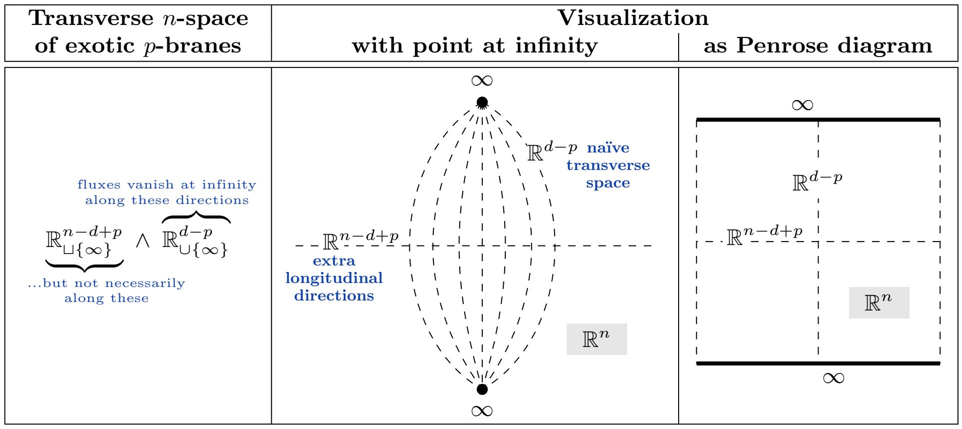
2.3.2 Exotic brane configurations via May-Segal's theorem

Cohomotopy charge in low codimension. The Pontrjagin theorem (§2.3.1) suggests that every solitonic brane seen by n-Cohomotopy "wants to be" a $d-n$-brane. Indeed, if the available transverse space is $< n$-dimensional as for exotic branes (e.g. the M9), then the *May-Segal theorem* [May72, Thm. 2.7][Segal73, Thm. 3] may be understood as saying that n-Cohomotopy still sees $(d-n)-$*branes*, but "delocalized" to look like exotic branes.



May-Segal's theorem indeed identifies the n-Cohomotopy moduli of $< n$-dimensional transverse spaces with the configurations of (unordered) points in $\mathbb{R}^n$ which are distinct as points in $\mathbb{R}^{d-p}$ (and as such look like transverse positions of flat p-branes) while their "core" may escape to infinity in the tangential direction, reflecting the fact that the n-Cohomotopy flux is not constrained to vanish at infinity in these directions.

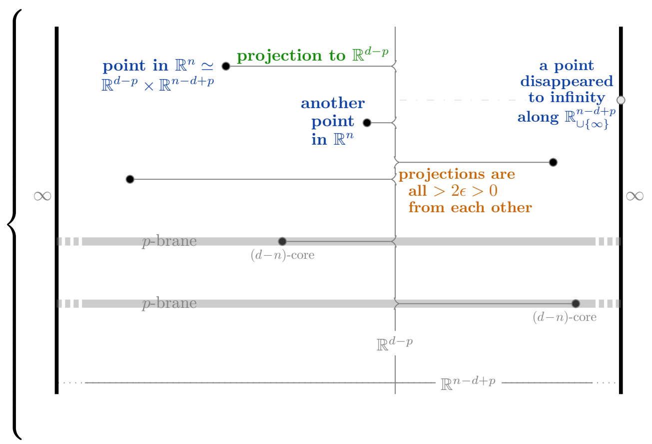
$$
\mathrm{Maps}^{*/}\Big(\mathbb{R}^{n-d+p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{d-p}_{\cup\{\infty\}},\, S^n\Big) \;=:\; \underbrace{\widetilde{\boldsymbol{\pi}}^n\Big(\mathbb{R}^{1,d-n}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{n-d+p}_{\sqcup\{\infty\}} \wedge \mathbb{R}^{d-p}_{\cup\{\infty\}}\Big)}_{\substack{n\text{-cohomotopical moduli space of} \\ \text{flat solitonic } p\text{-brane worldvolumes} \\ \text{with } < n\text{-dim. transverse space}}}
$$

$$\wr$$

$$
\mathrm{Maps}^{*/}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}},\, S^n\Big)
$$

$$
\Big\uparrow \wr \;\; \text{May-Segal theorem} \;/\; \text{cohomotopy charge map}
$$

$$
\underbrace{\mathrm{Conf}\Big(\mathbb{R}^{d-p}_{\sqcup\{\infty\}},\, \mathbb{R}^{p-d+n}_{\cup\{\infty\}}\Big)}_{\substack{\text{configuration space of points} \\ \text{in transverse space } \mathbb{R}^{d-p} \\ \text{with "labels" in } \mathbb{R}^{n-d+p}_{\cup\{\infty\}}}} =
$$



Notice the dichotomy: If branes *can not escape to* ∞ then their *fluxes vanish at* ∞ and vice versa. This is why in passing from the Cohomotopy charge to the corresponding brane configurations the subscripts swap as $(-)_{\sqcup\{\infty\}} \leftrightarrow (-)_{\cup\{\infty\}}$.

configuration space of points

$\mathrm{Conf}\big(\mathbb{R}^{d-p}_{\sqcup\{\infty\}}, \mathbb{R}^{n-d+p}_{\cup\{\infty\}}\big)$ is the pointed space of
• un-ordered tuples of points in $\mathbb{R}^n \simeq \mathbb{R}^{d-p} \times \mathbb{R}^{n-d+p}$ — as such they look like flat solitonic $d-n$-branes.
• which have pairwise distinct projections to $\mathbb{R}^{d-p}$ — as such they look like flat solitonic p-branes
• and may escape to or emerge from ∞ along $\mathbb{R}^{n-d+p}_{\cup\{\infty\}}$ — like partially de-localized $d-n$-brane solitons

(NB: These configuration spaces are connected: The moduli are all in higher homotopy, invisible to traditional treatment.)

Pontrjagin's Cohomotopy charge map still exhibits the equivalence of the May-Segal theorem, now known as the inverse "electric field map" [Segal73, §1][McD75, §1] or "scanning map" and evaluated on the configuration space by [Segal73, §3], assigning to each point in the configuration the unit $(d-p)$-cohomotopy charge of a solitonic p-brane, but regarded after inclusion into the cohomotopy charge space of solitonic $(d-n)$-branes:

$$
\begin{tikzcd}
\underset{\substack{\text{moduli space} \\ \text{of solitonic} \\ p\text{-branes}}}{\mathrm{Conf}\Big(\mathbb{R}^{d-p}, \mathbb{R}^{n-d+p}_{\cup\{\infty\}}\Big)} \arrow[rr, bend left, "n\text{-cohomotopy charge map for solitonic } p > d-n\text{-branes}"] & \widetilde{\boldsymbol{\pi}}^{d-p}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}}\Big) \arrow[d, equal] & \widetilde{\boldsymbol{\pi}}^{n}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}}\Big) \arrow[d, equal] \\
\underset{\substack{\text{single-brane} \\ \text{subspace}}}{\mathbb{R}^{d-p} \times \mathbb{R}^{n-d+p}} \arrow[u, hook] \arrow[r, "\substack{\text{pure solitonic} \\ p\text{-brane charge}}"] & \mathrm{Maps}^{*/}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}}, \mathbb{R}^{d-p}_{\cup\{\infty\}}\Big) \arrow[r, "\substack{\text{regarded as solitonic} \\ d-n\text{-brane charge}}"] & \mathrm{Maps}^{*/}\Big(\mathbb{R}^{d-p}_{\cup\{\infty\}}, \mathbb{R}^{n}_{\cup\{\infty\}}\Big) \\
(x, y) \arrow[r, mapsto] & \left(x' \mapsto \begin{cases} \dfrac{x'-x}{\exp\left(\frac{-1}{\epsilon-|x'-x|^2}\right)} & \text{if } |x'-x| < \epsilon \\ \infty & \text{otherwise} \end{cases}\right) \arrow[r, mapsto] & \left(x' \mapsto \begin{cases} \dfrac{(x'-x,\, y)}{\exp\left(\frac{-1}{\epsilon-|x'-x|^2}\right)} & \text{if } |x'-x| < \epsilon \\ \infty & \text{otherwise} \end{cases}\right)
\end{tikzcd}
\tag{109}
$$