Differential cohomology theory is Cohesive homotopy theory

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Abstract

Differential cohomology is a refinement of bare cohomology by differential geometry. It lifts the classical theory of principal connections, Yang-Mills fields and Chern-Weil theory to higher connections and higher gauge fields with underlying classes in generalized cohomology theories. The latter are famously characterized by the generalized Eilenberg-Steenrod axioms and represented by spectra. A few years back Simons and Sullivan asked if there is also an axiomatic characterization of differential generalized cohomology. We survey here [Schreiber 13, Bunke-Nikolaus-Völkl 13] a faithful such axiomatization by sheaves of spectra/homotopy types which are "cohesive". As an application we indicate the abstract characterization of differential moduli stacks such as higher (intermediate) Jacobian stacks.

Contents

1 Ordinary differential cohomology

For X a manifold and $A \in \mathrm{Ab}$ an abelian group, then the ordinary cohomology groups $H^n(X, A)$ are invariants of the underlying homotopy type of X, in particular they are *homotopy invariant* in that the canonical maps

$$H^n(X, A) \xrightarrow{\simeq} H^n(X \times \mathbb{R}, A)$$

are equivalences. In order to bring the actual geometry of manifolds into the picture, consider

Definition 1.1. Let S denote the *site* (Grothendieck topology) of

$$S = \begin{cases} & \{\text{smooth manifolds}\} \\ \text{or} & \{\text{complex analytic manifolds}\} \\ \text{or} & \{\text{super-manifolds}\} \\ \text{or} & \{\text{formal manifolds }\} \\ \text{or} & \{\text{formal super-manifolds}\} \\ & \\ \text{or} & \begin{array}{l} \text{any locally étale-contractible site with terminal object } * \\ \text{such that } \mathrm{Hom}(*, -) \text{ preserves split hypercovers} \end{array} \end{cases}$$

Then for $\mathbf{A} \in \mathrm{Ab}(\mathrm{Sh}(S))$ a sheaf of abelian groups, there are the *abelian sheaf cohomology groups* $H^n(X, \mathbf{A})$ which need not be homotopy invariant. Only when $\mathbf{A} := \mathrm{LConst}(A)$ is locally constant then its sheaf cohomology reproduces ordinary cohomology:

$$H^n(X, \mathrm{LConst} A) \simeq H^n(X, A) .$$

More generally for $A_\bullet \in \mathrm{Ch}_\bullet(\mathrm{Sh}(S))$ a chain complex of abelian sheaves, then there is the abelian sheaf hypercohomology

$$H^n(X, A_\bullet) := H^0(X, \mathbf{B}^n A_\bullet) \simeq \mathbb{R}\mathrm{Hom}(\mathbb{Z}[X], A_\bullet[-n]) .$$

Example. Write

$$\mathbb{G}_a = \left\{ \begin{array}{ll} (\underline{\mathbb{R}}, +) & \text{for } S = \{\text{smooth manifolds}\} \\ (\underline{\mathbb{C}}, +) & \text{for } S = \{\text{complex analytic manifolds}\} \end{array} \right. .$$

for the sheaf of (smooth, or holomorphic, etc.) functions and write

$$\flat \mathbb{G}_a := \mathrm{LConst}(\mathbb{G}_a(*))$$

for the sheaf of locally constant functions. Write

$$\Omega^\bullet \in \mathrm{Ch}^{-\bullet}(\mathrm{Sh}(S)) = \mathrm{Ch}_\bullet(\mathrm{Sh}(S))$$

for the (smooth, or holomorphic, etc.) de Rham complex. The Poincaré lemma says that

$$H^n(X, \Omega^\bullet) \simeq H^n(X, \flat \mathbb{G}_a)$$

In fact the local quasi-isomorphism

$$\flat \mathbb{G}_a \xrightarrow{\simeq} \Omega^\bullet$$

exhibits (just) a resolution, but this particular resolution serves to naturally induce the Hodge filtration:

$$\begin{tikzcd} \cdots \arrow[r, dotted] & F^{p+1}\Omega^\bullet \arrow[r] \arrow[dr, dashed] & F^p \Omega^\bullet \arrow[r] \arrow[d, dashed] & F^{p-1}\Omega^\bullet \arrow[r, dotted] \arrow[dl, dashed] & \cdots \\ & & \Omega^\bullet & & \end{tikzcd}$$

given by the degree-filtration of the de Rham complex:

$$F^p \Omega^\bullet := \Omega^{\bullet \geq p} .$$

Using this we find "genuinely geometric" cohomology groups of differential forms, such as

$$H^n(X, \Omega^{\bullet \geq n}) \simeq \Omega^n_{\mathrm{cl}}(X) .$$

Specifically in the complex-analytic case the above filtration reproduces the traditional Hodge filtration

$$H^k(X, \Omega^{\bullet \geq p}) \simeq \bigoplus_{k-q \geq p} H^{k-q,q}(X)$$

and thus the group of Hodge cocycles is given by the following fiber product of sheaf hypercohomology groups:

$$\mathrm{Hdg}^p(X) := H^{p,p}(X)_{\mathrm{integral}} \simeq H^{2p}(X, \mathbb{Z}) \underset{H^{2p}(X, \flat \mathbb{G}_a)}{\times} H^{2p}(X, \Omega^{\bullet \geq p}) .$$

(see e.g. [Esnault-Viehweg 88] for review of this and some of the following)

Definition 1.2. Pulling back the above Hodge filtration along the exponential sequence

$$\mathbb{Z} \xhookrightarrow{\mathrm{ch}_{\mathbb{Z}}} \mathbb{G}_a \xrightarrow{\exp\left(\frac{i}{\hbar}(-)\right)} \mathbb{G}_m$$

we obtain a tower of homotopy pullbacks in $\mathrm{Ch}_\bullet(\mathrm{Sh}(S))$:

$$\begin{tikzcd}
(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}} \arrow[r] \arrow[d, dashed] & \mathbf{B}^{n+1}\Omega^{\bullet \geq n+1} \arrow[d, dashed] \\
F^3(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}^{n+1}\Omega^{\bullet \geq 3} \arrow[d] \\
F^2(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}^{n+1}\Omega^{\bullet \geq 2} \arrow[d] \\
\mathbf{B}^n\mathbb{G}_m \arrow[r, "\theta"] \arrow[d] & \mathbf{B}^{n+1}\Omega^{\bullet \geq 1} \arrow[d] \\
\mathbf{B}^{n+1}\mathbb{Z} \arrow[r, "\mathrm{ch}"] & \mathbf{B}^{n+1}\Omega^{\bullet} \simeq \mathbf{B}^{n+1}\flat\mathbb{G}_a
\end{tikzcd}$$

Proposition 1.3. *These homotopy fiber products are given by the Deligne complexes:*

$$F^p(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}} \simeq \left(\mathbb{Z} \hookrightarrow \Omega^0 \xrightarrow{d_{\mathrm{dR}}} \Omega^1 \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^{p-1} \longrightarrow 0 \longrightarrow \cdots \right)$$

(with $\mathbb{Z}$ in degree $n+1$).

[Fiorenza-Schreiber-Stasheff 10]

Examples.

1. $\mathbf{B}\mathbb{G}_m$ modulates line bundles;
2. $(\mathbf{B}\mathbb{G}_m)_{\mathrm{conn}}$ modulates line bundles with connection;
3. $(\mathbf{B}^2\mathbb{G}_m)_{\mathrm{conn}}$ modulates bundle gerbes with connection and curving;
4. $F^2(\mathbf{B}^2\mathbb{G}_m)_{\mathrm{conn}}$ modulates bundle gerbes with connection but without curving,
   the symmetries of these are given by Courant algebroids;
5. generally $(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}}$ modulates line n-bundles with n-connection;
   the symmetries of these are higher Kostant-Souriau quantomorphism group extensions;
6. $H^\bullet(-, (\mathbf{B}^\bullet\mathbb{G}_m)_{\mathrm{conn}})$ is called *ordinary differential cohomology*;
   the deformation theory of ordinary differential cohomology (before analytification, in positive characteristic) is given by Artin-Mazur formal groups;
7. on a complex manifold X the fiber

   $$J^p(X) \longrightarrow H^0(X, F^p(\mathbf{B}^{2p}\mathbb{G}_m)_{\mathrm{conn}}) \longrightarrow \mathrm{Hdg}^p(X)$$

   is (the abelian group underlying) the pth *higher Jacobian* ("intermediate Jacobian") of X.

[Fiorenza-Rogers-Schreiber 13]

2 Chern-Weil secondary invariants

A central purpose of ordinary differential cohomology is to approximate non-abelian differential cohomology; this is the content of Chern-Weil theory. To capture this write

$$\mathrm{sSet}(S) := \mathrm{sSet}(\mathrm{Sh}(S))$$

for the homotopy theory of simplicial sheaves on S,
i.e. sheaves of homotopy types with weak equivalences the local weak homotopy equivalences.

Example. For a Lie group G with Lie algebra $\mathfrak{g}$, the system of simplicial nerves $N(-)$ of quotient groupoids $(-)//(-)$ given by the gauge action of G-valued functions on $\mathfrak{g}$-valued 1-forms

$$X \mapsto N\left(\Omega^1(X,\mathfrak{g})//C^\infty(X,G)\right)$$

defines a simplicial sheaf

$$(\mathbf{B}G)_{\mathrm{conn}} \simeq \Omega^1(-,\mathfrak{g})//G \quad \in \mathrm{sSh}(X)\,.$$

This modulates G-principal connections:

$$H(X, \mathbf{B}G_{\mathrm{conn}}) \simeq \{G\text{-principal connections on } X\}_\sim\,.$$

Proposition 2.1. *For $\mathfrak{a}$ an L_∞-algebroid and $\mu : \mathfrak{a} \longrightarrow \mathbb{R}[n+1]$ a cocycle which is "transgressive" and has integral periods, then there is canonical Lie integration to a morphism in* $\mathrm{sSh}(S)$

$$\tau_n \exp(\mathfrak{a})_{\mathrm{conn}} \longrightarrow (\mathbf{B}^{n+1}\mathbb{G}_m)_{\mathrm{conn}}\,.$$

Examples.

1. For $\mu = \langle -,[-,-]\rangle : \mathbf{B}\mathfrak{g} \to \mathbb{R}[-3]$ the canonical 3-cocycle on a semisimple Lie algebra, this yields

$$\mathbf{L}_{\mathrm{CS}} : (\mathbf{B}G)_{\mathrm{conn}} \longrightarrow (\mathbf{B}^3\mathbb{G}_m)_{\mathrm{conn}}$$

   that sends G-principal connections ∇ to differential cocycles represented by $(\mathrm{c}_2, \langle F_\nabla \wedge F_\nabla\rangle)$. The homotopy fiber modulates String 2-connections:

$$\mathbf{B}\mathrm{String}_{\mathrm{conn}} \longrightarrow \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \xrightarrow{\mathbf{L}_{\mathrm{CS}}} (\mathbf{B}^3\mathbb{G}_a)_{\mathrm{conn}}\,.$$

   Here $\mathbf{L}_{\mathrm{CS}}$ is a full de-transgression of the Chern-Simons secondary invariant:

   - transgressed to the circle it yields the WZW gerbe:

$$\begin{tikzcd} G \arrow[r] \arrow[rrr, bend right, "\mathbf{L}_{\mathrm{WZW}}"'] & {[S^1, \mathbf{B}G_{\mathrm{conn}}]} \arrow[r, "{[S^1,\mathbf{L}_{\mathrm{CS}}]}"] & {[S^1, (\mathbf{B}^3\mathbb{G}_m)_{\mathrm{conn}}]} \arrow[r, "\exp(\frac{i}{\hbar}\int_{S^1}(-))"] & (\mathbf{B}^2\mathbb{G}_m)_{\mathrm{conn}} \end{tikzcd}$$

   - transgressed to 2d-manifold Σ it yields a multiple of the θ-bundle on the moduli of G-principal connections: $\begin{tikzcd} \mathrm{Conn}_G(\Sigma) \arrow[r] \arrow[rrr, bend right] & {[\Sigma, \mathbf{B}G_{\mathrm{conn}}]} \arrow[r, "{[\Sigma,\mathbf{L}_{\mathrm{CS}}]}"] & {[\Sigma, (\mathbf{B}^3\mathbb{G}_m)_{\mathrm{conn}}]} \arrow[r, "\exp(\frac{i}{\hbar}\int_{\Sigma}(-))"] & (\mathbf{B}\mathbb{G}_m)_{\mathrm{conn}} \end{tikzcd}$
   - transgressed to a 3-manifold Σ it yields the Chern-Simons secondary invariant.

2. for $\mu = \langle -,[-,-],[-,-],[-,-]\rangle$ on $\mathfrak{a} = \mathbf{B}\mathfrak{string}$ this yields the de-transgressed Lagrangian for 7d String-Chern-Simons theory [Fiorenza-Sati-Schreiber 12]

$$\mathbf{L}_{\mathrm{CS}_7} : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to (\mathbf{B}^7\mathbb{G}_m)_{\mathrm{conn}}$$

3. For $\pi : \mathfrak{P} \to \mathbb{R}[-2]$ the canonical 2-cocycle on a Poisson Lie algebroid, then the induced

$$\mathrm{SymplGrpd}(X) \longrightarrow F^2(\mathbf{B}^2\mathbb{G}_m)_{\mathrm{conn}}$$

   is the prequantized symplectic groupoid of X [Bongers 14].

[Fiorenza-Schreiber-Stasheff 10, Fiorenza-Sati-Schreiber 13]

3 Simons-Sullivan conjecture

Theorem 3.1 ([Simons-Sullivan 07]). *On smooth manifolds, the functor $H(-,(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}})$ is uniquely characterized by sitting in a hexagon of presheaves of abelian groups*

$$\begin{tikzcd}
 & H(-,\Omega^n)/\mathrm{im}(d_{\mathrm{dR}}) \arrow[rr, "d_{\mathrm{dR}}"] \arrow[dr] & & H(-,\Omega^{n+1}_{\mathrm{cl}}) \arrow[dr] & \\
H(-,\mathbf{B}^n\flat\mathbb{G}_a) \arrow[ur] \arrow[dr] & & H(-,(\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}}) \arrow[ur, "\theta"] \arrow[dr] & & H(-,\mathbf{B}^{n+1}\flat\mathbb{G}_a) \\
 & H(-,\mathbf{B}^n\flat\mathbb{G}_m) \arrow[rr, "\beta"'] \arrow[ur] & & H(-,\mathbf{B}^{n+1}\mathbb{Z}) \arrow[ur, "\mathrm{ch}"] &
\end{tikzcd}$$

where the diagonals and the two boundaries are exact sequences.

The meaning of the hexagon is this:

$$\begin{tikzcd}
 & \boxed{\text{connect. on trivial bundles}} \arrow[rr, "\text{de Rham differential}"] \arrow[dr, "\text{regard as}"] & & \boxed{\text{curvature forms}} \arrow[dr, "\text{de Rham theorem}"] & \\
\boxed{\text{closed differ. forms}} \arrow[ur, "\text{regard as}"] \arrow[dr, "\text{regard as}"'] & & \boxed{\text{connections on geometric bundles}} \arrow[ur, "\text{curvature}"] \arrow[dr, "\text{topol. class}"'] & & \boxed{\text{ration. bundles}} \\
 & \boxed{\text{flat connect.}} \arrow[rr, "\text{Bockstein homom./ comparison map}"] \arrow[ur, "\text{regard as}"'] & & \boxed{\text{shape of bundles}} \arrow[ur, "\text{Chern character}"'] &
\end{tikzcd}$$

The outer parts of the hexagon has a classical generalization from ordinary to generalized Eilenberg-Steenrod-type cohomology (e.g. K-theory, elliptic cohomology, ... cobordism cohomology). By Brown's representability theorem, generalized ES-type cohomology is what is represented by spectra $E \in \mathrm{Spectra}$ instead of just chain complexes: $E^\bullet(-) = H^\bullet(-,E)$.

Question [Simons-Sullivan 07]: *Does such a hexagon also characterize differential generalized cohomology?*
or: *What* **is** *differential generalized cohomology, axiomatically?*

Actually, there is more "generalization" to cohomology than just ES-type generalization:

Generalizations of ordinary cohomology	**needed for**
Eilenberg-Steenrod-type abelian generalized cohomology	type II superstring RR-fields in twisted KR-theory, higher geometric quantization
non-abelian cohomology	Chern-Simons theory, Wess-Zumino-Witten theory, modular functors, equivariant elliptic cohomology
geometry other than plain smooth	Kähler geometric quantization, supersymmetric field theory, Artin-Mazur deformation theory
twisted cohomology	quantum anomaly cancellation, covariant quantization of higher gauge fields

For modern applications we need all of this. And here is how:

4 Cohesive homotopy theory

So we need for a given sheaf of spectra $\hat{E} \in \mathrm{Spectra}(\mathrm{Sh}(S))$ that there are natural spectra

$$\flat\hat{E}, \Pi\hat{E} \in \ \mathrm{Spectra} \xhookrightarrow{\mathrm{LConst}} \mathrm{Spectra}(\mathrm{Sh}(S))$$

equipped with natural maps

$$\begin{tikzcd}
 & & \hat{E} \arrow[dr, "\text{project}"] & & \\
\text{flat cocycles} & \flat\hat{E} \arrow[ur, "\text{include}"] \arrow[rr] & & \Pi\hat{E} & \text{underlying cocycles}
\end{tikzcd}$$

that induce homotopy exact hexagons which for $\hat{E} = (\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}}$ reproduce theorem 3.1 above.

Theorem 4.1 ([Schreiber 13])**.** *For S as in def. 1.1, the derived global section functor $\Gamma : \mathrm{sSh}(S) \to \mathrm{sSet}$ is "cohesive" in that it extends to a quadruple of derived adjoints*

$$\begin{tikzcd}
\mathrm{sSh}(S) \arrow[r, shift left=6, "\times" description] \arrow[r, "\Gamma"] & \mathrm{sSet} \arrow[l, hook, shift right=3, "\mathrm{LConst}"'] \arrow[l, hook, shift left=3]
\end{tikzcd}$$

with the bottom right adjoint homotopy fully faithful and the top left adjoint preserving products.

Definition 4.2. Write $\boxed{(\Pi \dashv \flat \dashv \sharp) \ : \ \mathrm{sSh}(S) \to \mathrm{sSh}(S)}$ for induced adjoint triple of derived endofunctors (e.g. $\flat = \mathrm{LConst} \circ \Gamma$).

For G in $\mathrm{Grp}(\mathrm{sSet}(S))$, write $G \xrightarrow{\theta_G} \flat_{\mathrm{dR}}G \longrightarrow \flat\mathbf{B}G \longrightarrow \mathbf{B}G$ for the homotopy fiber sequence of the $\flat$-counit on the delooping.

Notice: It follows that $\flat \circ \flat_{\mathrm{dR}}A \simeq *$; hence $\flat_{\mathrm{dR}}A$ is "purely differential".

Proposition 4.3 ([Schreiber 13])**.**

for G a Lie group with Lie algebra $\mathfrak{g}$	*for $G = \mathbf{B}^n\mathbb{G}_m$*
$\Pi(\mathbf{B}G) \simeq BG$ *and* $\flat(\mathbf{B}G) \simeq K(G,1)$	$\Pi(\mathbf{B}^n\mathbb{G}_m) \simeq K(\mathbb{Z}, n+1)$
$\flat_{\mathrm{dR}}G = \{$*sheaf of flat $\mathfrak{g}$-valued diff. forms*$\}$	$\flat_{\mathrm{dR}}\mathbf{B}^n\mathbb{G}_m \simeq \mathbf{B}^{n+1}\Omega^{\bullet \geq 1}$
θ_G *is the Maurer-Cartan form*	$\theta_{\mathbf{B}^n\mathbb{G}_m}$ *is the Chern character from def. 1.2*
$[X, \mathbf{B}G]$ *is the moduli stack of G-principal bundles*	$[X, \mathbf{B}^n\mathbb{G}_m]$ *is the higher Picard stack*
$\sharp_1[X, \flat\mathbf{B}G] \underset{\sharp_1[X,\mathbf{B}G]}{\times} [X, \mathbf{B}G]$ *is moduli stack of flat connections*	*(details below in 5)*

Theorem 4.4 ([Bunke-Nikolaus-Völkl 13])**.** *For $\hat{E}$ a spectrum object in any cohesive homotopy theory as in theorem 4.1, then the canonical hexagon*

$$\begin{tikzcd}
 & \Pi_{\mathrm{dR}}\hat{E} \arrow[rr, "\mathbf{d}"] \arrow[dr] & & \flat_{\mathrm{dR}}\hat{E} \arrow[dr] & \\
\flat\Pi_{\mathrm{dR}}\hat{E} \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur, "\theta_{\hat{E}}"] \arrow[dr] & & \Pi\flat_{\mathrm{dR}}\hat{E} \\
 & \flat\hat{E} \arrow[ur] \arrow[rr] & & \Pi\hat{E} \arrow[ur, "\mathrm{ch}_E := \Pi\theta_{\hat{E}}"] &
\end{tikzcd}$$

formed from homotopy-exact diagonals consists of homotopy fiber sequences.
Moreover, both squares are homotopy Cartesian and hence the outer hexagon uniquely determines $\hat{E}$.

And by prop. 4.3: For $\hat{E} \simeq (\mathbf{B}^n\mathbb{G}_m)_{\mathrm{conn}}$, this reproduces on cohomology groups the hexagon of theorem 3.1.

5 Differential moduli stacks

Central to the applications in physics (Chern-Simons and self-dual higher gauge theory [Witten 96, Witten 99]) and mathematics (θ-characteristics [Hopkins-Singer 02]) is the construction of moduli stacks of differential cocycles.

Problem. For $\hat{E}$ a cohesive (stable) homotopy type in general the mapping stack $[X, \hat{E}]$ is not *quite* the moduli stack of $\hat{E}$-cocycles on X. Induce from the cohesive axioms the existence of correct moduli stacks.

Definition 5.1. A Hodge filtration on $\hat{E} \in \mathrm{Spectra}(\mathrm{Sh}(S))$ is is a filtration $F^\bullet \flat_{\mathrm{dR}} \hat{E}$ such that

1. each stage has the same image under Π;
2. each stage is in the kernel of $\flat$.

Proposition 5.2. *The induced sequence of homotopy fiber products*

$$F^p \hat{E} := \Pi \hat{E} \underset{\Pi \flat_{\mathrm{dR}} \hat{E}}{\times} F^p \flat_{\mathrm{dR}} \hat{E}$$

exhibits $(\Pi \dashv \flat)$*-fractures as in theorem 4.4*

$$\begin{tikzcd}[column sep=small] & F^p \flat_{\mathrm{dR}} \hat{E} \arrow[dr] & \\ F^p \hat{E} \arrow[ur] \arrow[dr] & & \Pi \flat_{\mathrm{dR}} \hat{E} \\ & \Pi \hat{E} \arrow[ur] & \end{tikzcd} \simeq \begin{tikzcd}[column sep=small] & \flat_{\mathrm{dR}} (F^p \hat{E}) \arrow[dr] & \\ F^p \hat{E} \arrow[ur] \arrow[dr] & & \Pi \flat_{\mathrm{dR}} (F^p \hat{E}) \\ & \Pi F^p (\hat{E}) \arrow[ur] & \end{tikzcd}$$

Definition 5.3. Denote the Moore-Postnikov tower of the $\sharp$-unit by:

$$\begin{tikzcd} & \vdots \arrow[d, dotted] & \\ & \sharp_3 X \arrow[d] \arrow[ddr] & \\ & \sharp_2 X \arrow[d] \arrow[dr] & \\ X \arrow[uur] \arrow[ur] \arrow[r] & \sharp_1 X \arrow[r, hook] & \sharp X \end{tikzcd}$$

Hence $\sharp_n X$ is the "n-image" of the $\sharp$-unit on X.

Definition 5.4. Given a $\hat{E}$-Hodge filtration, def. 5.1, and given any $X \in \mathrm{sSet}(S)$ with k the lowest number such that $H(X, F^{k+1} \flat_{\mathrm{dR}} \hat{E}) \simeq 0$, then the *differential moduli* of $\hat{E}$ on X is the iterated homotopy fiber product

$$\hat{E}(X) := \sharp_1 [X, F^k \hat{E}] \underset{\sharp_1 [X, F^{k-1} \hat{E}]}{\times} \sharp_2 [X, F^{k-1} \hat{E}] \underset{\sharp_2 [X, F^{k-2} \hat{E}]}{\times} \sharp_3 [X, F^{k-2} \hat{E}] \underset{\sharp_3 [X, F^{k-3} \hat{E}]}{\times} \cdots .$$

Proposition 5.5. *The underlying homotopy type of $\hat{E}(X)$ is that of $\hat{E}$-cocyles on X in kth Hodge filtration stage:*

$$\flat(\hat{E}(X)) \simeq \flat [X, F^k \hat{E}] \simeq \mathbf{H}(X, F^k \hat{E}) .$$

Proof. Use that cohesion implies $\flat \circ \sharp_n \simeq \flat$ for all n, and that $\flat$ preserves homotopy limits. □

Proposition 5.6. *For $\hat{E} = (\mathbf{B}^n \mathbb{G}_m)_{\mathrm{conn}}$ this reproduces the Artin-Mazur moduli:*

$$(\mathbf{B}^n \mathbb{G}_m)_{\mathrm{conn}}(X) \;:\; U \mapsto \{U\text{-parameterized Deligne cocycles on } X\ \}$$

[Fiorenza-Rogers-Schreiber 13]

Example. *The homotopy fiber of*

$$\hat{E}(X) \xrightarrow{\simeq} (\Pi \hat{E})(X) \underset{(\Pi \flat_{\mathrm{dR}} \hat{E})(X)}{\times} (\flat_{\mathrm{dR}} \hat{E})(X) \longrightarrow \tau_0 (\Pi \hat{E})(X) \underset{\tau_0 (\Pi \flat_{\mathrm{dR}} \hat{E})(X)}{\times} \tau_0 (\flat_{\mathrm{dR}} \hat{E})(X) ,$$

is a stack whose 0-truncation is is the higher Jacobian $J^{k+1}(X)$ with its Griffiths complex structure.

6 Outlook

Proposition 6.1. *Parameterized spectra in* $\mathrm{Sh}(S)$ *are a cohesive homotopy theory over bare parameterized spectra. Now* $\hat{E}(X)$ *is the twisted differential* E*-cohomology spectra* $E^{\bullet+\tau}(X)$ *parameterized over the moduli stack of twists* τ *on* X *([Schreiber 13], section 4.2).*

Such twisted differential cohomology is demanded by the fine structure of string theory [Distler-Freed-Moore 09]...

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`http://ncatlab.org/schreiber/show/Differential+cohomology+is+Cohesive+homotopy+theory`

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