Parametrised homotopy theory and gauge enhancement

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joint work with H. Sati & U. Schreiber
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Motivation

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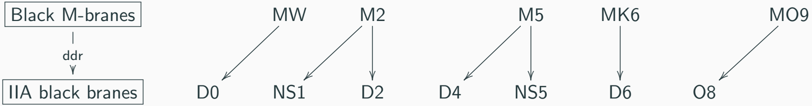

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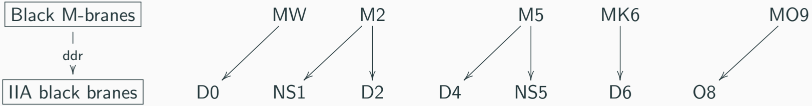

exhibiting a duality between M-theory and type IIA black branes.

String ending on a D-brane $\Rightarrow$ $U(1)$-gauge theory on worldvolume.
N-coincident D-branes $\Rightarrow$ $U(1)^N$ *enhances* to $U(N)$.

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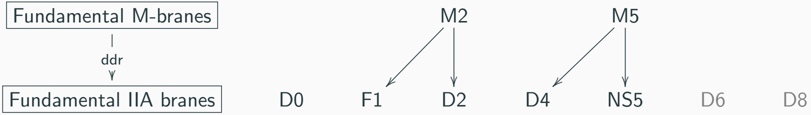

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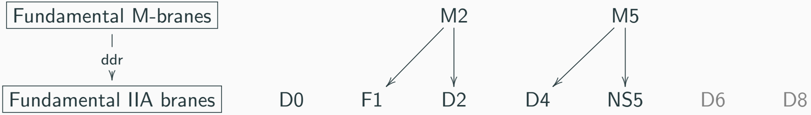

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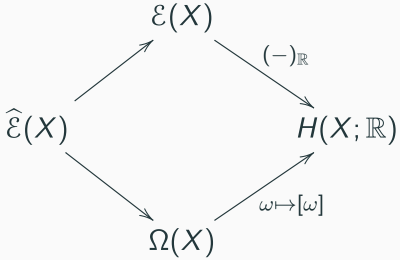
A (partial) answer: through the M-brane charge coefficients

Dirac: charges live in (twisted) differential cohomology



where a current $\widehat{j}_W \in \widehat{\mathcal{E}}(X)$ associated to a brane $W \hookrightarrow X$ determines

- a flux form $F \in \Omega_{cl}(X)$;
- a cohomology class $\lambda \in \mathcal{E}(X)$; and
- a quantisation condition $\lambda_{\mathbb{R}} = [F]$.

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- produces eoms for G_4, G_7 flux forms;
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- produces eoms for G_4, G_7 flux forms;
- equivariant enhancement at ADE singularities makes *black branes* appear, providing a unified black/fundamental brane perspective (Huerta–Sati–Schreiber); and
- exhibits, rationally, the twisted K-theory charge of fundamental IIA D-branes after double dimensional reduction (BM–Sati–Schreiber)

Cohomology

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As $n \to \infty$, $\Omega^n X$ becomes ever more commutative, so that *abelian* homotopical groups are infinite loop spaces.

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The levels of a spectrum $P = \{P_n\}$ are each infinite loop spaces

$$P_n \cong \Omega P_{n+1} \cong \Omega^2 P_{n+2} \cong \cdots \cong \Omega^k P_{n+k} \quad \text{for all } k,$$

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Spectra organise into a homotopy theory, encoded by the *stable homotopy category* Ho(Spectra).

There is an adjunction ($\simeq$ duality)

$$\mathrm{Ho(Spaces)} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Ho(Spectra)}, \quad \text{so that } (\underbrace{\Sigma^\infty_+ X \to \mathcal{E}}_{\text{map of spectra}}) \simeq (\underbrace{X \to \Omega^\infty \mathcal{E}}_{\text{map of spaces}}),$$

where

- $\Sigma^\infty_+ \colon X \mapsto \mathbb{S}[X_+]$ takes free homotopical abelian groups
- $\Omega^\infty \colon P \mapsto P_0$ forgets the homotopical abelian group structure.

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This is the "gauged version" of the more concrete

$$\mathrm{Sets} \underset{U}{\overset{\mathbb{Z}[-]}{\underset{\perp}{\rightleftarrows}}} \mathrm{AbGrps}, \quad \text{so that } (\underbrace{\mathbb{Z}[X] \to A}_{\text{map of ab grps}}) \simeq (\underbrace{X \to A}_{\text{map of sets}}).$$

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	nonlinear	**linear**	**parametrised linear**
geometry	manifold	vector space	vector bundle
homotopy	space	spectrum	parametrised spectrum

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Taking $\Omega_X^n Y$ as $n \to \infty$ gives a family of increasingly-commutative homotopical groups parametrised by X.

X-parametrised spectra are mathematical objects that encode fibrewise infinite loop spaces (over X). There is a corresponding *X-parametrised stable homotopy category* $\mathrm{Ho}(\mathrm{Spectra}_{/X})$, together with parametrised stabilisation adjunctions, for all X.

For X connected, the fibres of an X-parametrised spectrum P are all equivalent:

$$x, y \in X \Longrightarrow x^* P \cong y^* P.$$

The data $P \to X$ is equivalent to specifying a homotopical action of ΩX on $x^* P$, so we can write $P \cong x^* P // \Omega X \to X$.

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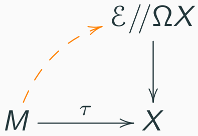

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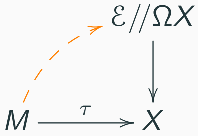

For trivial twists (τ null-homotopic) this recovers ordinary $\mathcal{E}$-cohomology.

Twisted K-theory

(Complex) K-theory is classified by a spectrum KU. Tensoring with complex lines equips KU with a $BU(1)$-action, giving rise to a parametrised spectrum

$$KU//BU(1) \longrightarrow B^2U(1)$$

over $B^2U(1)$. Since $B^2U(1)$ is the classifying space for $U(1)$-gerbes, $KU//BU(1) \to B^2U(1)$ is the moduli object for K-theory twisted by gerbes.

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$$f\colon X \to Y \text{ represents an isomorphism in Ho(Spaces)} \Leftrightarrow$$
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$$f\colon \mathcal{E}/\!/\Omega X \to \mathcal{F}/\!/\Omega X \text{ induces an isomorphism in } \mathrm{Ho}(\text{Spectra}_{/X}) \Leftrightarrow$$
$$f \text{ induces an isomorphism on all fibre spectra} \Leftrightarrow$$
$$\underbrace{\mathcal{E}^\bullet \text{ and } \mathcal{F}^\bullet \text{ are isomorphic}}_{\text{stable}}\ \underbrace{\text{as } \Omega X\text{-modules}}_{\text{unstable}}.$$

A large part of what makes homotopy theory difficult are the torsion subgroups of the $\pi_\bullet$'s.

So we discard the torsion and pass to *rational homotopy theory* (via the assignment $\pi_\bullet \mapsto \pi_\bullet \otimes_{\mathbb{Z}} \mathbb{Q}$) . This makes the story significantly simpler, in fact *algebraic* (due to Quillen & Sullivan).

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Unstable $$\mathrm{Ho}(\mathrm{Spaces})_{\mathbb{Q},\mathrm{nil},\mathrm{fin}} \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} \mathrm{Ho}(\mathrm{dgcAlg})^{\mathrm{op}}_{\mathrm{conn},\mathrm{fin}}$$

Stable $$\mathrm{Ho}(\mathrm{Spectra})_{\mathbb{Q},\mathrm{fin}} \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} \mathrm{Ho}(\mathrm{Ch}^{\mathbb{Q}})^{\mathrm{op}}_{\mathrm{fin}}$$

identifying full subcategories of (un)stable rational homotopy types with algebraic objects.

The existence of *minimal models* in the Sullivan theory allows us to read off rational cohomology as well as $\pi_\bullet \otimes_{\mathbb{Z}} \mathbb{Q}$ from algebraic data.

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Theorem [BM]
Let A be a minimal model for the rational homotopy type of X. If X is 1-connected, there is an equivalence of categories

$$\mathrm{Ho}(\mathrm{Spectra}_{/X})_{\mathbb{Q},\mathrm{fin},\mathrm{bbl}} \underset{\longleftarrow}{\xrightarrow{\ \cong\ }} \mathrm{Ho}(A\text{-}\mathrm{Mod})^{\mathrm{op}}_{\mathrm{fin},\mathrm{bbl}}$$

homotopy theory	**stable**	**parametrised stable**	**unstable**
ordinary	spectra	parametrised spectra	spaces
rational	cochain complexes	dg modules	dgc algebras

Many natural constructions in homotopy theory have nice algebraic descriptions, for instance

- stabilisation (Σ^∞_+ and its parametrised versions) forgets algebraic structure
- destabilisation (Ω^∞ and its parametrised versions) sends dg modules to free algebras
- pullback of parametrised spectra is pushforward of dg modules

Example: rational twisted connective K-theory

A minimal model for the parametrised spectrum $ku//BU(1)$ is

$$\mathbb{Q}[h_3] \otimes \langle \omega_{2k} \mid k \in \mathbb{N} \rangle \Big/ \begin{pmatrix} dh_3 = 0 & d\omega_0 = 0 \\ d\omega_{2k+2} = h_3 \wedge \omega_{2k} & \end{pmatrix} .$$

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A minimal model for *2-shifted* twisted K-theory $\Sigma^2 ku//BU(1)$ is

$$\mathbb{Q}[h_3] \otimes \langle \omega_{2k+2} \mid k \in \mathbb{N} \rangle \Big/ \begin{pmatrix} dh_3 = 0 & d\omega_2 = 0 \\ d\omega_{2k+4} = h_3 \wedge \omega_{2k+2} & \end{pmatrix} .$$

Back to M-theory

Working now in the *rational approximation*, we have access to algebraic models of unstable/stable/parametrised homotopy types.

Cocycles in generalised cohomology are controlled, after rationalisation, by their *flux forms*. These flux forms satisfy twisted Bianchi identites / eoms, identifying them as cocycles in rational generalised cohomology:

Flux forms	**Twisted Bianchi identity**	**Rational cocycle for**
M-branes	$dG_4 = 0$ $dG_7 = -\frac{1}{2} G_4 \wedge G_4$	degree 4 cohomotopy
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Local supersymmetry $\Rightarrow$ *super* flux forms encoding cocycles in *super* rational homotopy theory via *super* dgc (& Lie) algebras.

Combined M2/M5 cocycle $\mu_{M2/M5} \colon \mathbb{R}^{10,1|\mathbf{32}} \to S^4$

A minimal model for S^4 is

$$\mathcal{O}(S^4) \;\simeq\; \mathbb{Q}[\omega_4, \omega_7] / \begin{pmatrix} d\omega_4 = 0 \\ d\omega_7 = -\tfrac{1}{2}\omega_4 \wedge \omega_4 \end{pmatrix},$$

and the super-translation Lie algebra encoding super Minkowski spacetime $\mathbb{R}^{10,1|\mathbf{32}}$ is

$$\mathcal{O}(\mathbb{R}^{10,1|\mathbf{32}}) \simeq \mathbb{Q}[(e^a)_{a=0}^{10}]\{(\psi^\alpha)_{\alpha=1}^{32}\} / \begin{pmatrix} de^a = \overline{\psi}\Gamma^a\psi \\ d\psi^\alpha = 0 \end{pmatrix}.$$

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The combined cocycle is the map $\mathcal{O}(\mathbb{R}^{10,1|\mathbf{32}}) \leftarrow \mathcal{O}(S^4)$ sending

$$\begin{aligned} \omega_4 &\longmapsto \tfrac{i}{2}(\overline{\psi}\Gamma_{a_1 a_2}\psi) \wedge e^{a_1} \wedge e^{a_2} \\ \omega_7 &\longmapsto \tfrac{1}{5!}(\overline{\psi}\Gamma_{a_1 \dots a_5}\psi) \wedge e^{a_1} \wedge \cdots \wedge e^{a_5} \end{aligned}$$

(Fiorenza–Sati–Schreiber)

Fiorenza–Sati–Schreiber: double dimensional reduction is encoded by an adjunction

$$\mathrm{Ho}(\mathrm{Spaces})_{/BS^1} \underset{\mathrm{Cyc}=[S^1,-]//S^1}{\overset{\mathrm{Ext}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Ho}(\mathrm{Spaces}).$$

This admits an algebraic description in (super) rational homotopy theory. Roughly:

- Ext sends $\mu\colon A \leftarrow \mathrm{CE}(\mathfrak{b}\mathbb{R})$ to the extension controlled by μ;
- Cyc sends a minimal model $\mathbb{Q}[\omega_i]/(d\omega_i = \cdots)$ to a minimal model with additional generators $s\omega_i$, $|s\omega_i| = |\omega_i| - 1$, and ω_2 of degree 2 (with prescribed differential)

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Example: the D0-brane 2-cocycle $\mu_{D0}\colon \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \to \mathfrak{b}\mathbb{R}$ is such that $\mathrm{Ext}(\mu_{D0}) = \mathbb{R}^{10,1|\mathbf{32}}$—this is D0-brane condensation.

We use this to build a diagram in super rational homotopy theory:

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\begin{array}{ccc}
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\widetilde{\mu_{M2/M5}}} & \\
\downarrow & & \searrow \\
\mathrm{Cyc}(\mathrm{Ext}(\mu_{D0})) & \xrightarrow{\mathrm{Cyc}(\mu_{M2/M5})} & \mathrm{Cyc}(S^4)
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The minimal model for $\mathrm{Cyc}(S^4)$ is

$$\mathbb{Q}[\omega_2, h_3, \omega_4, \omega_6, h_7] \Big/ \begin{pmatrix} dh_3 = 0 & dh_7 = \omega_2 \wedge \omega_6 - \frac{1}{2}\omega_4 \wedge \omega_4 \\ d\omega_2 = 0 & \\ d\omega_4 = h_3 \wedge \omega_2 & \\ d\omega_6 = h_3 \wedge \omega_4 & \end{pmatrix},$$

and the combined cocycle $\widetilde{\mu_{M2/M5}}$ sends ω_2, h_3, ω_4, ω_6 and h_7 to the D0, F1, D2, D4 and NS5 cocycles respectively (FSS).

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but what about the D6 and D8?

There is an equivariant enhancement of $\mu_{M2/M5}$ at ADE subgroups making black M-branes appear (Huerta–Sati–Schreiber).

The A series actions through an S^1-action, which on spacetime we identify with the M-theory circle fibre. As $n \to \infty$, the A_n-actions exhaust that S^1 fibre:

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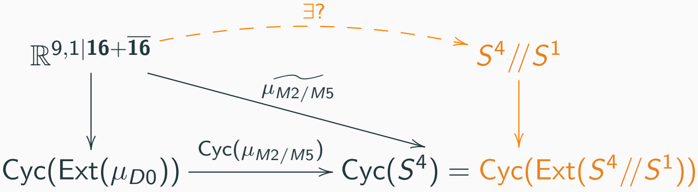
Equivalently, in terms of our previous diagram



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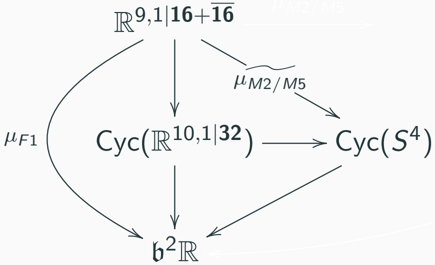
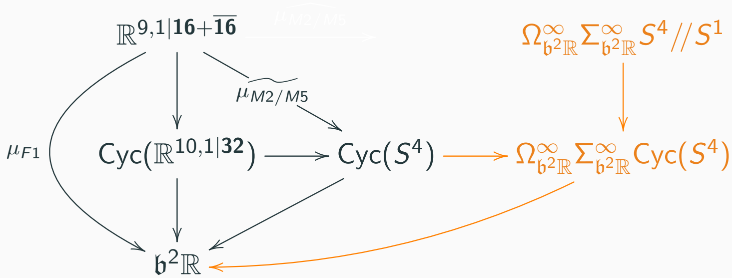
...but the fact that $\widetilde{\mu_{M2/M5}}$ produces the D0, D2 and D4 cocycles is due to a truncated copy of rational (2-shifted) twisted K-theory living inside $\mathrm{Cyc}(S^4)$!

Let's make some room



Let's make some room—by passing to parametrised spectra, as the first stage of *homotopical perturbation theory*



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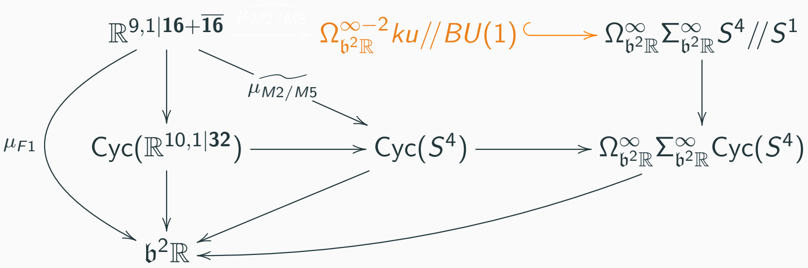

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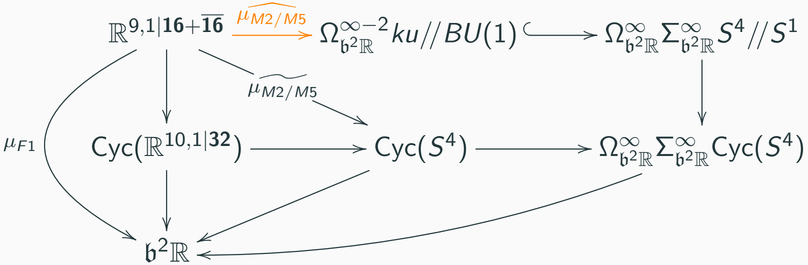

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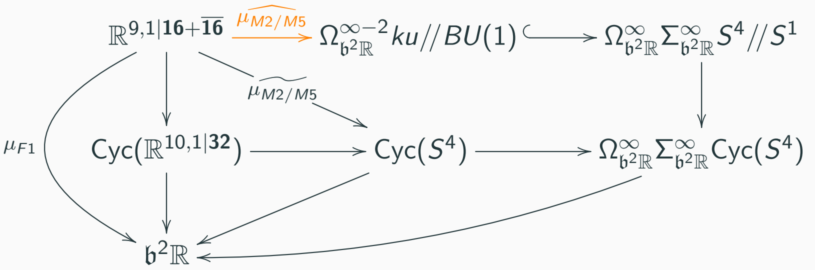

The cocycle $\widehat{\mu_{M2/M5}}$ is of the form

$$\widehat{\mu_{M2/M5}} = \begin{cases} d\mu_{F1} = 0 & d\mu_{D0} = 0 \\ d\mu_{D(2p+2)} = \mu_{F1} \wedge \mu_{D(2p)} & 0 \leq p \leq 4 \end{cases}$$

and exhibits the enhancement of $\widetilde{\mu_{M2/M5}}$ by the missing D6 and D8 cocycles!

(though the NS5 has disappeared)

In summary:

> *perturbative gauge enhancement of the double dimensional reduction of the combined S^4-valued M2/M5 cocycle is exhibited by lifting through the fibrewise stabilisation of the A-type orbispace of S^4*

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Outstanding questions:

1. what is the non-rational lift?

homotopy theory	$\Rightarrow$	rational homotopy theory
???	$\Rightarrow$	super rational homotopy theory

2. what is the role of homotopical perturbation theory (namely, the Goodwillie calculus of functors)?

Thank you for your attention!