Differential-topological review of Dirac flux quantization of the electromagnetic (EM) field.

The EM flux density (Maxwell-Faraday tensor) F_2 (6) is not the full content of the EM-field.

$$F_2 \in \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}}$$

First of all, there is also an integral class χ which coincides with the flux density in real cohomology.

$$[\chi] \in H^2(X;\mathbb{Z})$$

This χ is the electromagnetic "instanton number":
- On the complement of a magnetic monopole worldline $X \equiv \mathbb{R}^{3,1}\backslash\mathbb{R}^{1,1}$, this χ is the integer charge of the monopole.
- On the one-point compactification of a planar type II superconductor $X \equiv \mathbb{R}^{1,1}\times\mathbb{R}^2_{\cup\{\infty\}}$, this χ is the integer number of Abrikosov vortices.

$$\begin{array}{ccc} & H^2(X;\mathbb{Z}) & [\chi] \\ & \downarrow \mathrm{ch} & \updownarrow \\ \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \twoheadrightarrow & H^2_{\mathrm{dR}}(X) & \mathrm{ch}[\chi] \\ F_2 \mapsto & [F_2] & \text{equality} \end{array}$$

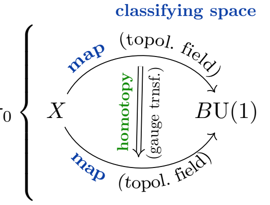
But the pair (F_2, χ) is still not the full EM field content: The remaining data is *how* F_2 and χ are identified.

To understand this, notice that ordinary cohomology groups have *classifying spaces*. In the case at hand, there is the space whose weak homotopy class is variously known as:

$$\underset{\text{infinite complex projective space}}{\mathbb{C}P^\infty} \underset{\text{hmtp}}{\simeq} \underset{\text{projective unitary group of sep. }\infty\text{d Hilbert space}}{PU(\mathcal{H})} \underset{\text{hmtp}}{\simeq} \underset{\text{classifying space of U(1)-princ. bundles}}{BU(1)} \underset{\text{hmtp}}{\simeq} \underset{\text{Eilenberg-MaxLane space}}{K(\mathbb{Z},2)}$$

$$H^2(X;\mathbb{Z}) \simeq \underbrace{\pi_0}_{\text{connected components}} \text{ of }\textbf{mapping space}\ \mathrm{Maps}\big(X, \underbrace{BU(1)}_{\textbf{classifying space}}\big)$$

$$= \pi_0 \left\{ \right.$$



$$\left. \right\}$$

and recall the notion of *higher homotopy groups* π_k of a simply connected space X, in degree $k \in \mathbb{N}_{\geq 2}$ these are abelian groups.

$$\underset{n\text{th homotopy group}}{\pi_k(X)} := \underset{\textbf{space of maps preserving base-point}}{\mathrm{Maps}^{*/}}\big(\underbrace{S^k}_{k\text{-sphere}}, X\big) \in \mathrm{AbGrp}$$

Now $BU(1)$ is special in that its homotopy groups are concentrated in degree 2, there being the integers.

$$\pi_k\big(BU(1)\big) \simeq \begin{cases} \mathbb{Z} & \text{if } k=2 \\ 0 & \text{otherwise} \end{cases}$$

In general, there is a (weakly homotopy-)unique connected space whose homotopy groups are concentrated in a single degree and there form an abelian group A, these are called the *Eilenberg-MacLane spaces* $K(A,n)$:

$$\pi_k\big(K(A,n)\big) \simeq \begin{cases} A & \text{if } k=n \\ 0 & \text{otherwise} \end{cases}$$

And these spaces happen to classify ordinary cohomology:

$$H^n(X;A) \simeq \pi_0\,\mathrm{Maps}\big(X, K(A,n)\big)$$

hence in particular also de Rham cohomology:

$$H^n_{\mathrm{dR}}(X) \simeq \pi_0\,\mathrm{Maps}\big(X, K(\mathbb{R},n)\big)$$

Using this, we can refine the integrality condition on cohomology classes to a gauge transformation of fields: Instead of asking the class of F_2 to equal the class of $\mathrm{ch}(\chi)$, we have a *homotopy* $\widehat{A}$ between them. This is equivalently [FSSt12, Prop. 3.2.26][FSS23-Char, Prop. 9.5] the final component of the EM-field, the *gauge potential* $\widehat{A}$.

$$\begin{array}{ccc} & \mathrm{Maps}\big(X, K(\mathbb{Z},2)\big) & \chi \\ & \downarrow \mathrm{ch} & \updownarrow \\ \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \xrightarrow{\eta^{\int}} & \mathrm{Maps}\big(X, K(\mathbb{R},2)\big) & \mathrm{ch}(\chi) \\ F_2 \mapsto & \eta^{\int}(F_2) & \overset{\widehat{A}}{\Longrightarrow}\ \text{homotopy} \end{array}$$

The equivalence classes of such "full EM-field" triples $(F_2, \widehat{A}, \chi)$ constitute the *differential cohomology* $\widehat{H}^2(X;\mathbb{Z})$. Turns out to be equivalent to isomorphism classes of U(1)-principal bundles with Chern class χ and connection $\widehat{A}$.

$$\begin{array}{ccc} \underset{\textbf{full EM-field is cocycle in ordinary differential cohomology}}{[\widehat{A}] \in \widehat{H}^2(X;\mathbb{Z})} & \xrightarrow{\chi} & \underset{\textbf{charge sector in ordinary cohomology}}{H^2(X;\mathbb{Z})} \\ \downarrow F_2 & & \\ \underset{\textbf{flux density differential form}}{\Omega^2_{\mathrm{dR}}(X)} & & \end{array}$$

The *reason* that this is the correct incarnation of the Maxwell field is that $(F_2, \widehat{A}, \chi)$ is exactly the required data to "cancel the anomaly" (cf. p. 22) of the Lorentz force coupling term (89) in the exponentiated action functional of an electron propagating in an EM background field.

$$\begin{array}{ccc} \widehat{H}^2(X;\mathbb{Z}) \times C^\infty(S^1, X) & \longrightarrow & \mathrm{U}(1) \\ (\widehat{A}, \gamma) & \mapsto & e^{2\pi \mathrm{i}\int_{S^1}\gamma^*\widehat{A}} \end{array}$$

Up to modernized language, this is the original observation of [Dirac1931] (cf. [Al85][Fr97, §16.4e] [Fr00, §2]).

The lesson is that:

> The usual differential forms entering the Lagrangian densities of (higher) gauge fields are not the full field content of the theory: Non-perturbatively, fields subsume maps to a classifying space, making the fields be cocycles in (generalized) differential cohomology thus enforcing a flux-quantization law on the differential form data.

We next explain how this works.