb{H}})_*"] & \Leftrightarrow & \Omega^\bullet_{\mathrm{dR}}\big(\Sigma^6\big) & \mathrm{CE}\big(\mathfrak{l}_{{}_{\mathbb{H}P^1}}\mathbb{C}P^3\big) \arrow[l, dashed] \arrow[d, "(\mathfrak{l}\, t_{\mathbb{H}})^*"] & \Leftrightarrow \\
\Sigma^{11} \arrow[r, dashed] & \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}\,\mathbb{H}P^1\big)_{\mathrm{clsd}} & & \Omega^\bullet_{\mathrm{dR}}\big(X^{11}\big) \arrow[u, "\phi^*"] & \mathrm{CE}\big(\mathfrak{l}\,\mathbb{H}P^1\big) \arrow[l, dashed] &
\end{tikzcd}$$

$$\begin{array}{c} F_2 \\ H_3 \end{array} \in \Omega^\bullet_{\mathrm{dR}}(\Sigma^6) \left| \begin{array}{lcl} \mathrm{d}\,F_2 & = & 0 \\ \mathrm{d}\,H_3 & = & G_4 + F_2\, F_2 \end{array} \right.$$

$$\begin{array}{c} G_4 \\ G_7 \end{array} \in \Omega^\bullet_{\mathrm{dR}}\big(X^{11}\big) \left| \begin{array}{lcl} \mathrm{d}\,G_4 & = & 0 \\ \mathrm{d}\,G_7 & = & \tfrac{1}{2} G_4\, G_4 \end{array} \right.$$

Aside: Projective Spaces and their Fibrations – Here we used the following classical facts. Consider:

division algebras $\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$ generically denoted $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

groups of units $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ understood with the multiplicative group structure

projective spaces $\mathbb{K}P^n := \big(\mathbb{K}^{n+1} \setminus \{0\}\big)/\mathbb{K}^\times$

higher spheres $S^n \simeq \big(\mathbb{R}^{n+1} \setminus \{0\}\big)/\mathbb{R}_{>0}$

$\mathbb{K}$-Hopf fibrations are the quotient co-projections induced by $\iota : \mathbb{R}_{>0} \hookrightarrow \mathbb{K}$

The classical Hopf fibrations $h_{\mathbb{K}}$ are:

$$\begin{tikzcd}
S^0 \simeq \mathbb{R}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\ker"] & \\
S^1 \arrow[d, two heads, "h_{\mathbb{R}}"] \simeq \big(\mathbb{R}^2\backslash\{0\}\big)/\mathbb{R}_{>0} & \arrow[d, two heads, "\iota_*"] \\
S^1 \simeq \underbrace{\big(\mathbb{R}^2\backslash\{0\}\big)/\mathbb{R}^\times}_{\mathbb{R}P^1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
S^1 \simeq \mathbb{C}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\ker"] & \\
S^3 \arrow[d, two heads, "h_{\mathbb{C}}"] \simeq \big(\mathbb{C}^2\backslash\{0\}\big)/\mathbb{R}_{>0} & \arrow[d, two heads, "\iota_*"] \\
S^2 \simeq \underbrace{\big(\mathbb{C}^2\backslash\{0\}\big)/\mathbb{C}^\times}_{\mathbb{C}P^1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
S^3 \simeq \mathbb{H}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\ker"] & \\
S^7 \arrow[d, two heads, "h_{\mathbb{H}}"] \simeq \big(\mathbb{H}^2\backslash\{0\}\big)/\mathbb{R}_{>0} & \arrow[d, two heads, "\iota_*"] \\
S^4 \simeq \underbrace{\big(\mathbb{H}^2\backslash\{0\}\big)/\mathbb{H}^\times}_{\mathbb{H}P^1} &
\end{tikzcd}$$

The Hopf fibrations in higher dimensions are the attaching maps exhibiting the topological cell-complex structure of projective spaces [94], from which the (cellular) cohomology follows readily.

$$\begin{tikzcd}
S\big(\mathbb{K}^{n+1}\big) \arrow[r] \arrow[d, "h_{\mathbb{K}}"'] & * \arrow[d] \arrow[dl, Rightarrow, "(\mathrm{po})"] \\
\mathbb{K}P^n \arrow[r, hook] & \mathbb{K}P^{n+1}
\end{tikzcd}$$

Further factor-fibrations arise by factoring the Hopf fibrations via the stage-wise quotienting along

$$\mathbb{R}_{>0} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}.$$

Notably, the classical quaternionic Hopf fibration $h_{\mathbb{H}}$ factors through a higher-dimensional complex Hopf fibration followed by the **Calabi-Penrose twistor fibration** $t_{\mathbb{H}}$ [39, §2].

$$\begin{tikzcd}
S^1 \simeq \mathbb{C}^\times/\mathbb{R}_{>0} \arrow[dr] & & \\
& S^7 \simeq \big(\mathbb{H}^2\backslash\{0\}\big)/\mathbb{R}_{>0} \arrow[dd, "h_{\mathbb{C}}"] \arrow[ddd, bend right=40, "h_{\mathbb{H}}"'] & \\
S^2 \simeq \mathbb{H}^\times/\mathbb{C}^\times \arrow[dr] & & \textbf{complex Hopf fibration} \\
& \mathbb{C}P^3 \simeq \big(\mathbb{H}^2\backslash\{0\}\big)/\mathbb{C}^\times \arrow[d, "t_{\mathbb{H}}"] & \\
\textbf{quaternionic Hopf fibration} & \mathbb{H}P^1 \simeq \big(\mathbb{H}^2\backslash\{0\}\big)/\mathbb{H}^\times & \textbf{Calabi-Penrose twistor fibration}
\end{tikzcd}$$

Equivariantization: Since the quotienting is by right actions, these fibrations are equivariant under the left action of

$$\mathrm{Spin}(5) \simeq \mathrm{Sp}(2) := \big\{ g \in \mathrm{GL}_2(\mathbb{H}) \,\big|\, g^\dagger \cdot g = \mathrm{e} \big\}\,.$$

For example, the involution $\sigma := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Sp}(2)$ swaps the two copies of $\mathbb{H}$:

$$\begin{tikzcd}
\mathbb{C}P^3 \arrow[rrr, "t_{\mathbb{H}}"] \arrow[ddd, "\sigma"] \arrow[dr, equal] & & & \mathbb{H}P^1 \arrow[ddd, "\sigma"] \\
& \big(\mathbb{H}\times\mathbb{H}\backslash\{0\}\big)/\mathbb{C}^\times \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \big(\mathbb{H}\times\mathbb{H}\backslash\{0\}\big)/\mathbb{H}^\times \arrow[ur, equal] \arrow[d, shift left] \arrow[d, shift right] & \\
& \big(\mathbb{H}\oplus\mathbb{H}\backslash\{0\}\big)/\mathbb{C}^\times \arrow[r] & \big(\mathbb{H}\oplus\mathbb{H}\backslash\{0\}\big)/\mathbb{H}^\times \arrow[dr, equal] & \\
\mathbb{C}P^3 \arrow[ur, equal] \arrow[rrr, "t_{\mathbb{H}}"'] & & & \mathbb{H}P^1
\end{tikzcd}$$

The resulting $\mathbb{Z}_2$-fixed locus is the 2-sphere:

$$\begin{tikzcd}
\big(\mathbb{C}P^3\big)^{\mathbb{Z}_2} \arrow[d, "(t_{\mathbb{H}})^{\mathbb{Z}_2}"] & \simeq & \big(\mathbb{H}\backslash\{0\}\big)/\mathbb{C}^\times \arrow[d] & \simeq & S^2 \arrow[d] \\
\big(\mathbb{H}P^1\big)^{\mathbb{Z}_2} & \simeq & \big(\mathbb{H}\backslash\{0\}\big)/\mathbb{H}^\times & \simeq & *
\end{tikzcd}$$

This is the 2-sphere coefficient that will end up being responsible for stabilizing anyons on orbi-worldvolumes! We next discuss how this comes about.

Aside: Implications of Hypothesis H, in view of traditional expectations for M-theory.

The plain Hypothesis H in the bulk says that the non-perturbative completion of the C-field in 11d supergravity involves a map χ from spacetime to the homotopy type of the 4-sphere, with the C-field gauge potentials $(\widehat{C}_3, \widehat{C}_6)$ exhibiting the flux densities (G_4, G_7) as $\mathbb{R}$-rational representatives of χ.

$$\begin{tikzcd}
& \mathrm{Maps}\big(X;\, S^4\big) \arrow[d, "\mathbf{ch}"] & \chi \arrow[d, mapsto] & \text{Cohomotopical charge sector} \\
\Omega^1_{\mathrm{dR}}\big(X;\, \mathfrak{l}S^4\big)_{\mathrm{clsd}} \arrow[r, "\eta^{\int}"] & \int\Omega^1_{\mathrm{dR}}\big(X;\, \mathfrak{l}S^4\big)_{\mathrm{clsd}} & \mathbf{ch}(\chi) & \text{character image} \\
\big(G_4,\, G_7\big) \arrow[r, mapsto] & \eta^{\int}(G_4, G_7) \arrow[ur, Rightarrow, bend left, "(\widehat{C}_3, \widehat{C}_6)"', "\text{gauge potentials}"] & & \\
\text{C-field flux densities} & & &
\end{tikzcd}$$

In other words, this is the postulate that the non-perturbative C-field is a cocycle in canonical, unstable *differential 4-Cohomotopy* $\widehat{\pi}^4$ [34, §4][52, §3.1][40, Ex. 9.3].

$$\begin{tikzcd}
& & \text{canonical differential non-abelian (unstable) 4-Cohomotopy} & \text{plain non-abelian (unstable) 4-Cohomotopy} \\
\big(\widehat{C}_3,\, \widehat{C}_6\big) \arrow[r, phantom, "\in"] & & \widehat{\pi}^4(X) \arrow[r, "\chi", "\text{topological sector}"'] \arrow[d, "(G_4, G_7)"', "\text{flux densities}"] & \pi^4(X) \\
\text{full nonperturbative 11d SuGra C-field} & & H_{\mathrm{dR}}\big(X;\, \mathfrak{l}S^4\big) & \\
& & \mathfrak{l}S^4\text{-valued de Rham cohomology} &
\end{tikzcd}$$

As an immediate plausibility check: This implies, from the well-known homotopy groups of spheres in low degrees, that:

integral quantization of charges carried by singular M5-brane branes *and*

$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{5,1}\big) \;=\; \pi^4\big(\mathbb{R}^{5,1} \times \mathbb{R}_+ \times S^4\big) \;=\; \pi^4(S^4) \;=\; \pi_4(S^4) \;=\; \mathbb{Z}$$

integral quantization of charges carried by singular M2-branes... plus a torsion-contribution (a first prediction of Hypothesis H).

$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{2,1}\big) \;=\; \pi^4\big(\mathbb{R}^{2,1} \times \mathbb{R}_+ \times S^7\big) \;=\; \pi^4(S^7) \;=\; \pi_7(S^4) \;=\; \mathbb{Z} \oplus \mathbb{Z}_{12}$$

Hypothesis H with curvature corrections. More generally, the curvature corrections from the coupling to the background gravity are postulated to be reflected in *tangentially twisted* 4-Cohomotopy [37], analogous to the well-known twisting of the RR-field flux-quantization in K-theory by its background B-field:

Hypothesis K

$$\begin{tikzcd}
& & \mathrm{KU}_0 /\!\!/ \mathrm{PU}(H) \arrow[d, "\text{twisted K-theory}"] \\
X^9 \arrow[urr, dashed, "\text{RR-field}"] \arrow[rr, dashed] \arrow[dr, dashed, "\text{twist by background B-field}"'] & & B\mathrm{PU}(H) \arrow[dl, "\sim"'] \\
& B^2\mathrm{U}(1) &
\end{tikzcd}$$

Hypothesis H

$$\begin{tikzcd}
& & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[d, "\text{twisted Cohomotopy}"] \\
T^2 \times X^8 \arrow[urr, dashed, "\text{C-field}"] \arrow[rr, dashed, "\text{Fivebrane structure}"] \arrow[dr, dashed, "\text{twist by background gravity}"'] & & B\widehat{\mathrm{Sp}(2)} \arrow[dl, dashed] \\
& B\mathrm{Spin}(8) &
\end{tikzcd}$$

To distinguish M2/M5-charge, the tangential twisting needs to preserve the $\mathbb{H}$-Hopf fibration ⇒ tangential $\mathrm{Sp}(2) \hookrightarrow \mathrm{Spin}(8)$-structure [37, §2.3]. With this, integrality of M2's Page charge & anomaly-cancellation of the M5's Hopf-WZ term follows from trivialization of the Euler 8-class, which means lift to the *Fivebrane* 6-group $\widehat{\mathrm{Sp}(2)} \to \mathrm{Sp}(2)$ [36, §4].

This implies [37, Prop. 3.13][36, Thm. 4.8]:

(i) half-integrally shifted quantization of M5-brane charge in curved backgrounds, *and*

(ii) integral quantization of the Page charge of M2-branes.

$$[\widetilde{G}_4] \;:=\; \underbrace{[G_4]}_{\text{C-field 4-flux}} + \tfrac{1}{2}\underbrace{\big(\tfrac{1}{2}p_1(TX^8)\big)}_{\text{integral Spin-Pontrjagin class}} \;\in\; H^4\big(X^8;\, \mathbb{Z}\big)$$

$$2[\widetilde{G}_7] \;:=\; 2\big([G_7] + \tfrac{1}{2}[H_3 \wedge \widetilde{G}_4]\big) \;\in\; H^7(\widehat{X}^8; \mathbb{Z})$$

Both of these quantization conditions on M-brane charge are thought to be crucial for M-theory to make any sense.

Previously, item (i) had remained enigmatic and item (ii) had remained wide open.

But there is more:

Provable implications from Hypothesis H of subtle effects expected in M-theory:

- half-integral shift of 4-flux	[37, Prop. 3.13]
- DMW anomaly cancellation	[37, Prop. 3.7]
- the C-field's "integral EoM"	[37, §3.6]
- M2 Page charge quantization	[36, Thm. 4.8]
- integrality of $\frac{1}{6}(G_4)^3$	[52, Rem. 2.9]
- M5-brane anomaly cancellation	[112]
- non-abelian gerbe field on M5	[38]

It is these results which suggest that Hypothesis H goes towards the correct flux-quantization law for the C-field in M-theory.

Orbi-worldvolumes and Equivariant charges. Flux-quantization generalizes to *orbifolds* [5] by generalizing the cohomology of the charges to *equivariant cohomology* [111].

In terms of classifying spaces this simply means that all spaces are now equipped with the action of a finite group G and all maps are required to be G-equivariant.
We take $G := \mathbb{Z}_2$ and the classifying fibration to be the **twistor fibration** $p := t_{\mathbb{H}}$ equivariant under swapping the $\mathbb{H}$-summands,

$$\begin{tikzcd}
\text{orbi-worldvolume} & \overset{G \circlearrowright}{\Sigma} \arrow[r, dashed, "\text{orbi-brane charges}"] \arrow[d, "\phi"'] & \overset{G \circlearrowright}{\mathcal{A}:} \arrow[r, equal] \arrow[d, "p"'] \arrow[dl, Rightarrow, dashed] \arrow[d, phantom, "\text{equivariant classifying fibration...}" near start, shift left=12ex] & \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{C}P^3} \arrow[d, "t_{\mathbb{H}}"'] & \text{...for equivariant twistorial Cohomotopy} \\
\text{orbi-spacetime} & \underset{G \circlearrowleft}{X} \arrow[r, dashed, "\text{orbi-bulk charges}"'] & \underset{G \circlearrowleft}{\mathcal{B}:} \arrow[r, equal] & \underset{\mathbb{Z}_2 \circlearrowleft}{S^4} &
\end{tikzcd}$$

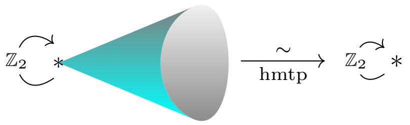
and the brane/bulk orbifold we take to be as on p. 3:

The orbi-brane diagram for a flat M5-brane wrapped on a trivial Seifert-fibered orbi-singularity. Shaded is the $\mathbb{Z}_2$-fixed locus/orbi-singularity.

We are adjoining the *point at infinity* to the space $\mathbb{R}^2_{\cup\{\infty\}} \underset{\text{homeo}}{\simeq} S^2$ which is thereby designated as transverse to any worldvolume solitons to be measured in reduced cohomology.

$$\begin{tikzcd}[column sep=small]
\overset{\mathbb{Z}_2 \circlearrowright}{\Sigma} \arrow[d, "\phi"'] & := & \mathbb{R}^{1,0} & \times & \mathbb{R}^2_{\cup\{\infty\}} & \times & S^1 & \times & \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{R}^2_{\mathrm{sgn}}} & & \\
\underset{\mathbb{Z}_2 \circlearrowleft}{X} & := & \mathbb{R}^{1,0} & \times & \mathbb{R}^2_{\cup\{\infty\}} & \times & S^1 & \times & \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{R}^2_{\mathrm{sgn}}} & \times & \mathbb{R}^5 \\
 & & \text{time} & & \text{trnsvrs space to solitons} & & \text{M/IIA-circle} & & \text{orbi-cone} & & \text{trnsvrs space to M5-brane}
\end{tikzcd}$$

But since the cone $\mathbb{Z}_2 \circlearrowright \mathbb{R}^2_{\mathrm{sgn}}$ is equivariantly contractible,



the inclusion of the $\mathbb{Z}_2$-fixed loci is actually a homotopy equivalence

$$\begin{tikzcd}
\Sigma^{\mathbb{Z}_2} \arrow[r, "\sim", "\text{hmtp}"'] \arrow[d, "\phi^{\mathbb{Z}_2}"'] & \overset{\mathbb{Z}_2 \circlearrowright}{\Sigma} \arrow[d, "\phi"] \\
X^{\mathbb{Z}_2} \arrow[r, "\sim", "\text{hmtp}"'] & \underset{G \circlearrowleft}{X}
\end{tikzcd}$$

Therefore our equivariant classifying maps are determined up to equivariant homotopy by their restriction to the fixed-locus and hence the charges are *localized on the orbi-singularity* where they take values in 2-Cohomotopy:

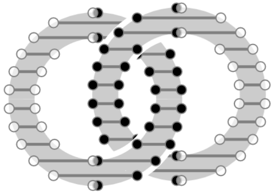
$$\left\{ \begin{tikzcd}
\overset{\mathbb{Z}_2 \circlearrowright}{\Sigma} \arrow[r, dashed] \arrow[d, "\phi"'] & \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{C}P^3} \arrow[d, "t_{\mathbb{H}}"] \arrow[dl, Rightarrow, dashed] \\
\underset{\mathbb{Z}_2 \circlearrowleft}{X} \arrow[r, dashed, "\text{charges on orbifold}"'] & \underset{\mathbb{Z}_2 \circlearrowleft}{S^4}
\end{tikzcd} \right\}
\simeq
\left\{ \begin{tikzcd}
\Sigma^{\mathbb{Z}_2} \arrow[r, dashed] \arrow[d, "\phi^{\mathbb{Z}_2}"'] & (\mathbb{C}P^3)^{\mathbb{Z}_2} \arrow[r, equal] \arrow[d, "t_{\mathbb{H}}^{\mathbb{Z}_2}"] \arrow[dl, Rightarrow, dashed] & S^2 \arrow[d] \\
X^{\mathbb{Z}_2} \arrow[r, dashed] & (S^4)^{\mathbb{Z}_2} \arrow[r, equal] & * \\
 & \text{charges localized on orbi-singularity} &
\end{tikzcd} \right\}
\simeq
\left\{ \begin{tikzcd}
\mathbb{R}^2_{\cup\{\infty\}} \times S^1 \arrow[r, dashed] & S^2 \\
\text{charges in 2-Cohomotopy of B-field solitons on M5 orbi-singularity} &
\end{tikzcd} \right\}$$

Moduli space of worldvolume solitons. To be precise, the solitonic charges are to be measured in the *reduced* 2-Cohomotopy classified by *pointed maps*, enforcing the condition that solitonic fields *vanish at infinity* [117, §2.2].

In the strongly-coupled situation, where the M/IIA circle de-compactifies to $\mathbb{R}^1$, the vanishing-at-infinity must also be applied here, whence the moduli space of topological solitons is the loop space of the reduced 2-Cohomotopy moduli of the transverse space:

$$\underset{\text{on M5 orbi-singularity}}{\overset{\text{moduli space of solitons}}{}} \quad \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}} \wedge S^1,\, S^2\big) \;\simeq\; \Omega\, \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \quad \text{loop space of moduli space of solitons on D4 orbi-singularity}$$

Outlook. Strinkingly, as we explain next, this is equivalently a space of *worldsheets of strings* in $\mathbb{R}^3$ with unit *charged endpoints* forming oriented *framed links*! [119]



Such link diagrams are just the envisioned topological quantum circuit protocols, and their framing regularizes the anyonic phase observables ("Wilson loop observables").

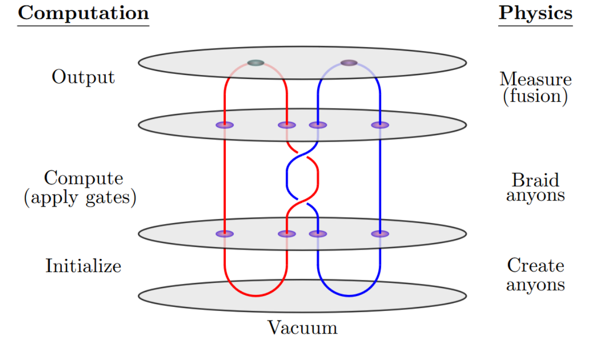


FIGURE 1. A single iteration of a topological quantum computation via braiding.

Figure from Rowell ([109], following [108, Fig. 2]).

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[5] For brevity we consider here only "very good" orbifolds, namely global quotients of manifolds by the action of a finite group G. This is sufficient for the present purpose and anyways the case understood by default in the string theory literature.

3 Cohomotopy Charge of Solitons

Remarkably, there is an equivalence between *Cohomotopy* of spacetime/worldvolumes and *Cobordism* classes of submanifolds behaving like solitonic branes carrying the corresponding Cohomotopy charge [113, §2.2] [110, §2.1]:

The **Pontrjagin theorem** [101][75, §IX] identifies the unstable n-Cohomotopy of a closed manifold with the cobordism classes of its normally framed submanifolds of co-dimension n.

framed unstable n-Cobordism of M^d

$$\begin{tikzcd}
\mathrm{Cob}^n_{\mathrm{Fr}}(M^d) \arrow[r, phantom, ":="] \arrow[rrr, bend left, "\text{assign Cohomotopy charge}"] & \mathrm{NFramedSubmflds}_{d-n}(M^d)_{/\mathrm{brdsm}} \arrow[r, shift left, "\text{directed asymptotic distance}"] \arrow[r, phantom, "\simeq"] & \mathrm{Maps}(M^d, S^n)_{/\mathrm{hmtpy}} \arrow[l, shift left, "\text{pre-image of regular value 0}"] \arrow[r, phantom, "=:"] & \pi^n(M^d). \arrow[lll, bend left, "\text{reconstruct submanifold from its charge}"]
\end{tikzcd}$$

unstable n-Cohomotopy of M^d

closed manifold

The **Cohomotopy charge** of a normally framed submanifold (aka *scanning map* or *Pontrjagin-Thom collapse*) is represented by mapping points of the ambient space to their directed distance if inside a tubular neighbourhood, else to ∞.

Conversely, every Cohomotopy class is represented by a smooth map with 0 a regular value, whose pre-image is a normally framed submanifold with that Cohomotopy charge.

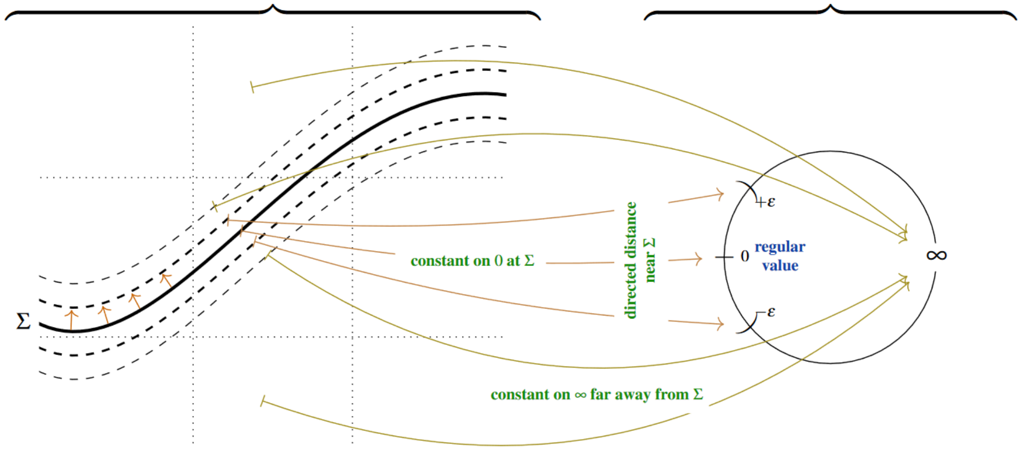
$$\begin{tikzcd}
\Sigma^{d-n} \arrow[r, hook] & M^d \arrow[r, "c"] & \mathbb{R}^n_{\mathrm{cpt}} = S^n
\end{tikzcd}$$

closed submanifold, normally framed

manifold

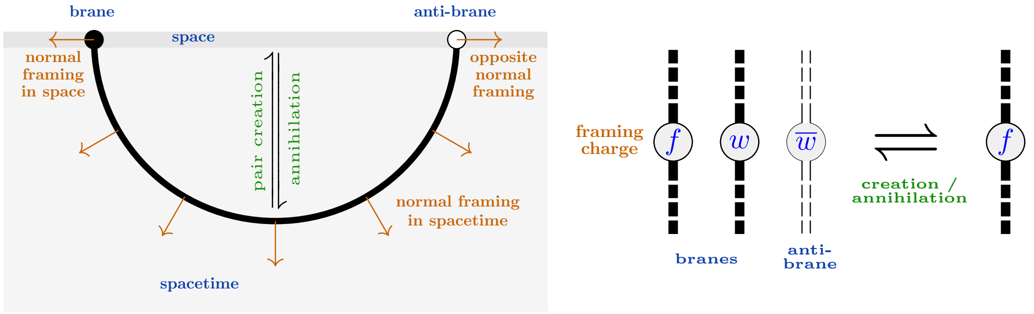
directed asymptotic distance from Σ $\simeq$ cocycle representing Cohomotopy charge of Σ

Cohomotopy classifying space (n-sphere)



Under this relation, homotopy of charge maps corresponds to nrml. framed **cobordism** of submnflds.

The cobordism relation exhibits a form of pair creation/annihilation of submanifolds carrying opposite Cohomotopy charges.



When making more ambient dimensions available, the cobordism classes eventually (quickly) exhibit **stabilization** on abelian cobordism cohomology groups. (This might relate *Hypothesis H* to Vafa's *cobordism conjecture* cf. [113, §4]).

make more ambient spatial dimensions available to bordisms ⟶

$$\begin{tikzcd}
\Sigma^{d-n} \subset M^d \arrow[r, hook, "(\mathrm{id},0)"] & M^d \times \mathbb{R}^1 \arrow[r, hook, "(\mathrm{id},0)"] & M^d \times \mathbb{R}^2 \arrow[r, hook, dotted] & M^d \times \mathbb{R}^\infty \\
\mathrm{Cob}^n_{\mathrm{Fr}}(M^d) \arrow[r, "\sigma"] \arrow[d, dashed, "\text{Cohomotopy charge map}"'] & \mathrm{Cob}^{n+1}_{\mathrm{Fr}}(M^d \times \mathbb{R}^1) \arrow[r, "\sigma"] \arrow[d, "\simeq"] & \mathrm{Cob}^{n+2}_{\mathrm{Fr}}(M^d \times \mathbb{R}^2) \arrow[r, dotted] \arrow[d, "\simeq"] & \widetilde{\mathrm{MFr}}^n(M^d_{\mathrm{cpt}}) \arrow[d, equal, "\text{stable Pontrjagin-Thom isom.}"] \\
\widetilde{\pi}^n(M^d_{\mathrm{cpt}}) \arrow[r, "(-)\wedge S^1"] & \widetilde{\pi}^{n+1}((M^d \times \mathbb{R}^1)_{\mathrm{cpt}}) \arrow[r, "(-)\wedge S^1"] & \widetilde{\pi}^{n+1}((M^d \times \mathbb{R}^2)_{\mathrm{cpt}}) \arrow[r, dotted] & \widetilde{\mathbb{S}}^n(M^d_{\mathrm{cpt}})
\end{tikzcd}$$

unstable framed Cobordism

stable framed Cobordism

unstable Cohomotopy

stable Cohomotopy

suspension homomorphism ⟶

This "linearized" Cohomotopy/Cobordism is a **form of K-theory**: algebraic K-theory over the "absolute base field $\mathbb{F}_1$" (cf. [17, Thm. 5.9][8, Cor. 2.25]).

non-abelian Cohomotopy

stable Cohomotopy

stable framed Cobordism

$$\begin{tikzcd}
\pi^\bullet \arrow[r, "\text{linearize}", "\text{(i.e.: stabilize)}"'] & \mathbb{S}^\bullet \arrow[rr, equal, "\text{Pontrjagin \& Thom}"] \arrow[dr, equal, "\text{Barratt-Priddy \& Quillen}"'] & & \mathrm{MFr}^\bullet \arrow[dl, equal] \\
& & K\mathbb{F}_1^\bullet &
\end{tikzcd}$$

algebraic K-theory of "field with one element"

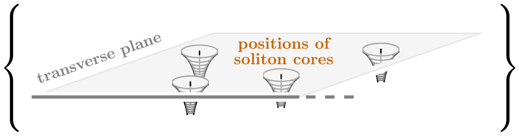
Thus flux quantization in Cohomotopy lifts to M-theory the same arguments that motivated topological K-theory in type II string theory: its character map reproduces the Bianchi identities & its equivalence relation models (anti-)brane pair-creation/annihilation.

Moduli space of soliton configurations. But the Pontrjagin theorem concerns only the total cohomotopical charge, identifying it with the *net* (anti-)brane content. Beyond that we have the whole *moduli space* of charges (considered now specialized to our 2D transverse space), and **Segal's theorem** [125] says that the cohomotopy charge map (scanning map) identifies this with a moduli space of brane positions, namely with the *group-completed configuration space of points* [19][138][70]:

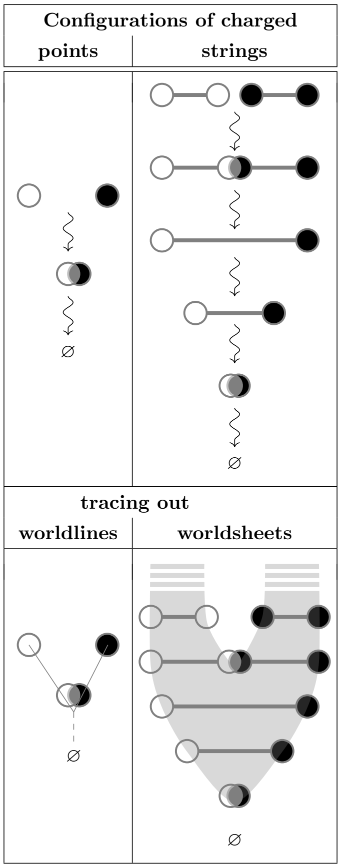
$$\begin{tikzcd}
\text{moduli space of solitonic brane charges} & \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \arrow[r, "{[-]}", "\text{net charge}"'] & \overset{\text{2-Cohomotopy}}{\pi^2\big(\mathbb{R}^2_{\cup\{\infty\}}\big)} \arrow[dr, "\sim", "\text{Hopf degree theorem}"'] & \\
& & & \mathbb{Z} \\
\text{moduli space of solitonic brane positions} & \underset{\text{group-completed}\quad\text{config space}}{\mathbb{G}\,\mathrm{Conf}(\mathbb{R}^2)} \arrow[uu, "\wr"', "\text{Segal theorem}"] \arrow[r, "{[-]}", "\text{net brane content}"'] & \underset{\text{2-Cobordism (unstable)}}{\mathrm{Cob}^2_{\mathrm{Fr}}(\mathbb{R}^2)} \arrow[uu, "\wr"', "\text{Pontrjagin theorem}"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

where the *configuration space of points* is the space of finite subsets of $\mathbb{R}^2$ – here understood as the space of positions of cores of solitons of unit charge $+1$,

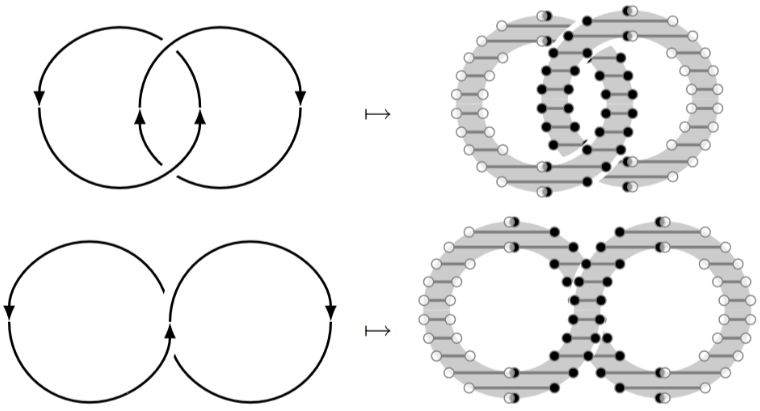
$\mathrm{Conf}(\mathbb{R}^2) \;=$



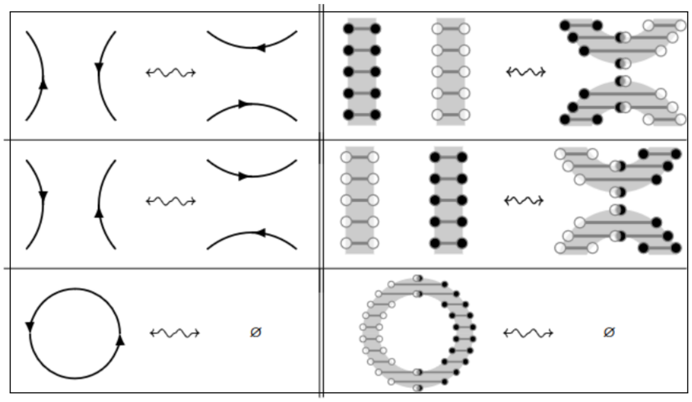
and its *group completion* $\mathbb{G}(-)$ is the topological completion of the topological partial monoid structure given by disjoint union of soliton configurations.
Naïvely this is given by including also **anti-solitons** in the form of configurations of ±*-charged points*, topologized such as to allow for their pair annihilation/creation as shown in the left column on the right.
Remarkably, closer analysis reveals [95] that the group completion $\mathbb{G}(-)$ produces configurations of **strings** (extending parallel to one axis in $\mathbb{R}^3$) **with charged endpoints** whose pair annihilation/creation is smeared-out to string worldsheets as shown in the right column.

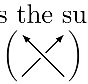
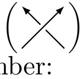


This means [119] that the **vacuum-to-vacuum soliton scattering processes**, forming the loop space $\Omega\,\mathbb{G}\,\mathrm{Conf}(\mathbb{R}^2)$, are identified with *framed links* ([96, pp 15]), for instance



subject to *link cobordism* (cf. [83]):



It follows [119, Thm 3.17] that the charge of a soliton scattering process L is the sum over crossings of the *crossing number* $\#$  $= +1$, $\#$  $= -1$, which equals the linking+framing number:

$$\Omega\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \xrightarrow{\sim} \Omega\mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}} S^2\big) \xrightarrow{[-]} \pi_3(S^2) \simeq \mathbb{Z}$$
$$L \xmapsto[\text{linking + framing number}]{\text{total crossing number =}} \#L$$

But this is precisely the **Wilson loop observable** of L **in** (abelian) **Chern-Simons theory**! [119, §4] As we explain next.

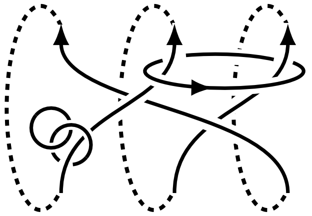
The k-Soliton sector. More generally, we may consider loops based in the kth connected component of the moduli space, corresponding to scattering process from k to k net number of solitons.

$$\begin{tikzcd}
\textbf{net charge } k & \textbf{Hopf degree } k \\
\mathbb{G}\mathrm{Conf}_{k}(\mathbb{R}^2) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Maps}^*_{k}\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \arrow[d, hook] \\
\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "\sim"] & \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big)
\end{tikzcd}$$

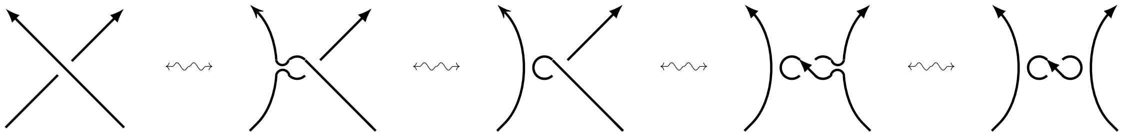
Since the double loop space $\mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big)$ admits the structure of a topological group, all these connected components have the same homotopy type, and hence these scattering processes L are again classified by the integer total crossing number $\#L$ which is the abelian Chern-Simons Wilson-loop observable.

$$\begin{tikzcd}
\Omega_{k}\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[d, two heads] \arrow[dr, "L \,\mapsto\, \#L"] & \\
\pi_0 \Omega_{k}\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "\sim"'] & \mathbb{Z}
\end{tikzcd}$$

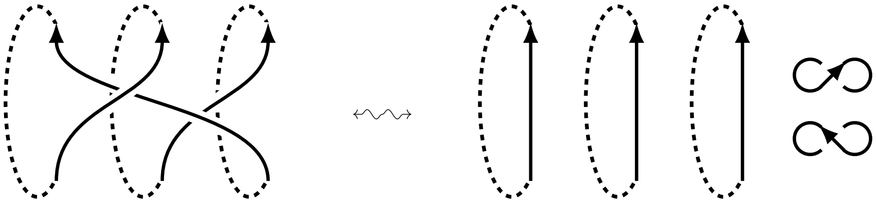
For instance, a generic $k = 3$ process looks like this:



and via the framed cobordism moves



it computes to the trivial scattering process accompanied by $\#L$ vacuuum pair braiding processes:



Chern-Simons level. We will see below further meanings of the number k:

This integer k is equivalently $\left\{ \begin{array}{l} \text{the } \textit{number} \text{ of fractional quasi-hole vortices in a quantum Hall system,} \\ \text{the } \textit{level} \text{ of their effective abelian Chern-Simons theory,} \\ \text{the } \textit{maximal denominator} \text{ for filling fractions of their quantum states.} \end{array} \right.$

Generally, we will recover in a novel *non-Lagrangian* way the features of quantum Chern-Simons theory that are traditionally argued starting with the kth multiple of the local Lagrangian density $a \wedge \mathrm{d}a$ for a gauge potential 1-form a.

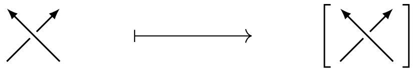
The situation on the 2-Sphere.
Furthermore consider k solitons on the actual 2-sphere S^2.
Here the 2-Cohomotopy moduli space satisfies (cf. [58]):

$$\pi_0 \Omega_{k}\, \mathrm{Maps}\big(S^2,\, S^2\big) \;\simeq\; \mathbb{Z}_{2|k|}\,,$$

and the long homotopy fiber sequence induced by point evaluation shows that the generator of this cyclic group is again identified with the basic half-braiding operation:

$$\mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big) \xrightarrow{\text{fiber of...}} \mathrm{Maps}\big(S^2, S^2\big) \xrightarrow{\substack{\text{point-}\\ \text{evaluation}}} S^2$$

$$\underbrace{\pi_2\big(S^2\big)}_{\mathbb{Z}} \xrightarrow{2k} \underbrace{\pi_0\Omega_{k}\mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big)}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_0\Omega_{k}\mathrm{Maps}\big(S^2, S^2\big)}_{\mathbb{Z}_{2|k|}} \longrightarrow \underbrace{\pi_1\big(S^2\big)}_{1}$$



With flux-quantized fields being equipped with a classifying space $\mathcal{A}$, there is a neat way to directly obtain the topological quantum observables – via the following observation:

Topological flux observables in Yang-Mills theory – Theorem [118]. For G-Yang-Mills theory on $\mathbb{R}^{1,1} \times \Sigma^2$, with a choice of Ad-invariant lattice $\Lambda \subset \mathfrak{g}$:
(i) Non-perturbative quantization of the algebra of flux observables through the closed surface Σ^2 is given by the group C^*-algebra $\mathbb{C}[-]$ of the Fréchet-Lie group of smooth maps $\Sigma^2 \to G \ltimes (\mathfrak{g}/\Lambda)$,
(ii) the corresponding group algebra of *topological* observables, depending only on the connected components of flux, coincides with the Pontrjagin homology algebra of pointed maps $(\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}} \to B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)$:

$$\underbrace{\mathbb{C}\Big[C^\infty(\Sigma^2, G) \ltimes C^\infty\big(\Sigma^2, (\mathfrak{g}/\Lambda)\big)\Big]}_{\text{non-perturbative quantum algebra of observables on flux through } \Sigma^2} \underset{\pi_0}{\rightsquigarrow} \underbrace{\mathbb{C}\Big[H^0(\Sigma^2; G) \ltimes H^1(\Sigma^2; \Lambda)\Big]}_{\text{sub-algebra of topological observables}} \simeq \underbrace{H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}}, B(G \ltimes (\mathfrak{g}/\Lambda)\big); \mathbb{C}\Big)}_{\text{Pontrjagin homology algebra of moduli space of soliton charges}}$$

For example in electromagnetism, with $G = \mathrm{U}(1)$ and $\Lambda := \mathbb{Z} \hookrightarrow \mathbb{R}$:

$$\mathbb{C}\Big[\underbrace{H^1(\Sigma^2;\mathbb{Z})}_{\text{electric}} \times \underbrace{H^1(\Sigma^2;\mathbb{Z})}_{\text{magentic}}\Big] \simeq H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}}, \underbrace{B\mathrm{U}(1) \times B\mathrm{U}(1)}_{\text{classifying space for Dirac flux quantization}}\big); \mathbb{C}\Big)$$

This allows to generalize:

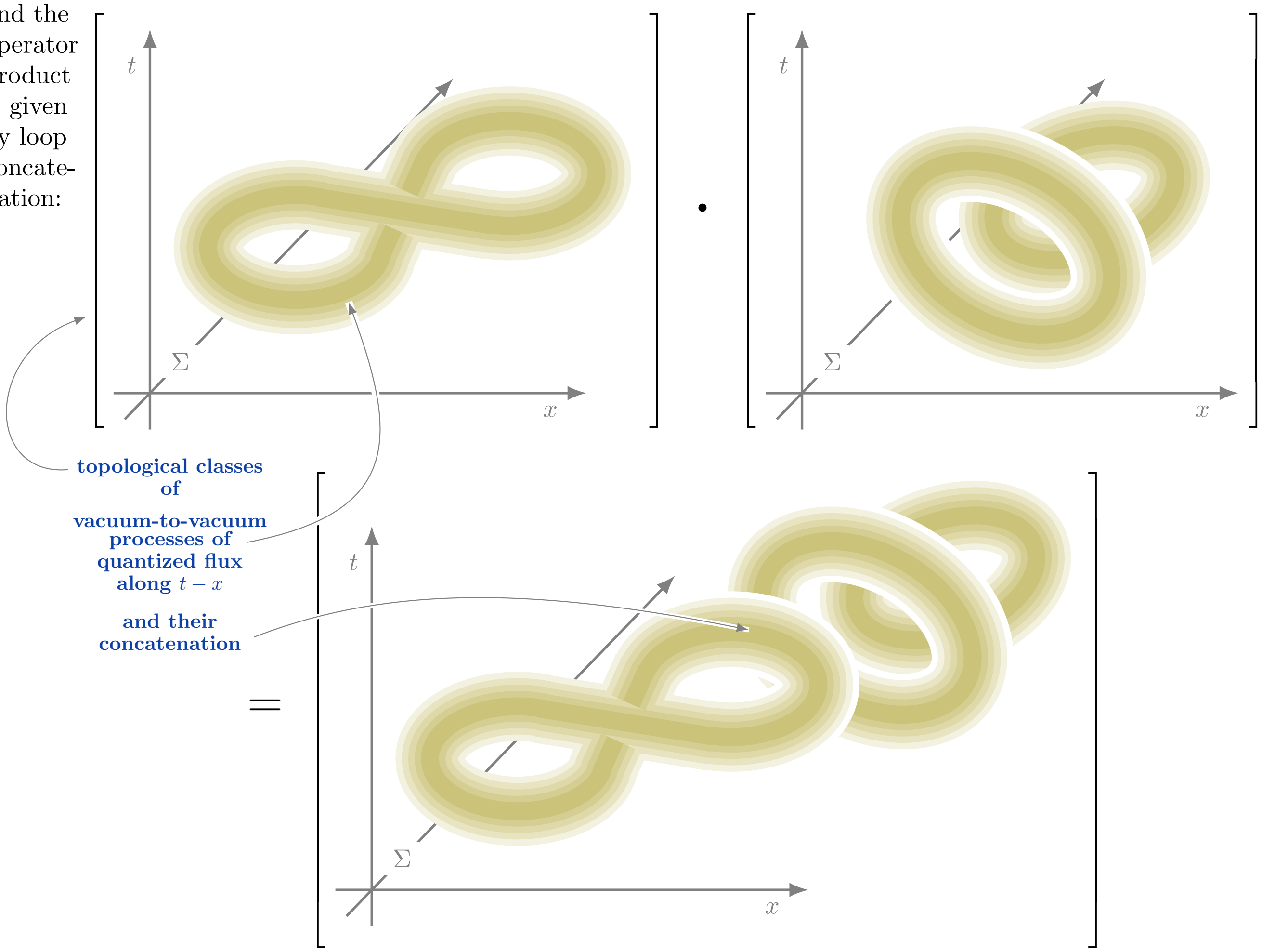
Topological flux observables of any higher gauge theory.

For a higher gauge theory flux-quantized in $\mathcal{A}$-cohomology the quantum algebra of topological flux observables on a spacetime of the form $\mathbb{R}^{1,1} \times \Sigma^{D-2}$ is the Pontrjagin homology algebra of the soliton moduli hence in deg $= 0$ is the group algebra of vacuum soliton processes "**on the light-cone**":

$$\begin{aligned} \mathrm{Obs}_\bullet &:= H_\bullet\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^{D-2})_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq H_\bullet\Big(\Omega\,\mathrm{Maps}(\Sigma^{D-2}, \mathcal{A}); \mathbb{C}\Big) \\ \mathrm{Obs}_0 &= \mathbb{C}\Big[\pi_0 \Omega\,\mathrm{Maps}(\Sigma^{D-2}, \mathcal{A})\Big] \end{aligned}$$

For note that the star-involution is given by the *combination* of
$\left\{\begin{array}{l}\text{- complex conjugation (time reversal)} \\ \text{- loop reversal (hence } x\text{-reversal)}\end{array}\right.$
where $\mathbb{R}^{1,1} \simeq \mathbb{R}\langle t, x\rangle$,

and the operator product is given by loop concatenation:



topological classes of vacuum-to-vacuum processes of quantized flux along $t - x$

and their concatenation

4 The topological Quantum States

To summarize so far, we have seen that the topological sector of the flux-quantized phase space of solitons on magnetized M5-probes Σ wrapping Seifert orbi-singularities is

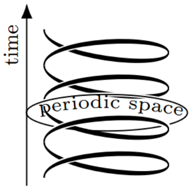
$$\begin{tikzcd}
\mathrm{Maps}\begin{pmatrix}\Sigma & \mathbb{C}P^3 \\ \downarrow, & \downarrow \\ X & S^4\end{pmatrix}^{\mathbb{Z}_2} \arrow[r, phantom, "\simeq"] & \mathrm{Maps}^{*/}\big(\mathbb{R}^2_{\cup\{\infty\}} \wedge S^1,\, S^2\big) \arrow[r, phantom, "\simeq"] & \Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "{[-]}"] & \pi_0\, \Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, phantom, "\simeq"] & \mathbb{Z} \\
\text{topological sector of flux-quantized phase space} & \text{2-Cohomotopy cocycle space} & L \arrow[rr, mapsto] & & \#L \\
 & & \text{loop space of group-completed configuration space} & & \text{net charge}
\end{tikzcd}$$

The topological quantum states of this system now follow [118][119, §4] by general algebraic quantum theory:

The gauge-invariant topological **observables** form the (higher) homology of this space $\mathrm{Obs}_\bullet := H_\bullet\big(\Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\, \mathbb{C}\big)$ making a (star-)algebra under concatenation (reversion) of loops — the *Pontrjagin algebra.*

$$\begin{tikzcd}
\Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "\text{loop reversal}", "\mathrm{rev}"'] & \Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \\
H_\bullet\big(\Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\, \mathbb{C}\big) \arrow[r, "\text{Pontr. antipode}", "\mathrm{rev}_*"'] \arrow[dr, "\text{Hermitian conjugation of quantum observables}"'] & H_\bullet\big(\Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\, \mathbb{C}\big) \arrow[d, "\overline{(-)}"', "\text{cmplx cnjgtn}"] \\
 & H_\bullet\big(\Omega_0\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2);\, \mathbb{C}\big)
\end{tikzcd}$$

This means that time-reversal goes along with reversal of looping around the M/IIA-circle, whence we are dealing with a version of *discrete light-cone quantization* in their topological sectors.



The basic ordinary (degree=0) observables detect the deformation class of a framed link L.

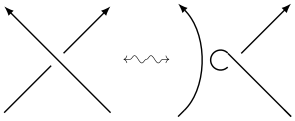
$$\begin{array}{ccccc}
\mathrm{Obs}_0 & \xrightarrow{\sim} & \mathbb{C}\Big[\pi_0\big(\Omega_0 \mathbb{G}\mathrm{Conf}(\mathbb{R}^2)\big)\Big] & \xrightarrow{\sim} & \mathbb{C}[\mathbb{Z}] \\
\mathcal{O}_L & := & \delta_{[L]} & = & \delta_{\#L} \\
\mathcal{O}_L \cdot \mathcal{O}_{L'} & = & \delta_{L \sqcup L'} & = & \delta_{\#L + \#L'}
\end{array}$$

Since these observables commute among each other, their *pure* topological quantum states are their (real & positive) algebra homomorphisms:

$$\begin{tikzcd}
\mathrm{PureQStates}_0 \arrow[d, hook] \arrow[r, phantom, "\simeq"] & \Big\{\rho \,:\, \mathrm{Obs}_0 \xrightarrow{\text{homo}} \mathbb{C} \,\Big|\, \rho \in \mathrm{MixedQStates}_0\Big\} \\
\mathrm{MixedQStates}_0 \arrow[r, phantom, ":="] & \Big\{\rho \,:\, \mathrm{Obs}_0 \xrightarrow{\text{linear}} \mathbb{C} \,\Big|\, \underset{\mathcal{O} \in \mathrm{Obs}_\bullet}{\forall} \Big(\underbrace{\rho(\mathcal{O}^*) = \rho(\mathcal{O})^*}_{\text{reality}},\ \underbrace{\rho(\mathcal{O}^* \cdot \mathcal{O}) \geq 0 \in \mathbb{R} \hookrightarrow \mathbb{C}}_{\text{(semi-)positivity}}\Big),\ \underbrace{\rho(1) = 1}_{\text{normalization}}\Big\}.
\end{tikzcd}$$

on commuting observables

Therefore pure topological states $|m\rangle$ are determined by an **anyonic phase** $\exp(\pi \mathrm{i}/m)$ assigned to any crossing,

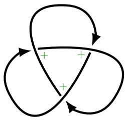


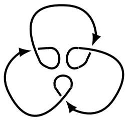
accumulating to the exponentiated crossing number

$$\begin{array}{ccc}
\mathrm{Obs}_0 & \xrightarrow{\langle m | - | m \rangle} & \mathbb{C} \\
\mathcal{O}_L & \longmapsto & e^{\frac{\pi \mathrm{i}}{m} \#L}
\end{array}$$

The resulting expectation values are [119, §4] just those of *Wilson loop observables* in *abelian Chern-Simons theory*, as **expected for abelian anyons**.

$$\langle m | \mathcal{O}_L | m \rangle \;=\; \exp\big(\tfrac{\pi \mathrm{i}}{m}\, \#L\big) \;=\; \exp\bigg(\frac{\pi \mathrm{i}}{m}\Big(\underbrace{\sum_{i \neq j \in \pi_0(L)} \mathrm{lnk}(L_i, L_j)}_{\text{linking numbers}} + \underbrace{\sum_{i \in \pi_0(L)} \mathrm{frm}(L_i)}_{\text{framing numbers}}\Big)\bigg)$$

For example: $\Big\langle m \Big|$ 

$\Big| m \Big\rangle \;=\; \Big\langle m \Big|$  $\Big| m \Big\rangle \;=\; \exp\big(\pi \mathrm{i}\, \tfrac{3}{m}\big)$

Applying the GNS-construction to such state produces a 1-dimensional Hilbert space $\overbrace{\mathbb{C}[\theta, \theta^{-1}]}^{\mathbb{C}[\mathbb{Z}]}/\big(e^{\pi \mathrm{i}/m} - \theta\big) \simeq \mathbb{C}$, which is as expected for the quantum states of abelian Chern-Simons theory on $\mathbb{R}^2_{\cup\{\infty\}}$. (More on this on p 17.)

Remark. At this point $m \in \mathbb{R} \neq 0$ may be irrational, but its rationality will be enforced by requiring compatibility with states on more general domain surfaces, see pp. 17 and p. 18.

Remark. These *solitonic* anyons are *not* yet the controllable/parameterized defect anyons that could be used for topological braid quantum gates operating by adiabatic movement of anyonic *defects* or (quasi-)*holes*. But the latter arise as defect points among the former, we come to this on p. 19.

Remark. The appearance of framed links along just the above lines is known in the condensed matter theory of anyonic defect lines in the 3D "8-band model" ([42, pp 15], following [128]): From this perspective, the Cohomotopy classifying space S^2 plays the role of the classifying space for electron band Hamiltonians on a crystal lattice.

Anyonic topological order on Flux-quantized M5-probes. We now identify the promised topological order on M5-probes flux-quantized in equivariant twistorial Cohomotopy, by considering M5s wrapping closed surfaces:

Anyonic quantum observables on closed surfaces. Consider now a *closed orientable surface* Σ^2_g of genus $g \in \mathbb{N}$ to replace the previous factor $\mathbb{R}^2_{\cup\{\infty\}}$ in the brane diagram:

$$\overset{\mathbb{Z}_2 \circlearrowright}{\Sigma^{1,5}} := \mathbb{R}^{1,0} \times \Sigma^2_g \times S^1 \times \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{R}^2_{\mathrm{sgn}}}$$

Directly analogous analysis as before gives that the topological quantum observables on the flux-quantized self-dual tensor field form the group algebra of the fundamental group of the 2-cohomotopy moduli space in the kth connected component

$$\mathrm{Obs}_0\big(\Sigma^2_g\big) \;:=\; H_0\Big(\Omega_k \,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big);\, \mathbb{C}\Big) \;\simeq\; \mathbb{C}\Big[\pi_0 \Omega_k \,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big)\Big]\,, \tag{3}$$

where $k \in \mathbb{N}$ is the degree of the classifying maps, corresponding under the Pontrjagin theorem to a net number of k (anti-)solitons on Σ^2_g.

Theorem (using [58, Thm 1][76, Thm 1][69, Cor 7.6]). This group of 2-cohomotopy charge sectors is identified as *twice* the integer Heisenberg group extension (cf. [78]) of $\mathbb{Z}^{2g}$ by $\mathbb{Z}_{2|k|}$ [6]:

$$\pi_0 \Omega_k \mathrm{Maps}\big(\Sigma^2_g,\, S^2\big) \;\simeq\; \left\{ \big(\vec a,\, \vec b,\, [n]\big) \in \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z}_{2|k|}\,, \;\; \begin{array}{l} \big(\vec a,\, \vec b,\, [n]\big) \cdot \big(\vec a',\, \vec b',\, [n']\big) \;= \\ \big(\vec a + \vec a',\, \vec b + \vec b',\, [n + n' + \vec a \cdot \vec b' - \vec a' \cdot \vec b]\big) \end{array} \right\} =: \widehat{\mathbb{Z}^{2g}}$$

Ground state degeneracy. Hence the observable group-algebra Obs_0 for $g = 1$, $\Sigma^2_1 = T^2$, has generators

$$\left\{ \begin{array}{rcl} W_a &:=& (1, 0, [0]) \\ W_b &:=& (0, 1, [0]) \\ \zeta &:=& (0, 0, [1]) \end{array} \right\} \;\; \text{subject to the relations} \;\; \left\{ \begin{array}{l} W_a \cdot W_b \;=\; \zeta^2\, W_b \cdot W_a \\ \zeta^{2k} = 1 \\ [\zeta, -] = 0 \end{array} \right\}.$$

This algebra is just the observable algebra expected [130, (5.28)] for anyonic topological order on the torus as described by abelian Chern-Simons theory at *level* k. The non-trivial irreps have:

- dimension k, this being the expected *ground state degeneracy* on the torus,
- are labeled by $\nu := p/k$, $p \in \{1, 2, \cdots, k\}$, as expected for fractional *filling factors*.

Hilbert space of quantum states on the torus

$$\mathcal{H}_{T^2} := \mathrm{Span}\Big(\big|[n]\big\rangle, [n] \in \mathbb{Z}_{|k|}\Big) \;\in\; \mathrm{Obs}_0(T^2)\mathrm{Modules}\,,\;\; \mathrm{dim}\big(\mathcal{H}_{T^2}\big) \;=\; k\,, \qquad \begin{array}{rcl} W_a\big|[n]\big\rangle &:=& e^{2\pi \mathrm{i} n \nu}\big|[n]\big\rangle \\ W_b\big|[n]\big\rangle &:=& \big|[n+1]\big\rangle \\ \zeta\big|[n]\big\rangle &:=& e^{\pi \mathrm{i} \nu}\big|[n]\big\rangle \end{array}$$

Modular equivariance. Strikingly, in this construction modular symmetry **is manifest**, since the looped mapping space is canonically acted on by the mapping class group MCG of Σ^2_g (cf. [31, §2.1]), simply by precomposition of maps! Inspection of the above theorem (cf. [58, bottom of p 153]) shows that this MCG-action action identifies indeed as the canonical action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\widehat{\mathbb{Z}^{2g}}$.

$$\begin{tikzcd}
\overbrace{\pi_0 \mathrm{Homeos}_{\mathrm{or}}\big(\Sigma^2_g\big)}^{\mathrm{MCG}(\Sigma^2_g)} \arrow[d, "{[31,\ \S 6.3]}"] \arrow[r, phantom, "\circlearrowright"] & \pi_0 \Omega_k \,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big) \arrow[d, phantom, "\wr\!\mid"] \\
\mathrm{Sp}_{2g}(\mathbb{Z}) \arrow[r, phantom, "\circlearrowright"] & \widehat{\mathbb{Z}^{2g}}
\end{tikzcd}$$

Hence we may ask for a lift of the $\widehat{\mathbb{Z}^{2g}}$ action on quantum states to an action of the semidirect product $\widehat{\mathbb{Z}^{2g}} \rtimes \mathrm{Sp}_{2g}(\mathbb{Z})$. For $g = 1$ and even k one readily checks that this gives the modular transformations of states known [85, pp 65] from abelian Chern-Simons theory:

modular action on observables and on states

$$m(W) \cdot m\Big(\big|[n]\big\rangle\Big) \;=\; m\Big(W\big|[n]\big\rangle\Big)\,, \qquad \forall \left\{ \begin{array}{rcl} m &\in& \mathrm{Sp}_{2g}(\mathbb{Z}) \\ W &\in& \widehat{\mathbb{Z}^{2g}} \\ \big|[n]\big\rangle &\in& \mathcal{H}_g \end{array} \right.$$

$$S\Big(\big|[n]\big\rangle\Big) \;=\; \frac{1}{\sqrt{|k|}} \sum_{[\widehat{n}]} e^{2\pi \mathrm{i} \frac{n\,\widehat{n}}{k}} \big|[\widehat{n}]\big\rangle\,, \qquad T\Big(\big|[n]\big\rangle\Big) \;=\; e^{\mathrm{i}\pi\frac{n^2}{k}} \big|[n]\big\rangle\,.$$

Generally, writing $(\vec e_i \in \mathbb{Z}^g)_{1=1}^g$ for the canonical basis vectors, the observable group-algebra Obs_0 for general g has generators

$$\left\{ \begin{array}{rcl} W^i_a &:=& (\vec e_i, 0, [0]) \\ W^i_b &:=& (0, \vec e_j, [0])\,, 1 \leq i \leq g \\ \zeta &:=& (0, 0, [1]) \end{array} \right\} \;\; \text{subject to the relations} \;\; \left\{ \begin{array}{l} W^i_a \cdot W^j_b \;=\; \delta^{ij} \zeta^2\, W^j_b \cdot W^i_a \\ \zeta^{2k} = 1 \\ \text{all other commutators vanish} \end{array} \right\}$$

Requiring the reps $\mathcal{H}_g$ of this algebra to analogously support modular equivariance requires them to have dimension $|k|^g$ — which is the result expected [85, p 40] for abelian topological order on Σ^2_g:

Hilbert space of quantum states on genus=g surface

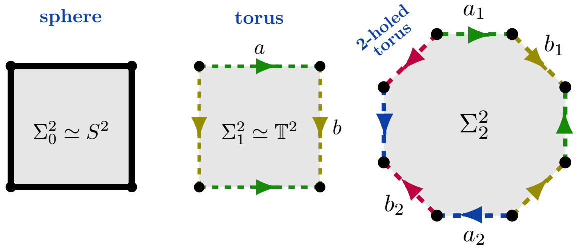
$$\mathcal{H}_{\Sigma^2_g} \;\in\; \mathrm{Obs}_0(\Sigma^2_g)\mathrm{Modules}\,, \quad \mathrm{dim}\big(\mathcal{H}_{\Sigma^2_g}\big) \;=\; |k|^g\,,$$

---

[6] Here $\mathbb{Z}_n := \mathbb{Z}/(n)$ (with $\mathbb{Z}_0 = \mathbb{Z}$) are the (in-)finite cyclic groups.

Here the generators $W^i_{a,b}$ correspond to the classical generators of the surface's fundamental group:

Oriented closed surfaces are all obtained (cf. [44, p 100]): by identifying in the regular $4g$-gon, for *genus* $g \in \mathbb{N}$:
(i) all boundary vertices with a single point
and, going clockwise for $r \in \{0, \cdots, g-1\}$,
(iia) the $4r+1$st boundary edge with the reverse of the $4r+3$rd,
(iib) the $4r+2$nd boundary edge with the reverse of the $4r+4$th.



In other words, the homotopy type of the surface sits in a (pointed) homotopy co-fiber sequence of this form:

$$S^1 \xrightarrow{\prod_i [a_i, b_i]} \bigvee_g \left(S^1_a \vee S^1_b\right) \longrightarrow \Sigma^2_g \xrightarrow{\delta} S^2$$

whence its fundamental group is the quotient of the free group on $2g$ generators $(a_i, b_i)_{i=1}^g$ by the normal subgroup generated by that polygon's boundary:

$$\pi_1\big(\Sigma^2_g\big) \;\simeq\; \big\langle a_1, b_1, \cdots, a_g, b_g \big\rangle \big/ \textstyle\prod_i [a_i, b_i]$$

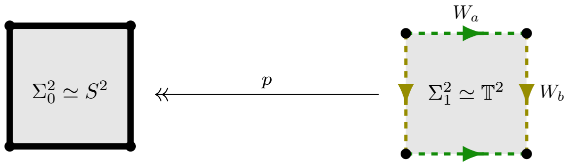
2-Cohomotopy moduli of oriented closed surfaces. Mapping this co-fiber sequence into S^2 and applying $\pi_0\Omega_k$, it collapses [58, Prop. 2] to twice [76, Thm 1] the integer Heisenberg central extension of $\mathbb{Z}^{2g}$ by $\mathbb{Z}_{2|g|}$:

$$1 \longrightarrow \underbrace{\pi_0\Omega_k \,\mathrm{Maps}\big(S^2,\, S^2\big)}_{\mathbb{Z}_{2|k|}} \xrightarrow{\;\delta^*\;} \underbrace{\pi_0\Omega_k \,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big)}_{\textbf{integer Heisenberg group}} \longrightarrow \underbrace{\pi_0\Omega_* \,\mathrm{Maps}^*\big(\bigvee_g (S^1_a \vee S^1_b),\, S^2\big)}_{\mathbb{Z}^{2g}} \longrightarrow 1\,.$$

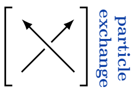
The phase generators. Hence these integer Heisenberg groups inject into each other as the surfaces are surjected onto each other by collapsing pairs of 1-cycles.

$$\begin{tikzcd}
\Sigma^2_g & \Sigma^2_{g+1} \arrow[l, two heads, "p"'] \\
\pi_0\Omega_k \,\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big) \arrow[r, "\pi_1(p^*{,}k)"] \arrow[d, phantom, "\wr"] & \pi_0\Omega_k \,\mathrm{Maps}\big(\Sigma^2_{g+1},\, S^2\big) \arrow[d, phantom, "\wr"] \\
\widehat{\mathbb{Z}^{2g}} \arrow[r, hook] & \widehat{\mathbb{Z}^{2(g+1)}}
\end{tikzcd}$$

Thereby their central generator ζ represents the previously identified half-braiding operation of solitons on these surfaces.
This is the "reason" for the central extension being by $\mathbb{Z}_{2|k|}$ instead of just $\mathbb{Z}_{|k|}$:
The phase generator ζ does not correspond to full rotations (such as around the square on the right) but to "*particle exchange*" by half-braiding — as expected for anyons.



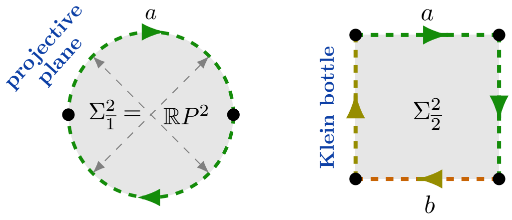
$$\mathbb{Z}_{2|k|} \hookrightarrow \widehat{\mathbb{Z}^2}$$



$$\longmapsto$$

phase generator $\zeta \mapsto e^{\pi \mathrm{i} \nu}$

Non-orientable closed surfaces are all obtained by identifying in the regular $2h$-gon, for *crosscap number* $h \in \mathbb{N}_{\geq 1}$:
(i) all boundary vertices with a single point
and, going clockwise for $r \in \{0, \cdots, h-1\}$,
(ii) the $2r+1$st boundary edge with the reverse of the $2r+2$nd



In other words, the homotopy type of the surface sits in a (pointed) homotopy co-fiber sequence of this form:

$$S^1 \xrightarrow{\prod_i a_i^2} \bigvee_h S^1 \longrightarrow \Sigma^2_h \xrightarrow{\delta} S^2$$

2-Cohomotopy moduli of non-orientable closed surfaces. Mapping this co-fiber sequence into S^2 and applying $\pi_0\Omega_k$, it induces [58, Prop. 3] an extension of $\mathbb{Z}^{h-1}$ by $\mathbb{Z}_2$ which as such is trivial [76, Thm. 2]:

$$1 \longrightarrow \underbrace{\mathrm{coker}\Big(\big(\Sigma \textstyle\prod_i a_i^2\big)^*\Big)}_{\mathbb{Z}_2} \xrightarrow{\;\delta^*\;} \underbrace{\pi_0\Omega_k \,\mathrm{Maps}^*\big(\Sigma^2_h,\, S^2\big)}_{\mathbb{Z}_2 \times \mathbb{Z}^{h-1}} \longrightarrow \underbrace{\ker\Big(\big(\textstyle\prod_i a_i^2\big)^*\Big)}_{\mathbb{Z}^{h-1}} \longrightarrow 1\,.$$

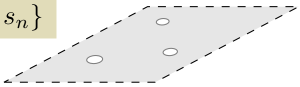
Again, the exponent appearing, $h-1$, is just that expected for abelian Chern-Simons ground state degeneracy, where (cf. [14, (73)]):

$$\dim\big(\mathcal{H}_{\Sigma^2_h}\big) \;=\; |k|^{h-1}$$

Defects via punctured worldvolumes. It is now immediate to bring *adiabatically movable defect anyons* into the picture, missing in traditional discussion but crucially needed for topological quantum gates (cf. [89, §3]). Namely we may simply further generalize the surfaces Σ^2_g to their *n-punctured* versions, obtained by deleting the positions of a subset of points – thus literally creating defects! That these defects are void of the dynamical solitons is elegantly enforced by identifying all their positions with the point-at-infinity.

$$\Sigma^2_{g,n} \;:=\; \Sigma^2_{g,n} \setminus \{s_1, \cdots s_n\}$$

for $\{s_1, \cdots s_n\} \subset \Sigma^2_g$



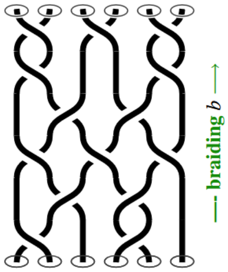
$$\big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}} \qquad \text{e.g.: } \mathbb{R}^2_{\cup\{\infty\}} \;\simeq\; \big(\Sigma^2_{0,1}\big)_{\cup\{\infty\}}$$

In this generality, our previous brane diagram now is:

$$\overset{\mathbb{Z}_2 \circlearrowright}{\Sigma^{1,6}} := \;\mathbb{R}^{1,0} \;\times\; \big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}} \;\times\; S^1 \;\times\; \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{R}^2_{\mathrm{sgn}}}\,,$$

and, by the same argument as before, the algebra of topological quantum observables on cohomotopically flux-quantized fields becomes:

$$\mathrm{Obs}_0\big(\Sigma^2_{g,n}\big) \;:=\; H_0\Big(\Omega_k \,\mathrm{Maps}^*\big((\Sigma^2_{g,n})_{\cup\{\infty\}},\, S^2\big);\, \mathbb{C}\Big)\,.$$



A more explicit description of this algebra of observables may not be available at the moment. But we can immediately see that these are quantum observables on defect anyons:

Braid group action... This algebra of observables is faithfully acted on by the mapping class group of the punctured surface – again simply by precomposition of maps. But, with punctures, that group is now an extension (cf. [86, Thm. 3.13]) of the plain mapping class group by the *surface braid group* that acts by ("adiabatically") moving the defects around each other!

$$1 \to \underset{\text{surface braid group}}{\mathrm{Br}_n(\Sigma^2_g)} \longrightarrow \underset{\substack{\text{mapping class group}\\ \text{of punctured surface}}}{\pi_0\mathrm{Homeos}^*_{\mathrm{or}}\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big)} \twoheadrightarrow \underset{\substack{\text{mapping class group}\\ \text{of plain surface}}}{\mathrm{MCG}(\Sigma^2_g)} \to 1$$

(In deducing this, we observed that $\mathrm{Homeos}^*\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big) \;\simeq\; \mathrm{Homeos}\big(\Sigma^2_{g,n}\big)$ since $(-)_{\cup\{\infty\}}$ is functorial on homeos.)

...on defect points. Concretely, we have with $\big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}} \;\simeq\; \Sigma^2_g \vee \bigvee_{n-1} S^1$ (cf. [61, p 11]):

$$\pi_1\mathrm{Maps}^*\Big(\big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}},\, S^2\Big) \;\simeq\; \pi_1\mathrm{Maps}^*\big(\Sigma^2_g,\, S^2\big) \times \mathbb{Z}^{n-1} \;\underset{g=0}{\simeq}\; \mathbb{Z}^n$$

Defect-braiding on M5s as a quantum-gravitational effect. Noting that the mapping class group is equivalently the group of large *diffeomorphisms* of the punctured surface (cf. [31, p 45]),

$$\pi_0\mathrm{Homeos}^*_{\mathrm{or}}\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big) \;\simeq\; \pi_0\,\mathrm{Diffeos}_{\mathrm{or}}\big(\Sigma^2_{g,n}\big)$$

we see that braiding of anyonic defects is reflected in equipping the moduli spaces of cohomotopical charges on the brane worldvolume with the action by diffeomorphisms, hence by passing to the action *groupoid* of moduli quotiented by diffeos.

$$\mathrm{GnrlCovariantModuli}(\Sigma) \simeq \mathrm{Moduli}(\Sigma) \sslash \mathrm{Diffeos}(\Sigma)$$

But this is the hallmark of *generally covariant* systems (cf. [27]), such as are our probe M5-branes.

Defect para-statistics. So the covariantized quantum observable algebra locally factors through the group algebra of the *wreath product* (cf. [10, §8]) of

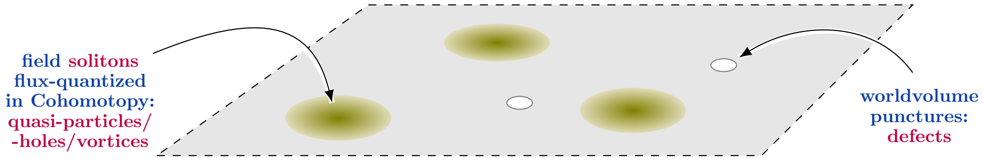
$$\begin{cases} 1.\ \text{the braid phase group } \mathbb{Z} \text{ of the solitonic anyons with} \\ 2.\ \text{the permutation group } \mathrm{Sym}_n \text{ of the defect anyons} \end{cases}$$

$$\mathbb{Z}^{n-1} \rtimes \mathrm{Br}_n \hookrightarrow \mathbb{Z}^n \rtimes \mathrm{Br}_n \twoheadrightarrow \mathbb{Z}^n \rtimes \mathrm{Sym}_n \;\simeq\; \Big\{\big((n_i)_{i=1}^n,\, \sigma\big) \;\big|\; \big((n_\bullet), \sigma\big)\cdot\big((n'_\bullet), \sigma'\big) \;=\; \big((n_\bullet + n'_{\sigma(\bullet)}),\, \sigma\sigma'\big)\Big\}$$

Just such *para-statistical* (cf. [134]) wreath-group statistics of defect anyons is seen in condensed matter [42].

Solitonic vs. Defect anyons. By the previous discussion, we are to think of $\mathrm{Obs}_0(\Sigma^2_{g,n})$ as the quantum observables on abelian *solitonic anyons* propagating on the punctured surface $\Sigma^2_{g,n}$. But the dependence of these observables on the external parameters of n defect positions makes them "collectively" represent braiding of defects.

anyons as seen in Cohomotopy	**nature**	**number**	**braiding**
solitonic anyons	concentrations of flux density	net charge, CS-level: k	by (LC-)time evolution
defect anyons	punctures in worldvolume	n in $\Sigma^2_{g,n}$	by worldvolume diffeomorphisms

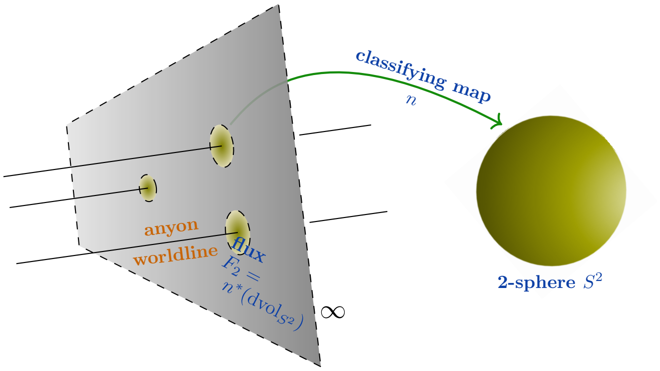


5 Conclusion: Better Anyon Theory

Conclusion – New theory of anyonic topological order, engineered on flux-quantized M5s. In summary, we have seen that global completion by flux-quantization of 11D supergravity with M5-probes (here: in equivariant twistorial cohomotopy – "Hypothesis H"), makes the quantized topological sector of the self-dual tensor field on M5-probes (wrapping Seifert orbi-singularities) reproduce key phenomena of abelian Chern-Simons theory thought of as an effective field theory for abelian anyons in fractional quantum Hall (FQH) systems:

(i) Flux tubes bound to anyons. The central assumption in the traditional heuristic understanding of the FQHE is that the anyonic solitons have flux quanta "attached" to them [126, pp 883]. It is crucially this assumption which motivates and justifies abelian Chern-Simons theory as an effective field theory for FQH anyons, since variation of the sum of the abelian Chern-Simons term with the standard source term predicts that the gauge field flux is localized at the source particles (cf. [130, (5.25)][139, (3.6)]).

In contrast, in the present approach **this effect is a consequence of cohomotopical flux-quantization**, via the Pontrjagin theorem: The classifying map of the 2-Cohomotopy charge identifies an open neighbourhood of each anyon with the 2-sphere minus its point at infinity, and the flux density F_2 is the pullback of the sphere's volume form along this map (cf. p 28), hence supported on just these open neighbourhoods.



(ii) Anyons subject to each other's Aharonov-Bohm phases.

Traditional discussion furthermore assumes from these attached flux tubes that the anyons must pick up Aharonov-Bohm quantum phases when circling around each other. While this is plausible, rigorous quantum field-theoretic derivation of this statement may not have found much attention.

In contrast, in the approach discussed here, this effect is again a direct consequence of cohomotopical flux-quantization, now via algebro-topological theorems of Segal and others, which serve to identify the cohomotopy charge moduli space with configuration spaces of soliton cores, whose fundamental group reflects the anyon braid phases (and thereby also the ground state degeneracy / topological order).

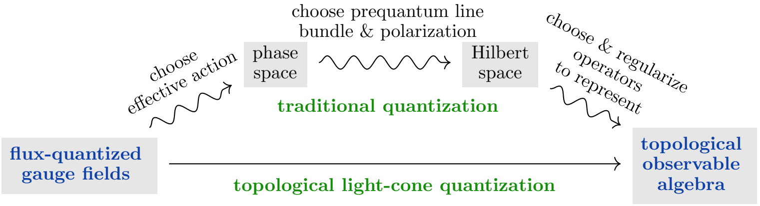
$$\begin{tikzcd}
\pi_0 \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, \boxed{S^2}\big) \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & \pi_0 \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, B^2\mathbb{Z}\big) \arrow[d, phantom, "\wr"] \\
\mathbb{Z} \arrow[r, phantom, "\text{same net charges...}"] & \mathbb{Z} \\
\pi_1 \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, \boxed{S^2}\big) \arrow[r] \arrow[d, phantom, "\wr"] & \pi_1 \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, B^2\mathbb{Z}\big) \arrow[d, phantom, "\wr"] \\
\underset{\text{config space}}{\pi_1 \mathbb{G}\mathrm{Conf}(\mathbb{R}^2)} \arrow[r, phantom, "\text{...but different moduli}"] & \underset{\text{no structure}}{1}
\end{tikzcd}$$

Note how both these effects come about by changing the traditional flux-quantization of the Chern-Simons field from the classifying space for complex line bundles to just its first "cell". This preserves the quantization of charges but makes their moduli exhibit anyonic effects.

$$\begin{tikzcd}
\underset{\substack{\text{classifying space} \\ \text{for 2-Cohomotopy}}}{\boxed{S^2}} \simeq \mathbb{C}P^1 \arrow[r, hook, "\text{1st cell inclusion}"'] & \mathbb{C}P^\infty \simeq \underset{\substack{\text{classifying space for} \\ \text{ordinary 2-cohomology}}}{B^2\mathbb{Z}}
\end{tikzcd}$$

(iii) Topological order. The traditional way of establishing topological order is by applying geometric quantization to Wilson line observables, with respect to some effective action, which is a somewhat convoluted process involving ad-hoc choices and regularizations.

In contrast, in the approach discussed here the quantum observables obtain immediately, without further choices, from the topological light-cone quantization of the flux-quanized moduli space (as its Pontrjagin homology algebra).



Here the looping Ω_k that drives this quantum dynamics reflects dependence of moduli on the M/IIA circle.(!)

(iv) Defect anyons — as opposed to the solitonic anyons tracing out "Wilson lines" — seem to have previously found little to no attention in quantum Hall theory in general and its effective abelian Chern-Simons theories in particular. And yet, it is only such classically parameterized and hence, in principle, externally controllable defect anyons which may support braid quantum gates as envision in topological quantum computation.

In our approach, defect braiding emerges just as readily as the solitonic anyons, as a mild kind of quantum gravitational effect on M5-worldvolumes having a punctured surface factor space. This may be seen as a theoretical prediction of defect anyons in quantum Hall systems which might inform future search for experimental realization.

Engineering of Anyons on M5-Branes via Flux-Quantization

Summary of results:

On super-space, the equations of motion
of **11D supergravity** with magnetic 1/2BPS **M5-brane** probes
are equivalent to these Bianchi identities on the super-flux densities:

$$
\begin{array}{lll}
\textbf{A-field} & \mathrm{d}\,F_2^s \;=\; 0 & \mathrm{d}\,G_4^s \;=\; 0 & \textbf{C-field} \\
\textbf{self-dual B-field} & \mathrm{d}\,H_3^s \;=\; \phi_s^* G_4^s + \theta\, F_2^s\, F_2^s & \mathrm{d}\,G_7^s \;=\; \tfrac{1}{2} G_4^s\, G_4^s & \textbf{dual C-field}
\end{array}
$$

$$
\textbf{M5 probe} \quad \Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow[\text{1/2BPS immersion}]{\phi_s} X^{1,10\,|\,\mathbf{32}} \quad \textbf{SuGra bulk}
$$

One admissible choice of **flux-quantization** law (the simplest in number of CW cells)
is **twistorial Cohomotopy**, where the charges are classified by dashed maps like this:

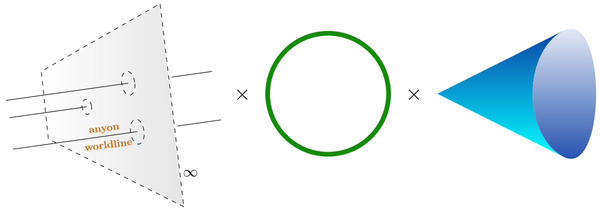
$$
\begin{tikzcd}
\textbf{M5-brane worldvolume} & \Sigma^{1,5} \arrow[rr, dashed, "\textbf{A- \& B-field charge}", "(a_1, b_2)"'] \arrow[d, "\phi", "\text{immersion}"'] & & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{H}}"] \\
\textbf{bulk spacetime} & X^{1,10} \arrow[rr, dashed, "\textbf{C-field charge}", "(c_3, c_6)"'] & & S^4
\end{tikzcd}
$$

For (very good) $G \subset \mathrm{Sp}(2)$-orbifold domains, these maps are to be G-equivariant.

This flux-quantization implies a list of topological effects expected in M-theory.
⇒ **Hypothesis H**: This is the right choice of flux-quantization for M-theory.

Choosing ("engineering") the M5-probe to be:

$$
\underset{\textbf{M5-brane probe worldvolume}}{\Sigma^{1,5}} \;=\; \underset{\textbf{2-brane worldvolume hosting anyonic solitons}}{\mathbb{R}^{1,0} \times \Sigma^2_{g,n}} \;\times\; \underset{\textbf{M-theory circle}}{S^1} \;\times\; \underset{\textbf{cone orbifold}}{\mathbb{R}^2_{\mathrm{sgn}} /\!\!/ \mathbb{Z}_2}
$$



the **moduli space of solitons** becomes:

$$
\mathrm{Moduli} \;\simeq\; \mathrm{Maps}^*\Big(\big(\mathbb{R}^1 \times \Sigma^2_{g,n}\big)_{\cup\{\infty\}},\, S^2 \Big)
$$

The algebra of **topological quantum observables** on theses solitons is:

$$
\underset{\textbf{topological quantum observables}}{\mathrm{Obs}_0} \;:=\; \underset{\textbf{Pontrjagin homology algebra}}{H_0\bigg(\mathrm{Maps}^*\Big(\big(\mathbb{R}^1 \times \Sigma^2_{g,n}\big)_{\cup\{\infty\}},\, S^2 \Big);\, \mathbb{C} \bigg)} \;\simeq\; \underset{\textbf{group algebra}}{\mathbb{C}\Big[\pi_1 \mathrm{Maps}^*\big(\Sigma^2_{g,n},\, S^2\big) \Big]},
$$

acted on by large diffeomorphisms (**general covariance** on the brane):

$$
1 \to \underset{\textbf{braid group}}{\mathrm{Br}_n(\Sigma^2_g)} \hookrightarrow \underset{\textbf{large diffeomorphism group}}{\pi_0 \mathrm{Homeos}^*_{\mathrm{or}}\big((\Sigma^2_{g,n})_{\cup\{\infty\}} \big)} \twoheadrightarrow \underset{\textbf{mapping class group}}{\mathrm{MCG}(\Sigma^2_g)} \to 1
$$

The corresponding **topological quantum states**:

on $\Sigma^2_{0,0} = S^2$	reflect abelian braiding of **solitonic anyons**
on $\Sigma^2_{g,0} = \Sigma^2_{1,0} \# \cdots \# \Sigma^2_{1,0}$	have k^g-fold degeneracy: **topological order**
on $\Sigma^2_{1,0} = \mathbb{T}^2$	exhibit irred $\mathrm{SL}_2(\mathbb{Z})$-**modular equivariance**
on $\Sigma^2_{0,n} = S^2 \setminus \{z^1, \cdots, z^n\}$	reflect abelian braiding of ***defect anyons***

} as for abelian **Chern-Simons QFT**

new & needed for
topological quantum gates!

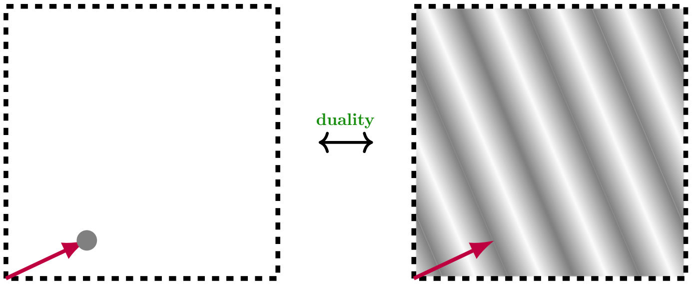
A broad lesson following immediately from our successful geometric engineering of topological qbits is the plausible existence of more exotic anyonic states than traditionally envisioned: Namely the “duality symmetry” [100][26, §6] of M-theory predicts that any geometrically engineered quantum system has “dual” incarnations with isomorphic quantum observables but entirely different geometric realization, where ordinary space is replaced by more abstract parameter spaces. Notably “T-duality” [133][35][51] applied to topological quantum materials has been argued [87][88][57] to exchange the roles of ordinary space with that of reciprocal “momentum space”.

(2) Novel experimental pathways towards anyons. Indeed, while anyonic solitons are traditionally envisioned as being localized in “position space” (meaning that the anyon cores are points in the plane of the crystal lattice) the physical principle behind topological quantum gates — namely [3][4][43, p 6][98, p 50] the *quantum adiabatic theorem* [107] — is unspecific to position space and only requires the material’s Hamiltonian to *depend on any continuous parameters* (such as external voltage or strain) varying in any abstract parameter space.

The general physical conditions for topological quantum gates given by the *quantum adiabatic theorem*, listed (a) - (e) on the right, are much more general than traditionally considered for anyon braid gates — the latter are only the special case where the parameters are configurations of points in the plane of the 2D crystal lattice.

(a) **Ground state degeneracy** (when frozen at absolute zero, the system still has more than one state to be in, even up to phase).
(b) **Spectral gap** (quanta of energy smaller than a given gap $\epsilon > 0$ cannot excite these ground states).
(c) **Control parameters** (the above properties hold for a range of continuously tunable external parameters).
(d) **Parameter topology** (there exist closed parameter paths that cannot be continuously contracted).
(e) **Local invariance** (continuously deformed parameter paths induce the same transformation on ground states).

This means that, in principle, the possibilities in which anyonic quantum states could arise in the laboratory are far more general than what has been explored to date.
Concretely, a key example of alternative parameters for ground states of a quantum material are points in their reciprocal *momentum space*: This is the space of (quasi-)momenta, hence of wave-vectors for plane quasi-particle waves going through the crystalline material.



Position space:
a point is a position
on the crystal lattice

Momentum space:
a point is a plane wave
on the crystal lattice

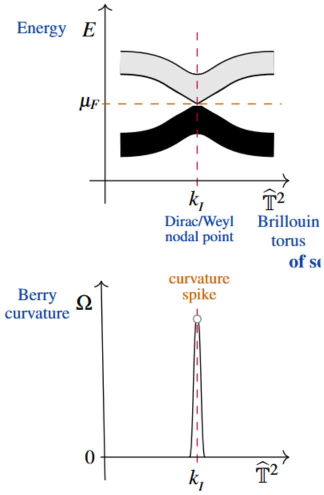
We have observed before that candidate anyon-like solitons localized (not in position space but) in momentum space are plausible both theoretically [116] as well as experimentally [142][129][67] and may have been hiding in plain sight: as band nodes of (interacting) topological semimetals.

Indeed, momentum space naturally features key properties that are typically assumed for anyon braid gates but remain elusive in position space:

(i) toroidal base topology is routinely assumed [135][137][80] in order to achieve the required ground-state degeneracy, but is quite unrealistic in position space, even more so when meant to be punctured by defect anyons — while the momentum space of a crystal is automatically a torus (the *Brillouin torus*).

(ii) stable defect points need special engineering in position space but arise automatically in momentum space in the guise of *band nodes* of topological semi-metals [116, Fig. 6]

(iii) defect point movement in a *controlled* way is necessary for braid gates but remains elusive in position space, while band nodes in momentum space have already shown to be movable in a varierty of systems, by tuning of external parameters (e.g. strain).



The geometric engineering of anyons discussed here goes towards providing also fundamental theoretical underpinning of the possibility of more “exotic” anyon realizations than have traditionally been envisioned.

A Background on Homotopy Theory

Some notions used in the main text, to establish notation and give basic pointers to the literature.

Homotopy theory (cf. [127]). For $f_0, f_1 : X \to Y$ a pair of continuous maps between (topological) spaces a *homotopy* $\eta : f_0 \Rightarrow f_1$ is a continous deformation between them: a continuous map $\eta : [0,1] \times X \to Y$ such that

$$\begin{array}{l} \eta(0,x) = f_0(x), \\ \eta(1,x) = f_1(x), \end{array} \quad \text{denoted} \quad \begin{tikzcd} X \arrow[r, bend left=50, "f_0"{name=U}] \arrow[r, bend right=50, "f_1"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\eta"] & Y \end{tikzcd}.$$

For example, a square "homotopy-commutative diagram"

$$\begin{tikzcd} \Sigma \arrow[r, dashed, "b"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"] \arrow[dl, phantom, "\swarrow \eta"] \\ X \arrow[r, dashed, "c"'] & \mathcal{B} \end{tikzcd}$$

means that

$$\begin{array}{l} \eta : [0,1] \times \Sigma \to \mathcal{B} \\ \eta(0,s) = p\big(b(s)\big), \\ \eta(1,s) = c\big(\phi(s)\big). \end{array}$$

If one declares – and we do – to work in a "convenient" full sub-category of all topological spaces (such as that of *compactly generated* or of *Delta-generated* topological spaces, cf. [114, p 21, 131]) then the topological space $\mathrm{Maps}(X,Y)$ of all continuous maps $X \to Y$ satisfies the adjointness relation $\big\{P \to \mathrm{Maps}(X,Y)\big\} \simeq \big\{P \times X \to Y\big\}$. For $P \equiv [0,1]$ this says that homotopies are equivalently paths in mapping spaces, and that homotopy-classes of maps are the mapping spaces' path-connected components: $\pi_0\, \mathrm{Maps}(X,Y) \simeq \mathrm{Maps}(X,Y)_{/\mathrm{hmtp}}$.

Since homotopies are maps themselves, there are homotopies-between-homotopies and ever higher-homotopies. Thereby topological spaces constitute a model for **higher categorical symmetry** namely for higher groupoids. As such, they represent both cohomology as well as higher gauge fields in the topological sector. [7]

cohomology	cocycle	coboundary	higher coboundary	...
homotopy	$X \xrightarrow{f} \mathcal{B}$	$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "f'"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\eta"] & \mathcal{B} \end{tikzcd}$	$\begin{tikzcd} X \arrow[r, bend left=60, "f"{name=U}] \arrow[r, bend right=60, "f'"'{name=D}] \arrow[from=U, to=D, Rightarrow, bend right=40, "\eta"'{name=L}] \arrow[from=U, to=D, Rightarrow, bend left=40, "\eta'"{name=R}] \arrow[from=L, to=R, phantom, "\Rrightarrow"] & \mathcal{B} \end{tikzcd}$	...
physics	field	gauge transf.	higher gauge transf.	...

In this vein, spaces are homotopy-*equivalent* $\mathcal{B} \simeq \mathcal{B}'$ if they are gauge equivalent namely if we have maps

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift left, "f"] & \mathcal{B}' \arrow[l, shift left, "g"] \end{tikzcd} \quad \text{with} \quad \begin{array}{l} g \circ f \Rightarrow \mathrm{id}_{\mathcal{B}} \\ f \circ g \Rightarrow \mathrm{id}_{\mathcal{B}'} \end{array}$$

For example $\mathbb{R}^n \simeq *$ in homotopy theory, reflecting the fact that there is no non-trivial topological sector for fields on $\mathbb{R}^n$.

For actually computing homotopy classes of maps — hence cohomology, hence gauge-equivalence classes of fields in the topological sector — tools from *model category theory* are indispensable, which largely say how to "absorb homotopies into spaces" (cf. [40, §1]).

E.g., if $p : \mathcal{A} \to \mathcal{B}$ is a *Serre fibration*, such as a fiber bundle, and Σ is a *cell complex*, such as a manifold, then sections-up-to-homotopy of p pulled back to Σ are homotopy equivalent to plain sections:

$$\left\{ \begin{tikzcd} \Sigma \arrow[r, dashed, "b"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"] \arrow[dl, phantom, "\swarrow \eta"] \\ X \arrow[r, "c"'] & \mathcal{B} \end{tikzcd} \right\}_{/\mathrm{hmtp}} \underset{\simeq}{\overset{\substack{\Sigma \in \mathrm{Cof} \\ p \in \mathrm{Fib}}}{}} \left\{ \begin{tikzcd} \Sigma \arrow[r, dashed, "b"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "p"] \arrow[dl, phantom, "="] \\ X \arrow[r, "c"'] & \mathcal{B} \end{tikzcd} \right\}_{/\mathrm{hmtp}}$$

Pointed homotopy theory (cf. [66, §3]). To reflect the condition that *solitonic fields are localized* in that they *vanish at infinity* we

- equip domain spaces X with a *point at infinity*, $\infty_X \in X$,
- equip classifying spaces $\mathcal{B}$ with a *point representing zero*, $0_{\mathcal{B}} \in \mathcal{B}$,
- require maps $f : (X, \infty_X) \to (\mathcal{B}, 0_{\mathcal{B}})$ to respect these base points

so that maps literally vanish at infinity

$$\begin{tikzcd} X \arrow[r, "c"] & \mathcal{B} \\ \{\infty_X\} \arrow[u, hook] \arrow[r, hook] & \{0_{\mathcal{B}}\} \arrow[u, hook] \end{tikzcd}.$$

For instance, to make fields on $\mathbb{R}^n$ vanish at infinity, we adjoin its would-be "point at infinity" to it (jargon: "one-point compactification") to obtain $\mathbb{R}^n_{\cup\{\infty\}} \simeq S^n$. On the other hand, if we want fields on some X without a vanishing condition, we may adjoin a *disjoint* point-at-infinity, then pointed maps $X_{\sqcup\{\infty\}} \to \mathcal{B}$ are ordinary $X \to \mathcal{B}$. E.g.:

$$\underset{\text{based loop space}}{\mathrm{Maps}^*(\mathbb{R}^1_{\cup\{\infty\}}, \mathcal{B}) = \Omega\mathcal{B}}, \qquad \underset{\text{free loop space}}{\mathrm{Maps}^*(S^1_{\sqcup\{\infty\}}, X) =: \mathcal{L}\mathcal{B}}, \qquad \underset{\text{maps out of contractible}}{\mathrm{Maps}^*(\mathbb{R}^1_{\sqcup\{\infty\}}, \mathcal{B}) = \mathcal{B}}$$

Given a pair of pointed spaces (X, ∞_X), (Y, ∞_Y), in their product space $X \times Y$ any point should be regarded as being at infinity which is so with respect to either factor space; this yields the *smash product*:

$$X \wedge Y := \frac{X \times Y}{\{\infty_X\} \times Y \cup X \times \{\infty_Y\}}$$

to which the sub-space $\mathrm{Maps}^*(-,-)$ of pointed maps is again adjoint:

$$\big\{P \xrightarrow{\mathrm{pntd}} \mathrm{Maps}^{*/}(X,Y)\big\} \simeq \big\{P \wedge X \xrightarrow{\mathrm{pntd}} Y\big\}.$$

For example, $S^n \wedge S^m \simeq \mathbb{R}^n_{\cup\{\infty\}} \wedge \mathbb{R}^m_{\cup\{\infty\}} \simeq (\mathbb{R}^n \times \mathbb{R}^m)_{\cup\{\infty\}} \simeq S^{n+m}$, so that for instance:

$$\mathrm{Maps}^*(X \wedge S^1, \mathcal{B}) \simeq \mathrm{Maps}^*\Big(S^1, \mathrm{Maps}^{*/}(X, \mathcal{B})\Big) =: \Omega\, \mathrm{Maps}^*(X, \mathcal{B}).$$

---

[7] Beyond the topological sector, full higher gauge fields are still represented by maps $X \to \mathcal{B}$ etc., only that now $\mathcal{B}$ is no longer just a topological space but a "smooth ∞-stack", cf. [33].

The differential character map $\mathbf{ch}_{\mathcal{A}}$, at the heart of flux-quantization in the generality of flux densities with non-linear Bianchi identities:

- takes maps into a classifying space $\mathcal{A}$ (classifying **charges**),
- to maps into the moduli ∞-stack of closed $\mathfrak{l}\mathcal{A}$-valued differential forms (classifying corresponding **flux densities**),
- thereby allowing **gauge potentials** to relate local flux densities to global charges.

$$\begin{tikzcd}
 & X \arrow[dl, "\text{charges}"'] \arrow[dr, "\text{fluxes}"] & \\
\mathcal{A} \arrow[rr, Rightarrow, "\text{gauge potentials}"] \arrow[dr, "\text{character}", "\mathbf{ch}_{\mathcal{A}}"'] & & \Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}} \arrow[dl, "\eta^{\int}"] \\
 & \int\Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}} &
\end{tikzcd}$$

At a high level, this $\mathbf{ch}_{\mathcal{A}}$ is readily described: It is the smooth differential-form model for the $\mathbb{R}$**-rationalization** of $\mathcal{A}$, followed by derived extension of scalars $\mathbb{Q} \to \mathbb{R}$ — as indicated in the following paragraphs.

But under the hood, this construction makes use of a fair bit of model category-theoretic rational-homotopy theory which we do not have space nor inclination to review here (all details in [40]), whence the following should be ignored by readers without serious background in (rational) homotopy theory — or else taken as motivation to learn it! (Start at [40, §1].) Here goes:

Fundamental theorem of homotopy theory. Regarding (classifying) spaces up to (weak) homotopy equivalence means equivalently to regard them as their ∞-*groupoids* (Kan simplicial sets) $\mathrm{Sing}(-)$ of points, paths, 2-paths, etc., in that there is a Quillen equivalence [40, Ex. 1.13]

$$\begin{tikzcd}
\mathrm{TopSp}_{\mathrm{Qu}} \arrow[r, phantom, "\simeq_{\mathrm{Qu}}"] \arrow[r, leftarrow, shift left=2] \arrow[r, shift right=2, "\mathrm{Sing}"'] & \Delta\mathrm{Set}_{\mathrm{Qu}}
\end{tikzcd}$$

Fundamental theorem of dg-algebraic rational homotopy theory. Sending simplicial sets to their dgc-algebras of simplex-wise $\mathbb{Q}$-polynomial differential forms ("piecewise linear", PL) is the left adjoint in a Quillen adjunction [40, Prop. 5.5]

$$\begin{tikzcd}
\big(\mathrm{dgcAlgs}^{\geq 0}\big)^{\mathrm{op}}_{\mathrm{proj}} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, leftarrow, shift left=2, "\Omega^\bullet_{\mathrm{P}\mathbb{Q}\mathrm{LdR}}"] \arrow[r, shift right=2, "\mathrm{Hom}\big((-),\,\Omega_{\mathrm{P}\mathbb{Q}\mathrm{LdR}}(\Delta^\bullet)\big)"'] & \Delta\mathrm{Sets}_{\mathrm{Qu}}
\end{tikzcd}$$

whose derived adjunction-unit models rationalization of (connected, nilpotent, $\mathbb{Q}$-finite) homotopy types $\mathcal{A}$ [40, Prop. 5.6].

$$\mathcal{A} \xrightarrow{\;\eta^{\mathbb{Q}}_{\mathcal{A}}\;} L^{\mathbb{Q}}\mathcal{A}$$

For $\mathbb{R}$-rational homotopy. The analogous Quillen adjunction with $\mathbb{R}$-polynomial forms

$$\begin{tikzcd}
\big(\mathrm{dgcAlgs}^{\geq 0}\big)^{\mathrm{op}}_{\mathrm{proj}} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, leftarrow, shift left=2, "\Omega^\bullet_{\mathrm{P}\mathbb{R}\mathrm{L}}"] \arrow[r, shift right=2, "\mathrm{Hom}\big((-),\,\Omega_{\mathrm{P}\mathbb{R}\mathrm{LdR}}(\Delta^\bullet)\big)"'] & \Delta\mathrm{Sets}_{\mathrm{Qu}}
\end{tikzcd}$$

models rationalization followed by derived extension of scalars from $\mathbb{Q}$ to $\mathbb{R}$ (no longer a localization but still denoted like one) [40, Prop. 5.8].

$$\mathcal{A} \xrightarrow{\;\eta^{\mathbb{Q}}_{\mathcal{A}}\;} L^{\mathbb{Q}}\mathcal{A} \longrightarrow L^{\mathbb{R}}\mathcal{A}$$

Now with $\mathbb{R}$-coefficients, we may equivalently use simplex-wise *smooth* differential forms (*piecewise smooth*, PS)

$$\begin{tikzcd}
\big(\mathrm{dgcAlgs}^{\geq 0}\big)^{\mathrm{op}}_{\mathrm{proj}} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, leftarrow, shift left=2, "\Omega^\bullet_{\mathrm{P}\mathbf{S}\mathrm{dR}}"] \arrow[r, shift right=2, "\mathrm{Hom}\big((-),\,\Omega_{\mathrm{P}\mathbf{S}\mathrm{dR}}(\Delta^\bullet)\big)"'] & \Delta\mathrm{Sets}_{\mathrm{Qu}}
\end{tikzcd}$$

In fact, we may equivalently use smooth differential forms on simplices times any $\mathbb{R}^n$ [40, Prop. 5.10].

$$\begin{tikzcd}
\big(\mathrm{dgcAlgs}^{\geq 0}\big)^{\mathrm{op}}_{\mathrm{proj}} \arrow[r, phantom, "\perp_{\mathrm{Qu}}"] \arrow[r, leftarrow, shift left=2, "\Omega^\bullet_{\mathrm{P}\mathbf{S}\mathrm{dR}}"] \arrow[r, shift right=2, "\mathrm{Hom}\big((-),\,\Omega_{\mathrm{P}\mathbf{S}\mathrm{dR}}(\mathbb{R}^n\times\Delta^\bullet)\big)"'] & \Delta\mathrm{Sets}_{\mathrm{Qu}}
\end{tikzcd}$$

Taking values in deformations of flux densities. Via the minimal Sullivan model $\mathrm{CE}(\mathfrak{l}\mathcal{A})$ of $\mathcal{A}$, this derived adjunction takes values in closed smooth $\mathfrak{l}\mathcal{A}$-valued differential forms [40, (9.9)]

$$\begin{aligned} &\Omega^1_{\mathrm{dR}}\big(\mathbb{R}^n \times \Delta^\bullet,\, \mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}} \\ &:= \mathrm{Hom}\Big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\, \Omega_{\mathrm{dR}}(\mathbb{R}^n \times \Delta^\bullet)\Big)\end{aligned}$$

which is the value on $\mathbb{R}^n$ of the homotopy-constant ∞-stack that is the *shape* $\int(-)$ of the sheaf of closed forms [114, Prop. 3.3.48]

$$\int\Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}} \;\in\; \mathrm{Sh}_\infty(\mathrm{CartSp})$$

In total, regarding also $\mathcal{A} \in \mathrm{Sh}_\infty(*) \xrightarrow{\mathrm{Disc}} \mathrm{Sh}_\infty(\mathrm{CartSp})$, this establishes the *differential character* map as promised [40, Def. 9.2]

$$\mathcal{A} \xrightarrow{\;\mathbf{ch}_{\mathcal{A}}\;} \int\Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{l}\mathcal{A}\big)_{\mathrm{clsd}}$$

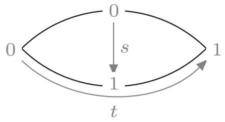
B Background on TED Cohomotopy

Gauge potentials in twistorial Cohomotopy — and the Green-Schwarz mechanism.

Consider the Whitehead L_∞-algebra of the twistor fibration $\mathbb{C}P^3 \xrightarrow{t_{\mathbb{H}}} \mathbb{H}P^1 \simeq S^4$,

$$\mathrm{CE}\big(\mathfrak{l}_{S^4}\mathbb{C}P^3\big) = \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} f_2 \\ h_3 \\ g_4 \\ g_7 \end{array}\right] \Big/ \left(\begin{array}{rcl} \mathrm{d}\,f_2 &=& 0 \\ \mathrm{d}\,h_3 &=& g_4 + f_2\,f_2 \\ \mathrm{d}\,g_4 &=& 0 \\ \mathrm{d}\,g_7 &=& \tfrac{1}{2}g_4\,g_4 \end{array}\right),$$

and bigons parameterized like this:



Theorem ([49, pp 23][50, §4.1]). Given a manifold U_i (generically: a coordinate chart):
(0.) Closed $\mathfrak{l}_{S^4}\mathbb{C}P^3$-valued differential forms are in natural bijection with **flux densities** of this form:

$$\left\{\begin{tikzcd} U_i \arrow[d, "{(F_2,H_3,G_4,G_7)}"] \\ \Omega^1_{\mathrm{dR}}\big(-;\mathfrak{l}_{S^4}\mathbb{C}P^3\big)_{\mathrm{clsd}} \end{tikzcd}\right\} \begin{array}{c} i_0\circ p_0 \,=\, \mathrm{id} \\ \overset{p_0}{\twoheadrightarrow} \\ \underset{i_0}{\hookleftarrow} \\ p_0\circ i_0 \,=\, \mathrm{id} \end{array} \left\{\begin{array}{l|l} F_2 \,\in\, \Omega^2_{\mathrm{dR}}(U_i) & \mathrm{d}\,F_2 \,=\, 0 \\ H_3 \,\in\, \Omega^3_{\mathrm{dR}}(U_i) & \mathrm{d}\,H_3 \,=\, G_4 \,+\, F_2\,F_2 \\ G_4 \,\in\, \Omega^4_{\mathrm{dR}}(U_i) & \mathrm{d}\,G_4 \,=\, 0 \\ G_7 \,\in\, \Omega^7_{\mathrm{dR}}(U_i) & \mathrm{d}\,G_7 \,=\, \tfrac{1}{2}\,G_4\,G_4 \end{array}\right\}$$

(1.) Given one of these, its set of coboundaries (null-concordances) naturally retracts onto the set of **gauge potentials** of this form:

$$\left\{\begin{tikzcd} U_i \arrow[rr] \arrow[d, "{(F_2,H_3,G_4,G_7)}"'] & & \ast \arrow[d, "0"] \arrow[dll, Rightarrow, dashed, "{(\widehat{F}_2,\widehat{H}_3,\widehat{G}_4,\widehat{G}_7)}"] \\ \Omega^1_{\mathrm{dR}}\big(-;\mathfrak{l}_{S^4}\mathbb{C}P^3\big)_{\mathrm{clsd}} \arrow[rr, "\eta^{\int}"] & & \int\Omega^1_{\mathrm{dR}}\big(-;\mathfrak{l}_{S^4}\mathbb{C}P^3\big)_{\mathrm{clsd}} \end{tikzcd}\right\} \begin{array}{c} \overset{p_1}{\twoheadrightarrow} \\ \underset{i_1}{\hookleftarrow} \\ p_1\circ i_1 \,=\, \mathrm{id} \end{array} \left\{\begin{array}{l|l} A_1 \,\in\, \Omega^1_{\mathrm{dR}}(U_i) & \mathrm{d}\,A_1 \,=\, F_2 \\ B_2 \,\in\, \Omega^2_{\mathrm{dR}}(U_i) & \mathrm{d}\,B_2 \,=\, H_3 - C_3 - A_1\,F_2 \\ C_3 \,\in\, \Omega^3_{\mathrm{dR}}(U_i) & \mathrm{d}\,C_3 \,=\, G_4 \\ C_6 \,\in\, \Omega^6_{\mathrm{dR}}(U_i) & \mathrm{d}\,G_6 \,=\, G_7 - \tfrac{1}{2}C_3\,G_4 \end{array}\right\}$$

$$\left(\begin{array}{rcl} \widehat{F}_2 &:=& t\,F_2 \,+\, \mathrm{d}t\,A_1 \\ \widehat{H}_3 &:=& t\,H_3 \,+\, \mathrm{d}t\,B_2 \,+\, (t^2-t)A_1F_2 \\ \widehat{G}_4 &:=& t\,G_4 \,+\, \mathrm{d}t\,C_3 \\ \widehat{G}_7 &:=& t^2\,G_7 \,+\, 2t\mathrm{d}t\,C_6 \end{array}\right) \begin{array}{c} \longmapsto \\ \longleftarrow\!\shortmid \end{array} \left(\begin{array}{rcl} A_1 &:=& \int_{[0,1]}\widehat{F}_2 \\ B_2 &:=& \int_{[0,1]}\Big(\widehat{H}_3 - \big(\int_{[0,-]}\widehat{F}_2\big)\widehat{F}_2\Big) \\ C_3 &:=& \int_{[0,1]}\widehat{G}_4 \\ C_6 &:=& \int_{[0,1]}\Big(\widehat{G}_7 - \tfrac{1}{2}\big(\int_{[0,-]}\widehat{G}_4\big)\widehat{G}_4\Big) \end{array}\right)$$

(2.) Given a pair of these, the set of higher coboundaries (2nd-order concordances) between them naturally retracts onto the set of **gauge transformations** of this form:

$$\left\{\begin{tikzcd}[column sep=huge] 0 \arrow[r, Rightarrow, bend left=50, ""{name=U, below}, "{(\widehat{F}_2,\widehat{H}_3,\widehat{G}_4,\widehat{G}_7)}"] \arrow[r, Rightarrow, bend right=50, ""{name=D}, "{(\widehat{F}'_2,\widehat{H}'_3,\widehat{G}'_4,\widehat{G}'_7)}"'] & (F_2,H_3,G_4,G_7) \arrow[from=U, to=D, Rightarrow, dashed, "{\left(\begin{smallmatrix}\widehat{\widehat{F}}_2,\,\widehat{\widehat{H}}_3\\ \widehat{\widehat{G}}_4,\,\widehat{\widehat{G}}_7\end{smallmatrix}\right)}"'] \end{tikzcd}\right\} \begin{array}{c} \overset{p_2}{\twoheadrightarrow} \\ \underset{i_2}{\hookleftarrow} \\ p_2\circ i_2 \,=\, \mathrm{id} \end{array} \left\{\begin{array}{l|l} \alpha_0 \,\in\, \Omega^0_{\mathrm{dR}}(U_i) & \mathrm{d}\,\alpha_0 \,=\, A'_1 - A_1 \\ \beta_1 \,\in\, \Omega^1_{\mathrm{dR}}(U_i) & \mathrm{d}\,\beta_1 \,=\, B'_2 - B_2 + \gamma_2 + \alpha_0\,F_2 \\ \gamma_2 \,\in\, \Omega^2_{\mathrm{dR}}(U_i) & \mathrm{d}\,\gamma_2 \,=\, C'_3 - C_3 \\ \gamma_5 \,\in\, \Omega^5_{\mathrm{dR}}(U_i) & \mathrm{d}\,\gamma_5 \,=\, C'_6 - C_6 - \tfrac{1}{2}C'_3\,C_3 \end{array}\right\}$$

$$\left(\begin{array}{rl} \widehat{\widehat{F}}_2 :=& t\,F_2 \,+\, \mathrm{d}t\,A_1 \,+\, s\,\mathrm{d}t\big(A'_1 - A_1\big) \,-\, \mathrm{d}s\,\mathrm{d}t\,\alpha_0 \\ \widehat{\widehat{H}}_3 :=& t\,H_3 \,+\, \mathrm{d}t\,B_2 \,+\, s\,\mathrm{d}t\,(B'_2 - B_2) \,-\, \mathrm{d}s\,\mathrm{d}t\,\beta_1 \\ & \quad + (t^2-t)A_1F_2 + (t^2-t)s(A'_1 - A_1)F_2 \\ & \qquad + (t^2-t)\mathrm{d}s\,\alpha_0F_2 \\ \widehat{\widehat{G}}_4 :=& t\,G_4 \,+\, \mathrm{d}t\,C_3 \,+\, s\,\mathrm{d}t\big(C'_3 - C_3\big) \,-\, \mathrm{d}s\,\mathrm{d}t\,\gamma_2 \\ \widehat{\widehat{G}}_7 :=& t^2\,G_7 \,+\, 2t\mathrm{d}t\,C_6 \,+\, 2st\mathrm{d}t(C'_6 - C_6) \\ & \quad - 2\mathrm{d}s\,t\mathrm{d}t\big(\gamma_5 \,+\, \tfrac{1}{2}\gamma_2\,C_3\big) \end{array}\right) \begin{array}{c} \longmapsto \\ \longleftarrow\!\shortmid \end{array} \left(\begin{array}{l} \alpha_0 := \int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{F}}_2 \\ \beta_1 := \int_{s\in[0,1]}\int_{t\in[0,1]}\Big(\widehat{\widehat{H}}_3 \,-\, \big(\int_{t'\in[0,-]}\widehat{\widehat{F}}_2\big)\widehat{\widehat{F}}_2\Big) \\ \gamma_2 := \int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{G}}_4 \\ \gamma_5 := \int_{s\in[0,1]}\int_{t\in[0,1]}\Big(\widehat{\widehat{G}}_7 - \tfrac{1}{2}\big(\int_{t'\in[0,-]}\widehat{\widehat{G}}_4\big)\widehat{\widehat{G}}_4\Big) \\ \qquad -\tfrac{1}{2}\gamma_2\,C_3 \end{array}\right)$$

Notice the expression for flux density subject to an (abelian) Green-Schwarz mechanism: $H_3 \,=\, \mathrm{d}\,B_2 \,+\, A_1F_2 \,+\, C_3$.

Proof. With the blue terms discarded, this is the statement of [49, pp 23][50, §4.1]. We compile the full argument:

To see that p_1 is well-defined:

- for C_3, C_6 this is [49, (70)],
- for A_1 it works just as for C_3,
- for B_2 we compute, in generalization of [50, below (138)], like this:

$$\begin{aligned}
\mathrm{d}\,B_2 \;&\equiv\; \mathrm{d}\textstyle\int_{[0,1]}\Big(\widehat{H}_3 - \big(\int_{[0,-]}\widehat{F}_2\big)\widehat{F}_2\Big) \\
&=\; \underbrace{\iota_1^*\Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]}\widehat{F}_2\big)\widehat{F}_2\Big)}_{H_3 - A_1\,F_2} - \underbrace{\iota_0^*\Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]}\widehat{F}_2\big)\widehat{F}_2\Big)}_{=0} - \textstyle\int_{[0,1]}\underbrace{\mathrm{d}\Big(\widehat{H}_3 - \big(\textstyle\int_{[0,-]}\widehat{F}_2\big)\widehat{F}_2\Big)}_{\widehat{G}_4} \\
&=\; H_3 \;-\; A_1\,F_2 \;-\; C_3\,.
\end{aligned}$$

To see that i_1 is well-defined:

- for $\widehat{G}_4, \widehat{G}_7$ this is [49, (72)],
- for $\widehat{F}_2$ it works just as for $\widehat{G}_4$,
- for $\widehat{H}_3$ we compute, in generalization of [50, further below (138)], as follows:

$$\left.\begin{aligned}
&\mathrm{d}\big(tH_3 + \mathrm{d}t\,B_2 + (t^2 - t)A_1F_2\big) \\
&= \mathrm{d}t\,H_3 + tG_4 + tF_2F_2 \\
&\quad - \mathrm{d}t\,H_3 + \mathrm{d}t\,C_3 + \mathrm{d}t\,A_1F_2 \\
&\quad + \mathrm{d}\big((t^2 - t)A_1F_2\big)
\end{aligned}\right\}\ \text{hence indeed: } \mathrm{d}\,\widehat{H}_3 \;=\; \underbrace{t\,G_4 + \mathrm{d}t\,C_3}_{\widehat{G}_4} + \overbrace{\underbrace{(tF_2 + \mathrm{d}t\,A_1)}_{\widehat{F}_2}\underbrace{(tF_2 + \mathrm{d}t\,A_1)}_{\widehat{F}_2}}^{=\mathrm{d}(t^2A_1F_2)}$$

  Moreover, it is immediate from inspection that $\iota_1^*\widehat{H}_3 \;=\; H_3$ and $\iota_0^*\widehat{H}_3 = 0$.

To see that $p_1 \circ i_1 = \mathrm{id}$:

- for C_3, C_6 this is [49, below (72)],
- for A_1 this works just as for C_3,
- for B_2 we immediately compute:

$$\textstyle\int_{[0,1]}\Big(\widehat{H}_3 - \big(\int_{[0,-]}\widehat{F}_2\big)F_2\Big) \;=\; \underbrace{\textstyle\int_{[0,1]}\mathrm{d}t\,B_2}_{B_2} - \textstyle\int_{[0,1]}\underbrace{tA_1\,\mathrm{d}t\,A_1}_{=0} \;=\; B_2\,.$$

To see that p_2 is well-defined:

- for $\widehat{\widehat{G}}_4, \widehat{\widehat{G}}_7$ this is [49, (74-5)],
- for $\widehat{\widehat{F}}_2$ this works just as for $\widehat{\widehat{F}}_2$,
- for $\widehat{\widehat{H}}_3$ we compute, in generalization of [50, below (140)], as follows:

$$\begin{aligned}
\mathrm{d}\beta_1 \;&\equiv\; \mathrm{d}\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\Big(\widehat{\widehat{H}}_3 - \big(\int_{t'\in[0,-]}\widehat{\widehat{F}}_2\big)\widehat{\widehat{F}}_2\Big) \\
&=\; \iota_{s=1}^*\textstyle\int_{t\in[0,1]}\big(\widehat{\widehat{H}}_3 - \cdots\big) - \iota_{s=0}^*\int_{t\in[0,1]}\big(\widehat{\widehat{H}}_3 - \cdots\big) - \int_{s\in[0,1]}\mathrm{d}\int_{t\in[0,1]}\big(\widehat{\widehat{H}}_3 - \cdots\big) \\
&=\; \textstyle\int_{t\in[0,1]}\iota_{s=1}^*\big(\widehat{\widehat{H}}_3 - \cdots\big) - \int_{t\in[0,1]}\iota_{s=0}^*\big(\widehat{\widehat{H}}_3 - \cdots\big) - \int_{s\in[0,1]}\iota_{t=1}^*\big(\widehat{\widehat{H}}_3 - \cdots\big) + \int_{s\in[0,1]}\int_{t\in[0,1]}\mathrm{d}\big(\widehat{\widehat{H}}_3 - \cdots\big) \\
&=\; \textstyle\int_{t\in[0,1]}\big(\widehat{H}'_3 - \cdots\big) - \int_{t\in[0,1]}\big(\widehat{H}_3 - \cdots\big) \;+\; \big(\int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{F}}_2\big)F_2 + \int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{G}}_4 \\
&=\; B'_2 - B_2 + \alpha_0\,F_2 + \gamma_2\,.
\end{aligned}$$

To see that i_2 is well-defined:

- for γ_2, γ_5 this is [49, (76)],
- for α_0 this works just as for γ_2,

- for β_1 we compute as follows:

$$
\begin{array}{ccl}
\mathrm{d}\,\big(t\,H_3 \,+\, \mathrm{d}t\,B_2 \,+\, s\,\mathrm{d}t\,(B_2' - B_2) \,-\, \mathrm{d}s\,\mathrm{d}t\,\beta_1\big) &=& \overbrace{t\,G_4 \,+\, \mathrm{d}t\,C_3 \,+\, s\,\mathrm{d}t(C_3' - C_3) \,-\, \mathrm{d}s\,\mathrm{d}t\,\gamma_2}^{\widehat{\widehat{G}}_4} \\
&& +\,t\,F_2F_2 \,+\, \mathrm{d}t\,A_1\,F_2 \,+\, s\,\mathrm{d}t(A_1' - A_1)F_2 \,-\, \mathrm{d}s\,\mathrm{d}t\,\alpha_0\,F_2 \\
&& \\
\mathrm{d}\left(\begin{array}{l} (t^2 - t)A_1\,F_2 \,+\, (t^2 - t)s(A_1' - A_1)F_2 \\ \quad +\,(t^2 - t)\,\mathrm{d}s\,\alpha_0\,F_2 \end{array}\right) &=& \overbrace{t^2\,F_2\,F_2 \,+\, 2t\mathrm{d}t\,A_1F_2 \,+\, 2t\mathrm{d}t\,s(A_1' - A_1) \,+\, 2t\mathrm{d}t\,\mathrm{d}s\,\alpha_0\,F_2}^{\widehat{\widehat{F}}_2\,\widehat{\widehat{F}}_2} \\
&& -\,t\,F_2\,F_2 \,-\, \mathrm{d}t\,A_1\,F_2 \,-\, \mathrm{d}t\,s(A_1' - A_1)F_2 \,-\, \mathrm{d}t\,\mathrm{d}s\,\alpha_0\,F_2 \\
\hline
\mathrm{d}\,\widehat{\widehat{H}}_3 &=& \widehat{\widehat{G}}_4 \,+\, \widehat{\widehat{F}}_2\,\widehat{\widehat{F}}_2\,.
\end{array}
$$

Moreover, it is immediate from inspection that $\iota^*_{s=0}\widehat{\widehat{H}}_3 \,=\, \widehat{H}_3$, $\iota^*_{s=1}\widehat{\widehat{H}}_3 \,=\, \widehat{H}_3'$ and $\iota^*_{t=0} \,=\, 0$, $\iota^*_{t=1} \,=\, H_3$.

To see that $p_2 \circ i_2 \,=\, \mathrm{id}$, we directly compute,
first

$$
\begin{aligned}
\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{G}}_4 \;&=\; \textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}(-\mathrm{d}s\,\mathrm{d}t\,\gamma_2) \;=\; \gamma_2 \\
\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{F}}_2 \;&=\; \textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}(-\mathrm{d}s\,\mathrm{d}t\,\alpha_0) \;=\; \alpha_0
\end{aligned}
$$

then

$$
\begin{aligned}
&\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\Big(\widehat{\widehat{G}}_7 - \tfrac{1}{2}\big(\int_{t'\in[0,t]}\widehat{\widehat{G}}_4\big)\widehat{\widehat{G}}_4\Big) \,-\, \tfrac{1}{2}\,\gamma_2\,C_3 \\
&= \textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\widehat{\widehat{G}}_7 \,-\, \tfrac{1}{2}\int_{s\in[0,1]}\int_{t\in[0,1]}\big(t\,C_3 + st(C_3' - C_2) + t\mathrm{d}s\,\gamma_2\big)\big(t\,G_4 \,+\, \mathrm{d}t\,C_3 \,+\, s\,\mathrm{d}t(C_3' - C_3) - \mathrm{d}s\,\mathrm{d}t\,\gamma_2\big) \\
&\quad -\tfrac{1}{2}\,\gamma_2\,C_3 \\
&= \big(\gamma_5 \,+\, \tfrac{1}{2}\gamma_2C_3\big)\underbrace{-\tfrac{1}{2}C_3\gamma_2 - \tfrac{1}{4}(C_3' - C_3)\gamma_2 + \tfrac{1}{2}\,\gamma_2\,C_3 + \tfrac{1}{4}\,\gamma_2\,(C_3' - C_3)}_{0} - \tfrac{1}{2}\,\gamma_2\,C_3 \\
&= \gamma_5
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&\textstyle\int_{s\in[0,1]}\,\int_{t\in[0,1]}\,\big(\int_{t'\in[0,-]}\,\widehat{\widehat{F}}_2\big)\widehat{\widehat{F}}_2 \\
&= \textstyle\int_{s\in[0,1]}\,\int_{t\in[0,1]}\,\big(tA_1 + st(A_1' - A_1) \,+\, t\mathrm{d}s\,\alpha_0\big)\big(t\,F_2 \,+\, \mathrm{d}t\,A_1 \,+\, s\,\mathrm{d}t(A_1' - A_1) \,-\, \mathrm{d}s\,\mathrm{d}t\,\alpha_0\big) \\
&= \tfrac{1}{2}A_1\alpha_0 + \tfrac{1}{4}(A_1' - A_1)\alpha_0 - \tfrac{1}{2}\alpha_0A_1 - \tfrac{1}{4}\alpha_0(A_1' - A_1) \\
&= 0
\end{aligned}
$$

so that also
$$
\textstyle\int_{s\in[0,1]}\int_{t\in[0,1]}\Big(\widehat{\widehat{H}}_3 - \big(\int_{t'\in[0,-]}\widehat{\widehat{F}}_2\big)\widehat{\widehat{F}}_2\Big) \;=\; \int_{s\in[0,1]}\int_{t\in[0,1]}(-\mathrm{d}s\,\mathrm{d}t\;\beta_1) \;=\; \beta_1\,.
$$

□

Cocycles in differential 2-Cohomotopy and the abelian Chern-Simons invariant on the 3-Sphere. Notice that the Bianchi identities encoded by 2-Cohomotopy are the characteristic property of the abelian Chern-Simons term:

$$\mathrm{CE}\big(\mathfrak{l}S^2\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_3 \end{bmatrix} \Big/ \left(\begin{aligned} \mathrm{d}\, f_2 \;&=\; 0 \\ \mathrm{d}\, h_3 \;&=\; f_2 f_2 \end{aligned} \right) \qquad \Rightarrow \qquad \Omega^1_{\mathrm{dR}}\big(X;\, \mathfrak{l}S^2\big)_{\mathrm{clsd}} \;\simeq\; \left\{ \begin{array}{l} F_2 \,\in\, \Omega^2_{\mathrm{dR}}(X) \\ H_3 \,\in\, \Omega^3_{\mathrm{dR}}(X) \end{array} \,\middle|\, \begin{array}{l} \mathrm{d}\, F_2 \;=\; 0 \\ \mathrm{d}\, H_3 \;=\; F_2\, F_2 \end{array} \right\}$$

We may bring this out more concretely:

Gauge-field configurations on $\mathbb{R}^3$ flux-quantized in 2-Cohomotopy and vanishing in a neighbourhood of infinity are cocycles in differential 2-Cohomotopy on $\mathbb{R}^3_{\cup\{\infty\}}$, hence dashed homotopies as shown on the right [117, §3.3].

$$\begin{tikzcd}
 & \mathbb{R}^3_{\cup\{\infty\}} \arrow[dl, dashed, "\text{flux density valued in } \mathfrak{l}S^2"', "{(F_2,H_3)}"] \arrow[dr, dashed, "\text{charge in 2-Cohomotopy}", "n"'] & \\
\Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^2\big)_{\mathrm{clsd}} \arrow[dr, "\eta^{\int}"'] \arrow[rr, phantom, "{\overset{(\mathbf{A}_1,\mathbf{B}_2)}{\Longleftarrow}} \atop {\text{gauge potentials in diff. 2-Cohomotopy}}"] & & \int S^2 \arrow[dl, "\mathbf{ch}"] \\
 & \int \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^2\big)_{\mathrm{clsd}} &
\end{tikzcd}$$

Theorem. For each $[n] \in \pi^2\big(\mathbb{R}^3_{\cup\{\infty\}}\big) \simeq \mathbb{Z}$ this exists with $H_3 = 0$ and $[n] = \int_{\mathbb{R}^3} A_1\, F_2$ the Chern-Simons invariant.

Lemma. *On a smooth manifold Σ,*
every cocycle α in rational 3-Cohomotopy
is represented by a globally defined differential form H_3,

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, dashed, "H_3"'] & \arrow[d, phantom, "\Leftarrow"] & \int \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^3\big)_{\mathrm{clsd}} \\
 & \Omega^1_{\mathrm{dR}}\big(-;\, \mathfrak{l}S^3\big)_{\mathrm{clsd}} \arrow[ur, "\eta^{\int}"'] &
\end{tikzcd}$$

Proof of the Lemma. Since $\mathfrak{l}S^3 \simeq \mathfrak{l}B^3\mathbb{Q}$ this is just the degree=3 case of the statement that cocycles in de Rham hyper-cohomology have global representatives on smooth manifolds (using partitions of unity). □

Proof of the Theorem. Stereographic projection provides a homeomorphism $\mathbb{R}^3_{\cup\{\infty\}} \xrightarrow{\sim} S^3$ which is smooth away from the point at infinity, which we may slightly deform to a smooth degree=1 map that is constant on a neighbourhood of infinity. Since $\pi^2(S^3) \simeq \pi_2(S^3) \simeq \mathbb{Z}$ we may find a smooth map $n : S^3 \to S^2$, with compact support away from the base point, so that $\mathbb{R}^3_{\cup\{\infty\}} \to S^3 \xrightarrow{n} S^2$ represents the charge $[n]$.

Now the 2-cohomotopical character map for charges on S^3, shown in black, factors as shown in blue (by naturality of rationalization), which furthermore factors as shown in orange (by the above Lemma).

$$\begin{tikzcd}
\mathbb{R}^3_{\cup\{\infty\}} \arrow[r] & S^3 \arrow[rr, "\eta^{\int}"] \arrow[d, "n\cdot\mathrm{dvol}_{S^3}"] & \arrow[dl, Rightarrow] & \int S^3 \arrow[rr, "n"] \arrow[d, "\mathbf{ch}_{S^3}"] & \arrow[dl, Rightarrow] & \int S^2 \arrow[d, "\mathbf{ch}_{S^2}"] \\
 & \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}S^3)_{\mathrm{clsd}} \arrow[rr, "\eta^{\int}"] & & \int \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}S^3)_{\mathrm{clsd}} \arrow[rr, "(\mathfrak{l}n)_*"] & & \int \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}S^2)_{\mathrm{clsd}}
\end{tikzcd} \tag{4}$$

Hence to get a differential cocycle as desired it is sufficient to exhibit gauge potentials (A_1, B_2) encoding a concordance filling the diagram on the right

$$\begin{tikzcd}
\mathbb{R}^3_{\cup\{\infty\}} \arrow[r] & S^3 \arrow[rr, "n"] \arrow[dr, "n\cdot\mathrm{dvol}_{S^3}"] \arrow[dd, "{(F_2,H_3=0)}"'] & & S^2 \arrow[r, "\eta^{\int}"] & \int S^2 \arrow[dd, "\mathbf{ch}_2"] \arrow[dll, Rightarrow, "(4)"] \\
 & \arrow[r, phantom, "\overset{(A_1,B_2)}{\Leftarrow}"] & \Omega^1_{\mathrm{dR}}(-;\, \mathfrak{l}S^3)_{\mathrm{clsd}} \arrow[drr, "(\mathfrak{l}n)_* \circ \eta^{\int}"] & & \\
 & \Omega^1_{\mathrm{dR}}(-;\, \mathfrak{l}S^2)_{\mathrm{clsd}} \arrow[rrr, "\eta^{\int}"'] & & & \int \Omega^1_{\mathrm{dR}}(-;\, \mathfrak{l}S^2)_{\mathrm{clsd}}
\end{tikzcd}$$

But, since $H^2_{\mathrm{dR}}(S^3) = 0$, and by the *Whitehead integral formula* (cf. [53, p 134][11, p 228][36, pp 19]) there exists:

$$\left\{ \begin{array}{lcl} A_1 & \in & \Omega^1_{\mathrm{dR}}(S^3) \\ B_2 & \in & \Omega^2_{\mathrm{dR}}(S^3) \end{array} \right. \quad \text{s.t.} \quad \begin{array}{lcl} \mathrm{d}\, A_1 & = & F_2 \;:=\; n^*\mathrm{dvol}_{S^2} \\ \mathrm{d}\, B_2 & = & n \cdot \mathrm{dvol}_{S^3} - A_1 F_2 \end{array}$$

From this we get the the desired concordance:

$$(0, n\cdot\mathrm{dvol}_{S^3}) \Rightarrow (F_2, 0) \quad : \quad \left\{ \begin{array}{lcl} \widehat{F}_2 & := & t\, F_2 \,+\, \mathrm{d}t\, A_1 \\ \widehat{H}_3 & := & (t-1) n\, \mathrm{dvol}_{S^3} \,+\, \mathrm{d}t\, B_2 \,+\, (t^2 - t) A_1 F_2\,. \end{array} \right. \qquad \begin{array}{r} (\widehat{F}_2, \widehat{H}_3)|_{t=0} \;=\; (0, n\cdot\mathrm{dvol}_{S^3}) \\ (\widehat{F}_2, \widehat{H}_3)|_{t=1} \;=\; (F_2, 0) \\ \mathrm{d}\,\widehat{F}_2 \;=\; 0\,,\; \mathrm{d}\,\widehat{H}_3 \;=\; \widehat{F}_2\, \widehat{F}_2\,. \end{array}$$

□

Cartesian M5-Probes charged in Cohomotopy. The equations of motion for a(n orbifolded) cartesian M5-probe demand that the flux $H_3 = \mathrm{const}$ [50, Ex. 3.14], and thus its solitonic vanishing-at-infinity implies $H_3 = 0$. The above theorem says that such solutions still support non-vanishing cohomotopical charge, in fact that the vanishing of H_3 forces the charge to be carried by the Chern-Simons invariant of the auxiliary gauge field $\mathbf{A}_1$ that is brought in by the cohomotopical flux quantization.

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